Bell's Theorem

Bachelor thesis Mathematics and Physics

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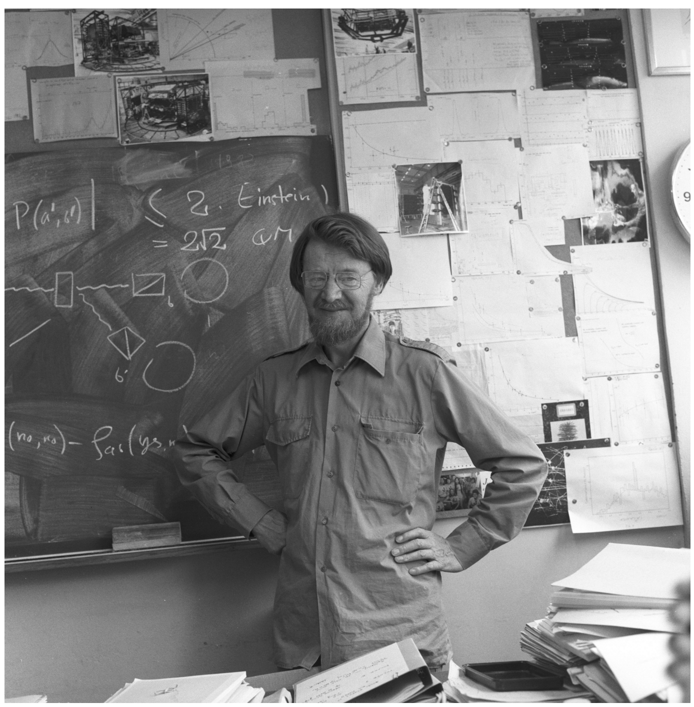

John Stewart Bell, 1982 [18].

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1 Introduction

Quantum mechanics is strange. But it works, scientifically as well as in practice. This appears, for example, from the correct predictions regarding atomic spectra and the omnipresent transistor. Perhaps quantum mechanics is right, but just not complete and maybe the missing pieces of the theory would make everything fall into place and would make quantum mechanics a bit less strange.

This, in a nutshell, is what led to the idea of 'hidden variables'. A hidden variable is supposed to give us a more precise description of the physical state of particles than quantum mechanics currently does. In 1964, John Bell published an article in which he showed that certain types of hidden variables would lead to a contradiction. This result is known as Bell's theorem and this is the central topic of this bachelor's thesis.

We will start by motivating the question *is quantum mechanics complete?* in order to show that quantum mechanics isn't just confusing for a bachelor student who first encounters concepts like true randomness and Heisenberg's uncertainty principle, but that there are genuine philosophical problems. We will give a short description of some of these problems and how they lead to the question of completeness in section 2.

Next, we will discuss an article from 1935 by Albert Einstein, Boris Podolsky and Nathan Rosen. Not only does Bell's article historically go back to the article of Einstein, Podolsky and Rosen (or, as the article is generally referred to, EPR), but also some of the concepts used in Bell's theorem are introduced in EPR. After having discussed this in section 3, we will turn to Bell's theorem in section 4.

Bell's theorem considers a hypothetical hidden variable theory which satisfies certain assumptions, which we have called Determinism, Independence and Locality. These assumptions roughly say that the hidden variable fixes the outcome of a measurement deterministically, that the hidden variable is independent of which specific measurement is performed and that two spatially seperated particles cannot influence each other instantaneously. From a classical point of view, these assumptions seem very reasonable. However, Bell's theorem shows that such a hidden variable theory *cannot* reproduce the results that are predicted by quantum mechanics.

In section 5 we try to strengthen Bell's theorem by showing that Independence and Locality are equivalent. Unfortunately, we have no succes here, and in the end it seems more likely that they are not equivalent after all.

Finally, some calculations on the quantum system we consider multiple times can be found in appendix A and some notes on special relativity and what it has to say about causality can be found in appendix B.

This thesis is written for the combined mathematics and physics bachelor program at the Radboud University. I would like to make a few remarks: one about the physics, one about the mathematics, and one about the writing.

Quantum mechanics is puzzling in many ways. As a student I often struggled with this; we learned how to use quantum mechanics, but I never felt like I really understood anything. Regarding this, reading and thinking about Bell's theorem and in particular the historical context preceding Bell's theorem was very helpful. I feel that it helped in gaining some understanding of quantum mechanics.

It may not be immediately obvious where the mathematics is in this thesis. I do not use, for example, any functional analysis, or other 'serious' mathematics - only a little bit of calculus and probability theory. The mathematics is mainly present, I would say, in the analytical approach of the subject: the precise definitions and the attempt to find out how various notions (such as relativity, locality and no-signalling) are exactly related.

Finally, I tried to write in such a way that the ideas involved are well-motivated. In the unlikely case that another bachelor student will read this, I hope this is a readable text.

2 The question of completeness

In the title of their 1935 article, Einstein, Podolsky and Rosen ask whether quantum mechanics can be considered as a complete description of physical reality [7]. For what reasons did they pose this question? In an article which, among other things, discusses the historic background of EPR, the philosopher Arthur Fine writes:

> 'Initially Einstein was enthusiastic about the quantum theory. By 1935, however, while recognizing the theory's significant achievements, his enthusiasm had given way to disappointment. His reservations were twofold. Firstly, [...] [t]he theory was simply silent about what, if anything, was likely to be true in the absence of observation. [...] Secondly, the quantum theory was essentially statistical. [...] Thus Einstein began to probe how strongly the quantum theory was tied to irrealism and indeterminism.' [9]

We will take a (very quick) look at the issues of irrealism and indeterminism and consider why we might desire determinism and realism in a physical theory. The following is not a historical account of how Einstein thought about these matters, but is merely meant to point out two philosophical problems that motivate the question *is quantum mechanics complete?*[1]

2.1 Determinism

Determinism has a certain appeal in physics, mainly because of the idea that everything happens for some reason, because of a certain cause. More precisely, the idea is that everything that happens, *must* happen and that all alternative happenstances are *impossible*. Being used to classical physics, it is difficult to imagine nature not deterministically; how can anything 'just' happen? If we perform an experiment and we get an unexpected result, we immediately ask ourselves *why did we get this result?* and we certainly expect this question to have an answer. If the result would be truly random, then (in the words of Nicolas Gisin, although he doesn't argue in favour of determinism):

> '[The result] is not predictable because, before it came into being, it just did not exist, it was not necessary, and it realisation is in fact an act of pure creation.' [11]

Hence a truly random result would seem to be nothing less than a small ontological miracle - because of this, one might expect that nature should be deterministic. However, in quantum mechanics there are, of course, random results. Thus you could wonder whether these random results are *truly* random or just seemingly so. In the latter case, this could be explained by a hidden variable theory that is deterministic but also reproduces the probabilistic results of quantum mechanics.

2.2 Realism

In quantum mechanics, the state of a particle is represented by its wave function. This allows us to model systems and predict outcomes of experiments. However, quantum mechanics is notoriously vague on what the actual *physics* of the particle is. It would seem fair to demand that there is something *real* that is represented by a wave function. But what is, for example, an atom? Is it a point-like particle, a wave-like particle, or something else? The fact that

[1] Although the following is a general discussion and I did not cite any specific results, I should note that writing this I used several articles from the Stanford Encyclopedia of Philosophy: Fine's article on EPR [9], an article on Bohmian mechanics [12], an article on the uncertainty principle [13] and an article on causal determinism [14].

quantum mechanics is quite silent on these matters, suggests that there might be a more complete discription of the state of particles than given by the wave function.

Something that is closely connected to this, is that in quantum mechanics a particle has no well-defined position and velocity at the same time. If a particle is something real (as opposed to just a theoretical model to explain certain physical phenomena), then you would expect it to be always at a particular point in spacetime and hence that it always has a certain position and velocity. Therefor one might wonder, is there a theory which reproduces the results of quantum mechanics and in which also every particle has, at any point in time, a certain position and velocity?

It may seem that proposing a theory in which particles have a certain position and velocity at the same time outright contradicts quantum mechanics, because of Heisenberg's uncertainty principle, but this doesn't need to be the case.[2] Suppose we measure subsequently the velocity and position of a particle. When the position is measured, the particle one way or another needs to affect our measurement device - for if the particle would leave the device unchanged, how would the device know the particle is there? The interaction can't leave the particle unchanged, because the effect from the particle on the measurement device requires a countereffect from the device on the particle. So during the measurement, the state of the particle changes; the measurement of the position may have altered the previously measured velocity. In this way, the particle *is* at a certain position with a certain velocity at any point in time, but it is impossible to *know* both the position and velocity.

Both the fact that quantum mechanics is not clear about what a particle really is and the fact that not all physical quantities of a particle have definite values, suggest that quantum mechanics might be incomplete.

2.3 Hidden variables

Certainly not everyone believes that quantum mechanics is incomplete, that physics should be about realism and determinism,[3] or that there exist hidden variables. If you suppose that quantum mechanics is actually complete, how do you answer to the above arguments for determinism and realism? There are as many answers as there are possible interpretations of quantum mechanics; this is beyond the scope of this thesis. The point is that there are good reasons to at least consider the possibility of hidden variables. Furthermore, Bell's theorem, the main topic of this thesis, is interesting regardless of whether you believe quantum mechanics is complete or not. If you believe that quantum mechanics is complete, than you might try to refine Bell's theorem in such a way that it rules out hidden variables alltogether; if you do have interest in hidden variable theories, Bell's theorem narrows down in which direction you should look.

[2]Note that the following is just an interpretation - one of several - of the uncertainty principle. There is no consensus about what the correct interpretation of the principle is, just as there is no consensus on the interpretation of quantum mechanics as a whole. For an article on the interpretation of the uncertainty principle (focusing on the views of Heisenberg and Bohr), see [13].

[3]Personally, I believe that determinism has no real advantage over probabilistic physics. I tried to sketch why random results are problematic, but philosophers have pointed out long ago that the idea of deterministic causality is also problematic. I suspect that any theory which tries to explain what physically interactions are fundamentally about, will run into philosophical issues.

However, irrealism I do consider a real problem. For if quantum mechanics is not about anything real, what is it about? You could claim that quantum mechanics is only a epistemic theory or that it is only about measurement, but supposedly there is something real behind what we know and measure. Yet what this real physics is, is not clear.

3 The Einstein-Podolsky-Rosen argument

EPR refers to the famous paper of Einstein, Podolsky and Rosen from 1935, titled 'Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?' In their article they conclude that 'the description of reality as given by a wave function is not complete' [7]. There are various versions of their argument formulated by, among others, Podolsky, Einstein, Bohr and Bell [9]. The argument as we will describe it is based on the version of Bell, which appeared in his 1964 paper. We follow this version instead of the original article (or any other version) because Bell's version is very clear. The argument goes as follows.

> **Argument of EPR.** Suppose we have a pair of electrons in the following (unnormalized) state:
> $$\Psi = |\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle \,. \tag{1}$$
> Furthermore, suppose we prepare the state Ψ in such a way that the electrons are moving in opposite directions. We wait until the electrons have moved some distance apart and measure the spin of the first electron. By doing so, we also get to know the spin of the second electron, since it is the same as the spin of the first.[4] Say, for example, the measurement on the first electron tells us that both electrons have spin up. Because the electrons are separated from each other by some distance, the measurement on the first electron cannot have had any influence on the second electron (at least not instantaneously). Thus the second electron must have been in a spin up state all along. The state Ψ doesn't give this information, so apparently it is an incomplete description of the system. We must conclude that the quantum mechanical description of the pair of electrons is incomplete.

The argument above may seem quite simple, but there are some subtleties that should be noted. Firstly, there is the fact that the two electrons are entangled. Secondly, there is an assumption being made regarding locality. The notions of entanglement and locality are not only important for EPR, but also for Bell's theorem, so we will discuss them with some care.

3.1 Entanglement

The pair of electrons we considered, is in a very specific state. Not every combined state of two electrons would work. Suppose, for example, that the two electrons are in the state $\Psi = (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle)$. In this case, a measurement of the spin of the first particle would tell us nothing about the spin of the second particle. For the argument of EPR, we specifically need an *entangled* state, that is, a combined state such that neither of the particles can be considered on their own.

Two particles are called entangled when their states can't be described independently from each other. We can describe entanglement mathematically using the tensor product. If one particle occupies a state in the Hilbert space H_1 and the other particle occupies a state in H_2, then the combined system of the two particles is a state in $H_1 \otimes H_2$. The combined state Ψ is called separable if we can write it as a product of individual states, that is, if $\Psi = \psi_1 \otimes \psi_2$ for some $\psi_1 \in H_1$ and some $\psi_2 \in H_2$. The combined state is called entangled if it can't be written as such, so if $\Psi \neq \psi_1 \otimes \psi_2$.

[4] The spin of the particles is in the same direction, regardless of the direction in which we measure the spin. The only condition is that the directions in which the spin is measured are the same for both particles. In other words, for an arbitrary orthonormal basis $|\rightarrow\rangle$, $|\leftarrow\rangle$ we have: $\Psi = |\rightarrow\rangle \otimes |\rightarrow\rangle + |\leftarrow\rangle \otimes |\leftarrow\rangle$. We prove this in example A.3.

3.2 Locality

In the way that quantum mechanics describes entanglement, it seems that an event at one place can have an instantaneous consequence at another place. If we think of the state (1) again, then we don't know whether the second particle has spin up or spin down. In fact, as we have just seen, we can't even describe the spin of one particle without mentioning the other particle as well. But if we would measure the spin of the the first particle and find that it has spin up, then we suddenly know that the second particle too has spin up - the state of the second particle seems to have changed instantaneously as a consequence of a measurement happening elsewhere. This kind of action is called non-local.

However, in EPR it's assumed that there can only be local action. In other words, they assume the particles can't influence each other instantaneously, because they are spatially separated. To see why this assumption is relevant, let's assume that non-local action *is* possible. If we measure the spin of the first electron and it turns out to be in a spin up state, then the second electron is also in a spin up state, but we *cannot* conclude that the second electron was in that state all along. The reason for this is that the measurement on the first particle may very well have influenced the state of the second particle, since we assumed non-local action is possible. So the assumption of locality in EPR is really necessary for their conclusion.

Is the assumption of locality in EPR right? Is nature local? One might point out that the theory of special relativity (SRT) rules non-locality out, since it prohibits any causal influence to travel faster than light.[5] However, locality in the sense of EPR and locality in the sense of SRT both have very specific meanings within their own context and so you can't directly infer something about EPR-locality from something about SRT-locality. (In the statement of Bell's theorem, we will give a mathematical definition of locality, so that there won't be any confusion as to what locality precisely means.) In fact - and perhaps surprisingly - it remains an open question whether relativity and quantum theory are compatible, because relativity suggests there is some kind of locality while quantum mechanics suggests there is some kind of non-locality.[6] In this discussion, the idea of 'no-signalling' plays an important role.

No-signalling

We will get briefly into no-signalling to explain the tension between quantum mechanics and relativity a little bit more and because we will need the concept later on in section 5.1. By signalling we simply mean sending some information from one point to another, but in the context of (non-)locality, we often specifically mean sending information faster than the speed of light. Thus the no-signalling principle, or in short just: 'no-signalling', states that super-luminal signalling is impossible.

If super-luminal signalling would be possible, it would surely violate SRT because it gives rise to certain paradoxes. Suppose that Alice sends Bob a message faster than the speed of light. Then the act of sending and the act of receiving would be two space-like seperated events and hence some observers would see Bob receiving a message before Alice has even sent it.

As we have already seen, the entangled state $\Psi = |\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle$ gives rise to some kind of non-local action (at least, in the current standard formulation of quantum mechanics - additional hidden variables might restore locality). Does that mean we can use this state Ψ to signal faster than light? To answer this, we first need to figure out how to describe signalling mathematically.

[5] This is explained in appendix B.

[6] For a discussion of this problem see for example the article 'Can quantum theory and special relativity peacefully coexist?' from Michiel Seevinck [19].

Suppose Alice tries to send a message to Bob faster than light using the state Ψ. Both of them measure one of the electrons in order to do this. Alice measures the spin of her electron in the direction a, Bob measures the spin of his electron in the direction b, and their results are f for Alice and g for Bob. Alice and Bob both know that the inital state of the electrons is Ψ and that is everything - they somehow need to use the variables a, b, f and g to send a message. Only one of those variables is controlled by Alice (her setting a); so if Bob receives a signal from Alice, that means he receives some information about the variable a. The only way for Bob to be able to attain this information would be if his result g somehow depends not only on his own setting b but also on Alice her setting a. More precisely, we can put this as follows.

> **Mathematical expression of signalling.** Bob can receive a signal from Alice (faster than the speed of light using the entangled state Ψ) if and only if the probability $P(\sigma_b = g|a, b)$ explicitly depends on a.

Here σ_b is the operator corresponding to Bob's measurement. Thus $P(\sigma_b = g|a, b)$ is the probability that Bob's measurement yields the result g, given the fact that Alice and Bob used the settings a and b. Note that the probability $P(\sigma_b = g|a, b)$ is a *marginal* probability that can be obtained from the *joint* probability as follows:

$$P(\sigma_b = g|a, b) = \sum_{f=\pm 1} P(\sigma_a = f, \sigma_b = g|a, b).$$

A simple calculation (done in example A.4 in the appendix) shows that

$$P(\sigma_b = +1|a, b) = P(\sigma_b = -1|a, b) = \frac{1}{2},$$

so the probabilities do *not* depend on a and hence it is *not* possible to use the entangled state Ψ to signal faster than the speed of light. Fortunately: there is no apparent contradiction between quantum mechanical non-locality and relativity, at least concerning this point.

Some authors go further and argue that because of this 'no-signalling property' there is no issue with the compatibility of quantum non-locality and relativity at all. One could argue for this by claiming that the no-signalling property is equivalent to the fact that no super-luminal causal influences are possible - which led us to the issue of compatibility in the first place.

$$\text{no super-luminal causal influences} \overset{?}{\Longleftrightarrow} \text{no super-luminal signalling}$$

The idea behind this is simple. Suppose a super-luminal causal influence is possible. Then presumably we could use this influence to send a message. For example, if we were able to switch a light on a distant planet on and off instantaneously (a super-luminal causal influence) then we could send a message using short and long light pulses (morse code). Conversely, suppose we would be able to send a message faster than light. Now again, presumably, there would be some kind of physical mechanism that transmits the message (like an electromagnetic wave, but then faster). This mechanism would need to work faster than light, hence there would be super-luminal causal influences.

However, there are problems with this. The above argument is very hand-waving and, as Seevinck puts it: 'it is highly questionable that special relativity is inextricably bound up with the impossibility of transmitting messages faster than the speed of light' [19]. Furthermore, some authors have objected that the no-signalling theorems in the literature (which try to prove that quantum entanglement cannot lead to super-luminal signalling regardless of the exact setup) are not completely sound. Seevinck writes that 'such theorems in fact presuppose locality of measurements and are therefore circular' [19] and according to John Bernard Kennedy 'the proofs assume the validity of the tensor product representation, and derivations of the tensor product representation assume the impossibility of signalling' [15]. Indeed, the calculation we have used in our no-signalling proof (exampe A.4) uses the tensor product as well.

Thus, as we said earlier, it remains an open problem whether quantum mechanics and relativity are compatible. For example, Joseph Berkovitz writes that 'the question of the compatibility of quantum mechanics with the special theory of relativity is very difficult to resolve' [3] and Seevinck starts his article by saying: 'This white paper aims to identify an open problem [...] - namely whether quantum theory and special relativity are formally compatible.' [19]

Back to EPR

To sum up, EPR shows using an entangled state that the following implication is true:

$$\text{QM is EPR-local}^7 \rightarrow \text{QM is incomplete.}$$

In EPR it is also assumed that quantum mechanics is local and therefore it is concluded that quantum mechanics is incomplete. However, nowadays most people think that quantum mechanics is actually non-local (because of Bell's theorem, which we will get to in a moment). Berkovitz, for example, says: 'Following Bell's work, a broad consensus has it that the quantum realm involves some type of non-locality' [3]. Still, EPR is of relevance because - among other things - it raises the problem of locality in the first place and it shows the questions about locality and completeness are connected. Bell's theorem, to which we will turn now, pursues this idea.

[7] That is: the measurement on one particle cannot influence another particle instantaneously.

4 Bell's theorem

The article from Einstein, Podolsky and Rosen was published in 1935. According to Fine, in the following years 'the EPR paradox was discussed at the level of a thought experiment whenever the conceptual difficulties of quantum theory became an issue' [9]. Almost thirty years later, in 1964, John Bell published an article called 'On the Einstein Podolsky Rosen paradox' in which he showed that certain types of hidden variable theories are impossible. In the introduction of his article, Bell writes:

> 'The paradox of Einstein, Podolsky and Rosen was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty.' [1]

We see that Bell's theorem does indeed go back to EPR and that Bell considered locality to be essential. It should be noted, however, that in Bell's theorem more assumptions are being made besides locality, so we should be careful with singling out locality as being at odds with quantum mechanics.

Bell's theorem actually refers to a family of similar results. There are two main versions of Bell's theorem, one concerning deterministic local hidden variables and the other concerning stochastic local hidden variables. We will focus on the first one, but also look at the second. Whenever we talk about Bell's theorem without further specification, we mean the first theorem.

Let us try to finally state Bell's theorem. Our formulation of the theorem follows that of Klaas Landsman [16, chapter 6], written in mathematical fashion (that is, every idea is defined precisely and the proofs are rigorous). At first glance, it might seem that the theorem is easier when it is somewhat more loosely stated in a more physical fashion, such as Bell himself does. However, the advantage of being very precise is that it becomes very clear what one's assumptions are - which is important, especially when dealing with subtle notions such as determinism and locality. In order not to loose ourselves in the details of definitions, we will first state and prove a preliminary version of Bell's theorem, in which we brush a few assumptions under the carpet. Later, we will make these asssumptions explicit.

4.1 The gist of Bell's theorem

Suppose we set up an experiment similar to the thought experiment in EPR. Again we will use pairs of electrons in the state

$$\Psi = |\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle \, ,$$

but this time we will measure the spin in different directions. Alice, who measures the spin of the first electron, will measure the spin with an angle A with respect to the z-axis. She has two choices for the angle; $A = a_1$ or $A = a_2$. Likewise, Bob, who measures the spin of the second electron can choose an angle $B = b_1$ or $B = b_2$. Together, A and B are the settings for the experiment. The outcomes of the experiment are denoted by F for Alice and G for Bob, so $F, G \in \{+1, -1\}$.

Quantum mechanics predicts that the outcomes for this entangled state are correlated as follows (for the calculations, see appendix A):

$$P(F = G|A, B) = \cos^2\left(\frac{A-B}{2}\right),$$

$$P(F \neq G|A, B) = \sin^2\left(\frac{A-B}{2}\right).$$

Now, suppose there is a hidden variable behind this experiment with the following properties:

- **Determinism.** There is a hidden variable Z taking values in the set X_Z, on which we have a probability measure μ. The hidden variable together with the settings completely determines the outcome of the experiment. Mathematically, this means there are functions

$$F_{A,B} : X_Z \to \{0, 1\} \text{ and}$$
$$G_{A,B} : X_Z \to \{0, 1\}.$$

- **Locality.** Bob's setting shouldn't influence Alice's measurement and vice versa:

$$F_{A,B} = F_A \text{ and } G_{A,B} = G_B.$$

- **Nature.** The outcomes of the experiment are correlated just as quantum mechanics predicts:

$$P_Z(F_A = G_B) = \cos^2\left(\frac{A-B}{2}\right),$$

$$P_Z(F_A \neq G_B) = \sin^2\left(\frac{A-B}{2}\right).$$

  These probabilities are induced by the measure μ on X_Z, for example:

$$P_Z(F_A = G_B) = \int_{X_Z} (\mathbb{1}_{F_A=0}\mathbb{1}_{G_B=0} + \mathbb{1}_{F_A=1}\mathbb{1}_{G_B=1})d\mu.$$

  Here the function $\mathbb{1}_{F_A=0}$ equals 1 if $F_A(z) = 0$ and equals 0 otherwise.

Now we can state Bell's theorem.

Theorem 1 (Bell's theorem, preliminary version)**.** *A hidden variable theory satisfying Determinism, Locality and Nature is impossible.*

Proof. Suppose that there is such a hidden variable theory. Determinism and Locality together give us functions $F_A, G_B : X_Z \to \{0, 1\}$. In the description of the experiment we mentioned that Alice and Bob could both choose two angles, so we have the functions $F_{a_1}, F_{a_2}, G_{b_1},$ and G_{b_2}. These four functions are all random variables on the space X_Z and take values in $\{+1, -1\}$. We claim (and will soon prove, see lemma 1) that these random variables, according to probability theory, must satisfy the following inequality, known as Bell's inequality:

$$C \equiv P(F_{a_1} = G_{b_1}) + P(F_{a_1} = G_{b_2}) + P(F_{a_2} = G_{b_1}) + P(F_{a_2} \neq G_{b_2}) \leq 3.$$

However, if we use Nature to evaluate the expression for C, we would get:

$$C = \cos^2\left(\frac{a_1-b_1}{2}\right) + \cos^2\left(\frac{a_1-b_2}{2}\right) + \cos^2\left(\frac{a_2-b_1}{2}\right) + \sin^2\left(\frac{a_2-b_2}{2}\right).$$

If we choose $a_1 = b_1 = 0$, $a_2 = 1/3\pi$ and $b_2 = 5/3\pi$, this becomes:

$$C = \cos^2(0) + \cos^2\left(\frac{5}{6}\pi\right) + \cos^2\left(\frac{1}{6}\pi\right) + \sin^2\left(\frac{4}{6}\pi\right) = 3.25.$$

We have a contradiction and this proves the theorem. □

Lemma 1. *Let (X, Σ, μ) be a probability space and let F_1, F_2, G_1 and G_2 be any random variables on this probability space taking values in $\{0,1\}$, that is:*

$$F_1, F_2, G_1, G_2 : X \to \{0,1\}.$$

These random variables satisfy:

$$C \equiv P(F_1 = G_1) + P(F_1 = G_2) + P(F_2 = G_1) + P(F_2 \neq G_2) \leq 3.$$

Proof. We can write C more explicitly as follows:

$$\begin{aligned}
C = &P(F_1 = 0, G_1 = 0) + P(F_1 = 1, G_1 = 1) \\
&+ P(F_1 = 0, G_2 = 0) + P(F_1 = 1, G_2 = 1) \\
&+ P(F_2 = 0, G_1 = 0) + P(F_2 = 1, G_1 = 1) \\
&+ P(F_2 = 0, G_2 = 1) + P(F_2 = 1, G_2 = 0)
\end{aligned}$$

$$\begin{aligned}
= \int_X \big(&\mathbb{1}_{F_1=0}\mathbb{1}_{G_1=0} + \mathbb{1}_{F_1=1}\mathbb{1}_{G_1=0} \\
&+ \mathbb{1}_{F_1=0}\mathbb{1}_{G_2=0} + \mathbb{1}_{F_1=1}\mathbb{1}_{G_2=0} \\
&+ \mathbb{1}_{F_2=0}\mathbb{1}_{G_1=0} + \mathbb{1}_{F_2=1}\mathbb{1}_{G_1=0} \\
&+ \mathbb{1}_{F_2=0}\mathbb{1}_{G_2=1} + \mathbb{1}_{F_2=1}\mathbb{1}_{G_2=0}\big)d\mu.
\end{aligned}$$

There are sixteen possible combined outcomes for (F_1, F_2, G_1, G_2). Each of these outcomes corresponds to a subset of X. For example, if all the random variables are zero, then the corresponding subset is $X_1 = \{x \in X | F_1(x) = 0, F_2(x) = 0, G_1(x) = 0, G_2(x) = 0\}$. If we evaluate the above integral for this subset, we find that it is equal to

$$\int_{X_1} 3 d\mu$$

and the same is true for each of the sixteen subsets, as can easily be checked. Thus we have:

$$C = \int_X 3 d\mu \leq 3.$$

□

The idea behind Bell's theorem

Imagine Alice and Bob going to a restaurant every evening for dinner. (I borrow this story from Gisin [11].) They go to restaurants in different cities. Alice lives in Falmouth and goes to restaurant F_1 or F_2; Bob lives in Glasgow and goes to restaurant G_1 or G_2. In each of the restaurants only two dishes are served: fish and chips, and curry. Every evening each restaurant serves only one dish, chosen seemingly randomly, but it's always either fish and chips or curry.

Suppose that Alice and Bob always get the same dish if Alice goes to restaurant F_1 and Bob goes to restaurant G_1. Apparently the two restaurants made an agreement to always serve the same dish; they both serve fish and chips or they both serve curry. Now suppose Alice and Bob also get the same dish if Alice goes to restaurant F_1 and Bob goes to restaurant G_2, or if Alice goes to F_2 and Bob G_1. Apparently all four restaurants made an agreement to serve the same dish; it would seem impossible that Alice and Bob would get different dishes if they went to F_2 and G_2. In other words, it seems no less than reasonable that the above correlation C is at most 3.

Back to the physics. In Bell's theorem Alice and Bob don't live in different cities but they measure the spin of different electrons. They don't choose between restaurants but between settings of the experiment, which comes down to choosing between the functions F_{a_1} and F_{a_2} for Alice and between G_{b_1} and G_{b_2} for Bob. Whatever they choose, they always get a seemingly random result - not fish and chips or curry, but $+1$ or -1. The physical system is analogous to our little story and therefore we might expect that $C \leq 3$ is also true for the experiment with the electrons. Indeed, in the proof of Bell's theorem we have shown that, if the electrons behave according to certain seemingly reasonable assumptions (Determinism and Locality), then the Bell inequality $C \leq 3$ must hold.

Yet quantum mechanics predicts and experiments confirm that in nature the Bell inequality is violated. In Bell's theorem we use the the assumption Nature to prove that the electrons can acquire $C > 3$. We have a contradiction.

Note that Bell's theorem is very general: it shows that *any* theory that describes nature correctly must violate at least one of the assumptions used to derive Bell's inequality.

4.2 Many assumptions

As we mentioned before, there are a few implicit assumptions in the above version of Bell's theorem. We will now take a more careful look at the assumptions in Bell's theorem. Our discussion will be based mainly on an article from Angel Valdenebro in which he tries to 'identify and discuss a complete set of assumptions valid to prove Bell's inequality.' [20]

Determinism

Our first assumption was Determinism and came down to the existence of functions F and G which determined the outcomes of the experiment. There are three assumptions which underly the existence of these functions, and only one of them is determinism[8]:

1. Realism
2. Determinism
3. Outcomes are single valued

Realism means, in the words of Valdenebro, that 'the outcome of a measurement is not created by the measurement, but corresponds to properties possessed by the measured system prior to the measurement.' [20] The functions F and G do indeed ensure that the outcomes are fixed before the measurement is actually performed (and thus the outcome is not created by the measurement).

Secondly, F and G of course also ensure that the outcome of the experiment is deterministic, in the sense that the outcome 'has a sufficient reason for being and being as it is, and not otherwise.' [14]

[8]Note that with determinism I mean the general term, while with Determinism I refer specifically to the assumption we defined as Determinism. The same goes for locality/Locality, etcetera.

The last assumption may come as a surprise. An electron can be in a superposition of spin up and spin down, but surely if we *measure* the spin it has only one value. However, in the many-worlds interpretation a measurement can have multiple outcomes [8]. Therefore the fact that F and G only give one outcome at a time is not completely trivial, but is an assumption.

I believe all three assumption are necessary to account for the existence of the functions F and G. However, one could argue that realism implies determinism - at least in the way these notions are defined here.

Independence

'Independence' is an assumption we haven't seen yet. It is connected with Determinism.

Continuing on the matter of determinism, if we are consistent, we should demand that also the settings and the variable Z are determined. That would mean that rather than just having functions F and G for the outcomes, we should also have functions A and B for the settings and a fucntion Z. The functions A, B, Z, F and G should all be defined on some space X (with a probability measure μ) which consists of possible states of the system. The system encompasses not only the entangled electrons but also the measurement devices; in this way the hidden variable theory (the states $x \in X$ are the hidden variables) describes the entire experiment including its settings, not just the entangled pair of electrons.

But now we run into trouble. What if A, B and Z are not stochastically independent? The variables A, B and Z could be correlated in such a way that:

- if the outcomes of Alice and Bob are the same, more often than not one of them would have chosen to measure a_1 or b_1;
- and if the outcomes of Alice and Bob are different, more often than not both of them would have chosen to measure a_2 and b_2.

This would allow the correlation

$$C = P(F_{a_1} = G_{b_1}) + P(F_{a_1} = G_{b_2}) + P(F_{a_2} = G_{b_1}) + P(F_{a_2} \neq G_{b_2})$$

to become larger than 3. So in order to derive the Bell inequality $C \leq 3$, we need to assume that A, B and Z are independent. (To be more specific, the pair (A, B) and Z need to be independent, as will become clear in the proof of Bell's theorem.)

What could cause the settings and Z to be dependent? It could be that a certain value of Z restricts the choices of Alice and Bob. In order for the settings and Z to be independent, Alice and Bob must thus be able to choose their settings freely.

This is of course suspicious: first we claimed that A and B are fixed deterministically and now we claim that Alice and Bob can freely choose their settings. This leads to an extensive philosophical debate, which is beyond the scope of this thesis. There are fortunately reasons to believe that our situation is not immediately contradictory. For example, Landsman and Valdenebro argue in favour of a form of compatibilism [16] [20], the latter of which writes that 'determinism does not forbid the existence of *stochastically* independent events' [20].

There is something else that could cause dependence, although it may seem very far-fetched: backward causation. If the settings would influence Z (which is, presumably, fixed before the settings), then the settings and Z would not be independent. The reason we mention backward causation is the same as why we mentioned the possibility of multi-valued outcomes: there is an interpretation of quantum mechanics in which there is backward causation, namely the transactional interpretation [6]. Note that backward causation could also be used to obtain $C > 3$ without causing (A, B) and Z to be dependent: the outcomes could have influence on the settings.

To sum up, Independence assumes that the settings (A, B) and the variable Z are dependent. This implies that Alice and Bob can freely choose their settings and there is no backwards causation.

Locality

Locality assumes that Bob's setting cannot influence Alice her result and vice versa. There are no 'hidden' assumptions behind this. In our discussion of locality in section 3.2 we already saw that SRT motivates this assumption and once we have formulated Locality mathematically we will consider the relation between Locality and SRT more detailed in section 4.4.

Nature

Nature is a bit different from the other assumptions in the way it is used. Determinism, Independence and Locality are used to prove Bell's inequality. Nature is subsequently used to show that Bell's inequality is violated - which leads to a contradiction.

Nature states that the outcomes are correlated as predicted by quantum mechanics. Just as with Locality, there are no hidden assumptions behind Nature. However, it is useful to take a look at the motivation behind the assumption of Nature.

Firstly, even though quantum mechanics might not be complete, it is generally assumed that the predictions of quantum are correct. In many cases this has been verified experimentally. Therefor it seems reasonable to assume that also the correlation between the entangled pair of electrons as predicted by quantum mechanics is correct, especially since the derivation of this correlation uses only quite elementary quantum mechanics.

If you are not convinced by this, there is a second way to motivate Nature: by experiment. You could simply carry out the thought experiment and see if this does indeed yield the familiar answer $P(F = G|a, b) = \cos^2\left(\frac{a-b}{2}\right)$.

I say simply, but in practice it is not so simple. In order for the experiment to satisfy Locality, choosing a setting on Alice's side and getting a result on Bob's side need to be spacelike seperated events (and vice versa). If this is not the case, Alice her setting could in theory influence Bob's outcome (via an unknown mechanism). You could imagine that even if nature would actually satisfy Bell's inequality, the inequality could still be exceeded in the experiment because of the influence of Alice's setting on Bob's result. This is called the locality loophole.

There is a second loophole: the detection loophole. The detection should be sufficiently efficient, so that it is impossible that the experimental finding of $C > 3$ could be caused by the fact that detection is more likely for certain states. However, 'typically, in experiments involving photons, most of the pairs produced fail to enter the analyzers. Furthermore, some photons that enter the analyzers will fail to be detected; in addition, the detector will occasionally register a detection even when no photon is detected.' [17]

Since these issues are only important in practical experiments and give no further insight in the theoretical understanding of Bell's theorem, we won't go into the practical details and merely quote from a recent article by Wayne Myrvold, Marco Genovese and Abner Shimony, who give an overview of Bell's theorem:

> 'Beginning in the 1970s, there has been a series of experiments of increasing sophistication to test whether the Bell inequalities are satisfied. [...] Until recently [...] each of these experiments was vulnerable to at least one of two loopholes, referred to as the communication, or locality loophole, and the detection loophole [...]. Finally, in 2015, experiments were performed that demonstrated violation of Bell inequalities with these loopholes blocked.' [17]

4.3 A formal statement of Bell's theorem

Keeping the preceding in mind, we can formulate the assumptions mathematically and reformulate Bell's theorem.

- **Determinism.** There is a set X of states with a probability measure μ. On this set, we have functions determining the settings and outcomes of the experiment:

$$
\begin{aligned}
&A : X \to X_A = \{a_1, a_2\}, \\
&B : X \to X_B = \{b_1, b_2\}, \\
&F : X \to \{+1, -1\} \text{ and} \\
&G : X \to \{+1, -1\}.
\end{aligned}
$$

- **Independence.** There is a variable taking values in X_Z which together with the settings determines the outcomes of the experiment; that is, there are functions

$$
\begin{aligned}
&Z : X \to X_Z, \\
&\hat{F} : X_A \times X_B \times X_Z \to \{+1, -1\} \text{ and} \\
&\hat{G} : X_A \times X_B \times X_Z \to \{+1, -1\},
\end{aligned}
$$

  such that $F = \hat{F} \circ (A, B, Z)$ and $G = \hat{G} \circ (A, B, Z)$. The random variables (A, B) and Z are stochastically independent.

- **Locality.** The setting of one measurement shouldn't influence the outcome of the other; there are functions

$$
\begin{aligned}
&\tilde{F} : X_A \times X_Z \to \{+1, -1\} \text{ and} \\
&\tilde{G} : X_B \times X_Z \to \{+1, -1\},
\end{aligned}
$$

  such that $\forall b \in X_B : \hat{F}(a, b, z) = \tilde{F}(a, z)$ and $\forall a \in X_A : \hat{G}(a, b, z) = \tilde{G}(b, z)$.

- **Nature.** The outcomes of the experiment are correlated just as quantum mechanics predicts:

$$
\begin{aligned}
P_X(F = G | A = a, B = b) &= \cos^2\left(\frac{a-b}{2}\right), \\
P_X(F \neq G | A = a, B = b) &= \sin^2\left(\frac{a-b}{2}\right).
\end{aligned}
$$

Theorem 2 (Bell's Theorem)**.** *A hidden variable theory satisfying Determinism, Independence, Locality and Nature is impossible.*

Proof. Notice that the probability measure μ on X induces another probability measure on X_Z in such a way that

$$P_Z(z) = P_X(Z = z) = P_X(Z^{-1}(\{z\})).$$

Using this, we can define the following random variables on X_Z that represent the outcomes:

$$
\begin{aligned}
\tilde{F}_a(z) &= \tilde{F}(a, z), \\
\tilde{G}_b(z) &= \tilde{F}(b, z).
\end{aligned}
$$

We need to show that these random variables are correlated as

$$P_Z(\tilde{F}_a = \tilde{G}_b) = P_X(F = G|A = a, B = b). \tag{2}$$

If we can show this, then we can finish the proof as in the preliminary version; if we again define

$$C \equiv P_Z(\tilde{F}_{a_1} = \tilde{G}_{b_1}) + P_Z(\tilde{F}_{a_1} = \tilde{G}_{b_2}) + P_Z(\tilde{F}_{a_2} = \tilde{G}_{b_1}) + P_Z(\tilde{F}_{a_2} \neq \tilde{G}_{b_2}),$$

then from lemma 1 it would follow that $C \leq 3$ and from Nature it would follow that $C > 3$, with the right choices for a_1, a_2, b_1 and b_2, which proves the theorem.

So if we can prove equation (2), we are done. Equation (2) follows from

$$P_Z(\tilde{F}_a = f, \tilde{G}_b = g) = P_X(F = f, G = g|A = a, B = b), \tag{3}$$

and equation (3) we will now prove. To start, we have:

$$\begin{aligned} P \coloneqq P_Z(\tilde{F}_a = f, \tilde{G}_b = g) &= P_Z(\{z \in X_Z | \tilde{F}_a(z) = f, \tilde{G}_b(z) = g\}) \\ &= P_Z(\{z \in X_Z | \hat{F}(a, b, z) = f, \hat{G}(a, b, z) = g\}). \end{aligned}$$

Here we have used the definition of $\tilde{F}_a$ and the fact that $\tilde{F}_a(z) = \hat{F}(a, b, z)$ for any b (and similarly for $\tilde{G}_b$ of course). If we now go from the probability space X_Z to the probability space X, this means:

$$P = P_X(Z \in \{z \in X_Z | \hat{F}(a, b, z) = f, \hat{G}(a, b, z) = g\}).$$

Since (A, B) and Z are independent, we may write this as:

$$P = \frac{P_X(Z \in \{z \in X_Z | \hat{F}(a, b, z) = f, \hat{G}(a, b, z) = g\}, A = a, B = b)}{P_X(A = a, B = b)},$$

which simplifies to

$$P = \frac{P_X(F = f, G = g, A = a, B = b)}{P_X(A = a, B = b)}.$$

To see why this is true, notice that

$$\begin{aligned} & x \in Z^{-1}\{z \in X_Z | \hat{F}(a, b, z) = f, \hat{G}(a, b, z) = g\} \cap A^{-1}(\{a\}) \cap B^{-1}(\{b\}) \\ \iff & \hat{F}(a, b, Z(x)) = f, \hat{G}(a, b, Z(x)) = g, A(x) = a, B(x) = b \\ \iff & \hat{F}(A(x), B(x), Z(x)) = f, \hat{G}(A(x), B(x), Z(x)) = g, A(x) = a, B(x) = b \\ \iff & F(x) = f, G(x) = g, A(x) = a, B(x) = b. \end{aligned}$$

Finally, we have

$$P = P_X(F = f, G = g|A = a, B = b).$$

□

4.4 The relation between Locality and SRT

In section 3.2 we have already looked at the tension between non-locality in quantum mechanics and SRT. Let us consider Locality and its relation with SRT. Casey Blood wrote an article in which he '[derives] Bell's locality condition from the relativity of simultaneity' [4]. In another article Gisin does essentially the same [10].

The arguments of Blood and Gisin come down to the following. Suppose that Alice's result is determined by a hidden variable, her setting, and possibly also Bob's setting. Furthermore, suppose the experiment is set up in such a way that Alice's measurement and Bob's measurement are spacelike seperated events. Then we could go to a frame in which Alice measures *before* Bob. It follows from this that her result cannot depend on Bob's setting after all.

It takes some care to make the argument of Blood and Gisin more precise. To begin with, suppose we have a hidden variable theory that satisfies Determinism, Independence and Nature. Suppose the measurements of Alice and Bob - performed at (x_A, t_A) and (x_B, t_B) - are spacelike seperated. Note that, although we talk about a spacelike seperation, we don't assume anything from special relativity: purely mathematically we can define the two events to be spacelike seperated if $(x_A - x_B)^2 - c^2(t_A - t_B)^2 > 0$.

Now we can state what we assume regarding relativity.

> **Relativity.** There are two frames of reference denoted by $\mathcal{S}$ and $\mathcal{S}'$ moving with respect to each other such that:
>
> 1. The coordinates $\vec{\boldsymbol{x}} = (x, y, z, ct)$ of the two frames are related by $\vec{\boldsymbol{x}}' = \Lambda\vec{\boldsymbol{x}}$, where Λ is a Lorentz transformation;
> 2. Alice measures before Bob in $\mathcal{S}$ and Bob measures before Alice in $\mathcal{S}'$;
> 3. The hidden variable theory is applicable in both frames.

The third assumption together with Determinism and Independence implies that the outcomes f and g in $\mathcal{S}$ and f' and g' in $\mathcal{S}'$ of the experiment are given by:

$$f = \hat{F}(a, b, z), \qquad f' = \hat{F}(a', b', z'),$$
$$g = \hat{G}(a, b, z), \qquad g' = \hat{G}(a', b', z').$$

At first sight, the settings might be different in the two frames. However, if we make sure that the direction in which the spin is measured lies in a plane perpendicular to the axis along which the frames move, then if follows from the first assumption that the settings are the same (see figure 1). Hence, we have:

$$f = \hat{F}(a, b, z), \qquad f' = \hat{F}(a, b, z'),$$
$$g = \hat{G}(a, b, z), \qquad g' = \hat{G}(a, b, z').$$

Finally, the second assumption together with the 'no backwards causation' imply that f cannot depend on b and g' cannot depend on a. In other words, $\hat{F}(a, b, z)$ is the same for every b and therefor we can define $\tilde{F}(a, z) := \hat{F}(a, b, z)$ and similarly $\tilde{G}(b, z') := \hat{G}(a, b, z')$. So we have functions $\tilde{F}$ and $\tilde{G}$ such that:

$$\forall b \in X_B : \hat{F}(a, b, z) = \tilde{F}(a, z),$$
$$\forall a \in X_A : \hat{G}(a, b, z') = \tilde{G}(b, z').$$

This is exactly what Locality states.

Corollary 1. *Determinism, Independence, Relativity and Nature are contradictory.*

A few notes are in order. Firstly, it may seem that corollary 1 above is a much nicer result that Bell's theorem (theorem 2). If we would choose to leave out one of the assumptions of Bell's theorem to avoid the contradiction, the choice would be between Determinism, Independence and Locality.[9] If we would choose to leave out one assumption of the corollary, one might be

[9] Formally, Nature could also be picked as the culprit, but Nature is quite well established.

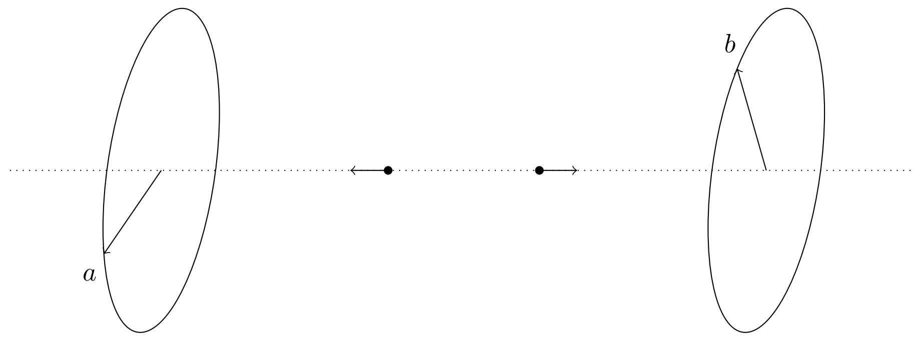

Figure 1: Two spin-half particles move apart from each other. The frames $\mathcal{S}$ and $\mathcal{S}'$ move along the dotted axis (say, the x-axis). The measurements are chosen in planes perpendicular to this axis. A Lorentz transformation only affects the x-coordinates and the time-coordinates, and this doesn't change the directions of the settings.

tempted to say that the choice is between Determinism and Independence - accepting Relativity as beyond doubt. However, Relativity is of course not completely undoubtable, at least not when considering what would fundamentally be possible or impossible. Therefor the corollary is not a 'better' result than Bell's theorem. In fact, looking at it from a purely logical angle, the corollary is a weaker statement, because Relativity is a stronger assumption than Locality: Relativity implies Locality, but Locality does not necessarily imply Relativity. The reason for including this corollary even though it is a weaker statement than Bell's theorem itself, is that conceptually it gives a bit more insight into what Bell's theorem means.

Secondly, note that the above argument hinges on the 'no backward causation' assumption and the fact that the functions $\hat{F}$ and $\hat{G}$ can be used regardsless whether Alice or Bob measures first. We could thus state Relativity differently without explicitly mentioning any inertial frames or transformations:

> **Relativity (II).** The hidden variable theory is applicable regardsless of who measures first.

Suppose Alice measures first. No backwards causation implies that Alice's result cannot depend on Bob's setting. That is, for every b in X_B the value of $\hat{F}(a, b, z)$ is the same. Thus Alice's result is local. Similarly, Bob's result it local.

This way of reasoning is certainly easier than the argument we gave above, but I am not convinced that it is also a better argument. This seems to capture the essence of the more elaborate argument above, but if someone would ask *why* the hidden variable theory is applicable regardsless of the time ordering, I think you still need Relativity (I) as an explanation.

4.5 Bell's second theorem

As we mentioned before, there is a second version of Bell's theorem. This theorem goes back to an article by Bell from 1975 [2] and also shows that a certain type of hidden variables is impossible. Now the assumption of Determinism is dropped and Locality is changed to 'Bell Locality'. We have focused on Bell's first theorem, mainly because we are interested in Locality and Independence as formulated in the first theorem, but for completeness we should also take a look at the second theorem. Our formulation and proof is once again based on the work of Landsman [16].

We no longer assume Determinism; A, B, F and G are no longer functions, but just used to denote the settings and outcomes. Also Z is no function, but just a random variable. We assume that both the marginal and joint probabilities are known (presumably by experiment).

To be more precise, we now have the following assumptions:

- **Stochastic Variables.** There is a variable Z on which the probabilities of certain outcomes depend. The variable takes values in the space X_Z with probability measure μ. The following probabilities are given:

$$\begin{aligned}&P(F = f, G = g|A = a, B = b, Z = z),\\&P(F = f|A = a, Z = z),\\&P(G = g|B = b, Z = z).\end{aligned} \tag{4}$$

- **Nature.** Without Z, we have the following probabilities:

$$\begin{aligned}P(F = G|A = a, B = b) &= \cos^2(\frac{a-b}{2}),\\P(F \neq G|A = a, B = b) &= \sin^2(\frac{a-b}{2}).\end{aligned}$$

- **Independence.** The probability measure is independent of the settings; for all settings we have:

$$\begin{aligned}&P(F = f, G = g|A = a, B = b)\\&\qquad = \int_{X_Z} P(F = f, G = g|A = a, B = b, Z = z)d\mu.\end{aligned}$$

- **Bell Locality.** The experiment is local in the following sense:

$$\begin{aligned}&P(F = f, G = g|A = a, B = b, Z = z)\\&\qquad = P(F = f|A = a, Z = z) \cdot P(G = g|B = b, Z = z).\end{aligned}$$

Note that Stochastic Variables assumes nothing physically speaking. We could let Z be the configuration of the stars and find the probabilities (4) by repeating the experiment many times. The physical assumptions necessary to derive Bell's inequality are entirely contained in Independence and Bell Locality. The reason we did include Stochastic Variables as an assumption is that mathematically speaking the existence of the space X_Z, the measure μ and the probabilities (4) *is* an assumption.

Theorem 3 (Bell's second theorem)**.** *A hidden variable theory satisfying Stochastic Variables, Nature, Independence and Bell Locality is impossible.*

Proof. Once again, we want to have four functions $\tilde{F}_{a_1}$, $\tilde{F}_{a_2}$, $\tilde{G}_{b_1}$ and $\tilde{G}_{b_2}$ on the same probability space $\tilde{X}$, so we can use lemma 1 to prove $C \leq 3$, where C is given by:

$$C \equiv P_{\tilde{X}}(\tilde{F}_{a_1} = \tilde{G}_{b_1}) + P_{\tilde{X}}(\tilde{F}_{a_1} = \tilde{G}_{b_2}) + P_{\tilde{X}}(\tilde{F}_{a_2} = \tilde{G}_{b_1}) + P_{\tilde{X}}(\tilde{F}_{a_2} \neq \tilde{G}_{b_2}).$$

Without Determinism, we don't have functions F and G, so we need to define the above space and functions ourselves. We do this as follows:

$$\begin{aligned}&\tilde{X} = [0,1] \times [0,1] \times X_Z\\&d\tilde{\mu}(s,t,z) = ds \cdot dt \cdot d\mu(z)\end{aligned}$$

So the unit intervals are equiped with the Lebesgue measure and $d\tilde{\mu}$ is induced by the product. On this space, we define

$$\begin{aligned}
\tilde{F}_a(s,t,z) &= +1 \qquad \text{if } s < P(F=1|A=a, Z=z) \text{ and} \\
\tilde{F}_a(s,t,z) &= -1 \qquad \text{if } s > P(F=1|A=a, Z=z).
\end{aligned}$$

The random variable $\tilde{G}_b(s,t,z)$ is defined in the same way.

Now we have the desired functions and together with lemma 1 it follows that we have $C \leq 3$. Next, we need to show that

$$P_{\tilde{X}}(\tilde{F}_a = \tilde{G}_b) = P(F = G|A = a, B = b), \tag{5}$$

because this together with Nature implies that $C > 3$ (for the right choices of a_1, a_2, b_1 and b_2).

Thus to complete the proof we need to show that

$$P \coloneqq P_{\tilde{X}}(\tilde{F}_a = f, \tilde{G}_b = g) = P(F = f, G = g|A = a, B = b),$$

from which equation 5 follows. By definition, we have:

$$P = \int_{\tilde{X}} \mathbb{1}_{\tilde{F}_a = f} \mathbb{1}_{\tilde{G}_b = g} d\tilde{\mu}(s,t,z).$$

Using the definitions of $\tilde{F}_a$ and $\tilde{G}_b$, the integral becomes:

$$P = \int_X P(F = f|A = a, Z = z) \cdot P(G = g|B = b, Z = z) \cdot d\mu(z).$$

Finally, if we use subsequently Bell Locality and Independence, we get:

$$\begin{aligned}
P &= \int_X P(F = f, G = g|A = a, B = b, Z = z) d\mu(z) \\
&= P(F = f, G = g|A = a, B = b).
\end{aligned}$$

$\square$

5 Are Locality and Independence equivalent?

Probably not.

Before trying to answer the question though, let us look at why it makes sense to pose the question in the first place. After all, at first sight Locality and Independence seem two very different things. The surmise that they actually are the same comes from a few paragraphs in the book *Bananaworld* by Jeffrey Bub. Originally, the goal of this thesis was to prove this surmise. This would allows us to strengthen Bell's theorem by removing one of the assumptions. Yet in the end an example came up which suggests (but does not prove entirely) that Locality and Independence are *not* equivalent.

5.1 The surmise

Consider, once again, the situation in which Alice and Bob measure the spin of a pair of electrons in the state $\Psi = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$. Would it be possible to *clone* one of the electrons? That is, is it possible to somehow prepare another electron in the exact same state of, say, the electron on which Bob performs his measurement, such that if we measured the spin of both the copy and the original, their spin would always be the same? Bub shows that given either the no-signalling principle or the principle of free choice (that is, the principle that Alice and Bob can freely choose their settings), this is impossible [5, chapter 4].

Suppose Alice tries to send Bob a message by using cloning. They agree in advance that if Alice measures the spin in the z-direction, that corresponds to sending a 0, and the y-direction corresponds to a 1. Bob can find out in which direction Alice measured the spin. First, he clones his electron many times. Then, he repeatedly measures the spin in the z-direction using the copies. If Alice signalled a 0, the result will always be the same; if Alice signalled a 1, the result will be ± 1 with probability $1/2$. Thus cloning leads to a violation of the no-signalling principle.

If Alice and Bob are far apart and measure quickly after each other, the two measurements are space-like seperated and we can switch to a reference frame in which Bob performs his measurement *before* Alice. If Bob gets the same result over and over again while measuring the spin of all the clones, then Alice is *forced* to measure in the z-direction - or it is at least very likely that she does so. Therefor this time cloning leads to a violation of the principle of free choice.

It seems that no-signalling and free choice are two sides of the same coin and that we can switch between them by switching between reference frames. This is what led to the surmise that, given special relativity, Locality (akin to no-signalling) and Independence (akin to free choice) are equivalent under a suitable relativistic assumption. However, the following shows that this is probably not the case.

5.2 A counterexample?

We can formulate a concrete model for the electron's spin with a hidden variable that satisfies Independence but not Locality. Accidently, this model was made up by Bell and served as an example of a hidden variable theory in his 1964 article [1].

Let us see how the model works. Consider an electron prepared in a spin up state $\vec{\boldsymbol{\Psi}}_0$. Suppose we want to measure the spin in some direction $\vec{\boldsymbol{a}}$ with an angle θ from $\vec{\boldsymbol{\Psi}}_0$ (see figure 2).

We know that the probability for the measurement of $A := \vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}}$ to yield $+1$ is $\cos^2 \theta/2$. As a first try to reproduce this result using a 'hidden' variable, we may introduce a unit vector $\vec{\boldsymbol{\lambda}}$ in the hemisphere $\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{\Psi}}_0 > 0$. Assume that $\vec{\boldsymbol{\lambda}}$ is uniformly distributed. Furthermore, suppose

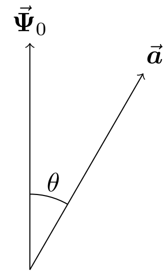

Figure 2: The setup.

that $A = +1$ if $\vec{\boldsymbol{\lambda}}$ is closer to $\vec{\boldsymbol{a}}$ than to $-\vec{\boldsymbol{a}}$ and that $A = -1$ otherwise. In other words: $A = \operatorname{sign}(\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{a}})$.

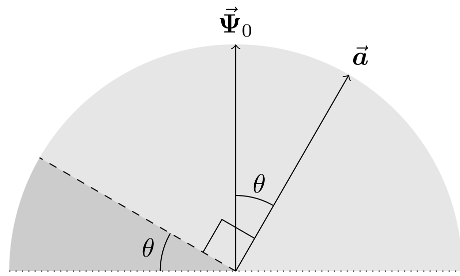

Figure 3: If $\vec{\boldsymbol{\lambda}}$ lies in the lighter grey, $A = +1$. If $\vec{\boldsymbol{\lambda}}$ lies in the darker grey, $A = -1$. The probability that, for example, $A = -1$ is given by the length of the dark grey arc divided by the length of the whole hemisphere, that is: $P = \theta/\pi$.

Looking at figure 3 above, one can see that $P(A = -1) = \theta/\pi$ and $P(A = +1) = 1 - \theta/\pi$. This isn't the right result, but it *does* give a probability as a function of θ using a hidden variable that determines the outcome of the measurement. Now we only need to change the function of θ slightly in order to obtain the correct result; for this we introduce another vector $\vec{\boldsymbol{a}}'$ in the model.

Let $\vec{\boldsymbol{a}}'$ be the unit vector rotated from $\vec{\boldsymbol{\Psi}}_0$ in the direction of $\vec{\boldsymbol{a}}$ with an angle θ', where

$$\theta' = \pi(1 - \cos^2 \theta/2).$$

Let the measurement be determined by

$$A = \operatorname{sign}(\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{a}}').$$

So the measurement is determined just like before, except that we changed $\vec{\boldsymbol{a}}$ to $\vec{\boldsymbol{a}}'$. Similarly, we now have:

$$P(A = +1) = 1 - \theta'/\pi.$$

If we substitue θ', we get the desired result:

$$P(A = +1) = 1 - (1 - \cos^2 \theta/2) = \cos^2 \theta/2.$$

There is a similar model for two entangled electrons. Suppose we measure their spins in the directions $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$. This time θ is the angle between $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{\lambda}}$ is uniformly distributed over all directions. The angle θ' is again given by $\theta' = \pi(1 - \cos^2 \theta/2)$ and the unit vector b' is obtained by a rotation with angle θ' from $\vec{\boldsymbol{a}}$. Finally, the measurements are determined as follows:

$$A \coloneqq \vec{\boldsymbol{\sigma}}_1 \cdot \vec{\boldsymbol{a}} = \operatorname{sign}(\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{a}}),$$
$$B \coloneqq \vec{\boldsymbol{\sigma}}_2 \cdot \vec{\boldsymbol{b}} = \operatorname{sign}(\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{b}}').$$

Note the slight asymmetry here. Because $\vec{\boldsymbol{\lambda}}$ is chosen uniformly over the whole circle, the marginal probabilities $P(A = \pm 1)$ and $P(B = \pm 1)$ are $1/2$, as they should be. Moreover, the joint probabilities are also as they should be. For example, as can be seen in figure 4:

$$P(A = +1, B = -1) = \frac{\theta'}{2\pi} = \frac{1}{2}(1 - \cos^2 \theta/2).$$

From symmetry (again, see figure 4) it then follows that:

$$P(A \neq B) = 1 - \cos^2 \theta/2 = \sin^2 \theta/2 \text{ and}$$
$$P(A = B) = \cos^2 \theta/2.$$

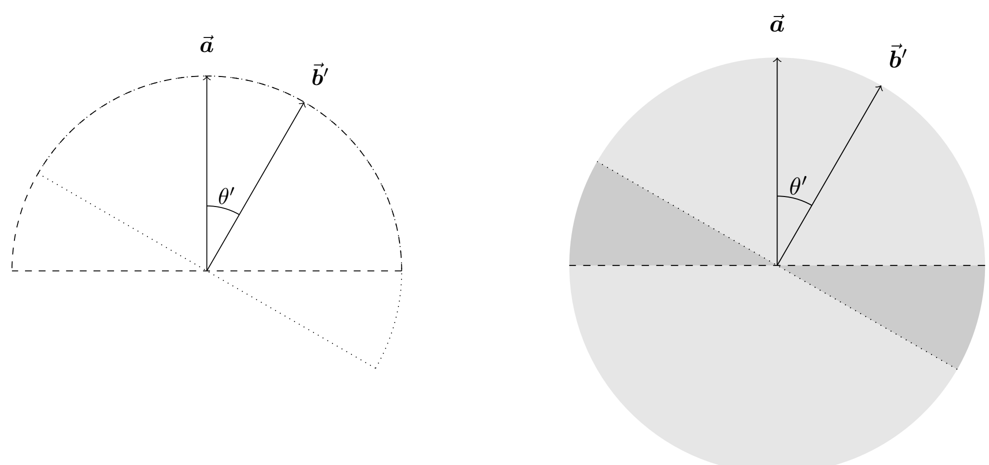

Figure 4: A representation of model and its probabilities. The left side is about the marginal probabilities: if $\vec{\boldsymbol{\lambda}}$ lies on the dashed hemishere, then $A = +1$ and if $\vec{\boldsymbol{\lambda}}$ lies on the dotted hemishere, then $B = +1$. The right side is about the joint probabilities: if $\vec{\boldsymbol{\lambda}}$ lies on the arc denoted by the lighter area on the top half, then $(A, B) = (+1, +1)$. If $\vec{\boldsymbol{\lambda}}$ lies on the arc denoted by the darker area on the right half, then $(A, B) = (-1, +1)$. *Et cetera.*

In this model the settings and the hidden variable are independent: they aren't related in any way. Thus the model satisfies Independence. However, the model is *not* Local, as the outcome B depends on θ' which in turn depends on both $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.

To complete this model as a counterexample, we should formulate it according to the definitions in section 4.3. The underlying state space X would simply be the Cartesian product of the settings and the hidden variable. The functions A, B and Z then are the projections and the functions F and G are defined by $\operatorname{sign}(\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{a}})$ and $\operatorname{sign}(\vec{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{b}}')$ as above. Notice that the functions F and G on X are actually the same as the functions $\hat{F}$ and $\hat{G}$ on $X_A \times X_B \times X_Z$.

We have given a counterexample that satisfies Determinism, Independence and Nature but does not satisfy Locality. Yet this does not prove entirely that Independence and Locality are not the same in the context of Bell's theorem. Mathematically, the counterexample suffices, but physically we may demand that the model should satisfy other properties as well to be realistic. Most importantly: this model is not invariant under Lorentz transformations. For this reason I would say that the above counterexample suggests, but does not prove, that Independence and Locality are not equivalent.

6 Conclusion and discussion

We have seen it may be worth considering if quantum mechanics can be extended with hidden variables, both because it's unclear what quantum mechanics is really about ontologically, and in order to resolve the apparent tension between quantum mechanics and SRT. Bell's theorem shows that the space for modifications is limited.

Can we draw more specific conclusions from Bell's theorem? It is alluring to single out one of the assumptions used to the derive Bell's inequality and to say that nature does not satisfy this specific assumption. That would be a very fundamental result.

Most authors seem to suspect that locality is wrong.[10] As we have already seen, Bell himself thought locality likely to be the wrong assumption: 'It is the requirement of locality [...] that creates the essential diffculty.' [1] Myrvold, Genovese and Shimony seem to agree with this: 'The distinctive assumption is the Principle of Local Causality, [...]. This condition can be maintained only if one of the supplementary assumptions is rejected.' [17]

Some authors are more confident and don't express any doubt. For example, Blood says that 'Bell showed theoretically, and the Aspect experiment confirmed that there could be no hidden variable theory which satisfied a locality condition,' [4] and according to Gisin, 'local variables cannot describe the quantum correlations observed in tests of Bell.' [10]

However, other authors are more cautious. Fine writes: '[Since] assumptions other then locality are needed in any derivation of the Bell inequalities [...] one should be cautious about singling out locality [...] as necessarily in conflict with the quantum theory, or refuted by experiment.' [9] Landsman writes: 'Hence Bell Locality is violated by quantum mechanics, but this does not imply that "quantum mechanics is nonlocal" (as some say). Bell's is a very specific locality condition invented as a constraint on hidden variable theories.' [16] Valdenebro remarks neutrally: 'Therefore, every interpretation of QM must consider one of these assumptions, at least, false.' [20]

Clearly, there is no consensus on this matter. I have tried to think of reasons to reject one assumption rather than the others grounded on what we have discussed in the previous sections, but didn't manage to come up with anything satisfactory. Perhaps a more thorough search in the literature dealing with the implications of Bell's theorem would bring something to light.

If the majority is right and nature is indeed non-local, then this leads to another issue: how can we reconcile this with SRT? We briefly discussed this in section 3.2. In section 4.4 we saw that Locality and SRT are connected (in particular, we showed that Determinism, Independence and Relativity together imply Locality), but how the two are *exactly* related is way beyond the scope of this thesis - and, as we mentioned before, this is an open problem.

What about the initial goal of this bachelor thesis, to prove the equivalence of Independence and Locality? Once again we can't say anything definitive, but the example of a model using hidden variables in section 5.2 suggests they are not equivalent and hence that, unfortunately, we cannot use this to strengthen Bell's theorem by reducing the number of assumptions.

Perhaps the main point - at least from the perspective of a bachelor student - of this thesis is not any specific result, but rather to get an acquaintance with some of the more philosophical issues surrounding quantum mechanics. I believe that learning about these issues doesn't make life more difficult, but actually helps to get a better feeling for quantum mechanics.

[10] It is not always clear which kind of locality one means: Locality as in Bell's first theorem, Bell Locality as in Bell's second theorem, or something else? Since Bell's first theorem uses specifically deterministic hidden variables, presumably most authors are thinking of Bell Locality.

A Spin

We will take a look at spin one-half particles and derive the mathematical results used in the thesis. It is well known that the operators corresponding to the spin in the direction of x, y or z is given by $S = \frac{1}{2}\hbar\sigma$, where σ is one of the Pauli spin operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here we have chosen, as is customary, to use the eigenvectors of S_z as a basis. Those eigenvectors are

$$|{\uparrow}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |{\downarrow}\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $|{\uparrow}\rangle$ corresponds to the eigenvalue $+1$ (or spin up) and $|{\downarrow}\rangle$ corresponds to -1 (or spin down).

In general, if one measures the spin in the direction $\vec{\boldsymbol{a}}$ (where $\vec{\boldsymbol{a}}$ is a unit vector), the corresponding operator is:

$$\vec{\boldsymbol{S}} \cdot \vec{\boldsymbol{a}} = \frac{1}{2}\hbar\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}} = \frac{1}{2}\hbar(a_x\sigma_x + a_y\sigma_y + a_z\sigma_z).$$

We need to find explicit expressions for $\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}}$ and it's eigenvectors (the factor $\frac{1}{2}\hbar$ is not important for our purposes, so we'll just leave it out). We can write any unit vector as

$$\vec{\boldsymbol{a}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},$$

where θ and ϕ are the usual sperical angles. Then $\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}}$ becomes:

$$\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{pmatrix}.$$

With a little bit of linear algebra you can find that the eigenvalues of this matrix are $+1$ and -1 and that the corresponding (unnormalized) eigenvectors are

$$\begin{pmatrix} e^{-i\phi}\sin\theta \\ 1-\cos\theta \end{pmatrix} \text{ and } \begin{pmatrix} -e^{-i\phi}\sin\theta \\ 1+\cos\theta \end{pmatrix}.$$

By using the trigonometric formulas $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\cos\theta = 1 - 2\sin^2(\theta/2)$, we can normalize and rearrange these vectors to obtain

$$|+\rangle \equiv \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \text{ and } |-\rangle \equiv \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix},$$

where the state $|+\rangle$ corresponds to the eigenvalue $+1$ (or spin up) and the state $|-\rangle$ corresponds to -1 (or spin down). Thus $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ are the eigenstates of σ_z and $|+\rangle$ and $|-\rangle$ are the eigenstates of $\vec{\boldsymbol{\sigma}} \cdot \vec{\boldsymbol{a}}$.

Example A.1. Before discussing entangled states, let's consider an example with just one particle, say, an electron. Suppose we prepare the electron in the state

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Next, we measure the spin in a direction rotated with an angle θ with respect to the z-axis. (We can always choose the coordinate system such that $\phi = 0$.) What is the probability that the measurement will give $+1$ as a result? Well, the eigenstate corresponding to this value is

$$|+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

and hence the probability is given by:

$$P(\sigma_\theta = +1) = |\langle +|\Psi_0\rangle|^2 = \cos^2\frac{\theta}{2}.$$

Example A.2. Now let's consider a pair of spin one-half particles in an entangled state:

$$\Psi_0 = \frac{|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle}{\sqrt{2}}.$$

We measure the spin σ_a of the first particle with an angle a with respect to the z-axis and the spin σ_b of the second particle with an angle b with respect to the z-axis (see figure 5). The eigenstates of σ_a and σ_b are denoted by $|+_a\rangle$, $|-_a\rangle$ and $|+_b\rangle$, $|-_b\rangle$ (see figure 5). The basis vectors are related in the following way (and similarly for b):

$$|\uparrow\rangle = \cos\frac{a}{2}|+_a\rangle - \sin\frac{a}{2}|-_a\rangle \text{ and } |\downarrow\rangle = \sin\frac{a}{2}|+_a\rangle + \cos\frac{a}{2}|-_a\rangle.$$

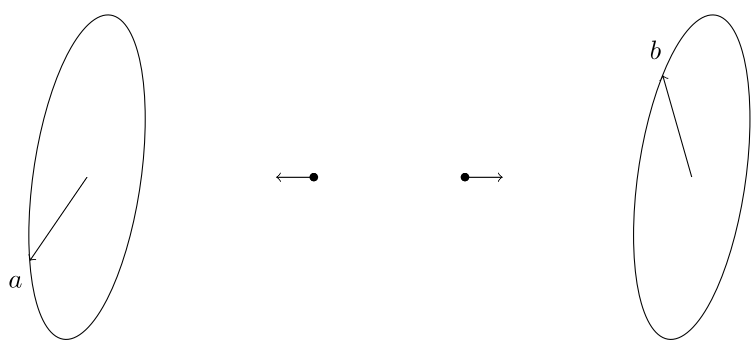

Figure 5: Two spin-half particles move apart from each other. The spin of both particles is being measured. If we chose the settings a and b in parallel planes then we can once again choose a coordinate system such that $\phi = 0$.

What is the probability that, for example, the first particle has spin up and the second has spin down? In terms of the eigenstates of σ_a and σ_b, Ψ_0 is given by:

$$\begin{aligned}
\sqrt{2}\cdot\Psi_0 = &\big(\cos\frac{a}{2}\cos\frac{b}{2} + \sin\frac{a}{2}\sin\frac{b}{2}\big)|+_a\rangle \otimes |+_b\rangle \\
&+ \big(\sin\frac{a}{2}\cos\frac{b}{2} - \cos\frac{a}{2}\sin\frac{b}{2}\big)|+_a\rangle \otimes |-_b\rangle \\
&+ \big(\cos\frac{a}{2}\sin\frac{b}{2} - \sin\frac{a}{2}\cos\frac{b}{2}\big)|-_a\rangle \otimes |+_b\rangle \\
&+ \big(\sin\frac{a}{2}\sin\frac{b}{2} + \cos\frac{a}{2}\cos\frac{b}{2}\big)|-_a\rangle \otimes |-_b\rangle.
\end{aligned} \tag{6}$$

If we project Ψ_0 on the state $|+_a\rangle \otimes |-_b\rangle$ we find that:

$$P(\sigma_a = +1, \sigma_b = -1) = \frac{1}{2}\left(\sin\frac{a}{2}\cos\frac{b}{2} - \cos\frac{a}{2}\sin\frac{b}{2}\right)^2.$$

Example A.3 (EPR)**.** In the thought experiment of EPR we measure the spin of both particles in the same direction. If this direction is rotated with some angle θ from the z-axis, are we still bound to find the particles with equal spin, or is it possible for them to have opposite spin? It turns out the spin of the particles must be the same, both up or both down, regardless of the direction in which we measure the spin. You can see this by taking equation 6 together with $\theta \equiv a = b$. The two middle terms drop out and the rest simplifies to:

$$\Psi_0 = \frac{|+\rangle \otimes |+\rangle + |-\rangle \otimes |-\rangle}{\sqrt{2}}.$$

Example A.4 (No-signalling)**.** We want to show that

$$P \equiv P(\sigma_b = g|a, b) \equiv \sum_{f=\pm 1} P(\sigma_a = f, \sigma_b = g|a, b)$$

is independent of a. In the case of $g = +1$, we would get:

$$P = |\langle +_a +_b |\Psi_0\rangle|^2 + |\langle -_a +_b |\Psi_0\rangle|^2,$$

where we use the notation $|+_a +_b\rangle \equiv |+_a\rangle \otimes |+_b\rangle$. Using equation 6 again and a little trigonometry, we see that this equals:

$$P = \frac{1}{2}\cdot\left(\cos\frac{a}{2}\cos\frac{b}{2} + \sin\frac{a}{2}\sin\frac{b}{2}\right)^2 + \frac{1}{2}\cdot\left(\cos\frac{a}{2}\sin\frac{b}{2} - \sin\frac{a}{2}\cos\frac{b}{2}\right)^2 = \frac{1}{2},$$

so $P(\sigma_b = g|a, b)$ is indeed independent of a.

Example A.5 (Bell's theorem)**.** The probabilities we use in the assumption Nature can be derived using equation 6:

$$\begin{aligned} P(\sigma_a = \sigma_b) &= P(\sigma_a = +1, \sigma_b = +1) + P(\sigma_a = -1, \sigma_b = -1) \\ &= |\langle +_a +_b |\Psi_0\rangle|^2 + |\langle -_a -_b |\Psi_0\rangle|^2 \\ &= \left(\cos\frac{a}{2}\cos\frac{b}{2} + \sin\frac{a}{2}\sin\frac{b}{2}\right)^2 \\ &= \cos^2\left(\frac{a}{2} - \frac{b}{2}\right). \end{aligned}$$

In the last step we used one of the trigonometric sum formulas. The probability that σ_a and σ_b don't give the same result is of course just $1 - P(\sigma_a = \sigma_b)$, so we have the following results:

$$\begin{aligned} P(\sigma_a = \sigma_b) &= \cos^2\left(\frac{a-b}{2}\right), \\ P(\sigma_a \neq \sigma_b) &= \sin^2\left(\frac{a-b}{2}\right). \end{aligned} \tag{7}$$

B Special relativity & causality

Let us show that causal influences cannot travel faster than the speed of light.

To start off we will introduce some notation and state some well-known facts about special relativity. A point in space-time is given by a four-vector with three spatial components and one time component:

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = (x, y, z, ct).$$

The scalar product of the four-vectors is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4,$$

and the distance squared between two points in space-time is given by $(\mathbf{x}-\mathbf{y})^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})$. In particular, the length squared of a four-vector is: $s = \mathbf{x} \cdot \mathbf{x}$.

The coordinates of two inertial frames $\mathcal{S}$ and $\mathcal{S}'$ can be related using the Lorentz transformation. Suppose a certain event (say, a small explosion) happens at the point $\mathbf{x}$ in $\mathcal{S}$ and at $\mathbf{x}'$ in $\mathcal{S}'$; furthermore, suppose $\mathcal{S}'$ is traveling with speed v with respect to $\mathcal{S}$ in the positive x-direction. Then the space-time coordinates are related as follows:

$$\mathbf{x}' = \Lambda \mathbf{x},$$

where Λ is called the Lorentz transformation and is given by:

$$\Lambda \equiv \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \text{ with } \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } \beta \equiv \frac{v^2}{c^2}.$$

The scalar product is invariant under Lorentz transformations, that is, $\mathbf{x}' \cdot \mathbf{y}' = \mathbf{x} \cdot \mathbf{y}$. We can use this to divide space-time into five regions, which we can show in a space-time diagram. Suppose you are at the origin $\mathcal{O}$ in an inertial frame $\mathcal{S}$. The vertical ct-axis shows the time and the horizontal xy-plane shows the position. (We'll forget about the z-direction for now, because we can't draw a fourth dimension.) If you hold up a light source, the light will move away from you with, of course, the speed of light; the path of the light through space-time satisfies $x^2 + y^2 = (ct)^2$ and therefor marks a cone in the space-time diagram. Similarly, if light reaches you from somewhere else, then the light will have reached you by traveling along a cone. A point on one of the cones is called **light-like** seperated from the origin and the two cones are called respectively the **future light cone** and the **past light cone** and their tips meet at the origin. See figure 6. Mathematically, we can describe the light cones as follows:

$$\text{future light cone} = \{\text{points in space-time such that } s = 0 \text{ and } t > 0\},$$
$$\text{past light cone} = \{\text{points in space-time such that } s = 0 \text{ and } t < 0\}.$$

A point inside the future light cone is a future event that you could reach by traveling with a speed less than the speed of light. Likewise, a point inside the past light cone is a past event from which you could have traveled, slower than light, to your present point in space-time, the origin. For every event $\mathbf{x}$ inside one of the light cones, there exists another inertial frame $\mathcal{S}'$ in which the event $\mathbf{x}'$ lies on the time-axis, that is, $\mathbf{x}' = (0, 0, 0, ct')$. Therefore the interior of the cones are called respectively the **absolute future** and **absolute past** and an event inside one of the cones is called **time-like** seperated from the origin. Mathematically, these are described as:

$$\text{absolute future} = \{\text{points in space-time such that } s < 0 \text{ and } t > 0\},$$
$$\text{absolute past} = \{\text{points in space-time such that } s < 0 \text{ and } t < 0\}.$$

Finally, there is the exterior of the light cones, simply called **elsewhere**, which is made up of all events in the future or past that are too far away to reach without exceeding the speed of light. In fact, for each event $\mathbf{x}$ outside the light cones there is an inertial frame $\mathcal{S}'$ in which the event $\mathbf{x}'$ is only spatially seperated from the origin, that is, $\mathbf{x}' = (x', y', z', 0)$, hence an event in the elsewhere region is called **space-like** seperated from the origin. Mathematically, we have:

$$\text{elsewhere} = \{\text{points in space-time such that } s > 0\}.$$

It is important to note that, although special relativity messes with our notions of space and time because of time dilation, length contraction and the relativity of simultaneity, the above five regions in space-time are invariant under Lorentz transformations. That means that, for example, a point on the future light cone in $\mathcal{S}$ lies on the future light cone in *every* inertial frame $\mathcal{S}'$. So all possible observers will agree on which events are space-like, light-like or time-like seperated. Even more importantly, if two events are time-like seperated, observers will agree on which event precedes the other.

As an example, we will prove that the absolute future is invariant under Lorentz transformations. That the other four regions are also invariant can be proved in a similar fashion.

Proposition 1. *Suppose that $\boldsymbol{x}$ is an event in the absolute future in the inertial frame $\mathcal{S}$. If $\mathcal{S}'$ is another inertial frame, than $\boldsymbol{x}'$ is also in the absolute future.*

Proof. We can assume that $\mathcal{S}'$ is moving in the x-direction with respect to $\mathcal{S}$. By definition of the absolute future, we know that $\mathbf{x} \cdot \mathbf{x} < 0$ and $t > 0$. Since the scalar product is invariant under Lorentz transformations, we also have $\mathbf{x}' \cdot \mathbf{x}' < 0$. Now we only need to show that $t' > 0$. According to the Lorentz transformation,

$$ct' = \gamma(ct - \beta x).$$

From $\mathbf{x} \cdot \mathbf{x} < 0$ we know that $ct > x$ and because $\mathcal{S}'$ can only move with a speed $v < c$ we also know that $\beta < 1$. Combining these two inequalities gives us $ct > \beta x$ or $ct - \beta x > 0$. So we have $t' > 0$ indeed and thus $\mathbf{x}'$ lies in the absolute future. □

If two events are time-like seperated, all observers will agree on which event happens first. If two events are space-like seperated, they do not:

Proposition 2. *Suppose two events, P and Q, are space-like seperated. Then the following statements are all true:*

- *There is a frame in which P happens before Q.*
- *There is a frame in which P and Q happen simultaneously.*
- *There is a frame in which P happens after Q.*

Before we prove this, let us consider what this means. If an event P would influence another event Q faster than the speed of light, these events would be space-like seperated. Therefor, there is a frame in which Q happens *before* P; in this frame the event P causes something in the past. Assuming that this is impossible, we must conclude that causal influences cannot travel faster than the speed of light.

Proof. Assume without loss of generality that we are in frame where P lies on the origin and Q lies on $\mathbf{x} = (x, 0, 0, ct)$ with $x > 0$. There are now three possibilities: $t < 0$, $t = 0$, and $t > 0$. Let us prove the proposition for the case $t > 0$ (for the other possibilities the proof is similar).

We are already in a frame in which P happens before Q, so we only need to find frames in which P and Q happen simultaneously and in which P happens after Q.

After a Lorentz transformation, the origin remains in place and the time coordinate of Q becomes:

$$ct' = \gamma(ct - \beta x).$$

We are allowed to chose any $0 \leq \beta < 1$. Since P and Q are space-like seperated, we know that $x^2 > (ct)^2$, so $\frac{ct}{x} < 1$. If we let $\beta = \frac{ct}{x}$, then we get $ct' = 0$, so P and Q happen simultaneously. If we make β slightly larger (that is, $\frac{ct}{x} < \beta < 1$), then $ct' < 0$, so P happens after Q. □

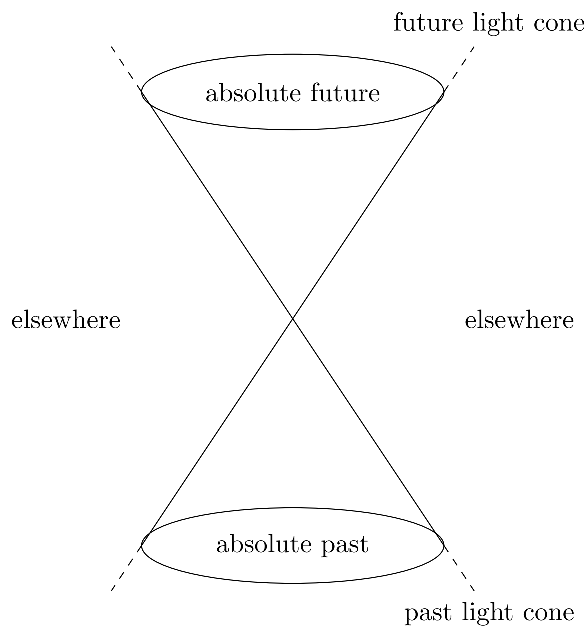

Figure 6: The five regions in space-time.

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