PLAYING BROUWER'S GAME

or

PLAYING CANTOR'S GAME IN BROUWER'S WAY

Nijmegen
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The propositions are divided into problems and theorems

(Plato ridicules the language of the geometers)

Speusippos: All propositions are theorems.

Menaechmos: All propositions are (construction) problems.

Proclos: both are right

(The discovery of theorems does not occur without recourse to "intelligible matter".

It is in imagination that constructions take place whereas the contents of our understanding all stand fixed without any generation or change.)

Cusanus:

Mens nostra quae mathematicalia fabricat ea, qui sui sunt officii, verius apud se habet quam sunt extra se ipsam

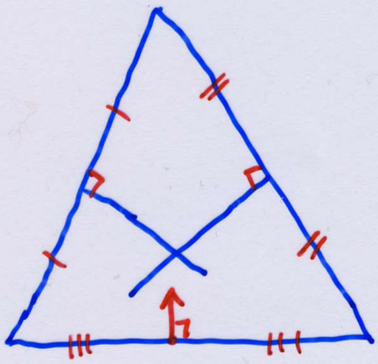
Kant

$$7 + 5 = 12$$

The outcome 12 can not be found from the definitions of 7, 5 and + alone

One has to invoke the help of intuition

1	2	3	4	5	6	7	1	2	3	4	5
x	x	x	x	x	x	x	x	x	x	x	x
							8	9	10	11	12



The third perpendicular bisector will go through the point of intersection of the first two.

Brouwer:

The statement of a theorem is an incomplete communication.

Its true and full meaning is revealed only by the proof

0, 1, 2, 3, ...

Every one knows this sequence as a sequence without end

One should preserve one's sense of wonder at this infinity and our ability to handle it.

Euclid IX, 20

There are infinitely many prime numbers

The prime numbers are more than any previously given multitude of prime numbers

$$F(n_0, n_1, \dots, n_k) \neq n_0$$

$$F(n_0, n_1, \dots, n_k) \neq n_1 \qquad F(n_0, n_1, \dots, n_k) \text{ prime}$$

$$\vdots$$

$$F(n_0, n_1, \dots, n_k) \neq n_k$$

Euclid's theorem is exemplary.
It shows us how to use the word "infinite"

(Is such coinage of terms logic ?)

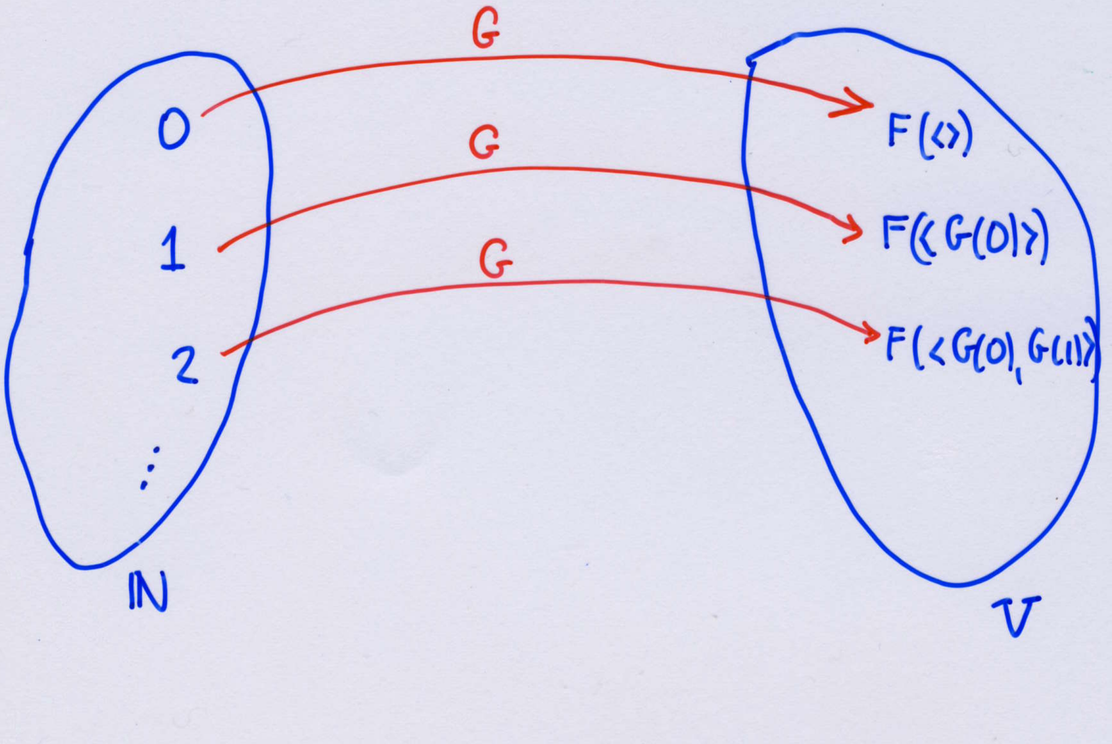
V, W sets

V is an infinite subset of W :

$$F(w_1, \dots, w_k) \neq w_1 \qquad F(w_1,\dots,w_k) \in V$$
$$\vdots$$
$$F(w_1, \dots, w_k) \neq w_k$$

Theorem: (ℕ is the smallest infinite set)

Let V be infinite subset of W
Then ℕ may be embedded into V



November 24, 1858

Eudoxos Eucl. El. V

$$A : B = C : D$$

For all m, n in $\mathbb{N}$

either $m \cdot A < n \cdot B$ and $m \cdot C < n \cdot D$

or $m \cdot A = n \cdot B$ and $m \cdot C = n \cdot D$

or $m \cdot A > n \cdot B$ and $m \cdot C > n \cdot D$

(Definition <u>in context</u>)

Dedekind :

$$A : B = \left\{ \frac{m}{n} \mid m \cdot A < n \cdot B \right\}$$

proportions / ratios as objects.

Cantor / Brouwer:

A real number is an infinite sequence

$x(0)$, $x(1)$, $x(2)$, of pairs of rationals

$x(0)$: $x'(0)$ $x''(0)$

$x(1)$: $x'(1)$ $x''(1)$

$\vdots$

$$x'(n) \le x'(n+1) \le x''(n+1) \le x''(n)$$

For each m, there exists n s.t.

$$x''(n) - x'(n) < \frac{1}{2^m}$$

$$x = x(0) \quad x(1) \quad x(2) \quad \ldots$$

The real number is continually growing and never finished.

$$\pi = 3,1415\ldots$$

$$x(n) = \left(-\tfrac{1}{n}, \tfrac{1}{n}\right) \quad \text{if } n < k_{99}$$

$$= \left(\tfrac{1}{k_{99}}, \tfrac{1}{k_{99}}\right) \quad \text{if } n \geq k_{99} \text{ and } k_{99} \text{ is even}$$

$$= \left(-\tfrac{1}{k_{99}}, -\tfrac{1}{k_{99}}\right) \quad \text{if } n \geq k_{99} \text{ and } k_{99} \text{ is odd}$$

$x \leq 0$ $\quad \forall n[n = k_{99} \to n \text{ is odd}]$?

$x \geq 0$ $\quad \forall n[n = k_{99} \to n \text{ is even}]$?

$x \leq 0$ or $x \geq 0$?

The exemplary proof of a statement $A \vee B$ provides one with a proof of A or with a proof of B.

Cantor 1873

The real numbers are more than every previously given infinite sequence of real numbers

Let $a_0, a_1, \dots$ be an infinite sequence of real numbers.

There exists $x = x(0), x(1), \dots$ such that $x \neq a_0, \quad x \neq a_1, \quad \dots$

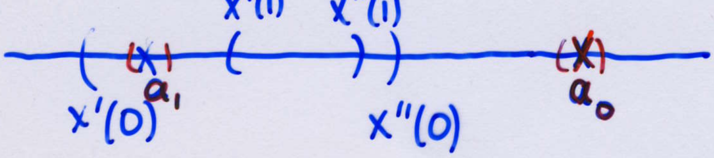
Choose $x(0)$ such that

$$a_0 \notin (x'(0), x''(0))$$

Choose $x(1)$ such that

$$a_1 \notin (x'(1), x''(1))$$

$\vdots$



The proof is beautiful and exemplary

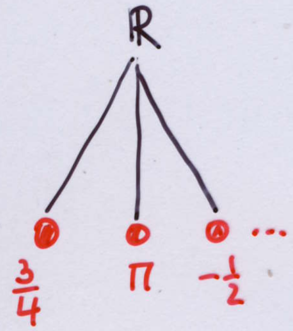
V, W $\quad$ V is an uncountable subset of W:

For every infinite sequence $w_0, w_1, \dots$

$F(w_0, w_1, \dots) \in V$

$F(w_0, w_1, \dots) \neq w_i$, $\quad$ for each i.

The set of all real numbers

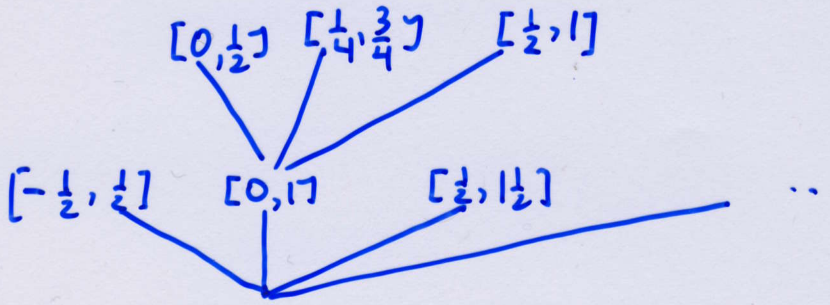


Cantor:

We form a set by taking together certain earlier constructed objects into a new whole

Brouwer:

Make the following picture:



The real numbers are the infinite paths in this tree

I. have to think of them as possible infinite paths

Let F be a function from $\mathbb{R}$ to $\mathbb{R}$
Then F must be continuous.

<u>Theorem</u>: „ $\{0,1\}^{\mathbb{N}}$ is the smallest uncountable set"

Let V be an uncountable subset of $\mathbb{R}$
Then $\{0,1\}^{\mathbb{N}}$ may be embedded into V.

<u>Proof</u>:

$$F(x_0, x_1, \dots) \# x_0$$
$$F(x_0, x_1, \dots) \# x_1 \qquad F(x_0, x_1, \dots) \in V$$
$$\vdots$$

$$\mathbb{F}(x_0, x_1, \dots x_m) = \{ F(x_0, x_1, \dots x_m, \dots) \mid (x_0, x_1 \dots x_m, \dots) \in \mathbb{R}^{\mathbb{N}} \}$$

If $F(x_0, x_1, \dots) \neq F(y_0, y_1, \dots)$, then
there exists m s.t. $\mathbb{F}(x_0, \dots, x_m) \cap \mathbb{F}(y_0, \dots, y_m) = \emptyset$

$$F(x_0, x_1, \dots, x_m, 0, 0, \dots) \#$$
$$F(x_0, \dots, x_m, F(x_0, \dots, x_m, 0, \dots), 0, 0, 0 \dots)$$

$$\{0,1\}^* \xrightarrow{G} \mathbb{R}^*$$
$$u \sqsubseteq t \longrightarrow G(u) \sqsubseteq G(t)$$
$$\mathbb{F}(G(u * \langle 0 \rangle)) \cap \mathbb{F}(G(u * \langle 1 \rangle)) = \emptyset$$