Quantization
and the
Resolvent Algebra

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Abstract

Let (X, σ) be our phase space, which we assume to be a possibly infinite-dimensional symplectic vector space admitting a unitary structure. We construct a so-called strict deformation quantization of (X, σ), which generalizes Weyl quantization, in such a way that the non-commutative C*-algebra obtained is the resolvent algebra $\mathcal{R}(X, \sigma)$, introduced in 2003 by Buchholz and Grundling. In the precise sense of strict deformation quantization, this resolvent algebra has a classical counterpart, which we call the commutative resolvent algebra. We describe this algebra in detail, and in particular compute its Gelfand spectrum.

Contents

1 Introduction

Initiated by the insights of Newton, mankind has developed a formalism to describe the world with incredible accuracy. A core concept behind this formalism is that any system with n degrees of freedom (for instance, a particle moving in n-dimensional space) can be described by only $2n$ parameters $x = (x_1, \ldots x_{2n})$. Any quantity that can be assigned to this system (its speed, its heat, et cetera) is therefore a function from $\mathbb{R}^{2n}$ to $\mathbb{R}$, called a **classical observable**. Prime examples of classical observables are momentum and position, which are usually defined as the coordinate functions on $\mathbb{R}^{2n}$. We will use a more general class of classical observables $p_x : \mathbb{R}^{2n} \to \mathbb{R}$, indexed by $x \in \mathbb{R}^{2n}$, defined by

$$p_x(y) := x \cdot y$$

for all $y \in \mathbb{R}^{2n}$.

Towards the end of the nineteenth century, it became clear that the formalism just mentioned was not the whole story. When looking at small or high-energetic systems, a wavefunction ψ on $\mathbb{R}^n$, rather than a point $y \in \mathbb{R}^{2n}$ fully describes the system. In quantum mechanics an observable is an **operator** that maps wavefunctions to wavefunctions. We define the momentum and position operators as

$$P_j \psi(u) := -i\hbar \frac{\partial \psi}{\partial u_j}(u), \quad Q_j \psi(u) := u_j \psi(u) \,, \tag{1}$$

respectively. Here $\hbar$ often denotes the reduced Planck constant, but in this context it can be any nonzero number. Mimicking our classical formalism, we will use a more general class of operators $\phi(x)$, indexed by $x \in \mathbb{R}^{2n}$, defined by

$$\phi(x) := \sum_{j=1}^{n} x_{2j-1} P_j + x_{2j} Q_j \,. \tag{2}$$

What distinguishes the quantum formalism from the classical formalism is that operators may not commute. In fact, we have

$$[\phi(x), \phi(y)] = -i\hbar \sigma_n(x, y) \mathbb{1} \tag{3}$$

for the standard symplectic form σ_n. One might see (3) as the defining relation of our formalism. Alternatively, one may regard (3) as the definition of σ_n, and verify that $(\mathbb{R}^{2n}, \sigma_n)$ is a symplectic space.

When describing quantum physics in a mathematically pleasing way, one often uses C*-algebras. The theory of C*-algebras is well developed, and thanks to that, many tools are readily available. To cast the relation (3) into the C*-algebraic framework, Weyl introduced the C*-algebra generated by $\left\{ e^{i\phi(x)} \mid x \in \mathbb{R}^{2n} \right\}$. This C*-algebra, known as the Weyl algebra or CCR-algebra, has long served quantum physisicts well, but we will nonetheless provide an alternative. Instead of forming complex exponentials of $\phi(x)$, we could form $g(\phi(x))$ for any $g \in C_0(\mathbb{R})$, meaning that g is continuous and vanishes at infinity. The C*-algebra

$$\mathcal{R}(\mathbb{R}^{2n}, \sigma_n) := C^* \left(g(\phi(x)) \mid x \in \mathbb{R}^m, g \in C_0(\mathbb{R}) \right)$$

is called the resolvent algebra, and was introduced by Buchholz and Grundling in 2003. Contrary to the Weyl algebra, the resolvent algebra is stable in time, as presented in [2] as one of the arguments in favor of the resolvent algebra. Adding to that, the present paper will show that the resolvent algebra is at least as appealing as the Weyl algebra, when it comes to classical physics.

One might expect that quantum physics would simply replace classical physics entirely, but this has not been the case. Even today most (quantum) physicists have to use the classical framework at some point. It is the task of the physicist to describe the world we experience, and the world we experience is (for all practical and some philosophical purposes) classical. Furthermore, progress in quantum physics is often motivated by our understanding of classical physics, and the models used in quantum physics are often derived from the analogous models in classical physics. It is therefore important to precisely relate the classical and quantum frameworks.

We have seen two examples of non-commutative C*-algebras, namely the Weyl algebra and the resolvent algebra. While non-commutative C*-algebras are used in quantum physics, commutative C*-algebras (containing classical observables, which are functions) are used to describe classical physics. But we can do more than embedding classical and quantum physics into the same theory, we can actually relate classical with quantum C*-algebras.

A **quantization map** is a linear map $Q_\hbar$ (for each $\hbar \neq 0$) assigning an operator (a quantum observable) to each classical observable. A quantization map should fulfill some demands, for instance that $Q_\hbar(fg)$ converges to $Q_\hbar(f)Q_\hbar(g)$ when $\hbar \to 0$. For different purposes, different demands on $Q_\hbar$ are set. Rieffel, in [11], [12] and [13], introduced a type of quantization that refers to C*-algebras. We are talking about **strict deformation quantization** as defined by [8], which fulfills about every demand known to precisely

relate classical physics to quantum physics.[1] The word 'deformation' means to suggest that a non-commutative C*-algebra is 'deformed' into a commutative C*-algebra when $\hbar \to 0$.

It is of no debate that we should 'quantize' classical position and momentum to their respective operators. In our notation, this means that we define

$$Q_\hbar(p_x) := \phi(x).$$

However, the definition of $Q_\hbar(f)$ for a general classical observable f is a choice made by the physicist. For example, the definition of $Q_\hbar(p_x p_y)$ is already nontrivial, as $p_x p_y = p_y p_x$ but $\phi(x)\phi(y) \neq \phi(y)\phi(x)$. Should we define $Q_\hbar(p_x p_y) = \phi(x)\phi(y)$ or rather $Q_\hbar(p_y p_x) = \phi(y)\phi(x)$? Our choice is to write

$$p_x p_y = \tfrac{1}{2}p_{x+y}^2 - \tfrac{1}{2}p_x^2 - \tfrac{1}{2}p_y^2,$$

and to agree that $Q_\hbar(g \circ p_x) = g(\phi(x))$ for any $x \in \mathbb{R}^{2n}$ and suitable function g. Using $g(t) = t^2$, we find

$$
\begin{aligned}
Q_\hbar(p_x p_y) &= \tfrac{1}{2}\phi(x+y)^2 - \tfrac{1}{2}\phi(x)^2 - \tfrac{1}{2}\phi(y)^2 \\
&= \tfrac{1}{2}(\phi(x)\phi(y) + \phi(y)\phi(x)).
\end{aligned}
\tag{4}
$$

One may recognise that we have ended up with Weyl quantization, as (4) is often used to introduce this quantization map. However, rather than (4), it is because of the rule $Q_\hbar(g \circ p_x) = g(Q_\hbar(p_x))$ that Weyl Quantization plays the leading part in this thesis.

Define the algebra of almost periodic functions as $C^*(e^{ip_x}|x \in \mathbb{R}^{2n})$. It can be intuitively expected that the Weyl algebra is obtained from the algebra of almost periodic functions, since $Q_\hbar(e^{ip_x}) = e^{i\phi(x)}$. Indeed, as proven in [1], the almost periodic functions form the classical counterpart of the Weyl algebra in the sense of strict deformation quantization. One may wonder if a similar result holds for the resolvent algebra. What is its classical counterpart?

The contribution of this thesis is the following. We define a new classical observable algebra called the commutative resolvent algebra as

$$C_\mathcal{R}(\mathbb{R}^{2n}) := C^* \left(g \circ p_x \mid g \in C_0(\mathbb{R}), \ x \in \mathbb{R}^{2n} \right),$$

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[1]It is slightly stronger than Rieffel's definition because the quantization map should also be *-preserving.

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we give a precise description of its structure, and show that it is the classical counterpart of the resolvent algebra $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$. Precisely stated, our main result is that Weyl quantization gives a strict deformation quantization of $C_{\mathcal{R}}(\mathbb{R}^{2n})$ and $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$.

Up to this point everything was done whilst assuming our phase space to be finite dimensional. We will also treat the general case, replacing $(\mathbb{R}^{2n}, \sigma_n)$ by a symplectic vector space (X, σ) admitting a unitary structure. We construct a strict deformation quantization that generalizes the prescription of Weyl, and we prove that in this sense $C_{\mathcal{R}}(X) := C^*(g \circ p_x \mid g \in C_0(\mathbb{R}), \ x \in X)$ is the classical counterpart of $\mathcal{R}(X, \sigma)$. In quantum field theory and the theory of multi-particle systems, the infinite-dimensional version of the resolvent algebra is the only interesting version. However, the key features of the resolvent algebra are already present in the finite case, and it will turn out to be a small step to generalize our results from $\mathbb{R}^{2n}$ to X.

We therefore first prove our result for finite dimensional phase spaces in Sections 2 to 4. More precisely, we define the commutative resolvent algebra in Section 2, and investigate the structure of this algebra in §2.1 and §2.2. The resolvent algebra is introduced in our own way in Section 3. We discuss strict deformation quantization in Section 4. We define our quantization map precisely in §4.1, and prove the key result $Q_\hbar(g \circ p_x) = g(Q_\hbar(p_x))$ in §4.2. Our main result is proven in §4.3 and §4.4.

The generalized version of the commutative resolvent algebra is given in Section 5 and the resolvent algebra in Section 6. Our main result is generalized in Section 7.

2 Commutative Resolvent Algebra: Finite Case

We define a commutative algebra, consisting of complex functions on the space $\mathbb{R}^m$. It will be defined as a C*-subalgebra of $C_b(\mathbb{R}^m)$, the algebra of bounded continuous functions. This C*-subalgebra will turn out to be the classical counterpart of the resolvent algebra on the phase space $\mathbb{R}^{2n}$, if $m = 2n$. This section allows for general $m \in \mathbb{N}$, as it stays in the classical context. We view $\mathbb{R}^m$ as an inner product space, with the standard inner product $x \cdot y$, $(x, y \in \mathbb{R}^m)$.

Definition 2.1. *For $\lambda \in \mathbb{R} \backslash \{0\}$ and $x \in \mathbb{R}^m$ define $h_x^\lambda(y) := 1/(i\lambda - x \cdot y)$. The **commutative resolvent algebra** over $\mathbb{R}^m$, denoted by $C_\mathcal{R}(\mathbb{R}^m)$, or simply by $C_\mathcal{R}$, is the C*-subalgebra of $C_b(\mathbb{R}^m)$ generated by the functions h_x^λ.*

This C*-algebra $C_\mathcal{R}$ is unital since $ih_0^1 = 1$. Let us write $h_x^\lambda = g^\lambda \circ p_x$ for $g^\lambda = 1/(i\lambda - \cdot)$ and $p_x(y) := x \cdot y$. The function p_x is surjective on its range, so the following very general observation applies to it.

Lemma 2.2. *Let $p : X \to Y$ be a surjection between topological spaces, and $A \subseteq C_b(Y)$ a *-subalgebra. Then its 'pull-back' $p^* : A \to C_b(X)$, $g \mapsto g \circ p$ is an isometric *-homomorphism.*

Proof. Because the operations addition, multiplication and involution on A and $C_b(X)$ are defined pointwise, these operations are preserved by p^*. Because p is surjective, we find

$$\sup_{x \in X} |g(p(x))| = \sup_{y \in Y} |g(y)| \,,$$

giving $\|g \circ p\|_\infty = \|g\|_\infty$. $\qquad\qquad\square$

We can apply Lemma 2.2 to give an equivalent definition of $C_\mathcal{R}$. The theorem of Stone-Weierstrass gives $C^*(g^\lambda | \lambda \in \mathbb{R} \backslash \{0\}) = C_0(\mathbb{R})$, implying $C^*(h_x^\lambda | \lambda \in \mathbb{R} \backslash \{0\}) = C_0(\mathbb{R}) \circ p_x$, for any x. Hence, $C_\mathcal{R}$ is the C*-algebra generated by $\{g \circ p_x \mid g \in C_0(\mathbb{R}), x \in \mathbb{R}^m\}$.

We will see that these $g \circ p_x$ generate more general functions $g \circ p$, when we generalize p_x by $p : \mathbb{R}^m \to \mathbb{R}^r$ and let $g \in C_0(\mathbb{R}^r)$ for any $r \in \{0, \dots m\}$. It will sometimes be useful to assume that g is a Schwartz function, by which we mean $g \in \mathcal{S}(\mathbb{R}^r)$. We discuss our conventions on the Schwartz space $\mathcal{S}(\mathbb{R}^r)$ in Appendix A.

Lemma 2.3. *For $j \in \{1, 2\}$, assume $\mathbb{R}^m \xrightarrow[\text{linear}]{p_j} \mathbb{R}^{r_j} \xrightarrow{g_j} \mathbb{C}$. There exists a linear surjection p and a complex function g such that*

   (i) $(g_1 \circ p_1)(g_2 \circ p_2) = g \circ p$,

   (ii) $\ker p = \ker p_1 \cap \ker p_2$,

   (iii) *if g_1 and g_2 both vanish at infinity, then so does g,*

   (iv) *if g_1 and g_2 both are Schwartz, then so is g.*

Proof. Let $\mathbb{R}^m = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ for subspaces $V_j \subseteq \mathbb{R}^m$ such that:

$$V_4 = \ker p_1 \cap \ker p_2\,,$$
$$V_3 \oplus V_4 = \ker p_1\,,$$
$$V_2 \oplus V_4 = \ker p_2\,. \tag{5}$$

Let $p : \mathbb{R}^m \to V_1 \oplus V_2 \oplus V_3$ be the canonical projection, which is linear and surjective. It has a linear section s_p, so $p \circ s_p = \mathrm{id}_{V_1 \oplus V_2 \oplus V_3}$. By virtue of (5), it is possible to write

$$g_1 \circ p_1 \circ s_p(v_1, v_2, v_3) = h_1(v_1, v_2)\,,$$
$$g_2 \circ p_2 \circ s_p(v_1, v_2, v_3) = h_2(v_1, v_3) \quad (v_j \in V_j)\,,$$

for functions h_1, h_2 on $V_1 \oplus V_2$ and $V_1 \oplus V_3$ respectively. Since $p_j \circ s_p \circ p = p_j$ for $j \in \{1, 2\}$, we have

$$(g_1 \circ p_1) \cdot (g_2 \circ p_2) = [(g_1 \circ p_1 \circ s_p)(g_2 \circ p_2 \circ s_p)] \circ p$$
$$\equiv g \circ p\,,$$

for some function g on $V_1 \oplus V_2 \oplus V_3$. This proves (i) and (ii). If g_1 and g_2 are Schwartz, then h_1 and h_2 are Schwartz as well. Therefore Lemma A.2 implies (iv). Since $\mathcal{S}$ is dense in C_0 with respect to $\|\cdot\|_\infty$ and multiplication is continuous in that same norm, (iv) implies (iii). $\square$

Because $C_{\mathcal{R}}(\mathbb{R}^m)$ is generated by the functions $g \circ p_x$, Lemma 2.3 implies that $C_{\mathcal{R}}(\mathbb{R}^m)$ contains various other functions $g \circ p$. It is appropriate to give them a name.

Definition 2.4. *A **dike** $g \circ p : \mathbb{R}^m \to \mathbb{C}$ is a composition of some linear surjective function $p : \mathbb{R}^m \to \mathbb{R}^r$ and some function $g \in C_0(\mathbb{R}^r)$.*

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When the C_0-condition on g is dropped, $g \circ p$ is called a cylindrical (or cylinder) function. Dikes for which g is Schwartz will be very useful when working with Weyl Quantization. We therefore define

$$\mathcal{S}_{\mathcal{R}}(\mathbb{R}^m) := \text{span}\left\{ g \circ p \mid p : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^r \text{ linear, } g \in \mathcal{S}(\mathbb{R}^r) \text{ for } 0 \leqslant r \leqslant m \right\}.$$

Any scalar multiplication of a dike is again a dike. Therefore, an arbitrary element of $\mathcal{S}_{\mathcal{R}} := \mathcal{S}_{\mathcal{R}}(\mathbb{R}^m)$ is just a finite sum of dikes.

Proposition 2.5. *The space $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^m)$ is a dense *-subalgebra of $C_{\mathcal{R}}(\mathbb{R}^m)$.*

Proof. We will show that any dike $g \circ p$ with $g \in \mathcal{S}(\mathbb{R}^r)$ is an element of $C_{\mathcal{R}}$. Because $\mathcal{S}(\mathbb{R}^r) = \mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R})$ with respect to the Schwartz topology, it is sufficient to assume $g = g_1 \otimes \cdots \otimes g_r$ for $g_j \in \mathcal{S}(\mathbb{R})$. If we define $p_j(x) := p(x)_j$, then $g \circ p = \prod g_j \circ p_j$. As $p_j \in (\mathbb{R}^m)^*$, there is an x_j such that $p_j = p_{x_j}$. It follows that $g \circ p \in C_{\mathcal{R}}$. We conclude that $\mathcal{S}_{\mathcal{R}} \subseteq C_{\mathcal{R}}$.
The set $\mathcal{S}_{\mathcal{R}}$ is clearly closed under linear combinations and involution. Furthermore, closure under multiplication follows by Lemma 2.3(i) and (iv), and we may conclude that $\mathcal{S}_{\mathcal{R}}$ is a *-subalgebra.
Finally, any generator h_x^λ is approximated by functions $g \circ p_x \in \mathcal{S}_{\mathcal{R}}$ where $g \in \mathcal{S}(\mathbb{R})$ approximates $g^\lambda = 1/(i\lambda - \cdot) \in C_0(\mathbb{R})$. This proves density. $\qquad\square$

This is all we need to know about the commutative resolvent algebra in order to discuss strict deformation quantization. However, there is much more to say about the structure of this intriguing C*-algebra, and an understanding of its structure yields a lot of intuition (if not knowledge) about the resolvent algebra on the quantum side.

The next two sections will give a precise description of $C_{\mathcal{R}}(\mathbb{R}^m)$, first by describing its elements, and second by describing its Gelfand spectrum Δ. At the end of this section it is established that Δ is a novel compactification of $\mathbb{R}^m$, which implies that the elements of $C_{\mathcal{R}}(\mathbb{R}^m)$ are precisely the continuous functions on Δ restricted to $\mathbb{R}^m$.

2.1 Function Spaces

If we want to understand $C_{\mathcal{R}}(\mathbb{R}^m)$, we will need to understand dikes. We have used the notation $g \circ p$, in which way we see this is the function $g \in C_0(\mathbb{R}^r)$ acting on the r directions picked out by p. Another notation provides more geometrical insight. If we use a projection $P : \mathbb{R}^m \to \mathbb{R}^m$, (that is, $P^* = P^2 = P$,) instead of p, and demand $g \in C_0(\text{ran } P)$ instead of $g \in C_0(\mathbb{R}^r)$, we

find that the collection of functions of the form $g \circ P$ is exactly the collection of dikes. Indeed, $\tilde{g} \circ p = g \circ P$ whenever

$$g = \tilde{g} \circ \gamma \,, \quad \operatorname{ran} P = (\ker p)^{\perp} \,, \quad p_{\restriction \operatorname{ran} P} = \gamma \circ P \,,$$

for a linear isomorphism $\gamma : \operatorname{ran} P \xrightarrow{\sim} \mathbb{R}^r$. Writing $g \circ P$ is a way to denote a dike 'independent of a choice of basis'.

In the rest of this section a dike is a composition $g \circ P$, consisting of a projection $P : \mathbb{R}^m \to \mathbb{R}^m$ and a function $g \in C_0(\operatorname{ran} P)$.

Before we begin the analysis, we give a geometrical interpretation of dikes. For $m = 2$ and $\operatorname{nul} P (= \dim \ker P) = 1$, the surface plot of the absolute value of $g \circ P$ resembles a physical dike with top height of $\|g\|_{\infty}$ stretching out indefinitely in the direction of $\ker P$ and - in the perpendicular direction - descending into the flat surrounding landscape. See Figures 1 and 2. The function g determines the shape of the dike and P determines the direction into which it extends. For general values of $\operatorname{nul} P$ and m, it is helpful to imagine an affine space of dimension $\operatorname{nul} P$, around which the support of $g \circ P$ is concentrated.
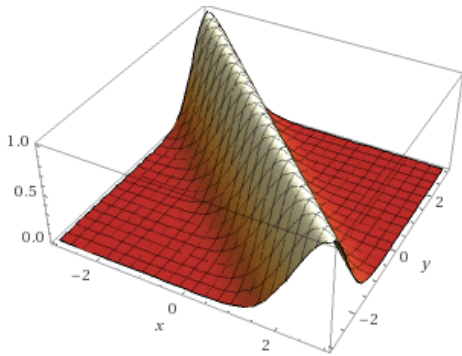

Figure 1: A dike

Figure 2: An actual dike

We have already seen a dense subset of $C_{\mathcal{R}}(\mathbb{R}^m)$, consisting solely of finite sums of dikes. The algebra $C_{\mathcal{R}}$ itself contains infinite sums that can be conditionally convergent. This already happens in the case that $m = 2$. In Figure 3 we have plotted a sum of two dikes with norm 1 and norm $\frac{1}{2}$, the norm of the sum being $\frac{3}{2}$. The region where this sum is greater than $1 + \epsilon$ (in absolute value) is compact for any $\epsilon > 0$, so we could subtract a C_0-function such that the result is bounded by 1. In this fashion, if we alternately add a dike and subtract a C_0-function (both with norm $1/n$), we can construct
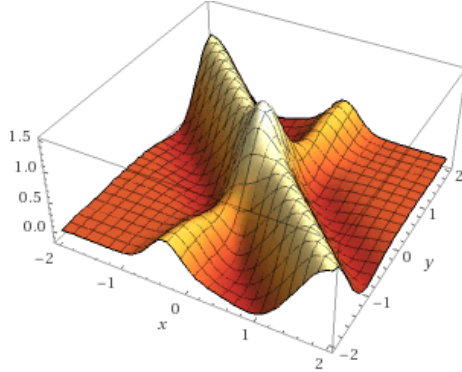
Figure 3: A sum of two dikes

an infinite sum that converges in $C_{\mathcal{R}}$, even though the sum of the subtracted C_0-functions is divergent.

In order to avoid conditionally convergent sums, we will define function spaces $C_r(\mathbb{R}^m)$, consisting of countable sums of dikes $g_k \circ P_k$ for which $\operatorname{nul} P_k = r$, modulo dikes $g \circ P$ with $\operatorname{nul} P < r$.

Definition 2.6. *For $0 \leqslant r \leqslant m$, define the spaces $C_r(\mathbb{R}^m)$ as follows. First, $C_0(\mathbb{R}^m)$ is the usual space of continuous functions vanishing at infinity (showing the consistency of our notation). Assuming $C_{r-1}(\mathbb{R}^m)$ is a C*-algebra, we denote the equivalence class of $f \in C_b(\mathbb{R}^m)$ in $C_b(\mathbb{R}^m)/C_r(\mathbb{R}^m)$ by $[f]_{r-1}$, and use the topology induced by*

$$\|[f]_{r-1}\|_{r-1} := \inf_{\varphi \in C_{r-1}} \|f - \varphi\|_\infty \ .$$

We define

$$C_r(\mathbb{R}^m) := \left\{ f \in C_b(\mathbb{R}^m) \ \middle| \ \begin{array}{l} [f]_{r-1} = \sum_k [g_k \circ P_k]_{r-1} \ \text{for } P_k \ \text{distinct (m-r)-} \\ \text{dimensional projections, and } g_k \in C_0(\operatorname{ran} P_k) \end{array} \right\},$$

where we use an arbitrary countable sum.

We often write $\|f\|_{r-1} := \|[f]_{r-1}\|_{r-1}$ for convenience. The function spaces C_r build up the commutative resolvent algebra, in the following precise way.

Theorem 2.7. *We have*

$$C_{\mathcal{R}}(\mathbb{R}^m) = C_m(\mathbb{R}^m).$$

Moreover, $C_0 \subset C_1 \subset \ldots \subset C_m$ is a chain of closed ideals in $C_{\mathcal{R}}$.

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The proof, given at the end of this section, uses an inductive argument to prove that each C_r is an algebra, for which the following lemma is important. We could prove this lemma using Lemma 2.3, but we will instead provide an independent proof, to give more insight.

Lemma 2.8. *Let $g \circ P \in C_s(\mathbb{R}^m)$ and $h \circ Q \in C_r(\mathbb{R}^m)$ be dikes. Then*

$$(g \circ P) \cdot (h \circ Q) \in C_{\min(s,r)}(\mathbb{R}^m) \,,$$

and if $s = r$ and $P \neq Q$, then $(g \circ P) \cdot (h \circ Q) \in C_{s-1}(\mathbb{R}^m)$.

Proof. Define R to be the projection onto $\operatorname{ran} P + \operatorname{ran} Q$. Then

$$\ker R = \ker P \cap \ker Q \,,$$

which gives $(g \circ P)(h \circ Q) = f \circ R$, where $f := (g \circ P)(h \circ Q)$. We claim that we can find C such that

$$\|Rx\| \leqslant C \max(\|Px\|, \|Qx\|) \quad \text{for all } x \,.$$

If this were not the case, we could find a sequence of x on the unit sphere such that the reverse inequality holds for increasing C. Then a convergent subsequence yields a contradiction. Now $\|Rx\| \to \infty$ implies

$$f(Rx) = g(Px)h(Qx) \to 0 \,.$$

Therefore, $f \in C_0(\operatorname{ran} R)$. From $\operatorname{nul} R \leqslant \min(\operatorname{nul} P, \operatorname{nul} Q)$ it follows that $f \circ R \in C_{min(s,r)}(\mathbb{R}^m)$. If $r = s$ and $P \neq Q$, then $\operatorname{nul} R < \operatorname{nul} P$, so $f \circ P \in C_{s-1}(\mathbb{R}^m)$. $\qquad\square$

From now on, we fix an $r \leqslant m$ such that C_s is an algebra for all $s \leqslant r$. We will specify the behaviour of an arbitrary function $f \in C_{r+1}$ at infinity. To this purpose, let $V + w \subseteq \mathbb{R}^m$ be an affine space, with space of directions $S(V) := \{v \in V \mid \|v\| = 1\}$ when $V \neq 0$, and $S(0) := \{0\}$. We equip $S(V)$ with the $\dim V$-dimensional Hausdorff measure μ. The convergence at infinity of f is captured by the following lemma.

Lemma 2.9. *Take $f \in C_s$ for $s \leqslant r + 1$. Then the limit*

$$f^{V,w}(v) := \lim_{t \to \infty} f(tv + w) \tag{6}$$

exists for all $v \in S(V)$ and hence defines a function $f^{V,w} : S(V) \to \mathbb{C}$. Furthermore, $f^{V,w}$ takes a constant value μ-almost everywhere.

Proof. We prove the lemma with induction to $s \leqslant r+1$, the case $s = 0$ being clear. Suppose the lemma holds for some $s \leqslant r$ and that $f \in C_{s+1}$. Writing $f_K := \sum_{k=1}^{K} g_k \circ P_k$ for the partial sums of f (meaning that $\|f_K - f\|_s \to 0$), we have a well-defined function $f_K^{V,w}$ with $f_K^{V,w} = c_K$ μ-a.e. for some $c_K \in \mathbb{C}$, just by comparing dimensions. Taking $\tilde{f}_K := f_K + \xi_K$ for the right $\xi_K \in C_s$, we can make sure that $\|\tilde{f}_K - f\|_\infty \to 0$. By the induction hypothesis $\tilde{f}_K^{V,w}$ is a well-defined function with $\tilde{f}_K^{V,w} = \tilde{c}_K$ μ-a.e., for some $\tilde{c}_K \in \mathbb{C}$. This sequence $(\tilde{c}_K)$ converges to some $c \in \mathbb{C}$ because $(\tilde{f}_K)$ is Cauchy in $\|\cdot\|_\infty$. If

$$\Gamma := \{ v \in S(V) \, | \, \forall K \colon \tilde{f}_K^{V,w}(v) = \tilde{c}_K \} \, ,$$

then $\mu(S(V) \setminus \Gamma) = 0$ by countable additivity of μ. Now for arbitrary $v \in \Gamma$ we have

$$\lim_{K \to \infty} \lim_{t \to \infty} \tilde{f}_K(tv + w) = \lim_{K \to \infty} \tilde{c}_K = c \, ,$$

and for any $v \in S(V)$ we have $\tilde{f}_K(tv + w) \to f(tv + w)$ uniformly in t. Therefore $f^{V,w}$ is a function with $f^{V,w} = c$ μ-a.e. $\qquad \square$

To stress that we will later quotient out $C_r(\mathbb{R}^m)$, we will now use the letter ξ for an element in $C_r(\mathbb{R}^m)$, contrasting the notation '$f \in C_{r+1}(\mathbb{R}^m)$'.

Corollary 2.10. *Let $W \subset \mathbb{R}^m$ be affine with $\dim W = r + 1$. For all $\epsilon > 0$ and $\xi \in C_r(\mathbb{R}^m)$ there exists an $x \in W$ with $|\xi(x)| < \epsilon$.*

Proof. Write $W = V + w$ so we can apply Lemma 2.9. With induction to $s < r + 1$ we obtain $\xi^{V,w} = 0$ μ-a.e. for all $\xi \in C_s$. The claim follows by taking $s = r$. $\qquad \square$

Corollary 2.11. *Let P be a projection with $\operatorname{nul} P = r+1$ and $g \in C_0(\operatorname{ran} P)$. Then $\|g \circ P\|_r = \|g \circ P\|_\infty = \|g\|_\infty$.*

Proof. It is easily seen that $\|g \circ P\|_r \leqslant \|g \circ P\|_\infty = \|g\|_\infty$, but we need Corollary 2.10 for $\|g\|_\infty \leqslant \|g \circ P\|_r$. Let $\xi \in C_r$ and x so that $|g(x)| = \|g\|_\infty$. Since $W := P^{-1}\{x\}$ is affine, we obtain for all $\epsilon > 0$ an x_0 with $|\xi(x_0)| < \epsilon$. Then $|g(Px_0) - \xi(x_0)| \geqslant \|g\|_\infty - \epsilon$. It follows that $\|g \circ P - \xi\|_\infty \geqslant \|g\|_\infty$. $\qquad \square$

We are now ready to prove the main result of §2.1.

Proof of Theorem 2.7. Using induction on $r \leqslant m$, we will prove the following claim:

$$C_r(\mathbb{R}^m) \text{ is a C*-subalgebra of } C_{\mathcal{R}}(\mathbb{R}^m). \tag{7}$$

If $r = 0$ this follows by applying the (locally compact version of the) Stone-Weierstrass theorem,[2] or by recalling that $\mathcal{S}_{\mathcal{R}} \subseteq C_{\mathcal{R}}$. Suppose now that (7) is true for a fixed $r < m$. Then C_b/C_r is a C*-algebra, in particular a Banach space, a fact we will use throughout the proof. Let $f \in C_{r+1}(\mathbb{R}^m)$, generically written as $[f]_r = \sum_{k \in \mathbb{N}} [g_k \circ P_k]_r$ for dikes $g_k \circ P_k$ with nul $P_k = r + 1$.

Lemma 2.12. *Under these conditions we have, for each $I \subseteq \mathbb{N}$,*

$$\left\| \sum_{k \in I} [g_k \circ P_k]_r \right\|_r = \sup_{k \in I} \|g_k\|_\infty . \tag{8}$$

Proof. By continuity of $\|\cdot\|_r$ on C_b/C_r, we only need to show (8) for every finite $I \subset \mathbb{N}$. We will use induction on $\#I$. Let $K \in I$ be such that $\sup_{k \in I} \|g_k\|_\infty = \|g_K\|_\infty$. Then by the induction hypothesis,

$$\left\| \sum_{K \neq k \in I} g_k \circ P_k \right\|_r \leqslant \|g_K\|_\infty .$$

Fix $\epsilon > 0$ and take $\xi \in C_r$ such that

$$\left\| \sum_{k \neq K} g_k \circ P_k - \xi \right\|_\infty \leqslant \|g_K\|_\infty + \epsilon . \tag{9}$$

So both $\sum_{k \neq K} g_k \circ P_k - \xi$ and $g_K \circ P_K$ are (almost) bounded by $\|g_K\|_\infty$, but their sum may be substantially larger at some region. It turns out that this region is small enough to be corrected for by a C_r-function. More precisely, we can find $\phi \in C_r(\mathbb{R}^n)$ such that

$$\left\| \sum_{k \in I} g_k \circ P_k - \xi - \phi \right\|_\infty \leqslant \|g_K\|_\infty + \epsilon .$$

Some analysis shows that

$$\phi = \left(\sum_{k \neq K} g_k \circ P_k - \xi \right) \frac{|g_K \circ P_K|}{\|g_K\|_\infty}$$

does the job. The fact that $\phi \in C_r$ follows from Lemma 2.8, using $P_k \neq P_K$, and C_r is closed. We conclude that $\left\| \sum_{k \in I} g_k \circ P_k \right\|_r \leqslant \|g_K\|_\infty$.

To attain $\|g_K\|_\infty$, we choose $x \in \operatorname{ran} P_K$ with $|g_K(x)| = \|g_K\|_\infty$. Fix $\epsilon > 0$

---

[2]Seperating $x, y \in \mathbb{R}^n$ is done by extending $e_1 := x - y$ to an orthogonal basis of $\mathbb{R}^n$. Then $h_{e_1}^1 \cdots h_{e_n}^1$ separates x and y.

and choose $\xi \in C_r$ to satisfy (9). Fix $\eta \in C_r$. Now $W = P_K^{-1}(x)$ is affine with dimension $r + 1$, so Corollary 2.10 gives an x_0 with $|\eta(x_0)| < \epsilon$ and $|g_K(P_K x_0)| = \|g_K\|_\infty$. Some more analysis yields

$$\left| \left(\sum_{k \in I} g_k \circ P_k - \xi - \phi - \eta \right) (x_0) \right| \geqslant \|g_K\|_\infty - \epsilon \, .$$

Letting $\epsilon \to 0$, we conclude that also $\left\| \sum_{k \in I} g_k \circ P_k \right\|_r \geqslant \|g_K\|_\infty$. Thus we have finished our inductive step, and the proposition follows. $\qquad\square$

Continuing the proof of Theorem 2.7, we observe that $\sum_{k=1}^\infty [g_k \circ P_k]$ converges unconditionally:

$$\left\| \sum_{k \geqslant K} \epsilon_k [g_k \circ P_k] \right\|_r = \sup_{k \geqslant K} \|\epsilon_k g_k\|_\infty = \left\| \sum_{k \geqslant K} [g_k \circ P_k] \right\|_r \to 0 \, .$$

Hence two converging sums such as in Definition 2.6 will add to another converging sum. It then follows that C_{r+1} is a vector space. Because of Lemma 2.8, the multiplication

$$[g_k \circ P_k]_r \cdot [g_k' \circ P_k']_r = [(g_k \circ P_k)(g_k' \circ P_k')]_r \in C_{r+1}$$

is well defined. Again by unconditional convergence, we have

$$\sum_k [g_k \circ P_k]_r \sum_k [g_k' \circ P_k']_r = \sum_{k,k'} [(g_k \circ P_k)(g_{k'}' \circ P_{k'}')]_r \in C_{r+1} \, .$$

Together with $(\sum_k [g_k \circ P_k])^* = \sum_k [\bar{g}_k \circ P_k]$, this implies that C_{r+1} is a *-algebra.

Let $(f^s)_{s \in \mathbb{N}} \subset C_{r+1}$ converge uniformly to f. Write $[f^s]_r = \sum_k [g_k^s \circ P_k^s]_r$ with g_k^s and P_k^s as usual. We can reshuffle the terms and add zeroes to obtain $\tilde{g}_\alpha^s$, P_α (for α in some countable set I) such that

$$\sum_{k \in \mathbb{N}} [g_k^s \circ P_k^s] = \sum_{\alpha \in I} [\tilde{g}_\alpha^s \circ P_\alpha] \, ,$$

for all $s \in \mathbb{N}$. Intuitively, we will let each $\tilde{g}_\alpha^s$ converge to some function g_α, thus obtaining f as the sum over all $[\tilde{g}_\alpha^s \circ P_\alpha]$, $\alpha \in I$. We can only do this because P_α does not depend on s anymore.

Lemma 2.12 displays an interplay between convergence of series and uniform

convergence of functions. For instance, (f^s) is Cauchy iff $(\tilde{g}^s_\alpha)$ is uniformly Cauchy:

$$\sup_{\alpha \in I} \left\| \tilde{g}^s_\alpha - \tilde{g}^t_\alpha \right\|_\infty = \left\| \sum_{\alpha \in I} [(\tilde{g}^s_\alpha - \tilde{g}^t_\alpha) \circ P_\alpha] \right\|_r = \left\| f^s - f^t \right\|_r \to 0 \,.$$

Thus we may define $g_\alpha := \lim \tilde{g}^s_\alpha \in C_0(\operatorname{ran} P_\alpha)$. It follows that $\tilde{g}^s_\alpha \to g_\alpha$ uniformly in α.

Using the just mentioned interplay, convergence of the series $\sum [\tilde{g}^s_\alpha \circ P_\alpha]$ implies $\|\tilde{g}^s_\alpha\|_\infty \to 0$ (for all s). Therefore $\|g_\alpha\|_\infty \to 0$, which in turn implies convergence of $\sum [g_\alpha \circ P_\alpha]$. We will write down the concluding step explicitly. Let rlim denote the limit in the quotient norm on C_b/C_r. Then

$$\left\| [f] - \sum_\alpha [g_\alpha \circ P_\alpha] \right\|_r = \left\| \operatorname*{rlim}_s \sum_\alpha [\tilde{g}^s_\alpha \circ P_\alpha] - \sum_\alpha [g_\alpha \circ P_\alpha] \right\|_r$$

$$= \lim_s \left\| \sum_\alpha [(\tilde{g}^s_\alpha - g_\alpha) \circ P_\alpha] \right\|_r$$

$$= \lim_s \, \sup_\alpha \|\tilde{g}^s_\alpha - g_\alpha\|_\infty = 0 \,,$$

and hence $f \in C_{r+1}(\mathbb{R}^n)$, giving us a C*-algebra.

Let P be a projection and take $g \in C_0(\operatorname{ran} P)$. It should be clear that

$$C^* \left(h^\lambda_{x \mid \operatorname{ran} P} \mid x \in \operatorname{ran} P, \lambda \right) \cong C^* \left(h^\lambda_x \mid x \in \operatorname{ran} P, \lambda \right) \,,$$

under $f \mapsto f \circ P$. Using the Stone-Weierstrass theorem, $C_0(\operatorname{ran} P)$ is contained in the left-hand-side. Therefore, $g \circ P$ is an element of the right-hand-side. Let $f \in C_{r+1}$ be arbitrary, written as

$$[f] = \sum [g_k \circ P_k] \in C_{r+1}/C_r \,,$$

with the usual conventions. Then all $g_k \circ P_k \in C_\mathcal{R}$, and thereby also the partial sums $f^K := \sum_{k=1}^K g_k \circ P_k \in C_\mathcal{R}$. Since $\left\| f^K - f \right\|_r \to 0$, we can find $\xi_K \in C_r \subseteq C_\mathcal{R}$ such that $\left\| f^K - \xi_K - f \right\|_\infty \to 0$. Hence, $f \in C_\mathcal{R}$.

Thus we have proven that $C_{r+1}(\mathbb{R}^m)$ is a C*-subalgebra of $C_\mathcal{R}$. By induction it follows that this holds for all $r < m$, and in particular we find $C_m(\mathbb{R}^m) \subseteq C_\mathcal{R}(\mathbb{R}^m)$.

The other inclusion follows if $h^\lambda_x \in C_m(\mathbb{R}^m)$ for all $\lambda \neq 0$, $x \in \mathbb{R}^m$. Define P as the projection on the span of x. Then $\ker P$ is m-dimensional when $x = 0$ and is $(m-1)$-dimensional otherwise. Since $g(Py) := h^\lambda_x(y)$ defines a function $g \in C_0(\operatorname{ran} P)$, we finally obtain $h^\lambda_x = g \circ P \in C_m(\mathbb{R}^m)$. $\qquad\square$

2.2 Gelfand Spectrum

We implicitly encountered characters of the commutative resolvent algebra in Lemma 2.9. Let us now define them precisely. For $V \subseteq \mathbb{R}^m$ linear, $w \in V^\perp$ and $f \in C_\mathcal{R}(\mathbb{R}^m)$, we have defined $f^{V,w} : S(V) \to \mathbb{C}$ in (6). Let $\chi(V + w)(f)$ be the unique $z \in \mathbb{C}$ such that $f^{V,w} = z$ almost everywhere.[3] A quick calculation shows that $\chi(V + w)$ is multiplicative and nonzero, hence $\chi(V + w) \in \Delta(C_\mathcal{R}(\mathbb{R}^m))$, where $\Delta(C_\mathcal{R}(\mathbb{R}^m))$ is the Gelfand spectrum of the commutative resolvent algebra, more briefly denoted by Δ, carrying the weak*-topology (i.e. the Gelfand topology). In practice the characters $\chi(V + w)$ are calculated on dikes, where they become rather simple.

Remark 2.13. *Let $f = g \circ P$ be a dike. If $V \subseteq \ker P$, then $f^{V,w}$ takes the constant value $g(Pw)$. If not, $V \cap \ker P$ is a proper linear subspace of V. For $v \in S(V) \setminus (V \cap \ker P)$ we obtain $f^{V,w}(v) = 0$. Hence*

$$V \subseteq \ker P \quad \Rightarrow \quad \chi(V + w)(f) = g(Pw)\,,$$
$$V \not\subseteq \ker P \quad \Rightarrow \quad \chi(V + w)(f) = 0\,.$$

What does it mean if a net $(\chi(V_\alpha + w_\alpha))$ weak*-converges to $\chi(V + w)$? In that case we have

$$\chi(V_\alpha + w_\alpha)(g \circ P_{V^\perp}) \to \chi(V + w)(g \circ P_{V^\perp}) = g(w)\,,$$

for any $g \in C_0(V^\perp)$. It follows that eventually (for all α bigger than a fixed α_0) we have $V_\alpha \subseteq V = \ker P_{V^\perp}$, with $P_{V^\perp}$ the projection onto $V^\perp$. Also, by choosing a sequence of g's with support closing in upon w, it follows that $P_{V^\perp} w_\alpha \to w$. Inspired by these results, we will prove that Δ is homeomorphic to the following space (see Theorem 2.18).

Definition 2.14. *We define the set*

$$\Omega := \left\{ V + w \mid V \subseteq \mathbb{R}^m \text{ linear, } w \in V^\perp \right\}\,,$$

*and say that a net $(V_\alpha + w_\alpha)_\alpha$ in Ω **is absorbed in** $V + w \in \Omega$ iff $P_{V^\perp} w_\alpha \to w$ and eventually $V_\alpha \subseteq V$.*

As a set, Ω is known by geometers as the affine Grassmanian $\mathrm{Graff}(\mathbb{R}^m)$, but we will endow Ω with a different topology. This topology is defined by a notion of convergence of nets that uses the notion of absorption of nets. By the previous discussion, if $\chi(V_\alpha + w_\alpha) \to \chi(V + w)$, then $V_\alpha + w_\alpha$ is absorbed in $V + w$. However, the converse is false, as manifested by the fact that all nets in Ω are absorbed in $\mathbb{R}^m + 0$.

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[3]The character $\chi(V + w)$ can be thought of as the 'mean value' on $V + w$.

Definition 2.15. *A net $(V_\alpha + w_\alpha)_\alpha$ in Ω **converges to** $V + w \in \Omega$ iff it is absorbed in $V + w$ and none of its subnets is absorbed in any $\tilde{V} + \tilde{w} \subsetneq V + w$.*

If a net converges to $V + w$, then any subnet also converges to $V + w$. Hence Definition 2.15 defines a topology on Ω. We have a topological embedding $\mathbb{R}^m \hookrightarrow \Omega$ by sending $w \mapsto \{0\} + w$, as a result of Definition 2.14.

Theorem 2.16. *The space Ω is a compactification of $\mathbb{R}^m$.*

Proof. Compactness follows from Definition 2.15. Indeed, to any net $(V_\alpha + w_\alpha)$ we can assign a $V + w \in \Omega$ such that some subnet $(V_\beta + w_\beta) \subseteq (V_\alpha + w_\alpha)$ is absorbed in $V + w$. Either $V_\beta + w_\beta \to V + w$ or a subsubnet $(V_\gamma + w_\gamma) \subseteq (V_\beta + w_\beta)$ is absorbed in a smaller dimensional affine space. The thus resulting chain of subnets has to stop somewhere, because $\dim V < \infty$, and gives us a convergent subnet of $(V_\alpha + w_\alpha)$.
To show that $\mathbb{R}^m$ is dense in Ω, let $V + w$ be arbitrary, and suppose that every $V' + w'$ with $\dim V' < \dim V$ lies in $\overline{\mathbb{R}^m}$, i.e. the closure of $\mathbb{R}^m$ in Ω. Then we can construct a sequence in $\overline{\mathbb{R}^m}$, converging to $V + w$, as follows. We choose $U \subset V$ with $\dim U = \dim V - 1$, some $u \in V \cap U^\perp$, and a sequence $(t_n) \subset \mathbb{R}$ without convergent subsequence. Then $U + t_n u \to V + w$. Applying induction to the dimension of V, it follows that $\overline{\mathbb{R}^m} = \Omega$. $\qquad\square$

The topology on Ω indeed matches the (weak*-)topology on Δ:

Lemma 2.17. *The function $\chi : \Omega \to \Delta$ is an embedding (i.e. a continuous open injection).*

Proof. We begin with injectivity. Let $\chi(V + w) = \chi(V' + w')$ for some $V + w$, $V' + w' \in \Omega$. Take a projection P onto $V^\perp$ and take a $g \in C_0(V^\perp)$ with $g(w) = 1$, and $g(v) < 1$ for all $v \neq w$. Now

$$\chi(V' + w')(g \circ P) = \chi(V + w)(g \circ P) = 1\,,$$

so $V' \subseteq V$ and $g(Pw') = 1$. By symmetry we obtain $V' = V$, and therefore $g(w') = 1$. It follows that $V + w = V' + w'$.
We are left to check that the maps χ and $\chi^{-1} : \chi(\Omega) \to \Omega$ preserve convergence of nets.
Suppose $\chi(V_\alpha + w_\alpha) \to \chi(V + w)$. As already discussed, $V_\alpha + w_\alpha$ is absorbed in $V + w$. Let $(V_\beta + w_\beta)$ be a subnet that is absorbed in $\tilde{V} + \tilde{w} \subsetneq V + w$. Take a dike $f = g \circ P_{\tilde{V}^\perp}$, where $g(\tilde{w}) = 1$, so

$$\lim_\beta \chi(V_\beta + w_\beta)(f) = \lim_\beta g(P_{\tilde{V}^\perp} w_\beta) = 1 \neq 0 = \chi(V + w)(f).$$

This contradicts $\chi(V_\alpha + w_\alpha) \to \chi(V + w)$. We conclude $V_\alpha + w_\alpha \to V + w$.

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Suppose conversely that $V_\alpha + w_\alpha \to V + w$. We would like to prove that

$$\chi(V_\alpha + w_\alpha)(f) \to \chi(V + w)(f)$$

for arbitrary $f \in C_{\mathcal{R}}(\mathbb{R}^m)$. Since sums of dikes lie densely in $C_{\mathcal{R}}$, we may assume $f = g \circ P$ is a dike. If $V \subseteq \ker P$, then we simply compute

$$\lim_\alpha |\chi(V_\alpha + w_\alpha)(f) - \chi(V + w)(f)| = \lim_\alpha |g(Pw_\alpha) - g(Pw)|$$
$$= |g(P \lim_\alpha P_{V^\perp} w_\alpha) - g(Pw)| = 0\,,$$

so we assume in the rest of the proof that $V \nsubseteq \ker P$. Since $\chi(V+w)(f) = 0$, it remains to show that $\chi(V_\alpha + w_\alpha)(f)$ converges to zero. We assume the contrary, which gives us a subnet $(V_\beta + w_\beta) \subseteq (V_\alpha + w_\alpha)$, such that all subnets $(V_\gamma + w_\gamma) \subseteq (V_\beta + w_\beta)$ have $\chi(V_\gamma + w_\gamma)(f) \nrightarrow 0$. Define $\tilde{V} := V \cap \ker P \subsetneq V$. As in the proof of Lemma 2.8, we have a constant C such that

$$\|P_{\tilde{V}^\perp} w_\gamma\| \leqslant C \max(\|Pw_\gamma\|, \|P_{V^\perp} w_\gamma\|)\,. \qquad (10)$$

To estimate the right-hand-side, firstly observe that $\lim_\gamma |\chi(V_\gamma + w_\gamma)(f)| \leqslant \lim_\gamma |g(Pw_\gamma)|$, if this limit exists. This means that $g(Pw_\gamma) \nrightarrow 0$, so (Pw_γ) has a bounded subnet. Secondly, observe that $P_{V^\perp} w_\gamma \to w$, so $(P_{V^\perp} w_\gamma)$ is eventually bounded. Now (10) implies that $(P_{\tilde{V}^\perp} w_\gamma)$ has a bounded subnet, and therefore a convergent subnet, denoted by $(P_{\tilde{V}^\perp} w_\delta)$. This net converges to some $\tilde{w} \in \tilde{V}^\perp \cap (V + w)$. Since $V_\delta + w_\delta$ is not absorbed in $\tilde{V} + \tilde{w}$, this implies that V_δ is not eventually in $\tilde{V}$. In other words, $(V_\delta + w_\delta)$ has a subnet $(V_\epsilon + w_\epsilon) \subseteq (V_\beta + w_\beta)$ such that $V_\epsilon \nsubseteq \tilde{V}$. But this cannot be, because $\chi(V_\epsilon + w_\epsilon)(f) \nrightarrow 0$. $\qquad \square$

Theorem 2.18. *The Gelfand spectrum of the commutative resolvent algebra $C_{\mathcal{R}}(\mathbb{R}^m)$ is homeomorphic to Ω, i.e. $\Delta(C_{\mathcal{R}}(\mathbb{R}^m)) \cong \Omega$ via the map χ.*

Proof. This relies on Lemma 2.17. Continuity of χ implies that its pullback,

$$\chi^* : C(\Delta) \to C(\Omega), \qquad f \mapsto f \circ \chi\,,$$

is a *-homomorphism. We are left to show injectivity and surjectivity of χ^*. Suppose $\chi^*(\hat{f}) = 0$ for some $\hat{f} \in C(\Delta)$, which is the Gelfand representation of $f \in C_{\mathcal{R}}(\mathbb{R}^m)$. For all $w \in \mathbb{R}^m$ we have

$$0 = \chi^*(\hat{f})(0 + w) = \chi(0 + w)(f) = f(w).$$

Hence χ^* is injective. If $g \in C(\Omega)$, then $g \circ \chi^{-1} \in C(\chi(\Omega))$. Since $\chi(\Omega)$ is a compact subset of the compact Hausdorff space Δ, we may use Urysohn's lemma to extend $g \circ \chi^{-1}$ to Δ. We obtain a function $h \in C(\Delta)$ such that $h \circ \chi = g$, completing the proof. $\qquad \square$

3 Resolvent Algebra: Finite Case

In this section we turn to quantum mechanics. Replacing the dimension m by $2n$, we will work with the space $\mathbb{R}^{2n}$, which we call phase space. This is a symplectic space with the symplectic form $\sigma_n(x, y) := x \cdot (J_n y)$, where

$$J_n := \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}.$$

For $x \in \mathbb{R}^{2n}$ we define the operators

$$\phi(x) := \sum_{j=1}^{n} x_{2j-1} P_j + x_{2j} Q_j, \quad \operatorname{dom}(\phi(x)) := \mathcal{S}(\mathbb{R}^n)$$

as unbounded operators in $H := L^2(\mathbb{R}^n)$, where P_j and Q_j are defined, initially on $\mathcal{S}(\mathbb{R}^n)$, by (1). From these definitions, the canonical commutation relation

$$[\phi(x), \phi(y)] \subseteq \frac{\hbar}{i} \sigma_n(x, y) \mathbb{1}$$

follows directly. Here $\hbar$ is a fixed nonzero constant, on which P_j and hence ϕ implicitly depend. The following lemma is crucial to the definition of the resolvent algebra, and also to the analysis in Section 4.

Lemma 3.1. *For a fixed $x \in \mathbb{R}^{2n}$, the operator $\phi(x)$ is essentially self-adjoint.*

Proof. To avoid unnecessary technicalities, all operators considered in this proof should be understood as maps $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. We claim that there exist unitaries $U_j \in B(H)$ such that $U_j(\mathcal{S}(\mathbb{R}^n)) \subseteq \mathcal{S}(\mathbb{R}^n)$ and

$$x_{2j-1} P_j + x_{2j} Q_j = a_j U_j^* Q_j U_j,$$

for some real a_j. If $x_{2j-1} = 0$, then $U_j = \mathbb{1}$ suffices, so suppose $x_{2j-1} \neq 0$. Abbreviating $c_j := x_{2j}/(2\hbar x_{2j-1})$, we define $\tilde{U}_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$\tilde{U}_j \psi(y) := \exp(i c_j y_j^2) \psi(y).$$

As $\tilde{U}_j$ is multiplication by a function with values in the unit circle, it extends to H and is unitary as such. We calculate

$$\tilde{U}_j^* P_j \tilde{U}_j \psi(t) = \frac{\hbar}{i} (2i c_j y_j \psi(y) + \partial_j \psi(y))$$

$$= \frac{1}{x_{2j-1}} (x_{2j} Q_j + x_{2j-1} P_j) \psi(y).$$

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Since $P_j = b_j \mathcal{F}_j^* Q_j \mathcal{F}_j$ with some $b_j \in \mathbb{R}$, and $\mathcal{F}_j$ the Fourier transform in the coordinate j, we find that $U_j = \mathcal{F}_j \tilde{U}_j$ suits our purposes. Define $U := U_1 \cdots U_n$. Because $[U_j, U_k] = [U_j, P_k] = 0$ for $j \neq k$, we find

$$U^* \sum a_j Q_j U = \phi(x) \, .$$

By pulling back a coordinate transformation that maps $\sum a_j e_j$ to $\|a\| \, e_1$, we find that $\sum a_j Q_j$ in turn is unitarily equivalent to $\|a\| \, Q_1$. Since Q_1 is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$, and unitary equivalence preserves essential self-adjointness (on a dense domain), $\phi(x)$ is essentially self-adjoint. $\qquad \square$

In what follows, we identify the operator $\phi(x)$ with its closure, so ϕ maps $\mathbb{R}^{2n}$ to unbounded self-adjoint operators in H. The resolvents of these self-adjoint operators allow for a definition of the resolvent algebra of $(\mathbb{R}^{2n}, \sigma_n)$. A definition of the resolvent algebra $\mathcal{R}(X, \sigma)$ in terms of generators and relations on a general phase space (X, σ), is given in Section 6. However, this definition is too abstract for our present purposes, and we will only need a concrete characterization of $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ as a subset of $B(H)$. Using Theorem 4.10 of their paper [2], Buchholz and Grundling proved that the Schrödinger representation $\pi_S^n : \mathcal{R}(\mathbb{R}^{2n}, \sigma_n) \to B(H)$ is faithful. We may therefore identify $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ with $\pi_S^n(\mathcal{R}(X, \sigma))$, and *define* it as follows.

Definition 3.2. *The **finite resolvent algebra** $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ is the C*-algebra generated by the operators $(i\lambda - \phi(x))^{-1} \in B(H)$ for every $x \in \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.*

In the next section we give a so-called strict deformation quantization

$$Q_\hbar^{2n} : \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n}) \to \mathcal{R}(\mathbb{R}^{2n}, \sigma_n) \, .$$

4   Quantization: Finite Case

In this section we achieve our goal for finite-dimensional phase spaces. Taking the resolvent algebra as quantum algebra, we give a *strict deformation quantization* of our commutative C*-algebra $C_{\mathcal{R}}(\mathbb{R}^{2n})$ of Section 2. The definition of strict deformation quantization is given next. Our definition is equivalent to Definition 1.1.2 of [8], and is stronger than any of the definitions of strict (deformation) quantization that occur in the excellent survey in [7].

Let $\tilde{A}_{\mathbb{R}}^0$ be a Poisson algebra that is densely contained in the self-adjoint part $A_{\mathbb{R}}^0$ of a commutative C*-algebra A^0. It follows that $\tilde{A}_{\mathbb{R}}^0$ is the real part of a *-algebra $\tilde{A}^0$, which in turn is dense in A^0. We can now give the anticipated definition.

Definition 4.1. *A **strict deformation quantization** of $\tilde{A}^0_{\mathbb{R}}$ consists of a subset $I \subseteq \mathbb{R}$ containing 0 as an accumulation point (meaning $0 \in I \cap \overline{I \setminus \{0\}}$), a collection of C*-algebras $\{A^\hbar\}_{\hbar \in I}$, and a collection of injective linear maps $\{Q_\hbar : \tilde{A}^0_{\mathbb{R}} \to A^\hbar_{\mathbb{R}}\}_{\hbar \in I}$, Q_0 being the identity map, such that for all $f, g \in \tilde{A}^0_{\mathbb{R}}$:*

$$\hbar \mapsto \|Q_\hbar(f)\| \quad \text{is continuous on } I, \tag{11}$$

$$\lim_{\hbar \to 0} \|Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)\| = 0, \tag{12}$$

$$\lim_{\hbar \to 0} \left\| \tfrac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\}) \right\| = 0, \tag{13}$$

and such that, extending $Q_\hbar$ to $Q_\hbar : \tilde{A}^0 \to A^\hbar$ by complex linearity, $Q_\hbar(\tilde{A}^0)$ is a dense *-subalgebra of $A^\hbar$, for each $\hbar \in I$.

For our convenience, we fix $\hbar \neq 0$, as we have done in Section 3. The map $Q_\hbar$ is called a *quantization map*. A standard example of a quantization map is Weyl quantization, denoted here by $Q_\hbar^{2n}$. (Keeping track of the phase space dimension $2n$ will be useful once we extend our results to infinite-dimensional symplectic spaces.) For a suitable function $f : \mathbb{R}^{2n} \to \mathbb{C}$, Weyl quantization is defined by

$$Q_\hbar^{2n}(f) := \int_{\mathbb{R}^{2n}} {\dj}y \, \hat{f}(y) e^{i\phi(y)}, \tag{14}$$

where $\hat{f}$ is the Fourier transform of f in the sense of Cordes, [4], which in general is not a function but a distribution. For example, the Fourier transform of $1_{\mathbb{R}^m}$ is a delta distribution. Also in keeping with Cordes, we denote ${\dj}y := (2\pi)^{-m/2} dy$ whenever y runs over $\mathbb{R}^m$. Notice that the $\hbar$-dependence of $Q_\hbar^{2n}$ comes from ϕ.

A suitable function in most contexts (for instance [6] and [8]) is a Schwartz function, $f \in \mathcal{S}(\mathbb{R}^{2n})$, but for more general f it is not immediately clear how the above integral is defined. Rieffel ([12]) works with Weyl quantization of functions in some bigger space, $\mathcal{B}(\mathbb{R}^{2n})$. We will work with the space $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$, for which we have $\mathcal{S} \subseteq \mathcal{S}_{\mathcal{R}} \subseteq \mathcal{B}$. In Section 4.1 we will define the integral in (14) for $f \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$, making $Q_\hbar^{2n}(f)$ an element of $B(H)$, and justifying our heuristic computations. For now, we just view (14) as a formal expression, and assume the basic rules of calculus apply to it.

Rieffel does not explicitly use (14), but uses an equivalent prescription. As explained in [13], Rieffel's results can be applied to show that Weyl quantization, when restricted to a *-subalgebra $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n}) \subseteq \mathcal{B}(\mathbb{R}^{2n})$, satisfies the first couple of requirements in Definition 4.1. For this version of Weyl quantization to be a strict deformation quantization, we only need to prove that

$Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}))$ is a dense *-subalgebra of $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$.

It turns out that Weyl quantization takes a concrete form on dikes, which will be used throughout this section. The especially appealing form of Weyl quantization on $g \circ p_x$ is discussed in §4.2. We use the notation $g \circ p$ from now on, contrary to Sections 2.1 and 2.2, because our definition of $Q_\hbar^{2n}$ is basis-dependent. Let $(e_1, \dots e_{2n})$ be the standard basis of $\mathbb{R}^{2n}$.

Proposition 4.2. *Let $(v_1, \dots v_r)$ be a basis of $V \subseteq \mathbb{R}^{2n}$. Define $p : \mathbb{R}^{2n} \to \mathbb{R}^r$ linearly by $v_j \mapsto e_j$, $p_{|V^\perp} = 0$. Define $B \in GL(\mathbb{R}^r)$ by its matrix elements $B_{jk} := v_j \cdot v_k$ with respect to the standard basis. Then for each $g \in \mathcal{S}(\mathbb{R}^r)$ we have*

$$Q_\hbar^{2n}(g \circ p) = \int_{\mathbb{R}^r} d\hskip-3pt\llap{/}\,x\, (g \circ B^{-1})\hat{}(x) e^{i \sum_{j=1}^r x_j \phi(v_j)}\,.$$

Proof. If we extend the basis (v_j) with $v_{r+1}, \dots, v_{2n} \in V^\perp$ to a basis of $\mathbb{R}^{2n}$, then we can define $R \in GL(\mathbb{R}^{2n})$ by $R : e_j \mapsto v_j$, and find

$$(g \circ p)\hat{}((R^t)^{-1}y) = |\det R|(g \circ p \circ R)\hat{}(y) = |\det R|(g \otimes 1)\hat{}(y)$$
$$= |\det R|\hat{g}(y_1, \dots, y_r)\delta(y_{r+1}, \dots, y_{2n})\,.$$

In the integral formula for Weyl quantization we can change variables by $y \mapsto (R^t)^{-1}y$ to obtain

$$Q_\hbar^{2n}(g \circ p) = \int_{\mathbb{R}^{2n}} d\hskip-3pt\llap{/}\,y\, |\det R|^{-1}(g \circ p)\hat{}((R^t)^{-1}y) e^{i\phi((R^t)^{-1}y)}$$
$$= \int_{\mathbb{R}^r} d\hskip-3pt\llap{/}\,x\, \hat{g}(x) e^{i\phi((R^t)^{-1}(x \oplus 0))}\,.$$

Since $R^t R$ has $\mathrm{span}\{e_1, \dots, e_r\}$ and $\mathrm{span}\{e_{r+1}, \dots, e_{2n}\}$ as invariant subspaces, we may write $R^t R = B \oplus C$ for $B \in GL(\mathbb{R}^r)$ and $C \in GL(\mathbb{R}^{2n-r})$. Indeed, $B_{jk} = v_j^t v_k$. Now it is worth changing variables once more, this time sending $x \mapsto Bx$, since $Bx \oplus 0 = R^t R(x \oplus 0)$. Because $B^t = B$, we find

$$Q_\hbar^{2n}(g \circ p) = \int_{\mathbb{R}^r} d\hskip-3pt\llap{/}\,x\, |\det B|\hat{g}(Bx) e^{i\phi(R(x \oplus 0))}$$
$$= \int_{\mathbb{R}^r} d\hskip-3pt\llap{/}\,x\, (g \circ B^{-1})\hat{}(x) e^{i \sum_{j=1}^r x_j \phi(v_j)}\,. \tag{15}$$

$\square$

4.1 The Operator-Valued Integral

We have done some calculus with integrals, while only using formal expressions. We will now show how to give a concrete meaning to these integrals, and hence to the results obtained so far.

Let $\{A_x\}_{x\in\mathbb{R}^r}$ be a family of bounded operators on H, and (K_j) an exhaustive sequence of compacts in $\mathbb{R}^r$. Whenever $x \mapsto A_x\psi$ is a continuous function $\mathbb{R}^r \to H$ for all $\psi \in H$, the Pettis integrals (as introduced in [10]) $\int_{K_j} A_x\psi dx$ exist, and we may define

$$\left(\int_{K_j} A_x dx\right)\psi := \int_{K_j} A_x\psi dx\,.$$

This gives a sequence $(\int_{K_j} A_x dx)$ of linear operators on H. If this is a Cauchy sequence in $B(H)$, then we define $\int_{\mathbb{R}^r} A_x dx$ as the limit of $\int_{K_j} A_x dx$, and call it the **operator-valued integral** of $\{A_x\}$.

The rules of calculus that we have used so far all hold for the Pettis integral on H, and using the operator norm on $B(H)$, we find that these rules hold for the operator-valued integral as well.

We apply the above general discussion to the situation at hand. The integrand of (15) is

$$A_x := (2\pi)^{-r/2}(g \circ B^{-1})\hat{\ }(x)e^{i\sum x_j\phi(\psi_j)}\,,$$

where $(g \circ B^{-1})\hat{\ }$ is Schwartz by Lemma A.1. For any $\psi \in H$, the function $x \mapsto A_x\psi$ is continuous by a multi-dimensional Stone's Theorem, called the Stone-Naimark-Ambrose-Godement Theorem. Hence the integral $\int_K A_x dx$ is defined for any compact K. We estimate

$$\left\|\int_K A_x\psi dx\right\|_2 \leqslant \int_K \đx\,|(g \circ B^{-1})\hat{\ }(x)|\,\left\|e^{i\sum x_j\phi(\psi_j)}\psi\right\|_2$$

$$\leqslant \|\psi\|_2 \int_K \đx\,|(g \circ B^{-1})\hat{\ }(x)|\,.$$

By this estimate, the sequence $(\int_{K_j} A_t dt)$ will be Cauchy in $B(H)$. Therefore (15), which equals $\int_{\mathbb{R}^r} A_x dx$, is defined. Following the proof of Proposition 4.2 backwards, we find that all integral expressions there can be defined as explained above for the bounded operator (15). Thus we have defined $Q_\hbar^{2n}(g \circ p)$ as a bounded operator on H.

Definition 4.3. *For a fixed $\hbar \neq 0$, Weyl quantization is the map*

$$Q_\hbar^{2n} : \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}) \to B(H) ,$$

$$Q_\hbar^{2n}(f) := \int_{\mathbb{R}^{2n}} d\hspace{-0.25em}\bar{}\hspace{0.1em}y \, \hat{f}(y) e^{i\phi(y)} ,$$

which is defined by the above discussion.

4.2 Weyl Quantization on Functions of One Variable

Let us apply Proposition 4.2 to the case $r = 1$. Fix a nonzero vector $x \in \mathbb{R}^{2n}$. Let the basis $(v_1, \ldots, v_r)$ consist solely of $v_1 = x/\|x\|^2$, and $p_x(y) := x \cdot y$. If $g \in \mathcal{S}(\mathbb{R})$, then the dike $g \circ p_x$ is quantized by the operator

$$Q_\hbar^{2n}(g \circ p_x) = \int_{\mathbb{R}} d\hspace{-0.25em}\bar{}\hspace{0.1em}y \, \hat{g}(y) e^{iy\phi(x)} \equiv \Phi_x(g) . \tag{16}$$

This defines a map $\Phi_x : \mathcal{S}(\mathbb{R}) \to B(H)$, which occurs in several places in the literature as a functional calculus of $\phi(x)$. It can be continuously extended.

Lemma 4.4. *There is a (norm-continuous) *-homomorphism $\Phi_x : C_0(\mathbb{R}) \to B(H)$ which equals the integral expression in (16) whenever $\hat{g} \in L^1(\mathbb{R})$.*

Proof. We can use (16) to show that Φ_x is a *-homomorphism on $\mathcal{S}(\mathbb{R})$. Indeed, when x is fixed, the operators $e^{iy\phi(x)}$ behave as the functions $t \mapsto e^{iyt}$ under the operations addition, involution and multiplication. By standard Fourier analysis it follows that $\Phi_{x\restriction\mathcal{S}(\mathbb{R})}$ preserves these operations.
It is known that the Fourier transform $\hat{\,} : C_0(\mathbb{R}) \to C^*(\mathbb{R})$ is a *-isomorphism. For a fixed $f \in L^1(\mathbb{R})$, define $\rho(f) := \int d\hspace{-0.25em}\bar{}\hspace{0.1em}y \, f(y) e^{iy\phi(x)}$, so that $\Phi_x(f) = \rho(\hat{f})$. Now $u(y) := e^{iy\phi(x)}$ is a unitary representation, and as such $\|\rho(f)\| \leqslant \|f\|_*$, by definition of the norm $\|\cdot\|_*$ on $C^*(\mathbb{R})$. (See for example [9], section C.18.) Extending ρ to all of $C^*(\mathbb{R})$, we can define $\Phi_x(f) := \rho(\hat{f})$ for $f \in C^*(\mathbb{R})$ and again find $\|\rho(f)\| \leqslant \|f\|_*$. We obtain

$$\|\Phi_x(f)\| = \left\|\rho(\hat{f})\right\| \leqslant \left\|\hat{f}\right\|_* = \|f\|_\infty ,$$

implying norm-continuity. Since Φ_x is a *-homomorphism on a dense domain and continous, it is a *-homomorphism on the whole of $C_0(\mathbb{R})$. $\qquad\square$

Proposition 4.5. $\Phi_x(1/(i\lambda - \cdot)) = R(\lambda, x)$.

Proof. The Fourier transform of $1/(i\lambda - \cdot)$ is:

$$\left(\frac{1}{i\lambda - \cdot}\right)^\frown(t) = -i\sqrt{2\pi}\,\mathrm{sgn}(\lambda)e^{\lambda t}\theta(-\lambda t)\,,$$

where θ is the Heaviside step function. The above function is clearly in $L^1(\mathbb{R})$. Applying (16) yields

$$\Phi_x\left(\frac{1}{i\lambda - \cdot}\right) = -i\mathrm{sgn}(\lambda)\int_{-\infty}^\infty e^{\lambda t}\theta(-\lambda t)e^{it\phi(x)}dt$$

$$= i\int_0^{-\mathrm{sgn}(\lambda)\infty} e^{\lambda t}e^{it\phi(x)}dt\,,$$

by distinguishing the two cases $\mathrm{sgn}(\lambda) = \pm 1$. Now a change of variables gives

$$\Phi_x\left(\frac{1}{i\lambda - \cdot}\right) = -i\int_0^{\mathrm{sgn}(\lambda)\infty} e^{-\lambda t}e^{-it\phi(x)}dt = R(\lambda, x)\,,$$

by the Laplace transform. See for instance [2], Corollary 4.4. $\qquad\square$

Let us stress the significance of the main result of §4.2. Weyl quantization, considered on functions of one variable, is the usual functional calculus of self-adjoint operators (for instance treated in [9]). To put it symbolically, we have

$$Q_\hbar^{2n}(g \circ p_x) = g(\phi(x))\,.$$

As explained in the introduction, this approach via dikes is an equivalent way to introduce Weyl quantization.

4.3   Dense Subalgebra of the Resolvent Algebra

We are now in a position to rewrite $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R})$. Recall that $\mathcal{S}_\mathcal{R}$ is the linear span of

$$\left\{g \circ p \mid p : \mathbb{R}^{2n} \to \mathbb{R}^r \text{ linear and surjective, } g \in \mathcal{S}(\mathbb{R}^r),\ r \leqslant 2n\right\}\,.$$

We once again fix a subspace $V \subseteq \mathbb{R}^{2n}$ and a basis $(v_1, \ldots, v_r)$ of V, more briefly denoted by (v_j). We say an operator A is (v_j)-Schwartz iff

$$A = \int_{\mathbb{R}^r} dx\,\hat{g}(x)e^{i\sum_{j=1}^r x_j\phi(v_j)} \text{ for some } g \in \mathcal{S}(\mathbb{R}^r)\,.$$

Loosely speaking, being (v_j)-Schwartz means that A has a symbol (see [4]) that is Schwartz in the direction of (v_j). This property should not depend on the choice of basis, and that is indeed the case.

Lemma 4.6. *Let (v_j) and (v'_j) be two bases of V. Then A is (v_j)-Schwartz iff A is (v'_j)-Schwartz, in which case we say A is V-Schwartz. Thus*

$$Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}) = \mathrm{span}\left\{A \mid A \text{ is } V\text{-Schwartz for some linear } V \subseteq \mathbb{R}^{2n}\right\}.$$

Proof. Suppose A is (v_j)-Schwartz. Then Proposition 4.2 lets us write $A = Q_\hbar^{2n}(g \circ p)$ for some $g \in \mathcal{S}(\mathbb{R}^r)$ and $p : v_j \mapsto e_j$. Now (pv'_j) forms a basis of $\mathbb{R}^r$, so there exists an $R \in GL(\mathbb{R}^r)$ such that $Rpv'_j = e_j$. Now $g \circ R^{-1} \in \mathcal{S}(\mathbb{R}^r)$ by Lemma A.1. It follows by Proposition 4.2 that $A = Q_\hbar^{2n}((g \circ R^{-1}) \circ (R \circ p))$ is (v'_j)-Schwartz.

Now we may safely use the statement "A is V-Schwartz". The last claim follows from Proposition 4.2, since for any $B \in GL(\mathbb{R}^r)$, g is Schwartz iff $g \circ B^{-1}$ is Schwartz (again by Lemma A.1). $\qquad\square$

The class of V-Schwartz operators ties together (a subset of) the compact operators[4] and the Schwartz functions[5]. What is useful for us, is that V-Schwartz operators form a *-algebra, as we will now prove.

Theorem 4.7. *The set $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}))$ is a *-algebra within $B(H)$.*

Proof. Closure under involution is easily checked, as

$$Q_\hbar^{2n}(f)^* = \int_{\mathbb{R}^{2n}} \!\!\!d\!\!\!/ y \, \overline{\hat{f}(y)} e^{-i\phi(y)} = \int_{\mathbb{R}^{2n}} \!\!\!d\!\!\!/ y \, \overline{\hat{f}(-y)} e^{i\phi(y)} = Q_\hbar^{2n}(f^*). \qquad (17)$$

The real problem here is closure under multiplication, but Lemma 4.6 provides a solution. Suppose $A_1, A_2 \in Q_\hbar^{2n}(\mathcal{S}_\mathcal{R})$ are V_1- and V_2-Schwartz, respectively. Then we may choose bases $(v_1^1, \ldots, v_{r_1}^1)$, $(v_1^2, \ldots, v_{r_2}^2)$ for V_1 and V_2 respectively, with the property that $v_j^1 = v_j^2$ for all $j \leqslant r := \dim V_1 \cap V_2$. For appropriate $g_k \in \mathcal{S}(\mathbb{R}^{r_k})$, we find

$$A_1 A_2 = \int \!\!\!d\!\!\!/ t \, g_1(t) e^{i \sum t_j \phi(v_j^1)} \int \!\!\!d\!\!\!/ s \, g_2(s) e^{i \sum s_j \phi(v_j^2)}$$

$$= \int \!\!\!d\!\!\!/ t d\!\!\!/ s \, g_1(t) g_2(s) f(t, s) e^{i \sum t_j \phi(v_j^1) + s_j \phi(v_j^2)},$$

---

[4]If V is nondegenerate, (meaning that $(V, \sigma_{\restriction V})$ is a symplectic space) then a V-Schwartz operator can be written as $A \otimes \mathbb{1}$, with compact operator A, for some appropriate factorization of H. We do not prove this fact, but it follows from our results.

[5]If V is fully degenerate, (meaning that $\sigma_{\restriction V} = 0$,) then a V-Schwartz operator is a Schwartz function applied (by functional calculus) to a set of commuting operators. Just as in the previous footnote, this fact brings some intuition, but it is not needed for our eventual goal.

where $f(t,s) = e^{\frac{i\hbar}{2}\sum_{j,k}\sigma(t_j v_j^1, s_k v_k^2)}$ by the Baker-Campbell-Hausdorff formula. We write $t^0 = (t_1,\ldots,t_r)$, $t^k = (t_{r+1},\ldots,t_{r_k})$. A change of variables $t^0 \mapsto t^0 - s^0$ gives

$$A_1 A_2 = \int \!\!\!\!/ dt \!\!\!\!/ ds\, g_1(t^0 - s^0, t^1) g_2(s) f(t^0 - s^0, t^1, s) e^{i\sum_{j=1}^{r_1} t_j \phi(v_j^1) + i\sum_{j=r+1}^{r_2} s_j \phi(v_j^2)}$$

$$= \int \!\!\!\!/ dt \!\!\!\!/ ds^2\, g(t, s^2) e^{i\sum_{j=1}^{r_1} t_j \phi(v_j^1) + i\sum_{j=r+1}^{r_2} s_j \phi(v_j^2)} \,,$$

where we define the function g by

$$g(x, y, z) := \int_{\mathbb{R}^r} \!\!\!\!/ dw\, g_1(x - w, y) g_2(w, z) f(x - w, y, z) \,,$$

for all $x \in \mathbb{R}^r$, $y \in \mathbb{R}^{r_1 - r}$, $z \in \mathbb{R}^{r_2 - r}$. We want to show that g is Schwartz. Using the Leibniz rule, $\partial^\beta g$ is a linear combination of functions of the form

$$h_\gamma(x, y, z) := \int \!\!\!\!/ dw\, \partial_x^{\gamma_1} \partial_y^{\gamma_2} g_1(x - w, y) \partial_z^{\gamma_3} g_2(w, z) \partial^{\beta - \gamma} f(x - w, y, z),$$

where $\partial_x^\alpha := \partial^{\alpha \oplus 0 \oplus 0}$ et cetera. We now investigate the absolute value of h_γ. Because of the form of f, we have $|\partial^{\beta-\gamma} f(x-w, y, z)| = |p_1(x-w)p_2(y)p_3(z)|$ for some polynomials p_j which implicitly depend on β and γ, but we leave out such β, γ dependence from now on. We can absorb the derivatives, polynomials and absolute value into the functions g_j, giving rapidly decreasing functions $\tilde{g}_j$, (see Appendix A,) such that

$$|h_\gamma(x, y, z)| = \int \!\!\!\!/ dw\, \tilde{g}_1(x - w, y) \tilde{g}_2(w, z) \,.$$

Therefore, $|h_\gamma|$ is rapidly decreasing by Lemma A.2, and consequently

$$\sup_x |x^\alpha \partial^\beta g| \leqslant \sum_\gamma c_\gamma \sup_x |x^\alpha h_\gamma(x)| < \infty \,,$$

proving that $g \in \mathcal{S}(R^{r_1 + r_2 - r})$.

Since $(v_1^1, \ldots v_{r_1}^1, v_{r+1}^2, \ldots v_{r_2}^2)$ is a basis of $V_1 + V_2$, it follows that $A_1 A_2$ is $V_1 + V_2$-Schwartz. By Lemma 4.6, $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R})$ is therefore a *-algebra. $\qquad \square$

Theorem 4.8. *We have* $\overline{Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}))} = \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$.

Proof. Since $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R})$ is a *-algebra, we want its closure to contain $R(\lambda, x)$. We know that $R(\lambda, x) = \Phi_x(1/(i\lambda - \cdot))$, by Proposition 4.5. Take a sequence (g_k) in $\mathcal{S}(\mathbb{R})$ converging to $1/(i\lambda - \cdot)$ in norm. Then $\Phi_x(g_k)$ converges

to $R(\lambda, x)$ by Lemma 4.4. Therefore, $R(\lambda, x) \in \overline{Q_\hbar^W(\mathcal{S_R})}$. It follows that $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n) \subseteq \overline{Q_\hbar^W(\mathcal{S_R})}$.

By Lemma 4.6 we are left to show that every V-Schwartz operator $A \in B(H)$ is contained in $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$. We do this by induction in $\dim V$. Let $(v_1, \ldots v_r)$ be a basis of some $V \subseteq \mathbb{R}^{2n}$, and fix the appropriate $g \in \mathcal{S}(\mathbb{R}^r)$ such that

$$A = \int_{\mathbb{R}^r} dx\, \hat{g}(x) e^{i \sum x_j \phi(v_j)}.$$

Let $f : \mathbb{R}^r \to S_1$ be such that $e^{i \sum x_j \phi(v_j)} = f(x) e^{i x_1 \phi(v_1)} e^{i \sum_{j=2}^r x_j \phi(v_j)}$. Keeping in mind that $\mathcal{S}(\mathbb{R}^r) = \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}^{r-1})$ with respect to the Schwartz topology, assume for now that $\hat{g}f = \hat{g}_1 \otimes \hat{g}_2$. In that case

$$A = \int_{\mathbb{R}^r} dx\, \hat{g}_1(x_1) \hat{g}_2(x_2, \ldots, x_r) e^{i x_1 \phi(v_1)} e^{i \sum_{j=2}^r x_j \phi(v_j)}$$

$$= \Phi_{v_1}(g_1) \int_{\mathbb{R}^{r-1}} dy\, \hat{g}_2(y) e^{i \sum_{j=1}^{r-1} y_j \phi(v_{j+1})}.$$

By the induction hypothesis the latter integral is in $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$, and by the Stone-Weierstrass theorem we can approximate g_1 by polynomials in $1/(i\lambda - \cdot)$. By Lemma 4.4 and Proposition 4.5, Φ_{v_1} maps these polynomials to $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$. By continuity of Φ_{v_1} it follows that also $\Phi_{v_1}(g_1) \in \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$. So $A \in \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ whenever $\hat{g}f$ is an elementary tensor or, by linearity of $\mathcal{R}$, is a finite sum of those. Suppose now that $\hat{g}f$ is not of this form, but $\hat{g}_k f$ is, so that $A_k := \int_{\mathbb{R}^r} dx\, \hat{g}_k(x) e^{i \sum x_j \phi(v_j)} \in \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$, and assume that $\hat{g}_k f \to \hat{g}f$ in the Schwartz topology. We find

$$\|A - A_k\| = \left\| \int dx\, (g - g_k)\hat{}(x) e^{i \sum x_j \phi(v_j)} \right\|$$

$$\leqslant \int dx\, |(g - g_k)\hat{}(x)| \left\| e^{i \sum x_j \phi(v_j)} \right\|$$

$$= \int dx\, |(\hat{g}f - \hat{g}_k f)(x)|.$$

The sequence $(\hat{g}_k f)$ converges to $\hat{g}f$ in the Schwartz topology, hence also in L^1-norm. This gives us $\|A - A_k\| \to 0$, proving that $A \in \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$. $\qquad \square$

4.4 Strict Deformation Quantization

At the beginning of Section 4 we have suggested that, because $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R})$ is a dense *-subalgebra of the resolvent algebra, we have achieved our goal in the finite case. However, giving a rigorous proof needs some careful work. We will begin by introducing the concepts needed for a precise formulation of the main result of Section 4.

The real part $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R}$ of $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$ consists of the functions f that satisfy $f^* = f$, where the involution is given by complex conjugation. Weyl quantization $Q_\hbar^{2n}$ is *-preserving, by (17). Therefore, $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R}) \subseteq \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)_\mathbb{R}$, and it makes sense to restrict $Q_\hbar^{2n}$ to a map $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R} \to \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)_\mathbb{R}$. The complex linear extension of this latter map is simply $Q_\hbar^{2n}$, because Weyl quantization is complex linear by definition.

On $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R}$ we can put the Poisson structure of $C^\infty(\mathbb{R}^{2n}; \mathbb{R})$, in effect defining

$$\{f, g\} := \sigma_n(\partial f, \partial g),$$

where $\partial f, \partial g$ are vector-valued functions to which we pointwise apply σ_n. Note that $\{f, g\}$ is just a polynomial of partial derivatives $\partial_j f, \partial_j g$. By applying the chain rule to a dike, we see that $\partial_j f, \partial_j g \in \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$, and hence $\{f, g\} \in \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R}$. This makes $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R}$ a Poisson algebra, and we may state the main result of this section.

Theorem 4.9. *Let $A^0 := C_\mathcal{R}(\mathbb{R}^{2n})$ and $A^\hbar := \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ for $\hbar > 0$. The set $I = [0, 1]$, together with the collection of C*-algebras $\{A^\hbar\}_{\hbar \in I}$, and the maps $Q_\hbar^{2n} : \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R} \to \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)_\mathbb{R}$, constitute a strict deformation quantization of $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})_\mathbb{R}$.*

Proof. Let us adopt some notation from the article [13] of Rieffel, because we will be using one of his results. We have the classical function algebra

$$\mathcal{B} := \left\{ f \in C^\infty(\mathbb{R}^{2n}) \mid \|\partial^\alpha f\|_\infty < \infty \text{ for all } \alpha \in \mathbb{N}^{2n} \right\},$$

on which Rieffel defines an alternative product, $\times_\hbar$, and norm, $\|\cdot\|_\hbar$, for every $\hbar \in I$. The $\|\cdot\|_\hbar$-completion of $\mathcal{B}$, equipped with linear and involutive structure from $\mathcal{B}$, and equipped with the product $\times_\hbar$, is denoted by $\overline{\mathcal{B}}_\hbar$. Rieffel proves that this is a C*-algebra. Now we shall stray from the course taken by Rieffel, and define $Q_\hbar^R : \mathcal{B} \hookrightarrow \overline{\mathcal{B}}_\hbar$ as the canonical embedding. Because the maps $Q_\hbar^R$ are *-preserving, just like $Q_\hbar^{2n}$, we may ask if these, together with I and $\{\overline{\mathcal{B}}_\hbar\}_\hbar$, constitute a strict deformation quantization. Indeed, the facts (1) and (2) on page 73 of [13] imply the axioms (11), (12) and (13) of

our Definition 4.1. By definition, each map $Q_\hbar^R$ is injective, linear and maps $\mathcal{B}$ to a dense *-subalgebra of $\overline{\mathcal{B}}_\hbar$.

Thus the maps $Q_\hbar^R$ give us a strict deformation quantization, which remains the case when they are composed with *-isomorphisms $\pi_\hbar$ (assuming $\pi_0 = \text{id}$). As explained in [13], the prescription

$$\pi_\hbar(f) := \int \mathrm{d}y \, f(y) e^{i\phi(y)} \qquad (f \in \mathcal{B})$$

defines an irreducible *-representation of $\overline{\mathcal{B}}_\hbar$ on H, for each $\hbar > 0$. When we define $Q_0^{2n} = \text{id}_{\mathcal{S}_\mathcal{R}}$ as well as $\pi_0 := \text{id}_\mathcal{B}$, it is clear that $\pi_\hbar \circ Q_{\hbar|\mathcal{S}_\mathcal{R}}^R = Q_\hbar^{2n}$ for each $\hbar \in I$. Restricting the quantization maps to a subalgebra has effect on neither their injectivity and linearity, nor on the axioms (11), (12) and (13). The only thing left to prove is that $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R})$ is a dense *-subalgebra of $A^\hbar$. This is exactly the statement that we have worked towards. For $\hbar = 0$ it follows from Proposition 2.5, and for $\hbar > 0$ it is the combination of Theorem 4.7 and Theorem 4.8. $\qquad\square$

We note here that even though Definition 4.1 (i.e. that of strict deformation quantization) does not explicitly demand $\{A^\hbar\}_\hbar$ to be a continuous field of C*-algebras, this existence is a consequence. In fact, the following corollary follows from Theorem 1.2.4 in [8].

Corollary 4.10. *Let $A^0 := C_\mathcal{R}(\mathbb{R}^{2n})$ and $A^\hbar := \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ for $\hbar > 0$. There exists a unique continuous field of C*-algebras $(C, \{A^\hbar, \varphi_\hbar\}_{\hbar \in [0,1]})$ whose collection of sections $\{\varphi_\hbar(A)\}_{\hbar \in [0,1]}$, $A \in C$, contains all $\{Q_\hbar(f)\}_{\hbar \in [0,1]}$ for $f \in \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$.*

The tuple $(C_\mathcal{R}(\mathbb{R}^{2n}), \mathcal{R}(\mathbb{R}^{2n}, \sigma_n))$ may be added to the list of existing strict deformation quantizations on $\mathbb{R}^{2n}$, using (some generalization of) Weyl quantization as quantization map. These can be fitted into the diagram

$$(C_0(\mathbb{R}^{2n}), K(H))$$
$$\downarrow$$
$$(C_\mathcal{R}(\mathbb{R}^{2n}), \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)) \qquad\qquad (\mathcal{W}(\mathbb{R}^{2n}, 0), \mathcal{W}(\mathbb{R}^{2n}, \sigma_n))$$
$$\searrow \qquad\qquad \swarrow$$
$$(C_u(\mathbb{R}^{2n}), \overline{\mathcal{B}}_\hbar)$$

in which arrows depict inclusion of the corresponding C*-algebras, $K(H)$ is the space of compact operators, $\mathcal{W}(\mathbb{R}^{2n}, 0)$ is the space of almost continuous functions (as in [1]), $\mathcal{W}(\mathbb{R}^{2n}, \sigma_n)$ is the Weyl algebra and $C_u(\mathbb{R}^{2n})$ is the space

of bounded uniformly continuous functions (as in [13]).

We have already foreshadowed that our work does not stop here, upon giving a strict deformation quantization involving the resolvent algebra on $\mathbb{R}^{2n}$, and will now advance to deal with the general version of the resolvent algebra. Still, the hardest part is now behind us. The general resolvent algebra is a direct limit of resolvent algebras on $\mathbb{R}^{2n}$ (as $n \to \infty$), allowing us to construct a quantization map which generalizes Weyl quantization.

5 Commutative Resolvent Algebra: General Case

Let X be a Pre-Hilbert space, which means a complex vector space (possibly infinite dimensional) with a Hermitian inner product $\langle \cdot, \cdot \rangle$, from which X derives its topology. We view X as a *real* vector space. By this we mean that, unless noted otherwise, we use the real structure on X. We can then view X as a symplectic space with symplectic form[6]

$$\sigma(x, y) := \mathrm{Im}\langle x, y \rangle \,.$$

It will prove useful to put a *real* inner product on X, which we do by defining

$$\langle x, y \rangle_{\mathbb{R}} := \mathrm{Re}\langle x, y \rangle \,.$$

When X has this structure, we refer to it as a symplectic space admitting a unitary structure. This is what Buchholz and Grundling usually assume for the symplectic vector space on which they define the resolvent algebra. It is this X that will replace our earlier $\mathbb{R}^{2n}$, and it should come as no surprise that we can define the commutative resolvent algebra in the following way.

Definition 5.1. *For $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in X$ define $h_x^\lambda(y) := 1/(i\lambda - \langle x, y \rangle_{\mathbb{R}})$. The **commutative resolvent algebra** over X, denoted by $C_{\mathcal{R}}(X)$, or simply by $C_{\mathcal{R}}$, is the C*-subalgebra of $C_b(X)$ generated by the functions h_x^λ.*

We will not go through the lengths of generalizing §2.1 and §2.2 to infinite dimensions, as this –in our opinion– will not yield much additional insight. Instead we briefly give a way to pass to $C_{\mathcal{R}}(X)$, starting from $C_{\mathcal{R}}(V)$ for finite dimensional $V \subseteq X$.

We can embed $C_{\mathcal{R}}(V) \hookrightarrow C_{\mathcal{R}}(W)$ whenever $V \subseteq W$ by composing with the projection onto V, thus sending $f \mapsto f \circ P_V$. This gives us a directed system $\{C_{\mathcal{R}}(V)\}_{V \subseteq X \text{ f.d.}}$ (f.d. meaning finite dimensional). Since $C_{\mathcal{R}}(X)$ is generated by the functions $h_x^\lambda = h_x^\lambda \circ P_{\mathrm{span}\{x\}}$, it is generated by the subalgebras $C_{\mathcal{R}}(\mathrm{span}\{x\})$. Therefore, $C_{\mathcal{R}}(X)$ is the direct limit of $\{C_{\mathcal{R}}(V)\}_{V \subseteq X \text{ f.d.}}$, in the sense of C*-algebras.

---

[6]As before we mean a bilinear anti-symmetric nondegenerate form. A form σ on a Banach space E under these assumptions is called a weak symplectic form, and is called strong iff $E \to E^*$, $x \mapsto \sigma(x, \cdot)$ is an isomorphism of Banach spaces. The Banach space $\overline{X}$ is in fact a Hilbert space, on which any weak symplectic form is a strong symplectic form. So the distinction is irrelevant here.

5.1 The Smooth Commutative Resolvent Algebra

In the finite case the commutative resolvent algebra is densely spanned by the functions $g \circ p$, which are basically functions that are Schwartz in, say, r directions and are constant in the other $2n - r$ directions. The infinite dimensional generalization thereof is close to the finite dimensional case. It consists of functions that are Schwartz in r directions, and are constant in the other (cofinitely many) directions. Now continuity is not yet garanteed, so we put this assumption on p, defining

$$
\mathcal{S}_{\mathcal{R}}(X) := \text{span} \left\{ g \circ p \; \middle| \; \begin{array}{l} p : X \twoheadrightarrow \mathbb{R}^r \text{ linear, continuous,} \\ g \in \mathcal{S}(\mathbb{R}^r) \text{ for } r \in \mathbb{N} \end{array} \right\} . \tag{18}
$$

When $X = \mathbb{R}^{2n}$, equation (18) coincides with our previous definition. For general X, we will regard $\mathcal{S}_{\mathcal{R}}(X)$ as the direct limit of finite dimensional versions, similar to the case of $C_{\mathcal{R}}$. But contrary to the case of $C_{\mathcal{R}}$, we will use a basis-dependent approach. We use the symplectic space $(\mathbb{R}^{2n}, \sigma_n)$ as defined in Section 3. If a linear isomorphism between two symplectic spaces preserves their symplectic forms, we call it a symplectomorphism, abbreviated 'sympl.'. For a complex-linear subspace $V \subseteq X$ (this implies that $(V, \sigma_{\restriction V})$ is a symplectic space) of real dimension $2n$, we define the space

$$
\mathcal{P}_{2n}^V := \left\{ p : X \to \mathbb{R}^{2n} \; \middle| \; p_{\restriction V} \text{ sympl.}, \; p_{\restriction V^\perp} = 0 \right\} ,
$$

and let $\mathcal{P}$ be the union of all possible $\mathcal{P}_{2n}^V$. For all $p \in \mathcal{P}_{2n}^V$ we define $s_p := p_{\restriction V}^{-1}$, and remark that s_p is a section of p. We also define

$$
\mathcal{S}_{\mathcal{R}}(X)^p := \left\{ f \circ p \; \middle| \; f \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n}) \right\} .
$$

Every $g \circ p$ in $\mathcal{S}_{\mathcal{R}}(X)$ can be written as $f \circ q$ for some $q \in \mathcal{P}_{2n}^V$ and $f \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$, for arbitrarily large V. Hence we find that

$$
\mathcal{S}_{\mathcal{R}}(X) = \bigcup_{p \in \mathcal{P}} \mathcal{S}_{\mathcal{R}}(X)^p . \tag{19}
$$

For any $p \in \mathcal{P}_{2n}^V$, the function algebra $\mathcal{S}_{\mathcal{R}}(X)^p$ is isometrically isomorphic to $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$, and so in a way $\mathcal{S}_{\mathcal{R}}(X)$ is built from the $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$'s. This is useful as we have already seen how to quantize $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$. Concretely, the isomorphism $\mathcal{S}_{\mathcal{R}}(X)^p \xrightarrow{\sim} \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$ that we use is the pull-back (see Lemma 2.2) of the symplectomorphism $p' : \mathbb{R}^{2n} \to V$, defined by

$$
p'x := i \; s_p(J_n x) . \tag{20}
$$

Here we use the complex linear structure on X when multiplying by i. Recall that J_n is the standard symplectic matrix, which can be viewed as the

analogue of $-i$. To justify our notation, note that p' is dual to p in the sense that $\langle p'x, v \rangle_{\mathbb{R}} = x \cdot pv$ for all $v \in V$ and $x \in \mathbb{R}^{2n}$.

An important property of the subspaces $\mathcal{S}_{\mathcal{R}}(X)^p$ is the following.

Lemma 5.2. *Assume $p \in \mathcal{P}_{2n}^V$ and $q \in \mathcal{P}_{2m}^W$. If $V \subseteq W$, then $\mathcal{S}_{\mathcal{R}}(X)^p \subseteq \mathcal{S}_{\mathcal{R}}(X)^q$.*

Proof. Let $f \circ p \in \mathcal{S}_{\mathcal{R}}(X)^p$ for $f \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$. We may assume that $f = g \circ e$ for some linear $e : \mathbb{R}^{2n} \twoheadrightarrow \mathbb{R}^r$ and $g \in \mathcal{S}(\mathbb{R}^r)$, giving the following diagram:

$$
\begin{array}{ccccccc}
W & \xrightarrow{\ p\ } & \mathbb{R}^{2n} & \xrightarrow{\ e\ } & \mathbb{R}^r & \xrightarrow{\ g\ } & \mathbb{C} \\
 & \searrow{\scriptstyle q} & \uparrow{\scriptstyle p \circ s_q} & & & & \\
 & & \mathbb{R}^{2m} & & & &
\end{array}
$$

From this diagram it can be seen that $e \circ p \circ s_q$ is surjective, giving $g \circ e \circ p \circ s_q \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2m})$. We conclude that

$$
\begin{aligned}
f \circ p &= g \circ e \circ p \\
&= g \circ e \circ p \circ s_q \circ q,
\end{aligned}
$$

which is in $\mathcal{S}_{\mathcal{R}}(X)^q$. $\qquad\square$

We may now write $\mathcal{S}_{\mathcal{R}}(X)^V := \mathcal{S}_{\mathcal{R}}(X)^p$ for a certain (and hence all) $p \in \mathcal{P}_{2n}^V$. We find that $I := \left\{ V \subseteq X \mid V \text{ is f.d. and complex linear} \right\}$ is a directed set, over which $\{\mathcal{S}_{\mathcal{R}}(X)^V\}_{V \in I}$ is a direct system, with inclusions as connecting maps. Because of (19), the direct limit of this system is $\mathcal{S}_{\mathcal{R}}(X)$.

As $C_{\mathcal{R}}(X)$ is the direct limit of $\{C_{\mathcal{R}}(V)\}_{V \in I}$, the next proposition follows directly from its finite dimensional analogue, Proposition 2.5.

Proposition 5.3. *The space $\mathcal{S}_{\mathcal{R}}(X)$ is a dense *-subalgebra of $C_{\mathcal{R}}(X)$.*

5.2 Poisson Structure

In this part we will investigate $\mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}$, the real part of $\mathcal{S}_{\mathcal{R}}(X)$, as a real *-algebra. We will give $\mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}$ a Poisson structure by noting that

$$
\mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}} = \bigcup_{p \in \mathcal{P}} \mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}^p, \quad \mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}^p \simeq \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})_{\mathbb{R}} \text{ when } p \in \mathcal{P}_{2n}^V.
$$

We can transfer the Poisson structure of $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})_{\mathbb{R}}$ to $\mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}^p$. Explicitly, the definition of the Poisson bracket $\{\cdot, \cdot\}_p$ on $\mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}^p$ reads

$$
\{f \circ p, g \circ p\}_p := \{f, g\} \circ p = \sigma_n(\partial f, \partial g) \circ p \quad (f, g \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})).
$$

Lemma 5.4. *Assume $p \in \mathcal{P}_{2n}^V$ and $q \in \mathcal{P}_{2m}^W$, and let $f, g \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$ be such that $f \circ p, g \circ p \in \mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}^p \cap \mathcal{S}_{\mathcal{R}}(X)_{\mathbb{R}}^q$. We then have*

$$\{f \circ p, g \circ p\}_p = \{f \circ p, g \circ p\}_q.$$

Proof. We first prove this when $V = W$. In this case, $f \circ p = f \circ T \circ q$ and $g \circ p = g \circ T \circ q$ for the symplectic isomorphism $T = p \circ s_q$ on $\mathbb{R}^{2n}$. The Poisson bracket is invariant under T, hence

$$
\begin{aligned}
\{f \circ p, g \circ p\}_q &= \{f \circ T, g \circ T\} \circ q \\
&= \{f, g\} \circ T \circ q \\
&= \{f \circ p, g \circ p\}_p.
\end{aligned}
$$

By Lemma 5.2, we may restrict ourselves to the case $V \subseteq W$. By the above, we may choose p and q to our liking. We can write $W = V \oplus U$ for $U := W \cap V^\perp$, and choose p, q such that $q_{|W} = p_{|V} \oplus \tilde{q}$ for some $\tilde{q} : U \xrightarrow{\sim} \mathbb{R}^{2m-2n}$. It follows that $h \circ p \circ s_q = h \otimes 1$ for any $h \in \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n})$, using the tensor product $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n}) \otimes \mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2m-2n})$. Because the Poisson bracket factors through this tensor product, we obtain

$$
\begin{aligned}
\{f \circ p, g \circ p\}_q &= \{f \otimes 1, g \otimes 1\} \circ q \\
&= (\{f, g\} \otimes 1) \circ q \\
&= \{f, g\} \circ p,
\end{aligned}
$$

which implies the lemma. $\qquad\square$

We are therefore able to define

$$\{f, g\} := \{f, g\}_p \qquad \text{whenever } f, g \in \mathcal{S}_{\mathcal{R}}(X)^p.$$

In order to prove that $\{\cdot, \cdot\}$ is a Poisson bracket, some conditions (bilinearity, antisymmetry, Leibniz rule, Jacobi identity) should hold. Because every triple of $f, g, h \in \mathcal{S}_{\mathcal{R}}(X)$ has a $p \in \mathcal{P}$ such that $f, g, h \in \mathcal{S}_{\mathcal{R}}(X)^p$, these conditions follow directly from those on $\{\cdot, \cdot\}_p$. This makes $\mathcal{S}_{\mathcal{R}}(X)$ a Poisson algebra, as desired.

6 Resolvent Algebra: General Case

The resolvent algebra was introduced by Buchholz and Grundling in [2]. We copy this definition (following [3]) to have an easy reference. As always, X is a symplectic space admitting a unitary structure.[7]

Definition 6.1. *Define $\mathcal{R}_0(X, \sigma)$ as the universal unital *-algebra generated by the set $\{R(\lambda, x) \mid \lambda \in \mathbb{R} \backslash \{0\}, \ x \in X\}$ and the relations*

$$R(\lambda, 0) = -\tfrac{i}{\lambda} \mathbb{1} \,,$$
$$R(\lambda, x) - R(\mu, x) = i(\mu - \lambda)R(\lambda, x)R(\mu, x) \,,$$
$$R(\lambda, x)^* = R(-\lambda, x) \,,$$
$$[R(\lambda, x), R(\mu, y)] = i\sigma(x, y)R(\lambda, x)R(\mu, y)^2 R(\lambda, x) \,,$$
$$\nu R(\nu\lambda, \nu x) = R(\lambda, x) \,,$$
$$R(\lambda, x)R(\mu, y) = R(\lambda + \mu, x + y)\big[R(\lambda, x)$$
$$+ R(\mu, y) + i\sigma(x, y)R(\lambda, x)^2 R(\mu, y)\big] \,.$$

Let $\mathfrak{S}$ denote the set of positive, normalized functionals (i.e. states) ω of $\mathcal{R}_0(X, \sigma)$. By Proposition 3.3 of [2], the corresponding GNS-representations (π_ω, H_ω) are uniformly bounded with respect to $\mathfrak{S}$. Now

$$\|A\| := \sup_{\omega \in \mathfrak{S}} \|\pi_\omega(A)\|_{H_\omega} \quad (A \in \mathcal{R}_0(X, \sigma))$$

defines a seminorm on $\mathcal{R}_0(X, \sigma)$, which allows for the following definition.

Definition 6.2. *The **resolvent algebra** $\mathcal{R}(X, \sigma)$ is the C*-completion of the quotient algebra $\mathcal{R}_0(X, \sigma)/\ker \|\cdot\|$.*

The functions h_x^λ were made to match with the generators $R(\lambda, x)$. Indeed, taking $\sigma = 0$ for a moment, it follows algebraically that the resolvent functions satisfy the above equations. Analogous to [1], one could generalise the definition of the resolvent algebra to spaces X with possibly degenerate σ, and thus validate the name 'commutative resolvent algebra'. However, this

---

[7]Looking at the original definition in [2], the reader may notice that $\mathcal{R}(X, \sigma)$ is defined there for any symplectic space (X, σ), not necessarily admitting a unitary structure. However, as shown in [14], a symplectic space without unitary structure can cause significant difficulties. As mentioned in [3], [14] and [15], these difficult symplectic spaces should be viewed as pathologies. Any symplectic space used in a physical application has a unitary structure. It is for this reason that a unitary structure is assumed on X in [3] and the successive articles on the resolvent algebra, and for the same reason we have assumed a unitary structure on X as well.

is not needed in the present analysis, and could make it confusing to refer to 'the resolvent algebra'. With this in mind, we stick with the terminology used in [2] and [3].

When our symplectic space X is $2n$-dimensional, we may define the Schrödinger representation $\pi_S^n : \mathcal{R}(X, \sigma) \to B(L^2(\mathbb{R}^n))$ by $\pi_S^n(R(\lambda, x)) := (i\lambda - \phi(x))^{-1}$. As mentioned in Section 3, this defines a faithful *-representation. Throughout Sections 3 and 4 we have used this to identify $\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$ with $\pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n))$. However, with our eye on defining a quantization map, we will henceforth make a clear distinction between the two.

7 Quantization: General Case

In this section we prove our main result, namely that a generalization of Weyl quantization provides a strict deformation quantization involving $C_{\mathcal{R}}(X)$ and $\mathcal{R}(X, \sigma)$. The quantization map $Q_\hbar^W$ that is meant to generalize $Q_\hbar^{2n}$ will be constructed using certain embeddings $\mathcal{S}_{\mathcal{R}}(\mathbb{R}^{2n}) \hookrightarrow \mathcal{S}_{\mathcal{R}}(X)$ and $\pi_S^n \mathcal{R}(\mathbb{R}^{2n}, \sigma_n) \hookrightarrow \mathcal{R}(X, \sigma)$, meant to reduce the case of X to the case of $\mathbb{R}^{2n}$. The first embedding is the inverse of p'^*, already isometric by Lemma 2.2. The second embedding we now construct.

For $\lambda \neq 0$ and $x \in \mathbb{R}^{2n}$, we define

$$i_p(\pi_S^n R(\lambda, x)) := R(\lambda, s_p(x)), \tag{21}$$

allowing for the following proposition.

Proposition 7.1. *The prescription* (21) *defines an isometric *-homomorphism*

$$i_p : \pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)) \to \mathcal{R}(X, \sigma).$$

Proof. We will show that i_p is the composition of three isometric *-homomorphisms. As we have seen, the Schrödinger representation π_S^n is faithful, hence

$$\pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)) \simeq \mathcal{R}(\mathbb{R}^{2n}, \sigma_n)$$

as C*-algebras. Let us define $\varphi(R(\lambda, x)) := R(\lambda, s_p(x)) \in \mathcal{R}(V, \sigma_{\restriction V})$ for $\lambda \neq 0$ and $x \in \mathbb{R}^{2n}$. The resolvent algebra $\mathcal{R}(X, \sigma)$ as defined in section 6 only depends on the linear and symplectic structure of (X, σ). Because s_p is a symplectic linear isomorphism, we find that φ extends to a *-isomorphism $\varphi : \mathcal{R}(\mathbb{R}^{2n}, \sigma_n) \to \mathcal{R}(V, \sigma_{\restriction V})$. Next, we denote by $\tau : \mathcal{R}(V, \sigma_{\restriction V}) \hookrightarrow \mathcal{R}(X, \sigma)$ the canonical embedding. By [2], Theorem 4.9(i), τ is an isometric *-homomorphism. Buchholz and Grundling use this fact to identify $\mathcal{R}(V, \sigma_{\restriction V})$ with $\tau(\mathcal{R}(V, \sigma_{\restriction V}))$. To clearly distinguish the two, we should replace $R(\lambda, s_p(x))$ in (21) by $\tau(R(\lambda, s_p(x)))$. In this light, we see that i_p extends to

$$i_p = \tau \circ \varphi \circ (\pi_S^n)^{-1},$$

which proves the claim. $\square$

We can now construct quantization maps on the individual subspaces $\mathcal{S}_{\mathcal{R}}(X)^p$ of $\mathcal{S}_{\mathcal{R}}(X)$.

Definition 7.2. *Define* $Q_\hbar^p : \mathcal{S}_{\mathcal{R}}(X)^p \to \mathcal{R}(X, \sigma)$ *by*

$$Q_\hbar^p := i_p \circ Q_\hbar^{2n} \circ p'^*,$$

where p'^* *is the pull-back of* p'.

One may wonder whether the maps $Q_\hbar^p$, for varying $p \in \mathcal{P}$, can be patched together. The answer is yes, they can.

Proposition 7.3. *Assume $p \in \mathcal{P}_{2n}^V$ and $q \in \mathcal{P}_{2m}^W$.*

(i) *If $V = W$, then $Q_\hbar^p = Q_\hbar^q$.*

(ii) *If $V \subseteq W$, then $Q_\hbar^p = Q_{\hbar \restriction \mathcal{S}_\mathcal{R}(X)^p}^q$.*

Proof of (i). By Lemma 5.2, we have $\mathcal{S}_\mathcal{R}(X)^p = \mathcal{S}_\mathcal{R}(X)^q$. We now have two isomorphisms $p'^*, q'^* : \mathcal{S}_\mathcal{R}(X)^p \to \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$, which are different in general, but can be related by the symplectic transformation $T := p \circ s_q$, or rather by $S := (T^t)^{-1} = J_n^{-1} T J_n$. Indeed, we find

$$
\begin{aligned}
p'(Sx) &= i \cdot s_p(T J_n x) \\
&= i \cdot s_p(p(s_q(J_n x))) \\
&= q'(x) \,.
\end{aligned}
$$

In terms of pull-backs, this implies $S^* \circ p'^* = q'^*$.
Inspired by this, we search for α such that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathcal{S}_\mathcal{R}(X)^p & \xrightarrow{p'^*} & \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}) & \xrightarrow{Q_\hbar^{2n}} & \pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)) \\
& \searrow{q'^*} & \downarrow{S^*} & & \downarrow{\alpha} \qquad \searrow{i_p} \\
& & \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}) & \xrightarrow{Q_\hbar^{2n}} & \pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)) \xrightarrow{i_q} \mathcal{R}(X, \sigma)
\end{array}
\tag{22}
$$

Because T^{-1} is a symplectic transformation, we find that $y \mapsto e^{i\phi(y)}$ and $y \mapsto e^{i\phi(T^{-1}y)}$ are both representations of the canonical commutation relations, in the sense of -for instance- [6]. By the Stone-von Neumann theorem, there exists a unitary $U \in B(H)$ such that

$$
e^{i\phi(T^{-1}y)} = U e^{i\phi(y)} U^*, \text{ for all } y \in \mathbb{R}^{2n} \,.
$$

This is just what we need. Define $\alpha \in \operatorname{Aut}(B(H))$ by $\alpha(a) := U a U^*$. From the definition of Weyl quantization, we then obtain

$$
\begin{aligned}
\alpha(Q_\hbar^{2n}(f)) &= \int \dslash y \, \hat{f}(y) U e^{i\phi(y)} U^* \\
&= \int \dslash y \, |\det T| \hat{f}(Ty) e^{i\phi(y)} \\
&= Q_\hbar^{2n}(f \circ S) \,,
\end{aligned}
$$

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from which we find commutativity of the middle square of (22). For commutativity of the right triangle we need $i_p = i_q \circ \alpha$, which we will now check for the resolvent $\pi_S^n R(\lambda, x)$ (fixing $\lambda \neq 0$, $x \in \mathbb{R}^{2n}$). Using the Laplace transformation, $\pi_S^n R(\lambda, x) = (i\lambda - \phi(x))^{-1}$ can be expressed as an integral over operators of the form $e^{i\phi(y)}$, and we thus obtain $\alpha(\pi_S^n R(\lambda, x)) = \pi_S^n R(\lambda, T^{-1}x)$. Because of this,

$$
\begin{aligned}
i_q(\alpha(\pi_S^n R(\lambda, x))) &= R(\lambda, s_q(T^{-1}x)) \\
&= R(\lambda, s_q(s_q^{-1}(s_p(x)))) \\
&= i_p(\pi_S^n R(\lambda, x)),
\end{aligned}
$$

which implies that $i_p - (i_q \circ \alpha)$ is zero on some generating elements. By Proposition 7.1 and by construction of α, the function $i_p - i_q \circ \alpha$ is a continuous *-homomorphism, hence it is zero on the whole of $\pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n))$. We now have commutativity of (22), which concludes the proof of (i). $\square$

Proof of (ii). By virtue of (i), we may choose $p \in \mathcal{P}_{2n}^V$ and $q \in \mathcal{P}_{2m}^W$ however we want. For this purpose, we first choose a complex orthonormal basis $(u_1, \ldots, u_m)$ of W such that its first n elements $u_1, \ldots, u_n$ form a $\mathbb{C}$-basis of V. Putting $v_j := iu_j$ gives us a symplectic basis $(u_1, v_1, \ldots, u_m, v_m)$ that is simultaneously orthonormal (with respect to $\langle \cdot, \cdot \rangle_\mathbb{R}$). Now define

$$
\begin{aligned}
p : u_j &\mapsto e_{2j-1}^n, & v_j &\mapsto e_{2j}^n & (j \in \{1, \ldots, n\}), \\
q : u_j &\mapsto e_{2j-1}^m, & v_j &\mapsto e_{2j}^m & (j \in \{1, \ldots, m\}).
\end{aligned}
$$

The maps $p_{|V}$ and $q_{|W}$ preserve the respective real inner products, and satisfy $p(iv) = J_n^t p(v)$ and $q(iw) = J_m^t q(w)$. Therefore, they are symplectomorphisms. As we can write $\mathbb{R}^{2m} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2m-2n}$, and $W = V \oplus U$ for $U := W \cap V^\perp$, we easily find $q_{|W} = p_{|V} \oplus \tilde{q}$ for some $\tilde{q} : U \to \mathbb{R}^{2m-2n}$.
As we take $f \in \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$, we would like to prove that the element

$$
Q_\hbar^p(f \circ p) = i_p(Q_\hbar^{2n}(f \circ p \circ p')))
$$

is equal to the element

$$
Q_\hbar^q(f \circ p) = i_q(Q_\hbar^{2m}(f \circ p \circ q'))).
$$

The identity $p(iv) = J_n^t p(v)$ implies $p' = i \cdot s_p \circ J_n = s_p$, and similarly we find $q' = s_q$. Now

$$
Q_\hbar^p(f \circ p) = i_p(Q_\hbar^{2n}(f))
$$

and

$$
Q_\hbar^q(f \circ p) = i_q(Q_\hbar^{2m}(f \circ p \circ s_q) = i_q(Q_\hbar^{2m}(f \otimes 1).
$$

Weyl quantization factors through tensor products, so we obtain $Q_\hbar^{2m}(f \otimes 1) = Q_\hbar^{2n}(f) \otimes \mathbb{1}$ as operators on $L^2(\mathbb{R}^m) = L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^{m-n})$. It remains to show that $i_q(a \otimes \mathbb{1}) = i_p(a)$ for any $a \in \pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n))$. As before, we only have to show this for $a = \pi_S^n R(\lambda, x)$, where $\lambda \neq 0$ and $x \in \mathbb{R}^{2n}$. Indeed,

$$
\begin{aligned}
i_q(\pi_S^n R(\lambda, x) \otimes 1) &= i_q(\pi_S^m R(\lambda, x \oplus 0)) \\
&= R(\lambda, s_q(x \oplus 0)) \\
&= R(\lambda, s_p(x)) \\
&= i_p(\pi_S^n R(\lambda, x)),
\end{aligned}
$$

where in the third equality we have used

$$
q(s_q(x \oplus 0)) = x \oplus 0 = p(s_p(x)) \oplus \tilde{q}(0) = q(s_p(x) + 0),
$$

hence $s_q(x \oplus 0) = s_p(x) \in W$ by injectivity of $q_{\restriction W}$. This finishes the proof. $\square$

Definition 7.2 has given us a family of maps $\{Q_\hbar^p \mid p \in \mathcal{P}\}$, defined on varying subsets of $\mathcal{S}_\mathcal{R}(X)$. Proposition 7.3 ensures that these maps coincide on the overlap of their domains. This enables the following definition.

Definition 7.4. *We define the map $Q_\hbar^W : \mathcal{S}_\mathcal{R}(X) \to \mathcal{R}(X, \sigma)$ by $Q_\hbar^W(f) := Q_\hbar^p(f)$ whenever $f \in \mathcal{S}_\mathcal{R}(X)^p$.*

Thus we have succeeded in defining a map which generalizes Weyl quantization to infinite dimensional phase space. By this we mean that, when $(X, \sigma) = (\mathbb{R}^{2n}, \sigma_n)$, we have $Q_\hbar^W = Q_\hbar^{2n}$. The only thing left to do is to prove that $Q_\hbar^W$ is the quantization map of a strict deformation quantization (as defined in Definition 4.1).

7.1 Strict Deformation Quantization

Recall that X is a symplectic space admitting a complex stucture. Also note that $Q_\hbar^{2n}$ is involutive, hence $Q_{\hbar \restriction \mathcal{S}_\mathcal{R}(X)^p}^W = i_p \circ Q_\hbar^{2n} \circ p'^*$ is involutive, and it makes sense to talk about $Q_\hbar^W : \mathcal{S}_\mathcal{R}(X)_\mathbb{R} \to \mathcal{R}(X, \sigma)_\mathbb{R}$. Finally, defining $Q_0^W := \mathrm{id}_{\mathcal{S}_\mathcal{R}(X)}$, our main result reads:

Theorem 7.5. *Let $A^0 := C_\mathcal{R}(X)$ and $A^\hbar := \mathcal{R}(X, \sigma)$ for $\hbar > 0$. The set $I = [0, 1]$, together with the collection of C*-algebras $\{A^\hbar\}_{\hbar \in I}$, and the maps $Q_\hbar^W : \mathcal{S}_\mathcal{R}(X)_\mathbb{R} \to \mathcal{R}(X, \sigma)_\mathbb{R}$, constitute a strict deformation quantization of $\mathcal{S}_\mathcal{R}(X)_\mathbb{R}$.*

Proof. Note that the maps $Q_\hbar^{2n} : \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}) \to \pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n))$ already form a strict deformation quantization of $\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$. So $Q_0^{2n} = \mathrm{id}$, $Q_\hbar^{2n}$ is linear,

injective, and satisfies (11),(12) and (13). Furthermore $Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}))$ is a dense *-subalgebra of $\pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n))$.

We have

$$Q_\hbar^W{}_{\restriction \mathcal{S}_\mathcal{R}(X)_p} = i_p \circ Q_\hbar^{2n} \circ p'^*$$

for all $p \in \mathcal{P}_{2n}^V$, where $p'^* : \mathcal{S}_\mathcal{R}(X)^p \to \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$ is an isometric *-isomorphism also leaving the Poisson structure invariant. The map $i_p : \pi_n^S(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)) \to \mathcal{R}(X, \sigma)$ is an isometric *-homomorphism.

By these considerations, $Q_\hbar^W$ is already linear and injective. For $f, g \in \mathcal{S}_\mathcal{R}(X)_\mathbb{R}$ we choose $p \in \mathcal{P}_{2n}^V$ such that $f, g \in \mathcal{S}_\mathcal{R}(X)_\mathbb{R}^p$. Because we have $p'^*(f), p'^*(g) \in \mathcal{S}_\mathcal{R}(\mathbb{R}^{2n})$, we find that

$$\begin{aligned}
\left\| Q_\hbar^W(f) \right\| &= \left\| i_p(Q_\hbar^{2n}(p'^*(f))) \right\| \\
&= \left\| Q_\hbar^{2n}(p'^*(f)) \right\|,
\end{aligned}$$

which is continuous as a function of $\hbar$. Also

$$\begin{aligned}
\lim_{\hbar \to 0} &\| Q_\hbar^W(f) Q_\hbar^W(g) - Q_\hbar^W(fg) \| \\
&= \lim_{\hbar \to 0} \left\| (Q_\hbar^{2n}(p'^*(f)) Q_\hbar^{2n}(p'^*(g))) - Q_\hbar^{2n}(p'^*(f) p'^*(g)) \right\| \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\hbar \to 0} &\| \tfrac{i}{\hbar}[Q_\hbar^W(f), Q_\hbar^W(g)] - Q_\hbar^W(\{f, g\}) \| \\
&= \lim_{\hbar \to 0} \left\| \tfrac{i}{\hbar}[Q_\hbar^{2n}(p'^*(f)), Q_\hbar^{2n}(p'^*(g))] - Q_\hbar^{2n}(p'^*\{f, g\}) \right\| \\
&= \lim_{\hbar \to 0} \left\| \tfrac{i}{\hbar}[Q_\hbar^{2n}(p'^*(f)), Q_\hbar^{2n}(p'^*(g))] - Q_\hbar^{2n}(\{p'^*(f), p'^*(g)\}) \right\| \\
&= 0,
\end{aligned}$$

proving (11), (12) and (13) for $Q_\hbar^W$.

Because the sets $Q_\hbar^W(\mathcal{S}_\mathcal{R}(X)^p) = i_p(Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)))$, labeled by $p \in \mathcal{P}$, form a net of *-algebras in $\mathcal{R}(X, \sigma)$, their union

$$Q_\hbar^W(\mathcal{S}_\mathcal{R}(X)) = \bigcup_{p \in \mathcal{P}} Q_\hbar^W(\mathcal{S}_\mathcal{R}(X)^p)$$

is a *-subalgebra of $\mathcal{R}(X, \sigma)$ as well.

What remains to show is density. We calculate

$$\overline{Q_\hbar^W(\mathcal{S}_\mathcal{R}(X))} \supseteq \bigcup_{p \in \mathcal{P}} \overline{Q_\hbar^W(\mathcal{S}_\mathcal{R}(X)^p)}$$

$$= \bigcup_{p \in \mathcal{P}} i_p(\overline{Q_\hbar^{2n}(\mathcal{S}_\mathcal{R}(\mathbb{R}^{2n}))})$$

$$= \bigcup_{p \in \mathcal{P}} i_p(\pi_S^n(\mathcal{R}(\mathbb{R}^{2n}, \sigma_n)))$$

$$= \bigcup_{\substack{V \subseteq X \text{ f.d} \\ \text{and } \mathbb{C}\text{-linear}}} \tau(\mathcal{R}(V, \sigma_{\upharpoonright V})),$$

using the notation from the proof of Proposition 7.1: τ is the natural embedding of $\mathcal{R}(V, \sigma_{\upharpoonright V})$ into $\mathcal{R}(X, \sigma)$. Thanks to [2], Theorem 4.9(ii), we know that $\mathcal{R}(X, \sigma)$ is the inductive limit (and therefore the closed union) of the net

$$\{\tau(\mathcal{R}(V, \sigma_{\upharpoonright V})) \mid V \subseteq X \text{ f.d. and nondegenerate}\},$$

of which

$$\{\tau(\mathcal{R}(V, \sigma_{\upharpoonright V})) \mid V \subseteq X \text{ f.d. and } \mathbb{C}\text{-linear}\}$$

is a subnet. Therefore, $\mathcal{R}(X, \sigma) \subseteq \overline{Q_\hbar^W(\mathcal{S}_\mathcal{R}(X))}$, which implies equality. $\qquad \square$

8 Discussion

We introduced a novel C*-algebra called the commutative resolvent algebra and gave a precise account of its structure. We have given a strict deformation quantization linking this algebra to the resolvent algebra, building on Rieffel's results on Weyl quantization. Subsequently, we have shown how to pass to infinite dimensional phase space, and have generalized our results, culminating in a strict deformation quantization.

We have achieved our goal, but we could have chosen other routes. For instance, we could have used the article [15] of Weaver, who constructs a very general strict deformation quantization, passing from $\mathbb{R}^{2n}$ to Hilbert spaces. We could also have used the framework of Werner given in [16]. In Werner's notation, our $C_{\mathcal{R}} \oplus \mathcal{R}$ is a pair. To give another approach, Binz, Honegger and Rieckers defined the Weyl algebra for a pre-symplectic space, before giving a strict deformation quantization in [1]. The same reasoning could have been applied to the resolvent algebra instead of the Weyl algebra. Finally, we could have used the Fock representation to directly define our quantization map. Be aware that the choices made in this thesis were deliberate, but nonetheless they were choices. The ideal route to quantization may depend on the application.

Our route was focused on $\mathbb{R}^{2n}$, in line with the philosophy that the heart of the problem is already present in the finite case. In the same way, we feel that the heart of the resolvent algebra is already present in the commutative resolvent algebra.

The commutative resolvent algebra helps to understand many features of the resolvent algebra. A helpful mindset to us was the following. We may only hope for something to hold in the resolvent algebra, if the analogous formulation holds in the commutative resolvent algebra. If it holds in the commutative resolvent algebra, and it does not seem to depend on its commutativity, we may as well write it down as a conjecture.

With the present paper, the analogy between the commutative resolvent algebra and the resolvent algebra is validated. The commutative algebra *is* the classical limit of the resolvent algebra. We are glad to say that applying the resolvent algebra to the classical world is now possible, and the result is as beautiful as we could have expected.

A   Schwartz functions

This section discusses the space $\mathcal{S}(\mathbb{R}^m)$ of Schwartz functions. There exist countless good introductions to this subject, like [4], with which we do not wish to compete. Here we just fix our notation, and prove two results which we have seen nowhere in quite the form we need.

For a multi-index $\alpha \in \mathbb{N}^m$ and a point $x \in \mathbb{R}^m$ we write $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ and $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$, where ∂_j is the partial derivative in the jth variable. We say a function $f : \mathbb{R}^m \to \mathbb{C}$ is **rapidly decreasing** if, for all α,

$$\sup_{x \in \mathbb{R}^m} |x^\alpha f(x)| < \infty \,.$$

A smooth function $f \in C^\infty(\mathbb{R}^m)$ is called **Schwartz** if all of its derivatives are rapidly decreasing. To make this definition a little more explicit, define the seminorms $\|\cdot\|_{\alpha,\beta}$ by

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^m} |x^\alpha \partial^\beta f(x)| \,,$$

for all $\alpha, \beta \in \mathbb{N}^m$. Now Schwartz functions are the elements of

$$\mathcal{S}(\mathbb{R}^m) := \left\{ f \in C^\infty(\mathbb{R}^m) \,\Big|\, \|f\|_{\alpha,\beta} < \infty \text{ for all } \alpha, \beta \right\} \,,$$

and the locally convex topology on $\mathcal{S}(\mathbb{R}^m)$ induced by the seminorms $\|\cdot\|_{\alpha,\beta}$ is called the **Schwartz topology**.

The following lemmma is indispensable for Section 4.

Lemma A.1. *If $g \in \mathcal{S}(\mathbb{R}^r)$ and $R \in GL(\mathbb{R}^r)$ then $g \circ R \in \mathcal{S}(\mathbb{R}^r)$.*

Proof. To prove this, we estimate $\|g \circ R\|_{\alpha,\beta} := \sup_x |x^\alpha \partial^\beta (g \circ R)(x)|$. Using the chain rule repeatedly we find that $\partial^\beta (g \circ R)$ is a linear combination of $\partial^\gamma g \circ R$, for multi-indices γ with $|\gamma| = |\delta|$. Furthermore, $\sup_x |x^\alpha \partial^\gamma g(Rx)| = \sup_x |(R^{-1}x)^\alpha \partial^\gamma g(x)|$, so we are left to estimate $|(R^{-1}x)^\alpha|$. This is possible, since $(R^{-1}x)^\alpha$ is a linear combination of x^δ, for $|\delta| = |\alpha|$. To summarize, there exists a finite collection of constants $c_{\gamma,\delta}$ such that

$$\|g \circ R\|_{\alpha,\beta} \leqslant \sum_{\gamma,\delta} c_{\gamma,\delta} \sup_x |x^\delta \partial^\gamma g(x)| = \sum_{\gamma,\delta} c_{\gamma,\delta} \|g\|_{\gamma,\delta} < \infty \,.$$

$\square$

Lemma A.1 tells us that the definition of $\mathcal{S}(\mathbb{R}^n)$ is independent of the basis, a fact which is used implicitly by many authors. How else can we justify the notation $\mathcal{S}(V)$, for a vector space V? This comes into play in the following lemma, where V_1, V_2, V_3 are finite dimensional vector spaces. By Lemma A.1 we may freely identify these with euclidean spaces.

Lemma A.2. *If $h_1 \in \mathcal{S}(V_1 \oplus V_2)$ and $h_2 \in \mathcal{S}(V_1 \oplus V_3)$ then the functions*

$$g(x, y, z) := h_1(x, y)h_2(x, z)$$
$$h(x, y, z) := [h_1(\cdot, y) * h_2(\cdot, z)](x)$$

are Schwartz as well: $g, h \in \mathcal{S}(V_1 \oplus V_2 \oplus V_3)$. If we only assume h_1, h_2 to be rapidly decreasing, then g, h are rapidly decreasing as well.

Proof. If $\alpha = \alpha_1 \oplus \alpha_2$ and $\beta = \beta_1 \oplus \beta_2$ then

$$\|f\|_{\alpha,\beta} = \sup_y |y^{\alpha_2}| \left\|\partial_y^{\beta_2} f(\cdot, y)\right\|_{\alpha_1,\beta_1} .$$

We use this repeatedly in the following, but now $\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3$ and $\beta = \beta_1 \oplus \beta_2 \oplus \beta_3$.

$$\begin{aligned}
\|h\|_{\alpha,\beta} &= \sup_{y,z} |y^{\alpha_2} z^{\alpha_3}| \left\|\partial_y^{\beta_2} \partial_z^{\beta_3} h(\cdot, y, z)\right\|_{\alpha_1,\beta_1} \\
&= \sup_{y,z} |y^{\alpha_2} z^{\alpha_3}| \left\|\partial_y^{\beta_2} h_1(\cdot, y) * \partial_z^{\beta_3} h_2(\cdot, z)\right\|_{\alpha_1,\beta_1} \\
&\leqslant \sum_{\gamma,\delta,\epsilon,\zeta} c_{\gamma,\delta,\epsilon,\zeta} \sup_y |y^{\alpha_2}| \left\|\partial_y^{\beta_2} h_1(\cdot, y)\right\|_{\gamma,\delta} \sup_z |z^{\alpha_3}| \left\|\partial_z^{\beta_3} h_2(\cdot, z)\right\|_{\epsilon,\zeta} < \infty,
\end{aligned}$$

for a finite set of constants $c_{\gamma,\delta,\epsilon,\zeta}$. If h_1, h_2 are only rapidly decreasing, we take $\beta = 0$ in the above calculation. Then $\beta_1 = \beta_2 = \beta_3 = 0$ and $c_{\gamma,\delta,\epsilon,\zeta} = 0$ whenever δ or ζ are nonzero. It follows that $\|h\|_{\alpha,0} < \infty$, meaning that h is rapidly dereasing. The same works for g, needing only a single nonzero $c_{\gamma,\delta,\epsilon,\zeta}$ in the last line. $\qquad\square$

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