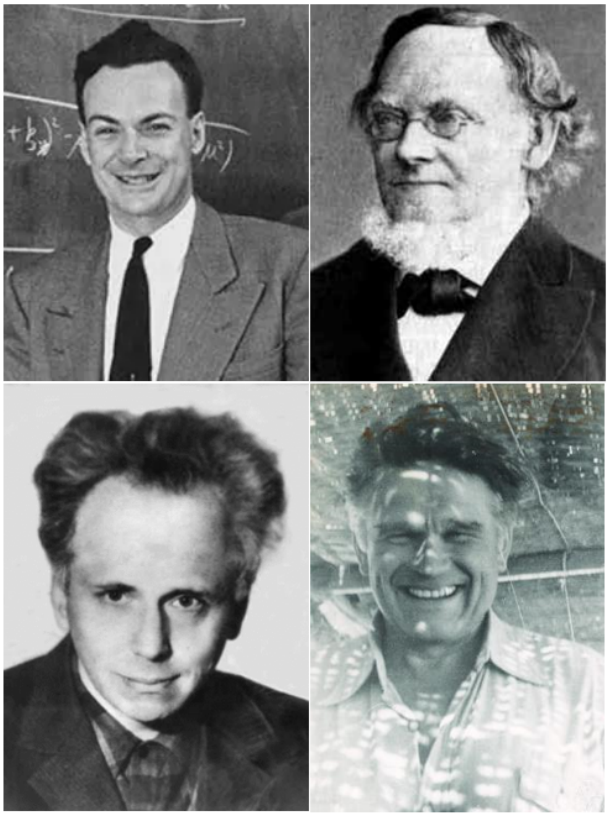
The Path to and from Functional Integration over Fermionic Fields

Master Research Internship and Thesis
NWI-NM086F
Physics and Astronomy: Particle and Astrophysics

Thomas A. L. J. Vissers
s4555724

Supervisor (Max Planck Institute for the History of Science): Dr. A.S. Blum
Supervisor (Radboud University): Prof. Dr. N.P. Landsman

Academic year 2023-2024



The Fermionic Path Integral and the Men Who Made It:
Feynman, Graßmann
Berezin, Faddeev
Each presents, in that order, the closest to a 'main character' of the chapters.

Abstract

The path integral formulation is widely recognized as an unmissable toolbox for doing quantum field theory. By integrating the exponential $e^{\frac{i}{\hbar}\int \mathscr{L} d^4x}$ over all possible evolutions of field configurations, with $\mathscr{L}$ the *classical* Lagrangian density, one can go on to derive scattering amplitudes and decay rates. However, it was not always this way. After its introduction to the broader public by Feynman in 1948, the path integral went through two 'dark decades', where it was neither well-known nor much used. An important factor in this relative radio silence was its initial inability to treat fermionic fields, that are generally thought not to have a classical Lagrangian description. The solution to this problem came about around the 1960s, in the form of integration over anticommuting Graßmann numbers. After introducing the path integral formulation and Graßmann's algebraic system, this thesis project investigates how this modern formulation of integration of path integration over fermionic fields came about. To this end, physics as well as historical methodology is employed through which several papers from British and Russian authors in the latter half of the 1950s are investigated. It is found that this path integral development is the end result of the many consecutive contributions of these authors, from introducing anticommuting numbers to defining explicit integration rules over them. Finally, it is shown how these results were used to gauge fix Yang-Mills theories and, later, prove the renormalizability thereof.

Contents

1 Introduction

When undergraduate students of physics are introduced to Quantum Mechanics (QM), they usually start with the relatively accessible wave mechanics formulation of Schrödinger. As they move on through their studies, they will eventually come to work in the more abstract language of operators on Hilbert spaces, characterized by the Dirac-von Neumann axioms. It is typically only graduate students taking an advanced course in Quantum Field Theory (QFT) that are familiarized with another influential formulation, namely that of path integrals. Inspired by earlier contributions of Paul A.M. Dirac (1902-1984), the path integral formulation was fully introduced by Richard P. Feynman (1918-1988) in his 1942 PhD-thesis, with a more complete article in *Reviews of Modern Physics* appearing in 1948 for a wider audience (Feynman, 1942)(Feynman, 1948). Feynman's approach focuses not so much on quantum states, but rather on transition amplitudes between them. In the quantum mechanical case, the probability of a particle at a position $\vec{x}_1$ at a time t_1 appearing at a position $\vec{x}_2$ at a later time t_2, is proportional to the absolute square of the associated amplitude. Feynman could calculate such problems by integrating the exponential $e^{\frac{i}{\hbar}S[x(t)]}$ over all possible paths $x(t)$ from $\vec{x}_1$ to $\vec{x}_2$, where $S[x(t)]$ is the (classical) action of a path under consideration. In QFT, the integration is instead over all possible field configurations between an initial and final state, which is why it is sometimes also referred to as the 'sum-over-histories' approach.

Nowadays, the path integral is generally accepted as a powerful tool, in many respects providing a convenient formulation of quantum theory. Yet, this was not always the case. While from roughly the 1970s onwards they enjoy ubiquitous use, the period before this could be referred to as its 'dark decades'. Throughout the 1950s and 1960s, path integrals were neither well known nor widely used. Other than novelty, an important contributing factor to their marginal position were the many problems path integrals still faced. Unitarity of the S-matrix in the path integral formulation was far from obvious, certain aspects of the integral were mathematically ill-defined, and spin could not be accounted for from the offset. An example of particular importance related to the latter case is that the path integral was unable to deal with integration over fermionic fields, evidently important for doing QFT (Weinberg, 1995, p. 376-377).

Therefore, much work still needed to be done. Although unitary of the S-matrix was not evident, it could at least be established indirectly from its mathematical equivalence to the operator formalism that follows from canonical quantization of fields. Secondly, despite still existing problems, the path integral now rests on a more rigorous mathematical basis than it used to due to the work of mathematical physicists, starting with Cécile A.P. DeWitt-Morette (1922-2017) already soon after Feynman's original paper (Morette, 1951). Nevertheless, the history of physics has plenty of examples of physicists not waiting around until every problem of mathematical rigor is solved when they are provided with a tool that *works*, so this is also unlikely to be the dominant reason for the dark decades. A pressing physical reason can thus be found in the aforementioned inability to deal with one-half of physical systems:

fermionic ones. It took until around the end of the 1960s for this problem to be solved. Its resolution came about through the implementation of the exterior (or Graßmann) algebra first introduced by Hermann G. Graßmann (1809-1877). Armed with this 'new' algebra, the path integral formulation has seen a great acceleration in its development and application range. It paved the way for Ludvig D. Faddeev (1934-2017) and Victor N. Popov (1937-1994) to in 1967, arguably, end this era by gauge fixing non-Abelian gauge theories in QFT, an instrumental step in the eventual establishment of the Standard Model of Particle Physics. Today, the use of the path integral is common practice in the field.

During these dark decades, however, it is far from entirely clear how these developments relating to the fermionic path integral, that eventually pushed the formulation into the mainstream, came about. The seemingly sudden appearance of century-old mathematics to solve the problem at hand is a clear illustration of this. This naturally leads to the question where this crucial step in the history of the path integral, and QFT as a whole, came from.

While such a question can be said to have intrinsic value by its contribution to our understanding of the history of (quantum) physics, it can be of further interest to the physicist too. It enriches the methodological apparatus of the physicist to show that the answers to problems presented in lectures do not come out of the blue. This is true because the intellectual journey towards these results, including the many failures along the way, can serve as a warning to practicing physicists, as well as expand the number of directions of one's thinking when confronted with a problem. Furthermore, from a philosophy of (mathematical) physics point of view, this story is another beautiful example of a familiar pattern in physics, whereby structures formerly residing in the domain of 'pure math' turn out to provide an elegant description of physical phenomena. This then also stimulates new developments in that area of mathematics, as is here already visible through the case of DeWitt-Morette. Another philosophical curiosity is that this story allows us to see the interpretation of a physical theory actively change as these developments are taking place. Lastly, an enquiry such as this is difficult to leave *only* to historians, as it simply takes Master's level physics knowledge to be able to understand and contextualize what people were doing during the 50s. For the physics student too, however, getting acquainted with how things were done back then is sure to carry plenty of lessons. Given all of the above reasons to pursue this topic, my own interest in history and/of physics and the foundations of the field, and me gladly taking on the challenge of grasping the technical details of the development of a to me previously unbeknownst but mystifying formulation, our research question shall now be introduced.

1.1 Research question

The research question of this thesis is formulated as follows:

How did the modern formulation of fermionic path integrals over anticommuting Graßmann variables come to be introduced around the late 1950s to allow for breakthroughs in the development of the path integral formulation of QFT?

Two things come to mind given the above question. For one, it presupposes an adequate degree of knowledge of the path integral formulation, Graßmann algebra, and QFT in general, most of which a starting Master's student in physics is unlikely to already possess. Secondly, the historical angle inherent to the question may lead one to wonder how a physics student is to go about answering it. The remaining two subsections of this Introduction therefore aim to clarify these matters, before getting into the main body of this thesis.

1.2 Approach

Earlier, it was stated that the above research question cannot just be left to the historians, due to the large amount of physics foreknowledge its answering requires. At the same time, it cannot trivially be left to a physics student with no formal training in historical research methodology of the field either.
Luckily, I was in the privileged position of being able to spend the first semester of this research internship at the *Max Planck Institute for the History of Science* (MPIWG) in Berlin, specifically in the Research Group titled *Historical Epistemology of the Final Theory Program*. While 'the Final Theory Program' will be recognized by the physicist as referring to the quest for a 'Theory of Everything', the former term may be less familiar. Historical epistemology as a discipline, however, is at the core of the MPIWG's activities. It can be seen as the combination of two different 20th century developments: the historical turn in the philosophy of science and the increasing focus on how knowledge is thought to come about ('epistemologization' in the French tradition) in the history of science. Historical epistemology is therefore not so much concerned with a rigid definition of what constitutes knowledge, but rather, *how* knowledge is generated throughout history. It investigates under what conditions things are even made into objects of knowledge in the first place. It came about as a 20th century reaction to the seeming inability of science to capture the world and all its phenomena as a unified system moving towards completion. Although experts disagree on its precise, detailed meaning, historian of science and former MPIWG-director Hans-Jörg Rheinberger (1946) characterizes historical epistemology by the question: 'under *what* conditions did *who* know *what* through an object?' (Rheinberger, 2010).
During my stay at the MPIWG and under the supervision of the research group's leader dr. Blum, I have been able to learn more about such methods from him and others. Moreover, I could use the MPIWG's extensive library and resources, and have followed a course given at the institute titled *Knowledge and Its Resources: Concepts, Methods, Historiographies*. Lastly, once back in the Netherlands, the new Radboud Center for Natural Philosophy (RCNP) also

provides a stimulating environment for discussing topics related to the history of physics. While all of this by no means makes me full-fledged historian, it did help me gain basic skills in addressing the historical part of the research question of this thesis.

Most of this historical research has been focused on reading not just secondary literature, but taking in the primary sources in the form of the papers at the end of the 1950s that tried to solve the inability of the path integral to describe fermionic fields. This adds another layer of difficulty. Notation may be wildly different from contemporary conventions, explanatory clarity may take a backseat in favor of brevity guidelines, and papers can be tied up in initially obscure and not always documented broader research contexts. To deal with these issues best as possible, an extensive survey of published articles on the topic at hand has been conducted. Many of these, while mentioned and read, have upon evaluation of their contributions not made the final cut. They do, however, help in uncovering this wider context. When writing about these articles, I have made efforts to communicate the meaning of their notation as much as possible, and I have often attempted to redo derivations on paper (some times more successful than others). All of this has ultimately culminated into a set of concise historical questions through which these papers are to be discussed later in this thesis, to draw as accurate historical conclusions as possible. Finally, it may be mentioned that rather than denoting sources such as these with square brackets and a number as is commonplace in physics theses (e.g., [9]), I consciously make use of APA (in-text) citations. This fits better with the historical character of the thesis, as page numbering will often be very important when, e.g., being concerned with where an author makes some crucial remark. Any factual statements made in this introduction are extensively elaborated upon later, as well as being sourced in this way.

Yet, historical methodology alone does not exhaust all required thought on the overall research approach to this thesis. The other big issue mentioned after the research question is the fact that the topic at hand demands a substantial amount of physics and mathematics knowledge that will very likely be unfamiliar to the average starting Master's student of physics (including, initially, myself). Since it ought to be written at that level, the following contains a large part dedicated to laying out this theoretical background. Aside from the core question what is required to understand fermionic path integrals, the approach to these parts are guided by three other goals. These goals exist also to assure some level of novelty in this large theoretical body, despite these results themselves being well-established. The first one is *clarity*. For one, didactics will play an important part in my career after my studies. But more directly relevant, this advances the understandability for the hypothetical average Master's student reading this thesis. The clarity manifests itself in, e.g., the attempt to provide clear commentary around new concepts and derivations, with a focus on how this may be done better than in many other bodies of literature. A second goal is that of *thoroughness of derivations*. I have found much existing literature to jump from premise to result rather quickly, with sometimes few of the logical steps in these derivations being provided or even explained. Nevertheless, these steps may often incorporate a surprising degree of

subtle mathematics. I have therefore often reproduced such derivations on my own to fill in these gaps and be more thorough than is usually seen. Thirdly and finally, some focus is put on *original analysis*. I make an effort to regularly take a step back and critically examine results, whether it be related to interpretation, consistency, assumptions or comparisons. It is my ambition that the manifestation of these goals in the text are evident to the reader. In this way, both the requirement of having the knowledge as well as some degree of novelty with regards to the theory can be satisfied.

Having addressed the approach to answering the research question, we shall now briefly consider the structure of this thesis that flows out of this, before moving on to the main body.

1.3 Thesis structure

The remainder of this thesis consists of chapters working towards an answer to the research question. In chapter 2, the necessary knowledge on the path integral formulation of QM and QFT is thoroughly described. Especially in the former case, we can stay close to Feynman's own work. After this, the chapter lays out the central problem of the path integral's initial inability to describe fermionic fields. To solve this problem, it is essential to understand the basics of Graßmann algebra. An exploration thereof, as well as to the historical context that comes with it, is provided in chapter 3. We then move to the crucial chapter 4, which fully describes how a number of individuals from the UK and the USSR eventually incorporated this algebra to solve the fermionic impasse of the path integral formulation. The most important identity to come out of this is derived at the end. Chapter 5 investigates further developments that were enabled by this new repertoire that would eventually end the dark decades for the path integral formulation. The primary focus is on the Faddeev-Popov procedure for gauge fixing non-Abelian gauge theories, directly using the results from the previous chapter. A conclusion then follows to summarize the entire story and concretely answer the research question. After this, we get to the discussion on the research in this thesis, to finally end with a bibliography containing all the (historical) sources. With this structure in mind, let us now proceed by discussing the path integral formulation.

2 The Path Integral Formulation of QM and QFT

This chapter lays the groundwork of this thesis by introducing the path integral formulation of quantum theory. First, the foundation of the formulation in quantum mechanics is introduced. After this, we show that this, on first sight, wildly different procedure, can indeed reproduce the familiar Schrödinger equation, and *vice versa*. We close our discussion on QM by considering how the path integral allows one to make calculations, considering perturbation theory in particular and providing the simplest and most direct case of the free particle as a concrete example. Next, we show how the generalization to QFT can be made, where paths are replaced with field configurations. We end this chapter by illuminating the advantages and problems of the formulation as it stood some time after its conception, with particular attention to its inability to describe fermionic systems.

2.1 The quantum mechanical origin of the path integral

Feynman is a man with many stories to his name, and the origin of his intuition into path integrals is no exception. As the disputed but famous story goes, Feynman, as a student, was attending a lecture on quantum mechanics wherein the double-slit experiment was being explained. The professor showed how it was the addition of the probability amplitudes of the waves going through both slits, not of the pure probabilities, that would allow one to derive the probability distribution of the intensity of signals at a certain position on the screen. Feynman supposedly asked the professor what to do if we were to add a third slit in the wall. The professor answered that one simply adds the amplitude of the wave through that slit to the sum, i.e., the amplitude associated with the extra particle path. Feynman continued his line of questioning, adding more slits and even more walls with slits (see figure 1), with the professor growing tired of having to repeat that one is simply to add the corresponding amplitudes. Eventually, Feynman is purported to have asked what happens if you have *an infinite number of consecutive walls, each with an infinite number of slits* (O'Dowd, 2017).

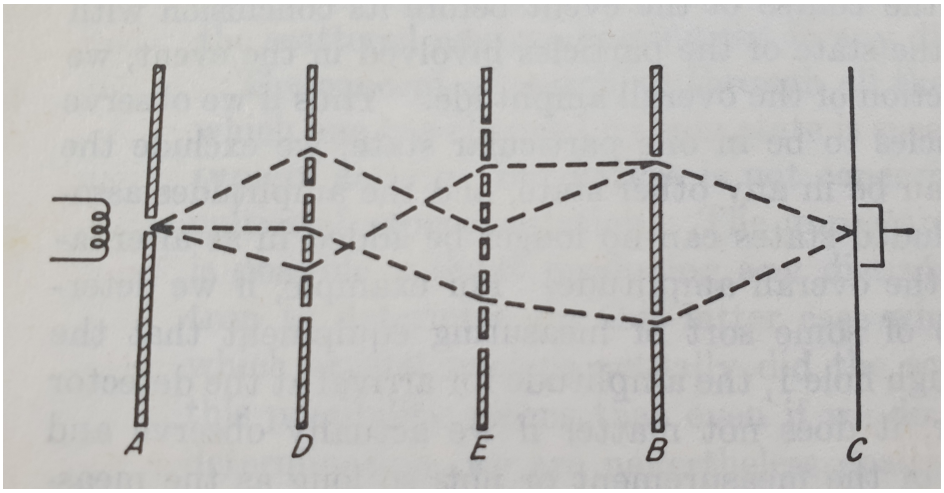


Figure 1: An arbitrary configuration of four walls A, D, E and B, each with a different number of slits. A source of, in this example, photons, is stationed on the left end, while on the right end C serves as the photographic screen. Three of the many possible paths are highlighted with dashed lines. To find the amplitude of a photon being detected at any position on C, the amplitudes of all possible paths need to be added (Feynman and Hibbs, 1965, p. 20).

True or not, the story works excellently to familiarize the reader with basic intuition behind the path integral formulation. To find the probability of a quantum particle appearing at a given position on the screen in the double-slit experiment, one needs only to add the amplitudes associated with each of the two paths and take, following Born's rule, the sum's squared modulus,[1] i.e., $P = |\phi_1 + \phi_2|^2$. This is then the probability distribution, and it can be integrated over any spatial interval to find the probability of finding the particle somewhere in that interval in any experiment. The implication of the story is now that if we want to find the probability of that particle at an initial position x_A at the time t_A, later appears at the particular position x_B at the time t_B, we must add and then take the squared modulus of the amplitudes of *all possible paths* between these two space-time coordinates, as though all of space is a slit. This intuition may aid in understanding the more formal ideas introduced later.

Feynman first wrote down these ideas in his PhD-thesis on "The principle of Least Action in Quantum Mechanics" (Feynman, 1942). While much was already in there, his ideas were further developed out and published for fellow physicists in a more condensed form in *Reviews of Modern Physics* six years later (Feynman, 1948). His most extensive body of work on the path integral appeared another 17 years later, in the form of a book written together

[1] While we are not currently discussing the effect of measurement, note that measuring which of the two slits the particle goes through will change the expression such that $P = |\phi_1|^2 + |\phi_2|^2$. While the study of decoherence is known to address this phenomenon, Feynman and Hibbs in their 1965 book differentiate between 'exclusive alternatives' and 'interfering alternatives'. If we can know which of the holes the particle goes through, the holes are exclusive alternatives to the particle, while if not, they are interfering alternatives.

with his doctoral student Albert Roach Hibbs (1924-2003) (Feynman and Hibbs, 1965). While the fundamental essence of the path integral does not change radically over the years, its range of application can be seen to expand greatly. The book, for example, contains an entire chapter on the uses of the path integral in statistical mechanics (Feynman and Hibbs, 1965, §10). Due to his methods being further matured, in the following three subsections, we shall mostly take Feynman's original presentations in the '48 paper and '65 book as our guidepost, with some occasional help from secondary sources.

With this established, we shall now consider the two postulates Feynman provided for the path integral formulation and all that follows from them. He introduces the first as follows:

> ***Postulate I***: *"If an ideal measurement is performed to determine whether a particle has a path lying in a region of spacetime, then the probability that the result will be affirmative is the absolute square of the sum of complex contributions, one from each path in the region."*
> (Feynman, 1948, p. 371)

With 'ideal measurement', Feynman intends to refer to a measurement whereby *"no further details could be obtained from the same measurement without further disturbance to the system"* (Feynman, 1948, p. 370).
Note that his postulate as presented above could also be argued to be a composition of two ones: the sum over paths yielding the probability amplitude on the one hand, and the Born rule on the other hand.
More importantly, the postulate introduces no restriction to the type of allowed paths. The sum includes everything from the classically taken path to an absurd path wherein the particle first goes to the Andromeda galaxy and then back. An important takeaway from this is that these paths are not physical: they violate locality. This is something that we shall refer back to later.

Another question one may ask about this postulate is how one performs a 'sum' over all paths. It would seem that for every path, one could always introduce a slight deformation so that the actual number of them is infinite. Feynman used calculus to find a way to deal with this. Let us first consider a particle moving in one dimension. First, we subdivide the time interval between the initial and final states into N equal small increments ϵ, i.e., $t_B - t_A = N\epsilon$. This is an example of a 'time-slicing procedure'. We shall ultimately take the limit as $\epsilon \to 0$, but for our current sake of understanding one may think of it as just 'a small increment of time'. Consequently, we can enumerate these discretized times with a subscript such that $t_{i+1} = t_i + \epsilon$, where logically $x_A = x_0$, $t_A = t_0$, $x_B = x_N$ and $t_B = t_N$. Consider now a particle in its initial state (x_A, t_A), as can be viewed in figure 2. After the time increment ϵ the particle has moved to the position x_1. One may note that figure 2 illustrates this using a straight line. This, as will now implicitly be described, is completely irrelevant with regards to the first postulate, but not so with regards to the second, and we shall thus come back to it later. For now, we must consider that we aim to sum over all possible paths. Since we have noted

that these paths are not physically restricted in any way, x_1 can take on any real number, including those that would require $v > c$ if the particle is to appear there after a time ϵ. After all, we are still discussing non-relativistic quantum mechanics. It follows that we must sum over all possible values of x_1 our particle might find itself after a time ϵ having started at x_A. Given the continuity of this variable, this sum becomes an integral, i.e., $\int_{-\infty}^{\infty} dx_1$. This step is then repeated $N-2$ more times, such that we end up with the product of $N-1$ such integrals $\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_{N-1}$. Given that x_0 and x_N themselves are determined by the initial and final state of the particle, they are not integrated over. Since the number of integrals is determined by our choice of ϵ, we require a normalization constant itself depending on ϵ to guarantee convergence of the path integral taken over a soon to be introduced integrand. The final step after this procedure is to take the limit $\epsilon \to 0$, as in this limit we truly obtain the sum over all paths. After all, if we were not to take the limit, we would not account for the many paths between any two particular values of x_i and x_{i+1}. Given that $t_B - t_A = N\epsilon$, the left hand side being constant, taking the limit as $\epsilon \to 0$ will simultaneously mean that $N \to \infty$, as may intuitively be expected (Feynman and Hibbs, 1965, p. 26-32).

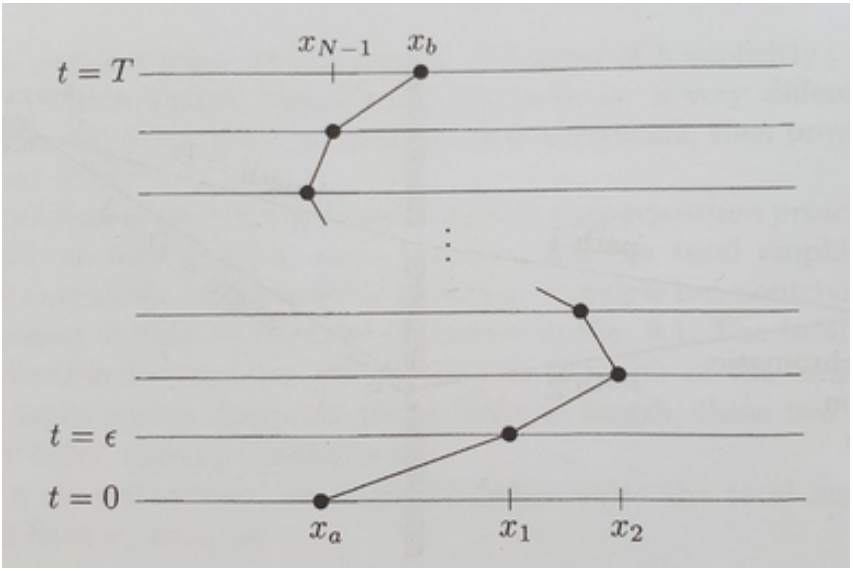


Figure 2: A graphical illustration of Feynman's approach to taking the sum over all paths for a particle moving in one dimension. At each time increment ϵ, a particle at a position x_i moves to a possible x_{i+1}. Ultimately, the limit $\epsilon \to 0$ is taken (Peskin and Schroeder, 2019, p. 278).

Of course, writing out the product of integrals all the time is rather tedious. In practice, the following notation is used (Feynman and Hibbs, 1965, p. 33-34):

$$\lim_{\epsilon \to 0} \frac{1}{C(\epsilon)} \int_{-\infty}^{\infty} \frac{dx_1}{C(\epsilon)} \int_{-\infty}^{\infty} \frac{dx_2}{C(\epsilon)} \cdots \int_{-\infty}^{\infty} \frac{dx_{N-1}}{C(\epsilon)} = \lim_{\epsilon \to 0} \frac{1}{C(\epsilon)} \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} \frac{dx_i}{C(\epsilon)} \equiv \int \mathscr{D}x(t) \tag{1}$$

Here, the symbol $\mathscr{D}$ is conventional for denoting the sum over all paths (Feynman and Hibbs, 1965, p. 34).

The splitting of the normalization factor is a convention adopted for convenience, its usefulness becoming clear in the later subchapter on calculations. Its value shall there also appear in a natural way. Note that when using the product symbol Π, it is here then meant as an instruction to repeat everything that follows for each i, including not just the dx_i but also the $C^{-1}(\epsilon)$. Of course, one could in principle just write $C^{-N}(\epsilon)$, or, while we are at it, redefine that as $C'(\epsilon)$.

A word of reflection on the above procedure is warranted. The method by Feynman described above is far from the only way to formulate the sum over all paths. This specific method is what is referred to as 'discrete lattice regularization', due to the need to regularize, i.e., prevent the integral from diverging, being fulfilled by means of working with discrete steps whose lattice spacing ϵ eventually goes to zero. Yet, this method is not without its mathematical problems, and a relation such as $t_{i+1} = t_i + \epsilon$ is not demanded by the theory but rather one possible construct to tackle the problem at hand. We might as well have introduced $t_{i+1=t_i+\ln(\delta)}$, since in the limit $\delta \to 0$ the lattice spacing still disappears. Another example of such a construct is the line segments between consecutive points x_i and x_{i+1}, which shall be addressed after the introduction of the second postulate.

The need of a second postulate is hinted at by the as of yet undefined 'contributions' that are spoken of in the first. In the above, it was discussed how a sum over all paths is taken. But to get the mass of some non-uniform plate, one does not just integrate over its surface, but one integrates its mass-density $\rho(x, y)$ over its surface. Similarly, to find the probability amplitude of a particle undergoing the process described in postulate I, we do not just integrate over all possible paths, but we integrate a particular complex-valued path-dependent object $\phi[x(t)]$ over all paths. The probability amplitude, whose square modulus provides the probability distribution, is referred to by Feynman as the 'kernel' $K(x_A, t_A, x_B, t_B)$ (Feynman and Hibbs, 1965, p. 26). This is because it carries the particle's state from (x_A, t_A) to (x_B, t_B). Consequently, $P(x_A, t_A, x_B, t_B) = |K(x_A, t_A, x_B, t_B)|^2$.
The second postulate, now, concerns the nature of these complex contributions $\phi[x(t)]$. Here, Feynman was heavily inspired by Dirac's 1933 attempt to find out what in quantum theory corresponds to the Lagrangian method in classical theory. In this paper, Dirac suggested that the unitary transformation mapping a state vector at a time t to a time $t + dt$ was analogous to the quantity $e^{\frac{i}{\hbar}Ldt}$, where L is the classical Lagrangian of the system (Dirac, 1933). Feynman decided to investigate what would happen if this analogy were to be regarded as an equality (Blum, 2017, p. 44-47), and this ultimately led to the content of the now to be introduced second postulate. He states the second postulate as follows:

> **Postulate II**: *"The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$; i.e., the time integral of the Lagrangian taken along the path."*
> (Feynman, 1948, p. 371)

Therefore, we can mathematically express this contribution $F[x(t)]$ as

$$F[x(t)] = e^{\frac{i}{\hbar}S[x(t)]}, \qquad S[x(t)] = \int_{t_A}^{t_B} L(\dot{x}(t), x(t), t)\,dt \tag{2}$$

where $S[x(t)]$ is thus the classical action along the path.
By now, one may wonder a to what the reason is for using square brackets. This is related to the mathematical concept of a *functional*. This is a function that, rather than taking in a number and outputting another one, takes in a *function* and then outputs a number. The square brackets then denote that the argument one is dealing with in the case of a functional is a function. Feynman's path integral is often referred to as a *functional integral*[2]. When one thinks about it, a regular old integral $G[f(x)] = \int f(x)\,dx$ is in this sense already a functional. Yet, in Feynman's path integral, the integrand $\phi[x(t)]$ is itself a functional, and we are then integrating over a space of functions. This idea of using such a mathematical construction was itself not new to physics. Two decades earlier, mathematician Norbert Wiener (1894-1964) had applied such a scheme to describe Brownian motion, where he integrated over the space of all Brownian paths, each being weighed by some probability measure (Cartier and DeWitt-Morette, 2006, p. 56-59). In fact, Feynman's and Wiener's integral can be related in an interesting way, something that we shall consider later.

Another point to note about the second postulate is that it follows from it that all paths contribute equally in magnitude, but not in phase. On first reading, one may wonder why an absurd path going to Andromeda and back would contribute equally to a more sensible one like the classical path. Yet, here we must reinvigorate the previous statement that the paths are not physical. The formulation ascribes no physical reality to any of these paths. Doing so in any way is to introduce an interpretation of the path integral on top of the formalism itself. While this prospect may be of interest to some, it is not forced upon us by theory.

Now that we have formulated both postulates, it should be noted that their combination is taken to constitute the whole of the path integral formulation of quantum mechanics, mathematically being fully equivalent to the others. We shall prove this later, although it may already be announced that this statement will be troubled by the subsequent inclusion of spin. In any case, we now have some conception of the meaning of 'quantization' of a classical theory through the path integral formulation. It means successfully taking a classical Lagrangian, plugging that into the phase factor, and integrating this over all possible paths.

When we combine postulate I and II, we find the following complete formula for the kernel of a particle initially being in the state (x_A, t_A) being later measured in the state (x_B, t_B):

$$K(A,B) = \int_A^B e^{\frac{i}{\hbar}S[x(t)]}\mathscr{D}x(t) \tag{3}$$

Where A and B are shorthand for denoting the initial states (x_A, t_A) and the final state (x_B, t_B), respectively.

[2]This becomes especially prevalent in the QFT-literature, where we will no longer be integrating over 'paths'.

In the abstract, the procedure for calculating this object is now quite straightforward: for every possible path, plug it into the Lagrangian such that the action can be calculated, and add up the exponentials for every such case. Some thought, however, needs to be put into how the action is to be calculated when we consider the concrete time-slicing method that has been employed above. The way we performed this functional integral was through the product of $N-1$ familiar integrals, in the sense that we integrate over numerical variables rather than functions. While this allows us to sum over all paths, we end up with a set of $N+1$ discrete points $(x_A, x_1, x_2, ..., x_{N-1}, x_B)$, yet the Lagrangian is a function of the *continuous* path $x(t)$. The action integral requires a path to exist between subsequent points x_i and x_{i+1}. Therefore, one is faced with the conundrum of how the action is now to be expressed given the discrete lattice regularization method.

To answer this problem, Feynman did two things. First, as stated before, he connected the discrete positions with lines. In the case of figure 2, these are straight lines, yet they need not be, since in the ultimate limit $\epsilon \to 0$ the length of all these line segments will approach zero too. We shall briefly discuss two possible types of line segments: straight line paths, i.e., paths linear in time, and paths that would be followed by a classical particle if it were to travel between two given points x_i and x_{i+1}.
The second thing that Feynman did was to split up the action into N contributions:

$$S = \sum_{i=0}^{N-1} S(x_i, x_{i+1}) \tag{4}$$

This is allowed because the action is an integral over time which we are slicing up. Therefore, the sum of these N integrals each over a small time interval from t_i to $t_i + \epsilon$ will, given the correct action, simply be equivalent to the one from t_A to t_B. This possibility will later allow us to connect the path integral formulation to the wave functions of ordinary QM.

We will first consider the case of straight lines. The action as formulated in postulate II can now be expressed as:

$$\int_{t_A}^{t_B} L(\dot{x}(t), x(t), t)\, dt \to \epsilon \sum_{i=0}^{N-1} L\left(\frac{x_{i+1}-x_i}{\epsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right) \tag{5}$$

Let us briefly explain where this new form comes from. First of all, the integral from t_A to t_B is, as mentioned, replaced by a sum of the action of each straight line path between consecutive points x_i and x_{i+1}. We must consider the effect of this on the arguments of the Lagrangian. Since the paths between the consecutive points are straight lines, the time derivatives of these paths are nothing more than $v_i = \frac{x_{i+1}-x_i}{t_{i+1}-t_i}$, the denominator being ϵ by definition. For the second argument, the average between the two points is taken. Of course, a linear formula $x_{i\to i+1}(t) = \left(\frac{x_{i+1}-x_i}{\epsilon}\right) t + x_i$ for the straight line between x_i and x_{i+1} could be constructed, but this would introduce time dependence due to which the Lagrangian could not be taken outside of the integral. Moreover, once the limit $\epsilon \to 0$ is taken, the approximation's effect will disappear too. Similar logic holds for the third argument, namely the time

variable that is replaced by the average time between two subsequent points $\frac{t_{i+1}+t_i}{2}$.
We notice now that given these arguments, the Lagrangian is time-independent, meaning that it can be taken outside of the integral. Since $\int_{t_i}^{t_{i+1}} dt = \epsilon$ for all i in the sum, we end up with the result as shown above.
Although rare, one might wonder what happens when one is presented with Lagrangians depending on higher derivatives of $x(t)$ than the velocity. Acceleration terms $\ddot{x}(t)$ in the Lagrangian, for example, would always be zero on the straight line trajectories between x_i and x_{i+1}. Therefore, these terms would not add to the contribution to the total action of paths that they could without the straight line approach. Moreover, the fact that the acceleration is infinite in the transition of the line segment between x_i and x_{i+1} to that between t_{i+1} and t_{i+2} could also spell trouble.
The trick Feynman deemed adequate here was to consider the average acceleration between two subsequent straight line segments (Feynman and Hibbs, 1965, p. 34). One can then substitute the following for the acceleration between a point x_{i-1} and x_i:

$$\ddot{x}(t) \rightarrow \frac{v_i - v_{i-1}}{\epsilon} = \frac{\frac{x_{i+1}-x_i}{\epsilon} - \frac{x_i - x_{i-1}}{\epsilon}}{\epsilon} = \frac{x_{i+1} - 2x_i + x_{i-1}}{\epsilon^2} \tag{6}$$

The same procedure could be repeated for higher derivatives than the second too.

Since it will be used in the following subchapters, we now state the full formula for the kernel in the form resulting from discrete lattice regularization, $\mathscr{D}x(t)$ written out:

$$K(A,B) = \lim_{\epsilon \to 0} \frac{1}{C(\epsilon)} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\epsilon \sum_{i=0}^{N-1} L\left(\frac{x_{i+1}-x_i}{\epsilon}, \frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right)} \frac{dx_k}{C(\epsilon)} \tag{7}$$

Where the subscript k was used for the integral product to avoid confusion with the subscript i used for the sum in the exponential.

Another way of addressing the problem of how to incorporate the action in Feynman's time-slicing method is by not using straight lines, but the classical path that a particle with a given Lagrangian would follow from (x_i, t_i) to (x_{i+1}, t_{i+1}). This was method that Feynman favored in his 1948 paper, although he does end up using straight lines later in the same paper (Feynman, 1948, p. 372). Nevertheless, while less explicit, there is certainly some (conceptual) merit to the use of the classical path. Since the principle of stationary action dictates that the classical path is the one for which the action is stationary, one can simply write the following:

$$S(x_i, x_{i+1}) = Stat\left(\int_{t_i}^{t_{i+1}} L(\dot{x}(t), x(t))\right) \tag{8}$$

That is, rather than introducing any specific type of path between consecutive points, one simply demands that each contribution to the total action is stationary. This is a compact way of writing things, both on paper and for a computer program. To further proceed, however, one would need the Lagrangian to subsequently solve the Euler-Lagrange equation(s)

$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$ to figure out the right paths between the x_i's and $x'_{i+1}s$. For our current purposes though, the insight that there are multiple ways of handling such problems will suffice.

In such ways, the formulation of the action has been made compatible with our currently introduced method of computing path integrals. Before moving on to performing a concrete computation in the next section, five important questions that the reader may have considered deserve an answer, as to complete our foundational understanding of the path integral formulation of quantum mechanics.

A first question might concern the empirical content of the formulation. It is clear how the mathematical toolkit introduced so far allows for measurable statistical predictions of positions of particles. What is not yet clear, is how this could be done for important and familiar physical quantities such as momentum, energy and angular momentum.
Feynman has a short and long answer to this question. The short one will ring a bell for those familiar with Bohmian mechanics, namely the idea that all physical measurements in reality come down to measurements of positions. Measuring mass on scales comes down to considering the position of the needle pointing to a particular number. Measuring the energy of an incoming cosmic ray ultimately comes down to mapping the positions at which we register shower particles in our detectors. And measuring momentum can be done through considering the positions of particles in a device it travels through. The argument is now that if all experimental knowledge can be derived from position measurements, a formulation that allows us to do the latter is complete in the sense that it can, in principle, account for all experimental knowledge as well as any other (Feynman, 1948, p. 371).
Much has been said about the above argument, but if one is inclined to accept it, it does seem to speak for the completeness of the path integral formulation with regard to its ability to make predictions for dynamical variables other than position. There is, however, also Feynman's longer and more direct answer. One can use the fact that it is possible to relate position to momentum mathematically through infinitesimal position measurements, as well as the familiar technique of Fourier transforms (Feynman and Hibbs, 1965, p. 95-118). The path integral can also be provided in a momentum representation. In this way, it is, in fact, possible to provide the kernel as a function of quantities such as momentum and energy using the sum over *paths*, or even sum over momenta too in a *phase space representation*. Up until now, by only using the sum over paths, we have just been using the *coordinate space representation*. The phase space integral was used by Feynman relatively early on too, appearing already in a 1951 in a paper on his work about operator calculus (Feynman, 1951, p. 125). It will naturally come about when showing how the path integral formulation follows from the Schrödinger equation, which is why we will discuss it there rather than here. In any case, while arguably not strictly necessary, the path integral can indeed explicitly answer questions as for example 'if a particle starts out with a momentum p_A at the time t_A, what is the probability that we will detect it with a momentum p_B at the later time t_B?'.

A second question may address whether the path integral formalism can be extended to an

arbitrary number of dimensions, seeing as only one dimension has been considered so far. Luckily, this can be done in a straightforward way. Using three Cartesian dimensions of space as an example we get the following Kernel (Feynman and Hibbs, 1965, p. 65-66):

$$K(A,B) = \int_A^B e^{\frac{i}{\hbar}S[x(t),y(t),z(t)]} \mathscr{D}x(t)\mathscr{D}y(t)\mathscr{D}z(t) \tag{9}$$

Where A and B now denote the initial particle state (x_A, y_A, z_A, t_A) and the final state (x_B, y_B, z_B, t_B). From here, the same time-slicing method used in the one-dimensional case can be employed. This triples the number of integrals, and keeping with the convention used for the normalization constant, we now find $3N$ factors of $C^{-1}(\epsilon)$.
While the above uses Cartesian coordinates, any coordinate system can be used. For example, in the two-dimensional case, Cartesian coordinates can also be replaced by functional integration over polar coordinates, i.e., $\mathscr{D}x(t)\mathscr{D}y(t) \rightarrow \mathscr{D}r(t)\mathscr{D}\theta(t)$. To make this a bit more concrete, in the example of Feynman's discrete lattice regularization, we can take two integrals $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \rightarrow \int_{-\infty}^{\infty}\int_0^{2\pi} rdrd\theta$, simultaneously changing the Cartesian variables for the corresponding action between t_i and t_{i+1} according to the familiar polar substitutions. This coordinate substitution was only published about 4 years after Feynman's and Hibbs' book, first appearing in the work of a British mathematician (Arthurs, 1969).
In conclusion, the path integral formulation of quantum mechanics is not restricted in the number of degrees of freedom it can describe, nor in the number of coordinate systems in which it can do so, at least not more so than in the case of any other formulation.

Our third and related question is about whether the above formulation also holds for many-particle systems. The answer is both yes and no, and we shall consider why.
First of all, we can simply reformulate the general kernel (3) with a many-particle Lagrangian. For a system of N interacting particles, we have:

$$L = \frac{m}{2}\sum_{j=1}^{N}\dot{x}_j^2(t) - \frac{1}{2}\sum_{j\neq l} V(x_j(t), x_l(t)) \tag{10}$$

Here, the j-subscript denotes the particles. An example of the above Lagrangian could be N moving charges. The division by two of the second term avoids double counting of the potential between two charges.
We can now write the kernel (3) as:

$$K(A,B) = \prod_{n=1}^{N}\left(\int_A^B \mathscr{D}x_n(t)\right) e^{\frac{i}{\hbar}\int_{t_A}^{t_B}\left(\frac{m}{2}\sum_{j=1}^{N}\dot{x}_j^2(t) - \frac{1}{2}\sum_{j\neq l} V(x_j(t),x_l(t))\right)dt} \tag{11}$$

Some things changed compared to the one-particle case.
For one, we now have $A = (x_{1,A}, x_{2,A}, ..., x_{N,A}, t_A)$, and likewise for B. After all, each particle j has some distinct starting position $x_{j,A}$ at t_A, and a final position $x_{j,B}$ at t_B.
Moreover, we have introduced a product over n to account for the N path integrals we now have, with brackets introduced to clarify what is repeated. Rather than looking at all possible

trajectories for one particle, the many-particle case considers the collective of all possible configurations of N paths.
As one may imagine, the above is hardly practical for systems with particle numbers of the order of Avogadro's constant. Moreover, one is unlikely to have full knowledge of the initial state of the system in such cases. One may then be interested in the probability of it ending up in a particular set of final states. Luckily, the path integral here can be applied to statistical mechanics, where the partition function corresponding to, e.g., the canonical ensemble, can be represented in terms of one (Feynman and Hibbs, 1965, p. 273-279). We shall say slightly more about this later, but this goes to show that the path integral does not lose its power when confronted with many-particle systems of any size.

This all sounds rather favorable for the formulation. Yet, when introducing the question as to whether it could handle many-particle systems, the answer provided was 'yes and no'. This 'no' part goes back to the problem at the core of the historical developments to be considered. The above generalization does not seem to incorporate any distinction between the case of identical bosons and identical fermions. Yet, as we are well aware, these very much exist. In fact, the many-particle kernel (11) only applies to bosons. Why this is the case, and how this could be solved, is something we will come back to extensively later in this thesis.

The fourth question we consider is that of mathematical legitimacy. The path integral is not your typical, well-defined object (Cartier and DeWitt-Morette, 2006, p. 3-7). For instance, the set of paths with continuous derivatives seems to be of measure zero when we take into account all possible path, the process of which can be handily visualized in figure 2. Moreover, the convergence of the path integral is far from obvious. As we shall see in the final subsection of this chapter, it has taken quite some effort to put the path integral on mathematically firm ground. In fact, this is still not unambiguously the case across the board.
One problematic aspect of the path integral is the imaginary unit i. The oscillatory behavior of the integrand due to the imaginary phase is more difficult to deal with than its real-valued counterpart more alike the Wiener integral or the partition function in statistical mechanics. Taking the Wiener integral as our example, we note that its integrand looks like $e^{-\int_{\tau_A}^{\tau_B} V(x[\tau]d\tau}$, with $V[x(\tau)]$ an unspecified functional. Now consider our path integral phase (2), and note that under a Wick rotation $t = i\tau$, it will look exactly like the Wiener integral. After all, by substituting $dt = id\tau$ in the action integral, the imaginary unit will combine with the one already in the phase to $i^2 = 1$. The Wick rotation is justified by analytic continuation. Without getting too much into the mathematical detail, one way to think about it would be to set up a contour integral that includes the whole imaginary and real axis. By Cauchy's integral theorem, the entire contour is zero, as well as contributions to the integral not on the axes under the right conditions. One can then switch the integral to one over the complex axis. More importantly, due to the Wick rotation, we have an integral over an exponential with a negative exponent. Its convergence behavior in the limit of large numbers is then much easier to establish. Another positive factor that will be more relevant in QFT, is that the Minkowski metric $ds^2 = -dt^2 + dx^2$ will become Euclidean under a Wick rotation, as

$ds^2 = -(i\,d\tau)^2 + dx^2 = d\tau^2 + dx^2$, which is much easier to deal with. In conclusion, it is possible to work with the path integral in a Wick-rotated setting and then just rotate back at the end. This has both mathematical advantages as well as laying bare the connection with the Wiener integral and even statistical mechanics (Cartier and DeWitt-Morette, 2006, p. ,3-4, 28-31, 362-363)(Feynman and Hibbs, 1965, p. 274).

A fifth and final question concerns the formulation's consistency with classical behavior in the classical limit. In other words, whether it adheres to the correspondence principle.
The ease of being able to verify this is usually considered one of the great strengths of the path integral formulation. The argument goes as follows. Consider this rather qualitative expression of the kernel:

$$K(A,B) \propto \sum_{paths} e^{\frac{i}{\hbar}S[x(t)]} \tag{12}$$

That is, the kernel of a particle starting at one point and later being measured at another is proportional to the sum of all paths between these points over the introduced exponential.
Let us consider this term for just a particular path $\mathscr{C}$ and, using Euler's formula, we can also write:

$$K_{\mathscr{C}}(A,B) \propto \cos\left(\frac{S_{\mathscr{C}}(A,B)}{\hbar}\right) + i\sin\left(\frac{S_{\mathscr{C}}(A,B)}{\hbar}\right) \tag{13}$$

In other words, we can write the contribution of each path as a vector in the complex plane.
We know that, in SI-units, $\hbar = 1.054 \cdot 10^{-34} J \cdot s$. We would also like to know the order of magnitude of the action on the classical scale. Consider we drop a raisin of mass $m = 1g$ to the floor from a height $h_0 = 1m$ above it, neglecting friction. The fall starts at $t = 0s$ and logically ends at $t = \sqrt{\frac{2\Delta h}{g}}$. The height as a function of time is $h(t) = h_0 - \frac{1}{2}gt^2$ Its initial velocity being zero, the raisin falls downward with a speed $\dot{h}(t) = gt$. Its Lagrangian is thus $L = T - V = \frac{1}{2}m\dot{h}^2(t) - mgh(t) = \frac{1}{2}mg^2t^2 - mg(h_0 - \frac{1}{2}gt^2)$. Thus, we ultimately find that the action of this raisin falling straight down to the ground is:

$$S_{clr} = \int_0^{\sqrt{\frac{2\Delta h}{g}}} \left(\frac{1}{2}mg^2t^2 - mg\left(h_0 - \frac{1}{2}gt^2\right)\right) dt = -\frac{m}{3}\sqrt{2gh_0^3} \approx -1.476 \cdot 10^{-3} J \cdot s \tag{14}$$

With S_{clr} being the 'classical raisin action'.
Unsurprisingly, the order of a classical action is much larger than that of $\hbar$, meaning the argument of our periodic functions, i.e., the phase, will be of order $\left|\frac{S_{clr}}{\hbar}\right| \sim 10^{31}$ radians. The fact that $\hbar$ is so much smaller than a typical value for the action on classical-like length and energy scales has another effect. Suppose, that in the above example, the raisin does not fall down in a straight line, but one with a slight curve. Perhaps we introduce a change $\vec{r}(t) = (0,0,h(t)) \rightarrow \vec{r}\,'(t) = (x(t),0,h(t))$, where $x(t)$ could be a very flat parabola. Let us suppose this slight alteration of the path leads to a slightly different action S'_{clr}, such that $\delta S = S'_{clr} - S_{clr} \approx 1.5 \cdot 10^{-6} J \cdot s$, i.e, about one-thousandth of S_{clr}. Yet, the relative smallness of $\hbar$ even compared to such a minute deviation of the path will result in an enormous phase change of the order $\frac{\delta S}{\hbar} \sim 10^{28}$ radians.

When we zoom out to see the bigger picture of this exercise, we may note that the arrows associated with the paths when represented in the complex plane point in wildly different directions even when these paths are very close together with respect to classical length scales. The practically random distribution of arrows makes it so that they sum up to zero[3]. This is decisively *not* the case when the action is of the order of $\hbar$ or smaller. Concretely, when $\frac{S}{\hbar} \sim 1$, small changes to S will shift the phase with only a fraction of a radian, and summing the arrows corresponding to different paths need not result in zero so easily.

Currently, we are still missing one essential ingredient in our analysis of the sum over all paths in classical-like settings. Namely, there is a particular path for which linear approximations to slight deviations from it will leave the action unchanged, so that they constructively add up. These are the paths for which the action is stationary. From the classical principle of stationary action, we know that classical particles always follow paths for which the action is stationary. Therefore, it is the classical path $x_{cl}(t)$ and paths very much near it that are the only ones to effectively contribute to the sum over all paths. They do not cancel each other out.
An elegant visualization of this process is the so-called Cornu spiral that can be seen in figure 3. To keep it orderly, it considers a situation wherein we have one wall with a large number of slits. The possible paths from the 'start' and 'end' point are then shown at the bottom of the figure. As was mentioned before, each of these paths corresponds to an arrow in the complex plane. These arrows are displayed in the top figure, and the colors can be used to see which arrows correspond to which paths. The length between the center of the two spirals is proportional to the kernel. After all, it is the sum of all these arrow contributions that yields the kernel. The important thing to notice is that when adding all these arrows to each other, the non-classical paths are spiraling around a point and in this sense cancel each other out, but the classical-like paths are putting in the work and adding up to contribute most to the kernel. At least, that is true for the classical regime. The above exposition is usually taken to the conclusion as an elegant demonstration of the correspondence principle. In Feynman's own words: "In this way the classical laws of motion arise from quantum laws." These classical laws can be derived from the principle of stationary action. This powerful principle had been known for centuries, but why it worked was unclear. And here, the lower-level quantum theory, formulated in terms of the path integral, is understood as the solution to this riddle. Because, as we have seen, it is only when the action is stationary that corresponding paths make significant contributions to the probability amplitude, and these corresponding paths are for all practical purposes the classical path (Feynman, 1948, p. 377-378)(Feynman and Hibbs, 1965, p. 29-31).

Nevertheless, I would argue that this view can be criticized. Certainly, one would be hard-pressed to brush off the equivalence of this crucial condition on the action in classical physics

[3]While I have tried to sketch a more intuitive physical picture here, this argument has formal backing through the application of the Riemann-Lebesgue lemma.

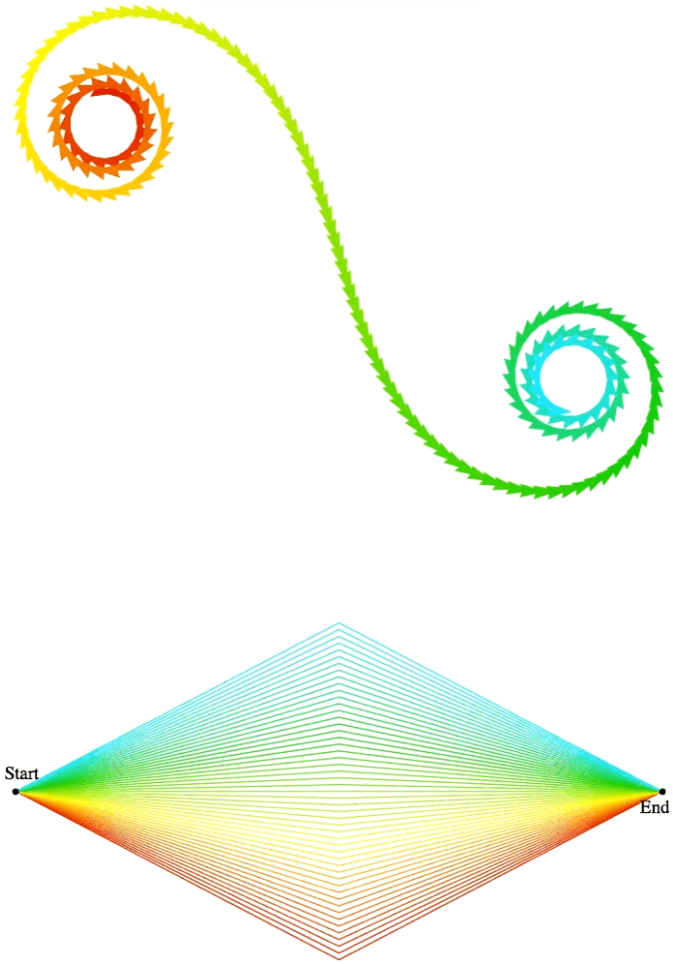


Figure 3: The bottom part of the image shows the possible paths with which a particle can go through a wall with many slits. Each path, being color-coded, corresponds through a complex arrow in the upper part of the image. Since the length of the sum of all arrows is proportional to the kernel, the near-classical paths can be seen to dominate in the classical regime (David, 2013).

and the action emerging from the path integral formulation in the classical regime as mere coincidence. Yet, it is not clear why the latter case should be logically connected to the former. When considering the classical regime whereby $v \ll c$, the predictive scheme of special relativity impeccably coincides with that of classical physics. But when we look at the predictive scheme of the path integral formulation, we do not find this to be the case. Its postulates tell us that the modulus squared of the sum over all paths is equal to the probability that a particle being at x_A at t_A will later at time t_B be found at x_B. It is true that in the classical regime, the classical-like paths contribute most to this probability. But it is still a probability. The conclusion that classically, action must be stationary, and the path associated with that must be taken by a particle, conflates contributions to a probability sum with what is physically real. It was mentioned before that the formalism ascribes no physical reality to the paths that are summed over. After all, many paths are highly unphysical, going to Andromeda and back in one increment of time ϵ. In that case, it is unclear why physical reality *should* be ascribed to another path that just happens to contribute more due to not being cancelled by its neighboring ones.
It might aid understanding to accentuate in what situation the correspondence argument *would* make sense. This would be the case where *the paths themselves* each carry a probability of occurring, and that as $h \to 0$ we find $P(x_{cl}) \to 1$. Yet, the postulates of the path integral formulation do not state this. The paths are not probabilities themselves, but they are equal contributions to a sum, that sum itself being related to the probability that a physically real particle can be found somewhere.
In conclusion, even though the value for the action whereby it is stationary is precisely the one that both stands at the basis of the classical principle of stationary action and dominates the contributions to the kernel in the classical regime, the postulates of the path integral formulation do not themselves provide the bridge with which this co-occurrence can be understood.

This concludes our discussion of the fundamentals of the path integral formulation of quantum mechanics. While this may have led to greater conceptual understanding thereof, it is not at all evident how this formulation is equivalent to that of quantum mechanics as one is usually introduced to it, nor is it clear how a complicated formula such as the full formula for the kernel resulting from discrete lattice regularization can be used in practice. The goal of the next two subchapters is, therefore, to gain insight into these respective issues.

2.2 Bridging the gap to the Schrödinger equation

Feynman himself was once quoted to have said that

> *"Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics."*
> (Feynman, 1965, p. 168)

It is known that, for example wave and matrix mechanics, have different strengths and weaknesses when tackling problems, supporting his case. I would argue, however, that explicit

understanding of the bridges between these representations is an essential part of this endeavour too, for it can reveal surprising translations and spark new tricks and insight. This will become evident in our attempt to connect the path integral to the more familiar formulation of quantum mechanics for a particle moving in a potential, and also help to understand steps that authors in chapter 4 are taking. We consider the one-particle system as it will convey to us everything of interest without needlessly complex bookkeeping, since the extension to many particles is straightforward in principle. We will mainly focus on the accessible wave mechanics of Schrödinger and show that the equation carrying his name, central to this formulation, can be linked to the path integral formulation. Proving the equivalence of the two requires us to show that the postulates of each are capable of producing the other. After all, if formulation A could successfully reproduce formulation B, but not the other way around, this would indicate not equivalence but rather that B is only a subset of A, i.e., A contains more information than B. In this subchapter, we will therefore take both routes. We will start by showing that the Schrödinger equation follows from the path integral. The wave function being its essential ingredient, this requires us to express it in our new language. After this, it will be shown how the path integral can also be constructed starting from the Schrödinger equation. Finally, we will focus on an especially important insight we gain from this, namely an alternative way of presenting the path integral.

2.2.1 Walking the path to the wave

Before the derivation can be displayed, it is first necessary to understand how the path integral can represent events occurring in succession. It may, for example, be the case that after the paths starting at position x_A at the time t_A converge at the the position x_B at time t_B, they go on to a further position x_C at the later time t_C. The total particle kernel we are now interested in is $K(x_A, t_A, x_C, t_C)$, or, more conveniently as expressed in the notation introduced before, $K(A, C)$. Since we know from equation (4) that the action can be expressed as a sum of contributions of the action between points on the total path, we can infer that $S[A, C] = S[A, B] + S[B, C]$. Then by definition of the path integral (3) and by the general algebraic properties of an exponential, we are led to believe the following (untrue!) statement:

$$K(A,C) = \int_A^C e^{\frac{i}{\hbar}(S[A,B]+S[B,C])} \mathscr{D}x(t) = \left(\int_A^B e^{\frac{i}{\hbar}S[A,B]} \mathscr{D}x_{A\to B}(t)\right)\left(\int_B^C e^{\frac{i}{\hbar}S[B,C]} \mathscr{D}x_{B\to C}(t)\right) \quad (15)$$

Here, the sum over all paths $\mathscr{D}x(t)$ has been split up in the part of a sum over all paths from A to B, i.e., $\mathscr{D}x_{A\to B}(t)$, and likewise from B to C. Note that with regards to the sum over all paths from x_A to x_B, the action term $S[B, C]$ is constant. Therefore, the first term above will by definition produce the kernel $K(A, B)$. The same goes for the sum over all paths form x_B to x_C, where $S[A, B]$ is constant and the result is the kernel $K(B, C)$.
The one key thing to keep in mind is that since x_B is the end point of the former kernel and the starting point of the latter, there will be no integral over the integration variable dx_B when performing the discrete lattice regularization method to calculate these kernels. Yet, if we desire to know the kernel $K(A, C)$, we must consider *all* paths, and it follows that we must

also take all possible values of x_B into account[4]. This what the above expression was missing and why it was wrong. Therefore, we arrive at the following correct expression (Feynman and Hibbs, 1965, p. 36-38):

$$K(A,C) = \int_{-\infty}^{\infty} K(A,B)K(B,C)dx_B \tag{16}$$

The essential takeaway here is that we must multiply the amplitudes for events following one another in time. We could, in principle, even keep adding an arbitrary number of further points after x_C. When the time interval between these points is infinitesimal, one will, in line with expectations, recover formula (7). After all, it will result into a product of kernels $K(x_j, t_j, x_{j+1}, t_j + \epsilon)$, each proportional to a single term in the sum of the exponential in (7).

With this knowledge, we can now connect the wave function $\psi(x,t)$ to the path integral formulation. This derivation is based on the one from Feynman and Hibbs, although it is done in a much more extensive way (Feynman and Hibbs, 1965, p. 76-78). It also contains some corrections of small mistakes therein.
One observation provides the key as to how this connection is to be made. We have learned that $|K(x',t',x,t)|^2$ is the probability that a particle starting from the configuration (x',t') will later be found in (x,t). From ordinary wave mechanics, we also know that $|\psi(x,t)|^2$ is the probability that a particle will be found in (x,t). The difference between the two, then, seems to be that the history of a particle's motion is irrelevant for the wave function, while specified for the kernel. The wave function can then be understood as a kernel in the case where it is of no interest where a particle came from. While deeming this characterization sufficient in the 1965 book, he goes on to give a direct expression for it in the 1948 paper. The expression is as follows:

$$\psi(x_j,t_j) = \lim_{\epsilon\to 0}\frac{1}{C(\epsilon)}\prod_{k=-\infty}^{j-1}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\sum_{i=-\infty}^{j-1} S(x_{i+1},x_I)}\frac{dx_k}{C(\epsilon)} \tag{17}$$

Where we can rename the j^{th} term (x_j,t_j) as (x,t).
The above is basically just the particle kernel (7) starting from some arbitrary position that we will call $x_{-\infty}$ an 'infinite' time ago, summed over all paths to arrive at a time t, e.g., the present, at a position x. Therefore, we can conclude that $\psi(x,t) = K(x_{k=-\infty}, t_{k=-\infty} = -\infty, x, t)$.
It is understandable why Feynman opted to drop the above direct expression in the 1965

[4]One may wonder, with regard to our earlier convention, why this integral over dx_B would not also come with a constant $C^{-1}(\epsilon)$. For this, consider the kernel expression resulting from discrete lattice regularization as expressed in (7). Note that other than the constant $C^{-1}(\epsilon)$ being attached to each dx_k, there is also a factor $C^{-1}(\epsilon)$ in front of the whole expression. This means that when we express the kernel $K(A,C)$ as a product of $K(A,B)$ and $K(B,C)$, both of these two kernels will introduce this front-factor. This means that we would have one more than when just directly calculating $K(A,C)$ according to (7). Yet, this excess $C^{-1}(\epsilon)$ is compensated by the integral over dx_B *not* carrying this constant. Therefore, the accounting of the constants in the convention as introduced before works out, and the expression for $K(A,C)$ below is correct.

book. Any path integral will still need the input of an initial position and the notion of 'an infinite time ago' is not necessary to define the wave function, and thus may serve to confuse readers. It suffices to just say that $\psi(x,t) = K(x't',x,t)$, where we can use the wave function if we do not care about the particle under consideration coming from $(x't')$.
One may ask if this is not a problem. After all, the initial configuration (x',t') will affect the expression for the wave function. Therefore, if I want to write down the wave function of a quantum system from the path integral, how do I know what initial configuration to use? This is, in fact, equivalent to the initial condition problem also present in wave mechanics. Consider for example the general solution of the Schrödinger equation in the time-independent case, $\psi(x,t) = e^{-\frac{iE(t-t')}{\hbar}}\psi(x,t')$. To calculate the wave function at later times, one still needs to be handed data on an initial state $\psi(x,t')$. To calculate the kernel from one state to a later one, one also needs to be handed data on an initial state (x',t').

Having identified the wave function with a kernel whose initial state is irrelevant, and knowing about the multiplicative property of kernels (16), we can conclude the following:

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x',t',x,t)\psi(x',t')\,dx' \tag{18}$$

Reminiscent of unitary operators in the familiar quantum mechanical formulations, we can say that the integral kernel carries the wave function at an initial state (x',t') to a later state (x,t). This expression will finally allow us to derive the Schrödinger equation from the path integral formulation. In a way, it is unsurprising that this puts us on the right path. Given an initial state, the Schrödinger equation tells us how it will evolve over time. The above integral does essentially the same.

Suppose the initial wave function of a quantum system is given to be $\psi(y,t)$. We are interested in the wave function at an infinitesimal time ϵ later, i.e., $\psi(x,t+\epsilon)$. In this case, the kernel carrying the one to the other is easy to identify, namely:

$$K(y,t,x,t+\epsilon) = \frac{1}{C(\epsilon)} e^{\frac{i}{\hbar}\epsilon L\left(\frac{x-y}{\epsilon},\frac{x+y}{2},t+\frac{\epsilon}{2}\right)} \tag{19}$$

The above kernel is a special case whereby the initial and final state are separated only by a single infinitesimal increment of time ϵ. The means that there are no intermediary dx_k to integrate over, since we go immediately from initial to final state, themselves never integrated over due to them being fixed. There is 'no path' in this path integral, so to say. This kernel is related to just one term in the action sum, which can be seen seen above. The arguments of the Lagrangian are simply a consequence of entering the initial and final state in the current case into the expressions given for the Lagrangian arguments in formula (7). Note that for the final time-dependence argument, here $t_{i+1} = t+\epsilon$ and $t_i = t$, such that $\frac{t_{i+1}+t_i}{2} = t+\frac{\epsilon}{2}$. Entering all this in formula (18) yields:

$$\psi(x,t+\epsilon) = \frac{1}{C(\epsilon)} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\epsilon L\left(\frac{x-y}{\epsilon},\frac{x+y}{2},t+\frac{\epsilon}{2}\right)} \psi(y,t)\,dy \tag{20}$$

As postulate II informs us, we now enter the *classical* Lagrangian for a particle with mass m in motion in the presence of a potential field. Using the coordinates as defined above, this gives us the following expression:

$$L = \frac{m(x-y)^2}{2\epsilon^2} - V\left(\frac{x+y}{2}, t+\frac{\epsilon}{2}\right) \tag{21}$$

Splitting the exponential into two parts corresponding to the kinetic and potential terms, the expression of the wave function becomes:

$$\psi(x, t+\epsilon) = \frac{1}{C(\epsilon)} \int_{-\infty}^{\infty} e^{\frac{im(x-y)^2}{2\hbar\epsilon}} e^{-\frac{i}{\hbar}\epsilon V\left(\frac{x+y}{2}, t+\frac{\epsilon}{2}\right)} \psi(y, t)\, dy \tag{22}$$

From here on out, a trick can be used the logic of which is reminiscent of that employed in the discussion on the classical limit at the end of chapter 4.1. Note that the denominator of the quotient in the kinetic exponential contains the infinitesimal interval ϵ. This means that unless $x - y$ is close to zero, the terms will, given the smooth behavior of the other terms, oscillate rapidly. As a result, the positive and negative contributions to the integral in the domain where $x - y$ is not close to zero will cancel one another.
This observation motivates the substitution $y = x + \xi$. Since this entails that $y - x = \xi$, we expect that only when ξ is small do we see major contributions to the integral[5]. The resulting expression becomes:

$$\psi(x, t+\epsilon) = \frac{1}{C(\epsilon)} \int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\epsilon}} e^{-\frac{i}{\hbar}\epsilon V\left(x+\frac{\xi}{2}, t+\frac{\epsilon}{2}\right)} \psi(x+\xi, t)\, d\xi \tag{23}$$

The smallness of ξ (and ϵ) allows us to think about expanding the various terms in the equation. Although we may wonder up to what order we should go. By the previous logic, we know that the significant contributions to the integral will be found when ξ is small, specifically of the order where $\xi^2 = \frac{2\hbar\epsilon}{m}$. After all, when ξ is around this value, i.e., when the exponent is of order 1, significant contributions can be expected. Consequently, $\epsilon \propto \xi^2$, and thus when expanding to the first order in ϵ we should do so to the second order in ξ. Since ϵ is infinitesimally small, we need not go further than the first order when expanding with respect to this parameter.

There are now two functions to consider in the integral, namely $\psi(x+\xi, t)$ and $V\left(x+\frac{\xi}{2}, t+\frac{\epsilon}{2}\right)$. Starting by expanding the former around x, we find:

$$\psi(x+\xi, t) = \psi(x, t) + \xi\frac{\partial\psi(x, t)}{\partial x} + \frac{1}{2}\xi^2\frac{\partial^2\psi(x, t)}{\partial x^2} + \mathcal{O}(\xi^3) \tag{24}$$

As for the potential, note that we can switch $V\left(x+\frac{\xi}{2}, t+\frac{\epsilon}{2}\right)$ for simply $V(x, t) + \mathcal{O}(\epsilon, \xi)$. This is true because we need not expand further than the first order in ϵ, and since there is an

[5] Since the term that appears in the kinetic exponential is $x - y$, one may wonder why we do not substitute $x - y = \xi$ instead. For one, since the term appearing in the exponential is squared, it will result in ξ^2 either way. But the added convenience lies in the fact that no minus sign will be produced when changing the variable of integration from dy to $d\xi$.

ϵ in front of the potential term in the exponential, the zeroth order approximation $V(x,t)$ is already of order one in ϵ. This also holds for a first order expansion yielding a ξ term, as $\xi\epsilon \propto \epsilon^{3/2}$, of greater order than 1, and thus the highest order next term. Thus, we can expand the exponential itself and find the following:

$$e^{-\frac{i}{\hbar}\epsilon V\left(x+\frac{\xi}{2},t+\frac{\epsilon}{2}\right)} = 1 - \frac{i}{\hbar}\epsilon V(x,t) + \mathcal{O}(\xi\epsilon) \tag{25}$$

Putting this together yields:

$$\psi(x,t+\epsilon) = \frac{1}{C(\epsilon)}\int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\epsilon}}\left(1 - \frac{i}{\hbar}\epsilon V(x,t) + \mathcal{O}(\xi\epsilon)\right)\left(\psi(x,t) + \xi\frac{\partial\psi(x,t)}{\partial x} + \frac{1}{2}\xi^2\frac{\partial^2\psi(x,t)}{\partial x^2} + \mathcal{O}(\xi^3)\right)d\xi \tag{26}$$

Note that this will require us to take a number of integrals of the particular form $\int_{-\infty}^{\infty}\xi^n e^{\frac{im\xi^2}{2\hbar\epsilon}}d\xi$, which reminds us of the well-known Gaussian standard integral:

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \tag{27}$$

Here, a is a constant for which $\Re(a) > 0$.
A useful corollary of this standard integral appears when we differentiate both sides of it with respect to a. When we do so, we find:

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \sqrt{\frac{\pi}{4a^3}} \tag{28}$$

Lastly, we also note the following:

$$\int_{-\infty}^{\infty} x^{2n+1}e^{-ax^2}dx = 0 \tag{29}$$

With $n \in \mathbb{N}$, such that all Gaussian integrals with uneven powers of x in front of the exponential are zero. This is true because x^{2n+1} is an odd function while e^{-ax^2} is even, so that the total is odd and the integral vanishes since the lower boundary is the negative of the upper one.
Notice that in our case, we find that $e^{\frac{im\xi^2}{2\hbar\epsilon}} = e^{-a\xi^2}$ if $a = \frac{m}{2i\hbar\epsilon}$, where we have conveniently used that $i = -i^{-1}$. We can now calculate our integrals. We multiply the terms in the brackets, pull out the factors without ξ-dependence and provide the results one by one below:

$$\psi(x,t)\int_{-\infty}^{\infty} e^{-\frac{m}{2i\hbar\epsilon}\xi^2}d\xi = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}\psi(x,t) \tag{30}$$

$$\frac{\partial\psi(x,t)}{\partial x}\int_{-\infty}^{\infty}\xi e^{-\frac{m}{2i\hbar\epsilon}\xi^2}d\xi = 0 \tag{31}$$

$$\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\xi^2 e^{-\frac{m}{2i\hbar\epsilon}\xi^2}d\xi = \frac{i\hbar\epsilon}{2m}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\frac{\partial^2\psi(x,t)}{\partial x^2} \tag{32}$$

$$-\frac{i\epsilon}{\hbar}V(x,t)\psi(x,t)\int_{-\infty}^{\infty}e^{-\frac{m}{2i\hbar\epsilon}\xi^2}d\xi=-\frac{i\epsilon}{\hbar}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}V(x,t)\psi(x,t) \tag{33}$$

$$-\frac{i\epsilon}{\hbar}V(x,t)\frac{\partial\psi(x,t)}{\partial x}\int_{-\infty}^{\infty}\xi e^{-\frac{m}{2i\hbar\epsilon}\xi^2}d\xi=0 \tag{34}$$

Note that we ignore the combination $-\frac{i\epsilon}{2\hbar}V(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\xi^2 e^{-\frac{m}{2i\hbar\epsilon}\xi^2}d\xi$, as we can infer that it is of order $\mathcal{O}(\epsilon^2)$, and therefore discardable.
Further, some brief justification may be provided for ignoring the rest terms in our integral. First, let us consider the multiplication of $\mathcal{O}(\xi\epsilon)\propto\xi\epsilon$ with the terms in the second pair of brackets of the integral. Note that only multiplication with the $\xi\frac{\partial\psi(x,t)}{\partial x}$ and $\mathcal{O}(\xi^3)\propto\xi^3$ term will not result in integrals uneven in ξ, i.e., they need not be zero. The first of these two can also be discarded due to its order. We have, after all, seen using the third integral in our list above that an integral with a factor $\epsilon\xi^2$ before the exponential produces a second order term in ϵ. The same goes for the second of these cases, i.e., a Gaussian integral over $\xi\epsilon\cdot\xi^3$ in front of the exponential. This results in a power $a^{-5/2}$ on the right-hand side and thus an even higher power of ϵ.
We reach similar conclusions when considering the multiplication of $\mathcal{O}(\xi^3)$ with the terms in the first pair of brackets. We have already commented on the pair $\mathcal{O}(\xi^3)\mathcal{O}(\xi\epsilon)$, and the earlier two terms in the left pair of brackets will result in an uneven power of ξ in front of the exponential, i.e., integrating over these terms will yield zero.

Putting everything together now, we find the following:

$$\psi(x,t+\epsilon)=\frac{1}{C(\epsilon)}\left(\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\psi(x,t)+\frac{i\hbar\epsilon}{2m}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\frac{\partial^2\psi(x,t)}{\partial x^2}-\frac{i\epsilon}{\hbar}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}V(x,t)\psi(x,t)\right) \tag{35}$$

At this point, we must finally uncover the identity of $C(\epsilon)$. There are multiple ways to do this throughout the literature, some surely more elegant than the following. Yet, it works just fine. Suppose that at this point we were to take the limit as $\epsilon\to 0$. In that case, the left-hand side would just be $\psi(x,t)$, and we need the right-hand side to match this. The first term in the brackets contains merely this function, and looks like $\frac{1}{C(\epsilon)}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\psi(x,t)$. The only way to prevent this term from going to zero in the limit, and make sure we just find $\psi(x,t)$ on the right-hand side too, is to demand that $\frac{1}{C(\epsilon)}\sqrt{\frac{2\pi i\hbar\epsilon}{m}}=1$, i.e.,

$$C(\epsilon)=\sqrt{\frac{2\pi i\hbar\epsilon}{m}} \tag{36}$$

This does indeed contain a dependence on ϵ, as was already argued in section 4.1. The other two terms in the brackets still vanish due to the ϵ factor in front of them, and we find that for this constant the equation is still correct when the limit is taken.
While we will not prove so explicitly, it should be mentioned that in the case of more than

one Euclidean dimension of space, we merely need to raise $C(\epsilon)$ to the number of dimensions, i.e., $C^D(\epsilon)$. A qualitative understanding of this can be gained by remembering formula (9). Due to the independent way in which the sum over all paths can be performed in each dimension, each will just spawn another $\sqrt{\frac{2\pi i\hbar\epsilon}{m}}$ which means that the just derived result for $C(\epsilon)$ can be generalized straightforwardly.

Armed with this fact, we now find that:

$$\psi(x,t+\epsilon) = \psi(x,t) + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{i\epsilon}{\hbar}V(x,t)\psi(x,t) \tag{37}$$

Now let us first move the function $\psi(x,t)$ to the left-hand side. Subsequently, we multiply both sides by $\frac{i\hbar}{\epsilon}$. This yields:

$$i\hbar\frac{\psi(x,t+\epsilon)-\psi(x,t)}{\epsilon} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) \tag{38}$$

Note that if at this point we finally take the limit as the small increment of time $\epsilon \to 0$, the left-hand side contains, by definition, the partial derivative with respect to the time-variable of the wave function. With this, we have finally retrieved the Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) \tag{39}$$

Therefore, we have proved that the path integral formulation contains Schrödinger's wave mechanics. But as was stated earlier, this bridging effort taught us more than just the this fact. We have found how the wave function can be understood in terms of the kernel, how the integral kernel (18) can be used to carry the particle wave function to a later time and accompanying position, and what the identity of the constant $C(\epsilon)$ is.
The next step, then, is to prove that we can do this the other way around too.

2.2.2 Riding the wave to the path

For the derivation the other way around, I have had some aid in the form of lecture notes on path integrals from a 2021 summer school, which I have extended upon and worked out in detail myself (Rischke, 2021, p. 7-9). A starting point is that it is now much more convenient to work with the Schrödinger picture in Dirac notation. In this language, the Schrödinger equation takes the following form:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{40}$$

Where the Hamiltonian operator is defined as $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, i.e., we consider a particle in motion subjected to a potential. Note that the Hamiltonian here is independent of time.
One might on first thought consider this a problem, in the sense that if the subset of Hamiltonians that is time-*independent* can already reproduce the path integral, perhaps the Schrödinger

picture as a whole, i.e., including time-*dependent* cases, contains more information than it. This worry, however, has been taken care of by the previous part, since there we have already shown the path integral to be able to produce the time-*dependent* Schrödinger equation with $\hat{H} = \hat{H}(x,t)$. Thus, the use of $\hat{H} = \hat{H}(x)$ entails no loss in generality.
The reason for using a time-independent Hamiltonian is that in this way, we can formulate the quantum state by having the familiar time-evolution operator act on a prior state:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_A)}|\psi(t_A)\rangle \equiv \hat{U}(t_A,t)|\psi(t_A)\rangle \tag{41}$$

The unitary time-evolution operator $\hat{U}(t_A,t)$ carries the state $|\psi\rangle$ from a time t_A to a later time t. Although an exponentiated differential operator differs from an integral operator like we saw in (18), this concept certainly reminds us of the path integral kernel. It may therefore not be a surprise that the use $\hat{U}(t_A,t)$ will be instrumental in the coming derivation.

Let us suppose that we want to carry the quantum state from an initial time t_A to a final time t_B, i.e., we wish to find the operator $\hat{U}(t_A,t_B)$.
To start our journey to the path integral, we first require the use of the Trotter product formula:

$$e^{A+B} = \lim_{N\to\infty}\left(e^{\frac{A}{N}}\cdot e^{\frac{B}{N}}\right)^N \tag{42}$$

Here, A and B are complex square matrices. The full proof of the formula shall not be given here, but a very brief description is as follows. The formula contains basically two steps: slicing up e^{A+B} in an infinite product of smaller contributions, and going from a sum of matrices in the exponential to a product of two exponentials, each with one matrix. Both steps can be shown to be valid by using expansions. An example in the second case would be that expanding e^{A+B} would also yield cross terms BA, which is not the case when multiplying the expansions of e^A and e^B individually. Yet, such as cross terms would be of the order N^{-2}, so when taking the limit the differences between the expressions caused by this disappear and the formula is valid. In physics terms, the non-commutativity of Hermitian operators is negligible in the limit where exponentials with extremely small arguments are multiplied.

We now apply the Trotter product formula to the expression of the unitary time-evolution operator. In order to do this, we reintroduce the time-slicing procedure $t_B - t_A = N\epsilon$ with $t_{i+1} = t_i + \epsilon$ and where $t_A = t_0$ and $t_B = t_N$. This leads to:

$$e^{-\frac{i(t_B-t_A)}{\hbar}\hat{H}} = \lim_{\epsilon\to 0}\prod_{i=1}^{N}\left(e^{-\frac{i\epsilon}{2\hbar m}\hat{p}^2}\cdot e^{-\frac{i\epsilon}{\hbar}V(\hat{x})}\right) \tag{43}$$

In other words, the limit of an infinite-entry product of N unitary operators (split into a kinetic and potential one), each carrying the quantum state forward with an infinitesimal time-increment ϵ, is equal to performing just one transformation from t_A to t_B. Again, by definition of $t_B - t_A = N\epsilon$, taking $\epsilon \to 0$ simultaneously implies that $N \to \infty$.
The above result, however, deserves a little extra explanation, as it is not immediately obvious how it follows from the Trotter product formula. First of all, we write out $\hat{H}$, such that

we can separate our exponential into the product of two, each with a matrix as its exponent. Secondly, $\epsilon \propto N^{-1}$, and we therefore also find matrices that are divided by a factor later taken to infinity as exponents. Finally, since the time-slicing is done in equal portions ϵ and each term k is the same due to the Hamiltonian being constant, there is no difference between $\lim_{N\to\infty}\left(e^{\frac{A}{N}}\right)^N$ and $\lim_{N\to\infty}\prod_{i=0}^{N} e^{\frac{A}{N}}$ in this case. Combining this with the aforementioned equivalence of $\epsilon \to 0$ and $N \to \infty$, we can see how the Trotter product formula leads us to the above result.

We know that 43 is just the unitary time-evolution operator. Therefore, it can also be written as follows:

$$\hat{U}(t_A, t_B) = \prod_{i=1}^{N} \hat{U}(t_{i-1}, t_i) = \lim_{\epsilon\to 0} \prod_{i=1}^{N} \left(e^{-\frac{i(t_i - t_{i-1})}{2\hbar m}\hat{p}^2} \cdot e^{-\frac{i(t_i - t_{i-1})}{\hbar} V(\hat{x})} \right) \tag{44}$$

We know from elementary quantum mechanics that if we specifically desire the amplitude that carries the quantum state not only from a time t_A to t_B but also from the specific position x_A to x_B, we are to take the inner product of it with the initial and final position, i.e.,

$$U(x_A, t_A, x_B, t_B) = \langle x_B | \hat{U}(t_A, t_B) | x_A \rangle \tag{45}$$

This, we could abbreviate with $U(A, B)$ like we did with the particle kernel when discussing the path integral formulation. It may not be a surprise that we are interested in developing this expression.
To kickstart this, however, one more trick will be needed beyond the identity already established in (43). This is the use of the identity operator that follows from the completeness relation of a Dirac orthonormalized continuous basis in position space:

$$\int_{-\infty}^{\infty} |x_i\rangle\langle x_i| dx_i = \hat{I} \tag{46}$$

The quantum state could of course be expressed as any linear combination of this continuous position basis.
At this point, we can combine all of the above ingredients in the following way. We are interested in the quantity (45), and will use it as a template for the expression below. For the time evolution operator, we substitute (43). Then, *before* every term corresponding to a i-value in (43), we insert the identity operator (46). This leads us to the following expression:

$$U(A, B) = \langle x_B | \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dx_N \left(|x_N\rangle\langle x_N| e^{-\frac{i\epsilon}{2\hbar m}\hat{p}^2} \cdot e^{-\frac{i\epsilon}{\hbar} V(\hat{x})} \right) \tag{47}$$

$$\left(|x_{N-1}\rangle\langle x_{N-1}| e^{-\frac{i\epsilon}{2\hbar m}\hat{p}^2} \cdot e^{-\frac{i\epsilon}{\hbar} V(\hat{x})} \right) \cdots \left(|x_1\rangle\langle x_1| e^{-\frac{i\epsilon}{2\hbar m}\hat{p}^2} \cdot e^{-\frac{i\epsilon}{\hbar} V(\hat{x})} \right) |x_A\rangle \tag{48}$$

The brackets show unitary operator terms where each time the identity operator has been inserted before these.
When we now expand $e^{-\frac{i\epsilon}{\hbar} V(\hat{x})}$, we see that the position operator will act on some position eigenstate $|x_\alpha\rangle$ in the next pair of brackets. This will simply produce the corresponding

eigenvalue x_α. Therefore, we can substitute $V(\hat{x}) \to V(x_\alpha)$ with α whatever label matches the one of the ket in front of it. Since it then just becomes a number, we move the potential exponentials to the left of the kinetic ones.

Still, it is not immediately clear how such integrals can be solved, at least not in the current position basis. The kinetic exponential in the inner product would have to be expanded, with momentum operators acting on $|x_i\rangle$. Luckily, there is a far easier procedure, which allows us to make use of the simply fact that having the momentum operator act on a momentum basis vector will simply return us the momentum eigenvalue, i.e., $\hat{p}|p_\alpha\rangle = p_\alpha|p_\alpha\rangle$. This is just the momentum version of the argument just used to substitute $V(\hat{x}) = V(x_\alpha)$.
To utilize this fact, this time we will use the identity operator following from the completeness relation of Dirac orthonormalized basis vectors in momentum space:

$$\int_{-\infty}^{\infty} |p_i\rangle\langle p_i| dp_i = \hat{I} \tag{49}$$

Again, we will insert this identity operator into our expression, this time *right after every kinetic exponential* $e^{-\frac{i\epsilon}{2\hbar m}\hat{p}^2}$. Expanding the kinetic exponential will make the momentum operators act on momentum kets, returning the respective eigenvalues. The effect on our expression is as follows:

$$U(A,B) = \langle x_B| \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_N \left(|x_N\rangle\langle x_N| e^{-\frac{i\epsilon}{\hbar}V(x_{N-1})} \cdot e^{-\frac{i\epsilon}{2\hbar m}p_N^2} |p_N\rangle\langle p_N| \right) \tag{50}$$

$$\cdots \left(|x_1\rangle\langle x_1| e^{-\frac{i\epsilon}{\hbar}V(x_A)} \cdot e^{-\frac{i\epsilon}{2\hbar m}p_1^2} |p_1\rangle\langle p_1| \right) |x_A\rangle \tag{51}$$

Again, every term has been put into brackets for clarity. Note that every bra $\langle p_i|$ at the end of a term forms an inner product with the subsequent ket $|x_{i-1}\rangle$ of the next.
So while there is a structure to it, the above nevertheless looks rather messy. Luckily, we can clean it up using the product operator:

$$U(A,B) = \langle x_B| \lim_{\epsilon\to 0} \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} dx_i \int_{-\infty}^{\infty} dp_i e^{-\frac{i\epsilon}{\hbar}V(x_{i-1})} \cdot e^{-\frac{i\epsilon}{2\hbar m}p_i^2} |x_i\rangle\langle x_i|p_i\rangle\langle p_i| \right) |x_A\rangle \tag{52}$$

Here, everything within the brackets ought to be repeated for all i-values. One can check that it reproduces the written-out version exactly. Note that by the above arguments, the x's and p's in the exponentials have now lost their hats by respectively acting on the position and momentum kets on their right. Because of that, they have now been removed from the inner products.

Having rid ourselves of operators and being left with familiar recognizable inner products, we can now proceed. Standard quantum mechanics tell us that[6]:

$$\langle x_i|p_i\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}x_i p_i} \tag{53}$$

[6]This identity can actually also be proved using the completeness relations.

And since $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$, it follows that:

$$\langle p_i|x_{i-1}\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-i}{\hbar}x_{i-1}p_i} \tag{54}$$

The first of these two inner products can be found in every i-term in $U(A,B)$ above. However, when we consider the second of these, we note that it is always a combination of the momentum bra at the end of any term i, and the position ket at the beginning of every subsequent term $i-1$. This is unproblematic in general, except for the term $i=1$ and $i=N$ which, respectively, have a momentum bra $\langle p_1|$ without a ket and a position ket $|x_N\rangle$ without a bra. This is where the $\langle x_B|$ all the way on the left and the $|x_A\rangle$ all the way on the right come in. These 'latch on' to, respectively, our braless position ket and our ketless momentum bra. One of the inner products this generates is quite trivial, the other is somewhat more subtle. Let us start with the latter.
For the case $\langle x_B|x_N\rangle$, we are dealing with the inner product of two orthonormal basis vectors. Specifically, they are both of the position basis, the only inner product with this property in the entire expression. Logically, we know that $\langle x_B|x_N\rangle = \delta(x_B - x_N)$. Since there is an integral over dx_N in our product, this subsequently entails $\int_{-\infty}^{\infty} f(x_N)\delta(x_B - x_N) = f(x_B)$. Therefore, if we just substitute $x_N = x_B$ where it appears, something we already knew anyway, we can now remove the integral over dx_N from our sum. Thanks to the delta function, it was easy to perform.
The more straightforward inner product we find is $\langle p_1|x_A\rangle$. We know that (or could also just define that) $x_A = x_0$, and therefore we find $\langle p_1|x_0\rangle$, which is just the $i=1$ case for the inner product $\langle p_i|x_{i-1}\rangle$ given above.

Therefore, in the end we get:

$$U(A,B) = \lim_{\epsilon\to 0} \prod_{k=1}^{N-1} \left(\int_{-\infty}^{\infty} dx_k \right) \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \frac{1}{2\pi\hbar} e^{-\frac{i\epsilon}{2\hbar m}p_i^2 + \frac{i}{\hbar}(x_i - x_{i-1})p_i} \cdot e^{-\frac{i\epsilon}{\hbar}V(x_{i-1})} dp_i \right) \tag{55}$$

Some changes have appeared. Firstly, we now have two product operators, with brackets demarcating what they act on. This is the case because the position integral corresponding to the N-th term, over dx_N, has now been performed. Therefore, the position integrals now only go up to $N-1$. For this reason, these now have their own product operator. The 'original' product operator from $i=1$ to $i=N$ still acts on the rest.
Moreover, the results of the inner products have now been added to the kinetic exponential, since these all have dependencies on momentum. Finally, due to the $(2\pi\hbar)^{-1/2}$ factor from each of the two inner products from all i, this product operator now also repeats the constant $(2\pi\hbar)^{-1}$ for all i.

We now turn our attention to just the momentum integrals. We ignore the potential exponentials for a moment, as they have no momentum-dependence. These momentum integrals look very much like the Gaussian integral (27) we saw in the previous derivation too[7].

[7] Technically, this integral is valid only when $\Re(b) \geq 0$, and $\Re(a) > 0$. Yet, our integral has complex coefficients. We require some modifications that one can come too using complex analysis and Fresnel integrals.

The difference is that this one is slightly more general, also containing a linear term:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \tag{57}$$

Looking back at our integrals of interest (55), this means that we can identify $a = \frac{i\epsilon}{2\hbar m}$ and $b = \frac{i}{\hbar}x_i - x_{i-1}$. Considering a single momentum integral in (55), we then find:

$$\int_{-\infty}^{\infty} \frac{1}{2\pi\hbar} e^{-\frac{i\epsilon}{2\hbar m}p_i^2 + \frac{i}{\hbar}(x_i - x_{i-1})p_i}dp_i = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}e^{\frac{im}{2\hbar\epsilon}(x_i - x_{i-1})^2} \tag{58}$$

Multiplying the exponential on the right-hand side with the potential exponential, and realizing that the result is subject to the product operator $\prod_{i=1}^{N}$ as we can see in (55), we can draw the following conclusion:

$$\prod_{i=1}^{N}\left(\sqrt{\frac{m}{2\pi i\hbar\epsilon}}e^{\frac{im}{2\hbar\epsilon}(x_i - x_{i-1})^2 - \frac{i\epsilon}{\hbar}V(x_{i-1})}\right) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{N}{2}} e^{\sum_{i=1}^{N}\left(\frac{im}{2\hbar\epsilon}(x_i - x_{i-1})^2 - \frac{i\epsilon}{\hbar}V(x_{i-1})\right)} \tag{59}$$

Where, thanks due the general multiplicative property of exponentials, we have turned the product of exponentials into a sum in the exponent.
As for the constant, we can immediately make one observation, namely that we can detect (36) in here. Both ways of the equivalence proof allow one to discover the identity of $C(\epsilon)$. Since there were N momentum integrals, we have found that $C(\epsilon)$ is now raised to the power of -N. Since we are working towards the path integral we might as well adopt the notation $C(\epsilon)$ in a similar way as it was introduced there, and instead of $C^{-N}(\epsilon)$, write one factor $\frac{1}{C(\epsilon)}$ for each value of k in the product found in (55). Considering our bookkeeping, we should remember, however, that there is one more factor $\frac{1}{C(\epsilon)}$ than there are integrals over dx_k (N terms against $N-1$ terms). We put the one 'excess' factor in the front.

Now substituting all of the above results back into (55), we find for the matrix element of the time-evolution operator that:

$$U(A,B) = \lim_{\epsilon\to 0}\frac{1}{C(\epsilon)}\prod_{k=1}^{N-1}\int_{-\infty}^{\infty} e^{\sum_{i=1}^{N}\left(\frac{im}{2\hbar\epsilon}(x_i - x_{i-1})^2 - \frac{i\epsilon}{\hbar}V(x_{i-1})\right)}\frac{dx_k}{C(\epsilon)} \tag{60}$$

And with this, the path integral really starts to show itself. For the finishing touch, we let the sum in the exponent start at $i = 0$ and end at $i = N-1$ instead, so that we should substitute $i \to i+1$ for the subscripts of the terms in the exponent. One final move then will be to

The result, however, ends up being the same as if we just take the Gaussian integral and substitute $a \to ia$ and $b \to ib$, namely:

$$\int_{-\infty}^{\infty} e^{-iax^2+ibx}dx = \sqrt{\frac{\pi}{ia}}e^{\frac{ib^2}{4a}} \tag{56}$$

Here $a, b \in \mathbb{R}$.

rearrange the terms in the sum so that we recover the phase $S/\hbar$. For this, we just remove $\frac{i\epsilon}{\hbar}$ from the brackets in the sum, which gives us:

$$\frac{i}{\hbar}\epsilon\left(\frac{1}{2}m\left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2 - V(x_i)\right) = \frac{i}{\hbar}\epsilon L\left(\frac{x_{i+1}-x_i}{\epsilon}, x_i\right) \tag{61}$$

Thus, we have found the Lagrangian for an infinitesimal time-displacement. Of course, we could just replace its second argument $x_i \to \frac{x_{i+1}+x_i}{2}$, so that it looks exactly like the Lagrangian we were introduced to. Since we eventually take the limit $\epsilon \to 0$, this is fine, as we argued back then that the difference between the two will disappear in that limit.
Incorporating these last little changes, we have:

$$U(A,B) = \lim_{\epsilon\to 0}\frac{1}{C(\epsilon)}\prod_{k=1}^{N-1}\int_{-\infty}^{\infty} e^{\sum_{i=0}^{N-1}\frac{i}{\hbar}\epsilon L\left(\frac{x_{i+1}-x_i}{\epsilon}, \frac{x_{i+1}+x_i}{2}\right)}\frac{dx_k}{C(\epsilon)} \tag{62}$$

And on the right-hand side we have recovered (7) exactly[8].
This means that the matrix element of the time-evolution operator $U(A,B)$ in the language of the Schrödinger picture in Dirac notation is equivalent to the kernel $K(A,B)$ in the way it was defined by Feynman[9]. Formally:

$$\langle x_B|\hat{U}(t_A,t_B)|x_A\rangle \equiv K(x_A,t_A,x_B,t_B) \tag{63}$$

Finally, we could multiply the Lagrangian with ϵ. Taking it to zero will recover the action functional $S[x(t)]$, such that we also recover the more general (3):

$$K(A,B) = \int_A^B e^{\frac{i}{\hbar}S[x(t)]}\mathscr{D}x(t) \tag{64}$$

This concludes the second direction of our proof. With this, it has been shown that the path integral formulation of quantum mechanics is indeed equivalent to the more familiar Schrödinger formulation for a particle moving in a potential. As was the case when going from the path integral to the Schrödinger equation, this derivation also provides us with more things of interest than just result of the proof itself. For one, we found the relationship between $U(A,B)$ and $K(A,B)$, we used new mathematical tricks, and most of all, we discovered a new way of formulating the path integral. This last point, however, will certainly need some elaboration, which we will provide in the next part.

2.2.3 The phase space representation of the path integral

In chapter 4.1, we discussed how besides the coordinate space path integral over $\mathscr{D}x(t)$ we have occupied ourselves with so far, it is also possible to adopt a phase space representation

[8] Except for the minor difference resulting from assuming a time-independent Hamiltonian at the beginning of our derivation, the consequence of which is that the Lagrangian is now also time-independent.

[9] The kernel is often understood as specifically an integral operator as opposed to an exponentiated differential one, but it was shown through the derivation of (18) how this all connects.

in which we *also* sum over all momenta, i.e., there is also $\mathscr{D}p(t)$. This is important, since the phase space representation will provide a natural bridge to the path integral formulation of quantum field theory.
Still, one may wonder about the claim that we supposedly already discovered this. The key lies in equation (55). Here, rather than calculating the N momentum integrals, we leave them be and just create one big exponential. This leaves us with:

$$K(A,B) = \lim_{\epsilon\to 0} \prod_{k=1}^{N-1} \prod_{i=1}^{N} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{i\epsilon}{2\hbar m}p_i^2 + \frac{i}{\hbar}(x_i - x_{i-1})p_i - \frac{i\epsilon}{\hbar}V(x_{i-1})} \frac{dp_i}{2\pi\hbar} dx_k \tag{65}$$

Where some things have been rearranged for convenience. Just like how in the coordinate space representation we wrote $\frac{dx_k}{C(\epsilon)}$, we now place the $2\pi\hbar$ factors beneath the dp_i's. Moreover, $U(A,B)$ has been replaced by $K(A,B)$ since we already discovered their equivalence anyway and we are now focused more so on extending our understanding of the path integral than on the Schrödinger picture.
Now, we will do some small manipulations of the exponent:

$$-\frac{i\epsilon}{2\hbar m}p_i^2 + \frac{i}{\hbar}(x_i - x_{i-1})p_i - \frac{i\epsilon}{\hbar}V(x_{i-1}) = \frac{i}{\hbar}\epsilon\left(\frac{x_i - x_{i-1}}{\epsilon}p_i - \left(\frac{p_i^2}{2m} + V(x_i)\right)\right) \tag{66}$$

One may already recognize some familiar expression in the brackets, which also explains the bracket placement. We will return to this point shortly.
One might also notice that we substituted $V(x_{i-1}) \to V(x_i)$. As has been mentioned before, in the limit as $\epsilon \to 0$, it does in this case not matter[10] whether we take as the argument x_i, x_{i-1}, $\frac{x_{i+1}+x_i}{2}$, or even $\frac{3x_{i-1}+x_i}{4}$. In this case, $V(x_i)$ is, however, the nicer choice.

Just like we did before, we can in principle multiply all of the exponentials together from $i = 1$ up and including $i = N$, and as a result have a sum in our exponent. This time, however, we are not doing the integrals over the momenta, so the product over the i's still remains for the $\int_{-\infty}^{\infty}\frac{dp_i}{2\pi\hbar}$ terms. Therefore, we use a new subscript j for the sum. We get:

$$K(A,B) = \lim_{\epsilon\to 0} \prod_{k=1}^{N-1} \prod_{i=1}^{N} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\sum_{j=1}^{N}\frac{i}{\hbar}\epsilon\left(\frac{x_j - x_{j-1}}{\epsilon}p_j - \left(\frac{p_j^2}{2m} + V(x_j)\right)\right)} \frac{dp_i}{2\pi\hbar} dx_k \tag{67}$$

[10]The reality is a little more complicated. In fact, these different possibilities give rise to the path integral version of the so-called 'ordering problem'. Let us consider how it manifests in the operator formalism. A term like xp in the classical Hamiltonian can be written as either xp or px yet it is still the same Hamiltonian, but due to the canonical commutation relations in the operator formulation of QM, this does not fly there: $\hat{x}\hat{p} \neq \hat{p}\hat{x}$. This leads to the question how one quantizes a Hamiltonian operator with such products, if the same Hamiltonian classically would lead to different quantum Hamiltonians. In the path integral, the same problem manifests in a different way. Currently, we changed from $H(p_i, x_{i-1})$ to $H(p_i, x_i)$. This difference becomes relevant when working with interactions originating form coupling to gauge fields or in the presence of curvilinear spaces. Usually the Weyl quantization procedure or 'midpoint rule', which averages the successive x-coordinates, is applied. Note that we have already been applying this rule by insisting on the position argument of the Lagrangian and Hamiltonian being $\frac{x_{i+1}-x_i}{2}$ many times before. In the current setting though, the ordering problem does not show itself.

We now return to the identification of the exponential. First of all, given the limit $\epsilon \to 0$, we identify $\frac{x_j - x_{j-1}}{\epsilon} = \dot{x}_j$ by definition. Thus, we are left with the product of velocity and momentum for each j-value. In the second pair of brackets, we find the sum of the classical kinetic energy and the potential energy, i.e., the classical Hamiltonian:

$$K(A,B) = \lim_{\epsilon \to 0} \prod_{k=1}^{N-1} \prod_{i=1}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\sum_{j=1}^{N} \frac{i}{\hbar} \epsilon (\dot{x}_j p_j - H(x_j, p_j))} \frac{dp_i}{2\pi\hbar} dx_k \tag{68}$$

This is the phase space representation of (7).

Again, we can multiple $\dot{x}_j p_j - H(x_j, p_j)$ by the ϵ in front of it and eventually take $\epsilon \to 0$. Thus, we can again replace this by a functional integral over time, just as we did in the coordinate space integral.
A final accompanying change, then, is to recognize that through the products of the integrals of position and momenta from $-\infty$ to $+\infty$, we are doing a sum over all paths and, this time, momenta too. Thus, we can reformulate our formula in terms of the general and briefer $\mathscr{D}$ notation. We define:

$$\int_A^B \mathscr{D}' x(t) \equiv \lim_{\epsilon \to 0} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty} dx_k \tag{69}$$

$$\int_A^B \mathscr{D} p(t) \equiv \lim_{\epsilon \to 0} \prod_{i=1}^{N} \int_{-\infty}^{\infty} \frac{p_k}{2\pi\hbar} \tag{70}$$

Here, we note three important differences.
Firstly, note the apostrophe in $\mathscr{D}' x(t)$. This is there for good reason, as in the phase space integral this term is different from the $\mathscr{D} x(t)$ one in coordinate space. After all, it does not come with the constants $C(\epsilon)$ now, which originate precisely because in the coordinate space integral we *do* integrate away the momenta, leaving us with a constant. Here, it is $\mathscr{D} p(t)$ which contains constants $(2\pi\hbar)^{-1}$ when written out in its discrete lattice regularization.
Secondly, there is a further difference, namely that the sum over all momenta ends at $i = N$, while the sum over all paths ends at $k = N - 1$. Mathematically, we saw how this logically came about. But there is a physical argument here too. The consequence of the $i = N$ term is integration over p_N, which in prior notation we could also write as p_B. We are thus *not* free to choose a definite final momentum, i.e., we are *not* writing $K(p_A, x_A, t_A, p_B, x_B, t_B)$. Instead, by integrating dp_B from $-\infty$ to $+\infty$, we consider all possible momenta for the system. This is no surprise, as the simultaneous specification of both x_B and p_B entails a violation of the uncertainty principle.
Lastly, we may wonder about the interpretation of the phase space kernel. In coordinate space we calculate as if a particle takes all possible paths between two points, which we could visualize in diagrams. Should we now consider the particle to take all possible paths, each path having to be considered with all possible momenta that the particle could have travelled along the path with? Perhaps we should speak of every possible path in phase space instead. But since there is no end point for the momentum, what do these paths look like? And classically, a *point* in phase space represents a path, and a line therefore does not. Therefore,

while mathematically just as solid, some of the conceptual clarity of the coordinate space representation of the path integral is surely lost here.

Concluding, however, we can write the coordinate space representation of the path integral as in (3):

$$K(A,B) = \int_A^B e^{\frac{i}{\hbar}\int_{t_A}^{t_B} L[\dot{x}(t),x(t)]dt} \mathscr{D}x(t) \tag{71}$$

While the phase space representation of the path integral looks like this instead:

$$K(A,B) = \int_A^B \int_A^B e^{\frac{i}{\hbar}\int_{t_A}^{t_B} [\dot{x}(t)p(t) - H[x(t),p(t)]]dt} \mathscr{D}'x(t)\mathscr{D}p(t) \tag{72}$$

The latter thus employs a Hamiltonian, replacing velocities by momenta.
For all of the above, extensions to more dimensions are straightforward, in line with what was described in chapter 4.1.

The phase space representation of the path integral also has its counterpart in quantum field theory, and as such it will return in some form there. For now, it is time to apply the path integral formulation to a concrete physical problem, to give an example of how one can actually calculate these seemingly complicated integrals. This is the topic of the next subchapter.

2.3 Calculating path integrals

While its formulation can be said to contain a certain elegance, calculations with the path integral are often rather laborious. While there are, especially in QFT, plenty of other advantages we will speak of in later subchapters, computational ease is something that is found more so in wave mechanics. Still, for now and later in this thesis, it is important to have some understanding of the ways in one can concretely use the path integral to handle physical problems. In order to achieve this, this subchapter will derive the kernel for a specific physical setting and discuss in general what and how physical systems can be analyzed by use of the path integral. For the former, we will consider the free particle. This is a rare case for which the path integral can be solved in a relatively straightforward matter, and without the need for lengthy advanced methods or approximations. For the latter, we will primarily discuss the application of perturbation theory to the path integral, an incredibly important approximation method for dealing with small deviations in a potential, that is also the key to the famous Feynman diagrams when used in QFT. While these two goals may seem remote, the free particle solution will be seen to play a pivotal role in the application of perturbation theory to the path integral.

2.3.1 The free particle

The following is, again, a derivation that is worked out further but is based on the treatment of Feynman and Hibbs (Feynman and Hibbs, 1965, p. 42-43).

The Lagrangian of a free particle is $L = \frac{1}{2}m\dot{x}^2$. Substituting this directly in formula (7) yields:

$$K(A,B) = \lim_{\epsilon\to 0}\frac{1}{C(\epsilon)}\prod_{k=1}^{N-1}\int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2}\frac{dx_k}{C(\epsilon)} \tag{73}$$

Where from the previous subchapter we know that each $C(\epsilon) = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}$.
The above still looks quite messy. In the following, we start by looking only at the term $k = 1$ and $i = 0, 1$, i.e., the values of i that will lead to x_1 terms in the exponential. After all, when $k = 1$, we integrate over dx_1. Lastly, we also drop the limit for later. We find the following term:

$$\frac{m}{2\pi i\hbar\epsilon}\int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}((x_1-x_0)^2+(x_2-x_1)^2)}dx_1 \tag{74}$$

Where we have multiplied the constant in the front and the one associated with dx_1.

In this way, the integral starts looking somewhat more manageable. When working out the brackets and moving the exponentials independent of x_1 out of the integral, we find:

$$\frac{m}{2\pi i\hbar\epsilon}e^{\frac{im}{2\hbar\epsilon}(x_0^2+x_2^2)}\int_{-\infty}^{\infty} e^{-\frac{m}{i\hbar\epsilon}x_1^2+\frac{m}{i\hbar\epsilon}(x_0+x_2)x_1}dx_1 \tag{75}$$

Note that we made use of the identity $i = -i^{-1}$ to change the signs of the two terms in the exponential in the integral.
More importantly, however, the integrand was written in a format allowing us to use the standard integral (57) introduced in the previous subchapter again. In our current case, $a = \frac{m}{i\hbar\epsilon}$ and $b = a(x_0 + x_2)$.
Using this identity to work out our integral equation, we eventually find the following term:

$$\sqrt{\frac{m}{2\pi i\hbar\cdot 2\epsilon}}e^{\frac{im}{2\hbar\cdot 2\epsilon}(x_2-x_0)^2} \tag{76}$$

Here, the explicit 2ϵ part rather than just putting a 4 at the beginning of the term might look awkward, but will make more sense after the next step.
Since ultimately we need to multiply every one of the integrals over the dx_k, and we have the result from one of them now, we now multiply that result with the second integral term, i.e., the one over dx_2. This will of course also come with another factor $C^{-1}(\epsilon)$. In addition, we also let this integral contain the entry of the exponential sum corresponding to $i = 2$, since this includes another factor x_2 that will have to be integrated over. The result is as follows:

$$\sqrt{\frac{m}{2\pi i\hbar\cdot 2\epsilon}}\cdot\sqrt{\frac{m}{2\pi\hbar\epsilon}}\int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\cdot 2\epsilon}(x_2-x_0)^2}\cdot e^{\frac{im}{2\hbar\epsilon}(x_3-x_2)^2}dx_2 \tag{77}$$

This expression, where we 'tacked on' the $k = 2$ and $i = 2$ terms to our existing $k = 1$ and $i = 0, 1$ result, looks structurally very similar to the integral that followed from the $k = 1$ and $i = 0, 1$ terms. When writing it as we did in the case of the term (75), we find:

$$\frac{m}{2\sqrt{2}\pi i\hbar\epsilon}e^{\frac{im}{2\hbar\cdot 2\epsilon}(x_0^2+2x_3^2)}\int_{-\infty}^{\infty} e^{-\frac{3m}{4i\hbar\epsilon}x_2^2+\frac{m}{2i\hbar\epsilon}(x_0+2x_3)x_2}dx_2 \tag{78}$$

Due to the extra factor of 2 in the denominator of the first exponential in the integral (77) from our previous calculation, the coefficients are different this time. Nevertheless, the form allows for the use of the standard integral (57) again, where this time $a' = \frac{3m}{4i\hbar\epsilon}$ and $b' = \frac{2}{3}a'(x_0 + 2x_3)$. When after using the standard integral, one carefully works out the new constant and the brackets in the exponential, this results in the expression:

$$\sqrt{\frac{m}{2\pi i\hbar \cdot 3\epsilon}} e^{\frac{im}{2\hbar \cdot 3\epsilon}(x_3 - x_0)^2} \tag{79}$$

This looks very similar to the previous result (76). The only changes are that x_2, now having been integrated over, is replaced by x_3, and that the factors 2ϵ in the coefficient and exponential have been replaced by 3ϵ. In fact, we notice a particular pattern:

$$\frac{1}{C(\epsilon)} \prod_{k=1}^{p-1} \int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}\sum_{i=0}^{p-1}(x_{i+1}-x_i)^2} \frac{dx_k}{C(\epsilon)} = \sqrt{\frac{m}{2\pi i\hbar \cdot p\epsilon}} e^{\frac{im}{2\hbar \cdot p\epsilon}(x_p - x_0)^2} \tag{80}$$

We can prove that this is true for all p using induction. The base case has already been verified, which leaves the induction step. Assuming that the above pattern holds for $k = p-1$, we show that it holds for $k = p$ too. We write:

$$\frac{1}{C(\epsilon)} \prod_{k=1}^{(p+1)-1} \int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}\sum_{i=0}^{(p+1)-1}(x_{i+1}-x_i)^2} \frac{dx_k}{C(\epsilon)} \tag{81}$$

We 'break off' the $k = p$ and $i = p$ part of the product and sum. Since we assumed the truth of our statement for the $p-1$ case, we can simply substitute according to equation (80). What is left is the integral over dx_p, the exponential corresponding to $i = p$ and the $C^{-1}(\epsilon) = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}$ corresponding to the $k = p$ term of the product. All together, we find:

$$\sqrt{\frac{m}{2\pi i\hbar\epsilon}} \cdot \sqrt{\frac{m}{2\pi i\hbar \cdot p\epsilon}} \int_{-\infty}^{\infty} e^{\frac{im}{2\hbar\epsilon}(x_{p+1}-x_p)^2} \cdot e^{\frac{im}{2\hbar \cdot p\epsilon}(x_p - x_0)^2} dx_p \tag{82}$$

From here on, we proceed as before. We work out the brackets, use the standard integral (57), use algebra to collect all terms neatly in both the exponential and the constant, and when working this out we ultimately find:

$$\sqrt{\frac{m}{2\pi i\hbar \cdot (p+1)\epsilon}} e^{\frac{im}{2\hbar \cdot (p+1)\epsilon}(x_{p+1} - x_0)^2} \tag{83}$$

Which completes the induction step. Therefore, we have proven equation (80) by induction. We can use this new relation to figure out what happens when $p = N$. What changes here, is that twice a term $N\epsilon$ will appear. However, by definition this is equal to $t_B - t_A$. Similarly, we know that $x_0 = x_A$ and $x_N = x_B$. Formally, we still need to take the limit $\epsilon \to 0$, but this will not affect our final result. In summary, we find that the kernel of our free particle K_{fp} is:

$$K_{fp}(A, B) = \sqrt{\frac{m}{2\pi i\hbar(t_B - t_A)}} e^{\frac{im(x_B - x_A)^2}{2\hbar(t_B - t_A)}} \tag{84}$$

One may recall from subchapter 4.1 that the generalization of this result to many dimensions is quite straightforward. The constant, now a power of 1/2, simply needs to be replaced with $D/2$, where D is the number of spatial dimensions.

To some, this way of presenting the free particle may not look all that familiar. After all, when solving the one-dimensional Schrödinger equation directly for the $V(x,t) = 0$ case, one finds:

$$\psi(x,t) = Ce^{\frac{i}{\hbar}(px-Et)} \tag{85}$$

With C an integration constant.
Moreover, our free particle result is clearly not normalizable. After all:

$$\int_{x_1}^{x_2} |K_{fp}(0,0;x,t)|^2 dx = \frac{m}{2\pi\hbar t}\int_{x_1}^{x_2} dx = \frac{m(x_2-x_1)}{2\pi\hbar t} \tag{86}$$

Here, we have a particle that starts at $t = 0$ at the origin so that $A = (0,0)$, and we consider the probability of it, after a time t, being found somewhere between $x = x_1$ and $x = x_2$, where $x_1, x_2 \in \mathbb{R}$. Clearly, since the interval $x_2 - x_1$ can be made arbitrarily large, the probability distribution is not normalized.

However, both observations, that of non-normalizability and of the unfamiliar form, can be addressed without much difficulty. The former is actually not surprising, as we already know that the free particle system is not normalizable in the usual way from wave mechanics. This follows trivially from the free particle solution provided above, which leaves one with a probability density $|C|^2 dx$. The path integral just reproduces this known fact.
This then leads us to the question of form, as our free particle kernel and the above wave function look rather different. In spite of looks, though, the two can be connected in a straightforward manner. First, let us remember that we can substitute $E = \frac{p^2}{2m}$. Moreover, we know that the general solution of the free particle wave function is given as a linear combination over all continuous momenta, i.e., we integrate the above wave function over dp from $-\infty$ to ∞. In the case where all momenta are taken as equally likely[11], we simply end up integrating $e^{\frac{i}{\hbar}\left(px-\frac{p^2}{2m}t\right)}$ over all momenta. This is the Gaussian integral (57) again, and accounting for the right constant, one can trivially show that we end up with K_{fp} after this.

[11] This has the effect of getting rid of the p-dependent weighing factor one can expect in the integral. This reminds us of our previous discussion of the phase space integral, where we also observed that the final momentum integral dp_B was integrated over all possible values. This is no coincidence. Here above, we show how the K_{fp} expression we derived from coordinate space can obtained by performing the integral over all momenta over the free particle wave function from the Schrödinger equation. But it can also be done the other way around, meaning that the general solution of the free particle from the Schrödinger equation, including the integral over momenta, can be retrieved from the path integral formulation. For this purpose, however, one needs to start from the phase space representation. Here, one performs all the integrals over the dx_k, resulting in δ-functions allowing one to determine the integrals over the dp_i. However, as explored in the previous subchapter, k goes from 1 to $N-1$ while i from 1 to N. This final integral over $dp_N \equiv dp_B$ retrieves the general solution one finds from wave mechanics.

This, then, concludes our discussion of the free particle, having derived the result and arguing its equivalence to the way one is more likely familiar with it. Yet, there are many more cases for the path integral to be confronted by that *do* have a potential energy term in the Lagrangian. For many, we require the use of perturbation theory. We cover this in the next part.

2.3.2 Perturbation theory and the path integral

Since we have established the equivalence between the Schrödinger equation and the path integral, it is to be expected that both are able to solve the same few cases analytically, while for many others requiring perturbation theory. Before getting into this, one should note that the typical introductory quantum systems such as the infinite square well, the harmonic oscillator and the hydrogen atom can therefore all be solved analytically with the path integral too.
A decent class of Lagrangians can, in fact, be solved analytically relatively easily. Feynman showed this for those of the following type:

$$L(\dot{x}(t), x(t), t) = a(t)\dot{x}^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t) \tag{87}$$

He refers to the integrals resulting from Lagrangians like this as Gaussian integrals (Feynman and Hibbs, 1965, p. 58). They include many of the well-known cases. As an example, to see that this Lagrangian also includes the case of the harmonic oscillator, take $b(t) = d(t) = e(t) = f(t) = 0$, $a(t) = m/2$ and $c(t) = m\omega^2$.
To come to this conclusion, Feynman uses a trick whereby he expresses a path in terms of a sum of the classical path and a path deviating from the classical path, i.e., $x(t) = x_{cl}(t) + \Delta x(t)$. Substituting this into the general path integral, he uses it to split off an exponential with an action purely depending on the classical path from the rest. The other term depending on $\Delta x(t)$ has integral boundaries that are both 0, because at the boundaries x_A and x_B, all paths will have the same value and thus there is no difference between $x(t)$ and the classical path $x_{cl}(t)$ with respect to these points. Therefore, that term is only a function of the initial and final *times*. Due to this, he arrives at the following expression for the kernel (Feynman and Hibbs, 1965, p. 59-60):

$$K(A, B) = e^{\frac{i}{\hbar}S[x_{cl}(t)]}\chi(t_A, t_B) \tag{88}$$

Here, χ is a function depending on the known coefficients of the Lagrangian and the known initial and final time. Other than that, one only needs the action corresponding to the classical path. He therefore shows that all kernels with Lagrangians of the type above can be determined exactly.
It should be noted that not *all* Lagrangians that allow for analytical solutions are of the above form. The hydrogen atom, for instance, is not. Since it took more than 30 years after the path integral's introduction to the wider community by Feynman in 1948 for it to successfully describe the hydrogen atom, this was not known at the time he wrote the book (Duru and Kleinert, 1979).

Thus, while there are quite some analytic solutions to many quantum systems with the path integral, many nevertheless require the use of approximation methods. These entail not *just* perturbative ones, as others, such as the semiclassical approximation dealing with slowly changing potentials, exist. Yet, perturbation theory is certainly the most prevalent. This is true all the more so in QFT, where applying perturbation theory to the path integral is the most straightforward way to derive Feynman diagrams.

One may remember the basis of perturbation theory in ordinary quantum mechanics. The central idea is as follows. Suppose a quantum particle is subject to a potential that looks quite similar to another potential for which we can solve the Schrödinger equation to arrive at the wave function and associated energy levels of a particle. The potential of the system under consideration, however, deviates slightly from this 'known' potential. Then we can approximate the wave function and associated energy levels of this system by using the deviation and the known solution of a quantum system where the deviation is absent, i.e., the unperturbed state (Griffiths, 2014, p. 251-322).

While applying perturbation theory to the path integral there are certainly differences in the way it plays out mathematically and even in its interpretation, we shall see that this basic philosophy remains the same. For one last time, we follow the methodology employed in Feynman & Hibbs, but it will include more explanation, derivations and reflections (Feynman and Hibbs, 1965, p. 120-125). One can in principle start both from the coordinate space or the phase space representation, but we will follow Feynman and consider the somewhat less messy coordinate space option. In the following, we will assume that the unperturbed potential is simply zero, with the perturbation now being a small potential.
Let us then consider the general one-particle path integral kernel in the coordinate space representation:

$$K(A,B) = \int_A^B e^{\frac{im}{2\hbar}\int_{t_A}^{t_B}\dot{x}^2(t)dt} e^{-\frac{i}{\hbar}\int_{t_A}^{t_B}V(x(t),t)dt}\mathscr{D}x(t) \tag{89}$$

Here, we have written out the Lagrangian and split the exponential in a purely kinetic and purely potential one. Both the velocity in the kinetic term and the position argument of the potential have been explicitly written as functions of time, as it is important to keep this in mind in this derivation. It also shows that the potential can of course admit a double time-dependence: both an explicit one and an implicit one in the change of position of a particle over time. An analogy would be that of a little ball moving on an electric stove, where we assume that the ball is always immediately in thermal equilibrium with the stove regardless of temperature varying on the stove's surface. The temperature of the ball is then both a function of time due to the stove heating up, as well as a function of position due to a point near a pit being hotter. But, since the ball is in motion on the stove, and this motion can be written as a function of time, we conclude that $T = T(x(t), y(t), t)$.
As was laid out, perturbation methods apply to small deviations in the potential. It is important to be clear about what must be small and how 'small' is to be quantified. These questions

can be succinctly answered by the condition below:

$$\left| \int_{t_A}^{t_B} V(x(t),t)dt \right| \ll \hbar \tag{90}$$

The key, thus, as Feynman and Hibbs note, is that the phase of the potential exponential is small (Feynman and Hibbs, 1965, p. 121). In that case, once we expand the exponential, it is actually the case that each subsequent order will contribute significantly less to the result than the previous. The expansion looks as follows:

$$e^{-\frac{i}{\hbar}\int_{t_A}^{t_B} V(x(t),t)dt} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \left(\int_{t_A}^{t_B} V(x(t),t)dt\right)^n \tag{91}$$

Substituting this into our kernel, we find:

$$K(A,B) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \int_A^B \left(e^{\frac{im}{2\hbar}\int_{t_A}^{t_B} \dot{x}^2(t)dt} \left(\int_{t_A}^{t_B} V(x(t'),t')dt'\right)^n \right) \mathscr{D}x(t) \tag{92}$$

For clarity's sake, brackets have been added inside the path integral, as these contain functions that are integrated over. Moreover, to avoid possible confusion between the two time integrals later on, the one over the potential has been labelled by a time variable t'.

Notation wise, we can denote each term as $K_n(A,B)$, the sum over all n yielding the full kernel:

$$K(A,B) = \sum_{n=0}^{\infty} K_n(A,B) \tag{93}$$

In the following, we will consider the terms $n = 0,1,2$ to get some idea of how these terms can be evaluated, but perhaps more importantly, how an interpretation related to scattering can be ascribed to each term.

The $n = 0$ term simply denotes the unperturbed result, just as we see in perturbation theory applied to ordinary quantum mechanics. In our case, this zeroth term of the expanded kernel (92) lacks a potential term in the path integral and thus simply reduces to:

$$K_0(A,B) = \int_A^B e^{\frac{im}{2\hbar}\int_{t_A}^{t_B} \dot{x}^2(t)dt} \mathscr{D}x(t) = K_{fp}(A,B) = \sqrt{\frac{m}{2\pi i\hbar(t_B - t_A)}} e^{\frac{im(x_B - x_A)^2}{2\hbar(t_B - t_A)}} \tag{94}$$

Which is, of course, our free particle kernel.
When we consider the $n = 1$ term, things get more interesting:

$$K_1(A,B) = -\frac{i}{\hbar}\int_A^B \left(e^{\frac{im}{2\hbar}\int_{t_A}^{t_B} \dot{x}^2(t)dt} \left(\int_{t_A}^{t_B} V(x(t'),t')dt'\right) \right) \mathscr{D}x(t) \tag{95}$$

Let us take a moment to consider what the above actually tells us. A first thing we immediately notice is the above described philosophy of perturbation theory in action: the free

particle kernel is still in here, but is now being changed in some way by the presence of the small potential. There is a way of writing the above that makes this point much more obvious. For this, we interchange the order of the time and path integration in $K_1(A,B)$ above, yielding:

$$K_1(A,B) = -\frac{i}{\hbar}\int_{t_A}^{t_B}\left(\int_A^B V(x(t'),t')e^{\frac{im}{2\hbar}\int_{t_A}^{t_B}\dot{x}^2(t)dt}\mathscr{D}x(t)\right)dt' \tag{96}$$

Feynman denotes the function in the brackets as $F(t')$, i.e.,

$$F(t') = \int_A^B V(x(t'),t')e^{\frac{im}{2\hbar}\int_{t_A}^{t_B}\dot{x}^2(t)dt}\mathscr{D}x(t) \tag{97}$$

and

$$K_1(A,B) = -\frac{i}{\hbar}\int_{t_A}^{t_B} F(t')dt' \tag{98}$$

In $F(t')$, the point about the free particle kernel being changed by the small potential becomes clear, and a nice way of handling and interpreting the expression will come with it. We note that $F(t')$ is just an expression for the sum over all paths given the free particle Lagrangian, except that for each path it is multiplied by $V(x(t'),t')$. In other words, each term in the sum associated with a particular path $x_{\mathscr{C}}(t)$ is 'weighted' by a scalar in form of the potential. The exact scalar value of this potential is, of course, found when evaluating it for the particular path $x_{\mathscr{C}}(t)$ under consideration at the particular time $t = t'$, i.e., $V(x_{\mathscr{C}}(t'),t')$. Therefore, the identification of this 'weighing factor' requires only one point on the path.

This insight can be used to derive an expression allowing for a convenient and elegant interpretation of the quantum mechanical perturbation series. Consider the formula for the kernel resulting from discrete lattice regularization (7).[12] Translating our above formula for $F(t')$ into these terms, we have:

$$F(t') = \lim_{\epsilon\to 0}\frac{1}{C(\epsilon)}\prod_{k=1}^{N-1}\int_{-\infty}^{\infty} V(x_p,t_p)e^{\frac{im}{2\hbar\epsilon}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2}\frac{dx_k}{C(\epsilon)} \tag{99}$$

This potential $V(x_p,t_p)$ is what we get when we express $V(x(t'),t')$ in the language of discrete lattice regularization. After all, we slice up the time in N intervals of ϵ. Since $t_A < t' < t_B$, we can express t' as the p-th time increment on the lattice. That is, given the limit $\epsilon \to 0$, there exists some $1 < p < N-1$ such that $p\epsilon = t' \equiv t_p$, corresponding to an x_p.
Consider now the integrals. We can simply perform the ones corresponding to dx_1 up and including dx_{p-1}. After all, $V(x_p,t_p)$ is a constant with respect to these integrals. This, by definition, results in $K_{fp}(A,P)$, where we have abbreviated $P = (x_p,t_p)$ as usual. Moreover, the same can be done for the integrals over dx_{p+1} up and including dx_{N-1}, which yield $K_{fp}(P,B)$. That leaves only the integral corresponding to dx_p, thus we find:

$$F(t') = \int_\infty^\infty K(A,P)V(x_p,t_p)K(P,B)dx_p \tag{100}$$

[12] Feynman and Hibbs directly write down this result, but I think it can more easily be seen in this way (Feynman and Hibbs, 1965, p. 122).

Two points of explaining still need to be done here. Firstly with regards to the bookkeeping of the constants $C^{-1}(\epsilon)$. All is well in this regard if we suppose that the constant which appeared after the limit is absorbed into the $K(A,P)$, and that the constant corresponding to the dx_p integral plays this role for $K(P,B)$. This is why the above expression shows dx_p rather than $\frac{dx_p}{C(\epsilon)}$. The second point one may wonder about concerns the bookkeeping of the sum in the exponential. The above expression does not show the term for $i = p$. Again, this term is absorbed into $K_{fp}(P,B)$. After all, $K(A,P)$ contains the terms $i = 0$ up and including $i = p-1$, while $K(P,B)$ contains the terms $i = p$ (the '$i = 0$' of this sum) up and including $i = N-1$.

Therefore, the above $F(t')$ is the correct result when taking the bookkeeping of our terms and constants into account. If we now remember that we stated $t_p \equiv t'$, correspondingly that $x_p \equiv x(t')$, and we write out A, P and B, we can identify the following result:

$$F(t') = \int_{-\infty}^{\infty} K_{fp}(x_A, t_A; x', t')V(x', t')K_{fp}(x', t'; x_B, t_B)dx' \tag{101}$$

Since similar, longer expressions will follow, we have introduced $x(t') \equiv x'$ as a shorthand. If one so desires, one could even write out the free particle kernels, yielding:

$$F(t') = \frac{m}{2\pi i\hbar\sqrt{(t_B - t')(t' - t_A)}} \int_{-\infty}^{\infty} V(x', t')e^{\frac{im}{2\hbar}\left(\frac{(x'-x_A)^2}{t'-t_A} + \frac{(x_B-x')^2}{t_B-t'}\right)}dx' \tag{102}$$

But since this is not exactly prettier and since we will soon see that (101) is not only a more general but also rather insightful way to write it down, we will stick to that. Before discussing the result, we note that for the first order of the perturbation series we have now found that:

$$K_1(A,B) = -\frac{i}{\hbar}\int_{t_A}^{t_B}\left(\int_{-\infty}^{\infty} K_{fp}(x_A, t_A; x', t')V(x', t')K_{fp}(x', t'; x_B, t_B)dx'\right)dt' \tag{103}$$

With this result, it is clear how one is to calculate $K_1(A,B)$. In the problem of interest, the form of the potential is known. Since we also know the free particle kernel, the calculation comes down to two integrals of a then known function $K_{fp}(x_A, t_A; x', t')V(x', t')K_{fp}(x', t'; x_B, t_B)$.

More interestingly though, let us spend a few words on the interpretation thereof. Feynman interprets the above result as a *scattering process* (Feynman and Hibbs, 1965, p. 122-123). In particular, it is taken to express a situation whereby a particle is at one point on its free particle trajectory scattered by the potential. This changes its trajectory, such that it follows a different free particle one afterwards. There is a way in which one can read this from the integrand of $K_1(A,B)$, i.e., $K_{fp}(x_A, t_A; x', t')V(x', t')K_{fp}(x', t'; x_B, t_B)$. Starting from left to right, we have a particle moving from (x_A, t_A) to (x', t') as a free particle, symbolized by $K_{fp}(x_A, t_A; x', t')$. Then, at (x', t'), it is scattered, which is expressed by $V(x', t')$. Finally, it proceeds as a free particle from (x', t') to its final state (x_B, t_B). Since in the expression $F(t')$ we integrate over all possible x', we can take this as taking into account all possible positions where the one-time scattering event can happen. Finally, to arrive at $K_1(A,B)$ the

time-variable t' is also integrated over, which allows us to consider all possible times between t_A and t_B where this scattering could happen. Note how space and time are treated on equal footing in this way.

The above description likely rings a bell to the reader familiar with calculating S-matrix elements from Feynman diagrams in the context of QFT. Indeed, that procedure is simply the QFT version of the very process laid out here. While in QFT, scattering often entails the creation and annihilation of different particles, the perturbation terms of the matrix element one is interested in can be derived from the consideration of all possible scattering events that can happen between the asymptotically free initial and final particle states.

This thought naturally leads us to consider the second order of the perturbation expansion (92). Because where $K_0(A,B)$ corresponds the particle moving from state A to B as a free particle without scattering, and $K_1(A,B)$ symbolizes the situation where a scattering event can happen at any time or place in-between A and B, we will find that $K_2(A,B)$ represents the situation one step up where two intermediary scattering events are considered. Logically, $K_N(A,B)$ then corresponds to N scattering events. A graphical illustration of this can be found in figure 4. The sum of all these possible number of scattering events each being possible at all places and all times, then yields the exact kernel of the system. Luckily, however, due to the smallness of the potential, these calculations will not need to go up to arbitrary order to be accurate enough for practical purposes.

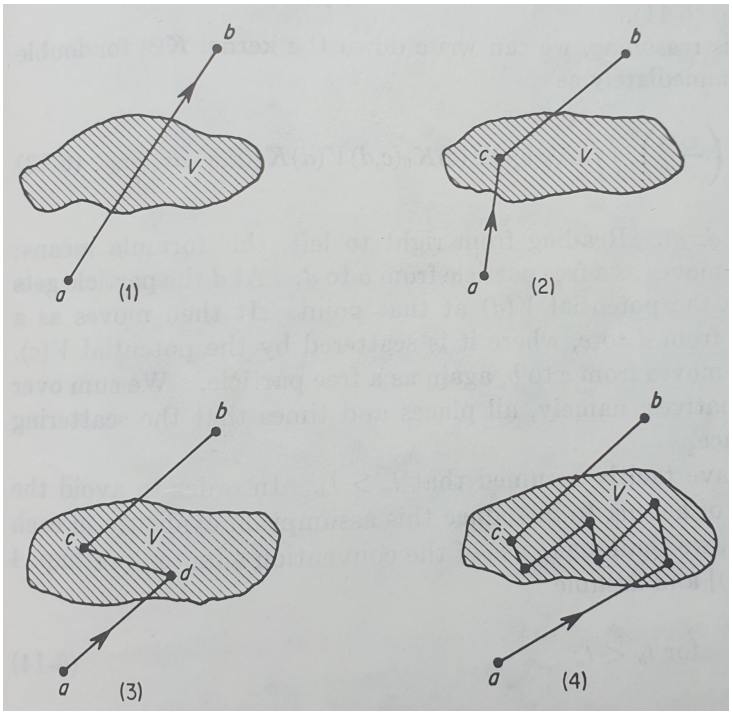


Figure 4: A graphical illustration of the scattering interpretation of the quantum mechanical path integral perturbation series in two spatial dimensions. Picture (1) represents $K_0(A,B)$, the unscattered free particle scenario. This role is played for $K_1(A,B)$ by picture (2), which represents the particle scattering and one point in space and time from its otherwise free particle paths before and after. Similarly, (3) displays $K_2(A,B)$ and (4) displays $K_6(A,B)$. One only need to count the number of scatterings to arrive at this conclusion. The potential may here be drawn for, e.g., the space-time region where it is strongest, but it can in principle be nonzero over all of space and time (Feynman and Hibbs, 1965, p. 123).

As an example of the generalized interpretation provided above, let us briefly consider how $K_2(A,B)$ is indeed of a mathematical form consistent with what we would expect from this interpretation. Following the above line of thought, we expect to find the integrand of $K_2(A,B)$ to look like $K_{fp}(x_A,t_A;x',t')V(x',t')K_{fp}(x',t';x",t")V(x",t")K_{fp}(x",t";x_B,t_B)$, where $(x",t")$ is the space-time coordinate where a second scattering event happens due to the potential $V(x",t")$ acting on the particle.

The $n=2$ term of (92) yields:

$$K_2(A,B) = -\frac{1}{\hbar^2}\int_{t_A}^{t_B}\left(\int_{t'}^{t_B}\left(\int_A^B V(x(t'),t')V(x(t"),t")e^{\frac{im}{2\hbar}\int_{t_A}^{t_B}\dot{x}^2(t)dt}\mathscr{D}x(t)\right)dt"\right)dt' \tag{104}$$

Here, the integration over paths and integration over times has already been exchanged, as before. Of course, when directly looking at (92), we see a term $\left(\int_{t_A}^{t_B}V(x(t),t)dt\right)^2$, but to use the integration variable exchange trick we get rid of the square and just consider the product

of the two integrals. Again, for the sake of clarity, we label one t' and the other $t"$.

A more substantial, but also the final significant, point to consider are the integration bounds on the time integral over dt' (from t_A to t_B) and those over $dt"$ (from t' to t_B). We assume that the 'second scattering' happens after the 'first scattering', such that we establish $t" > t'$, a dependency relation between the integrals' bounds. If we do not introduce this condition, we have double counting. The point is that two scatterings must happen. If we consider all the times t' at which a first one can possibly happen and given each of these, all the possible later times $t"$ for a second one, we have exhausted all physical options for two scattering events to happen. Adding on top of that situations where $t" < t'$ would lead to more scatterings than that, i.e., we would make a bookkeeping error leading to double counting.
However, while this makes perfect sense in our interpretive scheme, the term we find in the original Taylor expansion is simply $\left(\int_{t_A}^{t_B} V(x(t),t)dt\right)^2$. Since we made a product out of this and, crucially, changed the integration bounds of one of the two resulting integrals, the result will be different from this original squared one in the perturbation series. Luckily, we can infer what this difference is by noting the following:

$$\int_{t_A}^{t_B}\int_{t_A}^{t_B}(...)dt"dt' = \int_{t_A}^{t_B}\int_{t_A}^{t'}(...)dt"dt' + \int_{t_A}^{t_B}\int_{t'}^{t_B}(...)dt"dt' = \int_{t_A}^{t_B}\int_{t"}^{t_B}(...)dt'dt" + \int_{t_A}^{t_B}\int_{t'}^{t_B}(...)dt"dt' \tag{105}$$

The first equality follows from general rules of integrals. The second equality is found when for the first of the two integrals on the left-hand side (with respect to the second equality) the order of integration is switched: the integral over $dt"$ is now performed before the one over dt'. Just as when doing a double integral over the area of a right-handed triangle, there is a dependency relation between the bounds, and this makes it so that we mathematically retain the same quantity if we change the boundary values as above.
Note that the integrand, in accordance with the $n = 2$ term of the perturbation series (92), is just $V(x(t"),t")V(x(t'),t')$ in all cases. The arising symmetry then, makes it so that if we were to interchange t' and $t"$ in the first double integral of the final rightest-hand side above, it would just be equal to the second one. Concretely, we get:

$$\int_{t_A}^{t_B}\int_{t_A}^{t_B} V(x(t"),t")V(x(t'),t')dt"dt' = 2\int_{t_A}^{t_B}\int_{t'}^{t_B} V(x(t"),t")V(x(t'),t')dt"dt' \tag{106}$$

And therefore, the second order term of the perturbation theory (104) contains an extra factor of 2 due to these integration bounds. The reason that we do not see it in (104), is that this 2 cancels against the factor $\frac{1}{2}$ that is introduced by the factor $\frac{1}{n!}$ in (92) for the case $n = 2$. In fact, for each order n, the term $\frac{1}{n!}$ is cancelled due to the emergence of a term $n!$ from repeating the above logic of the integration bounds for higher orders. With this, we have fully accounted for (104). To those familiar with QFT, the above reasoning was likely recognizable, as identical reasoning appears when dealing with perturbative methods in QFT.

At this point, we are as good as done. The above expression (104) of $K_2(A,B)$, by the exact same argument that made use of the discrete lattice regularization form employed for

$K_1(A,B)$, will simply allow us to do the integrals for all dx_k except those corresponding to x' and $x"$ in the lattice. It follows that:

$$K_2(A,B) = -\frac{1}{\hbar^2}\int_{t'}^{t_B}\int_{t_A}^{t_B} F(t',t")\,dt'\,dt" \tag{107}$$

with

$$F(t',t") = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{fp}(x_A,t_A;x',t')V(x',t')K_{fp}(x',t';x",t")V(x",t")K_{fp}(x",t";x_B,t_B)dx' \tag{108}$$

And thus, what we have shown is in line with what was expected from the interpretation provided by Feynman. The same goes for all higher orders. With that, we have solved our quest as to how terms in the perturbation series can both be calculated and interpreted, the latter paving the way for a quantum field theoretical approach.

This, finally, completes our discussion on the path integral formulation of quantum mechanics. We have gained a broad understanding of the fundamentals of the formulation, shown the equivalence with the more familiar formulation of Schrödinger and described how calculations can be done with the path integral. The latter has been demonstrated directly for the case of the free particle, which could subsequently be used when analyzing how terms in the crucially important perturbation series of a path integral with a Lagrangian including a small potential can be handled.
The establishment of these facts, and the techniques discovered along the way, provide us with the ability to understand the development of the description of fermionic systems in quantum mechanics through the path integral. This does, however, not yet translate directly to the QFT-context. While the next subchapter will not be a full repetition of the foregoing, we will discuss how the quantum mechanical path integral carries over to quantum field theory. In fact, the foregoing makes it so that this endeavour ends up being *relatively* straightforward, as has, e.g., already been foreshadowed when discussing the scattering interpretation of the perturbation terms above.

2.4 The path integral in quantum field theory

While QM had been well-established by the time Feynman wrote his thesis and published his 1948 article, QFT and, especially, the Standard Model were still in full development. The path integral formulation has played an instrumental role in this, and chapter 5 will directly show important examples of this.
In this subchapter, we will therefore introduce the path integral formulation of QFT. First, the fundamentals will be laid out and it will be laid out how things change in comparison to the foregoing subchapters. As we will see, assuming some basic QFT knowledge, nothing here will be all too surprising. Then, in analogy to the free particle case and to see how the QFT path integral can be used in practice, we will calculate it for the case of the free Klein-Gordon field. Finally, we turn to what perturbation theory now looks like.

In this subchapter, we will leave (Feynman and Hibbs, 1965) as our main guidepost behind. Their book contains a chapter of quantum electrodynamics, and some attention will be paid to this in chapter 5. The pair, however, does not formally treat QFT. This is not because they did not know how to do this, as the general setup was already around by the time. See for example the later to be covered (Matthews and Salam, 1955) in chapter 4. Therefore, we will here base us on a number of secondary sources as our goal here is to establish a working knowledge for later chapters, with (Peskin and Schroeder, 2019, §9.2). The next subchapter will comment some more on the historical state of affairs until the mid-50s.

On a physical level, the biggest change is that we are now dealing with relativistic fields and their spacetime configurations rather than nonrelativistic particles and their paths. This ontological shift will inform most that follows below. For now, we will just speak of real scalar fields, but generalizations are possible, some of which will be discussed down the line.
The familiar one-particle Lagrangian now becomes a field Lagrangian, such as the one corresponding to the classical Klein-Gordon field[13]. In fact, to treat space and time on equal footing, we exchange the Lagrangian for the Lagrangian density such that $L = \int \mathscr{L} d\vec{x}$, with $d\vec{x}$ denoting that we integrate over all three spatial dimensions. Moreover, this Lagrangian density is subject to more constraints than it was in the QM-case before, viz., we want them to contain only local interactions (e.g., $\phi^4(x)$, rather than $\phi(x)^2\phi(y^2)$) and to correspond to renormalizable theories (e.g., not all powers $\phi^n(x)$ are allowed).
At the same time, the path integral is now taken over the (functional) space of all field configurations, rather than all paths. The term 'path' integral thus becomes somewhat redundant, and many textbooks switch to the more general term 'functional integral' instead (Peskin and Schroeder, 2019, p. 282). Other than naming convention, a final side-point is that one may note that this new integration measure still lends itself to a somewhat concrete interpretation, albeit more abstract than particle paths. 'We must sum over all paths the particle can take' just becomes 'we must sum over all configurations the field can be in'. We will see later in this thesis whether this view holds up.
The above implies the following main changes:

$$\vec{x}(t) \rightarrow \phi(t,\vec{x}), \qquad \vec{p}(t) \rightarrow \pi(t,\vec{x}) \equiv \frac{\partial \mathscr{L}}{\partial(\partial_t \phi(t,\vec{x}))} \tag{109}$$

Thus, rather than position and conjugate momentum, we now have the 'position' field and its conjugate 'momentum' field. Because a field has not just one, but an infinite number of degrees of freedom, it is also a function of three-dimensional space. Due to special relativity being built into QFT, the fields are therefore often also written as a function of spacetime four-vectors $\phi(x^\mu)$, or even abbreviated as $\phi(x)$.
It straightforwardly follows that the QFT version of (3) now becomes

$$K(A,B) = \int_A^B e^{\frac{i}{\hbar}\int_{t_A}^{t_B}\int \mathscr{L}(\partial_\mu\phi,\phi)d\vec{x}dt}\mathscr{D}\phi(t,\vec{x}) \tag{110}$$

[13] One may at this point already start to wonder how the Dirac field, which is generally not understood to have a classical counterpart, can then be worked with in the path integral formalism. This observation is in fact fundamental for the central problem covered in this thesis, and it will get extensive coverage later.

and similarly, the phase space representation (72) of the functional integral is now

$$K(A,B) = \int_A^B \int_A^B e^{\frac{i}{\hbar}\int_{t_A}^{t_B}\int(\pi\dot{\phi}-\mathscr{H}(\phi,\vec{\nabla},\pi))d\vec{x}dt}\mathscr{D}\phi(t,\vec{x})\mathscr{D}\pi(t,\vec{x}) \tag{111}$$

where $\mathscr{H}$ is the Hamiltonian density, i.e., $H = \int \mathscr{H} d\vec{x}$.
First, a few smaller comments are in order. For one, note that initial and final state A and B now no longer refer to points in time and space for a particle, but to an initial and final field configuration. For example, $A = (t_A, \vec{x}_A) \rightarrow \phi_A(t_A, \vec{x})$. If one wants to phrase the connection to the QM path integral as it was discussed in a very explicit manner, one could say that each of the (now not one but infinite number of) degrees of freedom of the field has a final 'position'. Secondly, $\mathscr{L}$ and $\mathscr{H}$ are predictably related through the QFT version of the Legendre transformation and, thirdly, the integral over $d\vec{x}$ in the exponent is meant to be taken over all space.

Given the above kernels, we may wonder whether there is, again, a way to express these functional integrals as a product of 'normal' ones so that we can directly calculate. Considering the coordinate space representation for a moment[14], we wish to find the analogue of the kernel resulting from discrete lattice regularization (7) for (110). The relativistic character of the treatment might already give a hint as to what must be done: we must 'slice space' in addition to time. Figure 5 below provides a visualization of this process, similar in function to figure 2.

[14]The Lagrangian method is preferred over the Hamiltonian one in QFT anyway, due to its manifest Lorentz invariance. While the Hamiltonian certainly has its use for the particle interpretation of fields in the canonical quantization approach, the preferred treatment of time in the definition of $\pi(t,\vec{x})$ above makes it less suitable for these relativistic requirements.

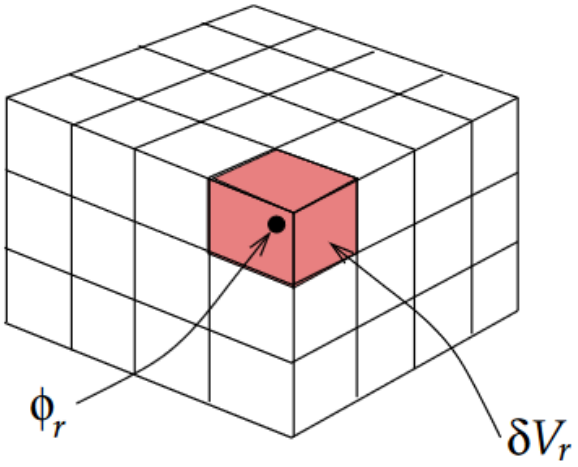


Figure 5: Three-dimensional space discretization can be visualized as above (Rischke, 2021, p. 79). We introduce M small boxes, or cells, with infinitesimal volumes δ (in the picture δV). The total volume is then $V = M\delta$, where at the end of the calculation we take $V, M \to \infty$ and $\delta \to 0$. Before the limit is taken, any real scalar field over space thus has M degrees of freedom, the field value at the r-th box being $\phi_r \equiv \phi(\vec{x}_r)$.

Let us consider a field at a specific snapshot in time, so that we only need to concern ourselves with its spatial component, i.e, $\phi(\vec{x})$. How are we now to perform an integral over all configurations of this field at this time? The answer starts by imposing a lattice structure on space, discretizing it into small cube-shaped cells with volume δS in the notation of figure 5. Let us now also suppose that the total volume of space considered is finite, denoting it with V. In that case, we are left with $V/\delta = M$ cells. As a consequence, the field ϕ now has M degrees of freedom. This is because these cells are the smallest unit of space where the field can take some value. Thus, we can now substitute $\phi(\vec{x}) \to \phi(\vec{x}_r) \equiv \phi_r, r \in \{1, \ldots, M\}$ with the latter being a subset of natural numbers. If the entire volume of space V is a cube, one could also have written $\phi(\vec{x}_r) = \phi(x_i, y_j, z_k)$ with $i, j, k \in \{1, \ldots, \sqrt[3]{M}\}$. To go back to the actual continuous field $\phi(\vec{r})$, we at the end of the calculation take the limits $\delta \to 0$ and $V \to \infty$, i.e., that of non-discretized space and infinite space, respectively. While these limits do not follow from each other individually, $M \to \infty$ follows from any one of them.

Heuristically, we may again look back at figure 2. There, after each time interval $\epsilon = \frac{t_B - t_A}{N}$, a straight-line path could be taken to anywhere on the one-dimensional spatial line x. In our new framework, we can think of this picture happening M times, i.e., at each spatial cell. The only difference is that we are now not dealing with paths. It is therefore not that after each subsequent ϵ we consider every possible element of the one-dimensional position line, but rather, that after each subsequent ϵ and at each spatial cell $\vec{x}_r$, we consider every possible value of the amplitude the field can take. This 'consideration' means, of course,

integration from $-\infty$ to $+\infty$. Given that we currently limited our discussion to real scalar fields, the underlying set is the same. Therefore, we just find that given a particular spatial cell, each time-increment introduces a new integral $\int_{-\infty}^{\infty}$ in the eventual $N-1$ integral product of them, the difference being an integral over dx_k before while over $d\phi(t_k, \vec{x}_r) \equiv d\phi_{k,r}$ currently ($k \in \{1, \dots, N-1\}$). Summarized, $\epsilon = \frac{t_B - t_A}{N}$ is to time-slicing as $\delta = \frac{V}{M}$ is to 'space-slicing'.
As one may suspect from the similarity between the QM and QFT discrete lattice regularization method, the relativistic symmetry of space and time also allows one to discretize *spacetime* as a whole. One would then consider spacetime volumes rather than talking separately about spatial cells and time increments. This methodology differs per source (Peskin and Schroeder, 2019, p. 285)(Rischke, 2021, p. 79). While this latter approach may be more proper with regard to the spirit of relativity, we opted for the more traditional one for didactic reasons.

Simply applying all the above lessons to (7) yields, simplified:

$$K(A,B) = \lim_{\epsilon,\delta \to 0} \prod_{k=1}^{N-1} \prod_{r=1}^{M} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\epsilon\delta \sum_{j=0}^{N-1} \sum_{l=1}^{M} \mathscr{L}\left(\frac{\phi_{j+1,l}-\phi_{j,l}}{\epsilon}, \frac{\phi_{j,l+1}-\phi_{j,l}}{\delta}, \phi_{j,l}\right)} d\phi_{k,r} \tag{112}$$

The result was to be expected from our discussion. Basically, we just get M times more integrals, as we are more or less doing what we did in the QM case in M spatial cells now. The 'simplified' adjective before the expression, though, applies to three things. For one, this is the discretized kernel for one-dimensional space, as to simply compare it to (7) and avoid needlessly cumbersome arguments for the Lagrangian density. That also means that the above relation implicitly redefines δ as a small spatial line segment rather than a volume. Secondly, as was the case in QM, one can argue about the regularization of the discretized argument $\phi_{j,l}$, in the sense that taking the average between two consecutive times or line segments could at this point just as well have been done. We will not repeat this discussion here though, and have just opted for the simplest option. Thirdly, we have left out the analogues of the constants $C^{-1}(\epsilon)$ that were so prominent in the quantum mechanical setting. The reason for this is that we will not need these and thus will not derive them later either, which means that they would just clutter the expression. We do not need them because, as we shall see in a moment, the practice of QFT will make sure that they always divide away.

The final point, however, brings us to the topic of the equivalence with the QFT-formulation students tend to be introduced to: the one following from canonical quantization, with its operator-valued fields, creation and annihilation operators and commutation relations. It was the proof of this equivalence that allowed us to identify $C(\epsilon)$, after all. The reason for not repeating the proof in the QFT case is that it is virtually identical (Rischke, 2021, p. 78-82), and would just take up space. Again, we would start from transition amplitudes carrying the field from one time to a later one $\langle \phi_B | e^{-\frac{i}{\hbar}\hat{H}(t_B - t_A)} | \phi_A \rangle$. We would then slice up the time and insert completeness relations $\int |\phi_{k,r}\rangle\langle\phi_{k,r}| d\phi_{k,r} = \hat{I}$ and its momentum field $|\pi_{k,r}\rangle$ counterpart. We would now just have more integrals, as this process happens for each one of M

spatial cells. Again, the relation (53) also has a field theoretical analogue and at that point we would be able to detect the Lagrangian density of the Klein-Gordon field in the exponent, the latter which will be discussed more later. Thus, we end up (111), and the functional integral formulation has been derived from canonical quantization. Needless to say, it can also be done the other way around[15].

We want this different formulation to again be able to calculate everything the other one can. In the context of QFT, it is therefore important to be able to deal with correlation functions, as these are instrumental to be able to derive scattering amplitudes and decay rates. Moreover, they will feature regularly in later chapters. Relating the expression for the n-point correlation function to the functional integral yields the following:

$$\langle 0|T(\hat{\phi}(x_1^\mu)\cdots\hat{\phi}(x_n^\mu))|0\rangle = \lim_{t_\pm\to\pm\infty} \frac{\int \phi(x_1^\mu)\cdots\phi(x_n^\mu)e^{\frac{i}{\hbar}S[\phi(x^\mu)]}\mathscr{D}\phi(x^\mu)}{\int e^{\frac{i}{\hbar}S[\phi(x^\mu)]}\mathscr{D}\phi(x^\mu)} \tag{113}$$

where here four-vector notation for the argument has been used for brevity. Note that this expression justifies our ignoring of the constants in our expression for the discrete lattice regularization: these will be the same for the numerator and denominator and therefore be divided out. Moreover, the S-matrix elements we can compute using this expression make use of asymptotically free states, so we now have the limit whereby we integrate over all times in addition to all space. This topic will briefly return in the next subchapter.
The object on the left should be recognizable to any student familiar with introductory QFT: the correlation function through which we can detect our Feynman propagators. On the right however, we find it expressed in the language of functional integrals. For the same reason as in the previous paragraph, we will not derive it explicitly, but refer to a source and briefly comment on it (Peskin and Schroeder, 2019, p. 283-284). The proof again uses completeness relations, but also another before-seen trick. Just like how we would sometimes have the path integral go to an intermediary point $B = (t_B, x_B)$ from A, before going to a final point C, this proof also splits up the functional integral so that intermediate field configurations are involved. These are the specific field configurations found on the left-hand side of (113). Using the eigenvalue equation in reverse, $\phi(\vec{x})|\phi\rangle = \hat{\phi}(\vec{x})|\phi\rangle$, and these Schrödinger picture operators become Heisenberg picture operators when squished between the operators $e^{\pm\frac{i}{\hbar}\hat{H}t}$, + left and - right. In this way, one can imagine how the fields $\phi(x_i^\mu)$ on the right-hand side above may be turned into the operators as seen on the left-hand side. The denominator then arises at the end as a normalization factor resulting from the partial overlap between the ground state and arbitrary field states, i.e., it acts as a normalization factor. In this rough way, the n-point correlation function can be expressed in terms of functional integrals.

One more useful property of the functional integral to be mentioned is its invariance to shift of the integration variables. A simple example would be to add some real-valued scalar function $F(x^\mu)$ to our real field, so that $\phi(x^\mu) \to \phi'(x^\mu) = \phi(x^\mu) + F(x^\mu)$. This will then change

[15] Dyson famously proved the equivalence of these formulations. For this, see (Schweber, 1994, 527-551).

$\mathcal{D}\phi \rightarrow \mathcal{D}\phi'$. Yet, if we think of the functional integral in terms of a product of regular integrals through discrete lattice regularization, we realize that this just shifts each variable $d\phi_{k,r}$ a bit, which is itself integrated from $-\infty$ to $+\infty$. The Jacobian is then equal to 1, and the (functional) integration measure does not change. More heuristically: if we are going to sum over all possible field configurations anyway, such shifts do not affect the functional integral, as 'everybody gets their turn anyway'.
This property is used plenty when derivations are performed with the functional integral (Peskin and Schroeder, 2019, p. 291,295). Its use will become more apparent in chapters 4 and 5.

At this point, it is finally time to introduce a Lagrangian density. One reason for this 'taking a while' is that in QFT, things are not so simple as in QM. In the latter, we could just focus on the one-particle Lagrangian, with a kinetic term minus an unspecified and free to choose potential. While in QFT not all 'potentials' are allowed, even the kinetic term depends on the type of field under consideration, e.g., the Klein-Gordon versus the Dirac field. To get a grip on the path integral, however, we will start with the easiest of all: the free real Klein-Gordon field. This field is associated with spinless particles, and once the corresponding interaction terms are added to its corresponding Klein-Gordon field, it describes real particles such as the pion or the Higgs particle. It is also a familiar choice to the student with basis QFT knowledge, as almost any source about it starts with it. A last point before proceeding is that now that we are moving to actual calculations, we will following the convention typical in the field and use natural units $\hbar = c = \mu_0 = \epsilon_0 = 1$. So far, SI-units have been used to keep the physics familiar and in plain sight, but in QFT, 'familiar' is often natural units, and the authors to be covered in chapter 4 and 5 almost exclusively use them.
The Lagrangian density of one free real Klein-Gordon field $\phi(x^\mu)$ can be written as

$$\mathscr{L}_{KG} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{114}$$

where if one applies the Euler-Lagrange equation

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathscr{L}}{\partial\phi} \tag{115}$$

this yields

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{116}$$

where the 'mostly minus' convention for the metric is used so that the contravariant derivative is $\partial^\mu = (\partial_t, -\vec{\nabla})$, and $\partial_\mu\partial^\mu = \partial_t^2 - \vec{\nabla}^2$.
Using the Legendre transformation

$$\mathscr{H}(\phi, \vec{\nabla}\phi, \pi) = \pi\dot{\phi} - \mathscr{L}(\phi, \partial_\mu\phi) \tag{117}$$

with the definition of $\pi(x^\mu)$ mentioned at the beginning of the subchapter, it is easy to show that the corresponding Hamiltonian density of the field is

$$\mathscr{H}_{KG} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{118}$$

Let us now substitute the Hamiltonian density in the phase space kernel (111), so that we find

$$K(A,B) = \int_A^B \int_A^B e^{i\int_{t_A}^{t_B}\int(\pi\dot{\phi}-\frac{1}{2}\pi^2-\frac{1}{2}(\vec{\nabla}\phi)^2-\frac{1}{2}m^2\phi^2)d\vec{x}dt}\mathscr{D}\phi\mathscr{D}\pi \tag{119}$$

Earlier, we have used (27) and (57) to deal with Gaussian integrals. The 'finishing touch' in this trilogy will be to add a final constant c:

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}-c} \tag{120}$$

Having set this up, a small first 'application' can be shown in the following way. We note that the phase space kernel for the Klein-Gordon Hamiltonian density is exactly of this Gaussian form in π. If we take $a = \frac{1}{2}$, $b = -\dot{\phi}$ and $c = \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$, and we ignore the constant as before, we will actually find that the above Gaussian integral is going to be proportional to the exponential of $\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$, which is exactly $\mathscr{L}_{KG}$. One might reasonably protest that the Gaussian standard integral is not defined for functional integration, yet it is being applied to the functional integral over $\mathscr{D}\pi$ above right now. Yet, discrete lattice regularization does not change the structure of the exponent, and we merely get a product of Gaussians resulting from momentum field integrals that will ultimately yield an exponential of the Lagrangian density in the limit. Therefore, by extension, (120) still holds in this case, and we have proven that the coordinate space representation of the functional integral follows from the phase space one for this Hamiltonian density.

More interesting, though, will be to directly calculate

$$K(A,B) = \int_A^B e^{i\int_{t_A}^{t_B}\int\left(\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2\right)d\vec{x}dt}\mathscr{D}\phi \tag{121}$$

This is the QFT-equivalent for the calculation of the free particle kernel, at least for 0-spin bosons. Working it out will be important for three reasons. For one, it is a direct demonstration of how a functional integral can be calculated in QFT. Secondly, its result will be important to contrast against a core development in the later story of this thesis. Finally, free field calculations provide the basis for perturbation theory, which means that by doing this we can unlock the full power of the widely relied upon perturbative methods in QFT for an important class of fields.

The derivation of the kernel of the free real Klein-Gordon field is usually done in a quite different way than we saw in its quantum mechanical counterpart in subchapter 2.3. We shall get to these much more efficient methods in a moment, but to connect these subchapters first show that it is in principle possible to simply use the same method to perform the derivation.
Consider (112) (and thus the case of one spatial dimension, since there are already plenty of terms and generalization is straightforward). Let us for a moment forget about the limit and pick out just the integral over $d\phi_{2,2}$ (corresponding to $k = r = 2$). With regards to the sum in

the exponent, we now just need the terms containing all occurring $\phi_{2,2}$. Therefore, we consider the $j = 1,2$ and $l = 1,2$ terms. As can be seen, the same step-by-step method is being set up as in the case of the quantum mechanical free particle. We will then find the following object:

$$\int_{-\infty}^{\infty} e^{i\epsilon\delta Q(\phi_{j,l})}\, d\phi_{2,2} \tag{122}$$

where

$$Q(\phi_{j,l}) = \frac{1}{2}\left(\frac{\phi_{2,2}-\phi_{1,2}}{\epsilon}\right)^2 + \frac{1}{2}\left(\frac{\phi_{3,2}-\phi_{2,2}}{\epsilon}\right)^2 - \frac{1}{2}\left(\frac{\phi_{2,2}-\phi_{2,1}}{\delta}\right)^2 - \frac{1}{2}\left(\frac{\phi_{2,3}-\phi_{2,2}}{\delta}\right)^2 - \frac{1}{2}m^2\phi_{2,2}^2 \tag{123}$$

Here, we have already thrown away mass terms such as $-\frac{1}{2}m^2\phi_{1,2}^2$, since we are integrating only over $d\phi_{2,2}$ for now. When working out all of the brackets, these will also produce terms such as $\frac{\phi_{1,2}^2}{2\epsilon^2}$ that will likewise be discarded. Rearranging in the suggestive Gaussian form, we find that

$$Q(\phi_{j,l}) = -\left(\frac{1}{\epsilon^2} - \frac{1}{\delta^2} - \frac{m^2}{2}\right)\phi_{2,2}^2 - \left(\frac{\phi_{1,2}+\phi_{3,2}}{\epsilon^2} + \frac{\phi_{2,1}+\phi_{2,3}}{\delta^2}\right)\phi_{2,2} \tag{124}$$

Substituting this into our integral yields

$$\int_{-\infty}^{\infty} e^{-i\left(\frac{\delta}{\epsilon}-\frac{\epsilon}{\delta}-\frac{\epsilon\delta m^2}{2}\right)\phi_{2,2}^2 - i\left(\frac{\delta(\phi_{1,2}+\phi_{3,2})}{\epsilon}+\frac{\epsilon(\phi_{2,1}+\phi_{2,3})}{\delta}\right)\phi_{2,2}}\, d\phi_{2,2} \tag{125}$$

At this point, we can invoke (57) and solve the integral:

$$\sqrt{\frac{i\pi}{\frac{\delta}{\epsilon}-\frac{\epsilon}{\delta}-\frac{\epsilon\delta m^2}{2}}}\, e^{\frac{i\left(\frac{\delta(\phi_{1,2}+\phi_{3,2})}{\epsilon}+\frac{\epsilon(\phi_{2,1}+\phi_{2,3})}{\delta}\right)^2}{\frac{4\delta}{\epsilon}-\frac{4\epsilon}{\delta}-2\epsilon\delta m^2}} \tag{126}$$

Note that π in this expression is the familiar constant, not the momentum field.
Clearly, the above is not quite as pretty as the analogous result that was found in the quantum mechanical case. But at least the proof of principle is here. Structurally, the two problems are the same, and we can see that there answers are subsequently similar as well in a to be expected way (Peskin and Schroeder, 2019, p. 286).

Luckily, however, there are ways to find compute the functional integral of the free Klein-Gordon field that are much simpler then the messy expression above[16]. We introduce one

[16]At this point an elephant in the room can be addressed. In general, direct computations as above are never done, and physicists opt fro the use of the 'generating functional', in scalar field theory defined as $Z[J] \equiv \int e^{i\int(\mathscr{L}+J(x^\mu)\phi(x^\mu))d^4x}\mathscr{D}\phi(x^\mu)$, with $\int d^4x$ being shorthand for $\iint d\vec{x}dt$ (nevermind $\int d\vec{x}$ itself already being shorthand). Here, a source term is added to the Lagrangian density, and one can use functional derivatives of the generating functional to derive correlation functions far more easily (Peskin and Schroeder, 2019, 289-292). This will be discussed some more in chapter 4. However, as important and interesting as this technique is, we strictly speaking do not need it here, as moreover this thesis is not a course in path integrals and the line needs to be drawn somewhere.

important one below, as it will be of importance in chapter 4 and 5.
Consider again the action

$$S = \int \left(\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right) d^4 x \tag{127}$$

We can use integration by parts by noting the product rule $\partial_\mu(\phi(\partial^\mu\phi)) = (\partial_\mu\phi)(\partial^\mu\phi) + \phi(\partial_\mu\partial^\mu\phi)$ so that we get instead

$$S = \frac{1}{2} \int (-\phi \partial_\mu \partial^\mu \phi - m^2 \phi^2) d^4 x + \frac{1}{2} \int \partial_\mu (\phi(\partial^\mu \phi)) d^4 x \tag{128}$$

Here, the second term is zero due to being a surface term, since it can through the divergence theorem be turned into an integral over the boundary of our infinite spacetime volume. For the first term, we can do a final bit of rearranging. As the squared mass is a scalar, we can write $\phi m \phi$, so that the integral in its entirety can be rewritten was

$$S = \frac{1}{2} \int \phi(-\Box - m^2)\phi d^4 x \tag{129}$$

with d'Alembert operator $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$.
We will now switch gears somewhat and exchange the functional integral we would have with this action for a very closely analogous set of integrals. This switch is written below:

$$\int e^{\frac{i}{2}\int \phi(-\Box - m^2)\phi \; d^4x} \mathscr{D}\phi \rightarrow \prod_{l=1}^{n} \left(\int du_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} u_i B_{ij} u_j} \tag{130}$$

This might seem like a strange thing move. Yet, it will make a lot of sense once we reach chapter 4 and, moreover, one can see that the two are not so different. The right-hand side is simply an easier way of writing the left-hand side after discrete lattice regularization, but it is all we need for now. The most radical departure is that the aforementioned operator has now been replaced by an nxn symmetric matrix B. This matrix operator is the analogue of the differential operator above. Solving the above product of integrals will therefore clearly tell us something about the free Klein-Gordon functional integral we have been attempting to address.

Our first step will be to diagonalize the matrix B. Given that it is Hermitian, the spectral theorem tells us that this is possible through unitary transformations. This process will also entail substituting $u_i = \sum_{i=1}^{n} U_{ik} v_k$ (and $u_j = \sum_{j=1}^{n} U_{jk} v_k$). Upon substitution, we will then find

$$\prod_{l=1}^{n} \left(\int du_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} u_i B_{ij} u_j} = \prod_{l=1}^{n} \left(\int dv_l \right) e^{-\sum_{k=1}^{n} \lambda_k v_k^2} \tag{131}$$

Here, the λ_k are then the eigenvalues of the matrix B.
While the above is general linear algebra that will seem familiar to anyone with a background in QM, this may have been a bit fast compared to earlier derivations. Firstly, we ought to

conserve some space for the many important things to come, but more practically, a similar derivation for fermionic fields will take place at the end of chapter 4. Due to its immense importance within this thesis, the choice was made to be as complete and precise as possible there, and not repeat the same thing here. For now, we will therefore do with this and look at the result.

Since each element of the sum can now be nicely paired to an element of the product, we can absorb the role of the sum into the product and rewrite the above to

$$\prod_{l=1}^{n}\left(\int dv_l\right)e^{-\sum_{k=1}^{n}\lambda_k v_k 2}=\prod_{l=1}^{n}\left(\int e^{\lambda_l v_l^2}dv_l\right) \tag{132}$$

These are integrals that we can handle. In fact, since we are still working on 'kernel' of the free real Klein-Gordon field, we could have expected such Gaussian integrals to come up again. Using (27) one more time, we find

$$\prod_{l=1}^{n}\left(\int e^{\lambda_l v_l^2}dv_l\right)=\prod_{l=1}^{n}\sqrt{\frac{\pi}{\lambda_l}} \tag{133}$$

We note that the denominator inside of the root contains the determinant of the matrix B, since that is equal to the product of its eigenvalues. Thus, we conclude:

$$\prod_{l=1}^{n}\left(\int du_l\right)e^{-\sum_{i=1}^{n}\sum_{j=1}^{n}u_i B_{ij}u_j}=\sqrt{\pi}(\det B)^{-\frac{1}{2}} \tag{134}$$

If desired, one can get rid of the constant $\sqrt{\pi}$ by inserting π in front of integral in the exponent on the left-hand side.
It is sometimes convenient (see chapter 5) to write this determinant as an exponential with the exponent being the trace of the logarithm of the matrix (Peskin and Schroeder, 2019, p. 304):

$$\det B=\prod_{l=1}^{n}\lambda_l=e^{\sum_{l=1}^{n}\log(\lambda_l)}=e^{Tr(\log(B))} \tag{135}$$

As mentioned before, our original functional integral of the free real Klein-Gordon field was precisely of this form, the difference being that the matrix B is the differential operator $(-\Box-m^2)$, and so we may write

$$\int e^{\frac{i}{2}\int\phi(-\Box-m^2)\phi\ d^4x}\mathscr{D}\phi\propto\ (\det(\Box+m^2))^{-\frac{1}{2}} \tag{136}$$

We provide a proportionality, neglecting the constant, because it would in practice be divided away anyway due to (113).
The above object is a 'functional determinant'. We will see it again in chapter 5. It can be written in terms of Feynman diagrams, e.g., through using the trace-format above (Peskin and Schroeder, 2019, p. 304-305).

Given that we have now addressed the functional integral of the free real Klein-Gordon field, the road is now open for treating perturbation theory. Let us extend the Klein-Gordon Lagrangian with the most straightforward interaction term that everybody knows: the ϕ^4-theory. This is not just a toy model, as this interaction prominently features in the Lagrangian density for the Higgs field. We write:

$$\mathscr{L} = \mathscr{L}_{free} + \mathscr{L}_{int} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{137}$$

Here $\lambda \in \Re$ is a coupling constant.
Like in the quantum mechanical case, let us consider the QFT-kernel and start by splitting the exponential containing the free Lagrangian density and the interacting one:

$$K(A,B) = \int_A^B e^{i\int_{t_A}^{t_B}\int \mathscr{L}_{free} d\vec{x}dt} \cdot e^{-i\int_{t_A}^{t_B}\int \frac{\lambda}{4!}\phi^4 d\vec{x}dt} \mathscr{D}\phi \tag{138}$$

Here, we write the somewhat longer $\int_{t_A}^{t_B}\int d\vec{x}dt$ rather than $\int d^4x$ because the time boundaries are relevant for the perturbation series.
Technically, we are already done now. We could now note that given the lack of field derivatives in the interaction term, $i\int \mathscr{L}_{int} d\vec{x} = -iH_{int}$. But that would result into the exact exponential that provides the basis for the $\hat{S}$-matrix, and that we have thereby shown that one can proceed with perturbation theory just as is done in the canonical quantization procedure. We will venture a bit further though, as seeing how the interpretation changes in the functional integral formulation from QM to QFT is important and interesting.

Our next step is to again expand the second exponential. Our 'smallness condition' now translates into

$$\left| \int_{t_A}^{t_B} \int \frac{\lambda}{4!}\phi^4 d\vec{x}dt \right| \ll 1 \tag{139}$$

where we should remember that the second integral is over all space.
Although technically we expand the entire integral, and we thus want its contributions to get progressively smaller with each order so as to be able to apply perturbative methods, some books (Peskin and Schroeder, 2019, p. 289) limit the condition to the coupling constant $\lambda \ll 1$. Moreover, now that we are working in natural units, the object must be sufficiently smaller than 1, rather than $\hbar$ as we saw in the quantum mechanical case. In SI-units, however, $\hbar$ would still play this role. Lastly, we may note that λ is a dimensionless coupling constant. The ϕ^4-interaction is a so-called 'marginal interaction', i.e., it is equally important at all energy scales. The above condition thus holds universally for this interaction, independent thereof[17]. Case in point, $\lambda \approx 1/8$ in the Lagrangian density of the Higgs field, so that higher powers rapidly make contributions be of lower orders of magnitude.

[17] Let $[Q]$ denote the mass-dimension of Q. Since the action $S = \int \mathscr{L} d^4x$ is dimensionless, and $[dx] = 1$, it follows that $[\mathscr{L}] = 4$. Given the $m^2\phi^2$ term in the Lagrangian density, we can infer that $[\phi] = 1$. This means that given a term $\lambda_n \phi^n$ in the Lagrangian density, $[\lambda_n] = 4 - n$. As can be seen, ϕ^4-theory thus leads to a

Upon expanding, we find

$$K(A,B) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_A^B \left(e^{i\int_{t_A}^{t_B} \int \mathscr{L}_{free} d\vec{x} dt} \left(\int_{t_A}^{t_B} \int \frac{\lambda}{4!} \phi^4(t', \vec{x}) d\vec{x} dt' \right)^n \right) \mathscr{D}\phi \tag{140}$$

with again

$$K(A,B) = \sum_{n=0}^{\infty} K_n(A,B) \tag{141}$$

Let us briefly assess the situation. Again, we want to find the kernel of a functional integral between an initial and final state, although we are now looking at a situation where a small potential is also present. In the quantum mechanical case, the order of the above sum corresponded to the number of scatterings a particle would undergo on the path from A to B. How does this picture hold up in QFT?

Below will treat the zeroth and first order, as these already provide us with the insight we need, and higher orders quickly lead to messy expressions. Derivations will not be done in full, as there are faster ways of doing these[18]. More on the full derivation of propagators from the functional integral can be found in (Peskin and Schroeder, 2019, p. 284-289).
Now first, consider the zeroth order:

$$K_0(A,B) = \int_A^B e^{i\int_{t_A}^{t_B} \int \mathscr{L}_{free} d\vec{x} dt} \mathscr{D}\phi = K_{ff}(A,B) \tag{142}$$

Here, $K_{ff}(A,B)$ stands for 'free field', as the analogue of the 'free particle' $K_{fp}(A,B)$ in the QM-case. Just as then, we know the expression for it, viz., a functional determinant.

The first order is already more interesting. We have

$$K_1(A,B) = -i \int_A^B \left(e^{i\int_{t_A}^{t_B} \int \mathscr{L}_{free} d\vec{x} dt} \left(\int_{t_A}^{t_B} \int \frac{\lambda}{4!} \phi^4(t', \vec{x}) d\vec{x} dt' \right) \right) \mathscr{D}\phi \tag{143}$$

To get the interpretation right, we cannot just look at the $K_n(A,B)$ terms though. As is typical of QFT, we need to be looking at correlation functions (113). Let us consider the two-point

dimensionless coupling constant. Yet, for $n > 4$, $[\lambda_n] < 0$. If we then have a process at energy scale E, where we get dimensionless combinations $\lambda_n E^{n-4}$, we note that we have positive powers of E in the $n > 4$ case. That means perturbation theory only works at *low* energies, since then the interaction term is small. If $n < 4$, we find that $\lambda_n E^{n-4}$ combinations will lead to negative power energies, meaning that at low energies, the interaction term is high. Thus, in that case, perturbation theory only works at *high* energies.

[18] It was previously mentioned how working with the generating functional is the generally preferred and more convenient method of deriving correlation functions through the functional integral formulation. This also holds for terms in the perturbation series. Other than this, one can work with discrete lattice regularization directly and moreover expand the discretized field into a Fourier series. This, however would take up a lot of space, and now that the basics of the path integral from the foregoing chapters are clear enough, and since we will not actually need much of this derivation later, we will instead refer to literature where this is clearly described.

correlation function below:

$$\langle 0|T(\hat{\phi}(x_1^\mu)\hat{\phi}(x_2^\mu))|0\rangle = \lim_{t_\pm \to \pm\infty} \frac{\int \phi(x_1^\mu)\phi(x_2^\mu) e^{i(S_{ff}+S_{int})}\mathscr{D}\phi(x^\mu)}{\int e^{i(S_{ff}+S_{int})}\mathscr{D}\phi(x^\mu)} \tag{144}$$

Since we are currently working on the first-order expansion of our perturbative series, we expand the exponential of the interacting part of the action and keep only the $n = 1$ term, realizing that we have the $n = 0$ term if we just plug in 1 for the brackets in the right-hand side of the expression below. We obtain

$$\lim_{t_\pm \to \pm\infty} \frac{\int \phi(x_1^\mu)\phi(x_2^\mu) e^{i(S_{ff}+S_{int})}\mathscr{D}\phi(x^\mu)}{\int e^{i(S_{ff}+S_{int})}\mathscr{D}\phi(x^\mu)} = \lim_{t_\pm \to \pm\infty} \frac{\int \phi_1\phi_2 \left(-i\int_{t_-}^{t_+}\int \frac{\lambda}{4!}\phi^4 d\vec{x}dt'\right) e^{i\int_{t_-}^{t_+}\int \mathscr{L}_{ff} d\vec{x}dt}\mathscr{D}\phi}{\int \left(-i\int_{t_-}^{t_+}\int \frac{\lambda}{4!}\phi^4 d\vec{x}dt'\right) e^{i\int_{t_-}^{t_+}\int \mathscr{L}_{ff} d\vec{x}dt}\mathscr{D}\phi} \tag{145}$$

where on the right-hand side, we have defined $\phi_1 \equiv \phi(x_1^\mu)$.
Due to the asymptotically free initial and final states that can be described through free field states, limits have been added. That means that the integral over space and time are now over all times and all space, respectively. Therefore, we might as well write one integral over the spacetime volume d^4x. Remembering this, we drop the limit for now, as we take it to be implicit in this notation for here. Lastly, like in QM, we switch the integral over spacetime with the functional integral over all field configurations. We find:

$$\lim_{t_\pm \to \pm\infty} \frac{\int \phi_1\phi_2 \left(-i\int_{t_-}^{t_+}\int \frac{\lambda}{4!}\phi^4 d\vec{x}dt'\right) e^{i\int_{t_-}^{t_+}\int \mathscr{L}_{ff} d\vec{x}dt}\mathscr{D}\phi}{\int \left(-i\int_{t_-}^{t_+}\int \frac{\lambda}{4!}\phi^4 d\vec{x}dt'\right) e^{i\int_{t_-}^{t_+}\int \mathscr{L}_{ff} d\vec{x}dt}\mathscr{D}\phi} = \frac{\int \left(-i\frac{\lambda}{4!}\int \phi_1\phi_2\phi^4 e^{i\int \mathscr{L}_{ff} d^4x}\mathscr{D}\phi\right) d^4x}{\int \left(-i\frac{\lambda}{4!}\int \phi^4 e^{i\int \mathscr{L}_{ff} d^4x}\mathscr{D}\phi\right) d^4x} \tag{146}$$

The next move is to make ϕ_1 and ϕ_2 part of the discrete lattice regularization. These will play the role of the potential in the quantum mechanical case, where this time a particular $d\phi_{k,r}$ integral will remain. Eventually, we will find that we can identify[19]

$$\frac{\int \phi_1\phi_2 e^{i\int \mathscr{L}_{ff} d^4x}\mathscr{D}\phi}{\int e^{i\int \mathscr{L}_{ff} d^4x}\mathscr{D}\phi} = D_F(x_1 - x_2) \tag{147}$$

Here, we abbreviated $x_1^\mu \equiv x_1$.
In other words, the zeroth order correlation function is just the Feynman propagator. As is well known, this can diagrammatically be drawn as a line between two external points. In the spirit of Feynman's diagrammatic particle processes interpretation: we have our 'unscattered particle'.

[19] Since we are not working in the operator formalism, we do not directly use Wick's theorem by considering all possible contractions, thus giving us Feynman propagators. Rather, in the functional integral formulation, a procedure analogous to this is used based on the properties of Gaussian integrals to systematically handle products of fields. This method results in the same combinatorial factors that Wick's theorem would provide in the operator formalism, and we can interpret the resulting terms as diagrams.

In the $n = 1$ case, we then find a number of these propagator terms. In the end, we get

$$\frac{\int \left(-i\frac{\lambda}{4!}\int \phi_1\phi_2\phi^4 e^{i\int \mathscr{L}_{ff}d^4x}\mathscr{D}\phi\right)d^4x}{\int \left(-i\frac{\lambda}{4!}\int \phi^4 e^{i\int \mathscr{L}_{ff}d^4x}\mathscr{D}\phi\right)d^4x} = \tag{148}$$

$$-\frac{i\lambda}{8}D_F(x_1-x_2)\int D_F^2(x-x)d^4x - \frac{i\lambda}{2}\int D_F(x_1-x)D_F(x-x)D_F(x_2-x)d^4x \tag{149}$$

The above terms can again be interpreted like a superposition of scattering events, but now in QFT, where much more is possible. After all, particles are not conserved and can partake in all kinds of (but limited) interactions that were not part of the quantum mechanical interpretation. This is the biggest difference between the work in subchapter 2.3 and the current situation.
The first term of our answer represents again a free particle propagation and, elsewhere, at some point and time, a vacuum bubble. After all, the latter is integrated over all time and space. The second term can be interpreted as a particle traveling from x_1 to x, then we have a closed loop interaction at x, after which the particle freely keeps travelling to x_2. Again, the integral over the spacetime volume d^4x tells us that we are dealing with a superposition such that this could happen at any place, any time. In this way, the perturbation series that can be derived from the functional integral forms the foundation of and rules for the famous Feynman diagrams. Treating quantum field theories is made enormously easier by just drawing all diagrams for a certain order and summing these, turning them into mathematical expressions for scattering amplitudes through the Feynman rules. In this way Feynman's path integral, and especially the diagrams he originally derived from this formulation, brought QFT 'to the masses'.

Now that we have a good view of what the basic changes for the functional integral formulation between quantum mechanics and quantum field theory entails, we are ready to proceed on our journey. In the next and final subsection of this chapter, we take stock of where we are.

2.5 The pros and (fermionic) cons of the path integral

In the foregoing subchapters, we have built a basic understanding of the functional integral formulation in quantum theory as it was around the mid-50s. We saw that it provides an alternative to the operator formulation resulting from canonical quantization. While zooming in on the content, however, some overview-context has been left undiscussed. We may wonder: why do physicists in some contexts opt to use one over the other? What are the unique advantages the functional integral brought to the table? Does its use also have drawbacks? And, most importantly for our purposes, what challenges did the development of the at that point still *"not readily accepted"* (C., 2011, p. 13-14) formulation still have to overcome?
Starting with its strengths, and moving on to its drawbacks, it is these questions that we will address in this final subchapter. While these lists will surely not cover every single positive or

negative thing one can say about the functional integral formulation, all the most prominent talking points on both sides will be touched upon. Crucially, we will end by laying out the initial inability of the functional integral to deal with fermionic fields. The process of solving this problem will inform most of the remainder of this thesis.

The functional integral has numerous advantages that eventually made it grow into the preferred formalism to do QFT in for many. Some of these we have already briefly passed by, while others may require some more explanation. Below, we list five of these most prominent advantages:

- *Built-in Lorentz covariance.* Because it works with the Lagrangian (density) as its main object, the functional integral provides a relativistically covariant framework. The operator formalism lacks this due to its insistence on the Hamiltonian, which in its definitions gives a preferred role to time over space. Since QFT, but not QM, has special relativity baked in, this advantage is prominent in QFt and one reason for its extensive use there especially (Peskin and Schroeder, 2019, p. 79).

- *Convenient generalizability to different quantum field theories.* The operator formalism starting from canonical quantization can look radically different depending on the field and Lagrangian under consideration. For example, quantization of the electromagnetic field is well-known to be extremely difficult in this formulation. The functional integral makes this comparatively easy to do, as we will see in chapter 5. In general, it is easy to apply it to all kinds of different quantum field theories. Especially in the case of non-Abelian gauge theories, the functional integral is a popular tool as it makes this process far more straightforward (Peskin and Schroeder, 2019, p. 79, 275).

- *Easy to work with (especially in QFT).* The functional integral is in many ways far less mathematically-demanding than the operator formalism, in particular within the confines of QFT. There are a number of examples for this. For one, working with regular commuting numbers and classical Lagrangians is simpler than an operator algebra. Renormalization is also less troublesome to formulate. Moreover, ordering ambiguities of quantities in classical expressions that otherwise have to be quantized do not appear in this way. Plenty of other examples could be given, with one famous historical case study being the relative ease with which Feynman calculated the Lamb shift on Bethe's request. He was able to do this much faster than others working with more traditional methods (Schweber, 1994, p. 206-247). Nevertheless, as will be mentioned below, some nuance and counterexamples can also be presented, especially concerning quantum *mechanics*.

- *Unified framework for quantum and statistical physics.* In the first subchapter, it was briefly mentioned that by switching to imaginary time, the functional integral resembles the partition function of statistical physics. In brief, we can derive statistical physics and do work in the field using much of the methodology here developed, again adding to its overall power. We therefore find a case of methodological unification between

two different fields of physics. As interesting as this is, we will not pursue this connection here further. The interested reader is referred to (Feynman and Hibbs, 1965, §10), where Feynman and Hibbs contemplate about this relation extensively.

- *Intuitive interpretation.* While this final entry is more subjective than the others, it is often expressed that the functional integral, especially in the quantum mechanical case of a very visualizable path integral, is a more intuitive way of doing quantum physics. It is an often expressed sentiment that bachelor students first learning about quantum mechanics come away from it wondering where between all Hilbert space vectors, commutation relations between hermitian operators and unitary transformations, the physics is to be found[20]. In the path integral, there are no operators. One can still use their familiar classical Lagrangian with commuting variables. We now just have to deal with that rather than one path, a particle between two points will in fact take every possible path. While we here certainly find a difference with classical physics, it is conceptually often not found to be an insurmountable one. Moreover, it is nicely visualizable and the classical limit argument shows a clear connection with the classical world.
  As a critical side note, I personally do not fully agree with this picture, for a number of reasons. Some have already been laid out in the foregoing, such as a possible critique on the validity of the argument on the classical limit. But moreover, as we will later see later in this thesis and explicitly mention, the clear pictorial interpretation of the path integral breaks down the deeper we get into it. The upcoming integral over fermionic fields, supplied with a Lagrangian containing anticommuting numbers, contains a real challenge to this. Lastly, this view of the path integral arguably undersells the intuitiveness of the operator formalism. That is not to say canonical quantization is not a significant paradigm shift in our physical thinking, but that the rift between it and classical physics is not as monumental as it is often thought to be. This argument will come by in chapter 4.5, when discussing the relationship between the often overlooked Poisson brackets in classical Hamiltonian mechanics and the Heisenberg equation of motion.

With five advantages of the functional integral formulation being clear, we will be democratic and turn our attention to five challenges now. More so here, the historical context will be accentuated. This is because while all of these problems were evident by the mid-50s, some of them have by now been solved, or at least greatly diminished.

- *Obscure unitarity of the $\hat{S}$-matrix.* The $\hat{S}$-matrix is an essential quantity in QFT. It describes how an asymptotically free initial state evolves into a final state after a scattering process, i.e., $\langle f|\hat{S}|i\rangle$. Mathematically, its unitarity means that $\hat{S}\hat{S}^\dagger = \hat{I}$. Physically, it is this property that ensures that the sum of the probability of all possible scattering processes is equal to unity. In the functional integration we are not dealing with operators, and showing this is notoriously far more difficult (Weinberg, 1995, p. 377).

[20] For the reader in need of a brief break, this April fool's video of a physics Youtube channel lays out this sentiment in a humorous manner `https://www.youtube.com/watch?v=jm7jVi8akcc`.

Nevertheless, since the formulation is equivalent to the operator formulation starting from canonical quantization, the fact that unitarity is still preserved is not surprising. Therefore, the impact of this disadvantage has more to do with an important property being opaque, rather than shaking the functional integral's foundations or showing that it cannot do certain things.

- *Mathematically ill-defined.* The path integral was anything but a mathematically rigorous approach to quantum physics upon its introduction. While the operator formalism had been rooted in solid mathematical ground thanks to the work of mathematical physicists like John von Neumann (1903-1957), Feynman came up with the path integral mostly because it just seemed to work. Yet some problems are immediately apparent. The problem of Lagrangians featuring accelerations was already described before. More generally, the Lagrangian the QM path integral features the velocity, yet many of the paths considered in the discrete lattice regularization will have all kinds of discontinuities, and one may wonder if it is at all certain that of all functions that have to be integrated over, the set of ones with continuous derivatives in $x(t)$ is not of measure zero. Moreover, how does one choose the correct time-slicing procedure and, perhaps even more important, how does one know if the product of integrals even uniquely converges in the limit anyway? Actual mathematicians will be able to, and have, pointed out many more problems with Feynman's path integrals. While this thesis is not about delving into these finer details, let us briefly comment on some developments.
  One extremely influential one in the development of a mathematically rigorous account of functional integrals was Cécile DeWitt-Morette (1922-2017). In fact, she was the first person to publish on the path integral after its '48 introduction to the broader public by Feynman (Morette, 1951). Together with a fellow mathematician Pierre Émile Cartier (1932), she wrote an entire book on the topic setting out to give a robust account to functional integration, as they for example aim to

  > *"...do away with N-tuple [the lattice discretization] integrals and to identify the function spaces which serve as domains of integration for functional integrals."*
  > (Cartier and DeWitt-Morette, 2006, p. 6)

  More than half a century after her first paper on the topic, she wrote in a small book contribution on the problems and possible remedies for the functional integral that

  > *"from a heuristic tool, functional integration is gradually becoming a mathematical tool. Path integrals are by now a well-defined, robust tool. [...] A number of functional integrals in Quantum Field Theory are mathematically reliable."*
  > (Morette-DeWitt, 2009, p. 246)

Therefore, while the mathematical underpinnings of the functional integration formulation remain a dodgy topic, much work has been done on the issue. Although where exactly on puts the emphasis with regard to the justification of a certain physical formalism is also a matter of philosophy, we can say that insofar empirical progress is not the sole measure of things, real mathematical progress has also been made.

- *Difficult to work with (especially in QM).* While, as the previous list stated, it is generally agreed upon that the functional integral formulation is the more convenient way of doing QFT, this conclusion need not to be extended to QM. If the path integral was truly so much easier to interpret and to work with, one would wonder why it has not completely replaced canonical quantization procedures. The free particle calculation in subchapter 2.3 already served as a foil to this notion. While it was by no means impossible, it cannot compete with just solving a wave equation. It has also been mentioned before how it took over three decades after Feynman's introduction of the formalism to finally solve the hydrogen atom system (Duru and Kleinert, 1979), which was not without reason. As a final example, spin operators are also relatively easy to work with, the spin-1/2 system often being used as the easiest possible instance of a quantum state in a superposition. Yet, as the next bullet point will address, doing calculations on spin is highly nontrival with the path integral. To close of with a point of nuance, the above separation between convenience with regard to QFT and the lack thereof concerning QM is not a black or white matter, as one might very well think of some cases where the operator formalism is easier in QFT and vice versa. Yet, it is not hard to conjure up plenty of examples supporting this way of thinking about the application of the formulations in general.

- *Hiding essential concepts.* Some concepts in quantum physics seem, at first glance, to be absent in the functional integral formulation. Here, we will specifically consider the case of quantum states and spin operators. Clearly, 'concepts' might even be putting it mildly, as these can be considered elements of the very ontology we engage with from the moment we start to learn QM.
  Let us first consider the quantum state $|\psi\rangle$. In QM, the state plays an essential role in the formalism for representing systems, doing calculations, and even being a widely discussed objects in interpretation debates. In QFT, however, the focus is more so on processes such as scatterings and decays, even in the operator formalism of canonical quantization. The state is of course still there, for example in calculating vacuum expectation values $\langle\Omega|T(\phi_1\phi_2)|\Omega\rangle$ as we saw earlier or 'creating a particle at position $\vec{x}$' with the Schrödinger picture operator $\hat{\phi}(\vec{x})|0\rangle = \int \frac{d\vec{p}}{(2\pi)^3}\frac{e^{-\vec{p}\cdot\vec{x}}}{2\omega_{\vec{p}}}|\vec{p}\rangle$, but the importance of its role in the formalism is greatly diminished. This shift in the *"paradigmatic problem that was to be calculated from this [theoretical] basis"* has been extensively explained in (Blum, 2017, p. 1-3).
  Feynman had, through his path integral, initially attempted to get rid of the quantum state entirely in this '42 thesis. However, as we know from the Feynman diagram approach, he still later came to realize that he still required the idea of asymptotic free

states at $t_{\pm} \to \pm\infty$ (Blum, 2017, p. 54-60). Therefore, while the functional integral does not fully abolish the state, it is much harder to find it in comparison with the operator formalism, especially in QM.
We now turn our attention to spin operators. Feynman and Hibbs write in the conclusion of their book that:

> *"With regard to application to quantum mechanics, path integrals suffer most grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital parat of real quantum-mechanical systems. It is a serious limitation that the half-integer spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication."*
> (Feynman and Hibbs, 1965, p. 355)

The above quotation is a goldmine for this thesis, so a few comments are in order. First of all, this paragraph might lead one to mix up two related but distinct problems. One problem, the one we aim to discuss right now, is that spin operators such as the Pauli matrices cannot be represented in the language of path integrals. This is a problem, as these allow one to make verifiable, empirical predictions about experimental systems involving spin. A second problem is that of how the functional integral differentiates between bosonic particles/fields and fermionic particles/fields. Clearly, the difference must manifest itself somewhere. On this, we will come to speak in the next point.
Secondly, note that they mention quaternions as noncommuting objects having the potential to describe half-integer spin particles. That they were thinking along these lines is certainly something to keep in mind for later chapters.
Thirdly, and finally, we observe the authors' conviction that the path integral just cannot deal with spin operators, and is in this sense incomplete compared to the operator formalism. While they were certainly right that it is hard to find them in there, they technically turned out to be wrong on this. With a quote as explicit as the above, many people quickly rose to the challenge (Schulman, 1968). The modern account of how spin operators feature in the language of functional integrals, is that it can be derived from classical geometric considerations (Altland and Simons, 2010, p. 134-142). In conclusion, it is possible to represent them, but it might raise an eyebrow that such a typical quantum property was that 'hidden' away. For both the case of quantum states and spin operators one may arguably prefer a formulation wherein these fruitful concepts and, depending on one's perspective, essential elements of the theory's ontology, are not so opaque.

That concludes our discussion of the first four challenges for the path integral. As with some of the advantages, it is clear that any such list contains some degree of subjectivity, with per-

ceptions also changing over the years. There is, however, one of the five problems still missing, and it happens to be one that is nowadays generally understood to have been definitively solved. This fifth and final problem is essentially the subject of the remainder of this thesis, and a mere bullet point would therefore not do it justice. Its description is therefore the logical endpoint of this entire chapter.

Before formulating the problem clearly, we need a brief recap. In QFT, there are two types of fields. On the one hand, there are bosonic fields. Bosonic field operators obey canonical commutation relations $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ and the quanta of the field will collectively obey Bose-Einstein statistics. Through the spin-statistics theorem, one can show that these will always have integer spin. On the other hand, we have fermionic fields. These obey canonical *anti*commutation relations $\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$ and their quanta will collectively obey Fermi-Dirac statistics. This leads to the famous Pauli exclusion principle for fermions. In the case of fermionic fields, the spin-statistics theorem tell us that we are dealing with objects that have half-integer spins (Peskin and Schroeder, 2019, p. 52-58).
The Klein-Gordon Lagrangian density that we saw before, corresponds to spin-zero particles such as the Higgs, i.e., to a subset of bosons. Fermions, most famously the electron, however, are described by the Dirac Lagrangian density, an integral part of the QED-Lagrangian density.

Now the functional integral readily generalizes to systems of identical bosons. Yet we may now wonder, what about systems with half-integer spins? For bosonic fields in the operator formalism, we saw that we could quantize the classical action by imposing commutation relations on classically commuting fields. But that correctly presupposes that we have a classical action (Blum, 2017, p. 60-61). One is then left to wonder how we go about this for Dirac fields. While we can impose anticommutation relations on these fields, what classical action forms their analogue? The Dirac equation has no classical real scalar field, but it deals with spinors. Although the formulation of this problem is hard to detect in any explicit manner in the book of Feynman and Hibbs, When you read between the lines, it appears that they were aware of it, writing

> *"...for a relativistic particle with spin (described by the Dirac equation), the quantity [...] cannot be described by a simple path integral based on any reasonable action. However, it is possible to calculate by other means, for example, from the Dirac equation."* (Feynman and Hibbs, 1965, p. 264)

Thus, solutions from the Dirac equations are invoked as an external addition to the path integral formulation to get things to work. Clearly, important quantum behavior such as the Pauli principle as not yet derivable from the integral itself. This might leave one to wonder how Feynman, famous for getting QED to work through the path integral, was still able to do this. After all, QED contains fermionic Dirac fields that the path integral was thus ill-equipped to handle. Remembering the minus signs resulting from the permutation of two electrons in a many-body wave function, and the anticommutation of the corresponding fields, it is

to be expected that perturbative expansions of correlation functions involving Dirac fields will involve a sum with pluses as well as minuses. What Feynman did was to basically ad hoc see whether things would work if he were to run with pluses or minuses in certain cases. He therefore derived his QED rules through what was later sometimes called 'the intuitive method', rather than being able to do so from first principles (Blum, 2017, p. 62-63).

Yet, this state of affairs is unsatisfactory. It makes the functional integral formulation weaker than the operator formulation based on canonical quantization. Therefore, we need a way to be able to do functional integrals over fermionic fields. In other words, can we find a classical counterpart for the action of a quantum mechanical spin-1/2 system?

Since this problem was most prominent in QFT, it is developments on it on this front that we will focus on. Nevertheless, it is good to realize that the problem also exists for the QM path integral. In fact, there is a nice way of visualizing our central problem in terms of paths in there, published in a journal based on physics education (Styer et al., 2002, p. 290-291). Early in this chapter, it was explained how we can find a generalized kernel for many particles (11). This works for bosons, but some additional rules need to be imposed for fermions. For this, consider figure 6 below:

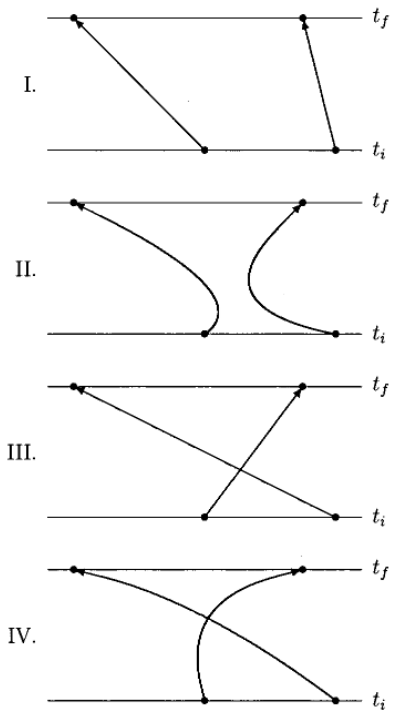


Figure 6: Four graphs showing different paths two identical fermions can take between an initial time t_i and a final time t_f. They provide a useful visualization for thinking about how Pauli's exclusion principle can be incorporated into the path integral formulation (Styer et al., 2002, p. 291).

Above, we find four different pairs of paths with which two identical fermions move from an initial time t_i (we usually wrote t_A) to a final time t_f (" t_B). The horizontal axis can be thought of as the one-dimensional x-axis. While in the first and second graph these paths do not intersect, this does happen in the latter two. The additional rule is now to insert a minus before amplitudes associated with interchanging paths. Thus, in the sum over paths, some contributions will now have an opposite sign as they would have had if we were dealing with bosons (or even distinguishable fermions such as in the case of a path integral for an electron *and* a muon). This procedure actually guarantees the Pauli principle. The paper explains why we can think of the rule in this way. First of all, by the Pauli principle we do not want two identical fermions to move to the exact same final state (x_f, t_f). The authors now ask us to, for example, slide the two particles at t_f toward each other to some middle x_M in picture I and III (in these two pictures, both have the same initial position). Given this limit, the amplitudes associated with I and III are identical. But this limit itself is forbidden, so only through a minus sign can we make sure the amplitudes of such a duo of path configurations cancel and we obey the Pauli principle (Styer et al., 2002, p. 290).
This minus sign insertion procedure, in fact, carries over to the Feynman rules, when adding diagrams that while different are indistinguishable. The question that remains is how the

path integral can itself incorporate this sign-bookkeeping, a troublesome task that presents itself in QFT too. We will eventually see how an algebra was devised that efficiently takes care of this.

In this chapter, we have thoroughly introduced the functional integral formulation. We started by going over the fundamentals in the case of quantum mechanics. After this, we showed how the formulation is equivalent to the more familiar operator methods resulting from canonical quantization. We moved on to show how the path integral can be used in practice, with specific attention to the free particle and its application in perturbation theory, the latter being found to have an elegant interpretation coming along. In the next step, we discussed how all of this carries over to quantum field theory, which can be done relatively straightforwardly but unlocks much more of the advantages of the now functional integral over fields. These advantages were then finally summarized in the final part, as well as the challenges and shortcomings of the formulation. We ended by laying out the functional integral's inability to deal with fermionic fields.

The remainder of this thesis will be occupied with showing how this problem was eventually solved during the latter half of the 1950s, and how this greatly strengthened the functional integral to definitively push it into the mainstream. Before we get to this, however, some more theoretical and historical context is required, albeit on first view quite different (and shorter) than that of this chapter. As it turned out that the anticommutative 'Graßmann algebra' was instrumental in fixing the functional integral's fermionic defficiency, it is this topic that we shall turn our attention to in the next chapter.

3 The History and Content of Graßmann Algebra

Graßmann numbers are the key to unlocking a path integral description for fermionic fields. However, since Hermann Günther Graßmann lived from 1809 until 1877, the mathematics named after him obviously has a history that goes back much further than their (re)appearance in addressing the problem at hand, which occupied people in mostly the 1950s. Therefore, in this chapter, first some historical background to the work of Graßmann is provided, based to a large degree on the book on the topic by historian of science Michael J. Crowe (1936). Later the focus turns more to the general workings of the algebra these Graßmann numbers are elements of. These investigations are not just interesting, but are relevant for providing context and knowledge contributing to a swifter understanding of the developments during the 1950s to be considered in the next chapter. For example, the mathematics will be recognizable, and we will also be able to explain why Graßmann's name did not immediately come up during this period.

3.1 A brief history of vector analysis

The study of vector quantities and their development into what most contemporary physics students are taught and generically understand them to be has, to a large degree, been a 19th century project. Numerous mathematicians and physicists contributed to the creation of distinct vector formalisms (Crowe, 1967). The competition between these has by some authors been dubbed the 'vector algebra war' (Chappell et al., 2015). Most of this subchapter will be dedicated to this 'warring period', in which Graßmann, while not necessarily the main character, played a role through the introduction of his own formalism.

Before getting to the 19th century though, it is important to understand that the vector concept did not appear out of nowhere, and that some of the formalisms alluded to directly build on earlier constructions. The earliest use of what we might consider 'vectorial quantities' dates back to ancient Greece, such as their use of the parallelogram law (reducing to the Pythagorean theorem in the rectangular case). Jumping all the way to the 17th century, we find an extensive appendix called *La Géométrie* in Descartes' famous *Discourse on the Method* (1637). In it, he introduces new ideas and methods combining algebra and geometry, rather than treating them independently [21]. Consider an operation like addition. Descartes notes that this operation need not be limited to the numbers of algebra, but can also apply to the length of lines and the areas of geometry. In fact, he argues that all basic arithmetic operations (addition, subtraction, multiplication, division, and root extraction) seen in basic algebra have a geometrical interpretation. We must realize that while these ideas were instrumental for the eventual development of the vector, we are here not yet dealing with the object in its modern sense. In modern terms, we can roughly say that *Descartes*'

[21] In reality, the history of mathematics is not a clear-cut, universally agreed upon story. While some authors indeed view the origin of the combination of algebra and geometry to lie in Descartes work (Chappell et al., 2015), others already consider some form of this to already be present in Ancient Greece (Blåsjö, 2016).

ideal was that the same operations used for scalars could also be used for vectors (Domski, 2022)(Crowe, 1967, p. 1-3).

While *La Géométrie* had its shortcomings, its vision was realized through the geometrical representation of complex numbers. These had themselves been around for centuries, through for example the work of Cardan. It was, however, only around the beginning of the 19th century that they, through the work of figures like Wessel, Argand and Gauß, grew to be represented as arrows in a plane with an imaginary and real axis (Crowe, 1967, p. 5-10). To see how this satisfies Descartes' aforementioned vision, consider the addition of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, resulting in $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$. Here, the '+' clearly acts both on the real numbers a_1, a_2, b_1 and b_2, but it is also applied to the (in this geometrical interpretation) 'vectorial' quantities z_1 and z_2 themselves. Geometrical addition is then expressed in terms of algebraic addition. Moreover, as may be expected, the result is identical to the parallelogram law for line segments. After all, $|z_1+z_2|^2 = (a_1+a_2)^2+(b_1+b_2)^2$, which is of course the Pythagorean theorem due to component-wise analysis of the complex numbers producing a rectangular form.

The success of complex numbers led to the ambition of finding a vector formalism applicable to three-dimensional space and suitable to all the sciences. This ambition kickstarts the vector algebra war that was to last for most of the remaining century.

A first major contender came in the form of the quaternionic vectors of William Rowan Hamilton (1805-1865) in 1843 (Crowe, 1967, p. 19-46). Being the same Hamilton we know from his formulation of classical mechanics and 'the Hamiltonian', he was already rather famous by the time he came up with them. His ideas were controversial, but broadly discussed, them gaining hardcore followers such as Peter Guthrie Tait[22] (1831-1901) as well as equally passionate detractors such as Oliver Heaviside[23] (1850-1925). In any case, the influence of Hamilton and his quaternions on the further development of vector analysis is direct and undeniable, so let us at least get a very slight idea of what it looked like.

Hamilton's idea was to generalize the complex numbers $z = a + bi$ through quaternions $q = a + bi + cj + dk$. This had the advantage of leaning on the solid mathematical work on the complex numbers and likewise adhering to Descartes' ideal. The $a + bi$ part can be viewed as an ordinary complex number, with the $cj+dk$ part as the extension. Consequently, $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$. One might then further wonder how i, j and k relate to each other. Firstly, these symbols mutually anticommute, e.g., $ij = -ji$. Secondly, multiplying two of them yields the remaining one, such that $ij = k$, $jk = i$ and $ki = j$. One may notice the same structure to it as the cross products of basis vectors, which as we will see is no co-

[22] In 1863, Tait wrote the following in the Proceedings of the Royal Society of Edinburgh: *"...the next grand extensions of mathematical physics will, in all likelihood, be furnished by quaternions."* (Crowe, 1967).

[23] In 1891, Heaviside wrote the following in 'The Electrician': *"But I came later to see that, so far as the vector analysis I required was concerned, the quanternion was not only not required, but was a positive evil of no inconsiderable magnitude..."* (Crowe, 1967).

incidence as quaternions inspired the development of this more familiar vector formalism. From these two rules, it follows that the product of the three of them will always produce positive or negative unity, e.g., $ikj = -ijk = -kk = -k^2 = 1$.
In the development of his quaternionic formalism, Hamilton invented words and notation still in use today. For example, the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ one comes across in calculus are derived from it. More importantly, he introduced the world 'scalar' and 'vector' to refer to the a and $bi + cj + dk$ part of a quaternion, respectively. Just as one may now write $Im(z) = b$ for a complex number, Hamilton wrote $Sq = a$ and $Vq = bi + cj + dk$ to refer to the scalar and vectorial component of any given quaternion. The Vq part was then meant to be used to encode points in three-dimensional space.
Using all of the above, consider the product of the vectorial part of two quaternions:

$$(Vq_1)(Vq_2) = (b_1 i + c_1 j + d_1 k)(b_2 i + c_2 j + d_2 k) = \tag{150}$$

$$-(b_1 b_2 + c_1 c_2 + d_1 d_2) + (c_1 d_2 - c_2 d_1) i + (b_2 d_1 - b_1 d_2) j + (b_1 c_2 - b_2 c_1) k \tag{151}$$

Now, one may notice that the above looks suspiciously like the sum of the dot and cross product of two 'familiar' vectors. This is no coincidence, as the quaternionic vector formalism influenced both William Kingdon Clifford's (1845-1879) geometric or Clifford algebra (1878) as well as Josiah Willard Gibbs' (1839-1903) and Heaviside's now mainstream vector calculus[24] (1881). Both of these directions took inspiration from Hamilton, but also wanted to solve some of the problems plaguing quaternions (Chappell et al., 2015, p. 1998-2003). The problem that Clifford (and to a lesser extent Gibbs and Heaviside too) tackled, as well as a brief description of his ideas, will be discussed later. The work of Gibbs and Heaviside did much to address issues with the efficiency of notation and applicability to physics. The latter was especially true for the up and coming theory of electromagnetism, with especially Heaviside doing much work to popularize vector calculus for this branch of physics. With success, as it emerged to mainstream. The anticommutativity of i, j and k in the quaternionic formalism was unfamiliar to many, as well as the necessity of inclusion of the scalar component a arguably being unnecessary. More importantly, in Hamilton's notation, the fact that $S(q_1 q_2)$ above has a negative sign is rather inconvenient. Suppose we represent the velocity of a particle with the following quaternion: $q_v = vi$, i.e., a particle traveling in one dimension. Then $q_v^2 = v^2 i^2 = -v^2$, and we are consequently are left with a negative kinetic energy. In other matters too, physics is troubled by the negative square of i, j and k (Chappell et al., 2015, 1998-1999).

The work that Gibbs and Heaviside did to right their wrongs with quaternions led to the vector calculus that fist year physics student are now taught all over the world. Gibbs introduced the familiar three-vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with $\hat{i}$, $\hat{j}$ and $\hat{k}$ being unit vectors. On quaternions, he wrote to fellow mathematician Victor Schlegel[25] (1843-1905) in 1888:

[24] The development of vector calculus by Gibbs and Heaviside was not a collaborative effort, but done mostly independently.

[25] More on him later!

""In regard to the product of [quaterionic] vectors, I saw that there were two important functions (or products) called the vector part & the scalar part of the product, but that the union of the two to form what was called the (whole) product did not advance the theory as an instrument of geometrical investigation... I therefore began to work out ab initio, ..., the three differential operations $\vec{\nabla}$ applied to a scalar, & the two operations to a vector,..."
(Crowe, 1967, p. 152)

Thus, one could say that Gibbs saw the product of two quaternions and felt that it was more useful to promote the scalar and vectorial part to two separate operations, viz., the dot product $\vec{r}_1 \cdot \vec{r}_2 (\cong -Sq_1 q_2)$ and the cross product $\vec{r}_1 \times \vec{r}_2 (\cong Vq_1 q_2)$. In addition, he made use of the familiar[26] $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$. In this formalism, there was no longer a negative square, as $\hat{i} \cdot \hat{i} = 1$. Moreover, Gibbs' vectors required no scalar part, and while the cross product anticommutes, the dot product restored familiar commutativity. In summary, the relatively easy to use notation and the aforementioned wide application in especially electromagnetism quickly led to the result whereby the vector algebra wars were definitively won by the vector calculus Gibbs and Heaviside (Crowe, 1967, p. 150-181).

Yet, even vector calculus has its shortcomings. For one, it is certainly not enough to describe all physical quantities. We also need Pauli matrices, Minkowski four-vectors, Weyl spinors, etc. While these are not necessarily difficult to use once one grasps the Gibbs-Heaviside three-vector, they make up many different mathematical objects nonetheless. This flies in the face of the ealier-stated ambition to find a vector formalism of universal applicability. A second point is that with regards to multiplication, vector calculus does depart from Descartes' ideal about (in contemporary terms) the same operations applied to scalars also being applicable to vectors. Not only have we exchanged multiplication for the dot and cross product, basic operations like division also do not apply to the Gibbs-Heaviside three-vectors. Lastly, the formalism does not distinguish between vectors that display different behavior. In particular, one can hardly tell whether one is dealing with a vector or pseudovector. As a brief refresher on this distinction, consider the following. When a particle in front of a mirror travels upwards, it also does so in its mirror reflection. The upward vector is invariant under this parity transformation. But now consider an electron rotating in clockwise fashion in front of a mirror. The magnetic field vector in the center of the electron's orbit points up. Yet, in the mirror image, the electron moves counterclockwise. Consequently, the magnetic field vector flips under this parity transformation, i.e., it now points downwards. While velocity and the magnetic field are represented by the same mathematical object, they behave differently under this transformation. We say that while velocity is a vector, the magnetic field is a *pseudovector* (or, equivalently, an *axial vector*). The Gibbs-Heaviside formalism not

[26] It should be mentioned that Tait, the hardened supporter of Hamilton's methods, had introduced the operator $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ as a part of quaternionic analysis. Maxwell even referred to him as the "Chief musician upon Nabla". Moreover, as Tait already often worked with just Sq or Vq, the way that he handled quaternions already started to look a little bit like modern vector calculus. This makes it easy to see how Gibbs and Heaviside could translate ∇ into their language, even though Tait was not happy with their vector calculus.

being able to account for this difference is, in fact, connected to what was stated earlier to be something addressed by Clifford. It is a symptom of a larger issue we shall return to in a later subsection, as it will be easier to explain by diving deeper into the content of formalisms (Chappell et al., 2015, p. 2001-2003).

The shortcomings of vector calculus may motivate one to still zoom in on the war somewhat further. While it is true that the largest 'front' in this war was that between quaternions and vector calculus, more 'factions' were involved. It turns out that already in 1844, just after Hamilton and his quaternions, another formalism was proposed that was praised by Gibbs as superior[27] to quaternions and was the most significant influence on Clifford's later efforts. While initially receiving little attention, the importance of Graßmann's exterior algebra (sometimes also just called 'Graßmann algebra') is, therefore, not to be underestimated. The next subsection will discuss the historical context of Graßmann's work, after which we will move on to a basic description of its content.

3.2 Graßmann and *die Ausdehnungslehre*

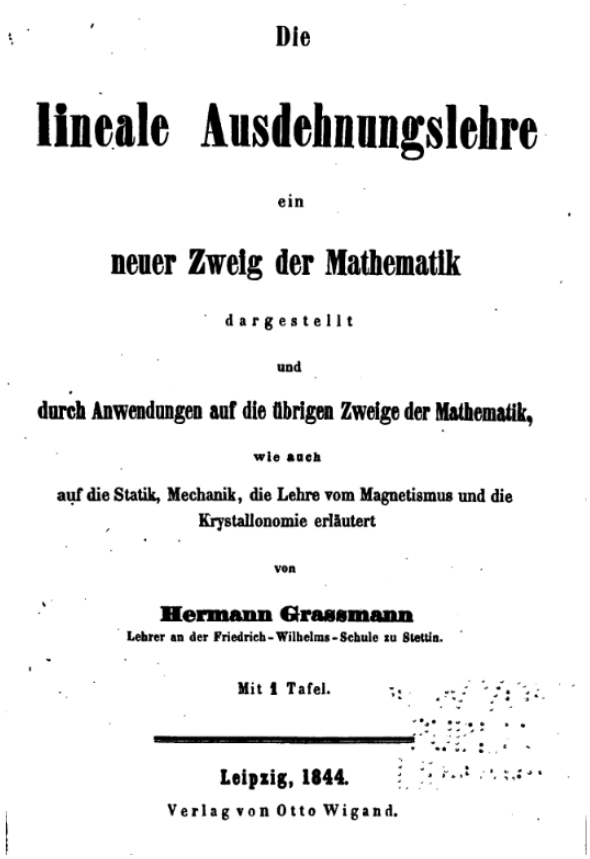
Graßmann was born in Stettin, Prussia (modern-day Poland) as the son of a father who was a moderately well-known theologian and math teacher at a local gymnasium. It would turn out that Graßmann (the son!) would take after his father. He first went to study theology in Berlin, after which he returned to Stettin and self-taught himself mathematics to also become a high school math teacher. In 1840, during the exams he needed to take to become one, he wrote a long essay on tidal physics. This was only published posthumously, but it already contained core ideas of his vector formalism. In fact, it contains the fundamentals of what would later become linear algebra, and it may have been the first body where these ideas were applied. With them, Graßmann was able to simplify many of efforts by Lagrange and Laplace on the topic (Crowe, 1967, p. 54-62).

Graßmann got the teaching position, and while being a teacher at the gymnasium in Stettin, published his mathematical magnum opus: the *Lineale Ausdehnungslehre*[28] (1944). The *Ausdehnungslehre* laid out Graßmann's vector formalism and contained the foundational concepts and ideas of what we now know as linear algebra. While the book is nowadays widely praised for being ahead of its time, introducing all kinds of innovations such as his formalism working for an arbitrary number of N dimensions, it was largely ignored at its time. Very few mathematicians read or where acquainted with it, bar some exceptions such

[27] While Gibbs did not use the work of Graßmann in the development of his vector calculus, he wrote the following in the same 1888 letter to Schlegel: *"I have no doubt that you consider, as I do, the methods of Graßmann to be superior to those of Hamilton. It thus seemed to me that it might be interesting to you to know how commencing with some knowledge of Hamilton's methods & influenced simply by a desire to obtain the simplest algebra for the expression of relations of geometric physics & calculus I was led essentially to Graßmann's algebra of vectors, independently of any influence from him or any one else."*

[28] In English: "The Linear Theory of Extension", which explains the denotation of his system as 'exterior algebra'.

as Möbius. It even got to the point that Graßmann's publisher let him know that 600 copies had been used as waste paper. Later editions did little to change this, and one may see why Graßmann eventually abandoned mathematics in favor of a (successful) career as a linguist (Crowe, 1967, p. 63-88).



Die

lineale Ausdehnungslehre

ein

neuer Zweig der Mathematik

dargestellt

und

durch Anwendungen auf die übrigen Zweige der Mathematik,

wie auch

auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert

von

Hermann Grassmann

Lehrer an der Friedrich-Wilhelms-Schule zu Stettin.

Mit 1 Tafel.

Leipzig, 1844.

Verlag von Otto Wigand.

Figure 7: A photocopy of the title page of the Ausdehnungslehre *(Graßmann, 1844).*

Given the paradoxical situation of the *Ausdehnungslehre* on the one hand here being claimed to be a highly innovative work of mathematics and on the other hand something that was mostly ignored, this invites the obvious question as to the reason for the latter. Crowe identifies two main reasons for this. Firstly, unlike figures like Hamilton, Graßmann was a high school teacher with no significant resume to speak of. One can imagine that this must not have put him at the top of everyone's reading list. But secondly, and perhaps even more importantly, Graßmann's mode of presentation in the *Ausdehnungslehre* has much to do with this. The book is notoriously obscure. It is highly abstract and contains few examples, which, especially in combination with its radically novel content and notation, does little to help accessibility. It was also very philosophical. The book contains much of Graßmann's philosophical thoughts on the subject matter[29]. This is of course anything but wrong per se,

[29] One interesting example of Graßmann's many philosophical analyses in his book is his separation of geom-

but by using this as a means of justifying his *lehre*, it could make it harder to resonate with the dispositions of many readers. Moreover, other than just *discussing* philosophical content, there was even an explicit philosophy to the very writing style of the book. Following the philosopher Friedrich Schleiermacher (1768-1834), Graßmann felt that his mathematical work had to be presented through a 'dialectic of contrast', meaning that the reader ought to be in a position where they can recognize always simultaneously the specific and the whole, the real and the formal, the discrete and continuous, etc. The uncommonness of this mode of presentation was yet another factor in the *Ausdehnunglehre's* scarcely received attention. It was only from the 1870s onwards that his work started gaining more recognition. One of his strongest supporters was the earlier mentioned Schlegel, also a colleague of Graßmann at the gymnasium. Schlegel published many papers using exterior algebra, and also wrote a biography on Graßmann that drew more attention to his ideas. Moreover, as was mentioned before, Clifford was also heavily inspired by Graßmann in his work (Crowe, 1967, p. 63-95). On his influence, mathematician Fearnley-Sander states the following:

> *"Even in those cases where forerunners may be discerned, his results, and especially his methods, were highly original. All mathematicians stand, as Newton said he did, on the shoulders of giants, but few have come closer than Hermann Graßmann to creating, single-handedly, a new subject."*
> (Fearnley-Sander, 1979, p. 816)

Yet even though the *Ausdehnungslehre* contained deep insights and later gained more recognition, Graßmann is not the 'household name' for fields of math like linear algebra that Newton and Leibniz are for calculus. Math education researcher Jean-Luc Dorier explains this through his description of Graßmann and the *Ausdehnungslehre* as an "isolated singularity" (Dorier, 1995). In Dorier's own words:

> *"In many ways, Graßmann's theory remains a singularity. Even if all its results correspond to modern concepts and theories, it contributed to the creation of very few of them. For the theory of vector spaces, it played an important role in the discovery of the axiomatic theory, but most of the concepts of this theory were reestablished independently of Graßmann's work."*(Dorier, 1995, p. 246)

Dorier goes on to explain that nevertheless, Graßmann's analyses still offer a valuable perspective on these topics with respect to our current understanding thereof.
The reason for the choice to cite Dorier's characterization of Graßmann's work as an isolated singularity here in full is not only because his own words are a clearer way of putting it

etry from the rest of mathematics, with his exterior algebra being the way to do geometry. In modern western epistemology, mathematics was usually classified as a strictly a priori science. This means that the truths of mathematics are not established through observations of reality, but rather through logical relations of ideas. A trivial example would be that the truth value of the sentence "a triangle has three corners" does not require any form of experience. Descartes, a well-known rationalist, also thought this way, and since he wanted to bring algebra and geometry together he certainly differs from Graßmann in this respect. Graßmann rejects the idea of geometry as an a priori science, as he thinks physical reality can at least partly be used to verify the theorems of geometry.

than mine would be, but also as a bit of foreshadowing for what is to come in the next chapter. Specifically, I will argue that the exact same thing happened regarding the use of 'Graßmann numbers' to address the problem of the path integral's inability to describe fermionic fields. The authors will be shown to independently rediscover an algebra isomorphic to that of Graßmann, and it is only later that it is identified to be as such.

Before we get to this though, we must first learn more about the exact contents of Graßmann's exterior algebra. The next subchapter will introduce this in its modern formulation, both for reasons for clarity as well as the fact that this formulation will make it easier to connect Graßmann algebra to the's fermionic Fock space discussed in the next chapter. This, however, leaves one with a final question related to this subchapter. The fact that a 'modern' formulation of Graßmann algebra exists, implies that the subject is nowadays introduced in a quite different manner than in the *Ausdehnungslehre*. The combination of significant progress in abstract algebraic methods over the last century, as well as the aforementioned obscurity of Graßmann's work, explains this rift. Yet, we may still wonder what his work concretely looked like. In the remainder of the current subchapter, we will therefore very briefly investigate two more things. Firstly, we will tip our to into the *Ausdehnungslehre* to 'see what the fuzz is all about'. More directly relevant for our story, we will fill in the explanatory gap between exterior algebra as introduced by Graßmann, and exterior algebra as can these days be found in mathematics textbooks (including in the work of the most well-known author in the next chapter).

In the *Ausdehnungslehre*, the objects of algebraic manipulation are called *extensive magnitudes*. These are abstract symbols that can stand for anything that can be assigned a numerical amount. We shall denote them below with a, b and c. Graßmann defines two types of *verknüpfungen* ('connections') between such magnitudes. The first is a *synthetic connection*, which he defines by means of the following two relations (Graßmann, 1844, p. 3-5):

$$a \frown b = b \frown a \tag{152}$$

$$(a \frown b) \frown c = a \frown (b \frown c) = a \frown b \frown c \tag{153}$$

Here, we recognize what we would now refer to as commutativity and associativity.
The second connection introduced by Graßmann is the *analytic connection*, defined through (Graßmann, 1844, p. 6-8):

$$(a \smile b) \frown b = a \tag{154}$$

$$a \smile b \smile c = a \smile c \smile b = a \smile (b \frown c) \tag{155}$$

$$a \smile (b \smile c) = a \smile b \frown c \tag{156}$$

While we need not do a deep dive into the *Ausdehnungslehre*, nor fully grasp the above formulae, it is interesting to see his notation and to note that Graßmann's description of how these magnitudes combine eventually leads him to a structure that we would now associate with the axioms of vector spaces. He is, in a different language, introducing concepts like

closure, distributivity, and the like. Later in the book, he also introduces the *wedge product* as a higher-dimensional generalization of multiplication, and this we will treat in the next subsection. Yet, looking inside the *Ausdehnungslehre*, one also sees far more philosophical elaboration than such formulae, seemingly supporting the earlier covered points on accessibility. In conclusion, while the spirit of his work is certainly still in there, the modern formulation of his exterior algebra looks quite different from what we find in his own book.

This, then, finally leaves us with the question where this modern formulation comes from. Geographically, the answer is mostly France. Especially important in the further development of exterior algebra[30] and its application to differential forms was the French mathematician Élie Joseph Cartan (1869-1951), who was aware of Graßmann's work (Bourbaki, 1974, p.663-664). An overview of the state of exterior algebra by the middle of the 20th century was written down in (Vivier, 1956). The many contributions of Cartan, as well as Graßmann and his soon to be discussed wedge product, are established from the offset (Vivier, 1956, p. 203-204). It is also this overview that is later directly cited as a "well-developed differential calculus on the Graßmann algebra" by a key author in the next chapter (Berezin, 1966, p. 49), who also did work on the translation of Graßmann's legacy into modern form himself. This, then, gives us some sense of the bridge between the work of Graßmann, undershadowed at its time, and the extensive use of it we shall see in the next chapters. Before continuing with introducing modern exterior algebra, we conclude this subsection with an in this regard perhaps fitting quote from Graßmann in the preface of the 1962 *Ausdehnungslehre*:

> *I remain completely confident that the labour I have expended on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another [referring to the gap between the first and second version] seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me (as I have until now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich them further, yet there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments. For truth is eternal and divine.*
> (Fearnley-Sander, 1979, p. 817, Translation of Graßmann.)

[30] In this regard, Henri Poincaré (1854-1912) and Giuseppe Peano (1848-1932) may also be mentioned (Bourbaki, 1974, p. 663-664).

3.3 The basics of exterior algebra

Let us start with two brief paragraphs on how the material below is to be covered.
This subchapter provides a basic understanding of Graßmann's exterior algebra. It is, as mentioned, based on contemporary economical secondary sources rather than the (in)famous *Ausdehnungslehre* itself. Most of the content below is derived from (Mac-Lane and Birkhoff, 1999, §16.5-16.10), with some visual help from (Chris, 2023). Our primary goal with exterior algebra here is to get a good grasp on its content insofar required for the problem of fermionic path integral description, and make the connection thereto as transparent as possible. We also go slightly beyond this and see what the underlying mathematical structure is about. This is best done with a modern formulation using tensor products that also show up in the next chapter. A logical place to start would be the definition of an exterior (or Graßmann) algebra. Yet, here another conundrum appears. On the one hand, a full formal treatment would take up some time and require definitions of definitions of definitions that are, like mathematical analysis for a calculus student who just wants to solve a simple integral, not a precondition for the goal at hand. On the other hand, leaping into applications without any idea of the structure one is standing on can also halt understanding. We will therefore go down the middle of the road, explaining necessary concepts albeit not with the rigor and precision an expert in the field would require.

With the previous paragraph in mind, let us proceed with a definition of exterior algebra (Mac-Lane and Birkhoff, 1999, Paraphrased from §16).

***Definition**: Let there be a tensor algebra over a vector space (as usual over a field F) $T(V)$. The **exterior algebra** $\Lambda(V)$ is then the quotient algebra of the tensor algebra by a particular ideal I, denoted $\Lambda(V) \equiv T(V)/I$. This ideal is two-sided and generated by all elements $v \otimes v \in T(V)$, with $v \in V$.*

The above contains plenty of mathematical jargon that might not be all that familiar to the average physics student. After this, a much more visual and application-focused picture will emerge, but let us briefly go through it. I assume familiarity with the definition of a vector space over a field. Let us start with the tensor product $v \otimes w$. This is an abstract operation we will not take apart further, but one can very bluntly think of it as a generalization of the multiplication of a column vector with a row vector, conserving properties such as distributivity. It is a mapping that takes a vector $v \in V$, $w \in W$, and outputs an element $v \otimes w \in V \otimes W$. Let us get a more concrete picture by example. Let us consider purely the vector space V of dimension 2, described by the orthonormal basis vectors $\hat{e}_1$ and $\hat{e}_2$. Now consider the tensor product of this vector space with itself, $V \otimes V$. The basis elements ('elementary tensors') in the tensor product of V with itself are $\hat{e}_1 \otimes \hat{e}_1$, $\hat{e}_1 \otimes \hat{e}_2$, $\hat{e}_2 \otimes \hat{e}_1$ and $\hat{e}_2 \otimes \hat{e}_2$. As usual, linear combinations can be formed between them. We can keep going and introduce $V \otimes V \otimes V$, containing 2^3 basis elements such as $\hat{e}_2 \otimes \hat{e}_1 \otimes \hat{e}_1$.
Now the tensor algebra $T(V)$ can be thought of as the direct sum of all these tensor products

of the vector space, i.e.,

$$T(V) = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus ... \tag{157}$$

The F can be viewed as the set of scalars.
An example of an element of the above tensor algebra would be $10 - 2\hat{e}_1 + 4\hat{e}_2 \otimes \hat{e}_1 - 3\hat{e}_1 \otimes \hat{e}_1 + 72\hat{e}_1 \otimes \hat{e}_2 \otimes \hat{e}_1 \otimes \hat{e}_1$, where I have taken one element F, one of V, two of $V \otimes V$ and one of $V \otimes V \otimes V \otimes V$.

From such a tensor algebra we can now derive the exterior algebra. For this, we still need to understand the idea of a quotient algebra, an ideal and the exterior product.
A *quotient algebra* can be thought of as a subalgebra of $T(V)$, incorporating only a subset of the elements of $T(V)$. The *ideal I* is then what we use to construct the quotient algebra, as it fills us in on what elements we are to remove from $T(V)$ to end up with the quotient algebra. It is a special subalgebra of $T(V)$ that always obeys *closure* and *absorption*. The former refers to the property that when applying the operation(s) associated with the tensor algebra to elements of the ideal, the resulting element will still be an element of the ideal. The latter refers to the property that when we operate on an element of the ideal with an element of the greater tensor algebra, the resulting element will still be an element of the ideal. If this happens regardless of whether we apply the element of $T(V)$ from the left or the right, the ideal is *two-sided*. A consequence of these properties is that we cannot 'accidentally' find elements of the ideal back in our quotient algebra after doing operations on elements of the later.

The above definition considered the ideal generated by all elements $v \otimes v \in T(V)$ with $v \in V$. For us, this means that our exterior algebra $\Lambda(V)$ is the quotient algebra of $T(V)$ from which we have removed all elements that are the tensor products of vectors with themselves, i.e., $v \otimes v$. This can be done by just setting $v \otimes v \equiv 0$.
Consider the effect of this on our previous example of the two-dimensional vector space V with the orthonormal basis vectors $\hat{e}_1$ and $\hat{e}_2$, and the tensor algebra we constructed from this. The scalars in F will be unaffected, and so too the vectors in V. However, when we remember that the elementary tensors of $V \otimes V$ were $\hat{e}_1 \otimes \hat{e}_1$, $\hat{e}_1 \otimes \hat{e}_2$, $\hat{e}_2 \otimes \hat{e}_1$ and $\hat{e}_2 \otimes \hat{e}_2$, we note that the first and last of these are not elements of the exterior algebra. Moreover, all elements of $V \otimes V \otimes V$ and those of higher orders go to zero. This is easy to see for elements such as $\hat{e}_2 \otimes \hat{e}_1 \otimes \hat{e}_1$, but not immediately obvious for other examples such as $\hat{e}_2 \otimes \hat{e}_1 \otimes \hat{e}_2$. This can, however, still be proven to be zero in the following way. Since $v \otimes v = 0$ for all $v \in V$, we can state the following:

$$(\hat{e}_1 + \hat{e}_2) \otimes (\hat{e}_1 + \hat{e}_2) = 0 \tag{158}$$

Due to the distributivity of the tensor product, we find:

$$\hat{e}_1 \otimes \hat{e}_1 + \hat{e}_1 \otimes \hat{e}_2 + \hat{e}_2 \otimes \hat{e}_1 + \hat{e}_2 \otimes \hat{e}_2 = 0 \tag{159}$$

Now note that again the first and last terms are already known to be zero. This, finally, results in:

$$\hat{e}_1 \otimes \hat{e}_2 = -\hat{e}_2 \otimes \hat{e}_1 \tag{160}$$

Therefore, due to imposing our condition $v \otimes v \equiv 0$, the tensor product is now anticommutative.

Because of that, we can say that $\hat{e}_2 \otimes \hat{e}_1 \otimes \hat{e}_2 = -\hat{e}_1 \otimes \hat{e}_2 \otimes \hat{e}_2 = 0$, and it follows that indeed all elements of $V \otimes V \otimes V$ and those of higher orders are zero.

For the case of the example above, we can thus conclude that our exterior algebra $\Lambda(V)$ has four basis elements: $\{1, \hat{e}_1, \hat{e}_2, \hat{e}_1 \otimes \hat{e}_2\}$, with the operation being anticommutative. Consistent with its roots in the tensor algebra, the elements of the exterior algebra are linear combinations of these basis elements. In just a moment, we will introduce a sensible vocabulary for these (basis) elements.
This then concludes our surface level detour into abstract algebra. We can now see that from the viewpoint of contemporary mathematics, exterior algebra does not come from nowhere and can be derived from existing concepts. Let us now fully switch back to operating as physicists and consider how the algebra can be employed.

First, note that by imposing the condition $v \otimes v \equiv 0$, we now have a tensor product subject to a restraint it is in general not restrained to. It is now also always anticommutative, which was not assumed for the general tensor algebra $T(V)$. Moreover, let us take a step back and realize that much of the above mathematical language was not around at the time of Graßmann, but what we did was derive an algebra that is isomorphic to his system. For these reasons, let us talk of a different operation and thereby introduce the key operation of exterior algebra: the *exterior* (or *wedge*, or sometimes, perhaps confusingly, also *outer*) *product*. The exterior product of two vectors v and w is denoted as $v \wedge w$. Luckily, it has a very clear geometrical interpretation. With that, we will be able to see how Graßmann's exterior algebra can be employed and visualized.

We will start simple with the two-dimensional case and slowly build up from there in complexity and dimensionality. For the sake of clarity, we will reintroduce the arrows for vectors. The exterior product of two vectors $\vec{v}$ and $\vec{w}$ is the oriented plane segment $\vec{v} \wedge \vec{w}$, called a *bivector*. The negative of this expression, i.e., $\vec{w} \wedge \vec{v}$ by the anticommutation rule, is the same plane segment with the opposite orientation. For a visual picture of this, see 8.

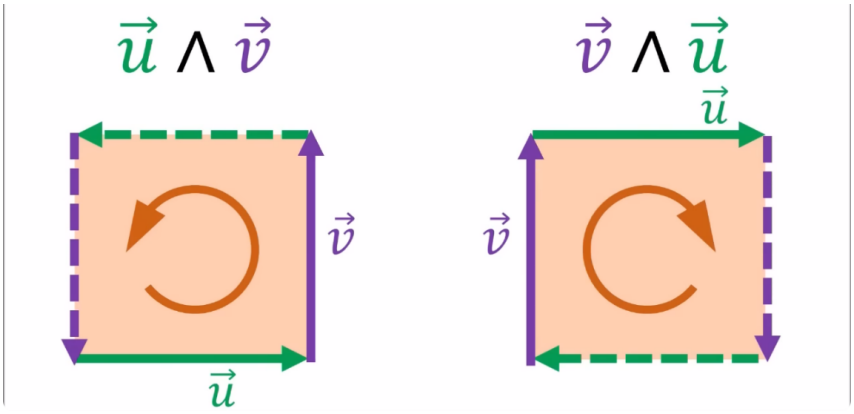


Figure 8: A simple visualization of the exterior product for vectors $\vec{u}$ and $\vec{v}$, resulting in two bivectors. Two possible orientations emerge that can be derived from following the direction of the first vector in the exterior product (Chris, 2023).

Bivectors are characterized by their magnitude (area) and their orientation[31]. The orientation of a bivector can be found simply by starting to walk around the area in the direction of the first vector in the exterior product. Algebraically, however, keeping track of the orientation is just a matter of keeping track of the order of the vectors in the exterior product (or, equivalently, whether or not there is a minus sign in front of it). Adding two bivectors can therefore add or subtract to the magnitude of the resulting bivector, depending on them having the same or different orientation respectively. The determination of the magnitude of a bivector can be done by taking the exterior product of vectors expanded in terms of their basis vectors. Returning to our earlier example of the two-dimensional vector space, we may have something like:

$$(a_1\hat{e}_1 + b_1\hat{e}_2) \wedge (a_2\hat{e}_1 + b_2\hat{e}_2) \tag{161}$$

Using the distributive property of the exterior product then yields:

$$a_1a_2(\hat{e}_1 \wedge \hat{e}_1) + a_1b_2(\hat{e}_1 \wedge \hat{e}_2) + a_2b_1(\hat{e}_2 \wedge \hat{e}_1) + a_2b_2(\hat{e}_2 \wedge \hat{e}_2) \tag{162}$$

We know that the exterior product of a vector with itself is zero: something that also makes sense with our geometric interpretation, as no parallelogram can be constructed from parallel vectors. Therefore, the first and last terms are zero. Moreover, we can flip the order of the basis vectors in the third term and gain a minus sign. This ultimately yields the following expression for the exterior product of two vectors written out in their basis vectors:

$$(a_1b_2 - a_2b_1)(\hat{e}_1 \wedge \hat{e}_2) \tag{163}$$

Here, the coefficient $a_1b_2 - a_2b_1$ acts as a sort of scaling factor for the unit bivector $\hat{e}_1 \wedge \hat{e}_2$. The coefficient is then equal to the total magnitude, or area, of the bivector. This is hardly

[31] A perhaps curious consequence of this is that the left bivector in 8 is equivalent to, for example, a counterclockwise oriented circle of the same area.

surprising, as this is in agreement with what we would expect.

While bivectors are certainly *Anschaulich*, one may wonder whether they contain any concrete physical application. They in fact do, as exterior products are sometimes argued to be a good replacement for cross products in physics. Consider angular momentum. While angular momentum concretely relates to rotation, we normally represent it with just another vector. The bivector, when for example displayed as an oriented circle, may be argued to more naturally represent the nature of angular momentum. Perhaps more importantly, angular momentum being a bivector removes the vector-pseudovector ambiguity. As opposed to vectors, bivectors *do* invert their orientation when mirrored, just like the pseudovectors appearing in physics. Graßmann may therefore be said to, at least in this respect, have found a better vector system with which to describe physical quantities.

We can climb one dimension higher and consider the three-dimensional case. While in one dimension our exterior algebra trivially has the two basis elements $\{1, \hat{e}_1\}$, and in two dimensions we saw that we had the four basis elements $\{1, \hat{e}_1, \hat{e}_2, \hat{e}_1 \wedge \hat{e}_2\}$, the three-dimensional case has the following eight basis elements: $\{1, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_1 \wedge \hat{e}_2, \hat{e}_2 \wedge \hat{e}_3, \hat{e}_3 \wedge \hat{e}_1, \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3\}$. That is, one scalar, three vectors, three bivectors and, finally, a *trivector*. Note that it simply follows from our abstract algebra discussions that these are simply the only eight basis elements we can construct for our exterior algebra given the vector space V with basis vectors $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$. Other orderings are just linear combinations of what is already in here. For example, $\hat{e}_2 \wedge \hat{e}_1 \wedge \hat{e}_3 = -\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3$, and therefore the former is not a linearly independent basis element. Also note that by the same logic seen for the tensor algebra, any element containing three wedge symbols would (perhaps after taking out a minus sign) contain an exterior product of identical basis vectors, and thus be zero.

Although the number of basis elements makes sense, the trivector $\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3$ may not yet. As one may guess, however, a trivector is simply a volume with two possible orientations. Like bivectors, they find use in physics, with monopole charge being an example of trivector. A visualization can be found in 9 below:

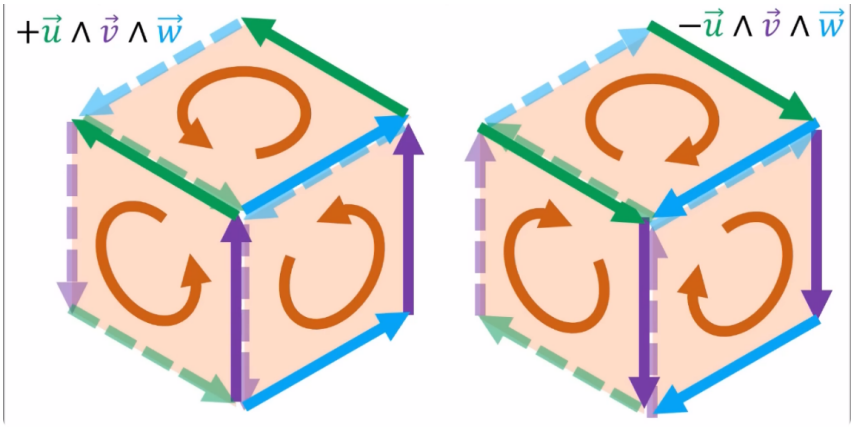


Figure 9: A simple visualization of a trivector with its two possible orientations. Note that overlapping edges of the squares on the cube will always have opposite orientations (Chris, 2023).

The above trivector can be thought of to have been conceived by taking the exterior product of a square and a vector in its perpendicular direction. From this, one does not construct a square, but a cube, where the orientation again follows the direction of the exterior product vectors in order.
This shows that the exterior product is not limited to two elements of identical dimensionality. In general, the exterior product outputs an object whose dimensionality is the sum of the operation's arguments.

Elements of our exterior algebra are usually not just vectors, but add objects with different dimensionality. For example, we may have the element $4-6\hat{e}_2-7\hat{e}_1\wedge\hat{e}_2+2\hat{e}_3\wedge\hat{e}_1-3\pi\hat{e}_1\wedge\hat{e}_2\wedge\hat{e}_3$. We call these elements of the exterior algebra *multivectors*. Although different in many ways, quaternions and exterior algebra are alike in that unlike vector calculus, they have no problem with adding objects of different dimensionality. This is just another, and certainly not less effective, way of doing physics.

At this point, we may finally say something about *Graßman numbers* or *Graßmann variables*. If we take the field F in the definition of the exterior algebra to be $\mathbb{C}$, then a Graßmann number is simply *an element of the exterior algebra*. Most important for now is to realize that these elements anticommute and are zero when operating upon themselves, as we saw with $\vec{v}\wedge\vec{w}=-\vec{w}\wedge\vec{v}$ and $\vec{v}\wedge\vec{v}=0$. With the foreknowledge we now possess, this is really all that is needed to be able to detect them in the next chapter.

With that, we now have a basic understanding of exterior algebra, as well as to how it came to be and how we arrived at its modern formulation. In combination with our familiarity with the path integral formulation, that means that we are now ready to get started on our historical quest. In the next chapter, we shall see how Graßmann's algebra was (re)discovered to solve the problem that was formulated in chapter 2.5.

4 The (Re)Discovery of Graßmann Algebra to Formulate Fermionic Path Integrals

In the previous two chapters, we have learned about the path integral formulation, Graßmann algebra and the central problem of why the path integral could not yet deal with fermions. In this chapter, the historical line that led to the eventual solution thereof will be investigated.

Before getting started on this, we will very briefly consider the standard narrative found in most textbooks. The Soviet physicist Felix Alexandrovich Berezin (1931-1980) is usually the name that appears with regard to the breakthrough in the problem of fermionic path integrals, as can, among others, be seen in (Peskin and Schroeder, 2019, §9.5)(Cartier and DeWitt-Morette, 2006, §9.3)(Weinberg, 1995, §9)(Itzykson and Zuber, 1980, §9)[32]. He published a small but decisive paper on the issue in the early sixties (Berezin, 1961), and further developed his work on this and other topics in a well-known book that came out a few years later (Berezin, 1966). Because of these works, he is credited with being the first to explicitly introduce 'Graßmann calculus', with which he derived the following result.

$$\prod_l \left(\int d\theta_l^* \, d\theta_l \right) e^{-\theta_i^* B_{ij} \theta_j} = \det B \tag{164}$$

Here, the summation convention is implied in the exponent, and the standardized notation of (Peskin and Schroeder, 2019, p. 301) has been employed. This integral, with θ^* and θ being Graßmann variables, and B a Hermitian matrix (both to be defined more clearly later), lies at the heart of the solution to our central problem. It is still used in this form. In fact, in most QFT applications, one does not need more than the above result when concerned with fermionic path integrals. After all, it is the type of Gaussian integral one gets for not just a free bosonic field, but also a free fermionic field. That means that if we can solve these for fermionic fields, we have all we need to apply perturbation theory and derive Feynman diagrams.
Another often used factoid related to this integral is that we note that its result is the inverse of its bosonic counterpart of the same form appearing in chapter 2.4. We will see how exactly this comes to be over the course of the papers discussed below, with a direct proof of it being included at the end of this chapter.

To the historian, however, this leaves many questions unanswered. We may wonder how Berezin came to this insight, whether he had any preceding influences and if he is indeed the sole name deserving of this discovery. As it turns out, many mathematicians and physicists had been working on the topic during the 50s. While Berezin and his Soviet influences will be considered, it turns out that lots of developments on this front took place in the United Kingdom. In fact, the search for a single person or country that can be credited with the discovery of Graßmannian path integrals will turn out to be a blurry one, with plenty of room

[32] As one may notice, Berezin has the tendency to be mentioned in the ninth chapter of books!

for priority disputes.

Therefore, this chapter will give an overview of all the noteworthy contributions to the eventual solution of our problem, ending with Berezin's 1961 paper on the issue. For each of these papers, the following questions will, insofar possible, be touched on each time:

- Who is the author? What is their background and who are they influenced by?
- What problem is the author trying to address in their paper? What are their goals?
- What novelty does the author bring? How do they attempt to solve the problem?
- What is the impact of the author's accomplishments? How do they compare to others?

Each question has a clear function in advancing our understanding of the papers, their context, and our evaluation of their relevance in the bigger picture. At the end of the chapter, some concluding remarks will be presented.
We will go through, in total, 6 key papers on the topic, starting in '55 and ending in '61, in order of submission date. These papers are interconnected and I have found them in particular to together make up a main post-Feynman storyline of how the problem of fermionic path integrals was eventually resolved. Of course, a boundary has to be drawn somewhere, and one can also name authors orbiting around this main body. These authors, for example, can be credited with contributions such as doing relatively early work on the theoretical basis of the path integral in QFT and with such or other research inspiring the authors in the main line, or themselves noticing the problem but not pursuing it further / doing so in a way that did not ultimately contribute to the main line of resolution of interest. Examples of such authors are (Schwinger, 1953)(Gel'fand and Minlos, 1954)(Fradkin, 1954)(Polkinghorne, 1955)(Davison, 1954)(Edwards and Peierls, 1954)(Burton and De Borde, 1956)(Symanzik, 1954). Some will be mentioned or (particularly Schwinger) briefly discussed in the next few subchapters. With our setup clear, let us proceed with the first author.

4.1 "The Representation of Green's Function in Quantum Electrodynamics in the Form of Continual Integrals" (1955) by I.M. Khalatnikov

Isaak Markovich Khalatnikov (1919-2021) was a Soviet physicist, working on many areas in the field. His doctoral advisor being the famous Lev Landau (1908-1968), quantum field theory was certainly among these[33]. In this 1955 paper, Khalatnikov aims to

> *"...present an expression for Green's function in the form of continual integrals in the space of ψ and A functions."* (Khalatnikov, 1955, p. 568)

[33] Landau is also credited for "participation in the investigation" leading to the to be discussed paper (Khalatnikov, 1955, p. 570).

This did not come out of nowhere. While perhaps not as well-known to your average physicist, the also widely acclaimed mathematicians Robert Adol'fovich Minlos (1931-2018) and, especially so, Israel Moiseevich Gel'fand (1913-2009), had published a paper doing precisely this for Bose fields earlier that year (Gel'fand and Minlos, 1954). Khalatnikov wanted to extend their work to electron fields, i.e., fields of the anticommutative fermions.

Khalatnikov's mission statement may warrant some commentary and backstory. First of all, Green's functions themselves are a well-known tool utilized in many subfields of physics, certainly including QFT. Once the Green's function associated with a differential operator acting on a field is known, we can easily solve for the field by having the Green's function act on the source term. For the path integral formulation, it would therefore be worthwhile to also express Green's functions in this way.
A particular fan of Green's functions in QFT was Julian Seymour Schwinger (1918-1994). While Feynman had formulated QED in terms of a functional integral, Schwinger preferred to formulate QFT in terms of functional differentials of fields. Therefore, we are not integrating over functions, but differentiating with respect to them, writing for example $\frac{\delta}{\delta\phi(x^\mu)}F[\phi(x^\mu)]$ where δ denotes that the derivative is taken with respect to a function like $\mathscr{D}$ does for functional integration. A typical way of taking a functional derivative is to add a small deviation function $\delta\phi(x^\mu)$ to $\phi(x^\mu)$, expanding the resulting functional $F[(\phi(x^\mu)+\delta\phi(x^\mu)]$ to first order in $\delta\phi(x^\mu)$ and considering this linear term's coefficient.
With that, Schwinger occupied a sort of 'third way', working neither with the operator formalism of canonical quantization, nor with the path integral, although it certainly shares some things with the latter. As Schwinger lays out in his Nobel lecture, both Feynman and him are inspired by Dirac's work on the quantum Lagrangian, although Schwinger took this in a differential rather than integral direction (Schwinger, 1965, p. 1-2). Central to his formalism is what is now known as the 'Schwinger action principle'

$$\delta\langle t_1|t_2\rangle = \langle t_1|\delta\hat{S}|t_2\rangle \tag{165}$$

Here, $\langle t_1|t_2\rangle$ is a transition function, and $\hat{S}$ is a quantum action *operator*, and thus not the classical action we see in Feynman's path integral. We therefore have a quantum principle of stationary action. We may wonder if we can also find a quantum version of the Euler-Lagrange equations through this. This is indeed the case, and these are called the Schwinger-Dyson equations. They are the equation of motion whose solutions are the Green's functions for quantum fields. It should however be noted that 'Green's functions' here may be more of a metaphor. These Green's functions are not literally those associated with, e.g., the Klein-Gordon or Dirac equation differential operators, as they are actually the 'Green's function' of a differential operator with an extra term. For example, we may have something like $\left(\Box + m^2 + \frac{\delta}{\delta J(x^\mu)}\right)G(x^\mu,(x')^\mu) = \delta(x^\mu-(x')^\mu)$, where the usual Klein-Gordon operator is extended by the functional derivative with respect to a source field[34]. They can be formulated

[34]Schwinger was particularly fond of using such source fields, and the source theory he later developed took up an important part of his work. Like Feynman's operator calculus, however, this idea did not catch on as much as others of his.

as follows (Peskin and Schroeder, 2019, p. 306-308):

$$T\left(\left(\frac{\delta}{\delta\phi(x^\mu)}\int d^4x'\mathscr{L}\right)\phi(x_1^\mu)\dots\phi(x_n^\mu)\right)=\sum_{i=1}^{n}T\left(\phi(x_1^\mu)\dots(i\delta(x^\mu-x_i^\mu)\dots\phi(x_n^\mu)\right) \tag{166}$$

Here, T denotes the usual time-ordering operator. We are thus left with a 'tower' of an infinite number of coupled functional differential equations, because every n-point Green's function involves the $(n+1)$-function, and so on. The terms on the right-hand side are the so-called 'contact terms'. These make up the essential difference with the classical Euler-Lagrange equations and encode the quantum nature of the equation.
While the Schwinger-Dyson equations are non-perturbative and exact, they cannot be solved analytically, but only approximated through perturbation theory. Dyson found how one can also derive the equations carrying his and Schwinger's name from the path integral formulation, and showed how the resulting Green's functions can be related to Feynman diagrams, i.e., terms of a perturbation series, through the LSZ reduction formula.
In the early days of the path integral, when it was not yet known that well and still plagued by some problems such as the one at the center of this thesis, Schwinger's methods as described above were more popular than they are now. They ultimately lost out to the easier to work with methods of Feynman, though[35]. Nevertheless, this all also explains some of the background of Khalatnikov's work, who at this time wanted to give formal solutions to the Schwinger-Dyson equations in terms of path integrals.

[35]The same goes for their person. While the charismatic Feynman is widely known, this seems to be much less so the case for Schwinger. This seems to be the case despite his enormous impact on the development of QED and other areas of physics, and his simultaneous Nobel prize with Feynman. To test this, I asked students at the end of their Master's in Astro and Particle Physics at my own university whether they know who Feynman is and whether they know who Schwinger is. The results can be found below in figure 10.

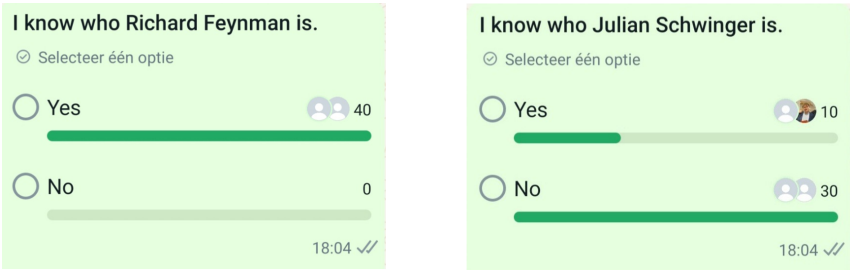


Figure 10: A poll ($N = 40$) under Master's students in Particle and Astrophysics near the end of their degree, on whether they know who Feynman and Schwinger are.

As can be seen, 100% of the $N = 40$ students who answered the poll knew who Feynman was, but only 25% could say the same about Schwinger. While the first result is no surprise, the second is rather shocking, even though it confirms the above hypothesis. The fact that three in four students even in a particle physics track do not know who Schwinger is, shows that his fame has fallen off as a result of his methods being somewhat marginalized by those of Feynman.
Of course, the above is by no means a scientific survey, as it is just a strawpoll in a *WhatsApp* group chat of one particular university. Nevertheless, the discrepancy is extremely strong, and it seems unlikely to fully disappear even if an international survey of the key demographic was taken.

With this context in mind, we can carry on to a second aspect of Khalatnikov's mission statement worthy of commentary. This has to do with what Khalatnikov calls 'continual integrals', and how these relate to path integrals. A continual integral can be viewed as a more mathematically formal generalization of the by now familiar path integral, viz., an integral over a function which is defined in terms of the infinite limit with respect to the number of regular integrals in a product[36], like in (1) (Berezin, 1966, p. 38). Thus, while the path integral evokes the image of a sum over physical paths or field configurations, the continual integral is devoid of such physical interpretations and is just about the general mathematical aspects of the object and procedure used to perform such a sum over paths. In other words, the path integral is a continual integral, but not every continual integral is a path integral like the one used by Feynman in quantum theory.
A last comment on the mission statement is that 'the space of ψ and A functions' refers to the electron and photon field, as the paper's title already establishes Khalatnikov is considering QED. With this, it is clear what he sets out to do.

To find the Green's function for a single electron,

When considering the electron field, Khalatnikov is confronted with the following exponent that we will turn our attention to first, given it is particularly relevant for our story. Resulting from turning the Schwinger-Dyson equation into an integral exponent by previous authors (Khalatnikov, 1955, p. 568), it can be written as:

$$\frac{1}{4}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' S_{F_{\alpha\beta}}(x-x') \frac{\delta^2}{\delta\psi_\alpha(x)\bar{\psi}_\beta(x')} \tag{167}$$

The spinor functions in the functional derivatives are stated to anticommute, which is by itself not yet surprising (Khalatnikov, 1955, p. 569).
Khalatnikov now introduces a coordinate lattice with N points, whereby $S_{F_{\alpha\beta}}(x_i - x_k) = S_{\alpha\beta}(ik)$. This is just a discretizaton process, as was also done when we introduced discrete lattice regularization, even though that was a different context. For Khalatnikov, this is useful to do as it allows for the use of sums instead of integrals, and partial derivatives rather than functional ones.
He then aims to circumvent the difficulty that comes with anticommuting spinors in the now partial derivative, by rewriting the above exponent as

$$\check{S}_{\alpha\beta}(ik) \frac{\delta^2}{\delta\psi_\alpha(i)\bar{\psi}_\beta(k)} \tag{168}$$

Note that the integral is now replaced by an implicit sum, as one can invoke the summation convention here.

[36] As we will see, it appears that while Western authors prefer to use 'functional integral', the Soviet authors seem to opt for 'continual integral' instead. One reason may be that the mathematically-minded Soviet authors we here discuss may have thought 'functional integral' to be too vague, as the object $\int_a^b f(x)dx$ is technically also a functional '$F(f(x))$', the difference being that the function is here integrated over rather than being the integration 'variable'.

The operator $\check{S}_{\alpha\beta}$ contains not just $S_{\alpha\beta}(ik)$ but, crucially, absorbs the anticommutative properties of the spinors. The spinors are now demoted to "ordinary functions which commute" (Khalatnikov, 1955, p. 569). The anticommutativity is instead 'moved to' a new matrix operator η_{ik}, whereby $\check{S}(ik) = S(ik)\eta_{ik}$. A crucial property of this matrix for our purposes is that $\eta_{ii} = -1$.

The Green's function for a single electron looks like (Khalatnikov, 1955, p. 568):

$$G_{\alpha\beta}(x, x') = \langle T(\psi_\alpha(x)\bar{\psi}_\beta(x')S/S_{vac}\rangle \tag{169}$$

To solve it, Khalatnikov needs the vacuum expectation value. This can be written as $S_{vac} = e^{-i\int\mathscr{H}(x^\mu)d^4x}$, which after entering the QED Hamiltonian density yields, among others, the exponential of (167). Again, looking at the part of the derivation most interesting to us, this leads to the inclusion of the following integral (Khalatnikov, 1955, p. 570):

$$I = \int e^{-\bar{\psi}(i)\check{S}^{-1}(ik)\psi(k)}\delta\psi\delta\bar{\psi}, \qquad \delta\psi = \prod_i d\psi(i), \;\; \delta\bar{\psi} = \prod_k d\bar{\psi}(k) \tag{170}$$

It is important to note that here the symbol δ is used to display the functional integration measure of the 'continual' integral, rather than the symbol $\mathscr{D}$ we used in chapter 2, which came around later.
One might also note that this continual integral looks very similar to (164). One crucial difference is that, as was just noted, ψ and $\bar{\psi}$ are regular, commuting functions. Still, this is not merely a regular bosonic path integral either, due to the matrix operator η_{ik} implicitly present in the exponent. Also note that Khalatnikov takes the integral over these functions, and that he, in familiar manner, writes them as the product of discretized parts.
While this takes more effort than later approaches, Khalatnikov can evaluate this integral, and does so by introducing a limiting trick and expanding the exponential.

$$I(\alpha) = \lim_{\alpha\to 0}\int e^{-\alpha\bar{\psi}(i)\psi(i)}\sum_{n=0}^{\infty}\frac{1}{n!}(\bar{\psi}(i)\check{S}^{-1}(ik)\psi(k))^n\delta\psi\delta\bar{\psi} \tag{171}$$

Thus, we can see that the extra exponential disappears in the limit, and the previous exponential is now in the form of a sum. Performing the leftover Gaussian integral, he then establishes

$$I(\alpha) = \lim_{\alpha\to 0}\frac{\pi^N}{\alpha^N}\sum_{n=0}^{\infty}(-1)^{n+P}\frac{1}{\alpha^N}S^{-1}(1i_1)\cdots S^{-1}(ni_n) \tag{172}$$

Here, the P is the parity of the permutation of the second indices. It is effectively a bookkeeping device for the minus signs related the matrix η_{ik}. Further, the above follows from the contraction of the $S^{-1}(ik)$ and η_{ik}, as results from the definition of $\check{S}(ik)$ above. What is left over now, however, is by definition the determinant, so that he establishes

$$I(\alpha) = \lim_{\alpha\to 0} = \frac{\pi^N}{\alpha^N}\det\left(\delta_{ik} + \frac{S^{-1}(ik)}{\alpha}\right) \tag{173}$$

While the Gaussian integral over the 'defermified fields' thus still gives a determinant, it is not quite the elegant expression as seen in (164). In fact, the determinant blows up in the limit introduced to derive it. For the resolution of this, he refers to another paper.

Returning to the vacuum expectation value, Khalatnikov does seem to be able to successfully calculate it, and he even writes it as the familiar seeming result below (Khalatnikov, 1955, p. 570):

$$S_{vac} = \frac{\int e^{i\int L(x,\psi,A)d^4x}\delta\psi\delta\bar{\psi}\delta A}{\int e^{i\int L_0(x,\psi,A)d^4x}\delta\psi\delta\bar{\psi}\delta A} \tag{174}$$

Here, L_0 is the Lagrangian of the electron's spinor field without interaction terms, while L does include interaction with the photon field.

After formulating S_{vac}, Khalatnikov can use it to derive the Green's function for a single electron using (169). Given that S_{vac} is indeed formulated in terms of continual integrals, Khalatnikov's efforts in trying to express Green's functions in QED in terms of continual integrals seem to be successful. These results are not new physics material, but reconstructions of perturbation results of QED in the orthodox formalism, incorporating the minus signs more directly present in the operator language of canonical quantization through different methods. He also considers anticommuting objects that are not operators. His methods are, however, far less efficient than later authors', and include awkward limiting procedures. Moreover, Graßmann numbers are not yet utilized in his derivation. Six years later, however, Berezin would do so, and Khalatnikov would in fact be the only citation in his paper (Berezin, 1961, p. 314). His influence on the man who would come to be known as the one to introduce Graßmann calculus to solve fermionic path integrals is, therefore, explicitly clear.

4.2 "Propagators of Quantized Field" (1955) by P.T. Matthews and A. Salam

In the same year as Khalatnikov the British physicist Paul Taunton Matthews (1919-1987) and one of his students, the Nobel prize winning Pakistani physicist Mohammed Abdus Salam (1926-1996), also published a paper on the issue (in some ways, as we shall see, also being comparable to Khalatnikov's in content) (Matthews and Salam, 1955). The doctoral advisor of both was Nicholas Kemmer (1911-1998), the Russian-British physicist famous for his work on the British nuclear program. Their paper was inspired not just by the work of Feynman[37], but also by the work of and discussion with Gordon Feldman (1929-2014), Rudolf Ernst Peierls (1907-1995) and in particular Samuel Frederick Edwards (1928-2015). The latter two had published a paper on finding the Green's function of a nucleon (i.e., a spin-1/2 particle) moving in an external field starting from Schwinger's functional equations a year earlier (Edwards and Peierls, 1954). This is not entirely surprising, as Edwards was a doctoral student of Schwinger.

[37] Feynman is, however, taken as the explicit starting point of their paper, unlike in the case of Khalatnikov.

At the beginning of their paper, Matthews and Salam state that fermionic fields have to anti-commute. This is again an interesting comment, as since they are talking about Feynman's functional integrals, they are *not* talking about *operator valued* fields. They continue with the following important remark:

> *"In view of the apparant difficulty of functional integrals over anticommuting functions, this direct approach has always been avoided in the past. It is the purpose of this paper to show that such integrals can be defined and evaluated, and lead in a simple and unified way to compact expressions for the many particle propagators, from which scattering cross sections and energy levels may, in principle, be deduced."* (Matthews and Salam, 1955, p. 121)

With this, the detected problem and subsequent goal of the paper is clear. Three further interesting claims are made by the authors. Firstly, they want to make explicit the connection between Schwinger's action principle and Feynman's path integral, noting that while Schwinger provides functional differential equations for propagators, Feynman provides the solutions to them in terms of functional integrals. This relationship can then be utilized Secondly, they state that while Feynman had taken Pauli's exclusion principle as a given and then just applied it to the path integral formalism, the principle will follow naturally "as a consequence of the anticommutation of the fields" in their case. Thirdly, they already claim that in the fermionic case, they will find a determinant, while they will find its inverse in the bosonic case (Matthews and Salam, 1955, p. 121).

As we are concerned with the relevance of these papers for the eventual solution of fermionic path integrals through Graßmann numbers, there is a relevant anecdote to note about this one. In letters between Peierls and Dyson, the latter objects to the publishing of Matthews' and Salam's paper, on the grounds of their results not being novel enough. He claims that these results are already implicit in Feynman and had been established by a Russian physicist. On their behalf, Peierls replies in the following way:

> *"In this I explain that we now accept the statement that the main result of the paper is implicit in Feynman's papers, though I still maintain that it is very hard to recognize and the result is also obtained in the Russian paper which you mentioned to me, but there it is derived in a rather round about way through the Schwinger equation. I still believe the Matthews-Salam paper is still worth publishing, but since some redrafting will be necessary to acknowledge the connection with earlier papers and to emphasize what is really new there would be a good deal of delay involved in having the amended version referred again and we are therefore proposing to withdraw the paper from the Physical Review and have it published in this country.*
> *I still believe that it should be published somewhere and I am strengthened in this both by the fact that the recent paper by Klein using functional methods still juggles around with the most complicated Schwingeresque formalism and still misses the simplicity of the whole matter and also by a remark in one of Feynman's earlier*

papers in which he says that nobody has ever worked out how to do the integration over spinor fields and that this ought to be done." (Lee, 2009, p. 483)

A few interesting statements appear in these two paragraphs. On the one hand, it is acknowledged by all parties involved that Matthews and Salam do not yet bring revolutionary results, as its implicit presence in Feynman's earlier work is acknowledged, as well as the work of a Russian author. The Russian author eventually cited in their work is Efim Samoilovich Fradkin (1924-1999), another physicist working on functional analysis in QFT (Matthews and Salam, 1955, p. 132). But his approach is, as stated by Peierls, more based on Schwinger's work. Nevertheless, it again shows the Russo-British occuptation with the issue at hand.
In any case, the innovation of using Graßmann variables is not yet present in the work of Matthews and Salam. This does not mean their influence is nill though, as their focus on the issue made it so that they are directly cited by later British authors continuing work on this issue.
A second interesting point is Feynman's very explicit statement about integrals over spinor fields, which I have unfortunately not been able to find in this language. The paragraph in Appendix C in a '51 paper may come closest (Feynman, 1951, p. 127). More importantly, the path integral's originator clearly posing the problem like this may have contributed to the many attempts to find a simple and satisfactory way of doing this.

With that background, let us go back to what Matthews and Salam actually do in their paper. They start by stating that the general sum over histories for an interacting nucleon (fermionic) and meson (bosonic) field is

$$(\xi', \sigma_1 | T(A(1), ..., B(n)) | \xi", \sigma_2) = \frac{1}{N} \int_{\xi"}^{\xi'} A(1) \cdots B(n) e^{i \int_{\sigma_2}^{\sigma_1} L(\phi, \psi, \bar{\psi}) d^4 x} \delta\phi \delta\bar{\psi} \delta\psi \tag{175}$$

Here, ξ' and $\xi"$ are "eigenvalues of a complete commuting set of operators, which specify the state of the system on the two surfaces", which is why they serve as the boundaries for the path integral. Then σ_1 and σ_2 are two space-like surface the Lagrangian is typically integrated over in QFT. The $A(1), ..., B(n)$ are a set of field operators that are time-ordered by $T()$. Lastly, ϕ and $\psi, \bar{\psi}$ are the meson and (anti)nucleon fields, respectively (Matthews and Salam, 1955, p. 122).
They go on to first derive the propagator of a free meson. Next up is the propagator of the quantized Dirac field ψ given the interaction with external bosonic field ϕ^{ex}. To find it, they must calculate the following propagator (where vacuum effects have been filtered out):

$$S(1, 1', \phi^{ex}) = \frac{1}{N(\phi^{ex})} \int \psi(1) \bar{\psi}(1') e^{-i I_n(\phi^{ex})} \delta\psi \delta\bar{\psi} \tag{176}$$

where,

$$N(\phi^{ex}) = \int e^{-i I_n(\phi^{ex})} \delta\psi \delta\bar{\psi} \tag{177}$$

Here, I_n is the action, which here depends on the external field ϕ^{ex}. Remember that this integral is to be taken over anticommuting fermionic fields. This is a problem, as no such integral is defined. While later authors will tackle this question more directly, the approach of

Matthews and Salam is to introduce a complete set of *anticommuting* functions $\psi_a(x) = \sum_n a_n \psi_n(x)$ and $\psi_b(x) = \sum_n b_n \psi_n(x)$. Thus, while a_n and b_n are regular c-numbers, the $\psi_n(x)$ anticommute, i.e., $\psi_n \psi_m = -\psi_m \psi_n$. These relations are then substituted into the propagator integral, leading to the expression (Matthews and Salam, 1955, p. 127):

$$S(1,1',\phi^{ex}) = \frac{1}{N(a)N(b)} \sum_i \psi_i(1)\bar{\psi}_i(1') \int (a^2+b^2) e^{-i(a^2+b^2)} da\, db = -i \sum_i \psi_i(1)\bar{\psi}_i(1') \quad (178)$$

Note that $N(\phi^{ex})$ is now separated into $N(a)N(b)$. More importantly, we see how the integral is not taken over anticommuting functions ψ_i, but rather, over the regular commuting c-numbers a and b. In this way, Matthews and Salam can express the Dirac propagator in terms of a well-defined integral. However, since the anticommutativity has to be moved somewhere, the result is a sum over the anticommuting 'eigenfunctions'. In this sense, what they are doing is similar to Khalatnikov: the anticommutativity is involved in the computation through the introduction of an object, but there is no direct integration over these anticommuting objects.

Now in the case of a two-nucleon propagator $S(1,2,1',2',\phi^{ex})$, summation over all permutations of these functions is required. Due to their anticommutativity, this sum will naturally include many terms with a minus sign. As we are well aware, the end result is again a determinant (Matthews and Salam, 1955, p. 129).

The final and most important case Matthews and Salam build up to is that of two interacting quantized fields. That is, the Bose field is no longer an external, unquantized field, and the factor $\mathscr{D}\phi$ now also appears in the propagator integral. The one-nucleon propagator becomes:

$$S'(1,1') = \frac{1}{N} \int \psi(1)\bar{\psi}(1') e^{-1(I_n(\phi)+I_m)} \delta\psi\delta\bar{\psi}\delta\phi \quad (179)$$

where,

$$N = \int e^{-1(I_n(\phi)+I_m)} \delta\psi\delta\bar{\psi}\delta\phi \quad (180)$$

Here, I_m and I_n are the *m*eson and *n*ucleon action, respectively. Also note that we now have a primed S, to differentiate the propagator in this case from the previous one with the external, unquantized bose field.
Now Feynman does the integration over ϕ first, and probably for this reason what Matthews and Salam do is already implicit in his work (Matthews and Salam, 1955, p. 130). They, however, argue that the integrals over $\psi, \bar{\psi}$ are more difficult and better to do first. They do this by simply expressing the above integral in terms of the results in the case of the external bose field. The end product is then:

$$S'(1,1') = \frac{\int S(1,1',\phi) N(\phi) e^{-iI_m} \delta\phi}{\int N(\phi) e^{-iI_m} \delta\phi} \quad (181)$$

Where the unprimed $S(1,1',\phi)$ was given before and $N(\phi)$ is another factor that can predictably be expressed in terms of results of their previous case. Therefore, implicitly, the

integral over fermionic fields is dealt with through their previous method of substituting in a complete anticommuting set and doing the actual integration over regular c-numbers.

The above formula can be generalized for an arbitrary number of nucleons and mesons, and determinants will appear again because of the sum over anticommuting functions (Matthews and Salam, 1955, p. 130). According to Matthews and Salam, the above expression $S'(1,1')$ is the main result of their paper. They do not carry out the final integration over ϕ, for which they point the aforementioned paper of Edwards and Peierls, as well as Fradkin's (Matthews and Salam, 1955, p. 132). This, however, is no trivial endeavour, and requires approximation to prevent divergences in the resulting series of that calculation.

Ultimately, Matthews and Salam seemingly successfully express the sought-after propagator in the language of functional integration, even though it concerns fermionic fields, and correctly point out the resulting determinant. The results are thereby not new physics, as one can already formulate these propagators in the operator language of canonical quantization, but they are a new way of expressing the same thing in the path integral formulation. At the same time, while the work of Matthews and Salam is a step forward, there was reason to continue research. One reason for this is that the expansion in terms of anticommutative functions, rather than directly integrating over such functions, is still far from efficient. Another reason is that they do not yet give us a path integral for fermionic fields as a first principle and telling us how to directly calculate it. Yet, they were thinking about the issue, and as we will see later, inspiring others to do the same. The next three authors to be considered thus directly take up their mantle, many citing them for this reason.

4.3 "Transition Amplitudes as Sums over Histories" (1956) by W. Tobocman

While Matthews and especially Salam are well-known figures, this is less so the case with the British authors we discuss from now on. Information on them is hard to come by, which makes it a bit harder to answer the first question at the beginning of this chapter in a way that goes beyond what is given in their papers. We shall nevertheless proceed with the information that is given to us.

For Tobocman specifically, we know first of all that he is aware of the work of Matthews and Salam, as they are included in this third citation (Tobocman, 1956, p. 1214). Just before doing so, he makes the historical point that Schrödinger already noted that in some cases that the solution to the equation bearing his name could be written as an exponential of the action, which is interesting as most narratives start with Dirac, if not Feynman directly.
At the end, we see that Tobocman has been a guest at the Instite for Advanced Study at the time of the writing of this paper. Like Matthews and Salam before him, he thanks (among others) Feldman for discussions surrounding his paper (Tobocman, 1956, p. 1228). Tobocman, however, is not just influenced, but also has a clear influence on later authors. The next

two authors directly cite him and continue his line of thinking.

As for the paper itself, Tobocman wants to start from the operator formalism of canonical quantization and then express transition amplitudes as functional integrals. He claims that once doing this, one will find Feynman's functional integral as long as one quantizes in terms of commutators. In the case of anticommutators, which makes more sense when describing fermionic fields, this 'Feynman principle' will however not follow (Tobocman, 1956, p. 1215).

Before getting into the why of this, we will first need to introduce a distinction made by Tobocman. He differentiates between the 'Feynman principle' and the 'sum over histories'. While these are nowadays sometimes used interchangeably, Tobocman claims that they only coincide in the case of bosons. He does not define these terms explicitly, but their meaning can be derived implicitly from the text. For example, Tobocman derives the phase space path integral (72) just as we did and then notes that *"...we have succeeded in writing the transition amplitude in the form of a sum over histories"* (Tobocman, 1956, p. 1217). It is only after introducing a Gaussian Hamiltonian that we get to the coordinate space representation, as, e.g. in our case, $H(x,p) = \frac{p^2}{2m} + V(x,t)$ allows one to perform the momentum integrals. The remaining coordinate space path integral with a Lagrangian in the exponent (3) is then what Tobocman refers to as *"the Feynman principle"* (Tobocman, 1956, p. 1217). When he later repeats the procedure of going from the operator formalism to, this time, fermionic QFT, we get a further hint. As we will see, he comes to the conclusion that time-slicing and then eventually taking the limit to zero defines the sum over history representation, but that this does not yield a Feynman principle because there is no exponential with an integral over the Lagrangian involved (Tobocman, 1956, p. 1222,1225). In this sense, we may say that the 'sum over histories' is whatever follows from the procedure whereby we start from the operator formalism, slice up the time, and see what we get when we later sum over all the terms, eventually taking the limit to zero. The Feynman principle stands on itself (as a possible result of this procedure) whereby it can be expressed as an exponential with the action in terms of an integral over the Lagrangian as its argument. Yet, if the path integral formulation is to be the equal of the canonical quantization operator formulation, we would want every possible quantum system to be addressable directly from a Feynman principle as starting point. If the two formulations are equivalent, that principle needs to be derivable from canonical quantization not just in the case of bosonic fields, but also in that of fermionic ones.

Now the most interesting part of the paper consists of Tobocman's attempt to construct the sum over histories starting from anticommutation relations associated with the Dirac field. After setting up the method, Tobocman makes the following observation:

> *"Since the operators in this theory anticommute with each other, they can be simultaneously diagonal only if their eigenvalues also anticommute. Thus we must use for the base field of our Hilbert space a field which has a noncommutative algebra rather than the field of complex numbers."* (Tobocman, 1956, p. 1219)

First, one may wonder why we would need operators to be simultaneously diagonalized.

For this, remember that one of the strengths of Feynman's path integral is the ability to use the classical action, no quantization required. Diagonal matrices act like normal c-numbers and always commute, in this sense being 'classical'. In the action, many operators can be present. Therefore, simultaneous diagonalization guarantees the classical action we want for our functional integral formulation of QFT.
Perhaps more interestingly, Tobocman's idea of replacing the field of complex numbers with an algebra that does not commute is a key step on the road to Graßmann integration, because that is precisely the kind of field[38] one integrates over in such cases.
In order to realize this, Tobocman replaces the components of Hilbert space vectors with infinite dimensional matrices. This takes after the occupation number representation, where we also find infinite rows with entries that can just be 0 or 1, as the exclusion principle forbids more in the case of fermions. He calls the kind of vector resulting from this a 'hypervector'. Similarly, he defines 'hypermatrices' as matrices whose elements are also matrices. The hypermatrices mutually anticommute, but square to the identity matrix.

After this complete and innovative setup, he then works out the transition amplitude through the familiar time-slicing procedure but even though he is left with an expression for the sum over histories, it does not contain the Feynman principle. The complete expression, including terms such as $\langle \zeta(0)\, t_0 | \zeta(1)\, t_1 \rangle$, are just expressed in terms of products and sums over hypervectors and hypermatrices[39]. Note, by the way, that this algebraic construction therefore also does not involve *integration over this noncommutative field* (Tobocman, 1956, p. 1222). While the sum over histories can be performed, and the moves Tobocman makes along the way are a huge step in the right direction, we therefore still do not have a path integral way of doing it. He himself concludes that a Feynman principle of transition amplitudes for Dirac fields cannot be given when trying to establish this from a sum over history procedure starting from the operator formalism. While perhaps a somewhat pessimistic conclusion, it motivated later authors to defy these odds after all.

4.4 "On Sums over Trajectories for Systems with Fermi Statistics" (1956) by D.J. Candlin

The third entry in the British line is by David John Candlin (1928-2019), who we will see makes very key contributions. Like Tobocman, he was a guest at the Princeton Institute for Advanced Studies. Unfortunately, he eventually quit physics and moved on to computing science.
In his paper, he cites Matthews and Salam as well as Tobocman, with particularly frequent reference to the latter. Candlin mentions how Tobocman showed how the path integral (in Tobocman's terms: the Feynman principle) arose from the opreator formalism when a system's conjugate variables satisfy the typical commutation relations. Now he sets out to do

[38] Note that we here refer to the mathematical notion of a field, not a physical quantum field.

[39] The analogue of such a term in our derivation of chapter 2.2 was a term like $\langle x_1 | e^{-\frac{i\epsilon}{2\hbar m}\hat{p}^2} \cdot e^{-\frac{i\epsilon}{\hbar}V(\hat{x})} | x_0 \rangle$. In Tobocman's term, however, the ζ's are hypervectors.

the same for anticommuting conjugate variables, where he explicitly mentions that this involves *"integration over anticommuting quantities"* (Candlin, 1956, p. 231-232).

Candlin starts by considering a single Fermi oscillator. In QFT, the vacuum can be pictured as at each mode in momentum-space being an oscillator. While one is usually led to think of a harmonic oscillator, the excitations of these correspond to bosons. Fermions then correspond to excitations of Fermi oscillators, which are described by a different Hamiltonian. Candlin provides the Hamiltonian $\hat{H} = m\eta^\dagger\eta$, where $\eta\eta^\dagger + \eta^\dagger\eta = 1$. These operators $\eta, \eta^\dagger$ now have the following matrix representations:

$$\eta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{182}$$

and

$$\eta^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{183}$$

Candlin explains that he ideally wants a completeness relation based on the eigenstates of these matrices, as these can then be inserted in the familiar way into the sum-over-histories approach. The trouble is that η and $\eta^\dagger$ cannot be diagonalized and thus have no eigenstates. However, this is (almost) possible if one uses eigenstates with anticommuting eigenvalues (Candlin, 1956, 232). He provides eigenstate equations $\eta|a\rangle = a|a\rangle$ and[40] $\eta^\dagger|a^*\rangle = a^*|a^*\rangle$. He now, crucially, notes the following:

> *"The quantities a and a^* are generalized numbers which all anticommute with each other and have zero squares. We shall call such quantities 'a-numbers'."*
> (Candlin, 1956, p. 232)

These 'a-numbers' are extremely important. While no rigorous mathematical definition lies beneath them at this point, such anticommuting numbers with zero squares will turn out to be isomorphic to the elements of the one-dimensional exterior algebra. While discussing this more extensively when we get to Berezin, this is easy to see when we interpret the operation between two a-numbers that are being squared as being the wedge product, i.e., $a^2 \cong a \wedge a = 0$. We also observe that the wedge product is anticommutative, just like the a-numbers.
Candlin was probably either not intimately aware of the work of Graßmann, or he did not think it necessary to connect this a-numbers with elements of the one-dimensional exterior algebra. After all, the latter is a much larger algebraic structure, while Candlin very simply needs anticommuting numbers with zero squares. The mathematical background of Berezin perhaps made him more inclined to embed these numbers in a pre-existing underlying mathematical structure. Nevertheless, it is of great importance to note that this is the earliest point where we see Graßmann numbers being explicitly introduced.

[40] Note that a^* here just refers to the eigenvalue of an eigenstate of $\eta^\dagger$, i.e., it does not denote the complex conjugate of a as usual.

Candlin solves the eigenstate equations and now, as stated, wants to sum over $|a^{(*)}\rangle\langle a^{(*)}|$ for all possible eigenvalues to find the identity operator. In other words, he wants to find a discrete version of (46) for the eigenstates with a-number eigenvalues. After all, we ourselves saw in chapter 2.2 how useful this is for getting from canonical transition amplitudes to the path integral. Candlin, however, states that getting the identity operator in this way is *not possible* due to the anticommutativity of the a-numbers. What *is* possible is to construct the unit operator multiplied by some a-number $\rho^{(*)}$, i.e., $\rho^{(*)}\hat{I}$. With this, Candlin essentially goes on to compute:

$$\langle\Phi(T)|\prod_{n=0}^{N-1}(1-i\epsilon H(t_n)|\Psi(0)\rangle \tag{184}$$

Here, Ψ is the initial state and Φ the final state. This is just the first order expansion in ϵ of the time-evolution operator we used in chapter 2.2, with eventually the limit $\epsilon \to 0$ being taken. Candlin largely goes through the same familiar sum over history steps, setting up the whole procedure without explicitly calculating the end result. Instead, he moves on to the more realistic case of the Dirac oscillator with the four operators $\psi_\alpha, \psi_\beta, \psi_\alpha^\dagger, \psi_\beta^\dagger$. In contrast to the Fermi oscillator, this model is a relativistic version of the harmonic oscillator (reproducing it in the classical limit), obeying Fermi-Dirac statistics (Moshinsky and Szczepaniak, 1989). Candlin now introduces the anticommutation relations and Hamiltonian as follows (Candlin, 1956, p. 235):

$$\{\psi_\alpha, \psi_\beta\} = 0 \tag{185}$$

$$\{\psi_\alpha, \psi_\beta^\dagger\} = \delta_{\alpha\beta} \tag{186}$$

$$\hat{H} = \frac{1}{2}(\psi_\alpha^\dagger Q_{\alpha\beta}\psi_\beta - \psi_\beta Q_{\beta\alpha}^*\psi_\alpha^\dagger) \tag{187}$$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta, which from now on may be assumed whenever this symbol occurs unless explicitly stating otherwise. Moreover, $\{A, B\} \equiv AB + BA$ is, of course, the anticommutator. The summation convention is adopted for α and β, and Q is a Hermitian matrix. Again, he defines eigenstates such as $|u\rangle = |u_4 u_3 u_2 u_1\rangle$ with double the degrees of freedom as in the Fermi case (as expected) and a-numbers as eigenvalues. He proceeds, like in the Fermi oscillator case, to set up the sum over histories approach starting from $\langle\Phi|\Psi\rangle$, but again does not directly calculate the whole. Interestingly, he now wants to consider a continuous realization of the completeness relation considered above, with an integral instead of a sum. Defining $\hat{F}$ as an operator of the system, he comes to the following expression:

$$\int_{-\lambda}^{\lambda}|u^*\rangle\langle u^*|\hat{F}|u\rangle\langle u|du^*du = \rho^*\rho\hat{F} + \mathcal{O}(\lambda^{-1}) \tag{188}$$

Here, the integration bounds λ are related to the numerical constant in front of the eigenstates. Since u and u^* are a-numbers, we here have an integral over anticommutative variables. This is another key step towards the eventual identity (164), yet it is still somewhat unclear. From the earlier stated completeness relation whereby $\sum|a^{(*)}\rangle\langle a^{(*)}| = \rho^{(*)}\hat{I}$ the right-hand side seemingly makes sense, yet it is still not quite clear how the actual integration over

these anticommutative variables is being performed in a rule-based manner (Candlin, 1956, p. 237).

Perhaps if Candlin had worked out his ideas in a more thorough way, we would nowadays speak of the 'Candlin integral' rather than the 'Berezin integral'. In fact, some would say the paper under consideration is already plenty of reason to do so. For one, this viewpoints floats around in small corners of the internet, including on his Wikipedia page[41]. But moreover, at least one well-known professional physicist appears to credit Candlin with fermionic integrals as well, namely Stanley Mandelstam (1928-2016), who in one of his papers stated the following:

> *"For an interpretation of the wave function of 'classical anticommuting variables', and for a treatment of the functional integral over such variables, we refer the reader to the work of Candlin." (Mandelstam, 1973, p. 449)*

Whether one is of the opinion that Candlin in this paper solves the problem of fermionic path integrals or not quite yet, it is undeniable that his work marks a significant step forward towards this end, by both introducing a-numbers and the idea of an integral being taken over them. Nevertheless, this could have been explained better, and it is moreover unclear where exactly the Feynman principle is at the end of his paper that he had set out to obtain for anticommuting fields. We will see how his work is taken up by the next author.

Finally we shall end with an interesting point Candlin makes in the discussion at the end of his paper that connects back to earlier themes in this thesis. Candlin notes that the interpretation of the path integral is altered as a conclusion of the work in this paper. He states:

> *"The presence of a-numbers seems to make it impossible to interpret the expression for the transition amplitude as a sum over trajectories. In our picture, we cannot say that the field has such a strength at such an intermediate time in a given classical trajectory, for obviously no a-number can be the result of a measurement."* (Candlin, 1956, p. 238)

The pretty picture at the beginning of this thesis where we sum over clear classical paths or field configurations is thus behind us. We are now summing of anticommutative numbers that do, as Candlin points out, not correspond to the numbers with which we represent measurement results of physical observables. All the way back in chapter 2.1 it was already noted that the classical limit interpretation is severely lacking, and in combination with the foregoing one may say that while the path integral demonstrably has many uses, the strength of a straightforward interpretation appears under further inspection not be one of them.

[41] Examples are to be found on the physics forum *StacksExchange* here `https://physics.stackexchange.com/questions/29475/what-happened-to-david-john-candlin` and Candlin's Wikipedia page here `https://en.wikipedia.org/wiki/David_John_Candlin`.

4.5 "The Feynman principle for a Fermi system" (1959) by J.L. Martin

Like Tobocman, almost nothing is known about J.L. Martin. At the time he was occupied with the subject at hand, he worked at the Tait Institute of Mathematical Physics in Edinburgh[42], to later move on to the Mathematics Division at the National Physical Laboratory in Teddington (London). His papers, published in the *Proceedings of the Royal Society A*, were communicated by Kemmer (the aformentioned doctoral advisor of Matthews and Salam). In these relevant works, Martin refers directly to Tobocman and Candlin, and aims to continue their line of work.

As one may note, we referred to 'papers' in the plural above. This is because Martin published his work in a 'part 1' and 'part 2' that immediately follow each other in the same journal, but are technically different papers (Martin, 1959a)(Martin, 1959b). As we will see, it makes sense content wise, but we care about the entire story. While the title of this subsection is that of the most directly relevant second paper, the first one is titled "generalized classical dynamics, and the 'classical analogue' of a Fermi oscillator". We will start with this first one and then flow into the second one.

Quantum physics is often thought (and taught!) to be a revolutionary departure from classical physics, where everything is turned on its head. While some key distinctions inevitably exist, others may upon inspection not be so 'strange'[43], and some 'strangeness' may be said to disappear if one looks at these theories through different formulations. It is another possible reason for Feynman's statement on the merit of knowing many formulations of the same thing at the beginning of chapter 2.2. Feynman's path integral is itself a good example, as it only requires the *classical* action and talks about physical paths, thereby arguably being closer to classical mechanics than its counterparts. Yet, perhaps an even better example can be found. In Hamiltonian classical mechanics, one of his equations of motion is that the time derivative of some observable quantity Q is equal to the Poisson bracket of that observable with the Hamiltonian of the system, i.e., $\frac{dQ}{dt} = (Q, H)$, where the normal brackets denote the Poisson brackets[44]. Heisenberg's equation of motion in his matrix mechanics formulation of quantum mechanics though, is $\frac{d\hat{Q}}{dt} = \frac{1}{i\hbar}[\hat{Q}, \hat{H}]$, with the rectangular brackets just being the familiar commutator. Therefore, we note that the jump from this particular formulation of classical mechanics to another particular one in quantum mechanics simply involves adding a factor $(i\hbar)^{-1}$ and replacing the Poisson brackets with a commutator. This is a rather simple quantization procedure, and certainly less radical than jumping from Newton's world of forces and $F = ma$ into Schrödinger's world of wave functions and $\hat{H}\psi = i\hbar\partial_t\psi$. We may say that Hamilton's equation of motion is the classical analogue of Heisenberg's equation of motion, where before we may have imagined that such an analogue did not exist.

[42] The institute is named after the very same Tait that was discussed in chapter 3!

[43] For example, the phenomenon of entanglement itself is not so strange. It is rather that the correlations between states are stronger than classically predicted that is strange.

[44] It is often the case that curly brackets are used for the Poisson brackets, but I (and, it turns out, also Martin) avoid these here because the same can be said for the anticommutator.

This small story is very relevant for the goal of Martin's first paper. In chapter 2.5 we discussed how half-integer spin objects are generally thought not to have classical analogues. Yet Martin, who has read Candlin's work, thinks that the Fermi oscillator there discussed very much does. This line of reasoning then eventually leads him to the formulation of a Feynman principle for fermionic fields.
Before we explain how he does this, let us first briefly remind ourselves of Tobocman's (who Martin has also read) distinction between the sum over histories and the Feynman principle. While Tobocman (and Candlin) carry out the sum over history procedure, they do not arrive at a Feynman principle for fermionic fields, i.e., they do not compute time-ordered correlation functions by from first principles taking an integral over fermionic fields with as its argument an exponential of the (integral over the) classical Lagrangian. If Martin is able to show that you can do that after all, and that that canonical quantization and path integral formulation can be connected even in the case of fermionic fields, that would be a big deal.

Let us consider Martin's first paper. He is very clear about what he wants to achieve in it, and wants to show that

> *"(a) it is possible to set up classical Hamiltonian dynamical systems in which fundamental sets of variables of canonical form cannot be found;*
> *(b) there exists a more general type of classical dynamics in which the fundamental variables are not c-numbers, but belong to some other ring, not necessarily commutative; and*
> *(c) such generalized systems can properly be regarded as 'classical analogues' of certain quantum systems, even though these quantum systems do not possess classical analogues in the accepted sense. This will be illustrated by the particular case of a Fermi oscillator."*
> (Martin, 1959a, p. 536)

We shall briefly consider the meaning of the above. In Hamiltonian mechanics, we are used to working with canonical coordinate pairs (q_1, p_1), (q_2, p_2), and so on. Martin thus claims in part (a) that such pairings are not possible in every physical system.
Claim (b) states that the complex numbers we usually work in are too restrictive to capture all classical physical systems. We should adopt a more general ring algebra, which will result in a 'dynamics algebra'. Elements of a ring, however, need not commute under multiplication, which is interesting for our fermionic purposes.
The final part then refers back to the earlier paragraph about classical analogues of quantum systems. Martin here claims that adopting the dynamics algebra will unlock descriptions of classical systems that will turn out to provide classical analogues to quantum mechanical systems thought not to have one, like the classical harmonic oscillator to the quantum harmonic oscillator.

Martin starts by briefly repeating the Poisson bracket and corresponding equation of mo-

tion in classical mechanics. He goes on to explain how one cannot always go from a Hamiltonian description to a Lagrangian one, in particular in cases where it is not possible to pair the canonical variables. That this is the case, is because going from 'H' to 'L' requires the inverse of a matrix[45], but this matrix is antisymmetric and is odd when the number of dynamical variables of the system is uneven. In the latter case, the matrix necessarily has a zero-determinant, i.e., it is singular, and singular matrices do not have inverses.
He also gives an example of this, which is that of a rotation system characterized by the three (=odd!) variables r_1, r_2 and r_3 with $(r_i, r_j) = \epsilon_{ijk} r_k$, where ϵ_{ijk} is the Levi-Civita symbol. The Hamiltonian may look like $H = \vec{r} \cdot \vec{\omega}(t)$ and the equation of motion found by entering this into Hamilton's equations is then $\dot{\vec{r}} = (\vec{r}, H) = \vec{\omega} \wedge \vec{r}$. This system, Martin states, then has no Lagrangian (Martin, 1959a, p. 537-538).

He now goes on to introduce his dynamics algebra. In particular, since he is interested in showing that the above system is actually the classical analogue of a Fermi system, he will use an anticommutative ring and refers to the system he will know develop as 'anticommuting classical dynamics'.
The dynamical variables here are defined as n symbols $\psi_1, ..., \psi_n$ that satisfy $\psi_i^2 = 0$ and $\psi_i \psi_j = -\psi_j \psi_i$. In this, we recognize Candlin's a-numbers.
One can make polynomials with complex coefficients out of these variables. What is new is that Martin defines explicit derivatives with respect to these anticommuting variables (Martin, 1959a, p. 539). He calls these 'partial antiderivatives' ∂_i. Since it is a linear operator we only need to state the rules for how it acts on monomials. It does so as follows:

$$(\partial_i \psi_j \psi_k \psi_l \cdots) = \delta_{ij} \psi_k \psi_l - \psi_j \delta_{ik} \psi_l + ... \tag{189}$$

$$(\psi_j \cdots \psi_m \psi_n) = \psi_j \cdots \psi_m \delta_{ni} - \psi_j \cdots \delta_{mi} \psi_n + ... \tag{190}$$

This definition makes intuitive sense, as it is just a product rule whereby every term will be zero unless the derivative ∂_i actually gets to act on ψ_i. More interestingly is that the whether we end up with a minus or plus sign now depends on the position of ψ_i in the monomial, as when interchanging two neighbouring ψ_k in the effort to move ψ_i to the derivate, will each time produce a minus sign. This is not something to worry about in the case of commuting complex numbers.

Martin moves on to lay out the Poisson bracket and equations of motion for this alternative classical dynamics, which is a straightforward modification of what he laid out at the start of the paper. However, due to the anticommutativity of the dynamical variables ψ_i, the matrix that was mentioned in footnote 45 is now guaranteed to have an even number of entries [*I don't always write down all of the math they do because it is not all equally relevant, but I may want to add a little proof here.*]. It is now possible to pair canonical variables, enabling us to

[45]The Poisson bracket is usually defined as $(f,g) \equiv \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$, but this can also be done in terms of a matrix. Martin writes $(u, v) = \sum_{r,s} \frac{\partial f}{\partial \phi_r} \alpha_{rs} \frac{\partial g}{\partial \phi_s}$, with the ϕ_i being the dynamical variables of the system and α_{rs} the matrix (also known as the 'Legendre matrix') under consideration.

always go from 'H' to 'L'.
Thus, he now returns to the example of the rotation system, this time with the three anticommuting variables ρ_1, ρ_2 and ρ_3 with $(\rho_i, \rho_j) = -\frac{1}{2} i\delta_{ij}$. Taking as the Hamiltonian $H = -i\omega_i(t)\rho_j\rho_k\epsilon_{ijk}$, the equations of motion become $\dot{\vec{\rho}} = (\vec{\rho}, H) = \vec{\omega} \wedge \vec{\rho}$. We now *can* formulate a Lagrangian through the usual procedure.

However, now another interesting part appears, as Martin attempts to quantize the above dynamics algebra. He does this precisely in the way described in the beginning of this subsection, by replacing the Poisson brackets with a commutator and tacking on $(i\hbar)^{-1}$. Since Martin adopts natural units ($\hbar = 1$), we find that $(q, p) = 1$ becomes $-i[\hat{q}, \hat{p}] = 1$. He now does this for the specific example of the anticommuting rotation system, where he replaces the Poisson brackets with an *anticommutator* instead, i.e., $(\rho_i, \rho_j) = -\frac{1}{2} i\delta_{ij}$ becomes $\{\hat{\rho}_i, \hat{\rho}_j\} = \frac{1}{2}\delta_{ij}$, where the minus sign disappears given the $-i$ factor involved in switching to the anticommutator. As it turns out, the matrix realization for $\hat{\rho}_i = \frac{1}{2}\sigma_i$, with σ_i a Pauli matrix. He states that the quantum system we now have has two independent states, it is a Fermi system.

Martin ends the first paper by laying out that this has achieved two things (Martin, 1959a, p. 542). For one, the rotating system that was introduced turns out to be a classical analogue of a Fermi system, i.e., against expectation, this exists after all. Secondly, the dynamics algebra employed by Martin makes it so that this classical rotation system has a Lagrangian formulation. Since we need the classical Lagrangian of a system to set up the Feynman principle with which its quantum version can be obtained, this means that the path integral representation for a Fermi system has been unlocked. How exactly this is done is what Martin describes in the second paper, that we now move on to.

Here, Martin starts by describing the Feynman principle as follows (Martin, 1959b, p. 543):

$$\langle q_1^{"} \ldots q_n^{"}, T | q_1^{'} \ldots q_n^{'}, 0 \rangle = \int_{q'}^{q^{"}} \delta q \int \delta\left(\frac{p}{2\pi}\right) e^{i\int_0^T (\sum_r p_r \dot{q}_r - H(q,p))dt} \tag{191}$$

He credits the above way of writing the path integral to Tobocman, who indeed writes the same thing (Tobocman, 1956, p. 1216). In the above $\delta q \equiv \mathscr{D} q(t)$, as the latter notation $\mathscr{D}$ was not yet introduced. While rather obvious, due to being so central to Martin's argument we repeat that the object $\sum_r p_r \dot{q}_r - H(q,p) = L(q, \dot{q})$, and wish to write a fermionic path integral in this way to realize the Feynman principle. Lastly, note that this is just the phase space formulation of the path integral (72).

Martin now introduces the canonical description of the Fermi system we aim to describe in the path integral formalism, namely

$$\{\psi_r, \psi_s\} = \{\pi_r, \pi_s\} = 0 \tag{192}$$

$$\{\psi_r, \pi_s\} = \delta_{rs} \tag{193}$$

$$\pi_r = \psi_r^\dagger \tag{194}$$

Like Candlin before, he wants to use the eigenvalues of a 'complete anticommuting set', but disagrees with Candlin who thought that this was impossible unless the identity operator is multiplied by some a-(Grassmann)-number. Martin points to the work of Schwinger. While the latter wanted to stay away from Feynman principles and did not take integrals over anticommuting quantities, he does some years earlier indeed already introduce the idea of complete anticommuting sets (Schwinger, 1953, p. 1284), which was after all also used by Matthews and Salam.
The above is all Martin claims to need to derive the Feynman principle of a Fermi system through a sum over histories procedure, thereby claiming to do what Tobocman and Candlin (who he cites explicitly with regard to this goal) did not (Martin, 1959b, p. 543).

To derive the desired result (and we will see why) Martin first needs a mathematical framework he calls 'eigensymbol theory' (Martin, 1959b, p. 544-545). He first introduces λ, an abstract symbol whose only property is that $\lambda^n = 0$. This symbol is used to provide an isomorphism to a usual column vector

$$|a\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \tag{195}$$

by writing this instead as

$$\sum_{r=0}^{n-1} a_r \lambda^r \tag{196}$$

Thus, one could say that in the language of eigensymbol theory, $|a\rangle$ now becomes $(\lambda|a\rangle$, where $(\lambda| = (1, \lambda, \ldots, \lambda^{n-1})$ is a row vector, where the curvy bracket is used to bring attention to the fact that its entries are not c-numbers as is the case with $|a\rangle$. Martin calls this the *"λ-representation of the vector $|a\rangle$ by the polynomial $(\lambda|a\rangle$"* (Martin, 1959b, p. 544). Similarly, we can write $\langle a|$ in the language of eigensymbol theory as $\langle a|\lambda)$, where we note that the $|\lambda)$ starts with λ^{n-1}, i.e., $\langle a|\lambda) = \sum_{r=0}^{n-1} a_r \lambda^{(n-1)-r}$.

With this basis established, two more useful operations in eigensymbol theory need to be laid out. First is multiplication by an nxn matrix $\hat{\Lambda}$ with ones on its subdiagonal, the rest being zero. For example if $n = 3$:

$$\hat{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{197}$$

Now consider the following expression:

$$(\lambda|\hat{\Lambda}|a\rangle = a_0\lambda + a_1\lambda + \ldots + a_{n-2}\lambda^{n-1} = \lambda(\lambda|a\rangle \tag{198}$$

This follows trivially from the above definitions and ordinary matrix multiplication. Note that the above expression should also yield a term $a_{n-1}\lambda^n$ in the sum, but $\lambda^n = 0$.
It follows that

$$(\lambda|\hat{\Lambda} = \lambda(\lambda| \tag{199}$$

Thus, while it is not, λ plays a role similar to that of an eigenvalue of the matrix operator $\hat{\Lambda}$. Similarly, one can find from the earlier introduced 'column vector' $|\lambda)$ that

$$\hat{\Lambda}|\lambda) = |\lambda)\lambda \tag{200}$$

With this in mind, the second useful operation to be defined is $\hat{S}_\lambda$. It is a linear operator that yields the coefficient in front of λ^{n-1} in any polynomial in λ. That is,

$$\hat{S}_\lambda \left(\sum_{r=0}^{n-1} a_r \lambda^r \right) = a_{n-1} \tag{201}$$

With this operator, we can derive an important result. Consider the product

$$\langle a|\lambda)(\lambda|b\rangle = \sum_{r=0}^{n-1} a_r b_r \lambda^{(n-1)-r} \lambda^r = \left(\sum_{r=0}^{n-1} a_r b_r \right) \lambda^{n-1} = \langle a|b\rangle \lambda^{n-1} \tag{202}$$

such that

$$\hat{S}_\lambda \langle a|\lambda)(\lambda|b\rangle = \langle a|b\rangle \tag{203}$$

Thus, it crucially follows that

$$\hat{S}_\lambda |\lambda)(\lambda| = 1 \tag{204}$$

And so, we find that $\hat{S}_\lambda$ acts like a sum over an 'eigenvector basis': Martin has a way to express a completeness relation like (46) over the abstract symbol λ. This will turn out to be important later.

With some basic eigensymbol theory covered, we will be able to see how Martin derives the Feynman principle for a Fermi system, introducing Graßmann numbers and implicit integration rules and comments on the way. To this end, he will reintroduce (192), explaining how he will now apply eigensymbol theory to these canonical anticommuting variables (Martin, 1959b, p. 546-548).
Martin introduces the dynamical variables for the Fermi system as:

$$\hat{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{205}$$

$$\hat{\pi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{206}$$

He now introduces an abstract symbol ψ whose property is that $\psi^2 = 0$. It is therefore like λ, except that the other n where it becomes zero is equal to 2. Similarly, a $2x1$ column vector

$|s\rangle$ whose elements are a and b is now written in eigensymbol theory as $(\psi|s\rangle = a + b\psi$.
It is now basically possible to establish all the same relations that we saw in the general case before, namely

$$(\psi| = (1, \psi), \quad |\psi) = \begin{pmatrix} \psi \\ 1 \end{pmatrix} \tag{207}$$

$$(\psi|\hat{\psi} = \psi(\psi|, \quad \hat{\psi}|\psi) = |\psi)\psi \tag{208}$$

$$\hat{S}_\psi(a + b\psi) = b, \quad \hat{S}_\psi|\psi)(\psi| = 1 \tag{209}$$

At this phase, a point of comparison with Tobocman and Candlin can be made. Unlike Tobocman, Martin remarks that he does not need to introduce hypervectors and hypermatrices for his anticommuting symbols (Martin, 1959b, p. 545). And unlike Candlin, the completeness relation in the final expression above is actually possible. According to Candlin, this was not the case, and the unit operator had to be multiplied by some a-number. This contradiction can be reconciled through the fact that $\hat{S}_\psi$ is not a sum as it is with Candlin. It is a different operation altogether, whose effect just resembles a sum when acting on $|\psi)(\psi|$. It should be noted that Martin also mentions that the first two of the above lines are already present in Candlin. This is indeed the case, due to Candlin's operators $\eta, \eta^\dagger$ and his eigenstates and their a-number eigenvalues $|a^{(*)}\rangle, a^{(*)}$ (Martin, 1959b, p. 546).

To complete the description of a Fermi system in eigensymbol theory though, Martin moves on to also introduce another symbol π that, in line with the anticommutation relations (192) he wants to apply eigensymbol theory to in the first place, anticommutes with ψ. The symbol itself naturally also has the property that $\pi^2 = 0$. Similarly, the same vector $|s\rangle$ can now be represented as $(\pi|s\rangle = a\pi + b$. Note, then, the interchange of the elements of $(\pi|$ compared to $(\psi|$. Repeating the above picture for ψ, we find:

$$(\pi| = (\pi, 1), \quad |\pi) = \begin{pmatrix} 1 \\ \pi \end{pmatrix} \tag{210}$$

$$(\pi|\hat{\pi} = \pi(\pi|, \quad \hat{\pi}|\pi) = |\pi)\pi \tag{211}$$

$$\hat{S}_\pi(a\pi + b) = a, \quad \hat{S}_\pi|\pi)(\pi| = 1 \tag{212}$$

An interesting relation Martin notes between ψ and π is that

$$(\psi|\hat{\pi}|s\rangle = \frac{d}{d\psi}(\psi|s\rangle \tag{213}$$

This is easily verified by, on the left-hand side, just doing matrix multiplications, and on the right side taking the derivative of $a + b\psi$. Thus, $\hat{pi}$ is represented by $d/d\psi$ in ψ-space representation. The analogy with the usual quantum mechanical $\hat{x}$ and $\hat{p}$ is not hard to see. It follows that one can also write

$$(\pi|\hat{\psi}|s\rangle = \frac{d}{d\pi}(\pi|s\rangle \tag{214}$$

Even more importantly for this derivation, Martin establishes that:

$$(\psi|\pi) = 1 + \psi\pi = e^{\psi\pi} \tag{215}$$

$$(\pi|\psi) = 1 + \pi\psi = e^{\pi\psi} \tag{216}$$

where we note that

$$1 + \psi\pi = 1 - \pi\psi = e^{-\pi\psi} = e^{\psi\pi} \tag{217}$$

The final line simply makes the anticommutating behavior of our symbols explicit. The first equality of the first two lines trivially follows from from matrix multiplication. More interestingly, the second equality follows since the expansion of $e^{\pm\psi\pi}$ terminates at second order since $\psi^2 = \pi^2 = 0$, and thus $1 \pm \psi\pi$ is all that is left.

Lastly, it should be stated that all of the above can, again in line with (192), be generalized to an arbitrary number of 'N' symbols ψ_k and π_k in a straightforward manner. For example, one will find that

$$(\psi_1 \ldots \psi_N|\pi_1 \ldots \pi_N) = e^{\sum_{k=1}^{N} \psi_k \pi_k} \tag{218}$$

and

$$\hat{S}_\pi \pi_1 \ldots \pi_N = 1 \tag{219}$$

where in case of the later, we note that $\hat{S}_\pi$ now picks out the coefficient in front of the full monomial $\pi_1 \ldots \pi_N$. All other terms in some polynomial in π, including something like $\pi_2 \ldots \pi_N$, will yield zero.

Martin now considers the transition amplitude (or "transformation function" as he calls it) in the spirit of Tobocman as at the start of the second paper. He writes that for *a small interval*[46] Δt we have:

$$(\psi', t + \Delta t|\psi", t) = \hat{S}_{\pi'}(\psi', t + \Delta t|\pi', t + \Delta t/2)(\pi', t + \Delta t/2|\psi", t) \tag{220}$$

Here, he has directly used $\hat{S}_\pi|\pi)(\pi| = 1$. This is not surprising, as the derivation of Feynman principles from sum over history approaches has until now always involved the use of such relations. Except in Martin's eigensymbol theory this is, unlike in the case of Candlin, now possible for the case of a fermionic system. Next, we have

$$\hat{S}_{\pi'}(\psi', t + \Delta t|\pi', t + \Delta t/2)(\pi', t + \Delta t/2|\psi", t) = \hat{S}_{\pi'} e^{\sum_k \psi'_k \pi'_k + \sum_k \pi'_k \psi"_k - i\Delta t H(\psi'_k, \pi'_k, t)} \tag{221}$$

Two relations are utilized in the above step. The first is that we note that

$$(\psi', t + \Delta t|\pi', t + \Delta t/2) = (\psi', t + \Delta t/2|(1 - i(\Delta t/2) H(\hat{\psi}_k, \hat{\pi}_k, t))|\pi', t + \Delta t/2) \tag{222}$$

$$= (\psi', t + \Delta t/2|\pi', t + \Delta t/2) e^{-iH(\psi'_k, \pi'_k, t)\Delta t/2} \tag{223}$$

[46] Supposedly Martin's 'ϵ', cf. chapter 2.

Where, due to having $\hat{\psi}_k$ and $\hat{\pi}_k$ act on their eigenvectors on the left and right respectively[47], we get the 'eigenvalue' symbols ψ and π, and thus an exponential with the classical Hamiltonian. The fact that the Hamiltonian in the above expression is followed by half of Δt rather than its double is due to the fact that other than $(\psi', t+\Delta t|\pi', t+\Delta t/2)$, we also get this exponential from $(\pi', t+\Delta t/2|\psi", t)$, and these add up.
The second relation used in the step is relation (218) above, which yields the exponentials with the sums. Martin's final step for the discrete case is then

$$\hat{S}_{\pi'} e^{\sum_k \psi'_k \pi'_k + \sum_k \pi'_k \psi"_k - i\Delta t H(\psi'_k, \pi'_k, t)} = \hat{S}_{\pi'} e^{\sum_k \psi'_k \pi'_k - i H(\psi'_k, \pi'_k, t)\Delta t} \tag{224}$$

Here, Martin seems to have first used the anticommutation relation between ψ_k and π_k so that $\sum_k \pi'_k \psi"_k = -\sum_k \psi"_k \pi'_k$. Then, he takes Δt 'outside of the brackets', like $A + \Delta t B = \Delta t(\frac{A}{\Delta t} + B)$. Subtracting the sums then results in $\sum_k \frac{\psi'_k - \psi"_k}{\Delta t}\pi'_k$ where we thus have a 'velocity' term in front of π'_k given the 'smallness' of Δt. This makes a lot of sense since if we want the Lagrangian, we will need something analogous to $\sum \dot{x}p - H$. Yet, Martin just writes ψ'_k for this 'velocity $\dot{\psi}_k$', probably at least partially since the 'time derivative of the abstract anticommuting symbol ψ_k' has not been given any sort of mathematical basis yet, let alone physical. Here, we shall honor Martin's original notation, yet this comment is good to keep in mind.
The last thing Martin does amounts to saying that $e^{ab} = ae^b$. This is the one step I just don't get, so will still have to look at it ($a = \Delta t$, $b = \sum_k \psi'_k \pi'_k - i H(\psi'_k, \pi'_k, t)$).

Bringing it all together, we have, in conclusion:

$$(\psi', t+\Delta t|\psi", t) = \hat{S}_{\pi'} e^{\sum_k \psi'_k \pi'_k - i H(\psi'_k, \pi'_k, t)\Delta t} \tag{225}$$

Which, Martin says, "on a suitable limiting process" yields the Feynman principle:

$$(\psi', T|\psi", 0) = \int_{\psi'}^{\psi"} \delta\psi' \int \delta\pi' e^{i\int_0^T L dt} \tag{226}$$

where,

$$L = i\sum_k \pi_k \psi_k - H(\psi_s, \pi_k, t) \tag{227}$$

We have now arrived at the main result. Before commenting, let us consider Martin's own concluding remarks after deriving it, as it contains some important remarks.

> *"The 'functional integrals' of [the result] are indefinitely repeated $\hat{S}_\psi$ and $\hat{S}_\pi$ operations; their resemblance to sums-over-histories is as superficial as the resemblance of $\hat{S}_\psi$ to a sum-over-eigenvalues. Nevertheless, there exists a complete theory of 'classical anticommuting dynamics' derived from the Lagrangian L (Martin, 1959a). It may be possible with the help of such a dynamics to reinterpret*

[47] Martin says a bit more about the algebra behind ordering the operators in $\hat{H}$ in the appendix such that the 'right' ones *can* add left and right, and shows with relative ease that this leads to the classical Hamiltonian (Martin, 1959b, p. 548-549).

[the result] as a sum-over-histories. This has yet to be considered." (Martin, 1959b, p. 458)

Martin goes on to provide the classical Lagrangian for the Fermi system that can be substituted into the result, viz., $L = i\pi\psi - \omega(\psi\pi - \frac{1}{2})$.

Now let us close by considering two aspects of Martin's above quote in particular. Firstly, since $\hat{S}$ was already seen to be working analogous to an ordinary sum, it makes heuristic sense that the limit thereof would turn into an integral. However, Martin is quick to note that it is, essentially, a fake integral, that ought not be taken literally. It is a wolf in sheeps' clothing, except it will turn out to be a very helpful wolf. While Berezin will be seen to take up this superficial symbol and give straightforward rules about how it can be computed for any given Lagrangian, Martin has already done so implicitly, as the $\hat{S}$ it is composed of is clearly defined. We will therefore see when discussing Berezin how his formal integral over anticommuting variables basically does exactly what Martin's $\hat{S}$ does. One could therefore say that other than the mainstream 'Berezin integral' or Mandelstam's 'Candlin integral', a case for the 'Martin integral' might just as well be made.
A second comment considers the interpretation that Martin is talking about. While most of the paper is an abstract exercise in manipulating the symbols of eigensymbol theory, he does connect it to the Fermi system at the end. Clearly, the algebra he here develops is well-suited to describe physical (fermionic) systems. Both his papers herein nicely come together. Fermi systems have classical analogues after all if anticommuting algebras are considered, and these lead to a classical Lagrangian that serve as the input for the path integral description of this quantum Fermi system, where we 'integrate over' these anticommuting dynamical variables. In this anticommuting algebra, called the eigensymbol algebra by Martin, we can indeed derive the Feynman principle for fermionic systems through the sum over histories procedure. In that sense, he finished what Tobocman started by considering that aside of an anticommuting algebra one also needs alternative operations (e.g. $\hat{S}$) to go alongside with it. The end point of this development and its physical application will become apparant in Berezin, where we will directly see fermionic fields being handled in this way. In any case, it is not hard to see why Martin is the most acknowledged of all the aforementioned authors by Berezin, as his contribution to the problem at hand above has been extremely important. With that, it is time to finally cover Berezin.

4.6 "Canonical operator transformation in representation of secondary quantization" (1961) by F.A. Berezin

At last (but not least), we have arrived at our subchapter on Berezin. As was laid out at the beginning of this chapter, he is the one commonly credited with the true formulation of using Graßmann numbers and integrating over them to solve the path integral's initial inability to address fermionic fields. As this subchapter will show, there is good reason for the historical narrative having turned out this way.

We will start with a brief introduction of Berezin the person. Together with Salam, he is probably the most well-known figure of all that have been covered in this chapter, and plenty has been written about him. We will then move on to dig into his work on the topic at hand, mainly through his short but dense article in the *Proceedings of the USSR Academy of Sciences* in 1961 (received in 1960, a year after Martin's articles), followed by his highly influential book containing a far more extensive and rigid mathematical basis of the theory, first published in Russian in 1965, with an English translation following the year after. This will also include the full realization of the derivation of (164).

The current paragraph is based on the brief scientific biographies about Berezin found in (Shifman, 2007) and (Karabegov et al., 2013). The former of these is a book chapter that was written by Minlos, who was well very acquainted with him (and refers to him by his nickname 'Alik').
Berezin was a mathematical physicist, and a key figure in the development of the (sub)field in the former USSR. His seminal works are to be found in a rigid development of (second) quantization and the introduction of supermathematics. While some words will be spent on this second pursuit in the next chapter, we here focus on his work on second quantization.
Although he performed well and was interested in mathematics from childhood onwards, Berezin, born to a Jewish mother, was initially denied an academic career on the basis of Stalinist state antisemitism[48]. Like Graßmann, whose works he would later take to new heights, he therefore spent the first years after his graduation as a high school teacher. Eventually though, due to the relative liberalization brought about by Khrushchev, and the advocacy of Gel'fand (whom Berezin had been a student of), he got a position at Moscow State University that he would hold for all his life. This position was at the chair of the 'theory of functions and functional analysis', so it is safe to say that he had a strong background in working with functionals. Gel'fand there also motivated Berezin to get into QFT. This set him on the path to his work on (second) quantization. The heart of Berezin's concept of quantization was to understand it as a mathematical deformation of the algebra of observables of classical physics, with Planck's constant being the deformation parameter. His work on this topic eventually led him to invoke Graßmann algebra to describe fermions in the theory of functionals, which in turn brought him to the creation of the aforementioned supermathematics to be discussed in the next chapter. The latter also left an influential legacy beyond just the 'Berezin integral' (Karabegov et al., 2013, p. 23-24).
While Berezin enjoyed an intellectually stimulating and collegial atmosphere at the mathematical seminars at the university, he unfortunately faced antisemitic discrimination from party bureaucrats above him. Despite his increasingly appreciated research output inside and outside of the USSR, he was withheld promotions, pay-raises and the opportunity to go on scientific trips abroad. Tragically, Berezin died young, drowning on a 1980 summer trip to east Siberia (Shifman, 2007, p. 47-50).

[48] Years earlier across the Atlantic, Feynman, also born into a Jewish family, similarly got denied entrance into Columbia University, based on quota for the number of Jews allowed to be accepted.

Let us now turn our attention to the highly relevant body of work Berezin has left us. Most about his work on fermionic path integrals can be learned from his 1965 book. On the one hand, this is for the positive reason that the book is an extensive and accessible body of work laying out, among many other things, the theory behind fermionic path integrals, and embedding it in a greater mathematical and physical project. On the other hand, there are the negative reasons that the 1961 article on the matter is only available in Russian as well as being rather short and dense. The former has been dealt with through use of the *DeepL* AI PDF-file translator, which while far from perfect from both a linguistic and typesetting perspective, at least makes the article somewhat decipherable. On the latter one may wonder why the article should be written in this way, even though the book shows Berezin can write much more clearly too. The answer to this is that it is a consequence of the journal *Doklady Akademii Nauk SSSR* (English: The Proceedings of the USSR Academy of Sciences) only allowing four pages (Karabegov et al., 2013, p. 10) for their articles. Nevertheless, some paragraphs will be spent on the article regardless. Not only did it precede the book by four years, it also already contains some novel results with respect to earlier articles in this chapter. Lastly, the article contains just one citation, namely that of Khalatnikov that we started with. Berezin after all, like Khalatnikov, is interested in 'continual integrals'.

The title of Berezin's article already lays out the goal: he wants to show how canonical transformations of operators can be represented in the method of second quantization. Canonical transformations just change the canonical coordinates one deals with in Hamiltonian mechanics. A straightforward example of classical physics is going from position and linear momentum to angular position and angular momentum. Such transformations are of course useful to be able to do in any mathematical formalism that aims to capture a physical theory. The 'method of second quantization' is, not coincidentally, also the title of Berezin's later book. It can, for now, be taken to mean that we wish to represent these transformations in the language of path (or 'continual') integrals. This is stated to be far more convenient than doing so in the orthodox operator formalism (Berezin, 1961, p. 311).

Berezin starts with the case for bosons, which he defines through operators $\hat{c}(\xi)$ and $\hat{c}^*(\xi')$ satisfying the commutation relation $[\hat{c}(\xi), \hat{c}^*(\xi')] = \delta(\xi, \xi')$, and then moves on to the case of fermions, which he defines through such operators satisfying the anticommutation relation $\{\hat{c}(\xi), \hat{c}^*(\xi')\} = \delta(\xi, \xi')$ instead. This is of course familiar, although one might note that the delta-function with two arguments is defined in terms of $\int \delta(\xi, \xi') f(\xi') d\xi' = f(\xi)$.
Considering the bosonic case, Berezin wants to move from the operator $A(c^*, c)$ in the language of functional integrals to its canonically transformed counterpart $\tilde{A}(a^*, a)$. We leave out the hat on A as the expression in terms of functional integrals will mean expression in terms of classical, non-operator quantities. He wants to look for this relation in the following form (Berezin, 1961, p. 312):

$$\tilde{A}(a^*, a) = \int K(a^*, a | c^*, c) A(c^*, c) \prod_{\xi} dc^*(\xi) dc(\xi) \tag{228}$$

We see how $A(c^*, c)$ and $\tilde{A}(a^*, a)$ are connected in a familiar way: by a path integral with the recognizable large product of integration variables and the integral kernel taking the operator in one pair of canonical coordinates to another. Given the canonical coordinates (a^*, a) and (c^*, c), the question that remains is how one finds the kernel. For bosons, it can be expressed as:

$$K(a^*, a|c^*, c) = (\det \pi \Psi^* \Psi)^{-1/2} e^{-\frac{1}{2}\left((\bar{\Phi}\Psi^{-1}b,b)+(\Phi\bar{\Psi}^{-1}b^*,b^*)-2(b^*,b)\right)} \tag{229}$$

Berezin proceeds to prove this result. This will not be copied here, as a somewhat similar and more important proof will be explicitly given here later. It suffices to know that $b(\xi)$ and $b^*(\xi)$ can be constructed as linear combinations of a, a^*, e.g., $b(\xi) = \int \Phi(\xi, \eta) a(\eta) d\eta + \int \Psi(\xi, \eta) a^*(\eta) d\eta - c(\xi)$, and that $(b_1, b_2) = \int b_1(\xi) b_2(\xi) d\xi$.

Next, he moves on to the fermionic case, and this is essentially where we will see the final ingredient for the completion of the fermionic path integral, the process of which we started with Khalatnikov, after which each subsequent author (not necessarily intentionally) took another step in this direction.
First, again, $\tilde{A}(a^*, a)$ is sought after in the form (228). This time, though, the kernel will be different. This is not surprising. Bosonic and fermionic operators behave differently, so $\tilde{A}(a^*, a)$ is unlikely be the same for both of these systems. Another major difference is that the integration variables are now anticommuting numbers. Rather than invoking some tricks to replace these with regular commuting c-numbers as we saw with earlier authors, Berezin bites the bullet and takes the radical step to explicitly define integration over these anticommuting numbers. He first introduces these numbers, but does not call them 'a-numbers' (Candlin) or 'elements of a noncommutative ring' (Martin). Rather, he refers to them as *generators of a Graßmann algebra* (Berezin, 1961, p. 314). For the first time, Graßmann is explicitly invoked, as Berezin recognizes the algebraic structure the numbers he requires can be coupled to. When covering his book, some more will be said on how he makes this connection. In any case, Berezin provides the anticommutation relation of these generators $x_1, \ldots, x_n$ as $\{dx_i, dx_k\} = \{dx_i, x_k\} = \{x_i, x_k\} = 0$. Since this automatically implies that $x_i^2 = 0$, we have our Graßmann numbers. Berezin also, like Martin, defines the left and right derivatives one can take of monomials of these variables. Lastly, he takes the novel step of directly defining an integration procedure over the Graßmann variables:

$$\int dx_k = 0\,, \qquad \int x_k dx_k = 1 \tag{230}$$

We will extensively discuss these 'integrals' in a minute, but let us first briefly conclude Berezin's article. After establishing them, Berezin uses them to derive the integral kernel in the fermionic expression for $\tilde{A}(a^*, a)$. He derives that (Berezin, 1966, p. 314):

$$K(a^*, a|c^*, c) = (\det \pi \Psi^* \Psi)^{1/2} e^{-\frac{1}{2}\left((\bar{\Phi}\Psi^{-1}b,b)-(\Phi\bar{\Psi}^{-1}b^*,b^*)+2(b^*,b)\right)} \tag{231}$$

Note that, compared to the bosonic kernel, some signs changed in the exponent. But, most importantly, we now have the inverse of the previous determinant in front of the exponential. As has been mentioned before, this is a key result distinguishing bosonic and fermionic

path integrals in QFT. We shall derive it later.
Berezin now concludes that he has successfully represented canonical transformations of operators as functional integrals, as he has shown how to do it for both bosons and fermions. He ends by remarking that the considerations of both cases are in this mathematical framework altogether almost identical(Berezin, 1966, p. 314).

Having gone through Berezin's 1961 article, and soon moving to his book where we can find a more extensive analysis of the path integrals over anticommuting fermionic fields here developed as well as the applications thereof, we first need a better understanding of the 'integrals' (230). While Berezin refers to them approximately as 'fermionic integrals over a Graßmann algebra', the above are now commonly known as 'Berezin integrals' or 'Graßmann integrals'. Much can be said about the above integration rules, and they lie at the basis for the derivation of equation (164). First, it is good to know that these are not your typical well-defined integrals that can be found all throughout any physics degree. The above identities are *defined* in this way, implying a certain freedom that is not there if one works with, e.g., the already rigidly defined Riemann integral. The symbol $\int$ here should therefore be taken to signify an *analogy*, rather than the real thing.

A next question may be why one would choose to define them in a way where they act just like derivatives. After all, the integral over 1 yields 0 and the one over x yields 1. Most (otherwise excellent) texts, including Berezin himself, do not comment on this much (e.g. (Grosche and Steiner, 2019, p. 56)), which may initially leave the reader with a feeling of arbitrarity. A good motivation, however, can be found in (Peskin and Schroeder, 2019, p. 299, 308), where the choice for this definition is linked to the in chapter 2.4 noted property that the path integral's functional measure $\mathscr{D}\phi$ is invariant upon a shift of integration variable $\phi(x^\mu) \rightarrow \phi'(x^\mu) = \phi(x^\mu) + \epsilon(x^\mu)$ with $\epsilon(x^\mu)$ an infinitesimal variation. Thus, $\mathscr{D}\phi = \mathscr{D}\phi'$ upon this shift. This may itself seem arbitrary on first hearing, but this property is essential in all kinds of path integral derivations in both the bosonic and fermionic case. This includes the derivation of conservation laws and the Schwinger-Dyson equations from the path integral.

Let us see how this shift-invariance demand concretely leads to the above definition of Berezin integration. First, note that since $x_k^2 = 0$, any polynomial in x_k will terminate after the linear order, and we can thus at most integrate over a linear formula $f(x_k) = a + bx_k$, with $a, b \in \mathbb{C}$. Since we can Taylor expand any function $f(x_k)$ into this linear polynomial, this is all that really needs to draw our interest. As one may note, the ease of manipulating the elements of the one-dimensional exterior algebra extends to these integrals as well. We thus want to

determine the integral[49]:

$$\int dx_k f(x_k) = \int dx_k (a + bx_k) \tag{232}$$

Now we invoke our demand of shift-invariance. We substitute $x_k \to x_k + x_i$ and thus, by our demand, we must find that

$$\int dx_k (a + bx_k) = \int dx_k (a + bx_k + bx_i) \tag{233}$$

The question now becomes: what linear function of a and b, resulting from this integral, has the property that such a shift leaves it unchanged? The only possible answer is the product of b with some constant (the latter of which we take as 1, as anything else would be redundant). Therefore, the imposed shift-invariance demand leads us to the conclusion that

$$\int dx_k (a + bx_k) = b \tag{234}$$

Now the only further assumption we need to come to (230) is linearity, so that we can say that

$$b = \int dx_k (a + bx_k) = a \int dx_k + b \int dx_k x_k \tag{235}$$

And thus, we have completed our motivation for the choice of definition (230).

The above can all be extended to a multiple integral which, as one may imagine, will certainly be of use when handling path integrals. The aforementioned ordering plays an important role here. We write

$$\iint dx_i dx_k x_k x_i = \int dx_i x_i = 1 \tag{236}$$

The important thing to note is that we start the integration with respect to the integration variable farthest *to the right* and, if the monomial in the integral contains this variable, it must be moved all the way to the left. These rules are essential for not getting the signs wrong. For example, had x_k and x_i above been interchanged, we would have had $\iint dx_i dx_k x_i x_k = -\iint dx_i dx_k x_k x_i = -1$. The same would of course hold had dx_k and dx_i been interchanged.
Let us summarize all these results for the case of two Graßmann variables x_k and x_i, with the most general (Taylor expanded) formula that can then be integrated over being $f(x_k, x_i) = a + bx_k + cx_i + dx_k x_i$, $a, b, c, d \in \mathbb{C}$, as follows:

$$\iint dx_i dx_k f(x_k, x_i) = \iint dx_i dx_k (a + bx_k + cx_i + dx_k x_i) \tag{237}$$

[49] One may note that while throughout this thesis, I have always put the function to be integrated over in front of the integration variable, this is not done below. The reason for the former is simply aesthetic preference, but from here on it will be more convenient and in line with general convention not do to this. This is because due to the anticommutativity of the symbols involved, ordering everything correctly will turn out to be very important, and it will be easier on both the reader and me to keep in line with existing convention on this.

$$= a \iint dx_i dx_k + b \iint dx_i dx_k x_k + c \iint dx_i dx_k x_i + d \iint dx_i dx_k x_k x_i \tag{238}$$

$$= 0 + b \int dx_i - c \int dx_k + d \int dx_i x_i = 0 + 0 + 0 + 1 = 1 \tag{239}$$

Where after the third equality, only the first-in-line of the two integrals is performed. While the above surely has an overly large number of intermediate steps, it is at least safe to say that no ambiguity in relation to this henceforth important operation remains. With the above, the extension to the case of n generators $x_1, \ldots, x_n$ of the Graßmann algebra follows trivially.
It is possible to also define complex Graßmann numbers, which come in handy when dealing with the Dirac field. They do, however, not change the above mathematics. As is typical in QFT, a dynamical variable and its complex conjugate can be treated as independent dynamical variables, so that in the case of a Graßmann variable z and its complex conjugate z^*, we just have $\iint dz^* dz\, zz^* = 1$. Lastly, we define $(z_k z_i)^* \equiv z_i^* z_k^*$, as with applying the dagger to a product of Hermitian operators.

More on what one can do with Berezin integrals will follow later. First, an interesting connection with the work of Martin can here be detected. He defined the *linear* operation

$$\hat{S}_\lambda \left(\sum_{r=0}^{n-1} a_r \lambda^r \right) = a_{n-1} \tag{240}$$

where $\hat{S}_\lambda$ was defined as picking out the coefficient in front of the λ^{n-1} symbol, the last nonzero one as $\lambda^n = 0$. When applying his eigensymbol theory to the Fermi system, this became

$$\hat{S}_\psi (a + b\psi) = b \tag{241}$$

thus picking out b, as this is now the coefficient of the last nonzero symbol, since $\psi^2 = 0$ in the Fermi system. He also showed this operation to be easily extendable to multiple symbols ψ_k, when one just needs a product of operators for each k. Lastly, we remember Martin's statement that *"The 'functional integrals' of [the results] are indefinitely repeated $\hat{S}_\psi$ and $\hat{S}_\pi$ operations..."* (Martin, 1959b, p. 458).

It is therefore clear to see that Martin effectively already introduced the Berezin integral, one of the inventions that made Berezin famous, in a different language. This is another argument for the possible case for the 'Martin integral'. In any case, it is easy to insist that the in general poorly known or credited Martin[50] is certainly deserving of more recognition than he got. Nevertheless, this is not to take away credit from Berezin, neither in person nor achievements. It is likely that while writing the '61 article, he was not yet aware of the work of Martin, as was also claimed in one of the biographical articles (Karabegov et al., 2013, p. 17-

[50] Unfortunately, multiple attempts to get in touch both with the *Tait Institute of Mathematical Physics* and the *National Physical Laboratory* of the UK, where Martin was active at the time of writing his papers, left me empty-handed. The result is that I do not even know the names his initials 'J.L.' are supposed to signify!

18)[51]. Once Berezin did learn about Martin, he cited and/or credited him (as well as other authors discussed in this chapter) on many occasions (Berezin, 1966, p. 225)(Berezin and Marinov, 1976, p. 9, 22)(Berezin, 1979, p. 1670). Moreover, aside from the independent discovery, Berezin contributed much more than just writing down the integral. He was the first to connect the work on anticommuting numbers attempting to describe fermionic systems to the larger algebraic structure of Graßmann algebra, and in his book he formulated a complete, consistent and extensive mathematical formalism for describing bosons and fermion using it. He also most clearly laid down the integration rules and concretely went on to use them to derive now elementary results such as (164). We can, therefore, simultaneously hold that, in particular, Martin deserves more credit, while continuing to believe the widely existing credit to Berezin's accomplishments is justified.

At this point it makes sense to also look at Berezin's book 'The Method of Second Quantization', to consider direct applications of his new integrals as well as get some grasp on the greater project Berezin is occupied with. For this, let us briefly recap the basis of second quantization.
As is well known, one step to get from QM to QFT involves the incorporation of special relativity. The other step is, of course, to take a different physical concept as the basis for quantization. In QM, the particle is essentially fundamental, and quantization entails the subjugation of particle properties such as x and p to canonical commutation relations. In QFT, fields, with their infinite number of degrees of freedom, make up the essential physical concept instead. The quantization procedure is now applied to canonical variables $\Phi(x^\mu)$ and $\Pi(x^\mu)$ of the classical field. This also explains the common understanding of second quantization as a misnomer. The 'second' in second quantization seems to imply that we are quantizing field variables after already having quantizated the 'particle variables', or, even worse, that the particle wave function now becomes an operator. In reality, it is now the classical field that serves as the quantization basis, rather than the classical particle. While Dirac came up with the term, the procedure comes from Ernst Pascual Jordon (1902-1980), who was an important figure in early QFT (Kuhlmann, 2020). It was later also developed by the famous Russian physicist Vladimir Aleksandrovich Fock (1898-1974), whose work Berezin, as one might have guessed, was certainly familiar with. He is taken as one of the starting points in the book (Berezin, 1966, p. vii-ix). We shall therefore briefly discuss the concept of a Fock space.

In QM, we generally want to know which particle is in which state. However, since all particles of the same type are fundamentally identical, we know that in QM we should not be able to distinguish between many-body quantum systems where two identical particles have been interchanged.[52] Thus, $|\Psi(\ldots,\vec{r}_i,\ldots,\vec{r}_j,\ldots)|^2 = |\Psi(\ldots,\vec{r}_j,\ldots,\vec{r}_i,\ldots)|^2$. Remember that it is

[51] In that article, which does describe Martin as a predecessor of Berezin and mentions his crediting of Martin, it is stated that *"Berezin learned about them [Martin's two articles] only around 1976"*. It is, however, likely that Berezin at least knew about him before that time, as Martin's first paper appears in the bibliography of Berezin's book (Berezin, 1966, p. 225).

[52] The reality is a bit more complicated than this argument. Nevertheless, due to its familiarity, we shall stick to it for now.

only the square of the absolute value of the wave function that is measurable[53]. Therefore, as for the wave function itself, it is allowed to either pick up a plus or minus sign upon exchange. Many of us are introduced to the concept of 'bosons' and 'fermions' by learning that bosonic identical particles will in this situation have the plus sign, while for fermions one we are left with a minus sign instead. Thus, we must be careful that any many-body wave function is constructed in such a way to obey these respective symmetrization requirements. In atomic physics one often learns that the permanent here comes in handy for bosons, while the Slater determinant does the job for fermions (Griffiths, 2014, p. 203-207).

In QFT, one is no longer interested in which state a particle is, as we are dealing with fields. Keeping in mind that we can view fields as an infinite set of oscillators, what *is* still interesting is the number of field excitations of particular energy level are present. In this sense we may still speak of particles. However, due to this description doing away with individuality of particles of shared typing entirely, we no longer need cumbersome symmetrization procedures to account for the nonexistence thereof. It is only the *number* of particles in a given state that is of interest.

This is where Fock states and the Fock space come in (Fock, 1932). A Fock state can be written as follows:

$$|n_1, n_2, \dots n_k, \dots\rangle \tag{242}$$

It means that there are n_1 particles in the state $|1\rangle$, n_2 particles in the state $|2\rangle$ and, in general, n_k particles in the state $|k\rangle$. One can therefore see why the Fock state is sometimes also referred to as the 'occupation number state'. The Fock state is associated with a particular particle number $N = \sum_k n_k$. This is, therefore, a very simple way to write down how many particles we have in what state.
It is also possible to immediately differentiate between bosons and fermions through Fock states in the following straightforward way: for bosons, $\forall_k n_k \in \mathbb{N}$ and for fermions, $\forall_k n_k \in \{0, 1\}$, i.e., we impose the Pauli exclusion principle by noting that in the case of fermions no more than one particle can be in any specific state $|k\rangle$.

Now of course in QFT, we are interested in the case where the number of particles is not a constant N, but can be changed. Essential to the second quantization formalism is therefore the manipulation of Fock states through creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ acting on them to change the particle numbers. For example,

$$\hat{a}_k^\dagger |n_1, n_2, \dots n_k, \dots\rangle = \sqrt{n_k + 1}\,|n_1, n_2, \dots n_k + 1, \dots\rangle \tag{243}$$

and

$$\hat{a}_k |n_1, n_2, \dots n_k, \dots\rangle = \sqrt{n_k - 1}\,|n_1, n_2, \dots n_k - 1, \dots\rangle \tag{244}$$

One could of course introduce specific bosonic and fermionic creation and annihilation operators, but for now grasping the general principle will do. We just need to keep in mind the

[53] Not in one measurement of course, but it will reveal itself after many of them.

difference that comes about when applying these operators for bosons and fermions. In case of the former, one can apply indefinitely many creation operators, but applying $\hat{a}_k$ to a Fock state where $n_k = 0$ will not result in another Fock state. There is, after all, nothing to annihilate in this case, and negative integers do not make sense here. In the case of fermions, all we need to add is that having creation operators $\hat{a}_k^\dagger$ act on states where $n_k \neq 0$ (in practice, $n_k = 1$) will also not yield another Fock state.

The idea of using these Fock states in combination with having creation and annihilation operators act upon them, leads to the concept of a Fock space. As we are well aware, we can describe quantum mechanical particles through vectors in a Hilbert space. Consider a one-particle Hilbert space of some type. To it correspond Fock states such as $|1,0,0,\ldots\rangle$, $|0,1,0,\ldots\rangle$, and so on. When we take the tensor product of the Hilbert space with itself, we are considering the space of two of such identical particles. This now corresponds to Fock states like $|1,1,0,\ldots\rangle$, $|1,0,1,\ldots\rangle$, and so forth.
Of course, in the above case one problem arises. In the case for two identical bosons, the space $H \otimes H$ should contain Fock states such as $|2,0,0,\ldots\rangle$. Yet, as established before, it should not in the case of identical fermions. The trick is to add an operator $\hat{S}_\nu$ that symmetrizes the tensor product of the Hilbert spaces if $\nu = b$, where 'b' stands for 'boson', and antisymmetrizes if $\nu = f$, where 'f' now predictably stands for 'fermion'. The result is that the space now only allows for admissable Fock states.
Another note is that we can define a zero-particle Hilbert space corresponding to the empty Fock state $|0,0,0,\ldots\rangle$ as just being the field of complex numbers. But with that, we have all the ingredients for our Fock space, and we can write it simply as

$$F_\nu(H) = \mathbb{C} \oplus H \oplus (\hat{S}_\nu(H \otimes H)) \oplus (\hat{S}_\nu(H \otimes H \otimes H)) \oplus \ldots \tag{245}$$

Here, $F_\nu(H)$ is the bosonic or femionic Fock space.
Therefore, the Fock space is the direct sum of the Hilbert space corresponding to n number of identical bosons or fermions, with that of the one corresponding to $n+1$ number of them, and so on. Its elements are therefore linear combinations of Fock states with different particle numbers, e.g., $2|1,0,0,\ldots\rangle - 3i|0,4,9,\ldots\rangle + (9-16i)|1,1,1,\ldots\rangle \in F_b(H)$, where in this example we note that this is a bosonic Fock space. We see that the Fock space is ideally suited for QFT, as it is a good way to deal with situations where the number of particles is not conserved, as it paradigmatically the case in QFT. With this, we know all we need to about the second quantization formalism to understand the background to Berezin and his book.

A final note before ending our Fock-prelude though is the suspicious structural equivalence of the above definition of the Fock space and the definition of the tensor algebra (157) in the previous chapter. There too, we start with a field and subsequently take the direct sums of tensor products of vector spaces. In that case, we got the Graßmann algebra by looking at a particular subset of the tensor algebra, just like here we get the fermionic Fock space by setting $\nu = f$ for the unspecified Fock space. That Graßmann algebra just so happens to be ideally suited as a mathematical structure for describing fermions, one of Berezin's central

theses, is therefore not entirely a coincidence, as one may already have had such suspicions based on the mathematical formulations of these concepts above.

With the context above, Berezin's title now makes more sense: he wants to discuss the method of second quantization from his point of view as a mathematical physicist. In the prelude already, he explains that it took a while for the method to become more popular, due to some remaining mathematical problems. These were, however, not the typical mathematical problems associated with physics in the form of differential equations. Algebraic topics are far more important and it is questions on them, from Lie group representations to measure theory, that compose the core of how Berezin discusses second quantization (Berezin, 1966, p. vii).
Earlier it was described how Berezin had an extensive background in functional analysis. One core message of his book may be said to be that the answer to the mathematical difficulties of the second quantization formalism is to realize the Fock space in terms of functionals (Berezin, 1966, p. viii). Operators and Hilbert space vectors are to be expressed using functionals, something we actually already saw happening in (228) and also happens all throughout this book (Berezin, 1966, p. 4). He introduces bosons as functionals in the space that have as their input regular functions of the complex numbers, while for fermions we have anticommuting functions instead. In Berezin's methodology, the boson-fermion distinction is everything but some afterthought past already having introduced the basic content of his mathematical underpinnings of quantum theory. Rather, they are an essential starting point, and I would argue that Berezin's work lends itself well for an ontological view wherein the boson-fermion categorization can be viewed as the most fundamental natural kinds for physical entities.

Berezin mentions two distinct advantages of realizing a Fock space in terms of functionals. The first is that these functionals do not just form some linear space, but also a multiplicative ring, simplifying computations. Specifically, multiplications of functionals corresponding to bosons commute, and to fermions anticommute, such that we have a commuting and anticommuting ring. While these are different, *"all the basic formulas for them surprisingly show an almost complete coincidence"* (Berezin, 1966, p. viii), which we will see later in the case of Gaussian path integrals. Moreover, the anticommutative ring for fermions will, as may be suspected, be a Graßmann algebra. A second advantage of Berezin's method has to do with the infinite number of degrees of freedom typical for QFT. This is the fact that a functional can be expressed as a function of an infinite number of variables. To get an intuitive feel for this, one may remember how in chapter 2 the functional $S[x(t)]$ is, through the discrete lattice regularization process, effectively turned into a function $S(x_1, \ldots, x_N)$ where at the end $N \to \infty$. This also allows for the interpretation of quantum mechanical problems with an infinite number of degrees of freedom (Berezin, 1966, p. viii).

In the next few chapters (sub)chapters Berezin works out his program, in part by finding functional representations of operators. For example, he expresses Fock states in terms of

functional as (Berezin, 1966, p. 5):

$$\Phi_{n_1,\dots,n_k,\dots} = \frac{1}{\sqrt{n_1!\cdots n_k!\cdots}}(a_1^*)^{n_1}\cdots(a_k^*)^{n_k}\cdots \tag{246}$$

While the coefficient is related to the ordering, the most interesting part above are the factors $(a_k^*)^{n_k}$ for some k witih $n_k \in \mathbb{N}$. These are (complex-valued or Graßmann-valued) functions of a Hilbert space, with n_k the number of identical particles in the state carrying that label. For fermions, $(a_k^*)^{n_k} = 0$ if $n_k > 1$.
Berezin also rigidly defines the continual integral, which was a more abstract and general version of the path integral introduced when discussing Khalatnikov (Berezin, 1966, p. 38).

After setting up the basics like those above, he moves on to apply these concepts to bosons and fermions, after which the next chapters consider how one applies this rigid and polished version of second quantization to linear canonical transformations, quadratic operators and Thirring's four-fermion model. In the former part on bosons, we also see Berezin use his methods to derive the bosonic Gaussian path integral that we saw in chapter 2.4 (Berezin, 1966, p.41-42).
When he gets to fermions, he (like in the paper) starts by introducing the Graßmann algebra with n generators $x_1,\dots,x_n$. Here, he cites the work of Vivier who was noted in the previous chapter for playing a role in developing Graßmann algebra in its modern form. He lays out the basics and then moves on to developing calculus on the Graßmann algebra, starting with introducing the left and right derivatives in the same way we saw with Martin (189). He goes a bit further though, by also introducing the Graßmann variants of concepts such as the chain rule or higher-order derivatives (Berezin, 1966, p. 51-52). He moves on to discuss the integrals on the Graßmann algebra (230), but again goes further by also considering, e.g., integration by parts.

Crucially, Berezin goes on to compute the Gaussian integral over Graßmann variables right after introducing these new techniques, formulating it as (Berezin, 1966, p. 56-57)

$$I = \int e^{\sum a_{ik}x_i x_k} dx_n \cdots dx_1 \tag{247}$$

Here, a_{ik} is a real, antisymmetric matrix, and the x_i are again the Graßmann variables.
As we are well aware, the computation of these integrals is incredibly important to our story, as being able to do so means that we can use perturbation theory for path integrals over fermionic fields. Berezin goes over it rather quickly, but do the centrality of this effort for our story with it being the 'endpoint' (164) formulated at the beginning of this chapter where we can finally do path integrals over fermionic fields, as well as this derivation when looked at in detail being an excellent demonstration of how the Berezin integral can be applied, the full derivation will be worked out here in detail. We will now turn our attention to this, after which some concluding remarks on the current chapter will follow.

As we want to prove (164), we will use that notation, as it is equivalent to Berezin's with just some differences in which (Greek) letters are used. Other than that, all that is needed for the derivation are the Berezin integration rules we discussed above (latter half of the derivation) and some messy but basic linear algebra required to show that the integral is invariant under unitary transformations of its variables (earlier half). Thus, we are to calculate the following integral over Graßmann variables $\theta_1, \theta_1^*, \ldots, \theta_n, \theta_n^*$:

$$I = \prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i^* B_{ij} \theta_j} \tag{248}$$

Here, the product operator Π acts again on what is in the brackets, so that we end up with $2n$ integrals over Graßmann variables.
Since B is an nxn Hermitian matrix, it follows from the spectral theorem that it can be diagonalized. We can write this as the unitary transformation $D = U^{-1}BU$, with D the diagonalized and U a unitary matrix so that $U^\dagger = U^{-1}$. Given the sums in the exponent above, we write $B_{ij} = \sum_{k=1}^{n} U_{ki} \lambda_k U_{kj}$. Here, $\lambda_1, \ldots, \lambda_n$ are the diagonal entries of D, i.e., they are the eigenvalues of B.
The exponent in I will get quite messy now, as substituting this B_{ij} will result in

$$\prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i^* B_{ij} \theta_j} = \prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \theta_i^* U_{ki} \lambda_k U_{kj} \theta_j} \tag{249}$$

Note that $U_{\alpha\beta} \in \mathbb{C}$, so these are just commuting numbers and we can make the following substitutions:

$$\psi_k = \sum_{j=1}^{n} U_{kj} \theta_j, \qquad \psi_k^* = \sum_{i=1}^{n} U_{ki} \theta_i^* \tag{250}$$

And with that we get the not so messy

$$\prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \theta_i^* U_{ki} \lambda_k U_{kj} \theta_j} = \prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{k=1}^{n} \psi_k^* \lambda_k \psi_k} \tag{251}$$

However, while the exponent is now simplified significantly, the change of variables that took also means that the integration variables must change[54]. If we write the above substitutions as a matrix equation $\psi = U\theta$ (and similarly for ψ^*), we have, equivalently, $\theta = U^{-1}\psi = U^\dagger \psi$. Thus, we find that

$$d\theta_l = \sum_{j=1}^{n} U_{jl} \, d\psi_j, \qquad d\theta_l^* = \sum_{i=1}^{n} U_{il} \, d\psi_i^* \tag{252}$$

Changing the integration variables now yields

$$\prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{k=1}^{n} \psi_k^* \lambda_k \psi_k} = \prod_{l=1}^{n} \left(\int \sum_{i=1}^{n} \sum_{j=1}^{n} U_{il} U_{jl} \, d\psi_i^* \, d\psi_j \right) e^{-\sum_{k=1}^{n} \psi_k^* \lambda_k \psi_k} \tag{253}$$

[54] One may rightfully wonder if substitution of integration variables is allowed for integration on the Graßmann algebra. Berezin addresses just this case and, not coincidentally, states that this is perfectly fine (Berezin, 1966, p. 55).

Note that if we let the product operator act on the brackets, we also get $2n$ sums

$$\prod_{l=1}^{n}\left(\int\sum_{i=1}^{n}\sum_{j=1}^{n}U_{il}U_{jl}d\psi_i^*d\psi_j\right)e^{-\sum_{k=1}^{n}\psi_k^*\lambda_k\psi_k}= \tag{254}$$

$$\int\left(\sum_{i_1=1}^{n}\sum_{j_1=1}^{n}U_{i_11}U_{j_11}d\psi_{i_1}^*d\psi_{j_1}\right)\cdots\left(\sum_{i_n=1}^{n}\sum_{j_n=1}^{n}U_{i_1n}U_{j_1n}d\psi_{i_n}^*d\psi_{j_n}\right)e^{-\sum_{k=1}^{n}\psi_k^*\lambda_k\psi_k} \tag{255}$$

At this point one may wonder how we get out of this mess of $2n$ sums and start applying the Berezin integral. Below I explain how, in a semi-qualitative way where I somewhat sacrifice the rigor of strict symbolic manipulation but hopefully create an easier intuitive understanding in line with the goal of comprehensibility.
Consider the expression above. In each pair of brackets, we will have a sum of n^2 terms. Multiplying all the pairs of brackets will then result in n^{2n} terms. Luckily, most of these are zero, for the following reason. Consider the term associated with $i_1 = r$ and $j_1 = s$ ($r,s \in \{1,\ldots,n\}$) in the first pair of brackets. This term will be $d\psi_r^*d\psi_s$ with some complex number coefficient. Now note that if we multiply out this particular term with all the other terms in the $n-1$ other pairs of brackets and look at the resulting integration variable monomials, most of these will contain another $d\psi_r^*$ or $d\psi_s$. Concretely, there will be far fewer monomials $d\psi_{s_1}d\psi_{s_2}\cdots d\psi_{s_n}$ where none of the subscripts will coincide, than those where at least one pair does. Yet, every time that does happen, the whole thing is zero. For example, $d\psi_4d\psi_{19}d\psi_4\cdots = -d\psi_4d\psi_4d\psi_{19}\cdots = -(d\psi_4)^2d\psi_{19}\cdots = 0$, with a minus sign in front due to the permutation of $d\psi_{19}$ and $d\psi_4$. After all: Graßmann variables square to zero. In conclusion: only a sum of terms where every integration variable subscript in the set $\{1,\ldots,n\}$ appears exactly once remains. The coefficients of these terms will be products of the complex number unitary matrix coefficients.
The next realization is then that using the anticommutativity of the Graßmann integration variables, all of these terms can be neatly ordered as $d\psi_1^*d\psi_1d\psi_2^*d\psi_2\cdots d\psi_n^*d\psi_n$. Two things here are important to note. Firstly, the thing in its totality can now be written as a sum of products of matrix elements, i.e., $(A-B+\ldots)d\psi_1^*d\psi_1d\psi_2^*d\psi_2\cdots d\psi_n^*d\psi_n$, where A and B look like, e.g., $A = U_{51}U_{91}U_{22}U_{71}\cdots$. Secondly, whether we find a plus or minus sign in front of coefficients like A and B, depends on whether an even or odd number of permutations of integration variables was required to produce the neatly ordered monomial above, respectively. Thus, rather than writing the coefficients as a messy sum with pluses and minuses, the sum of coefficients can neatly be expressed through a sum operator with as two of its arguments the Levi-Civita symbols $\epsilon_{i_1\cdots i_n}$ and $\epsilon_{j_1\cdots j_n}$.
Incorporating this story into our expression leaves us with

$$\int\left(\sum_{i_1=1}^{n}\sum_{j_1=1}^{n}U_{i_11}U_{j_11}d\psi_{i_1}^*d\psi_{j_1}\right)\cdots\left(\sum_{i_n=1}^{n}\sum_{j_n=1}^{n}U_{i_1n}U_{j_1n}d\psi_{i_n}^*d\psi_{j_n}\right)e^{-\sum_{k=1}^{n}\psi_k^*\lambda_k\psi_k}= \tag{256}$$

$$C\int d\psi_1^*d\psi_1d\psi_2^*d\psi_2\cdots d\psi_n^*d\psi_ne^{-\sum_{k=1}^{n}\psi_k^*\lambda_k\psi_k} \tag{257}$$

where

$$C = \left(\sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} U_{i_1 1} \cdots U_{i_1 n} \, \epsilon_{i_1 \cdots i_n} \right) \left(\sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} U_{j_1 1} \cdots U_{j_1 n} \, \epsilon_{j_1 \cdots j_n} \right) \tag{258}$$

Here, C has consciously been written in a suggestive form, as in the brackets we find *the definition of the determinant of an nxn matrix*. We have

$$C = \det(U^\dagger) \cdot \det(U) = \det(U^\dagger U) = \det(I) = 1 \tag{259}$$

Here, the unitary of U, the fact that we integrate over pairs of Graßmann variables and their conjugates, and general relations from linear algebra were used. Thus, we can conclude that

$$\prod_{l=1}^{n} \left(\int d\theta_l^* \, d\theta_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i^* B_{ij} \theta_j} = \prod_{l=1}^{n} \left(\int d\psi_l^* \, d\psi_l \right) e^{-\sum_{k=1}^{n} \psi_k^* \lambda_k \psi_k} \tag{260}$$

In a nutshell, we have established that the integral is invariant under unitary transformations of the variables. With it established, we can now get to the 'second half' where the actual integration over the Graßmann variables happens. The process will turn out to be rather easy, due to the convenient properties of Graßmann numbers.

As a first step, we use the Taylor series for the exponential in the integral. Before doing so, we interchange the ψ_k^* and ψ_k in the exponent to get rid of the minus sign, making the series expansion a bit simpler. We find that

$$\prod_{l=1}^{n} \left(\int d\psi_l^* \, d\psi_l \right) e^{-\sum_{k=1}^{n} \psi_k^* \lambda_k \psi_k} = \prod_{l=1}^{n} \left(\int d\psi_l^* \, d\psi_l \right) \sum_{m=0}^{\infty} \left(\frac{\left(\sum_{k=1}^{n} \psi_k \lambda_k \psi_k^* \right)^m}{m!} \right) \tag{261}$$

To greatly simplify the above, let us consider the object $\sum_{k=1}^{n} \psi_k \lambda_k \psi_k^* = \lambda_1 \psi_1 \psi_1^* + \ldots + \lambda_n \psi_n \psi_n^*$. This is the case $m = 1$. Now consider that in the case $m = 2$ we have $(\lambda_1 \psi_1 \psi_1^* + \ldots + \lambda_n \psi_n \psi_n^*)^2$. We therefore get n^2 terms with each term (aside from a complex coefficient that is the product of eigenvalues) having some combination $\psi_i \psi_i^* \psi_j \psi_j^*$, $i, j \in \{1, \ldots, n\}$. This way of thinking helps one to see the following three cases.

- $m < n$: In this case, we get monomials of Graßmann variables that are integrated over the integration variables $d\psi_1^* d\psi_1 \cdots d\psi_n^* d\psi_n$. Yet, because we have a product of less Graßmann variables than there are integration variables, these monomials will always miss at least one Graßmann variable ψ_r that *is* present among the product of integration variables $d\psi_1 \cdots d\psi_r \cdots d\psi_n$ (neglecting the conjugate counterparts for brevity). But since the Berezin integral is defined as $\int d\psi (a + b\psi) = b$, and in such cases we effectively have $b = 0$, this will just render the integral zero.

- $m > n$: In this case, $(\psi_1 + \ldots + \psi_n)^m$ (again neglecting eigenvalues and conjugate counterparts for brevity) will produce monomials where one Graßmann variable appears at least twice. This results in a squared variable, which is defined to always be zero. Therefore, all entries of the sum over m in this range disappear.

- $m = n$: Given the two earlier entries we realize that in this case there will be some number of monomials with both enough Graßmann variables for each to be integrated over *and* not too much so that the monomial is zero because of the presence of more than one of a particular variable in the monomial. Thus, let us rewrite the above formula for this case and go further from here.

$$\prod_{l=1}^{n}\left(\int d\psi_l^* d\psi_l\right)\sum_{m=0}^{\infty}\left(\frac{\left(\sum_{k=1}^{n}\psi_k\lambda_k\psi_k^*\right)^m}{m!}\right)=\int d\psi_1^* d\psi_1\cdots d\psi_n^* d\psi_n\frac{(\psi_1\lambda_1\psi_1^*+\ldots+\psi_n\lambda_n\psi_n^*)^n}{n!} \tag{262}$$

The object $(\psi_1\lambda_1\psi_1^*+\ldots+\psi_n\lambda_n\psi_n^*)^n$ contains two pieces of information we still need. First, for reasons clear from the above list, we need to know how many of the n^n monomials this object produces contains precisely one of each of the n Graßmann variables. Imagine that one has n baskets, each with n differently colored marbles. One wonders in how many ways it is possible to take just one marble from each basket, and end up with n differently colored marbles at the end. For the first basket, it we have n options, as it does not matter yet which we pick. For the second basket, we have $n-1$ options, as it is only the color that we already picked from the previous basket that we do not want from this one. Given the two already taken colors, there are $n-2$ options left for the third basket, and so on. Since the 'baskets' are just the (n-times multiplied) sums over k, and the 'marbles' are just the Graßmann variables with the n'colors' being the n subscripts, the situations are identical, and we conclude that there are $n!$ of the monomials we are looking for. Conveniently, this $n!$ cancels against the $\frac{1}{n!}$ in our expression.
But hold on! Even if we have $n!$ monomials, we cannot just add them up to get a $n!$ coefficient, as these monomials will not all be neatly ordered like $\psi_1\lambda_1\psi_1^*\psi_2\lambda_2\psi_2^*\cdots\psi_n\lambda_n\psi_n^*$. All possible orderings of the subscripts will appear. Of course we interchange these variables using the anticommutation relation, but if the number of permutations is odd this will leave the total resulting monomial with a minus sign, and we will not get $n!$ as a coefficient.
Luckily, there is a second important piece of information we can draw from the object $(\psi_1\lambda_1\psi_1^*+\ldots+\psi_n\lambda_n\psi_n^*)^n$. Namely, because all individual terms $\psi_r\lambda_r\psi_r^*$ contain *two* Graßmann variables, and we always want to move these terms as a whole, *there are no even permutations in this case*. In other words, these terms as a whole commute, i.e., $(\psi_r\lambda_r\psi_r^*)(\psi_s\lambda_s\psi_s^*)=(\psi_s\lambda_s\psi_s^*)(\psi_r\lambda_r\psi_r^*)$. More generally, if a,b,c,d are Graßmann numbers, then $(ab)(cd)=-acbd=cabd=-cadb=(cd)(ab)$. Therefore, in general, products of two Graßmann numbers commute. The consequence here is that all of the $n!$ monomials and their different ordering can be ordered in any way we want as long as we respect the terms they are made up of. This means that rather than having to do integrals for $n!$ monomials, we only have to do them for one with a coefficient $n!$ in front that cancels with its already present inverse.

Thus, we can now write

$$\int d\psi_1^* d\psi_1\cdots d\psi_n^* d\psi_n\frac{(\psi_1\lambda_1\psi_1^*+\ldots+\psi_n\lambda_n\psi_n^*)^n}{n!}=\int d\psi_1^* d\psi_1\cdots d\psi_n^* d\psi_n\psi_n\lambda_n\psi_n^*\cdots\psi_1\lambda_1\psi_1^* \tag{263}$$

Where the terms have been ordered such that we can directly apply (236). Therefore, what we end up with is simply the product of all the λ's

$$\int d\psi_1^* d\psi_1 \cdots d\psi_n^* d\psi_n \psi_n \lambda_n \psi_n^* \cdots \psi_1 \lambda_1 \psi_1^* = \prod_{k=1}^{n} \lambda_k \tag{264}$$

Let us remember that these are the eigenvalues of the nxn Hermitian matrix B. Moreover, the determinant of a matrix is the product of its eigenvalues. With that, we arrive at our final result:

$$\prod_{l=1}^{n} \left(\int d\theta_l^* d\theta_l \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i^* B_{ij} \theta_j} = \det B \tag{265}$$

This is, of course, precisely (164).

While the above has been somewhat lengthy and detailed due to it being intended as a first exposition for starting Master's students, one may note that the underlying rules are, in fact, rather easy, and one will probably be quick to get used to them. Getting used to them is important, as in QFT, a one-dimension Graßmann algebra and how Gaussian integrals can be done in terms of that is all that is needed to correctly and reliably derive propagators of fermionic fields. After all, free fermion propagators require the computation of Gaussian integrals like we just handled. With these Gaussian (free propagator) integrals lying at the basis of perturbation theory, this result is essentially the key to unlock the full power of perturbative methods of fermionic fields, including the Feynman diagrams that can be derived from them.
Moreover the aforementioned near complete coincidence of the basic formulas for bosons and fermions that Berezin spoke of can be perfectly exemplified here. After all, the Gaussian integral over bosonic fields computed in chapter 2.4 results in the inverse of the above result for fermionic fields. This is a quite remarkable result. In fact, symmetries like this inspired Berezin's program of 'supermathematics', although this is a story for the next chapter.

For now, let us round off by concluding that Berezin's contribution to the problem of fermionic path integrals has been enormous, and marks the endpoint of that story. While, as discussed, the integral over anticommuting functions is not unambiguously attributable to him, it can hardly be denied that he is responsible for embedding it into a greater mathematical structure underlying second quantization, expanded greatly on its properties and uses, made the link with Graßmann algebra explicit, introduced the clearest and now mainstream notation and used all of this to directly derive the now indispensible result (164).

In this chapter, we have started by considering the standard narrative on the origin of fermionic path integrals. While naming some adjacent authors, we then systematically went through six bodies of work around the second half of the fifties, that were found to be most important in this development. Each time, I have aimed to answer the question posed about them at the beginning of the chapter. What was found, is that in general it can be said that against the standard narrative, the fermionic path integral was not the result of one author in one

country, nor did that particular author even claim anything of the sort. In reality, we saw a gradual line of innovation upon innovation of many authors taking as a starting point the work of predecessors. One could imagine assigning several phases to this development. For example, we have the detection of the problem, the introduction of anticommuting functions, turning integrals over anticommuting functions into ones over real number commuting functions, introducing a minimal algebra of anticommuting numbers, and developing rules for explicit differentiation and, in particular, integration over them.
It has been argued that, especially in the case of figures like Martin and Candlin, the contributions of a number of these authors have been severely underappreciated. Moreover, the developments on this topic have not only played out in the Soviet Union, as the United Kingdom also had scientists making major contributions.

At the end of this thesis, further concluding remarks on the subject will be made. For now, one main question remains, namely, what developments were made possible by the innovations laid out in this chapter. The impact of them turns out to be momentous. This will be the topic of the next and final chapter.

5 From Ghosts to Strings: Reaping the Rewards of the Fermionic Path Integral

The previous chapter showed us how after a long chain of contributions, the problem of fermionic path integrals was ultimately solved. This culminated in a calculation of the Gaussian integral over Graßmann variables (164). Now, it is time to cash in and see what we can do with this.

In this chapter, we will start by considering two very direct applications of this new toolbox. The first of these will be the functional integral over the Dirac field. Being able to set this up directly and derive the Feynman rules in this way is a clear step forward, as the Graßmann variables take care of any minus signs between diagrams in the sum. After all, as was seen in chapter 2.5, Feynman himself was stuck on this issue, and had to resort to artificially inserting minus signs to get his QED to work. The second application we shall look at is the quantization of (non)-Abelian gauge theories. Both cases are convincing demonstrations of the strength of the path integral formulation, but especially the non-Abelian case will directly use integrals over Graßmann fields in the form of the famous Faddeev-Popov trick, and also leave some interesting new 'fields' in its wake. This second application can also be taken to usher in the end of the dark ages of the path integral formulation. That is why the third and final topic to be discussed in this chapter will be about the developments from the 70s onwards. Albeit in a more qualitiative matter, we shall briefly discuss the renormalization of non-Abelian gauge theories and supersymmetry.

5.1 The functional integral of the Dirac field

With Graßmann algebra at our disposal, we may now wonder how we can represent, e.g., the Dirac field. We may suspect that the left-hand side of the Gaussian fermionic integral (164) may be written as a general functional integral of a fermionic field, as with equation (110) for bosons. This, as it turns out, is indeed the case.

First, we have to move from a Graßmann variable to a Graßmann field. The following draws heavily on (Peskin and Schroeder, 2019, p. 301-303). Moreover, for the remainder of this chapter, we shall again write $x^\mu \equiv x$ for the four-vector, as we will be needing these a lot. A Graßmann field $\xi(x)$ can then be written as

$$\xi(x) = \sum_i \xi_i \chi(x) \tag{266}$$

Here, the coefficients ξ_i are Graßmann variables as we have seen many times up until now. The $\chi(x)$ on the other hand, are complex number functions.

Due to the Dirac spinors featuring in the Dirac equation, it seems unlikely that the Dirac field can be represented in this case. This suspicion is indeed correct, as the right way of

doing so involves replacing the $\chi(x)$ as here defined with Dirac spinors. With that, we now have a theoretical representation for the Dirac field and its adjoint $\psi(x)$ and $\bar{\psi}(x)$: they are implicitly sums of products of Graßmann numbers and spinors. We do not explicitly need this decomposition from here on out, but it is good to note that this is the underlying basis of a Graßmann field.

Thanks to the power of anticommuting numbers, we can now represent a spin-1/2 system through a classical Lagrangian, which was such a problem before. Let us write down the Lagrangian density of the free Dirac field:

$$\mathscr{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{267}$$

where $\partial\!\!\!/ \equiv \gamma_\mu \partial^\mu$, that is, the contraction of the gamma matrices with the contravariant derivative. The abbreviation is called the 'Feynman slash notation'.
With this, we can now write down the two-point correlation function (113) for the free Dirac field

$$\langle 0|T(\psi(x_1)\bar{\psi}(x_2))|0\rangle = \lim_{t_\pm \to \pm\infty} \frac{\int \psi(x_1)\bar{\psi}(x_2) e^{i\int \bar{\psi}(i\partial\!\!\!/ - m)\psi \ d^4x} \mathscr{D}\bar{\psi}\mathscr{D}\psi}{\int e^{i\int \bar{\psi}(i\partial\!\!\!/ - m)\psi \ d^4x} \mathscr{D}\bar{\psi}\mathscr{D}\psi} \tag{268}$$

Consider first the denominator. The hard work for figuring out what this is has already been done in chapter 2.4 and 4.6. Just like how equation (134) gave us equation (136) in the bosonic case as we simply replaced the matrix by a differential operator, equation (164) will now give us

$$\int e^{i\int \bar{\psi}(i\partial\!\!\!/ - m)\psi \ d^4x} \mathscr{D}\bar{\psi}\mathscr{D}\psi \propto \det(i\partial\!\!\!/ - m) \tag{269}$$

After all, the product of Graßmann variables that is integrated over to find the Gaussian integral over (164) is just the result of the discrete lattice regularizaton of $\mathscr{D}\bar{\psi}\mathscr{D}\psi$. Another close similarity (one can see why this distinction was such a central starting point to Berezin) between our previous bosonic discussions and the fermionic one now, is that the correlation function giving rise to the Feynman propagator for the free Klein-Gordon field (147) now translates into the Dirac Feynman propagator

$$\lim_{t_\pm \to \pm\infty} \frac{\int \psi(x_1)\bar{\psi}(x_2) e^{i\int \bar{\psi}(i\partial\!\!\!/ - m)\psi \ d^4x} \mathscr{D}\bar{\psi}\mathscr{D}\psi}{\int e^{i\int \bar{\psi}(i\partial\!\!\!/ - m)\psi \ d^4x} \mathscr{D}\bar{\psi}\mathscr{D}\psi} = S_F(x_1 - x_2) \tag{270}$$

This means that if one now possesses the Feynman rule of the photon propagator too (see next subchapter), the full capacity of perturbative quantum electrodynamics is unlocked. Let us write down its full Lagrangian density:

$$\mathscr{L}_{QED} = \mathscr{L}_D + \mathscr{L}_M + \mathscr{L}_{int} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu \tag{271}$$

Here, $\mathscr{L}_M$ is the Lagrangian density of electromagnetism, i.e., it corresponds to the Maxwell action. Accordingly, $F_{\mu\nu} \equiv \partial_\mu A_\mu - \partial_\nu A_\mu$ is the electromagnetic field tensor, with A_μ the electromagnetic four-potential. The final term $\mathscr{L}_{int}$ represents the interaction Lagrangian density of QED, with the coupling constant e. It can be viewed as the most famous Feynman diagram in existence: the coupling of the photon propagator to that of the electron and positron.

Perturbation theory will now require the expansion of the interaction part of the action in the correlation function expression (113). Given the Feynman propagators of the Dirac and electromagnetic field, this allows one to derive the Feynman rules for QED in the expected fashion. Due to the functional integral over the Dirac field being expressed in terms of the anticommutative Graßmann algebra, the required minus signs for adding up Feynman diagrams in the perturbative contributions are now automatically taken care of. In other words, no arbitrarily imposed bookkeeping regime for minus signs is required. Thanks to the power of the functional integral formulation and its extension with Berezin's integration rules for fermionic fields, the entire process has now become easier than any other known formulation could provide. Let us therefore end the chapter with the full functional integral kernel[55] for QED, in the language of how this was introduced for real scalar fields by expression (110) (Feynman and Hibbs, 1965, p. 236)(Feynman, 1949):

$$K(A,B) = \mathcal{N} \iiint e^{i\int(\bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu})d^4x} \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}\mathbf{A} \tag{272}$$

we note that $\mathcal{N}$ is a normalization constant and $\mathscr{D}\mathbf{A} \equiv \mathscr{D}A^0\mathscr{D}A^1\mathscr{D}A^2\mathscr{D}A^3$, as this is the functional integral over the four-potential. This comes down to being a more complicated application of expression (9).
With this, we have completed our discussion of the functional integral of the Dirac field.

5.2 Context to Faddeev Popov quantization

When describing perturbative QED above, we assumed that the photon propagator was already known. Yet, this is no trivial result. Canonical quantization of $A_\mu(x)$ is (in)famously extremely difficult. This difficulty originates from the gauge invariance involved in the electromagnetic field (Peskin and Schroeder, 2019, p. 79). As we will see, the path integral formulation is much better equipped to derive the photon propagator and thereby find out how to quantize electrodynamics. In this subchapter, we will consider this problem and see how it can be addressed with the help of the functional integral. After this, these techniques will be applied to deal with similar problems in non-Abelian gauge theories, specifically Yang-Mills theory (all to be defined shortly). In the latter case, the integral over fermionic fields will play a major role.

The mathematical tricks below, including those using Berezin integrals, have first been introduced by Ludvig Dmitrievich Faddeev (1934-2017) and Victor Nikolaevich Popov (1937-1994), and the objects carrying their name will thus appear later (Faddeev and Popov, 1967).

[55] The notion of the kernel is at this point somewhat obsolete. Yet, we will express it in the tradition of chapter 2 and that of Feynman and Hibbs, who also expressed the path integral for the Maxwell action in this way (Feynman and Hibbs, 1965, p. 236), just this once. In reality, we are expression the denominator of the general expression for calculating time-ordered correlation functions in any QFT in the language of functional integrals. It is then also good to keep in mind that we are dealing with asymptotically free states, so that the initial and final state A and B are implicitly defined in the limits.

They were both aware of Berezin's work and had even published with him before (Berezin and Faddeev, 1961). The background to their work can actually be found in the 1962 Conference on Relativistic Theories of Gravitation in Jabłonna (near Warsaw, Poland). This was an early conference that gave an impulse to the field of quantum gravity. Feynman and Bryce Seligman DeWitt[56] (1923-2004) played an important role in this. Starting with a conference talk by Feynman, they realized that Yang-Mills theory with zero mass is quite similar to their attempted model at quantum gravity. Yet, both contain divergences resulting from closed loops in their associated diagrams. While the two were able to combat this to some degree, Faddeev and Popov took up their mantle to, in the context of Yang-Mills theory, *"propose a sipmle method for calculation of the contribution from arbitrary diagrams."* (Faddeev and Popov, 1967, p. 29). This, essentially, involved addressing problems resulting from gauge freedom as we will see below. Thus, two different lines of development in physics here, namely that of quantum gravity and our story so far, interacted and fruitfully cross-pollinated.
As in the Soviet style we have seen before, the paper of Faddeev and Popov was extremely short, although not unclear [57]. It will be cited below, yet at times the help of secondary sources will also be enlisted (Peskin and Schroeder, 2019, p. 295-297).

First, some key terms will be defined. We remember that in classical electromagnetism, we have

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{273}$$

with $\vec{B}$ the magnetic field and $\vec{A}$ its associated vector potential (in fact, the latter three components of the four-potential used earlier).
Since in general the curl of the gradient of some function $\Lambda(\vec{x})$ is zero, i.e., $\vec{\nabla} \times (\vec{\nabla}\Lambda) = 0$, we note that we can add any such term to the vector potential yet not change the magnetic field:

$$\vec{B} = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda) \tag{274}$$

From the classical idea that only the electric and magnetic field are observable, but not their potentials, this means that the addition of this new term has no observable effects on the physics[58]. We can simply choose whatever is most convenient for us. The transformation $\vec{A} \rightarrow \vec{A} = \vec{\nabla}\Lambda$ is then called a *gauge transformation*, and by selecting a particular function Λ through some condition we are imposing a choice of gauge.

While this is the source of the 'gauge terminology' in QFT, it there takes on a much deeper meaning than just a mathematical manipulation. By demanding that the Lagrangian density of a quantum field theory is invariant under local gauge transformations, one is actually

[56] As we have briefly discussed the work of Cécile DeWitt-Morette before, note that Bryce Dewitt is 'the husband of'!

[57] It helps that, thanks to a reading group session I was able to participate in during my time at the MPIWG thanks to dr. Blum, I had read and discussed the transcript of Feynman's talk at the conference on quantum gravity. This made it easier to contextualize the work of Faddeev and Popov.

[58] This assumption no longer holds in quantum mechanics, as it was shown through the Aharonov-Bohm effect that a change in potential can affect charged particles even if the fields themselves do not change.

able to derive the underlying structure of such theories. A theory making use of this principle is then a *gauge theory*. One example is QED. We note that the free Dirac Lagrangian density (267) is invariant under a *global* phase transformation $\psi \to e^{i\alpha}\psi$ (and, consequently, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$). Since $\alpha \in \Re$, these are just numbers. We can multiply the exponentials and just get 1, so that the total Dirac Lagrangian density is unchanged. A *local* phase transformation now adds some extra, viz., that the number α is a function of the spacetime four-vector x, i.e.,

$$\psi \to e^{i\alpha(x)}\psi, \quad \bar{\psi} \to e^{-i\alpha(x)}\bar{\psi} \tag{275}$$

Under this transformation, it seems that $\mathscr{L}_D$ is *not* invariant, due to the derivative involved in $\partial\!\!\!/$ now also acting on $\alpha(x)$. This produces an extra term $(\partial_\mu \alpha(x))\bar{\psi}\gamma^\mu\psi$. Yet, the principle telling telling us to demand that this is the case. The way out is now to introduce a new field called a *gauge field* that transforms in such a way that it cancels the extra term that comes about from this transformation. This, wonderfully, turns out to be exactly the electromagnetic field $A_\mu(x)$, which then must transform according to the rule $A^\alpha_\mu(x) = A+\mu(x)+\frac{1}{e}\partial_\mu\alpha(x)$. Thus, one can derive the QED Lagrangian and the Maxwell and interaction terms by demanding that the Dirac Lagrangian is invariant under a local phase transformation. Since the latter is a number, we could also refer to it as a $1x1$ matrix. In fact, we could write the gauge transformations discussed above as a multiplication by such a matrix, namely $\psi \to U\psi$, with U satisfying the property that $U^\dagger U = 1$. If we now consider these matrices as a representation of a group, then the collective of them forms what is called the $U(1)$ group, or the 'unitary group of dimension 1'. The elements of this group commute because they are numbers. In group theory, a group whose elements commute is called an 'Abelian group'. Thus, we can summarize by saying that QED has the special $U(1)$ gauge symmetry, and we can refer to it as an *Abelian gauge theory*.

The above terminology seems a bit much given we are just multiplying by numbers. The reason for adopting it anyway is that it provides a language suitable for all the other Standard Model interactions too. For example, the strong interaction has a Lagrangian density that obeys the $SU(3)$ (special unitary group of dimension 3) symmetry. The 'special' means that the determinant of the matrices that from the representation of this group is always equal to 1. Keeping in with the above logic, this means that we can derive the gluon field by demanding invariance of the quark Lagrangian upon applying a local transformation of an exponential with this times matrices, rather than numbers, as its argument. Although understanding the Standard Model in this way only came around during the 70s, the principle of local gauge invariance provides a powerful tool for deriving its interactions. With this context, we can turn our attention to the quantization of the electromagnetic field.

5.3 Gauge fixing the electromagnetic field

With the jargon contained in the rest of this chapter cleared up, let us turn our attention to the quantization of the Abelian fields first. As a pratical goal, we want to know how the photon propagator can be derived, something that was stated to be extremely difficult in the

operator formalism. It is therefore good to state that what follows below does not necessitate the methodology we employ, but in any case, it will serve as a good introduction to the more difficult case that follows.

Above, the motivation behind why it is that the electromagnetic field leaves the physics unvariant under a transformation

$$A_\mu^\alpha(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \tag{276}$$

was explained. It can therefore be thought of as the QFT-version of the gauge freedom we had in choosing some function $\lambda(\vec{x})$ to our liking in classical electromagnetism. This is by itself fine, but introduces a problem into the functional integral. To see this, let us first realize that to derive the photon propagator, we need to calculate correlation functions like (268). Since we know that the fields in the numerator will then be products of $A(x_1)A(x_2)\cdots$, we abbreviate this with the functional quantity $\mathcal{O}[A]$, so that

$$\langle 0|T(\hat{\mathcal{O}}[\hat{A}])|0\rangle = \lim_{t_\pm \to \pm\infty} \frac{\int \mathcal{O}[A] e^{-\frac{i}{4}\int F_{\mu\nu}F^{\mu\nu} d^4x} \mathscr{D}\mathbf{A}}{\int e^{-\frac{i}{4}\int F_{\mu\nu}F^{\mu\nu} d^4x} \mathscr{D}\mathbf{A}} \tag{277}$$

Combining the wish to compute this expression with the gauge freedom inherent through the transformation rule of the electromagnetic field, the main problem presents itself. That is, we now have a degenerate functional integral, infinitely degenerate even. The reason is that because of the gauge freedom in $\mathscr{D}\mathbf{A}$, we are constantly integrating over physically equivalent field configurations. After all, any two different choices of gauge for $A_\mu(x)$ will yield the same physics, so we are 'double counting' the contribution of that field configuration. Since one can theoretically write down an infinite number of gauges for the same field configuration, the consequent degeneracy of this freedom is infinite, and our functional integral will diverge. In the jargon, we say that if two fields can be related through some gauge transformation, they share a *gauge orbit*. In the words of Faddeev and Popov:

> *"In fact, we can say, using the natural geometrical language, that the integrand is constant on the "orbits" $A_\mu \to A_\mu^\Omega$ of the gauge group in the manifold of all fields $A_\mu(x)$."* (Faddeev and Popov, 1967, p. 30)

In the above, their notation has been cast into ours. The Ω represents a certain choice of gauge.
In our integration, we therefore want to 'pass' each gauge orbit only once, i.e., we wish to integrate only over all physically inequivalent fields. This will hopefully leave us with a diverging functional integral once more.

Many terms in the form of 'gauge x' have been introduced, but we need one more. Namely, to solve the aforementioned problem, we need to *gauge fix* our electromagnetic potential field. If we can in some way force the functional integral to only consider field configurations of

the field for one specific choice of gauge, we can remove the infinite degeneracy. There is a useful trick for this. Let us introduce the following functional:

$$G[A] = \partial^\mu A_\mu(x) - w(x) \tag{278}$$

Here, $w(x)$ can be any scalar function. We now want $G[A] = 0$ to be a gauge fixing condition. The result can be related to the Lorentz gauge as introduced in classical electromagnetism. A clever way to incorporate this into the functional integral, and in that way force it to only consider physically inequivalent field configurations, is to then embed this function into a delta-*functional* $\delta(G[A])$ in the functional integral. To think intuitively about this, one may think back to the discrete lattice regulariztion in QFT introduced in chapter 2.4, in the sense that we have a product of delta functions at each spacetime point.
Yet, we cannot just plug this delta-functional into our correlation function expression. This is where the so-called *first Faddeev Popov trick* comes in[59] (Faddeev and Popov, 1967, p. 30):

$$\int \delta(G[A^\alpha]) \det\left(\frac{\delta[A^\alpha]}{\delta\alpha}\right) \mathscr{D}\alpha = 1 \tag{279}$$

Here, A^α denotes our earlier introduced expression for the gauge transformed electromagnetic field (276), so that $G[A^\alpha(x)] = \partial^\mu(A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)) + w(x) = \partial^\mu A_\mu(x) + \frac{1}{e}\Box\alpha(x) + w(x)$. Moreover, I wrote G with square brackets as it is technically also a functional, and it would make little sense to functionally differentiate it otherwise[60]. Moreover, the above equation contains a functional determinant, as we have seen from earlier chapters.

The good thing about '1' is that it can always be inserted into any equation. Yet, the first Faddeev Popov trick kind of 'falls out of the air' if we were to stoically proceed from here. As has often been the case, lattice discretization whereby we express the functional integral as a product of many integrals over some $d\alpha_i$ will aid us. We can roughly express it as

$$\prod_{i,j=1}^{N} \left(\int \delta^{(n)}(G_i(\alpha_j)) \det\left(\frac{\partial G_i}{\partial \alpha_j}\right) d\alpha_i \right) = 1 \tag{280}$$

Here, we are working with n-dimensional vectors, which explains the breaking up of the δ-functional and the fact that we do not just discretize in index i but also need to account for vector components through j. The functional derivative becomes a regular partial one relating to the change of G_i, a function of numerous α_j's, in one specific spacetime degree of freedom of the discretized field. The easiest way to make this intuitively 'click' is to just consider $N = 1$ for both indices. The above will then turn into

$$\int_{-\infty}^{\infty} \delta(G(\alpha)) \frac{dG(\alpha)}{d\alpha} d\alpha = 1 \tag{281}$$

[59] This is again translated into our notation, with the help of secondary literature (Peskin and Schroeder, 2019, p. 295).

[60] This is therefore an easily forgivable yet small notational mixup in Peskin & Schroeder.

where since G is now a function of only one variable, $\partial \to d$, and the determinant of a one-dimensional matrix is just the element itself.
Consider the standard scaling property of the δ-function:

$$\int_{-\infty}^{\infty} \delta(g(x)) f(x) dx = \sum_i \frac{f(x_i)}{|g'(x_i)|} \tag{282}$$

where the x_i are the roots of $g(x)$.
With regards to the right-hand side, in our situation, we have $f(x) \to \frac{dG(\alpha)}{d\alpha}$ and $g'(x) \to \frac{d}{d\alpha}G(\alpha)$. Thus, whatever the root may be α_i may be, the numerator and denominator are both the derivative of $G(\alpha)$ evaluated at $\alpha = \alpha_i$. Therefore, we have 1. Do note though, that we have implicitly assumed that $G(\alpha)$ has only one root, which is the shortcoming of this intuitive $N = 1$ case.

Nevertheless, having motivated the origin of the Faddeev Popov trick, we shall proceed. We shall substitute it in the correlation function (277). Specifically, we will consider the numerator of this expression. Once we know that, we can figure out the denominator by substituting $[A](A) = 1$. Moreover, since we do not need to do anything with the Maxwell Lagrangian density right now, We shall write $S_M[A] = \int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}d^4x$ for brevity. We then find

$$\int \mathcal{O}[A] e^{iS_M[A]} \mathcal{D}\mathbf{A} = \int \left(\int \delta(G[A^\alpha]) \det\left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right) \mathcal{O}[A] e^{iS_M[A]} \mathcal{D}\alpha \right) \mathcal{D}\mathbf{A} \tag{283}$$

There are a number of manipulations we can now do. Let us start by considering the determinant inside of the integral. Earlier, we wrote that $G[A^\alpha(x)] = \partial^\mu A_\mu(x) + \frac{1}{e}\Box\alpha(x) + w(x)$. That means that our functional derivative is not actually that difficult at all, since

$$\frac{\delta}{\delta\alpha(x)} G[A^\alpha] = \frac{1}{e}\Box \tag{284}$$

Therefore, we end up with a functional determinant. Given our discussion on such an object in chapter 2.4, this will be relevant in the eventual derivation of the photon propagator. For the remainder of this derivation, however, it is mostly just useful to know that the determinant is not a function of any of the fields the functional integral is taken over. Because of that, we can move it out of there. This step is good to keep in mind, as this will not be so easy when discussing the non-Abelian Yang-Mills theory later on.

The next move is somewhat subtle, but functionally and conceptually important. Note that we now have three functionals of the field: the gauge transformed functional quantity $G[A^\alpha]$ in the δ-function and the untransformed $\mathcal{O}[A]$ and $S[A]$. Then, there is also the actual integration measure $\mathcal{D}\mathbf{A}$. We can now gauge transform all of these quantities. For the action, we know that it is gauge invariant, as the electromagnetic field was derived by demanding this for the Dirac action. We can therefore just substitute $S[A] \to S[A^\alpha]$ without any changes. The same can be said for the measure $\mathcal{D}\mathbf{A} \to \mathcal{D}\mathbf{A}^\alpha$, as we have accentuated many times that it is

shift-invariant. Lastly, we have $\mathcal{O}[A]$. Here, we do not have an a priori reason to say that it is gauge invariant (although in practice typical correlation functions will be). Therefore, one assumption behind the current proof is that this quantity is, in fact, gauge invariant. If it is, everything is now 'on one line', and the δ-function can do what it was set out to do to only consider physically inequivalent field configurations. With everything being A^α now comes to convenience that it now acts as a dummy function, and it no longer matters for the integral if we write A or A^α since the whole thing is gauge invariant anyway, therefore, at this point we may as well switch back to A altogether. There is, however, still the issue of $\int \mathcal{D}\alpha$, which now seems to be left as a divergent constant. Yet, we remember that all of the above still holds if $\mathcal{O}[A] = 1$, which means that the denominator of our time-ordered correlation function (277) will also carry this same factor. This means that $\mathcal{D}\alpha$ will at the end of the calculation just be divided out.

Technically, we are now done. Yet, there is a trick for writing the above in a far more appealing way, whereby this whole gauge fixing process ultimately manifests itself as an extension to the action. Let us first take stock of where our expression is currently at. In the following, we ignore $\mathcal{D}\alpha$ because of the above argument, although we should keep in mind that it is technically still there. We write:

$$\int \mathcal{O}[A] e^{iS_M[A]} \mathcal{D}\mathbf{A} = \det\left(\frac{1}{e}\Box\right) \int \delta(G[A]) \mathcal{O}[A] e^{iS_M[A]} \mathcal{D}\mathbf{A} \tag{285}$$

For this trick, it is useful to remember our definition (278), since it involves using this free function $w(x)$. It entails the following:

$$\int e^{-i\int \frac{w^2}{2\xi} d^4x} \mathcal{D}w \int \mathcal{O}[A] e^{iS_M[A]} \mathcal{D}\mathbf{A} = \det\left(\frac{1}{e}\Box\right) \int \left(\int \delta(G[A]) \mathcal{O}[A] e^{-i\int \frac{w^2}{2\xi} d^4x} e^{iS_M[A]} \mathcal{D}w \right) \mathcal{D}\mathbf{A} \tag{286}$$

That is, we add a functional integral over $w(x)$ on both sides of our equation. On the left hand side, there is no further presence of $w(x)$ at all, so we just wrote it to the left of the main expression to be calculated. Since it is a Gaussian integral, it will result into some function ξ. Here, ξ is an as of yet unspecified number, we have the freedom to choose it as we see fit later. In any case, we move the resulting constant on the left-hand side to the right-hand side and call it $N(\xi)$. We know how to calculate it, but this would be a wasteful exercise, as it will again likewise be produced when applying the same trick to the denominator of (277).

The more interesting side is the right-hand side itself. We again note that the δ-functional reads $\delta(\partial^\mu A_\mu(x) - w(x))$. We can then take the following step:

$$\int \delta(\partial^\mu A_\mu - w) e^{-i\int \frac{w^2}{2\xi} d^4x} \mathcal{D}w = e^{-i\int \frac{1}{2\xi}(\partial^\mu A_\mu)^2 d^4x} \tag{287}$$

Here we simply used the definition of the δ-functional.
But this means that our original right-hand side is now contains the product $\exp\left(-i\int \frac{1}{2\xi}(\partial^\mu A_\mu)^2 d^4x\right)$.

$\exp(iS_M[A])$. We can therefore add the exponents, and define the gauge fixed Maxwell action $S_{GFM}[A]$ as follows

$$S_{GFM}[A] = \int \mathscr{L}_{GFM}[A]d^4x = \int \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)d^4x \tag{288}$$

This means that we can now finally express our initial integral as

$$\int \mathscr{O}[A]e^{iS_M[A]}\mathscr{D}\mathbf{A} = N(\xi)\det\left(\frac{1}{e}\Box\right)\int \mathscr{O}[A]e^{iS_{GFM}[A]}\mathscr{D}\mathbf{A} \tag{289}$$

and the full gauge fixed correlation function (277) as:

$$\langle 0|T(\hat{\mathscr{O}}[\hat{A}])|0\rangle = \lim_{t_\pm\to\pm\infty}\frac{\int \mathscr{O}[A]e^{iS_{GFM}[A]}\mathscr{D}\mathbf{A}}{\int e^{iS_{GFM}[A]}\mathscr{D}\mathbf{A}} \tag{290}$$

In brief, thanks to Faddeev and Popov, we have found a modified Maxwell action that prevents diverging functional integrals due to the infinite degeneracy caused by the gauge freedom of the electromagnetic field. From here on out, it is relatively easy to derive the photon propagator: one can simply use the same perturbative methods as discussed in chapter 2.4. We will not derive it explicitly, but state the result here[61] (Peskin and Schroeder, 2019, p. 297):

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{291}$$

The most interesting part of this for our current purposes is to note the added bonus of being able to choose whatever value of ξ makes one's work easier in any given context. After all, the functional integration over $\mathscr{D}w$ with the Gaussian including ξ was done on both sides of our equation. Three defined use cases are (Peskin and Schroeder, 2019, p. 297, 513):

- $\xi = 0$: Landau gauge
- $\xi = 1$: Feynman-'t Hooft gauge
- $\xi = 3$: Yennie gauge

As may be expected, each has its own particular calculations where it is useful to adopt.

With that, we have finished our discussion of the quantization of the electromagnetic field and the resulting photon propagator. With the Faddeev Popov trick to fix the gauge, one can proceed as one always does with the path integral to derive the propagator. This is one instantiation of the often mentioned advantage that the path integral has the great advantage of readily generalizing to different gauge theories. That introduces the question of whether this same gauge fixing method also works for non-Abelian gauge theories. With a slightly extended toolbox, including fermionic path integrals, we shall see that it does, although we pay a spooky price.

[61] To give some idea of where this comes from: one can use integration by parts to rewrite the action S_{GFM} in the same structure as was done in chapter 2.4 (operator between the fields). Then, one can go to momentum space (explaining the occurrence of k over x) and use the Euler-Lagrange equation to solve for the photon propagator.

5.4 Gauge fixing the Yang-Mills field

In 1954 Chen-Ning Yang (1922) and Robert Laurence Mills (1927-1999) attempted to extend the idea of local phase invariance for QED, with its underlying $U(1)$ gauge symmetry, to a proton-neutron system. These gauge transformations this time collectively made up the $SU(2)$ Lie group. As mentioned before, the representation of this group consists of $2x2$ *unitary* matrices with unity determinant (the *special* part). Crucially, the elements of this group do *not* commute, i.e., we are now dealing with an example of a non-Abelian gauge theory. The Yang-Mills model for the proton and neutron itself ultimately did not work out, but their idea of demanding invariance under local gauge transformations of $SU(n)$ groups turned out to be very influential, and had an important role in the eventual creation of the $SU(3)$ symmetry of the strong force and the $SU(2)_L \otimes U(1)$ symmetry of the electroweak force.

We, however, will now work with the general Yang-Mills Lagrangian density to see how we can fix the degeneracy again originating from gauge freedom of the involved field. Much of this derivation will be very similar to the electromagnetic case, until it will not be. Again, we base ourselves on Faddeev and Popov, but most certainly get some help from secondary literature (Peskin and Schroeder, 2019, p. 512-517). We introduce $\mathscr{L}_{YM}$ as follows:

$$\mathscr{L}_{YM}(A) = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} \tag{292}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{293}$$

while there are certainly some structural similarities with the Maxwell action, we are now dealing with matrix-valued fields running over the generators of the Lie group. We can write $A_\mu = A^a_\mu t_a$, with A_μ our vector field and t_a the group elements, e.g., the Pauli matrices in the case of $SU(2)$. The structure constants f^{abc} relate these generators through $[t^a, t^b] = i f^{abc} t^c$. Lastly, g, rather than e, is now our coupling constant (Peskin and Schroeder, 2019, p. 486-491). From here on, we proceed as before.

First, we again introduce the gauge transformed field as

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c = A^a_\mu + \frac{1}{g} D_\mu \alpha^a \tag{294}$$

Like in the QED case, this transformation rule for the gauge field results from demanding local gauge invariance, this time under $SU(n)$ transformations. It looks similar, except for the addition of the third term in the second equality, which will be the source of extra trouble later. The symbol $D_\mu \alpha^a \equiv \partial_\mu \alpha^a + g f^{abc} A^b_\mu \alpha^c$ (it thus implicitly employs the above introduced structure constant relation).
The gauge freedom involved here will again lead to the same problem of infinite degeneracy due to the functional integral including all physically equivalent field configurations on identical gauge orbits. This leads to divergences and there an inability to calculate correlation functions. We again set out to fix the gauge.

For this, we introduce a functional $G[A]$ that, combined with a δ-functional, can get us on the correct road to this end. We have

$$G[A] = \partial^\mu A^a_\mu + w^a \tag{295}$$

and, gauge transformed,

$$G[A^\alpha] = \partial^\mu A^a_\mu + \frac{1}{g} D_\mu \alpha^a + w^a \tag{296}$$

We also again want to calculate the same object at the end:

$$\langle 0|T(\hat{\mathcal{O}}[\hat{A}])|0\rangle = \lim_{t_\pm \to \pm\infty} \frac{\int \mathcal{O}[A] e^{-\frac{i}{4}\int F^{\mu\nu}_a F^a_{\mu\nu} d^4x} \mathcal{D}A}{\int e^{-\frac{i}{4}\int F^{\mu\nu}_a F^a_{\mu\nu} d^4x} \mathcal{D}A} \tag{297}$$

From now on, we will again abbreviate the exponent by just writing the action $S_{YM}[A]$, and work with the numerator first and foremost. Immediately using the first Faddeev Popov trick on the numerator, we may write

$$\int \mathcal{O}[A] e^{i\int S_{YM}[A]} \mathcal{D}A = \int \left(\int \delta(G[A^\alpha]) \det\left(\frac{\delta G[A^\alpha]}{\delta \alpha} \right) \mathcal{O}[A] e^{iS_{YM}[A]} \mathcal{D}\alpha \right) \mathcal{D}A \tag{298}$$

Retracing our previous strategy, we will now consider the functional derivative in the determinant. Taking it amounts to

$$\frac{\delta}{\delta \alpha(x)} G[A^\alpha] = \frac{1}{g} \partial^\mu D_\mu \tag{299}$$

While at first sight it may seem like we are proceeding as in the Abelian case, here we are actually confronted with the troublesome third term in the gauge transformed field (294). As we could read from its definition just now, the covariant derivative D_μ contains the field variable A_μ. Thus, we can take the determinant out of the $\mathcal{D}\alpha$ functional integral and at least use that, but not out of the $\mathcal{D}A$ one. In the words of Faddeev and Popov:

> *"It is the nontriviality of [the functional determinant] which distinguishes the theories of Yang-Mills and gravitational fields from quantum electrodynamics."*
> (Faddeev and Popov, 1967, p. 30)

To keep oversight, we will briefly postpone this issue and continue with other steps that are the same as in the former case, to return to it later.

Three familiar steps are now in order. Firstly, we have some fields that have been gauge transformed and some that have not been. Assuming $\mathcal{O}[A]$ is a gauge invariant quantity, we can once more gauge transform these quantities without changing anything about the functional integral, and are consequently left with a dummy index A^α that we will replace with A. A second move we can already make is to isolate the factor $\int \mathcal{D}\alpha$ and ignore it for the remainder of the derivation, as it will ultimately cancel yet again in the fraction of (297).

Thirdly and finally for now, we can functionally integrate both sides of our equation over a Gaussian distribution in w again. Because we went from $A^\alpha \to A$, we just have $\delta(\partial^\mu A^a_\mu + w^a)$, and the δ-functional over the functional integral $\mathscr{D}w$ will set $w^2 = (\partial^\mu A^\alpha_\mu)^2$ on the right-hand side. It is then once more convenient to multiply the exponentials.
Combining these three familiar steps, we find:

$$\int \mathscr{O}[A] e^{i\int S_{YM}[A]} \mathscr{D}A = N(\xi) \int \det\left(\frac{1}{g}\partial^\mu D_\mu\right) \mathscr{O}[A] e^{i\int\left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right)d^4x} \tag{300}$$

Now the million dollar question is how we can rid ourselves of the functional determinant. The answers to this riddle was a great innovation of the short '67 paper by Faddeev and Popov. Given the end of the previous and start of this current chapter, we may actually be able to guess it. After all, we have a determinant to the power of unity (and not, e.g., -1/2). That, of course, reminds us of the fermionic Gaussian integral (269). While this was a proportionality rather than an equality, any constants will be divided out by the denominator of (297), so this will not matter. We therefore introduce the *second Faddeev Popov trick* as[62]

$$\det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int \mathscr{D}\bar{c}\mathscr{D}c\; e^{i\int d^4x\; \bar{c}(-\partial^\mu D_\mu)c} \tag{301}$$

These functional integrals are, as anticipated, taken over anticommuting Graßmann fields. We do not need to motivate the second Faddeev Popov rule, as we have essentially already done so in another context.
To obey Lorentz invariance, these are (complex) scalars fields $\bar{c}(x)$ and $c(x)$. Furthermore, we note that the inverse coupling constant $\frac{1}{g}$ is missing. This is because it has been absorded into the definition of the new fields.
Before thinking more about the possible nature of these fields, let us first finish our process of gauge fixing the Yang-Mills field so that we can compute correlation functions. Using the second Faddeev Popov trick, we can again multiply exponentials. This then allows us to define a new action like we did in the electromagnetic case, now with one more term. We write this gauge fixed Yang-Mills action S_{GFYM} as

$$S_{GFYM}[A,\bar{c},c] = \int d^4x\; \mathscr{L}_{GFYM}[A,\bar{c},c] = \int d^4x\; \left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}(-\partial^\mu D_\mu)c\right) \tag{302}$$

Given our modified action, we can formulate the functional integral we started with as

$$\int \mathscr{D}A\, \mathscr{O}[A] e^{iS_{YM}[A]} = N(\xi) \iiint \mathscr{D}A\mathscr{D}\bar{c}\mathscr{D}c\; \mathscr{O}[A] e^{iS_{GFYM}[A,\bar{c},c]} \tag{303}$$

and the full gauge fixed correlation function (297) as:

$$\langle 0|T(\hat{\mathscr{O}}[\hat{A}])|0\rangle = \lim_{t_\pm \to \pm\infty} \frac{\int \mathscr{D}A\mathscr{D}\bar{c}\mathscr{D}c\; \mathscr{O}[A] e^{iS_{GFYM}[A,\bar{c},c]}}{\int \mathscr{D}A\mathscr{D}\bar{c}\mathscr{D}c\; e^{iS_{GFYM}[A,\bar{c},c]}} \tag{304}$$

[62] Since we are now dealing with anticommuting objects again, the integration measures will be put in the front as to avoid any confusion due to possible minus signs arbitrarity.

That is, Faddeev and Popov succesfully fixed the gauge and prevented diverging correlation functions due to the infinite degeneracy caused by the gauge freedom of the Yang-Mills field. The result is that one must now use a modified action, this time extended not just by another term of the same field (again giving us a convenient choice of gauge through ξ), but by two new fields. While it is unsurprisingly the case that we can use perturbation theory to derive the propagators of this modified Yang-Mills theory, the question remains what the physics behind the fermionic fields $\bar{c}(x)$ and $c(x)$ and their Feynman rules entails.

Ultimately, there is no physical reality to these fields. In fact, they are called *ghost fields* or *Faddeev Popov ghosts* for this reason. We can say this because the ghosts violate the spin-statistics theorem discussed in chapter 2.5. On the one hand, these fields obey an anticommutation relation, and their ghost particle quanta will collectively obey Fermi-Dirac statistics. On the other hand, they were introduced as a scalar field, which has the property of having zero spin. Thus, we have fields with integer spin while satisfying anticommutation relations, i.e., the ghost fields are incompatible with the spin-statistics theorem. Moreover, in derived Feynman diagrams, ghost particles always only apppear as internal lines. This also makes them unobservable in principle. In conclusion, the ghost quanta are thus not physical particles, they are an instrument, a mathematical artifact used to enable perturbative methods for Yang-Mills theories. Nevertheless, they are benevolent ghosts. While not fully getting into this, it is good to know that they play an important role in the so-called *BRST* symmetry. The ghosts act as 'negative degrees of freedom' to effectively annul the effects of unphysical polarization states of gauge bosons, a property also exploited by the authors in the next subsection. The underlying reason for this symmetry can be found in a geometrical analysis of gauge fields.

This, then, concludes our discussion of the quantization of the Yang-Mills theory by fixing the gauge. The functional integral formulation once again readily generalizes to even non-Abelian gauge theories, and the power of this method was therefore increasingly being noticed by physicists around the early end-60s early-70s. As we have seen, this would not have been possible without the advent of the fermionic path integral. In the next and final subchapter, we shall (more qualitatively) investigate how the functional integral, with special attention to the fermionic version, truly broke into the mainstream, and consider two important research fields.

5.5 Beyond the dark ages

By solving the degeneracy problem that came with the gauge freedom inherent in gauge theories, Faddeev and Popov successfully applied the functional integral to non-Abelian gauge theories. This had a significant effect on the status of the functional integral formulation. Nobel prize winner (together with Salam) on the electroweak unification Steven Weinberg (1933-2021) refers to this as follows:

"The path integral approach was revived in the late 1960s, when Faddeev and

> *Popov and DeWitt showed how to apply it to non-Abelian gauge theories and general relativity. For most theorists, the turning point came in 1971, when 't Hooft used path integral methods to derive the Feynman rules for spontaneously broken gauge theories. [...] Since then, the path integral methods described here have become an indispensable part of the equipment of all physicists who make use of quantum field theory."*
> (Weinberg, 1995, p. 376-377)

The word 'revived' seemingly implies that the path integral formulation was 'dead' before. Although we have seen that there were still quite some papers published on it, it is nevertheless true that it was not used all that much, certainly not the degree it has been since the 1970s. More interesting for our current purposes, however, is the second sentence of the quote. We shall turn our attention to it now.

Around the 1970s, Gerardus 't Hooft (1946) and his doctoral advisor Martinus Justinus Godefridus Veltman (1931-2021) successfully proved that Yang-Mills theory is renormalizable, i.e., they were able to deal with the infinities that show up when using perturbation theory with it using dimensional regularization (Veltman, 1968)('t Hooft, 1971b)('t Hooft, 1971a)('t Hooft and Veltman, 1972). Since the electroweak interactions are just a type of Yang-Mills theory, they received the 1999 Nobel prize for their elucidation thereof. To achieve this, extensive use was made of the functional integral ('t Hooft, 1971b, p. 180)('t Hooft, 1971a, p. 167-169), as well as the previously discussed tricks and ghost fields of Faddeev and Popov ('t Hooft, 1971b, p. 197)[p. 173]('t Hooft, 1971a). Since the now renormalized Yang-Mills theory lies at the basis of the Standard Model of particle physics, this work has been extremely important. Most interesting for our purposes that it was made possible through the functional integral, now that it was finally able to incorporate fermionic fields. Initially, colleagues (including Weinberg) had been highly sceptical of Veltman's and 't Hooft's use of these methods (Close, 2011, p. 203-229). But, luckily, they pushed ahead. Faddeev and Popov had already made an important contribution by gauge fixing the Yang-Mills field. Add to that the mainstream attention the work of 't Hooft and Veltman got, one can understand how this was indeed the "turning point" that definitively ended the dark decades for the path integral formulation.

The functional integral, and its variety over fermionic fields using Graßmann algebra, are still widely used in contemporary theoretical physics. We shall consider one example, namely that of supersymmetry. It has particularly nice continuity with regards to our story, seeing as after his work on second quantization, Berezin went on to play an important role in its conception.
Supersymmetry is an extension of the Standard Model by a new (spacetime) symmetry whereby each elementary particle is accompanied by a 'superpartner'. In the case of a fermion, this partner is a boson, and vice versa. As they could potentially be very helpful with addressing the hierarchy problem or acting as dark matter candidates, these superpartners have long been sought after. Yet, as of now, none have been found. Supersymmetry does not constitute one particular theory, but may rather be thought of as a property contained in many

Beyond the Standard Model theories. String theory is an example of this.

The history of supersymmetry intertwines with the story of Berezin. As is stated Mikhail Shifman (1949), a theoretical physicist having done much work on supersymmetry, in a book on the topic:

> *"Next, it is worth mentioning [...] Felix Berezin, an outstanding mathematician who created a mathematical apparatus used in supersymmetric field theories. The integral over the anticommuting Graßmann variables that he introduced in the 1960s paved the way for the path integral formulation of quantum field theory with fermions, the heart of modern supersymmetric field theories and superstrings."*
> (Shifman and Kane, 2000, p. 4)

Let us remember the centrality of the fermion-boson distinction in the system devised by Berezin in his book on Second Quantization discussed in chapter 4.6. There, it was also mentioned that Berezin was the father of *supermathematics*. On the topic, he reflects in the following way:

> *"The striking coincidence of the main formulas of the operator calculus in the Fermi and Bose variants of the second quantization method... led to the idea of the possibility of a generalization of all the main notions of analysis so that generators of a Graßmann algebra would be on an equal footing with real or complex variables."*
> (Karabegov et al., 2013, *Translated from Berezin.*)

This 'generalization' constitutes supermathematics. The mathematical system underlying many supersymmetric models thus directly flowed out of Berezin's work on applying Graßmann algebra to fermionic systems. About this system one could say, somewhat simplified[63], that it is one mathematical framework that can describe both types of fields in nature (bosonic and fermionic ones) rather than using complex numbers for the former and Graßmann numbers for the latter. A 'superfield' $\Phi(x,\theta)$, for example, would be a function of both a complex-valued object x and a Graßmann-valued θ. Unsurprisingly, supersymmetric extensions of the Standard Model often make use of functional integrals. These may feature integration measures over both bosonic and fermionic fields, so that the Berezin integral is an essential ingredient in the supermathematical description of functional integrals in supersymmetric models. One can in any case see how such a mathematical system would be well-suited for supersymmetry, given its core idea just mentioned.

Many of the early contributors to supersymmetry (including, again, Salam) were keenly aware of Berezin and his supermathematics. For example, physicists Vladimir Akulov and Dmitrij Volkov make reference to the idea of supermathematics and explicitly thank and cite Berezin

[63] Slightly more technically, it is a $\mathbb{Z}_2$-graded Lie Algebra. The algebra then contains *both* commuting and anticommuting elements, or an even and odd subalgebra describing, respectively, bosons and fermions.

in their paper (Akulov and Volkov, 1974, p. 29, 35). Even in the work of well-known contemporary theoretical physicists working on supersymmetric theories, such as Edward Witten (1951) through string theory, handily make use of functional integrals over Graßmann variables (Witten, 1988, p. 540). It shows that the functional integral, Graßmann's algebra, and the results of the fruitful combination of the two, are alive and well in contemporary mathematics and physics.

Thus, we now have a clearer image of what followed from the invention of the fermionic functional integral over Graßmann-valued fields. We saw that it provides a way to express half-integer spin fields such as the Dirac field, allows us to fix the gauge for Abelian and non-Abelian theories alike, and it is even an instrumental part of the mathematical formalism underlying work on supersymmetry. Most importantly, the ability to describe fermionic fields can through what it made possible be seen to have initiated the end of the dark decades of the path integral formulation, propelling it into the mainstream ever since. With that, we have come to the end of our long story. It is therefore time to move to the conclusion of this thesis, to zoom out and harvest.

6 Conclusion

The research question of this thesis was introduced as follows:

How did the modern formulation of fermionic path integrals over anticommuting Graßmann variables come to be introduced around the late 1950s to allow for breakthroughs in the development of the path integral formulation of QFT?

To come to a concise answer to it, we first summarize our findings throughout this body of work.

We started with chapter 2, where Feynman's path integral formulation that the question is about was extensively laid out. This started with the foundations, after which proofs were given to show the equivalence to the operator formalism following from canonical quantization. We then derived the free particle kernel, that would subsequently play an important role in the application of perturbative methods to the path integral. The step to QFT was then made, where we showed how the foregoing changed in the context of relativistic fields. This provided the tools to do and understand later calculations. The chapter was closed with a list of strengths and challenges for the path integral. Most important was that it was held back through its inability to deal with systems of fermions, as these did not have a classical counterpart that could serve as the Lagrangian for the path integral.
We then moved on to the third chapter, that was about the background and content of the Graßmann variables that would ultimately be employed to solve this 'fermionic impasse'. This started with a brief history of vector analysis, to eventually end up at Graßmann and his *Ausdehnungslehre*. While much of his work was ignored during most of his life, we saw that it was eventually revived, as his anticommuting and zero square algebra turned out to be a suitable language for describing fermionic systems.

With adequate knowledge of the path integral and Graßmann's anticommuting algebra, we could get to work on the historical core of the research question in chapter 4. It was established that the mainstream historical narrative to the introduction of fermionic path integrals often came down to crediting Berezin and introducing the integral named after him. Careful literature study, however, resulted in the discussion of six different key articles that played a role in getting to this end state. Each was analyzed through a historical approach centering around four questions, that can here be summarized as the 'who', 'why', 'how' and 'what then' of the article. The considered contributions were by Khalatnikov (1955), Matthews & Salam (1955), Tobocman (1956), Candlin (1956), Martin (1959) and, finally, Berezin (1961, 1965). Through this list, we saw that the search for a formulation of the fermionic path integral was mainly a British and Russian endeavour, despite the American origin of the path integral itself with Feynman. Within each country, we observed a significant amount of continuity in the research of each consecutive author, although Berezin was even well aware of his British counterparts. A very nice conclusion that can be drawn based on the work of these

authors is that each was able to bring something new to the table, with the endpoint of these developments being our modern formulation of dealing with fermionic fields by describing these in terms of Graßmann variables.

Khalatnikov aimed to derive a functional integral representation of the electron propagator but, by starting from an approach closer to Schwinger's differential methods, first had had to work with functional derivatives of Dirac spinors. The anticommuting behavior therein was then absorbed into a matrix. The functional integral was then itself done using regular commuting fields, with the matrix taking account of the minus sign bookkeeping.
The work of Matthews and Salam was not incomparable to that of Khalatnikov. The two saw the problem with integrals over fermionic fields clearly. Their way of deriving the one-particle electron propagator was to expand the Dirac field in a sum of the product of commuting coefficients and anticommuting functions. The functional integral is then performed over the regular-numbered coefficients, with the permutations over the sum of anticommuting functionals doing the rest.
Tobocman argues that these indirect methods, as well as his own arguments, nevertheless show that it is not possible to lay down a Feynman principle for fermionic fields. The sum-over-histories approach does not yield an exponential with a classical action as its phase. In his approach to showing this, he also introduces the idea of anticommuting eigenvalues of operators. His work can be seen as a bridge from the earlier two authors to the latter three.
Candlin (and, later, Martin) takes up this challenge. He introduces eigenstates of the Dirac oscillator with anticommuting eigenvalues. He calls these a-numbers, and like Graßmann numbers, they mutually anticommute and square to zero. He does hold that it is not possible to set up a completeness relation using these states, although he does find an approximate action by using the sum-over-histories regardless. At the end, Candlin ends by giving an integral over anticommuting numbers as a continuous realization of sums over them, but does not comment on it much further.
Martin goes on to establish a system that can be viewed as the classical analogue of a fermionic system, using his 'eigensymbol' theory that like a-numbers is isomorphic to those of Graßmann. He defines a completeness relation whereby a unit operator indeed comes out, and successfully derives a classical fermionic Lagrangian. He also introduces both derivatives over anticommuting values as well as integrals. He characterizes his functional integrals as an indefinite repetition of an operation $\hat{S}$, which is effectively the Berezin integral.
Finally, Berezin was seen to recognize Graßmann's exterior algebra in all of this, and formulated a rigid mathematical framework to deal with bosonic and fermionic fields using functional integrals. He introduced the Berezin integral over Graßmann variables, including the rules about how it can be used. He also explicitly derived (164). His clear and all-encompassing formulation is the endpoint of the development of fermionic integrals and is still used today, as well as having the most direct influence on later developments.

The process from clearly formulating the central problem and using indirect ways of adressing it to ultimately ending up with a new 'integral' over anticommuting Graßmann-valued

fields, and all inbetween steps such as the introducing of anticommuting numbers, show the gradual process towards the contemporary path integral formulation for bosonic *and* fermionic fields. The potential impact of this on the mainstream narrative will be covered in the discussion, but we can conclude that the reality was that the fermionic path integral was a British-Russian effort that had many contributors, without one name appearing and solving it all himself.

In chapter 5, we saw that the advent of the fermionic functional integral, in particular its formulation by Berezin, has had great impact on the development of both the path integral formulation and QFT as a whole. Not only was it easier to directly express the Dirac field, but more revolutionary, Faddeev and Popov were able to use these results to gauge fix Yang-Mills theory. Having dealt with the degeneracy resulting from the gauge freedom in that theory, Yang-Mills could finally be successfully quantized by means of the path integral formulation. Their 'trick', after all, employed exactly the path integral over Graßmann variables (164) derived by Berezin. These developments made their way to 't Hooft and Veltman, who used (fermionic) path integral methods to prove that Yang-Mills theory is renormalizable. This definitively ended the 'dark decades' of the path integral formulation, which is today used ubiquitously in fields from statistical mechanics to supersymmetry.

In summary, we can answer the research question as follows. The modern formulation of fermionic path integrals over anticommuting Graßmann variables came to be introduced around the late 1950 through the work of British and Russian mathematicians and physicists who initially wanted to be able to describe the Dirac field with it. Through the efforts of numerous subsequent authors, they eventually (re)discovered that Graßmann numbers were a suitable means of doing so, and later extended this with calculus for these numbers. By expressing the classical Lagrangian as well as the integration measure in terms of this anticommuting algebra, they were able to handily perform integrals over fermionic fields. These could be directly used to deal with the degeneracy resulting from the gauge freedom inherent in Yang-Mills theory, which in turn later led to the renormalization proof thereof. These results thus strengthened the capabilities of the path integral as well as paved the way to the construction of the Standard Model of particle physics.

While this answers the research question, there is plenty of room for several categories of reflection. This will now be done in the discussion.

7 Discussion

This final chapter will contain a few evaluative undertakings. We shall start by considering what may conceivably follow from this research. After this, its methodological limitations will be discussed. These shall subsequently be turned into suggestions for future research. The chapter ends with personal evaluation and acknowledgements.

7.1 Implications of the research

The most direct implication that arguably flows out of this thesis, is a new way in which the story of the discovery of the fermionic path integral in terms of Graßmann numbers ought to be told. As was shown, current textbooks most often leave it at mentioning Berezin. As follows from the conclusion, however, multiple authors contributed to this final result in incremental but important ways.

The most concrete example that could be discussed is the attribution of the 'Berezin integral'. After all, most textbooks are understandably not concerned with a deep dive into this history, but just introduce the integration rules for Graßmann variables, equation (164), and get on with it. Therefore, with regard to this history, the *Berezin integral* is what most will be acquainted with. This is simply because this is all they will be confronted with in their studies and research insofar the history of these ideas is concerned. Thus, to the degree that we are engaged with the attributions of the contributions discussed in this thesis, textbook references such as these are very impactful.

Given the fourth chapter, it is certainly not an indefensible position that instead of 'Berezin integral' or the sometimes appearing 'Graßmann integral', it should be called the 'Candlin integral' (as did Mandelstam) or 'Martin integral'. While Candlin was the first to explicitly put an integral over anticommuting variables on paper, Martin was the first to work out the calculus to an adequate extent, defining the integral through his operator $\hat{S}$. I believe a genuine case can be made for all four of them. Perhaps the most neutral option is to use 'Graßmann integral', as he is the mind behind the algebraic structure but is not involved in the potential priority dispute on who actually first devised the integral on that structure. The later, however, can also be submitted as a reason *not* to name it after him. If one wants to divide the credit among the three devisors of the integral, the historical chronology might lead one to the name 'CMB-integral'.[64] Ultimately, there is no objective answer to this question, as it simply depends on one's own standards related to from what point onward what any of these authors devise 'fits the bill'.
If I were to have to give my own opinion on the issue, I would certainly say that at the very least Martin deserves some recognition for his work. While Berezin clearly went further and provided a more elegant mathematical structure, Martin, as we saw, already provides all the tools needed to perform functional integration over fermionic fields, and in a highly com-

[64] Although, perhaps astrophysicists would not like this much.

prehensible manner too. While Candlin gets at least some recognition through the work of Mandelstam who is himself a well-known physicist, Martin is never cited on the topic outside of the works of Berezin himself. In any case, that is one reason for there being no need for a Newton-Leibniz style ugly priority dispute. This is not only the case due to the magnitude of the discovery which, while being high, nevertheless not being the creation of all of calculus. But it is also good to reiterate that, as we discussed, Berezin indeed gave generous references to these authors, with even his fellow countrymen explicitly recognizing Martin as his predecessor (Karabegov et al., 2013, p. 15-16).

With the attribution-dilemma of the functional integral over fermionic fields aside, there are conceivably some more minor implications of the research that may also briefly be mentioned. One of these concerns the interpretation of the path integral. Throughout the thesis, this was somewhat of a side-issue that turned up every so often. The instigator of this is the, as described in chapter 2.1 and 2.5, commonly held belief that the path integral can be characterized as a very intuitive formulation of quantum theory, a clearly visualizable one, and one where the classical limit is exceptionally insightful. Generally, I have been critical of this. As argued in chapter 2.1, I believe that the postulates of the path integral formulation, when taken to their logical conclusion, do not at all make the classical limit as obvious as it is often taken to be. The core of this argument was that this belief confuses the dominant contribution of near-classical paths *to the probability amplitude* of a particle to, after starting out at a spacetime point A, appear at B after measurement, with the likelihood of the manifestation of these particular paths in physical reality itself. The postulates of the path integral are, after all, not about 'which path is a particle likely to take'.
While the topic appeared at some other places too, one other that may be mentioned here is a point made by Candlin. As was discussed, he noted that his anticommuting, zero square a-numbers cannot be the outcome of a measurement of field strength (or any other observable for that matter). This connects the interpretation debate to the topic of this thesis, because it shows that the introduction of anti-commuting numbers to describe fermionic path integral problematizes the interpretation thereof. An implication of this research can therefore be a more critical look regarding the interpretation of the path integral formulation.

A final small implication I wish to mention concerns the more didactical goals of clarity, thoroughness of derivations and original analysis mentioned in the Approach section at the beginning of this thesis. These were specifically devised for chapters containing primarily theoretical background necessary to answer the research question, without themselves containing novel results *per se*. It is my hope that the reader has come across some fruitful descriptions that contribute to the realization of these goals. Examples of these attempts include the visualization of the abstract exterior algebra for physicists in chapter 3.3 in relation to clarity, the proof of the equivalence of formulations in chapter 2.2 in relation to thoroughness and the interpretation issue in relation to original analysis. At least some positive feedback has been received on this front, from one Bachelor's student who was able to use and cite this work for his own thesis and emailed me that he found it to be the "clearest

explanation of the path integral that he had seen so far". Nevertheless, $N = 1$, and perhaps he had not consulted enough other sources! On a more serious note, there are also critical remarks to be made that extend to these goals, as will be discussed near the end of this chapter.

In summary, the research in this thesis may have implications for the attribution of recognition relating to the invention of functional integration over fermionic fields, the interpretation debate surrounding the path integral formulation and the way in which required foreknowledge on these topics could be conveyed. With that clear, we now move to the limitations that were also present in the research.

7.2 Limitations of this research

In this subsection, the focus will be on limitations of this research project relating to its *historical methodology*. Limitations of other type will be discussed in the final subsection.

In the Approach section, as well as in the beginning of chapter 4, we described that the story of the invention of the fermionic path integral would be researched mainly through a set of carefully crafted questions to be applied to the selected articles. This by itself I believe to be an adequate methodology, as these questions were designed to be able to analyze the primary sources through a historical lens as best as possible. Its effectiveness was further enhanced by a partial presence at the MPIWG during this project, allowing for very fruitful discussions, supplementary relevant courses and otherwise unavailable resources such as its extensive and highly-specialized library.

This single-handed focus on interrogating the primary sources did, however, have its limitations. As was also mentioned, many of these papers were written in a history of physics context one is not intimately familiar with. These contain sometimes dense and difficult physics in (out)dated notation, while simultaneously being a niche in the sense that not much supporting secondary literature is available that directly engages with the primary literature. This all combines to, sometimes, make it difficult to understand everything there is to understand about these papers and their context, as well as being a more time-consuming endeavour than the mere number of pages that tend to be involved may have one suspect[65].

Related limitations of this research regarding historical methodology comes in the form of the absence of methods such as interviews and extensive archival research of correspondences. Not only could such undertakings aid in understanding and contextualizing the examined articles, it would also have the epistemic advantage of exchanging my fallible interpretation for the direct experiences of the people involved[66]. Some exploratory efforts have been made in both categories. For example, to get more information on Martin, I have

[65] Naturally, this also has to do with the level of experience of the researcher involved, which is in this case still very much in development.

[66] Of course, people's memory is also fallible, and historians still need to be highly critical of possible personal incentives of interviewees, such as a desire to be featured in history in a certain way. Yet even in given these

on multiple occasions reached out to the two institutes he was aligned with at the time of writing his articles: the *Tait Institute of Mathematical Physics* and the *National Physical Laboratory* in the UK. Unfortunately, neither could help. I also had access to a letter archive. This yielded some results, as with the example of Peierls in chapter 4.2, but in general many of the discussed authors were hard to find. These outcomes are related to the fact that the authors discussed in this thesis are all no longer alive and, in some cases, were not that well-known. Direct interviews were therefore not possible, and especially in the case of Tobocman, Candlin and Martin, it is hard to even find who one would reach out to to learn more about them. In the case of the Russian authors, the language barrier also makes for a higher threshold for such enquiries.

This is all not to say that it would be impossible to find some relevant letters or knowledgeable descendants of these authors open to conversation, but more so that these research methods would in this context be especially time-consuming, and therefore be better suited for another research project. This then brings us to the next subsection.

7.3 Suggestions for further research

Directly continuing with the final remark in the previous subsection: one first suggestion for possible future research would be to do extensive archival research and conduct interviews with people involved. Even if no direct writings of someone like Tobocman can be found, this may still be done through the letters of other, related experts. For example, we were able to extract some greater context to the work of Matthews and Salam not through their own letters, but through those of Peierls and Dyson. Moreover, given the post-1955 time period we focused on, it is not at all unlikely that more information on the discussed authors could be retrieved from interviews with descendants, younger research colleagues and formal inquiries at the institutes were they worked. For example, Candlin passed away as recently as 2019, so many people who directly knew him are bound to be alive and well.
Efforts making use of these suggested methods could, for reasons already suggested in the previous subsection, greatly contribute to a more in-depth answer to the research question than could be reached in the confines of this thesis.

If one wants to focus on the developments after the introduction of the fermionic path integral, the possibilities become even greater. Whether it would be 't Hooft who renormalized Yang-Mills using path integral methods or Witten doing so with supersymmetry, many of these physicists are both alive and active. This also relates to the possibilities for future research on adjacent topics to the one in this thesis. The following list contains, perhaps, a 'lucky 7' suggestions:

- How exactly did research on quantum gravity and on the fermionic path integral cross-pollinate, as we saw in our discussion of the conference on Relativistic Theories of

issues, being able to speak with the researchers directly involved would undeniably be a great asset to the research of this thesis.

Gravitation and the motivation of Faddeev and Popov?

- What mathematical inventions have resulted from efforts of putting the path integral formulation on solid mathematical ground?
- How did the mainstream interpretation of the path integral end up being what it is, and what role has it played in the debate on the interpretation of quantum mechanics?
- To what degree did Berezin's supermathematics determine the developmental trajectory of supersymmetry?
- Why did Feynman eventually become less-enthusiastic about his own increasingly popular formulation, and why did the time-ordered operator calculus he thought to be more promising instead never catch on like that? (Feynman and Hibbs, 1965, p. 355-356)(Feynman, 1951)(Blum, 2017, p. 63)
- Who was, in detail, responsible for what modification of exterior algebra since its conception by Graßmann, to eventually arrive at the modern day formulations of it?
- Why did the path integral formulation, despite its dominant position in advanced QFT and beyond, never become fully integrated into the QM-curriculum at most universities?

An even more ambitious topic would be expand the current research to eventually encompass a full account of the development of the path integral formulation in the 'dark decades' between Feynman's thesis and ending at Faddeev Popov quantization or 't Hooft's renormalization of Yang-Mills theory. Several periods could be distinguished, as well as several lines of development (e.g., mathematical rigor, fermionic systems, application to QM-systems, etc.). While this would be an enormous undertaking, nothing on that scope currently exists.

Lastly, it is also not unrealistic to say that a more experienced historical researcher, better schooled in (especially around 1950s) QFT, could probably extract more useful information from the papers discussed in this thesis. One final suggestion could thus be to apply the methodology of this research project again, using this written thesis as a stepping stone and, hopefully, adequate introduction. This, however, already touches on personal evaluation. This will be the final subsection that we now move to.

7.4 Personal evaluation and acknowledgements

To end this thesis, I will conduct some introspection and evaluate both on the many things I learned during this thesis as well as lessons for what could be improved on. I will end with a word gratitude.

Working on this research project the past academic year has been an immensely valuable

experience, whether it be with regard to knowledge, competencies or not directly physics-related points. Starting with the first category, I have without a doubt been able to pick up on much knowledge of physics and mathematics. Before starting this project, I had only the most superficial understanding of the path integral formulation, and I doubt that I had even heard of exterior algebra. Moreover, it has greatly added to existing knowledge I had of quantum physics by being able to consider things from the perspective of another formulation, as well as being introduced to some new applications and topics in the field. For example, I did not know too much about gauge fixing, and found this topic to be clearer in the context of the path integral. Lastly, new physics knowledge did not only come through direct research that ended up in this thesis. I also read plenty that did not end up being included but was nevertheless useful, as well as being able to participate in interesting colloquia at the MPIWG, following the course 'Cultures of Modern Physics' and 'Knowledge and its Resources' in Berlin and participating in the reading group sessions of the RCNP.

As for competencies, I have been able to expand these in both the area of physics as well as history. Starting with the former, doing previously unfamiliar derivations myself and discovering new techniques along the way has added to my physical repertoire. As a simple example, while I knew about the completeness relations, being able to practice applying these myself in this way has certainly made it easier to recognize when this could be a useful trick in other settings. More novel to me were the historical research methods. While I obviously still have much to learn, it was a good experience to learn to think like a historian and know what kind of questions to ask when confronted with a given text. More indirectly, seeing how people approached their research projects at the MPIWG and chatting with them about this was also very much valuable. This ranges from the type of questions they ask to smart ways of finding the right papers.

The most prominent lessons unrelated to physics itself manifest in the time I was able to spend in Berlin. Some of these have little to nothing to do with the research project, such as being immersed in another culture and 'building a new life' in a city where you initially do not know anyone. Yet, such social experiences are undeniably valuable. Somewhat closer to doing research itself was seeing how an institute other than *Radboud University*, where I have spent my entire academic life, operates. Other categories include practicing scientific writing, working in two research groups and thinking about how to convey new concepts as clearly as possible given a particular target audience.

Of course, there are also lessons to be learned from what could have gone better. For example, I could have reached out to even more possible contact points to get information on Martin. However, to me, by far the most prevalent lesson of this research project is time-management, from which I feel most of the problems this body of work (still) has stem. While on the one hand any thesis can always be better and a line has to be drawn at some point, I do think I should have better anticipated time-constraints which have led me to not being able to do everything I would have wanted to. I also tended to be too optimistic on how

much I could do in a given time, where I should have known better. As for anyone, these time-constraints were not without reason, so I shall limit myself to three important ones. For one, given the combination of two Master's programs, there was still other academic work that needed to be done that could draw away attention from this research projects at inopportune moments. A second factor was the amount of time spent on my student job but especially post-graduation job hunt, which was eventually successful, yet this position did already demand substantial time investment especially in the final period of this research project. This job also came with a required graduation deadline at the end of June, the strictness of which I had not had in mind most of my time working on this project. And while one could always think of external reasons, I should also say that settling in Berlin and back home afterwards took me some time to adjust.

While this has had only a very limited impact on writing the content I had set out to write, it ultimately did affect my ability to reflect on what was already in place. I would have liked to fine-tune these chapters much more and, in particular, would have wanted to take far more time to carefully go through the many insightful points of feedback I have received.
Through better time-management, many of these issues could have been mitigated. This does not just mean 'being more productive', but also making different choices within the research project. For example, I spent a lot of time reading about Clifford algebra and had written many notes on it, to ultimately not really do much with it because its relevance was very much secondary. When the deadline approached, I could also have chosen to drop, e.g., some of the less relevant parts of chapters 2, 3 and 5, instead investing this in perfecting what is there and especially allocate optimal attention to the historical core of chapter 4.
Therefore, I have been able to draw concrete lessons out of this about carefully considering what amount of time one *realistically* has, and then choosing accordingly, rather than not daring to make a choice meaning that time will choose for you. And time, as it turns out, does not necessarily make the best choice either. As a final note on the topic, perhaps the project can be further developed in my spare time, which would ultimately entail giving more attention to these issues after all.

To close, I would like to extend some words of gratitude.

The first half of this research project took place at the MPIWG, under the supervision of **dr. Alexander Blum**. You were always open for a chat, whether it would be for a quick question or long technical discussion. Thanks to your general involvement and enthusiasm I always left those with fruitful new ideas about physics, history and clever methods you proposed. I am very grateful for being able to have been a part of the research group on the final theory program and all that it entailed, from interesting guest lectures to fun group activities.

The second half of the project took place at the new RCNP, under the supervision of **prof. dr. Klaas Landsman**. It was always a fun start of the week to work with you and the others at the office on Mondays. Our conversations on the varying topics related to this thesis

have been indispensable to this project and incredibly instructive for myself. It has been an honour to have played a small role in the first year of the RCNP, and there is no doubt in my mind that it will grow into a successful, intellectually stimulating and fun research group.

Lastly, I want to thank my **girlfriend, family and friends** for their support and patience in this ninth and final year of my studies. With that, it is now time for a new phase of life.

8 Bibliography

In this section, all of the sources used throughout this thesis can be found in APA-style.

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