INTUITIONISM

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1. Intuitionism is a constructivist view of mathematics, but not the only one. It is the only one which has been inspired by a general philosophical point of view (as far as I can see).

2. I shall present an overview of its principles and methodology. Familiar ground for most of you, but I shall provide some annotations.

3. Intuitionism started in 1907. Basic Philosophy unchanged, the mathematical consequences developed over the years in Brouwers thinking.

4. Brouwer summarized in "first and second act". We consider a late formulation of "the first act

first half { The first act of intuitionism separates mathematics from mathematical language, in particular from the phenomena of language which are recognized by theoretical logic and recognizes that intuitionist mathematics is an essentially languageless activity of the mind .....

5. This is the first half and states that mathematics does not need language, and that logic is tied to language (in Brouwers view of logic); more of this later

6. The first act continues:

second half { ... activity of mind having its origin in the perception of a move of time, that is, the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory.

and Brouwer adds:

addit. The two-ity thus born is divested of all quality, there remains the empty form of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.

7. This may be clarified by paraphrasing it, borrowing concepts from Husserls phenomenology as explained by Mark van Atten.

(a) A philosophy which only talks about objects is incomplete, since it does leave out the subject which experiences / becomes conscious of / perceives objects.

(b) To experience an object is an act.
This activity presupposes an awareness of the move of time, a "before" and an "after" the experience of something.
This is Brouwers two-ity.
Time is very abstract here, it is certainly not physical time.

(c) The subject (in Brouwers language "creating subject) is not a psychological subject, not simply a subjects as we are — rather, it is an abstraction, something very basic at the root of subjectivity.
Illustration:
The idea that we may die, so that our activity comes to an end, is part of our awareness of ourselves as psychological subjects, but ~~does~~ is not ~~not belong~~ part of to the idea of the creating subject.
Husserls terminology here would be to speak of of a transcendental subject, transcending the our ~~being~~ role psychological subjects in the world.

You might also say: the transcendental subject is the most fundamental and irreducible aspect of our subjectivity. By its acts the TS can create math. objects.

8. Consequences.

(a) language, that is our need for language as an aid to memory and as a means of communicating properly belongs to the psychological subject (hence: languageless activity!)

(b) The first act by repetition gives us natural numbers and induction.

9. Actualist or ultra-intuitionist critique

Intuitionism treats all nat nrs as the same type of construct. But our ways of grasping 3, 100, 9^{9^9} are not the same. Some numbers we can write down with exponents for example might be "too large" to be viewed as natural numbers reached by chasing after successors.

This is a psychologist criticism.

10 Implementing actualist views is notoriously difficult; conceptually the idea of a "too large number" smacks of a sorites paradox. No satisfactory implementation is known to me (A bit like non-standard model of arithmetic)
(Polynomial arithmetic not a partial implementation)

Here the "transcendental" view helps: the theory becomes simpler by corresponding on the transcendental subject.

II. Logic

BHK -interpretation.

mathematical acts are constructions

proofs are mathematical acts.

a statement (to be proved) represents
a condition to be fulfilled by a proof / construction.

~~So~~
A logical law is to an intuitionist just
a construction of a very general nature.

P proves $A \to B$ if any proof q of A
is transformed into a proof $p(q)$ of B.

$A \to (B \to A)$:
any proof of A, q
is transformed into a proof of $B \to A$
by a constant function q'' s.t. $q''(q') = q$ for all q'

For ~~any~~ all math. statement A, B the same ~~type~~ sort of
construction works — this observation gives us the logical law.

12 Second act. also falls apart into two clauses.

The second act of intuitionism recognizes the possibility of generating new mathematical entities: firstly in the form of infinitely proceeding sequences $P_0, P_1, P_2 \ldots$ whose terms are chosen more or less freely from mathematical entities previously acquired ...

and secondly in the form of mathematical species ...

The first part of the second act introduces choice sequences; the second part species; properties of elements in a previously given domain.

13 Originally Brouwer held that the awareness of time created both the two-ity and the irreducible notion of the continuum. Like Borel.
Never retracted.
Mathematically unmanageable.
Introd. of choice sequences (inspired also by Borel!)

14 Choice sequences of nat nrs.

Created by the CS.; processes.
for ever unfinished.
at any moment only finite segment known
+ whatever restrictions on future choices the creating subject has imposed until now.

Borderline case: Lawlike: a sequence completely fixed by a recipe. ~~We have~~ Does lawlike really mean algorithm? Perhaps. More later.

15 Maybe forever unfinished,
but with definite identity

16 Singling out particular kinds of choice sequences.

a) No restrictions ; lawless LS
b) Lawlike LL already mentioned.
c) hesitant sequences HS.
d) general sequences: all sort of restrictions permitted provided continuation is guaranteed.

Holistic view:
there is only one meaningful domain of choice sequences, that is, over which you can quantify.

Illustration of differences

LL $\forall\alpha\,\exists a\,(\alpha = a)$

LS $\forall\alpha\,\neg\exists a\,(\alpha = a)$

HS $\forall\alpha\,\neg\neg\exists a\,(\alpha = a) \wedge \neg\forall\alpha\,\exists a\,(\alpha = a)$

let us look at this.

How do we find such properties?

17. Methodology : informal rigour

Not a deep notion.
Need for it because I hold that it is not always easy (for us) to discern what belongs to the transcendental aspect of subjectivity, and what is psychological.
This rift requires a methodology. ?
But the only thing we have is informal rigour:
take your informally described ~~ob~~ math. objects seriously and ask yourself what it means to be "given" an object of a certain kind.

18.) "It can only be the case that" — argument

~~Usual~~ We most often have to do with implications

if we prove A, then ...

how can we prove A, what does it mean to prove something of the form A.

Sometimes rather crude motivation "Argument from impotence"

e.g.

UP. $\forall X \exists n\, A(X,n) \rightarrow \exists n \forall X\, A(X,n)$

But the typical choice-sequence pr. more sophisticated, though one still has to use a little jump from belief to conviction (or call it a strong interpretation.

19) Continuity, e.g. for lawless sequences

20) Bar theorem

21) The need for complicated notions

a) lawless: nice for substitutional validity but not for analysis.

b) restrictions on α ~~must~~ ~~can~~ must possibly depend on other choice parameters

22 Creating Subject arguments.

Assumption: the CS can "take stock" at regular intervals of its actions, so to speak punctuate ~~continuous~~ the continuous flow of time.

$\vdash_n A$: the CS has experienced (prove) the truth of A at stage n.

Exp(n, A) might as notation be preferable $\vdash$ is already overloaded — but too unwieldy.

(1) $\vdash_n A \vee \neg \vdash_n A$.

(2) $\vdash_n A \rightarrow \vdash_{n+m} A$. truth is stable.

(3) $\exists n(\vdash_n A) \rightarrow A$ reflection on activity!

$\exists n(\vdash_n A) \rightarrow \neg\neg A$ (or $A \rightarrow \neg\neg \exists n \vdash_n A$)

Somewhat stronger

(3*) $A \leftrightarrow \exists n(\vdash_n A)$. "truth is experienced truth"

(1), (2), (3) suffice for Brouwers prototype argument showing that there is a real x s.t.

$?(x \neq 0 \rightarrow x \# 0)$ cannot be shown.

3* seems to suggest that no matter what statement you consider, the CS will attempt to prove it.

The very first formal statement (by Kreisel) in (1)–(3) actually relativized things

$\vdash_n^B A$

"in the course of attempts to establish B, the CS experiences the truth of A"

Then $\exists n\ (\vdash_n^B B) \leftrightarrow B$

certainly holds. Does this caution help with the difficulties I am going to signal? I doubt it.

23. The typical argument:

Let $? A \vee \neg A$.

Define x^A by a sequence of rationals r_n^A:

$r_n^A = 0$ if $\neg \vdash_n A$ and $\neg \vdash_n \neg A$.

$r_n^A = 2^{-m}$ if $\vdash_m A$ or $\vdash_m \neg A$ for $m \leq n$, m the first such.

Easy to see: This is a choice sequence

$x^A \neq 0$.

for if $x^A = 0$, then $\forall n (\neg \vdash_n A \wedge \neg \vdash_n \neg A)$

$\neg A \wedge \neg\neg A$ contradictory.

but $? x^A \# 0$

because if we can tell how, we know we have decided A.

Very convincing.

24. Narrowing it down.

A single conclusion at a time, is that plausible?

It would mean things like

$\forall m\, Am \rightarrow \forall m \exists n (\vdash_n A(m))$

a commitment we don't like. But let us accept for the time being.

Diagonalization trouble.

Consider conclusions of the form

$\vdash_p \forall n \exists! m\, B(n, m)$. (B describes a function f_B)

Let us enumerate them as

$f_0, f_1, f_2 \ldots$

Then define

$f(n) = f_n(n)+1$

Obviously a (choice) function; but there is no stage where we can draw the conclusion

$\forall n \exists! m (\cancel{f(n)=}\ \underset{m}{} f_n(n)+1)$

So in constructing its own activity, the CS ought to distinguish levels.

$\#$ objects by reference to

25) Let us reject the "atmost one conclusion at a time" hypothesis.

But how much is implicitly evident whenever A has become evident?

$$\vdash_m A \wedge B \Longrightarrow \vdash_m A \text{ and } \vdash_m B$$

defensible, as well as

$$\vdash_m \forall n A(n) \longrightarrow \vdash_m A(n) \text{ for all } n.$$

But not everything that follows from A by e.g. pred. logic is evident at the same time.

The diagonalization paradox is blocked, but we run into another kind of trouble.

26) Lawlike, what is it?