Radboud University Nijmegen

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The History of the Riemann–Roch and Hirzebruch–Riemann–Roch Theorem

Thesis MSc Mathematics

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Abstract

In this thesis, we investigate the mathematical development of the Hirzebruch–Riemann–Roch theorem and its predecessor, the Riemann–Roch theorem. First, we describe the emergence of the theory of Abelian integrals. Then we study how the Riemann–Roch theorem originated from the theory of Abelian integrals. Lastly, we we examine the synthesis of new mathematical methods that culminated in the Hirzebruch–Riemann–Roch theorem.

Contents

1 Introduction

The Riemann–Roch theorem (Theorem 4.7) forms one of the pillars of the theory of Riemann surfaces and is the result of a synthesis between algebra, analysis and topology. The first part of this thesis describes how the Riemann–Roch theorem formed by describing its historic roots in the theory of Abelian integrals. The second part describes how mathematicians tried to generalise Riemann's theory to any dimension. It describes a split between the different branches of mathematics and shows how the reunion of them gave the necessary insights to put the pieces together. The resulting Hirzebruch–Riemann–Roch theorem (Theorem 9.18) was not only a milestone in intself but had an important impact on later mathematics. In particular, Grothendiek generalised it to the Grothendiek–Riemann–Roch theorem (see §10), inventing K-theory along the way. Subsequently, the quest to generalise the Hirzebruch–Riemann–Roch theorem to differentiable manifolds (it was first proven for algebraic varieties) gave the fundamental insights to formulate and prove its far reaching generalisation: the Atiyah–Singer index theorem (an outline of this development is given in [At88]). The last theorem was actually the start of this project. But by asking questions I went far back in time into a very beautiful story.

In this thesis we will only cover the mathematical history of the subject. We will not cover the lives of the mathemacians. Although it would give even more perspective on the history of the mathematics, this is not within the limits of this thesis. The main sources of this thesis are of course the original works in which the mathematics was published, but also the secondary sources that already existed served as a great guide where to search. Most secondary sources used are cited in the text. I would highly recommend, reading this thesis, to also take a look in the original works, with this thesis serving as a guide. As one of our key players, Abel, stated[1] [Sø10]:

> "It seems to me that if one wants to make progress in mathematics one should study the masters and not the pupils."

In particular, I would recommend the works of Jacobi, Riemann, Severi, Serre and Hirzebruch, but also the historical or biographical memoirs written by the great players of this story (e.g. [At77], [At96], [CE97], [Hi96], [Ho49] and [Lef68]).

This thesis can roughly be divided into two parts. The first part covers the history of the Riemann–Roch theorem (Theorem 4.7) ending at chapter 4. The second part is on the history of the Hirzebruch–Riemann–Roch theorem (Theorem 9.18). There are some gaps between these two parts that we won't cover in detail, such as the important work of the Italian geometers (Enriques, Castelnuovo and Severi), Lefschetz, Hodge and Kodaira. Some important results and references are given in §5.4. The work of Kodaira is sketched in §9.1.

First, we explore the path that led to the development of Riemann's theorem (Theorem 4.2) and its extension by Roch (Theorem 4.7). In particular,

[1] Translation from [Sø10]. Original: "il me parait que si l'on veut faire des progres dans les mathématiques il faut étudier les maitres et non pas les écoliers."

we explore why Riemann developed his inequality. Although Riemann's paper [Ri57] is known today for its revolutionary methods in the theory of complex functions, at the time it was important because it solved a then famous, but now practically forgotten inversion problem of Jacobi (§3.3). This problem was posed in the context of Abel's addition theorem (§3.2). We will investigate the early history of the addition theorem (§2.5), which led to Abel's general addition theorem, and explain why people were interested in addition theorems in the first place (§2.3). We will then investigate Jacobi's reasons for posing his inversion problem and see why Riemann developed his inequality (§4).

In the second part (from §5 onwards) we will describe how the different aspects of the Riemann–Roch formula (Theorem 4.7) were generalised. First, we give a brief outline of the algebraic attempts to generalise the Riemann–Roch theorem just after Riemann (§5), then we will develop, in their historic context, the necessary tools to formulate the Hirzebruch–Riemann–Roch theorem. In particular we will treat the development of sheaf cohomology.

Enjoy!

2 The emergence of Abelian integrals and Addition formulas

2.1 Paracentric isochrone

In 1689, Leibniz published an article titled "*The Isochronic Line, on which a heavy object descends without acceleration, and the controversy with mr. Abbate de Conti.*"[2] [Lei89] In this article, Leibniz considered a curve, called the isochronic line, to support his earlier arguments that the *quantity of potential* is conserved. This argument seemed to oppose the Cartesian view that the *quantity of motion* was conserved. Today we call the quantity of potential the *energy* of the system, while the conservation of the quantity of motion has developed into the conservation of *momentum.*

Let's interpret what is meant by the title with the isochronic line (see Figure 1).

Definition 2.1. The *isochronic line* is a plane-curve such that, if a heavy object is restricted to descend along this curve and the only external force is gravity as measured on earth, the object experiences no vertical acceleration.
Alternatively, Leibniz noted, an object having no vertical acceleration may be interpreted as moving away from (or towards) a horizontal line at a constant speed.

With the interpretation that the object on a paracentric isochrone (Definition 2.1) moves away from (or towards) a horizontal line, it may be natural

[2] My translation with the assistance of ChatGPT. Original: "De Linea Isochrona, in qua Grave sine Acceleratione descendit, et de Controversia cum dn. Abbate de Conti."

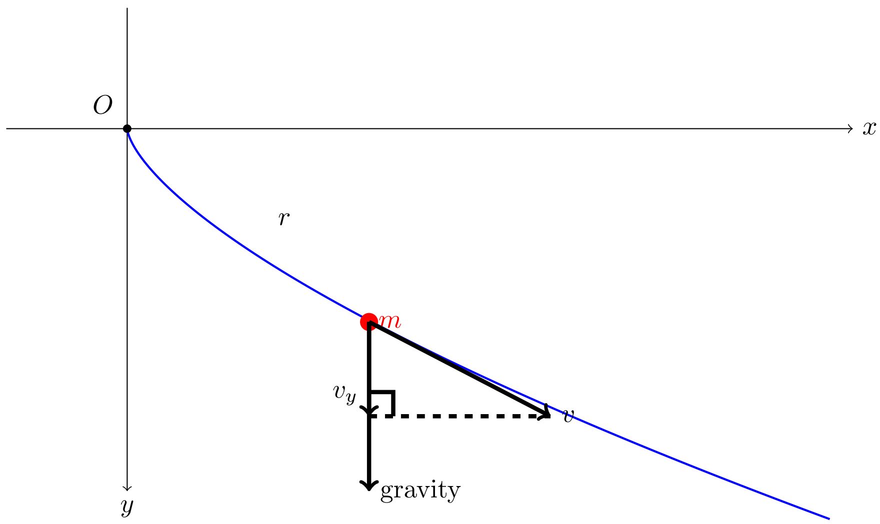

Figure 1: The isochronic line. The curve along which an object descends due to gravity such that its vertical velocity, v_y, remains constant.

to pose the following alternative problem, which Leibniz did at the end of his article:

> "If anyone, however, complains that their already settled solution has been taken away from them, let them seek another isochronous one nearby, in which, unlike before, the weight does not uniformly deviate from the horizontal (or approach it), but from a specific point. Hence, the problem will be to find a line along which the falling weight uniformly deviates from the given point or approaches it."[3]

This line came to be called the *paracentric isochrone* (see Figure 2 and for an original drawing by Bernoulli see Figure 3).[4]

Definition 2.2. The *paracentric isochrone* is a plane-curve such that, if a heavy object is restricted to descend along this curve and the only external force is

[3] My translation using ChatGPT. Original: "Si quis tamen praereptam sibi jam solutionem queratur, petarit aliam isochronam huic vicinam quaerere, in qua non, ut hactenus, grave uniformiter recedat ab horizontali (vel ad eam accedat), sed a certo puncto. Unde problema erit tale, invenire lineam, in qua descendens grave recedat uniformiter a puncto date, vel ad ipsam accedat."

[4] Here ds etc. are seen by Jacob Bernoulli as the infinitesimals of Leibniz. They can also be interpreted as a paramater free way to write a differential equation. If we have an expression with squares such as $ds^2 = dx^2 + dy^2$, it means that if the quantities s, x and y depend on a parameter such as time t, then $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$. Note we work here with quantities (or variables) instead of functions. s can depend different on x than on y, giving 2 different functions.

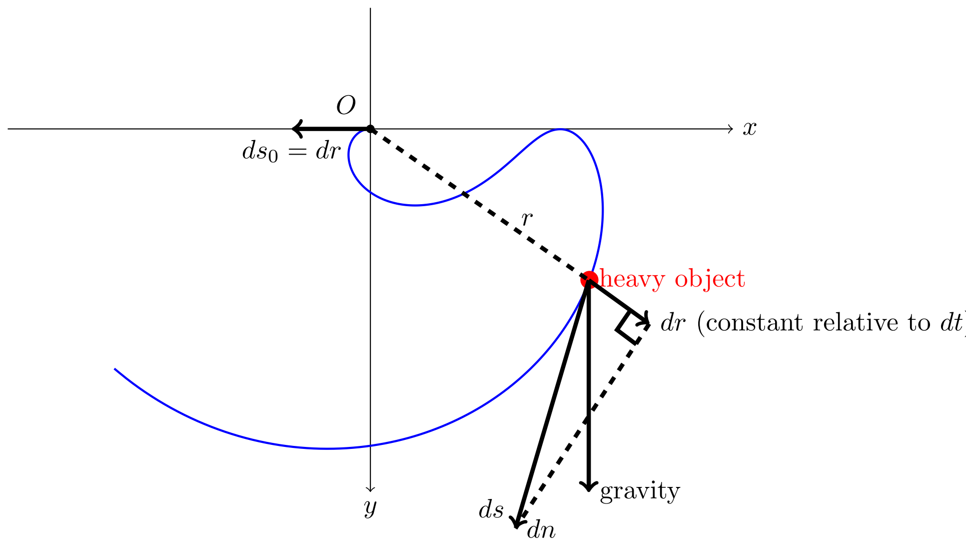

Figure 2: Paracentric isochrone with specific point $(0, 0)$ and the y-axis pointing downwards. This is the curve along which an object descends due to gravity such that the distance to the origin increases at a constant velocity, i.e. dr is constant relative to dt.

gravity as measured on earth, the object moves away from a specific point at a constant speed.

This problem of finding the paracentric isochrone became one of the favourite problems of Jacob and Johann Bernoulli. Jacob found a way to construct the curve using another curve in his article dated June 1694: "*The solution to the Leibnizian problem. On the Curve of Uniform Approach and Recession from a Given Point, through the rectification of the Elastic Curve.*"[5] [Ber94a] In his own words the paracentric isochrone was "The most elegant problem of Leibniz."[6]

Jacob says that in a "Gallorum Diario", Johan Bernoulli derived the differential equation corresponding to the problem. Below we give a derivation along the lines of Jacob Bernoulli. Let the paracentric isochrone be represented by cartesian coordinates (x, y), the y-direction pointing downwards. Let the origin be the specific point from which the weight deviates (or which it approaches) and let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point (x, y) on the curve, also called the radius. Let $ds^2 = dx^2 + dy^2$ represent an infinitesimal

[5] My translation with the assistance of ChatGPT. Original: "Solutio Problematis Leibniztiani. De Curva Accessus et Recessus aequabilis a puncto dato, mediante rectificatione Curvae Elasticae."

[6] My translation with the assistance of ChatGPT. Original: "elegantissimi problematis Leibnitiani"

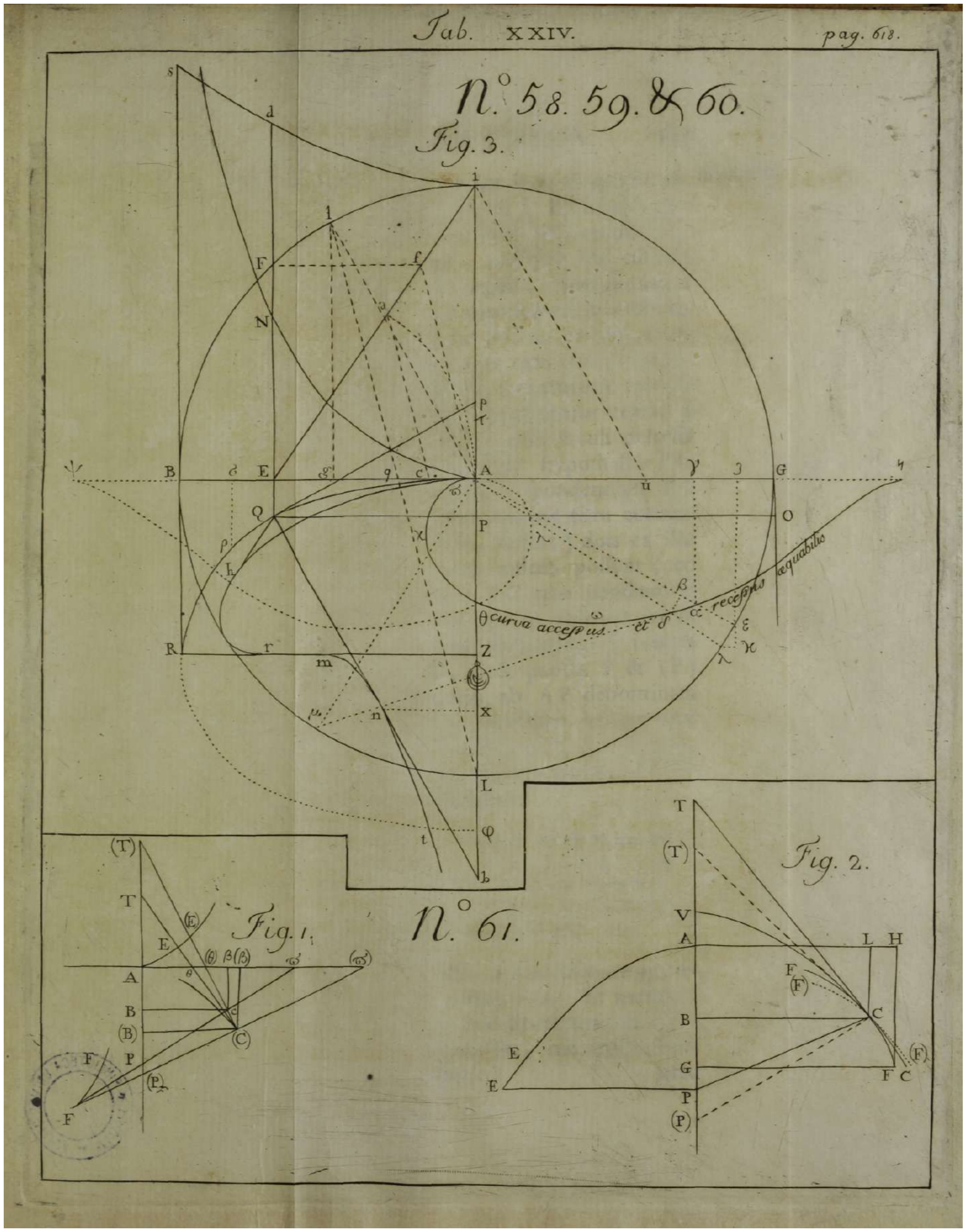

Figure 3: In fig. 3 the paracentric isochrone (*curva accessus et recessus aequabilis*) is drawn. [Ber44]

arc length. For clarity, we may write e.g. ds_0 for the infinitesimal at the origin and ds_p for the infinitesimal at an arbitrary point p. Let t be the time and define the constant a such that $2ga$ is the square of the speed of deviation or approach. $\left(\frac{dr}{dt}\right)^2 = 2ga$. The defining property of the curve (Definition 2.2) is that the speed of deviation from the origin is constant, i.e. $\frac{dr}{dt} = constant$, which is equivalent to saying a is a constant.

Theorem 2.3. *The paracentric isochrone satisfies the differential equation*

$$(xdx + ydy)\sqrt{y} = (xdy - ydx)\sqrt{a}, \tag{2.1}$$

Proof. Jacob Bernoulli coined a law of a descending mass, stating that the square of the velocity gained is proportional to the difference in altitude, which follows directly from our conservation of energy. Namely we get $g\Delta y = \frac{1}{2}\Delta\left(\frac{ds}{dt}\right)^2$, so

$$ydt^2 = \frac{1}{2g}(ds_p^2 - ds_0^2). \tag{2.2}$$

At the origin the velocity is in the same direction as dr, so $ds_0 = dr$. Therefore, $ds_0^2 = dr^2 = 2gadt^2$. Equation (2.2) gives now

$$\begin{aligned} ydt^2 &= \frac{1}{2g}ds_p^2 - adt^2, \\ (a+y)dt^2 &= \frac{1}{2g}ds_p^2. \end{aligned} \tag{2.3}$$

This gives, with $dr^2 = a^2dt^2$, that

$$\begin{aligned} 2gads_p^2 = a(a+y)dt^2 &= (a+y)dr^2, \\ a(ds_p^2 - dr^2) &= ydr^2. \end{aligned} \tag{2.4}$$

From $dr = \frac{xdx+ydy}{\sqrt{x^2+y^2}} = \frac{xdx+ydy}{r}$ we derive that at $p = (x, y)$ we have

$$\begin{aligned} r^2(ds_p^2 - dr^2) &= (x^2+y^2)(dx^2+dy^2) - (xdx+ydy)^2 \\ = x^2dy^2 + y^2dx^2 - 2xydxdy &= (xdy - ydx)^2. \end{aligned} \tag{2.5}$$

Equations (2.4) and (2.5) give

$$y(xdx + ydy)^2 = yr^2dr^2 = ar^2(ds_p^2 - dr^2) = a(xdy - ydx)^2. \tag{2.6}$$

Theorem 2.3 follows. □

Jacob Bernoulli found a clever way to change the variables of equation (2.1) such that the differential equation becomes separable. Define the new variables $w := \frac{a}{r}x$ and $z := \frac{a}{r}y$. Geometrically w and z can be interpreted as the coordinates on the circle when the curve is projected by the line from the origin through the curve on the circle of radius a.

Theorem 2.4. *With $w = \frac{a}{r}x$, $z = \frac{a}{r}y$, the differential equation of the paracentric isochrone transforms to the seperated differential equation*

$$\frac{dr}{\sqrt{r}} = \frac{a}{w}\frac{dz}{\sqrt{z}} = \frac{adz}{\sqrt{a^2z - z^3}} \tag{2.7}$$

Proof. Define $dn = \sqrt{ds^2 - dr^2}$, measuring the change of (x, y) in the direction normal to the radius. The differential equation (2.1) of the paracentric isochrone (as can be seen using equation (2.4)) is equivalent to

$$\frac{dr^2}{dn^2} = \frac{a}{y} = \frac{a^2}{rz} \tag{2.8}$$

Let dN measure the change of (z, w) in the direction tangent to the circle at (z, w). By proportionality we have $dn = \frac{r}{a}dN$. We can draw similar triangles in which the sides dN, dz correspond to a, w respectively, since the angle between the normal and the z direction is equal to the angle of the radius and the w-direction and the change dz is the projection on the z-direction of the change dN and w is the projection of the radius on the horizontal axis. So we conclude that $dN = \frac{a}{w}dz$. Therefore $dn = \frac{r}{w}dz$. So our differential equation can be transformed into

$$\frac{dr}{\sqrt{r}} = \frac{adz}{w\sqrt{z}}. \tag{2.9}$$

We can rewrite this in terms of z by noting that $w^2 + z^2 = \frac{a^2(x^2+y^2)}{r^2} = a^2$. □

The left-hand side of (2.9) can be integrated. Analogously, we may attempt to integrate the right-hand side with respect to $\frac{dz}{\sqrt{z}} = 2d\sqrt{z}$, as Jacob Bernoulli did.

Theorem 2.5. *Let $az = u^2$, with $u \geq 0$. Then the differential equation (2.1) is equivalent with*[7]

$$\sqrt{ar} = \int \frac{a^2du}{\sqrt{a^4 - u^4}} \tag{2.10}$$

Proof. We have $adz = 2udu$ and so $\frac{adz}{2\sqrt{z}} = \sqrt{a}du$. Therefore equation (2.7) can be transformed into

$$\sqrt{r} = \int \frac{dr}{2\sqrt{r}} = \int \frac{\sqrt{a}du}{w} = \int \frac{\sqrt{a}du}{\sqrt{a^2 - \frac{u^4}{a^2}}}, \tag{2.11}$$

where we used $w^2 + z^2 = a^2$ again. The statement follows. □

[7] Here the integral is an indefinite integral: $z = \int ydx$ would mean that z is a solution to $\frac{dz}{dx} = y$. In terms this functions this means that if f and F represent the dependance on x of the quantities y and z respectively, we get the indefinite integral $F(x) = \int f(x)dx$ as a solution to $F' = f$. Note therefore the integral $\int ydx$ can still be regarded as a function of x.

2.2 Lemniscate

Jacob Bernoulli continued his study in an article dated September 1694, called "*Uniform Approach and Recession, Through the Rectification of a certain Algebraic Curve: Additions to the June Solution.*"[8] First Jacob discusses the pros and cons of methods of constructing curves. This helps us understands why he continues to investigate the problem of the paracentric isochrone. He recognized three methods of constructing these curves which he ordered from least to best. For this ordering he uses criterea of how practical these methods are.

The first construction is through the surface areas, i.e. through integrals. But he says it is better done through the rectification of an algebraic curve (i.e. by using the arclength of another curve), since curves can be rectified more accurately and with more speed and efficiency by means of a thread wrapped around them, than surface area can be measured. Alternatively he considers curves that are not constructed by rectification or squaring, but simply through the description of a curve, such that on a dense set, the points can be geometrically found (using straight edge and compass), e.g. via the logarithmic description (he refers to the functional equation of the logarithm). But the best method, he says, is by means of a curve which nature itself produces on the command of the geometer. The reason is that in contrast to the previous methods, this does not need the invention of many points, which is a slow process. An example of such a curve is the catenary, which is the curve nature attains if you hang a chain on its two ends.

Jacob said the problem of the paracentric isochrone deserved to be constructed by all three methods. He had constructed it by integration. He was not able to find a construction through the last method, since he would do it using the elasticity property (he noted in [Ber94a] that the integral is the rectification of a curve with such a property) and he did not find the right tension or stretching forces in nature, and even if he found them, he would not be certain of this law. So the best method seems to him the second method. And he was successful!

Jacob constructs an algebraic curve of which the arc length is given by[9]

$$\int \frac{a^2 du}{\sqrt{a^4 - u^4}}, \tag{2.12}$$

where a is a constant. In other words, this integral gives the rectification of an algebraic curve. In [Ber94a] he showed the paracentric isochrone can be constructed using this integral. So he solved it using the second method.

Bernoulli does not just state the equation of the curve and derives its arc length.

[8] My translation with the assistance of ChatGPT. Original: "Constructio Curvae. Accessus et Recessus aequabilis, Ope Rectificationis Curvae Cujusdam algebraicae: Addenda nuperae solutioni Mensis Junii."

[9] Here the integral should be interpreted as an indefinite integral that represents a quantity s. It is defined as a solution to the differential equation $\frac{ds}{du} = \frac{a^2}{\sqrt{a^4 - u^4}}$. The arc length between points on the curve corresponding to u_0 and u_1 is $|s(u_1) - s(u_0)|$

He derives the curve from its integral. However, how he found some steps is still obscure and I will try to make each step more natural to see. Bernoulli gives the following derivation of the algebraic curve of which the arc length is given by (2.12).

Theorem 2.6. *The integral*

$$\int \frac{a^2 du}{\sqrt{a^4 - u^4}}. \tag{2.13}$$

gives the arc length of an algebraic curve given by the equation

$$x^2 + y^2 = a\sqrt{x^2 - y^2}, \tag{2.14}$$

where $u^2 = x^2 + y^2$.

Proof. We are searching to split ds^2 into $dx^2 + dy^2$. By a generalised version of the partial fraction decomposition

$$\frac{1}{1-u^4} = \frac{1 - p(u)(1-u^2)}{2(1-u^2)} + \frac{1 + p(u)(1+u^2)}{2(1+u^2)}, \tag{2.15}$$

with p an arbitrary function, Jacob is able to decompose the square of the integrand of (2.13) such that the numerators are perfect squares,

$$ds^2 = \frac{a^4}{a^4 u^4} du^2 = \frac{a^4 - 4a^2u^2 + 4u^4}{2(a^4 - a^2u^2)} du^2 + \frac{a^4 + 4a^2u^2 + 4u^4}{2(a^4 - a^2u^2)} du^2. \tag{2.16}$$

In order to get the dx and dy we need to take the square roots of the terms:

$$dx = \frac{a^2 - 2u^2}{\sqrt{2a^4 - 2a^2u^2}} du, \quad dy = \frac{a^2 + 2u^2}{\sqrt{2a^4 + 2a^2u^2}} du. \tag{2.17}$$

These terms we can integrate by trigonometric or hyperbolic substitutions to

$$y := \frac{u\sqrt{a^2 - u^2}}{a\sqrt{2}}, \quad x := \frac{u\sqrt{a^2 + u^2}}{a\sqrt{2}}. \tag{2.18}$$

The equations 2.18 give two equations

$$2a^2y^2 = a^2u^2 - u^4, \quad 2a^2x^2 = a^2u^2 + u^4. \tag{2.19}$$

Now we can eliminate u by adding and substracting the equations, which give

$$x^2 + y^2 = u^2 \tag{2.20}$$

and

$$a^2(x^2 - y^2) = u^4 = (x^2 + y^2)^2. \tag{2.21}$$

This gives the equation of the curve of which integral 2.13 determines the arc length. □

Jacob also gives a name to this algebraic curve in [Ber94b]:
"quaeque circum axem BG (see fig 3) constituta formam refert jacentis notis notae octonarii ∞, seu complicatae in nodum fasciae, sive lemnici, d'un neud de ruban Gallis."[10]

Definition 2.7. The curve given by the equation

$$x^2 + y^2 = a\sqrt{x^2 - y^2} \tag{2.22}$$

is called the *lemniscate*.

2.3 Addition theorems

In the introduction we noted the importance of Abel's addition theorem. We will now investigate simple addition theorems that we are familiar with. The addition theorem of the sine and cosine in particular motivated Euler to discover the addition theorems for the lemniscate. The addition theorem of the lemniscate was the first in a series of addition theorems with led to Abel's far reaching generalisation.

The addition formulas of the sine and cosine,

$$c(\varphi + \theta) = c(\varphi)c(\theta) - s(\varphi)s(\theta), \tag{2.23}$$

$$s(\varphi + \theta) = c(\varphi)s(\theta) + c(\theta)s(\varphi), \tag{2.24}$$

with c and s real functions, enable us to algebraically determine the values of c and s on a dense set, if we assume that the functions s and c have a period equal to 2π. Together with the assumption of continuity, the addition formula and the period of 2π, therefore, define the sine and cosine.

We will demonstrate the procedure using the simpler example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies

$$f(x + y) = f(x)f(y). \tag{2.25}$$

This example is even very similar to the trigonometric functions if you consider it in the form

$$\cos(\varphi + \theta) + i\sin(\varphi + \theta) = (\cos(\varphi) + i\sin(\varphi))\,(\cos(\theta) + i\sin(\theta))\,. \tag{2.26}$$

Taking $y = x$ we derive a duplication formula

$$f(2x) = f(x)^2. \tag{2.27}$$

Therefore f is positive and rewriting this in a half argument formula we can inductively derive[11]

$$f\left(\frac{1}{2^n}\right) = f(1)^{\frac{1}{2^n}}, \tag{2.28}$$

[10] My translation with the assistance of ChatGPT: "which placed around the axis BG looks like the lying sign 8, ∞, or as a ribbon twisted into a knot, that is, a lemniscate, or as the French call it, a ribbon knot."

[11] We could have taken $y = (n-1)x$ to determine all values $f(\frac{1}{n})$ immediately using induction, but this procedure will give values not constructable by straightedge and compass.

From the addition formula (take e.g. $x = \frac{k}{2^n}$ and $y = \frac{1}{2^n}$) we can inductively derive for all positive dyadic numbers the equation

$$f\left(\frac{k}{2^n}\right) = f(1)^{\frac{k}{2^n}} \tag{2.29}$$

(dyadic numbers are numbers of the form $x = \frac{k}{2^n}$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$). Taking $y = 0$, we see $f(0) = 1$, which gives the reflection formula $f(-x) = \frac{1}{f(x)}$ by taking $y = -x$. So the function f can be determined from the value $f(1)$ on all dyadic numbers. Since these numbers lie densely in $\mathbb{R}$, continuity of f implies that we can determine the value of f everywhere using the value on $f(1)$:

$$f(x) = f(1)^x. \tag{2.30}$$

The procedure above can be generalized to many functional equations which often arise as addition formulas (formulas determining the value at $x + y$ from the values at x and y). They are important since they often enable us to algebraically determine the values of the corresponding functions on a dense set. Combined with the condition of continuity, they uniquely define the function (often up to a parameter, which in the case above was $f(1)$).[Cau21] To the mathematicians of the 18th century, these formulas held special importance, as they often allowed the construction of values of functions on a dense set using only straightedge and compass. This motivated the search for addition formulas for functions associated with other curves then the cicle. The first curve for which a simple addition formula was found was the lemniscate.

2.4 Addition formulas arising from integrals of algebraic functions

Transcendental functions are functions that are not algebraic. The first transcendental functions known to humans are the trigonometric functions and their inverses, but also areas under curves. With the invention of calculus people were able to say much more about functions that arise as areas under curves, i.e. as integrals.

The sine and cosine do not arise from integrals of algebraic functions, but their inverses do. Namely, they are defined to give the arc length of the algebraic curve $x^2 + y^2 = 1$. For example we have $\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$. Using calculus we can determine the addition formula for the sine and cosine from the integral expressions of their inverses. We will illustrate this type of argument with a simpler example. The question is then if we can do something analogous for our lemniscate integral. To demonstrate this we go back again to our earlier example $f(x + y) = f(x)f(y)$. Like the sine and cosine, the inverse of such an f arises as an integral of an algebraic function. Let g be an inverse of f (the domain of g being equal to the range of f). These inverses g are multiples of

$$\ln(x) := \int_1^x \frac{dt}{t}. \tag{2.31}$$

For these inverse functions g, the addition formulas appear in the form

$$g(x) + g(y) = g(xy). \tag{2.32}$$

Formulas similar to these, in which a sum of the function of some arguments is given by the function at an algebraic expression of the arguments, we will also call addition formulas.

The integral formula enables us to derive the addition formula. However, it is often easier to find the duplication formula. For example, using the substitution $u^2 = t$, which gives $2udu = dt$, so $2\frac{du}{u} = \frac{dt}{t}$, we get the duplication formula

$$2\int_1^x \frac{du}{u} = \int_1^{x^2} \frac{dt}{t}, \tag{2.33}$$

so $2\ln(x) = \ln(x^2)$.

In order to find the full addition formula, we have to do more work. First we transform the integral

$$\int_1^{xy} \frac{dt}{t} \tag{2.34}$$

by a substitution $u = \frac{t}{x}$, to get

$$\int_1^{xy} \frac{dt}{t} = \int_{\frac{1}{x}}^{y} \frac{du}{u}. \tag{2.35}$$

So $\ln(xy) = \ln(y) - \ln(\frac{1}{x})$. We can transform $\ln(\frac{1}{x})$ by the substitution $ut = 1$, so $tdu + udt = 0$, i.e. $\frac{du}{u} = -\frac{dt}{t}$. This gives

$$-\int_1^{\frac{1}{x}} \frac{dt}{t} = \int_1^x \frac{du}{u}, \tag{2.36}$$

so $-\ln(\frac{1}{x} = \ln(x)$. All in all[12], $\ln(xy) = \ln(x) + \ln(y)$.

Analogously for the inverse of the sine, the formula

$$\arcsin(u) + \arcsin(v) = \arcsin(w) \tag{2.37}$$

can be found where $w = u\sqrt{1-v^2} + v\sqrt{1-u^2}$. This can be seen to be equivalent to the addition formula of the sine using the identity $\cos^2\varphi + \sin^2\varphi = 1$. Is it possible to find such a formula for the lemniscate integral?

[12]The essence of this formula was discovered in [SV47] in the form of an equivalent theorem.

2.5 Measuring the lemniscate

The arc length of the lemniscate is given by a transcedental function arising from an integral of an algebraic function (Theorem 2.6). Setting $a = 1$ this integral is

$$\int \frac{1}{\sqrt{1-u^4}}. \tag{2.38}$$

Like the elliptic and hyperbolic integrals, the integral associated with the lemniscate doesn't seem able to being transformed into an integral of a rational function like the circle integrals. In order to determine its values exactly however, it would be nice to have an addition formula for its inverse analogous to the addition formulas for the sine and cosine, as explained in section 2.3. As we saw for the logarithm in section 2.4, instead of looking directly for the addition formula, a duplication formula can often be found using simpler substitutions. Indeed, in part II of his article titled "*Method for Measuring the Lemniscate*"[13] [Fa18], Fagnano discovered the duplication formula for the lemniscate. In this paper he proved several theorems on the lemniscate integral including its relations with the elliptic and hyperbolic integrals. How he found this particular duplication formula remains unknown, although people have speculated in an attempt to reproduce his reasoning [Si59].

Theorem 2.8 (Fagnano)**.** *If*

$$z = \frac{2u\sqrt{1-u^4}}{\sqrt{1+u^4}}, \tag{2.39}$$

then

$$\frac{dz}{\sqrt{1-z^4}} = \frac{2du}{\sqrt{1-u^4}}. \tag{2.40}$$

Euler was ordered to edit the works of Fagnano. Upon encountering this theorem of Fagnano, he immediately recognized its significance. He reasoned by anology of the addition formulas of the sine and cosine that there could be an addition formula, and he found it by trial and error. [Eu61]

Theorem 2.9 (Euler)**.** *If*

$$x = \frac{c\sqrt{1-u^4} + u\sqrt{1-c^4}}{\sqrt{1+c^2u^2}}, \tag{2.41}$$

then

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dc}{\sqrt{1-c^4}} + \frac{du}{\sqrt{1-u^4}}. \tag{2.42}$$

[13] My translation with the assistance of ChatGPT. Original: "Metodo per misurare la lemniscata"

In subsequent papers, Euler was able to find addition formulas for elliptic and hyperbolic integrals. Here there is still an algebraic relation between the variables like equation (2.41), but the sum of the integrals will be equal to an integral plus an integral of an algebraic function.

The integrands of these lemniscate, elliptic and hyperbolic integrals are irrational only due to a square root under which lies a polynomial of at most the 4th degree. These functions where classified into three different kinds by Legendre in [Lef11], since they could be transformed into three standard forms by rational transformations. Altogether they came to be known as *elliptic integrals.*

3 Abel's theory

3.1 Abel's integrals

In 1826, Abel presented a memoir to the French Academy of Sciences [Ab26] on a far reaching generalisation of an addition formula for a very broad class of integrals. However, this paper was not taken seriously until Jacobi wrote a letter to the Academy. In the meantime, Abel had published a simpler piece covering the addition formula for a special class called hyperelliptic functions in Crelle's journal [Ab26]. This work along with its succesors had the greatest historical impact.
The general class of integrals Abel considered are integrals of the form

$$\psi(x) := \int f(x,y)dx, \tag{3.1}$$

where x and y are complex variables that are related by an algebraic equation $F(x,y)=0$ and f is a rational function. We will call these integrals *Abelian integrals.* The integral ψ was thought of by Abel as a multivalued complex function satisfying $\psi'(x)=f(x,y)$, where y was considered a multivalued function of x. But since to any x there can correspond multiple y we have to refine the meaning of this indefinite integral. Before we refine this meaning that, note elliptic integrals form a special case: elliptic integrals such as the lemniscate integrals are integrals $\int f(x,y)dx$ such that x and y are related by an equation $y^2-p(x)=0$, where p is a polynomial of at most the 4th degree. In case of the the lemniscate integral we have that $y^2-(1-x^4)=0$ and the integral is given by

$$\psi(x) = \int \frac{1}{y}dx. \tag{3.2}$$

According to Gray[Gr15], "it is clear that in the first phase of the creation of a theory of elliptic functions there was no theory of complex integrals. Furthermore, the fact that Cauchy's theory of complex integrals made no mention of multi-valued integrands was to drive people, including Jacobi in the 1830s, to seek other foundations for elliptic functions altogether, and to abandon the starting point of elliptic integrals."

So Abel did not have a theory of complex integrals. However, we can give these integrals exact meaning. Abel considers the algebraic equation $F(x, y) = 0$ as a polynomial equation $p_x(y) = 0$ in the variable y with coefficients depending on x. Let the degree of this polynomial be n. For any x, the polynomial equation has n solutions of y with two exceptions. The leading coefficient can become 0, which can also only happen a number of points of at most the degree of the leading coefficient. The other exception is when there is multiple root, i,e. when $p'_x(y) = F_y(x, y) = 0$. Abel called these n solutions the *different forms for the function y*.[14] We can also regard y as a *multivalued function* of x. We can represent a multivalued function set theoretically by a relation, i.e. a set of pairs (x, y). So when y is considered a multivalued function of x it is given by the set of pairs

$$\{(x, y) \in \mathbb{C} \times \mathbb{C} | F(x, y) = 0\}. \tag{3.3}$$

In order to get a notion of differentiation and integration, the multivalued functions need to be locally single valued functions. We will achieve this using the implicit function theorem[15]. Let

$$D := \{(x, y) \in \mathbb{C} \times \mathbb{C} | p_x(y) = 0,\ p'_x(y) \neq 0\}. \tag{3.4}$$

Take a point $(x_0, y_0) \in D$. At this place $F_y(x_0, y_0) \neq 0$, so we can apply the implicit function theorem: There exists a unique differentiable function φ such that $y_0 = \varphi(x_0)$ and $f(x, \varphi(x)) = 0$ in a neighbourhood U of x_0.

We can define any integral locally: Define the *local integral* of f on U_0, ω_0, to be a singlevalued function of x defined upto a constant by the differential equation $\omega'_0(x) = f(x, \varphi(x))$.

To define the integral globally we want to glue these local integrals together. We say two pairs of a neighbourhood and a local integral are glueable, and we will write $(U, \omega_0) \sim (U', \omega_{0'})$, if $U \cap U' \neq \emptyset$ and the corresponding local expressions of y and the integral match on their overlap, i.e. $\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}$ and $\omega_0|_{U \cap U'} = \omega_{0'}|_{U \cap U'}$. Now we define a new relation on the local integrals, $(U, \omega_0) \simeq (U', \omega_{0'})$, if there exists a sequence of glueable local integrals connecting the two:

$$(U, \omega_0) \sim (U_1, \omega_1) \sim \cdots \sim (U_{n-1}, \omega_{n-1}) \sim (U', \omega_{0'}). \tag{3.5}$$

For our remaining purposes we omit the neighbourhoods in our notation of (U, ω_0) and we say $\omega_{0'}$ is a *prolongation* of ω_0 if $\omega_0 \simeq \omega_{0'}$.

[14] Abel stated: "Cette equation (...) donne pour la fonction y un nombre n de formes différentes."[Ab26]

[15] The implicit function theorem was not known to Abel in this form and is due to Cauchy who published it after Abel's paper [Ab26]. Mathematicians before Cauchy had an intuitive notion of the implicit function theorem as can be seen in the work of Newton and Lagrange used special cases of the implicit function theorem. An exposition of the history of the implicit function theorem can be found in [KP13].

Now we can define a multivalued function Ω, consisting of the prolongations of an initial local integral ω_0, depending on a pair of two complex variables (x, y), via the set of pairs

$$\{(x'_0, y'_0), \omega_{0'}(x'_0) | (x'_0, y'_0) \in D,\ \omega_{0'} \simeq \omega_0\} \tag{3.6}$$

Since y is a multivalued function of x we can also create the multivalued function $\psi(x) = \Omega(x, y(x))$. In the eyes of Abel, but even in the eyes of Riemann, the integral (depending on the initial local integral) should be intepreted as a quantity

$$\int f(x, y)dx \tag{3.7}$$

which we can regard both as a function Ω of (x, y) and as a function ψ of x. Abel considered the integral mainly as a function ψ of x, Riemann uses both intepretations, but we will see in §4.2 that Ω can directly be translated into a function on T. Note that we only defined the integral as a function of x on D, but it can be extended continuously to a multivalued function on $\mathbb{C}$ with possibly a pole or a logarithmic infinity. We can show that $\int f(x, y)dx$ is defined up to a constant by considering the different choices of intitial local integral.

3.2 Abel's addition theorem

Abel considers the integral as a function of x:

$$\psi(x) = \int f(x, y)dx, \tag{3.8}$$

where x and y satisy $F(x, y) = 0$ and the integral should be intepreted as defined in §3.1.

Using a partial fraction decomposition we see that, if f is a rational function of x only, an integral $\psi(x)$ can be expressed as a sum of a rational function and a logarithm of a rational function. A sum[16] $\psi(x_1) + \cdots + \psi(x_\mu)$ of these integrals is therefore expressible in the form

$$\psi(x_1) + \psi(x_2) + \cdots + \psi(x_\mu) = u(x_1, x_2, \ldots, x_\mu) + \ln v(x_1, x_2, \ldots, x_\mu), \tag{3.9}$$

where u and v are algebraic functions. A sum $\psi(x_1) + \cdots + \psi(x_\mu)$ of similar elliptic integrals is also expressible in the same form, provided a certain algebraic relation is established between the variables. In the example of two lemniscate integrals we can establish the relation $x_1\sqrt{1 - x_2^4} + x_2\sqrt{1 - x_1^4} = 0$ by setting $x = 0$ in Theorem 2.9 to get that $\psi(x_1) + \psi(x_2) = \mathit{constant}$. This analogy led Abel to investigate if analogous properties hold for more general integrals [Ab26]. He arrived at the following theorem to which I will refer to as *Abel's relations theorem* according to [Kl04]:

[16] Abel distinguishes the integrals by a subscript, i.e. writes $\psi_1 x_1, \psi_2, \ldots, \psi_\mu x_\mu$ to denote the integrals. Using this subscript he is able to consider integrals where different integration constants are chosen. This is not important for the resulting theorems since the constants can be absorbed in the functions on the right hand side of equations (3.10) and (3.11)

Theorem 3.1 (Abel's relations theorem)**.** *Let $i = 1, \ldots, n$ and let x_i and y_i be complex variables related by an algebraic equation $F(x_i, y_i) = 0$ and let f be a rational function. Furthermore, let there be μ integrals $\psi(x_i) = \int f(x_i, y_i)dx_i$ for $i = 1, \ldots, \mu$. Then there exists a particular number p of certain algebraic relations constraining the variables $x_1, \ldots, x_\mu$ such that*

$$\psi(x_1) + \psi(x_2) + \cdots + \psi(x_\mu) = u(x_1, x_2, \ldots, x_\mu) + \ln v(x_1, x_2, \ldots, x_\mu), \quad (3.10)$$

where u and v are algebraic functions. The number p does not depend on the number of integrals, but only on the nature of the particular integrals being considered.

For elliptic integrals such as the lemniscate integral, $p = 1$. The number p is what is now known as the genus of the curve $F(x, y) = 0$.

Abel's relations theorem (3.1) also holds if the integrals are multiplied by arbitrary rational numbers. From this it follows the following theorem, which we will call *Abel's addition theorem* or simply *Abel's theorem*:

Corollary 3.2 (Abel's addition theorem)**.** *Let $i = 1, \ldots, n$ and let x_i and y_i be complex variables related by an algebraic equation $F(x_i, y_i) = 0$ and f be a rational function. Let there be α integrals $\psi(x_i) = \int f(x_i, y_i)dx_i$ for $i = 1, \ldots, \alpha$ and let $h_1, h_2 \ldots, h_\alpha \in \mathbb{Q}$. Then there exist a particular number p of algebraic functions $x'_1, x'_2, \ldots, x'_p$ of the $x_1, x_2, \ldots, x_\alpha$ such that*

$$\begin{aligned} &h_1\psi(x_1) + h_2\psi(x_2) + \cdots + h_\alpha\psi(x_\alpha) \\ &= V(x'_1, x'_2, \ldots, x'_p) + \psi(x'_1) + \psi(x'_2) + \cdots + \psi(x'_p), \end{aligned} \quad (3.11)$$

where V is the sum of an algebraic function and the logarithm of an algebraic function.

An important special case is when $h_1, h_2, \ldots, h_\alpha = 1$, which includes the addition formulas of the elliptic functions (for elliptic functions $p = 1$). In the particular case of the lemniscate, $V = constant$.

The way Abel came up with theorems had been investigated by e.g. Brill and Noether [BN94]. Abel's addition theorem was not just important since it generalised the addition theorem of elliptic functions to a very broad class of integrals, the Abelian integrals. It also connected algebra and analysis, and it would later appear that Abel's theorem had deep connections with algebraic geometry and topology. This theorem kick started the development of the theory of Abelian integrals, which had a great historical impact as we will see in §3.3 and §4. The importance of his addition theorem was noted particulary by Jacobi who said [Jac32]: "It is pleasing to bestow the name of the Abelian theorem on the preceding theorem as a most beautiful monument to the admirable genius snatched away by premature death."[17] But it was also noted by other mathe-

[17] My translation with the assistance of ChatGPT. Original: "Theorematі antecedenti ut monumento pulcherrimo ingenii admirabilis morte praematuri abrepti theorematis Abeliani nomen omponere placet."

maticians: Legendre called it a [Bi73] "monument lasting longer than bronze"[18] and Picard said that "perhaps never in the history of science has so important a proposition been obtained using such simple considerations."[19]

3.3 Jacobi's inversion problem

In the first paragraph of his paper, where Jacobi poses his famous inversion problem [Jac32], he motivates the search for the inverses to Abelian integrals. He begins by recalling his and Abel's successes in inverting elliptic integrals:

> "However, the nature and character of these transcendents could not be fully understood by considering only this transcendent $\Pi(x)$ (an elliptic integral) (...) Instead, one had to consider the function for which $\Pi(x)$ is the inverse."[20]

Jacobi notes this idea can be motivated by the analogy with the trigonometric functions, which represent at the same time a special case:

> "For indeed, if we consider the analogy with trigonometric functions, into which elliptic functions degenerate in a special case, we also see that, by setting[21]
>
> $$u(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}, \tag{3.12}$$
>
> Analysts consider the interval x as a function of the integral u, which they call the sine. We know that this function enjoys very important properties, which make its use and application exceedingly frequent throughout analysis. For example, to mention just a few, for any value of the argument u, the function has a unique and determined value; it can be expanded into a series progressing according to powers of u, which converges for all real and imaginary values of the argument; it can be factored into linear factors determined by the values of u for which the function vanishes; finally, *it possesses all the properties of a polynomial of* u. On the other hand, analysts consider the function u merely as the inverse of the function $x = \sin(u)$, saying it is that for which $\sin = x$, or writing $u = \arcsin(x)$. This function, however, cannot be expanded into an always-convergent series, nor is it determined, but has an infinite number of values,

[18] My translation with the assistance of chatGPT. Original: "monumentum aere perennius"

[19] Translation from [Bi73]. Original: [Pi93] Sous cette forme, le tliéoreme parait tout à fait élémentaire, et il n'y a peut-être pas, dans l'histoire de la Science, de proposition aussi importante obtenuc à l'aide de considérations aussi simples.

[20] My translation with the assistance of ChatGPT. Original: "Neque tamen harum transcendentium indoles atque natura plane pernocsi poterat, consderando hanc solam transcendentem $\Pi(x)$ (...) sed considerari debuit functio, euius ipsa $\Pi(x)$ inversa est"

[21] Here I changed the variable of integration according to modern notation.

since its nature is that of the *root of an algebraic equation of infinite order*, $u = \sin(x)$. Hence, neither a special name nor a suitable symbol was thought necessary to assign to it."[22]

Jacobi and Abel discovered that elliptic inverses do indeed enjoy "all the properties of a rational function." Jacobi refers to his *Fundamenta nova theoriae functionum ellipticarum* [Jac29]. Abel's researches are mostly layed out in his *Recherches sur les fonctions elliptiques* [Ab27].

In paragraph 3 of [Jac32], Jacobi poses the more general question:

> "what are the more general cases of those functions, whose inverses are the Abelian transcendentals, and how does the Abelian theorem relate to these?"[23]

Jacobi only considers the special class of Abelian integrals called hyperelliptic integrals. They are integrals of the form

$$\int f(x,s)dx, \tag{3.14}$$

where x and s are related by the polynomial equation $F(x,s) = s^2 - X(x) = 0$, where X is a polynomial and f is a rational function. Jacobi wrote upper and lower bounds at the integral. It clear from [Jac32] what he means by this, however it could be interpreted as follows: Let

$$\psi(x) = \int f(x,s)dx \tag{3.15}$$

where x and s are related by the polynomial equation $F(x,s) = 0$, as defined in

[22]My translation with the assistance of ChatGPT. Original: "Entenim si analogiam functionum trigonometricarum respicimus, in quas casu speciali functiones ellipticae abeunt, etiam videmus, posito

$$u = \int_0^x \frac{dx}{\sqrt{1-x^2}}, \tag{3.13}$$

considerari ab Analystis intervallum x tanquam functionem integralis u, oui nomen sinus tribuunt. Quam functionem scimus proprietatibus gravissimis gaudere, quae eius isum et applicatiunem per totam analysin frequentissimam reddunt. Quippe quae, ut de aliss taceam, pro quolibet valore argumenti u valorem unicum ac determinatum habet; evolvi potest in seriem secundum dignitates ipsius u progredientem, quae pro omnibus argumenti valoribus et realibus et imaginariis convergit; discerpi potest in factores lineares, qui determinantur valoribus ipsius u, pro quibus functio evanescit; denique *gaudet illa proprietatibus omnibus functionis ipsius u rationalis integrae.* E contra functionem u considerant analystae tantum ut inversam functionis $x = \sin(u)$, dicentes eam esse cuius *sinus* $= x$ aut scribentes $u =$ *arcus sinus* x; neque ea functio ullo modo evolvi potest in seriem sempet convergentem, neque determinata est, sed nemerum valorum infinitum habet, quippe cuius eadem est natura atque *radicis aequationis algebraicae ordinis infiniti*, $u = \sin(x)$. Unde nec nomen nec signum peculiare ei tribuere, idoneum putabatur."

[23]My translation with the assistance of ChatGPT. Original: "quaenam sint casu generaliori functiones illae, quarum inversae sunt transcendentes Abelianae, et quomodo de hisce exhibitum audiat theorema Abelianum."

3.1. Then we could define

$$\int_a^b f(x,s)dx := \psi(b) - \psi(a).$$

To answer the question above, Jacobi reformulates Abel's addition theorem into a system of linearly independend simultaneous equations because "the Abelian theorem, if you wish to correctly grasp its power and nature, should be proposed in the following manner:"

Theorem 3.3. *"Let X be a polynomial function of the fifth or sixth order and let*[24]

$$\int_0^x \frac{dt}{\sqrt{X(t)}} = \Phi(x), \int_0^x \frac{tdt}{\sqrt{X(t)}} = \Phi_1(x). \tag{3.16}$$

Then there are two simultaneous equations,

$$\begin{aligned} \Phi(a) + \Phi(b) &= \Phi(x) + \Phi(y) + \Phi(z) \\ \Phi_1(a) + \Phi_1(b) &= \Phi_1(x) + \Phi_1(y) + \Phi_1(z). \end{aligned} \tag{3.17}$$

such that the quantities a and b are algebraically determined by the given quantities x, y, and z."[25]

Jacobi sets out to propose the inverse functions related to the integrals of equation (3.16) and provide an addition formula for them as a consequence of Abel's theorem. He only suggests what mapping these inverse functions should inverte but he does not prove the existence of these functions. The existence of these functions will be called Jacobi's inversion problem.

He proceeds for the more general class of hyperelliptic integrals: Let X denote a polynomial function of order $(2m-1)$ or $2m$, and define for $i \in \{0, 1, \ldots, m-2\}$ the integrals

$$\Phi_i(x) := \int_0^x \frac{t^i dt}{\sqrt{X(t)}}. \tag{3.18}$$

In the hyperelliptic case the number p from Abel's theorem 3.2 (the genus) is equal to $m-1$. According to Abel's theorem 3.2 we have the system of

[24] The integral was originally notated as $\int_0^x \frac{dx}{\sqrt{X}}$

[25] My translation with the assistance of ChatGPT. Orginal: "Designante X functionem ipsius x integram rationalem ordinis quinti aut sexti, sit

$$\int_0^x \frac{dt}{\sqrt{X(t)}} = \Phi(x), \int_0^x \frac{tdt}{\sqrt{X(t)}} = \Phi_1(x),$$

propositis duabus simul aequationibus,

$$\begin{aligned} \Phi(a) + \Phi(b) &= \Phi(x) + \Phi(y) + \Phi(z) \\ \Phi_1(a) + \Phi_1(b) &= \Phi_1(x) + \Phi_1(y) + \Phi_1(z). \end{aligned}$$

quantitates a, b e datis quantitatibus x, y, z algebraice determinantur."

simultaneous equations

$$\sum_{i=0}^{m-2} \Phi_0(y_i) = \sum_{i=0}^{m-2} \Phi_0(x_i) + \sum_{i=0}^{m-2} \Phi_0(x'_i)$$
$$\sum_{i=0}^{m-2} \Phi_1(y_i) = \sum_{i=0}^{m-2} \Phi_1(x_i) + \sum_{i=0}^{m-2} \Phi_1(x'_i)$$
$$\vdots$$
$$\sum_{i=0}^{m-2} \Phi_{m-2}(y_i) = \sum_{i=0}^{m-2} \Phi_{m-2}(x_i) + \sum_{i=0}^{m-2} \Phi_{m-2}(x'_i) \tag{3.19}$$

such that the y_i are determined algebraically by the x_i and the x'_i. Now define

$$(u_0, u_1, ..., u_{m-2}) := \left(\sum_{i=0}^{m-2} \Phi_0(x_i), \sum_{i=0}^{m-2} \Phi_1(x_i), \ldots, \sum_{i=0}^{m-2} \Phi_{m-2}(x_i)\right),$$
$$(u'_0, u'_1, ..., u'_{m-2}) := \left(\sum_{i=0}^{m-2} \Phi_0(x'_i), \sum_{i=0}^{m-2} \Phi_1(x'_i), \ldots, \sum_{i=0}^{m-2} \Phi_{m-2}(x'_i)\right), \tag{3.20}$$

and

$$(u_0 + u'_0, u_1 + u'_1, ..., u_{m-2} + u'_{m-2}) = \left(\sum_{i=0}^{m-2} \Phi_0(y_i), \sum_{i=0}^{m-2} \Phi_1(y_i), \ldots, \sum_{i=0}^{m-2} \Phi_{m-2}(y_i)\right). \tag{3.21}$$

Now Jacobi proposes the following inverse functions:

$$x_i = \lambda_i(u_0, u_1, \ldots, u_{m-2})$$
$$x'_i = \lambda_i(u'_0, u'_1, \ldots, u'_{m-2})$$
$$y_i = \lambda_i(u_0 + u'_0, u_1 + u'_1, \ldots, u_{m-2} + u'_{m-2}) \tag{3.22}$$

Thus, Abel's theorem relates to these inverse functions as follows:

Theorem 3.4 (Abels theorem inverted by Jacobi)**.** *Let X denote a polynomial of x of order $(2m-1)$ or $2m$, and let*

$$\Phi_i(x) := \int_0^x \frac{t^i dt}{\sqrt{X(t)}}. \tag{3.23}$$

With these given, let there be $m-1$ functions $x_0, x_1, \ldots, x_{m-2}$, each depending on the $m-1$ quantities $u_0, u_1, \ldots, u_{m-2}$ in such a way that the following equations hold simultaneously:

$$(u_0, u_1, ..., u_{m-2}) = \left(\sum_{i=0}^{m-2} \Phi_0(x_i), \sum_{i=0}^{m-2} \Phi_1(x_i), ..., \sum_{i=0}^{m-2} \Phi_{m-2}(x_i)\right), \tag{3.24}$$

and the functions are given by:

$$x_i = \lambda_i(u_0, u_1, \ldots, u_{m-2}). \tag{3.25}$$

The functions

$$\begin{aligned} \lambda(u_0 + u_0',\ u_1 + u_1',\ \ldots,\ u_{m-2} + u_{m-2}'), \\ \lambda_1(u_0 + u_0',\ u_1 + u_1',\ \ldots,\ u_{m-2} + u_{m-2}'), \\ \vdots \\ \lambda_{m-2}(u_0 + u_0',\ u_1 + u_1',\ \ldots,\ u_{m-2} + u_{m-2}') \end{aligned} \tag{3.26}$$

can be expressed algebraically by the functions

$$\begin{aligned} \lambda(u_0,\ u_1,\ \ldots,\ u_{m-2}), \\ \lambda_1(u_0,\ u_1,\ \ldots,\ u_{m-2}), \\ \vdots \\ \lambda_{m-2}(u_0,\ u_1,\ \ldots,\ u_{m-2}), \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \lambda(u_0',\ u_1',\ \ldots,\ u_{m-2}'), \\ \lambda_1(u_0',\ u_1',\ \ldots,\ u_{m-2}'), \\ \vdots \\ \lambda_{m-2}(u_0',\ u_1',\ \ldots,\ u_{m-2}'). \end{aligned} \tag{3.28}$$

Jacobi, therefore, answered part of his question: how does the Abelian theorem relate to the inverse functions of Abelian integrals? However, the possibility of indeed inverting these integrals is not yet clear and will be called by the name: *Jacobi's inversion problem.* That is, can we invert the mapping

$$(x_1, \ldots, x_p) \mapsto \left(\sum_{i=0}^{m-2} \Psi_0(x_i), \sum_{i=0}^{m-2} \Psi_1(x_i), \ldots, \sum_{i=0}^{m-2} \Psi_{m-2}(x_i) \right)? \tag{3.29}$$

We will see in the next section that Riemann is able to prove this is possible in the general case of Abelian integrals using revolutionary methods.

4 From Jacobi's inversion problem to Riemann–Roch

> "The period we reach now is without any doubt the most important of all in the history of algebraic geometry to this day. It is entirely stamped by the work of one man, Bernhard Riemann, one of the

greatest mathematicians who ever lived, and also one of those who have had, most profoundly, the perception (or divination) of the essential *unity* of mathematics."[26] - Dieudonné

Our goal is to describe the history of Riemann–Roch (Theorem 4.7). And although Riemann is now famous, among other things, for his contribution to this theorem, i.e. Riemann's inequality (Theorem 4.2), this was not his goal. Rather, this inequality was a byproduct of his reasoning in order to achieve the solution to Jacobi's inversion problem. Jacobi's inversion problem gave Riemann the right ingredients. Riemann's inequality turns out to be an immediate consequence of these. Even more, as we will see, the attainment of equality sheds some light on the uniqueness of the inversion. As the first ingredient, Riemann considered the *totality of Abelian integrals* belonging to a complex polynomial equation $F(x, y) = 0$, rather than restricting attention to a single integral. These integrals are all integrals of rational functions f,

$$\int f(x, y)dx, \tag{4.1}$$

as defined in §3.1, such that x and y are related by the same polynomial equation $F(x, y) = 0$. We will see this equation defines a Riemann surface. This ingredient was already known to Jacobi, as he considers the inversion of the system of integrals $\Phi_i(x) = \int_0^x \frac{t^i dt}{\sqrt{X(t)}}$ (equation (3.18)), which turn out to span all everywhere finite integrals with variables connected by $y^2 - X(x) = 0$. This consideration of the totality of Abelian integrals with the variables connected by the same algebraic equation is a direct consequence of Jacobi's efforts to invert the hyperelliptic integrals. The second ingredient is Riemann's original contribution: Riemann developed a geometric interpretation of complex function theory, the theory of Riemann surfaces. This geometric theory enabled him to add a third ingredient: defining functions on the surface through a boundary value problem. By a clever topological procedure, Riemann used this boundary problem to relate spaces of integrals on the surface to the topology of the surface. We will now treat these ingredients in more detail.

4.1 Riemann's physical perspective on complex functions

In his doctoral disseration [Ri51], Riemann revolutionized our understanding of complex functions. Inspired by Gauss, who represented complex numbers as coordinates in a plane, Riemann viewed complex functions as relationships between surfaces, treating them as functions of position. This approach enabled him to reduce arguments in complex function theory, which were traditionally

[26]Translation from [Die85]. Original[Die74]: "La période à laquelle nous arrivons est sans aucun doute la plus importante de toute l'histoire de la Géométrie algébrique à ce jour. Elle est tout entière marquée par l'œuvre d'un seul homme, Bernhard Riemann, un des plus grands mathématiciens qui ait jamais vécu, un de ceux aussi qui ont eu le plus profondément le sentiment (ou la divination) de l'*unité* essentielle de la mathématique."

based on the simple algebraical and analytical laws, to arguments of analysis situs, what is now known as topology.

In his dissertation, Riemann deals only with meromorphic functions, because he argues, these are exactly the functions that arise from a combination of the simple laws of dependence between variables arising from algebraic operations. Therefore, if we refer to a function, as Riemann did, we mean a meromorphic function.

In paragraph 20 of his dissertation, Riemann emphasizes how these simple laws motivated the introduction of complex numbers:

> "The introduction of complex quantities into mathematics has its origin and immediate purpose in the theory of simple[27] laws of dependence between variables arising from algebraic operations. For if the field of these laws of dependence is extended by permitting the variable quantities to have complex values, a formerly hidden harmony and regularity emerges."[28]

On the other hand, Riemann noted in his table of contents "the earlier method of determining a function through operations on quantities contains superfluous components." However, "via the treatment carried out here, the extent of the matter that determines a function is reduced to the necessary minimum."[29] Here Riemann refers to his method of defining complex functions through the boundary value problem corresponding to the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{4.2}$$

Although Riemann motivates the well-posedness of this problem in his paper [Ri51] by Dirichlet's principle, Klein argues [Kl82b] it is the physical interpretation of functions representing flows of fluids that convinced Riemann of the well-posedness of this problem:

[27] Here Riemann considers the elementary operations to be addition, substraction, multiplication, division, integration and differentiation: and simpler dependence indicates that fewer elementary operations are required. In fact, all functions used up to now in analysis can be defined via a finite number of these operations.

[28] Translation from [Ri92]. Orginal [Ri51]: "Die Einführung der complexen Grössen in die Mathematik hat ihren Ursprung und nächsten Zweck in der Theorie einfacher (Wir betrachten hier als Elementaroperationen Addition und Subtraction, Multiplication und Division, Integration und Differentiation, und ein Abhängigkeits gesetz als desto einfacher, durch je weniger Elementaroperationen die Abhängigkeit bedingt wird. In der That lassen sich durch eine endliche Anzahl dieser Operationen alle bis jetzt in der Analysis benutzten Functionen definiren.) durch Grössenoperationen ausgedrückter Abhängigkeitsgesetze zwischen veränderlichen Grössen. Wendet man nämlich diese Abhängigkeitsgesetze in einem erweiterten Umfange an, indem man den veränderlichen Grössen, auf welche sie sich beziehen, complexe Werthe giebt, so tritt eine sonst versteckt bleibende Harmonie und Regelmässigkeit hervor."

[29] Translation from [Ri92]. Orginal [Ri51]: "Die frühere Bestimmungsweise einer Function durch Grössenoperationen enthält überflüssige Bestandtheile. Durch die hier durchgeführten Betrachtungen ist der Umfang der Bestimmungsstücke einer Function auf das nothwendige Mass zurückgeführt."

> "But I have no doubt he started from precisely those physical problems [treated in Klein's book], and then, in order to give what was physically evident the support of mathematical reasoning, he afterwards substituted Dirichlet's principle. Anyone who clearly understands the conditions under which Riemann worked in Göttingen, anyone who has followed Riemann's speculations as they have come down on us, partly in fragments, will, I think, share my opinion."[30]

In his book, Klein tries to follow Riemann's "true train of thought" developing his function theory.

Dirichlet's principle, in the form Riemann used, states that given an arbitrary function $\beta : T \to \mathbb{R}$ on a compact surface T (a surface that we will specify in §4.2), there is a function $u : T \to \mathbb{R}$ depending on the complex variable $z = x + iy$ that minimizes a certain surface integral

$$\int_T \left(\left(\frac{\partial u}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right) dT, \tag{4.3}$$

and that we can use this minimizer to obtain a solution to the boundary problem (equation 4.2). Here does dT denote a surface element. Even though Riemann attempted to use Dirichlet's principle to support the mathematical reasoning, this principle was later critizised by Weierstrass [Weie95] as it was unclear whether the infinum of Dirichlet's integral was actually attained by any function, i.e. whether there is a function minimzing this integral. This issue was later resolved by Hilbert [Hil04]. Prior to Hilbert, Schwarz and Neumann invented their alternating process, which provided the mathematical support to what was physically evident [Sc90][Ne70][Ne84].

4.2 The concept of a Riemann surface

The Abelian integrals (see §3.1) take the form[31]

$$\int f(s, z) dz,$$

where f is a rational function and s and z are related by a polynomial equation $F(s, z) = 0$. In his article where he solves the inversion problem [Ri57], Riemann considered an irreducible polynomial equation $F(s, z) = 0$ of two complex

[30] Translation from [Kl82b]. Original [Kl82a]: "Aber ich kann nicht zweifeln, dass er genau von jenen physikalischen Problemen ausgegangen ist, und ihnen nur hinterher, um die physikalische Evidenz durch einen mathematischen Schluss zu stützen, das Dirichlet'sche Princip substituirt hat. Wer sich die Bedingungen klar macht, unter denen Riemann in Göttingen arbeitete, wer die Speculationen Riemann's verfolgt hat, wie sie zum Theil in Fragmenten auf uns gekommen sind, wird, denke ich, diese Meinung theilen."

[31] Note that Riemann not only changes the names of the variables $s = y$, $z = x$ with respect to Abel's notation. He also interchanges the places of s, z in the rational function f, i.e. Abel wrote $f(x, y)$ for the rational function and Riemann $f(s, z)$. The integral should be interpreted in the sense of §3.1.

variables. If s is thereby considered a function of z, this function may be multi-valued. Look, for example, at the equation $s^2 - z = 0$. Here s is a two-valued function of z except at $z = 0$. Consider $F(s,z)$ to be a polynomial $p_z(s)$ in s with coefficients being polynomials in z and let's call the degree of p_z to be n. If the leading coefficient is non-zero, s has n solutions. If these solutions are different, we have that $\frac{\partial F}{\partial s} \neq 0$. So by the implicit function theorem s is locally a holomorphic function of z (as we saw in §3.1).

Riemann, inspired by Gauss, saw the complex numbers as a plane. At places where $\frac{\partial F}{\partial s} \neq 0$, since s is locally a holomorphic function of z, Riemann visualized the n different solutions of z to form n different layers. These layers are called the *branches of* s. The branches come together at points in which s is a multiple root of the polynomial p_z, which points are called *branch points*. Let's once again consider the example of the multivalued function s given by $s^2 - z = 0$. We see that $z = 0$ is a branch point and at any other place s forms two branches.

It may happen that the leading coefficient of p_z becomes 0; then s has a pole at a branch. An example of this phenomenon is given by the equation $sz - 1 = 0$. Here, s has a pole at $z = 0$. So in general, s is locally a meromorphic function of z except at the branch points.

Riemann imagined a surface formed by the branches and the branch points. Riemann understood the notion of a surface in an intuitive sense. We will, however, make Riemann's surface, which is represented by the multi-valued function s, more precise. Let's keep $F(s,z)$ as above and keep the degree of p_z equal to n. Let's define the set

$$U := \left\{ (s,z) \in \mathbb{C} \times \mathbb{C} | F(s,z) = 0, \ \frac{\partial F}{\partial s} \neq 0 \right\}.$$

Let's also define the function $\varphi : (s,z) \mapsto z$. We already saw that locally s is a holomorphic function on U, so locally φ is invertible and so the restrictions of φ to sufficiently small neighbourhoods can be used to define an atlas on U. We can now extend this surface to a surface T containing the branch points and poles of s (for details, see e.g. theorem 8.9 of [Fo81]). This surface is a connected one-dimensional complex manifold. Such manifold is now called a *Riemann surface*.

Any meromorphic function of s and z is now a meromorphic function on T. In particular the integral,

$$\int f(s,z) dz,$$

as defined in §3.1 where f is a rational function, is a multivalued function on T. Riemann calls the different values of $\int f(s,z)dz$ on a point T different *prolongations* of the integral. Geometrically these give rise to different sheets, since locally they are single valued function on T (see §3.1). Riemann uses the terms "Zweig" and "Blat" interchangeably. In accordance with modern terminology I will consider a branch as a special case of a sheet, when the two sheets come together in a branch point [Wey55].

The first modern definition of a manifold using charts was published by Weyl in 1913 in his book on Riemann surfaces [Wey13].

4.3 Integrals defined through a boundary value problem.

> "The principal significance of the inversion problem to us today lies primarily, not in its intrinsic value, but in the splendid developments created by Riemann and Weierstrass in their efforts to solve the problem."[32] - Weyl

In 1857, Riemann published his paper solving the inversion problem of Jacobi [Ri57]. In order to solve the inversion problem, he had to develop his theory of integrals on Riemann surfaces. In paragraph 2 of the introduction he noted:

> "When examining the functions arising from the integration of closed[33] differentials, some of the propositions belonging to the *analysis situs* are almost indispensable. With this name used by Leibniz, albeit perhaps not entirely in the same sense, a part of the doctrine of continuous quantities cannot be considered as existing independently of position and measurable by each other, but rather, completely disregarding the proportions, only subjects their position and surface situation to investigation."[34]

Closed differentials on the xy-plane are differentials $gdx + hdy$ such that $\frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$ and which correspond physically to fluid flows. Complex functions correspond to closed differentials since they satisfy the Cauchy–Riemann equations. Namely, let $f(z) = u(z)+v(z)i$ and $z = x+yi$, then the Cauchy–Riemann equations (4.2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{4.4}$$

are equivalent to saying the differential fdz is closed.

We will now illustrate how Riemann derived his topological theorems on the integrals of T. Riemann, in fact, considered a more general surface than a surface arising from a equation $F(s, z) = 0$. But for simplicity we keep considering the Riemann surface corresponding to such an equation. Riemann first defined

[32] Translation from [Wey55]. Original [Wey13]: "Die große Bedeutung des Umkehrproblems liegt für uns Heutige nicht nur (und wohl nicht einmal überwiegend) in seinem Wert an sich als in den großartigen Gedankenreihen, zu deren Schöpfung Riemann und Weierstraßdurch die Bemühungen um seine Lösung getrieben wurden."

[33] Originally translated as 'complete'.

[34] Translation from [Ri92]. Original [Ri57]: "Bei der Untersuchung der Functionen, welche aus der Integration vollständiger Differentialien entstehen, sind einige der analysis situs angehorige Sätze fast unentbehrlich. Mit diesem von Leibnitz, wenn auch vielleicht nicht ganz in derselben Bedeutung, gebrauchten Namen darf wohl ein Theil der Lehre von den stetigen Grössen bezeichnet werden, welcher die Grössen nicht als unabhängig von der Lage existirend und durch einander messbar betrachtet ; sondern von den Massverhältnissen ganz absehend, nur ihre Orts- und Gebietsverhältnisse der Untersuchung unterwirft."

integrals of meromorphic functions f on T as integrals of the corresponding closed differentials fdz along a path. He considered the starting point of the paths to be fixed. This way the integral becomes a multivalued function of the end point, the multiple values depending on the path taking. Two paths with the same end and starting point give the same value if they enclose a simply connected domain. Central to his theory is his alternative definition of the integral as a boundary value problem, so this is the definition on which we will eleborate.

First of all, Riemann regarded the surface T as a closed (i.e. compact and without a boundary) surface T by adding the points corresponging to $z = \infty$, Using $2p$ cuts, Riemann cut the surface into one simply connected piece T'. The cuts then form the boundary of the surface. Riemann calls the $2p$ cuts needed to cut the surface into one simply connected piece the connectivity of the surface. The number p was later called the *genus* by Clebsch [Cl68]. To illustrate this procedure of cutting up a surface, take a torus $T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$. This is a closed surface. If we now cut the surface using a cut along the real and a cut along the imaginary axis, we obtain a simply connected surface

$$T' = \{q \in \mathbb{C} \mid 0 \leq Re(q), Im(q) \leq 1\}. \tag{4.5}$$

We noted in §3.1 that an Abelian integral can be a multivalued function of s and z. Therefore, it is multivalued on T. Locally, however, the different local integrals in the definition of §3.1 corresponding to one pair (s, z) define different sheets. For any Abelian integral, Riemann chooses one sheet so the Abelian integral becomes single valued on the surface T'. The values of different prolongations of the Abelian integral can only differ by constants, since their derivatives with respect to z are equal. The value of an integral on the new surface on the two sides of the cut differs therefore by a number depending on the specific cut, known as the *modulus of periodicity.* To be more precise, the map $\varphi : (s, z) \mapsto z$ induces an orientation on T. Choosing a direction of cutting, the left-hand sides of the cut is then called *positive* and the other side negative. The moduli of periodicity is then the value of at the positive side of the cut minus the value at the negative side of the cut. These moduli of periodicity correspond to the difference in the value of the Abelian integral if we traced its values along a loop on T that only crosses that cut once and that crosses no other cuts.

The cuts form the boundary of the simply connected surface T'. Therefore, by considering an Abelian integral a single valued function $\omega : T' \to \mathbb{C}$ on the simply connected surface T', Riemann shows that ω satisfies the Cauchy Riemann equations (where $\omega = u + vi$)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{4.6}$$

and that u satisfies the periodic boundary conditions

$$\begin{aligned} u^+ &= u^- + Re(k^{(\nu)}), \\ \frac{dv^+}{dz} &= \frac{dv^-}{dz} \end{aligned} \tag{4.7}$$

with $\nu = 1, 2, \ldots 2p$. Here u^+ and u^- denote the value of u at the positive and negative side of the νth cut respectively (same for v). The constant k^ν denotes the modulus of periodicity of ω at the νth cut. Riemann showed that, up to a constant and the conditions on its poles or logarithmic infinities, any ω is defined by this boundary value problem. Therefore, Riemann showed that ω is defined by the real parts of its moduli of periodicity and the condition on its poles and logarithmic infinities. This function can then be extended to a multivalued function on the uncut surface T. The values of the different sheets of the multivalued integral above the same point can only differ by combinations of the moduli of periodicity. Therefore any Abelian integral is defined by its moduli of periodicity.

From now on we will consider integrals as defined by it's boundary value problem. This definition of an Abelian integral has an immediate generalisation to any compact Riemann surface (a connected one-dimensional complex manifold). We will call a multivalued function defined by the boundary value problem just described an *integral* on the surface.

Let's consider the torus $T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$. We will show the multivalued function $\omega : T \to \mathbb{C}$ given by $z \mapsto az + b$, where a and b are constants is an integral on the torus. Firstly, ω is complex differentiable and therefore satisfies the Cauchy–Riemann equations (4.2). The first cut we make is from 0 to 1, and the second from 0 to i, the resulting surface is a simply connected surface $T' = \{q \in \mathbb{C} | 0 \leq Re(q), Im(q) \leq 1\}$. The line from 0 to 1 is the positive edge of the first cut, since the surface lies along the left-hand side of this edge with respect to the direction of cutting. The line from i to $i+1$ is the negative edge if the first cut. The line from 1 to $1+i$ is the positive edge and the line from 0 to i is the negative edge of the second cut. The moduli of periodicity of ω are $k^{(1)} = az + b - (a(z+i) + b) = -ia$ and $k^{(2)} = a(z+i) + b - (az + b) = a$. These are indeed constants, so ω is indeed a solution to the Riemann's boundary value problem (equation (4.6)). Therefore we call ω an integral.

Since the solution of the boundary value problem (equation (4.6)) is unique, all everywhere finite integrals, i.e. without poles or a logarighmic infinities, are determined upto a constant by the real parts of the moduli of periodicity $k^{(\nu)}$. We find in our example that a can be determined from $Re(a) = Re(k^{(2)})$ and $image(a) = Re(k^{(1)})$, so all everywhere finite integrals are of the form $z \mapsto az+b$.

4.4 The space of integrals on a surface

Riemann now goes on to consider the space of integrals on T (the totality of integrals), which forms a complex vector space. First he divides the integrals into three kinds. *Integrals of the first kind* are everywhere finite, *integrals of* *the second kind* may have poles and *integrals of the third kind* may have a logarithmic discontinuity, i.e. they are integrals of a meromorphic function with a simple pole.

The number of conditions that determine the integrals of the first kind, i.e. the number of moduli of periodicity $2p$, depends only on the topology of the surface T. From this Riemann derives that all integrals w of the first kind are of the form:

$$w = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_p w_p + const., \tag{4.8}$$

where $w_1, w_2, \ldots w_p$ together with the constant term give a basis of the integrals of the first kind and $\alpha_1, \alpha_2, \ldots \alpha_p$ are complex numbers. This can be reformulated in the following theorem, of which we give a proof along the lines of Riemann:

Theorem 4.1. *On a compact Riemann surface of genus p, the dimension of the space of integrals of the first kind is* $p + 1$.

Proof. Let w and $w_1, w_2, \ldots, w_q$ be integrals of the first kind connected by $w = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_q w_q + const.$, where $\alpha_1, \alpha_2, \ldots \alpha_q$ are complex numbers, and let's call $k^{(\nu)}$ and $k_1^{(\nu)}, k_2^{(\nu)}, \ldots k_q^{(\nu)}$ the moduli of periodicity of w and $w_1, w_2, \ldots, w_q$. Denote $\alpha_\mu = \gamma_\mu + \delta_\mu i$ for $\mu = 1, \ldots, q$. We then have the following system of equations for the:

$$\begin{pmatrix} Re(k^{(1)}) \\ \vdots \\ Re(k^{(2p)}) \end{pmatrix} = \begin{pmatrix} Re(k_1^{(1)}) & \cdots & Re(k_q^{(1)}) & -Im(k_1^{(1)}) & \cdots & -Im(k_q^{(1)}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Re(k_1^{(2p)}) & \cdots & Re(k_q^{(2p)}) & -Im(k_1^{(2p)}) & \cdots & -Im(k_q^{(2p)}) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_q \\ \delta_1 \\ \vdots \\ \delta_q \end{pmatrix} \tag{4.9}$$

If $q < p$, since the possible numbers of $(Re(k^{(1)}), \ldots, Re(k^{(2p)})$ form a vectorspace of a higher dimension then the possible numbers of $(\gamma_1, \ldots \gamma_q \delta_1 \ldots \delta_q)$, we see that there is a combination of $(Re(k^{(1)}), \ldots, Re(k^{(2p)})$ that is not a solution of the system (4.9). These real parts of the moduli of periodicity give an integral linearly independend of $w_1, \ldots, w_p$ and a constant function. Therefore we can find integrals $w_1, \ldots, w_p$ such that they are, together with a constant function, linearly independend. Let's choose such integrals.

If there exist $\gamma_1, \ldots, \gamma_p, \delta_1, \ldots, \delta_p$ such that $Re(k^{(1)}), \ldots, Re(k^{(2p)}) = 0$. Since the real parts of the moduli of periodicity determine w uniquely upto a constant we get that w is a constant function. Therefore $w_1, \ldots w_p$ and a constant function are linearly dependend, which is contrary to our assumption. Therefore the kernel of our matrix is 0, so it's surjective (since it's a square matrix). So any combination of $(Re(k^{(1)}), \ldots, Re(k^{(2p)})$ can be attained and thefore any integral can be determined as a linear combination of $w_1, \ldots, w_p$ and a constant function. This proves Riemann's theorem. $\square$

More generally, if ω is an integral that has a simple pole at points $\epsilon_1, \ldots, \epsilon_m$, and $t_1, \ldots, t_m$ are arbitrary functions which only have a simple pole on these points respectively, then ω is of the form (Riemann writes ω to denote a general integral and w to denote an integral of the first kind):

$$\omega = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_m t_m + \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_p w_p + const. \quad (4.10)$$

We can view the space of meromorphic functions on T as a subspace of the space of integrals, with functions on T being integrals whose moduli of periodicity are 0. This gives the following system of $2p$ equations with $p+m$ variables: Denote the modulus of periodicity of a w_μ at the νth cut by $k_\mu^{(\nu)}$, the modulus of periodicity of a t_k at the νth cut by $\ell_k^{(\nu)}$ and the modulus of periodicity of ω at the νth cut by $r^{(\nu)}$. The moduli of periodicity $r^{(\nu)}$ of ω are given by the equations

$$r^{(\nu)} = \sum_k \beta_k \ell_k^{(\nu)} + \sum_\mu \alpha_\mu k_\mu^{(\nu)}. \quad (4.11)$$

Setting the moduli of periodicity of ω equal to zero gives a system of equations whose solution space gives the space of meromorphic functions on T,

$$\begin{pmatrix} \ell_1^{(1)} & \cdots & \ell_m^{(1)} & k_1^{(1)} & \cdots & k_p^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \ell_1^{(2p)} & \cdots & \ell_m^{(2p)} & k_1^{(2p)} & \cdots & k_p^{(2p)} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix} = 0. \quad (4.12)$$

Since this is a $2p \times (p+m)$ system of equations, the dimension of the solution space of equation (4.30) is at least $m-p$. Including the additive constant from equation (4.10), the space of meromorphic functions is at least $m-p+1$. This is called Riemann's inequality. In fact, Riemann notes that a pole of the nth order can be viewed as n coincident simple poles (section 2 of [Ri57]), so there is an inequality for poles of any order.

Theorem 4.2 (Riemann's inequality)**.** *Let T be a compact Riemann surface of genus p. The space of meromorphic functions on T that may only have poles at designated points and designated maximal order, such that the sum of the maximal orders is m, has a dimension of at least $m-p+1$.*

As we see, proving Riemann's inequality was a small step when the ingredients were there.

4.5 Divisors

In 1882, Dedekind and Weber introduced methods of algebraic number theory into Riemann's theory [DW82]. By working strictly in an algebraic setting

Dedekind and Weber were able to make the non-rigorous methods of Riemann fully rigorous and were able to reveal a deep analogy between number fields and function fields. In our thesis one object introduced in their work is of particular importance: the notion of a divisor.

Definition 4.3. Let T be a compact Riemann surface. A *divisor* D on T is defined to be a finite formal sum

$$D = \sum_{p \in T} a_p p, \tag{4.13}$$

with $a_p \in \mathbb{Z}$. We also define $\deg D \coloneqq \sum_{p \in T} a_p$.

The notion of a divisor enables us to encode the poles and zeroes of a meromorphic function f, counted with multiplicity. Namely define for any f not everywhere vanishing

$$\operatorname{div}(f) \coloneqq \sum_{p \in T} \operatorname{ord}_p(f) p, \tag{4.14}$$

where $\operatorname{ord}_p(f)$ is the order of the zero at p or minus the order of the pole at p. The number of points p at which $\operatorname{ord}_p(f) > 0$, i.e., at which f has a zero, is finite. Namely, suppose f has an infinite number of zeroes. Then, by compactness of T, the set of zeroes of f has a limit point. By the identity theorem f vanishes everywhere, which is a contradiction with our assumption. So f has indeed a finite number of zeroes. Similarly we can show f has a finite number of poles by considering the zeroes of $\frac{1}{f}$. Therefore $div(f)$ indeed defines a divisor.

We can now define the space of functions with poles no worse than described by D to be

$$L(D) \coloneqq \{f \in \mathcal{M}(T) \mid \operatorname{div}(f) + D \geq 0\}, \tag{4.15}$$

where $\mathcal{M}(T)$ is the set of meromorphic functions on T. The inequality is defined coefficientwise, that is,

$$\sum_{p \in T} a_p p \geq \sum_{p \in T} b_p p \tag{4.16}$$

if and only if $a_p \geq b_p$ for all $p \in T$. Using this notation we are able to reformulate Riemann's inequality symbolically.

Theorem 4.4. *If T is a compact Riemann surface of genus p and if D is a divisor, then*

$$\dim L(D) \geq \deg(D) - p + 1. \tag{4.17}$$

In fact, this is a slight generalisation of Riemann's original statement (Theorem 4.2), since he only considers $D \geq 0$. That is, Riemann only designates the possible poles. If there is a coefficient of a point p in D that is less then 0, then the functions $L(D)$ have a zero at p. This gives rise to extra equations. Details of this generalisation can be found in [Wey55].

Now we will develop some extra notation to be able to set Roch's addition (Theorem 4.7) in this modern framework of divisors. In their work Dedekind Weber, Dedekind and Weber lack a differential structure to perfom integration, therefore they reformulated everything in terms of differentials. The differentials of Dedekind and Weber would not be introduced analytically, but in a fully algebraic setting. Nowadays, differentials can also rigorously described in an analytic way via differential 1-forms. Let a differential 1-form η be locally expressed via $f_i dz_i$, with z_i being the ith coordinate of the atlas $\{(U_i, \varphi_i)\}_{i \in I}$, and with f_i being meromorphic. If $p \in U_i$ define $\text{ord}_p(\eta) = \text{ord}_p(f_i)$, so that we can define

$$\Omega^1(D) := \{\eta \in \Omega^1 | \sum_{p \in T} \text{ord}_p(\eta) + D \geq 0\}, \tag{4.18}$$

where Ω^1 denotes the set of all meromorphic 1-forms.

4.6 Riemann's solution to the inversion problem

In part 2 of his paper [Ri57], Riemann addresses the inversion problem. Riemann starts with the everywhere finite integrals $u_1, u_2, \ldots, u_p$ such that together with a nonzero constant function these integrals form a basis of the space of integrals (this is possible by Theorem 4.1). Riemann sets out to invert the tuple of sums

$$(\epsilon_1, \ldots, \epsilon_p) := \left(\sum_{\nu=1}^{p} u_1(q_\nu), ..., \sum_{\nu=1}^{p} u_p(q_\nu) \right), \tag{4.19}$$

that depends on p points $q_1, \ldots, q_p$ on the surface. That is, he wants to find functions $\lambda_1, \ldots, \lambda_p : \mathbb{C}^p \to T$ such that for all ν

$$\lambda_\nu(\epsilon_1, \ldots, \epsilon_p) = q_\nu. \tag{4.20}$$

This is the right generalisation of the inverse functions Jacobi wanted to find in §3.3. To any point q_ν correspond multiple prolongations of the integrals. These different choices of prolongations give different $\epsilon_1, \ldots, \epsilon_p$ corresponding to the same points $q_1, \ldots, q_p$. Therefore we define the following congruence:

Definition 4.5. Tuples $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_p)$ are called *congruent* (with respect to the choice of integrals $u_1, u_2, \ldots, u_p$ and therefore giving per integral u_π the $2p$ moduli of periodicity $k_\pi^{(\nu)}$), written

$$(a_1, \ldots, a_p) \equiv (b_1, \ldots, b_p), \tag{4.21}$$

if there are $m_1, \ldots m_p \in \mathbb{Z}$ such that for all π we have $a_\pi = b_\pi + \sum_{\nu=1}^{2p} m_\nu k_\pi^{(\nu)}$

Using the definition of the moduli of periodicity, we see that any congruent tuple $(\epsilon_1, \ldots, \epsilon_p) \equiv (\epsilon'_1, \ldots, \epsilon'_p)$ give for any ν that

$$\lambda_\nu(\epsilon_1, \ldots, \epsilon_p) = q_\nu = \lambda_\nu(\epsilon'_1, \ldots, \epsilon'_p), \tag{4.22}$$

so the moduli of periodicity give the periods of the inverse functions λ_ν.

We will only give a very short sketch of Riemann's solution of the inversion problem. Riemann inverts the integrals $u_1, u_1, \ldots u_p$ by cutting up the surface again into a simply connected surface and he chooses one sheet per integral. To invert the integrals, it is then enough to show that for any tuple of constants $(e_1, e_2, \ldots, e_p)$ there exists a tuple $q_1, \ldots q_\nu$ such that $(e_1, e_2, \ldots, e_p)$ is congruent to a tuple $(\sum_{\nu=1}^p u_1(q_\nu), ..., \sum_{\nu=1}^p u_p(q_\nu))$, since $(e_1, e_2, \ldots, e_p)$ is then equal to $(\sum_{\nu=1}^p u_1(q_\nu), ..., \sum_{\nu=1}^p u_p(q_\nu))$ in another choice of sheets.

In order to solve the inversion problem, Riemann defines a theta function that depends on complex variables $v_1, \ldots v_p$ and complex coefficients $a_{\mu,\mu'}$, where $\mu, \mu' = 1, 2, \ldots, p$, ensuring that $a_{\mu,\mu'} = a_{\mu',\mu}$ and ensuring that for all $m_1, \ldots m_p \in \mathbb{Z}$ the real part of $a_{\mu,\mu'} m_\mu m_{\mu'}$ is negative. Up to a constant factor, the theta function is defined by the $2p$ relations

$$\theta(v_1, v_2, \ldots, v_\mu, \ldots, v_p) = \theta(v_1, v_2, \ldots, v_\mu + \pi i, \ldots, v_p), \tag{4.23}$$

$$\theta(v_1, v_2, \ldots, v_p) = e^{2v_\mu + a_{\mu,\mu}} \theta(v_1 + a_{1,\mu}, v_2 + a_{2,\mu}, \ldots, v_p + a_{p,\mu}). \tag{4.24}$$

Riemann also shows that such a function exists.

Riemann cuts the surface T into a simply connected surface T' using $2p$ specific cuts $a_1, \ldots a_p$ and $b_1, \ldots, b_p$. He selects a basis $u_1, u_2, \ldots u_p$ of everywhere finite integrals such that the moduli of periodicity of u_μ at a_μ is equal to πi and equal to 0 at the other a. The moduli of periodicity of u_μ at b_ν are called $a_{\mu,\nu}$. Riemann defines the coefficients of the theta function to be equal to $a_{\mu,\nu}$ and shows this is possible by his specific choice of his cuts a_ν and b_ν. The integrals $u_1, u_2 \ldots u_p$ are holomorphic on T'.

Let $(e_1, e_2, \ldots, e_p)$ denote arbitrary constants for which we want to find a $\lambda_\nu(e_1, e_2, \ldots, e_p)$. Since the u_μ are holomorphic functions on T', the composition $\theta(u_1 - e_1, u_2 - e_2, \ldots, u_p - e_p)$ becomes a holomorphic function on T'. By the choice of moduli of periodicity of the u and the defining relations of the theta function, $\theta(u_1 - e_1, u_2 - e_2, \ldots, u_p - e_p)$ is only discontinuous at the cuts b_ν (when considered a function on T). By considering the integral $\int d \log \theta$ along the boundary of the simply connected piece consisting of both sides of the cuts, Riemann shows that $\theta(u_1 - e_1, u_2 - e_2, \ldots, u_p - e_p)$ has p zeroes $\eta_1, \eta_2 \ldots, \eta_p$.

Next, Riemann cuts up the surface T' even further into a surface T^* such that $\log \theta$ can be regarded as single valued through a choice of branch. He goes on to analyse the integrals $\int \log \theta du_\mu$ along the boundary of T^*. Riemann shows that

$$\sum_{\nu=1}^{p} u_\mu(\eta_\nu) - e_\mu + k_\mu \equiv 0, \tag{4.25}$$

where the k_μ are constants that only depends on the original choice of u_μ and $\equiv$ is the congruence from Definition 4.5.

Riemann then shows he can make a new choice of u_μ such that each new u_μ differs by a constant from the old u_μ and $k_\mu = 0$. The new integrals $u_1, \ldots, u_p$ still form, together with a constant function, a basis of the space of integrals. Using this choice of additive constants, Riemann solves the inversion problem in paragraph 24. The above proof fails, however, when θ is the zero function. In that case Riemann shows there are infinitely many solutions. Riemann refined his treatment of the second case in a later article [Ri65]. To summarize:

Theorem 4.6 (Jacobi's inversion problem)**.** *There exist integrals of the first kind $u_1, \ldots u_p$ such that, together with a constant, these form a basis of the space of integrals of the first kind on a Riemann surface and any arbitrarily given tuple $(e_1, \ldots, e_p)$ is congruent to either a unique tuple of the form*

$$\left(\sum_{\nu=1}^{p} u_1(\eta_\nu), \ldots, \sum_{\nu=1}^{p} u_p(\eta_\nu) \right), \tag{4.26}$$

or to infinitely many.

In other words, either there exist unique functions $\lambda_1, \ldots, \lambda_p : \mathbb{C}^p \to T$ such that

$$(e_1, \ldots e_p) = \left(\sum_{\nu=1}^{p} u_1(\lambda_\nu(e_1, \ldots, e_p)), \ldots, \sum_{\nu=1}^{p} u_p(\lambda_\nu(e_1, \ldots, e_p)) \right), \tag{4.27}$$

or there are infinitely many such functions.

Riemann did not directly use Riemann's inequality (Theorem 4.2) to solve Jacobi's inversion problem. However in solving the inversion problem, Riemann examined the equations associated with the $2p$ moduli of periodicity, which are often linearly independent. Riemann's inequality serves as a simple application of these equations. However, Riemann observed that infinite solutions to the inversion problem arise when the equality in his inequality is not attained due to the presence of a non-constant meromorphic function on T with fewer than $p+1$ poles. (paragraph 10, 16, 23 & 24 of [Ri57]). This stresses the importance of his inequality and the search for the exact conditions when the equality in his inequality is violated. And although Riemann aimed to prove the inversion problem, it were the developments, such as Riemann's inequality, for which this work is famous today.

4.7 Work of Roch

In paragraph 10 of [Ri57], Riemann provided a condition for the existence of a non-constant meromorphic function on a surface T that has a simple pole at fewer then $p + 1$ points. The corresponding divisor has $\deg D < p + 1$. This would imply that some of the equations in the proof of his inequality (Theorem 4.2) (arising from setting the moduli of periodicity equal to 0) are, in that case, dependent on each other. Namely, $\dim L(D) \geq 2$, since $L(D)$ contains a constant and a non-constant function, and so $\dim L(D)$ is more then the lowerbound

$$\deg(D) - p + 1 < p + 1 - p + 1 = 2. \tag{4.28}$$

The condition on the existence of such a function was about zeroes of the second order of the integrals, i.e., on the first order zeroes of the corresponding differentials. Roch, a student of Riemann, recognized this and, utilizing Riemann's tools, began to investigate the precise relationship between dimension of the space of functions with specific poles and the dimension of integrals with designated zeros in his 1865 paper [Roc65]. We will now treat this paper in detail.

Roch considers the same cuts $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$ as we saw Riemann do in §4.6 to reduce the surface T to a simply connected surface T'. By strategically selecting the integrals t_k and $w_\mu = u_\mu$ to ensure that many moduli of periodicity already vanish, Roch simplifies equation (4.30) to a $p \times m$ system. To achieve this, he first chooses a new basis of the space of integrals of the first kind u_μ such that it's moduli of periodicity are equal to πi at a_μ, equal to zero at the other a_ν, and equal to $a_{\mu,\nu}$ at b_ν (like we saw Riemann do in §4.6). Let's denote the modulus of periodicity of t_k at a_ν by $\tau_{k,\nu}$ and at b_ν by $\rho_{k,\nu}$. By taking new integrals of the second kind $t'_k = t_k - \frac{1}{\pi i}(\tau_{k,\nu} u_1 + \cdots + \tau_{k,p} u_p)$ which moduli of periodicity are equal to zero, we get the new system of equations determining the space of meromorphic functions

$$\begin{pmatrix} & & & \pi i & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \pi i \\ \rho_{1,1} & \cdots & \rho_{m,1} & a_{1,1} & \cdots & a_{p,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{1,p} & \cdots & \rho_{m,p} & a_{p,1} & \cdots & a_{p,p} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix} = 0. \tag{4.29}$$

This gives $\alpha_1, \ldots, \alpha_m = 0$, so we get a reduced $p \times m$ system of equations

$$\begin{pmatrix} \rho_{1,1} & \cdots & \rho_{m,1} \\ \vdots & \ddots & \vdots \\ \rho_{1,p} & \cdots & \rho_{m,p} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = 0. \tag{4.30}$$

Now Roch considers the same p integrals $\int_{\partial T'} u_\mu dv$ along the boundary of T' consisting of the cuts as Riemann did in paragraph 20 of [Ri57]. Here is v is an arbitrary integral and therefore in the form

$$v = \beta_1 t'_1 + \cdots + \beta_m t'_m + const. \tag{4.31}$$

and we call it's moduli of periodicity A_ν and B_ν at a_ν and b_ν respectively. The integral $\int_{\partial T'} u_\mu dv$ was only understood intuitively by Riemann and Roch, but it can now also be interpreted as a combination of a path integral and a Riemann–Stieltjes integral.

Riemann showed these p integrals have expressions in terms of the moduli of periodicity of s if the cuts are chosen in a specific way, namely start with an arbitrary loop a_1 not dividing the surface in two pieces. Let b_1 go from the positive side of a_1 to the corresponding point on the negative side of a_1. Then we inductively define the next pair by taking a new cut consisting of a loop $a_{\nu+1}$ and a continuation part c_ν linking it to the previous cuts $a_1, b_1, \ldots, a_\nu, b_\nu$, and a loop $b_{\nu+1}$ going from the positive side of $a_{\nu+1}$ to the corresponding point on the negative side of $a_{\nu+1}$. The integral along the boundary is an integral along the positive edges minus an integral along the negative edges. Their difference of u_μ along these edges is the moduli of periodicity corresponding to the cuts. So we get

$$\int_{\partial T'} u_\mu dv = \int_{a_\mu} \pi i dv + \int_{b_\nu} A_\nu dv. \tag{4.32}$$

But the integral $\int dv$ along a curve just equals the difference of v between the values at the starting point and end point of the curve. Since any loop a_ν starts and ends at the opposite side of the loop b_ν, the difference of v between the values at the starting and end point of a_ν is the moduli of periodicity of v at b_ν, and vice versa. So we have

$$\int_{\partial T'} u_\mu dv = \pi i B_\mu + \sum_{\nu=1}^{p} A_\nu a_{\mu,\nu}. \tag{4.33}$$

Roch noted that for each $\mu = 1, \ldots, p$ the integral equals the sum of integrals among the poles ϵ_k of v, that Roch assumes for simplicity to not lie at the branch points or at infinity (the points ϵ_k are the poles of t_k as defined in §4.3). This gives an expression for $\int_{\partial T'} u_\mu dv$ in terms of the differentials of u_μ using Cauchy's integral formula. Namely, if we write the u_μ as an Abelian integral[35] $u_\mu = \int f_\mu(s,z)ds$ where f is a rational function. Since locally u_μ and t_k are singlevalued functions of z in a neighbourhood $\epsilon_k = (s_k, z_k)$, we can expand them in powers of $z - z_k$: we get $u_k(z) = u_\mu(z_k) + f(s_k, z_k)(z - z_k) + (z - z_k)^2 g(z)$ and $t_k = \frac{1}{z - z_k} + h(z)$, where g and h are holomorphic functions in a neighbourhood of ϵ_k. Then we get that the integral among the pole ϵ_k is equal to

$$\int u_\mu dv = \int \left(-\frac{u(z_k)}{(z - z_k)^2} - \frac{f(s_k, z_k)}{(z - z_k)} + j(z) \right) dz = -2\pi i f(s_k, z_k) \tag{4.34}$$

with j a holomorphic function in a neighbourhood of z_k. Therefore we have

$$\int_{\partial T'} u_\mu dv = -2\pi i \sum_{k=1}^{m} \beta_k f_\mu(s_k, z_k). \tag{4.35}$$

[35]Roch even wrote f in the form $f(s,z) = \frac{\varphi(s,z)}{\frac{\partial F(s,z)}{\partial s}}$, where φ is a polynomial. This expression was proved by Riemann in §9 of [Ri57].

The vanishing of the moduli of periodicity of v provides Roch with a system of equations related to the values of the differentials of u_μ:

$$\sum_{k=1}^{m} \beta_k f_\mu(s_k, z_k) = 0, \tag{4.36}$$

or equivalently

$$\begin{pmatrix} f_1(s_1, z_1) & \cdots & f_1(s_m, z_m) \\ \vdots & \ddots & \vdots \\ f_p(s_1, z_1) & \cdots & f_p(s_m, s_m) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = 0. \tag{4.37}$$

The space of differentials with zeroes at least at the points ϵ_k are the solutions to the transposed system

$$\begin{pmatrix} f_1(s_1, z_1) & \cdots & f_p(s_1, z_1) \\ \vdots & \ddots & \vdots \\ f_1(s_m, z_m) & \cdots & f_p(s_m, s_m) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{pmatrix} = 0. \tag{4.38}$$

The solution of this system of equations is the space of differentials that vanish at least in the points ϵ_k. We will call the dimension of the solution space of equation (4.38) q. By observing the system (4.38) is the transpose of system (4.37), we can relate q to the dimension of the space of functions which have at most simple poles in the points ϵ_k. Namely, the dimension of solution space to the system of equations (4.37) is $m - p + q$. Including the additive constant from equation (4.10), this gave Roch's addition to the Riemann–Roch theorem: $\dim L(D) = m - p + 1 + q$.

Theorem 4.7 (Riemann–Roch)**.** *Let T be a Riemann surface of genus p. Let the dimension of the space of meromorphic one-forms that vanish at least in m designated points be q. Then the dimension of the space of functions that have at most simple poles in those points is equal to $m - p + 1 + q$. That is using the notation of equation (§4.5)*

$$\dim L(D) = \deg(D) - p + 1 + \dim \Omega^1(-D).$$

Riemann's inequality (Theorem 4.2) is a direct consequence of the Riemann–Roch theorem (Theorem 4.7) since $\dim \Omega^1(-D) \geq 0$.

4.8 The arithmetical determination of the genus.

In paragraph 7 of [Ri57], Riemann calculates the genus p using the number of sheets and the number of branch points. Consider the equation $F(s, z) = 0$. We saw in §4.2 that the number of sheets (except at branch points) is equal to degree of F with respect to the variable s, that we defined as n.

Let's consider a banch point (z_0, s_0) connecting two branches. Let s_1 and s_2 be the two values corresponding to a value of z in these two branches. Making a loop in the z-plane enclosing only this branch point results in interchanging the values of s_1 and s_2. We will call such branch point a *simple branch point*.

Consider for example the equation $F(s, z) = s^2 - z = 0$, the branch point $(0, 0)$ and the two pairs $(1, 1)$ and $(-1, 1)$. Consider the loop $t \mapsto e^{2it}$ with $t \in [0, \pi]$ in the z-plane. The corresponding loops on the Riemann surface are $t \mapsto (\pm e^{it}, e^{2it})$ and send $s = -1$ to $s = 1$ and vice versa, i.e. they interchange s_1 and s_2.

Riemann considers all other branch points as coinciding simple branch points. There are two cases: Let $a_1, \dots a_\mu$ be simple branch branch points on a anti-clockwise loop such that a_j interchanges s_j and s_{j+1}. A circuit enclosing all branch points results in an anticlockwise loop results in an transformation of $s_1, s_2, \dots s_{\mu+1}$ to $s_2, s_3, \dots, s_1$. A branch point with this transformation is said to be of order μ and is considered a coincidence of μ simple branch points. Any other coincidence of branch points interchanging different sheets can be transformed to this case by relabeling the branches. The other case is that two branch points interchange the same two branches. A loop in the z-plane around these two points will not result in an interchange of branch. If they coincide we consider the sheets as disconnected. Indeed, the extension of U used in §4.2 gives rise to two points corresponding to the coinciding branch points in the two different sheets. Therefore we say that the branch points cancel. Coinciding branch points therefore either cancel or give rise to an higher order branch points.

Riemann defines w to be the number of branch points (considering a branch point of higher order as multiple simple branch points and excluding the canceling the canceling branch points) and r the number of pairs of canceling branch points. Riemann proved the following theorem relating the number of sheets and the number of branch points to the genus p.

Theorem 4.8. *Consider the Riemann surface T defined by a multivalued function s of z given by a polynomial equation $F(s, z) = 0$. Let p be the genus of T and let n be the degree of F with respect to s and let w equal the number of true branch points. Then the following relation holds:*

$$w - 2n = 2(p - 1). \tag{4.39}$$

This formula was later generalised to the Riemann–Hurwitz formula [Hur93].

Using this formula, Riemann also derived a formula including the degree of F with respect to z, which we call m.

Theorem 4.9. *Consider the Riemann surface T defined by a multivalued function s of z given by a polynomial equation $F(s, z) = 0$. Let p be the genus of T and let n and m be the degree of F with respect to s and z respectively, and let r equal the number of pairs of canceling branch points. Then the following relation holds:*

$$p = (n - 1)(m - 1) - r. \tag{4.40}$$

Consider the example of the Riemann surface corresponding to hyperelliptic integrals, i.e. let $F(s,z) = s^2 - p(z) = 0$ with p a polynomial of degree m. We will show how to determine the genus using both formulas. Considered as a polynomial in s, F has a multiple root in at $s = 0$ or $s = \infty$. To $s = 0$ correspond $p(z) = 0$, and the behaviour at $s = \infty$ depends on the degree of p. The degree of p is equal to m, i.e. the degree of F with respect to z. Let's consider the case in which p has m distinct roots. Then there are m simple branch points at $z \neq \infty$. If m is even, we can consider the point $(z,s) = (\infty,\infty)$ as a coincidence of $r = \frac{m}{2}$ pairs of canceling simple branch points, so in this case $w = m$. If m is odd, we can consider the point $(z,s) = (\infty,\infty)$ as a coincidence of $r = \frac{m-1}{2}$ pairs of canceling simple branch points and a true simple branch point, so in this case $w = m+1$.

The formula of Theorem 4.8 gives for the m even $m - 2 \cdot 2 = 2p - 2$, so $p = \frac{m}{2} - 1$. For m odd it gives $m + 1 - 2 \cdot 2 = 2p - 2$ so $p = \frac{m-1}{2}$. The formula of Theorem 4.9 gives for m even $p = (2-1)(m-1) - \frac{m}{2} = \frac{m}{2} - 1$ and for m odd $p = (2-1)(m-1) - \frac{m-1}{2} = \frac{m-1}{2}$. This is in accordance with our statement in §3.3.

5 Developments initiated by Riemann's theory

Riemann achieved a synthesis between topological, algebra-geometric and analytical (also called transcendental) methods in his theory of Riemann surfaces. In the remainder of this thesis, we will investigate how the concepts of Riemann's theory developed in order to generalise his theory to multiple dimensions. In particular we follow the development of the Hirzebruch–Riemann–Roch theorem (Theorem 9.18), a generalisation of the Riemann–Roch theorem. We will focus on the immediate history of the Hirzebruch–Riemann–Roch theorem. But first, we will give a brief outline of the developments between Riemann's theory and the Hirzebruch–Riemann–Roch theorem.

5.1 Projective geometry as a natural framework

In 1863, Clebsch and Roch began making connections between Riemann's theory and projective geometry[36] [Cl64][Die89]. This way the point at infinity Riemann used to close his complex plane (see §4.3) could be treated algebraically and in a symmetric way. Furthermore, it suggested the right generalisation of the point at infinity to algebraic varieties of higher dimension. Before we will elaborate on the introduction of projective geometry, let's define what is meant by a complex projective space:

Definition 5.1. We write $x \sim y$, with $x, y \in \mathbb{C}^{n+1}$, if there exists a $c \in \mathbb{C} \setminus \{0\}$

[36] I was not able to find the paper of Roch nor a reference of the paper of Roch, but Dieudonné stated in [Die89] (without reference) that Roch also made connections between Riemann's theory and projective geometry.

such that $x = cy$. The *complex*[37] *projective space* is then defined to be

$$\mathbb{P}^n := \mathbb{C}^{n+1} \setminus \{0\}/\sim . \tag{5.1}$$

The elements of $\mathbb{P}^n$ that are represented by $(x_0, x_1 \ldots, x_n) \in \mathbb{C}^{n+1}$ are written $[x_0 : x_1 : \cdots : x_n]$.

The colons in $[x_0 : x_1 \cdots : x_n]$ indicate that it's the ratio between the coordinates that matter. Clebsch and Roch observed that there is a natural embedding $\mathbb{C}^n \subset \mathbb{P}^n$ via

$$[x_1, x_2, \ldots, x_n] \mapsto [1 : x_1 : x_2 \cdots : x_n]. \tag{5.2}$$

In the special case of the complex plane $\mathbb{C}$ we have

$$z \mapsto [1 : z] \tag{5.3}$$

and the point $[0 : 1]$ can be thought of as the point at infinity. The behaviour in a neighbourhood of $\infty = [0 : 1]$ can now be treated in a symmetric way by considering the open neighbourhood

$$\{[z : 1] \in \mathbb{P} \mid z \in \mathbb{C}\}. \tag{5.4}$$

Riemann considered a complex polynomial equation $F(s, z) = 0$ (see 4.2). The only polynomials well defined in the projective space are the *homogenous polynomials*, i.e. each term has equal degree. In order to transform a polynomial equation between complex variables $F(x_1, \ldots, x_n) = 0$ to a homogeneous polynomial equation of the projective coordinates $f(x_0, x_1 \ldots, x_n) = 0$ we set

$$f(x_0, x_1, x_2 \ldots x_n) := x_0^n F\left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \ldots, \frac{x_n}{x_0}\right). \tag{5.5}$$

In the special case where the multivalued functions s of z defined via $F(s, z) = 0$, the points corresponding to $z = \infty$ are the solutions of $f(0, s, 1) = 0$, where $f(x_0, s, z)$ is the homogeneous polynomial corresponding to $F(s, z)$ via equation (5.5). A set is called a *projective curve* if it is the zero set of a homogeneous polynomial of three variables in $\mathbb{P}^2$.

Riemann studied the genus as a topological and therefore also as a complex diffeomorphic invariant (although not in that language). In the algebraic setting of the transformations between two projective curves given by equations of the form $f(x, y, z) = 0$, where f is a homogeneous polynomial in $\mathbb{P}^2$, complex diffeomorphic transformations are exactly the *birational transformations*, that is, the rational transformations with a rational inverse on a dense set (only on a dense set since this is not possible to algebraically distinguish the singular points, i.e., the coinciding branch points of Riemann described in §4.8). In section 11 of [Ri57], Riemann provides a proof for the non-projective setting). A quantity

[37] We only need the complex numbers, but the complex numbers can be replaced with any field.

invariant under birational transformations is called a birational invariant. Note that every projective transformation, i.e. a transformation induced by a linear transformation, is a birational transformation. Therefore all birational invariants are projective invariants, but not the other way around. The study of the algebraic geometers after Riemann focused on birational invariants.

5.2 Generalising the genus

In Riemann's theory, there is only one important invariant of a Riemann surface: the genus p. The genus can be described in multiple ways: First of all, the genus can be described topologically, and this may be the most fundamental way of looking at it. The genus p of a closed surface can be defined as half the number of cuts needed to cut a surface into one simply connected piece. The second way Riemann described the genus is analytical. He described the genus p via the dimension of the space of integrals of the first kind, which is $p+1$ (Theorem 4.1). Lastly the genus could be described algebraically by an arithmetical formula (Theorem 4.9). This was reformulated by Clebsch [Cl64] to a formula for a projective curve defined by a homogeneous polynomial of degree n with d double points (the double points are the canceling branch points of Riemann):

$$p = \frac{(n-1)(n-2)}{2} - d. \tag{5.6}$$

These three descriptions of the genus have all been subject to attempts of generalisation, with success. In 1870, Betti, inspired by Riemann, published a paper [Bet70] in which he generalised Riemann's topological description of the genus into what now are called Betti numbers. On an n-dimensional manifold there are Betti numbers p_m for $0 \leq m \leq n$. These topological ideas were taken up by Poincaré in his famous papers on "analysis situs" [Po04]. Poincaré's papers on analysis situs were motivated both by Riemann's theory and his investigations into the theory of partial differential equations. In his first paper, he gave an alternative definition of the Betti numbers that Poincaré believed to be equivalent. However, as Heegaard pointed out in [He98], these two definitions of the Betti numbers can differ. This motivated Poincaré to write his two supplements [Po04]. In his second supplement, he discovered torsion numbers. The word torsion refers to the fact that it only appears in non-orientable manifolds, that are twisted in some way, like the Möbius band.

Later it was discovered that the Betti and torsion numbers were better expressed in terms of homology groups. This was discovered by E. Noether [No26] and independently by Vietoris [Vi27] (details on the discovery of homology groups are given in [Hi96]). Hopf, unaware of the papers by Vietoris, states in one of the first articles using the group viewpoint [Ho28] that:

> "I was able to make my original proof of this generalisation of the Euler–Poincaré formula much clearer in the course of a lecture I gave in Göttingen in the summer of 1928 by using group-theoretical

concepts under the influence of Miss. E. Noether to make it much clearer and simpler."[38]

In [Hi96], Hirzebruch cites a letter of Vietoris where he states that the group-theoretical viewpoint was not entirely new but that the older topologist considered this viewpoint superfluous:

> "Of course, topologists already knew before this work that they were dealing with Abelian groups when adding cycle classes. However, because they knew that these groups were characterised by rank and elementary divisors (torsion numbers), they considered it superfluous to deal with the groups. S. Lefschetz still takes this view in his excellent book Topology (1930). He writes there on p. 29: 'Indeed everything that follows in this section can be, and frequently is, translated into the theory of groups. It is of course a mere question of a different terminology.'[39]"

Later however, Lefschetz emphasised at the end of a congress [Dic81] "the great value that Emmy Noether's ideas had for the development of modern topology."

The second description of the genus was through the dimension of the space of integrals of the first kind (Theorem 4.1). In 1868, Clebsch adjusted this concept of the description of the genus through the dimension of the space of integrals to apply for projective surfaces in a short note, where he also noted the invariance of this genus [Cl68]. Clebsch' generalisation was to consider the dimension of the space of double integrals of the first kind (i.e. everywhere finite integrals)

$$\int\int f(x,y,z)dydz, \tag{5.7}$$

where x, y and z are complex numbers that are related by a polynomial equation $F(x,y,z) = 0$. Here the integral could be interpreted analogously[40] to the

[38]My translation using the assistance of DeepL. Original: "Meinen ursprünglichen Beweis dieser Verallgemeinerung der Euler–Poincaréschen Formel konnte ich im Verlauf einer im Sommer 1928 in Göttingen von mir gehaltenen Vorlesung durch Heranziehung gruppentheoretischer Begriffe unter dem Einfluß von Fräulein E. Noether wesentlich durchsichtiger und einfacher gestalten."

[39]My translation using the assitance of DeepL. Original: 'Selbstverständlich wußten die Topologen schon vor diesen Arbeiten, daßsie es bei der Addition von Zykelklassen mit Abelschen Gruppen zu tun hatten. Weil sie aber wußten, daßdiese Gruppen durch Rang- und Elementarteiler (Torsionszahlen) charakterisiert sind, hielten sie die Beschäftigung mit den Gruppen für überflüssig. Diesen Standpunkt vertritt noch S. Lefschetz in seinem ausgezeichneten Buch Topology (1930). Er schreibt dort auf S. 29: 'Indeed everything that follows in this section can be, and frequently is, translated into the theory of groups. It is of course a mere question of a different terminology''

[40]To be more precise, if $\frac{\partial F}{\partial x} \neq 0$ at (y_0, z_0), we can define an implicit function φ in a neighbourhood of (y_0, z_0) such that $F(\varphi(y,z), y, z) = 0$. Then the local integrals $\omega : \mathbb{C}^2 \to \mathbb{C}$ can be defined via the differential equation $\frac{\partial^2 \omega}{\partial x \partial y} = f$. The gluing process is identical to the process of §3.1.

interpration of §3.1. If you read Clebsch' note, however, you won't see any reference to integrals. In contrast to Riemann, who used geometry as a tool in the theory of Abelian integrals, Clebsch used the theory of Abelian integrals as a tool in geometry, as can for example be seen from the title of [Cl64]: "On the application of *Abel*ian functions in geometry[41]". In order to be able read Clebsch' paper we have to go back to §9 of Riemann's paper on Abelian functions [Ri57]. Here Riemann proved that the integrals of the first kind are of the form

$$\int \frac{\varphi(s,z)}{\frac{\partial F}{\partial s}} ds \tag{5.8}$$

where F is the polynomial of degree n in s and m in z that restricts the complex variables s and z via $F(s,z) = 0$ and φ is a polynomial of degree $n-2$ in z and $m-2$ in s that vanishes in the singularities of $F(s,z) = 0$ (in Riemann's case the singularities are the pairs of cancelling branch points). The dimension of the space of polynomials φ of degree $n-2$ in z and $m-2$ in s that vanish in the singularities of $F(s,z) = 0$ is therefore equal to the dimension of the space of integrals of the first kind minus one, due to the integration constant, and hence is equal to the genus. In the projective case, for a surface in $\mathbb{P}^3$ given by a polynomial equation of degree n, the space of polynomials that corresponds with the double integrals of the first kind (or 2-forms of the first kind) contains the polynomials of degree $n-4$ that vanish in the singularities of the surface (see [No70]). That is why Clebsch talks about surfaces of order $n-4$ passing through the singularities in [Cl68]. The double integrals were explicitly studied in [No70]. In this paper, M. Noether proved the invariance of the dimension of the space of n-fold integrals of the first kind. This theorem was announced in [No69]. Interpreted in the sense of Clebsch, we could also think of this definition of the genus as a geometrical one.

By considering the dimension of spaces of polynomials, and therefore not counting the degree of freedom created by the integration constant, Clebsch effectively considered the space of differential 1-forms of the first kind, i.e., holomorphic 1-forms (Clebsch did not have the concept of a differential form). In fact, when the Italian algebraic geometers were talking about "the number of integrals" they meant the dimension of the space of the corresponding polynomials and therefore the dimension of the space of differential forms.

We will now treat the third definition of a genus: via the arithmetical formula. In his note [Cl68], Clebsch already noted that, if there are no singularities, his geometrical genus of a surface given by an equation $f(w,x,y,z) = 0$ of an nth degree homogeneous polynomial f, is equal to

$$\frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3}, \tag{5.9}$$

since this is the number of coefficients in a homogenous polynomial of degree $n-4$. That there are no singularities means that there are no restrictions

[41] My translation using Google Translate. Original: Ueber die Anwendung der *Abel*schen Functionen in der Geometrie

on the homogeneous polynomials other than the degree. Like in the case of a curve, this genus is lower than $\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}$ when the surface has singularities. Building further on results obtained by M. Noether in [No70], Cayley collected the different kinds of singularities into the formula and obtained a formula that was true if the order n of the surface was sufficiently high [Cay71]. In this paper, Cayley also showed this formula failed in some cases. Nevertheless, in his example Cayley showed that for certain surfaces that give a curve of genus q when intersected with a plane, the formula gives exactly $-q$, an invariant of the curve. This result was generalised by M. Noether in [No71]. Furthermore, Zeuthen showed in 1871 that the formula also determined a birational invariant [Ze71b], following an announcement [Ze71a].

The definition of the genus by Clebsch defined via polynomials of degree $n-4$ vanishing in the singularities of the surface, geometrically representing surfaces that pass through the singularities of the surface, came to be called the geometric genus p_g. The arithmetical formula shown to be invariant by Zeuthen came to be called the arithmetical genus p_a. It turned out that the most important invariant of the surface was their difference $q = p_g - p_a$, the *irregulariy* of the surface. (for more details see [Ba13] and [CE97]).

The breakthrough towards the unification of these concepts of the genus came in 1884 when Picard began to study the simple integrals of the first kind (i.e. everywhere finite integrals)

$$\int P(x,y,z)dx + Q(x,y,z)dy, \tag{5.10}$$

where x, y and z are complex variables related by a polynomial equation $f(x,y,z) = 0$ and P and Q are rational functions [Pi84a]. The integral can be interpreted analogously[42] to the interpretation of Abelian integrals in §3.1. Subsequently, Poincaré and Picard published two other notes [Po84][Pi84b] on the subject and the second note of Picard, he announced his first [Pi85] of multiple memoirs on the subject (see [Pi80]). The Italian geometers Enriques, Castelnuovo and Severi showed that the dimension of the space of the corresponding differential 1-forms of the first kind[43] was equal to the irregularity $q = p_g - p_a$ of the surface [Ba13]. The proofs of the Italian geometers were not rigorous, but this was normal in Italian algebraic geometry (see [GB12] or [TF22]). Enriques once stated that intuition is the aristocratic way of discovery, rigour the plebian way [Ho49]. And according to Goodstein and Babbitt [GB12]:

> "Severi, perhaps more than any other major mathematician of his day, stated more true theorems whose proofs were 'irreparable' by

[42]To be more precise, if $\frac{\partial F}{\partial z} \neq 0$, we can define an implicit function φ in a neighbourhood of (x_0, y_0) such that $f(x,y,\varphi(x,y)) = 0$. Then the local integrals $\omega : \mathbb{C}^2 \to \mathbb{C}$ can be defined via the system of differential equations $\frac{\partial \omega}{\partial x} = P$ and $\frac{\partial \omega}{\partial y} = Q$. The gluing process is identical to the process of S3.1

[43]When the Italian geometers said that they were counting the number of linearly independent integrals they actually meant the number of differential 1-forms, without having the terminology for it. In doing so, they excluded the constant of integration.

> modern standards or 'almost true' theorems that required modifications to make them true or that were just plain false 'theorems'."

The first rigorous (and the first analytic) proof of the equality between the dimension of the space of simple integrals of the first kind and the irregularity came from Poincaré [Po10].

The arithmetic genus was generalised in 1909 by Severi to k-dimensional algebraic surfaces [Se09]. Severi defined the arithmetic genus algebraically and conjectured on the last page an expression of the algebraic genus in terms of the dimension of the spaces of differential forms of the first kind[44], namely

$$p_a = i_k - i_{k-1} + i_{k-2} - \cdots + (\pm 1)^{k-1} i_1, \tag{5.11}$$

where i_ℓ is the dimension of the space of holomorphic ℓ-forms. Thereby Severi generalised the statement for surfaces that $p_a = p_g - q$, where q was shown to be equal to the dimension of the space of simple integrals of the first kind[45].

5.3 Unity in mathematics

Riemann, in his theory of Riemann surfaces, fully grasped the unity of mathematics. As Lefschetz stated [Lef29]:

> "When A is a curve, we are dealing with systems of points on the curve (linear or algebraic series, etc.), and one encounters a chapter of Geometry that has reached a high degree of perfection in all its phases: purely geometric, transcendental (abelian integrals), and topological (Riemann surfaces).[46]"

For higher dimensions however, mathematicians had difficulty generalising the unity of Riemann's theory, as Lefschetz continued to explain:

> "For surfaces, the approach via algebraic geometry has been far more successful than strictly transcendental methods. This is hardly surprising when one considers how recent and incomplete our knowledge still is concerning analytic functions of several variables and topology in more than two dimensions.[47]"

[44] As stated before, Severi called the dimension of the space of differential forms the number of integrals.

[45] Again, we neglect the degree of freedom arising from the integration constant.

[46] See next footnote.

[47] My translation using the assistance of DeepL. Original: 'Lorsque A est une courbe, on a affaire aux systèmes de points sur la courbe (séries linéaires ou algébriques, etc.), et l'on se trouve en présence d'un chapitre de la Géométrie qui a atteint un haut degré de perfectionnement sous toutes ses phases: purement géométrique, transcendante (intégrales abéliennes), topologique (surfaces de Riemann). Pour les surfaces, l'attaque par voie algébro-géométrique a eu bien plus de succès que les méthodes strictement transcendantes. On ne peut guère s'en étonner quand on se rappelle combien sont encore récentes et incomplètes nos connaissances sur les fonctions analytiques de plusieurs variables et sur la topologie à plus de deux dimension.'

Around 1900, we have the Italian school of algebraic geometry, the analytical works of Picard, and the topological works of Poincaré. Although they were all inspired by Riemann's theory, they diverged. But as Baker put it [Ba13]:

> "As so often happens in the progress of science, these have come about by the union of two subjects, which, though their origin was largely identical, had drifted somewhat apart."

This was also noted by the Italian geometers Castelnuovo and Enriques, but they noted [CE97]:

> "However, one must remember that such a vast horizon cannot be embraced from a single point of view. On the contrary, it is through the combined efforts of several researchers following different paths, through the skillful application of all the resources that geometry and analysis place at our disposal today, that we may hope to deepen the theory of surfaces and enrich it with new discoveries.[48]"

Indeed, the unification of these methods turned out to be fruitful. In 1910 and 1911 Poincaré published two memoirs [Po10][Po10] rediscovering results of the Italian geometers using analysis and topology in a very direct way. Poincaré's research was continued by Lefschetz [Lef17][Lef21][Lef24] who greatly simplied Poincaré's discoveries, introducing new analytical and topological tools. According to Lefschetz [Lef29]:

> "Through this approach we arrive in a simple and quick way to the very core of surface theory.[49]"

Although Lefschetz brought back the unity in the theory, Riemann's theory was not yet developed to the same degree of perfection as it had for complex curves. Our main goal for the rest of this thesis is to describe the history of the Hirzebruch–Riemann–Roch theorem. We will only describe the immediate history and we will see that the unity of mathematics plays a big role. To bridge the gap towards these developments, I will try to give references to the secondary literature in §5.4.

5.4 References to help bridge the gap

In this section we saw how the genus was generalised in different ways to higher dimensional varieties. An outline of the developments by the Italian algebraic geometers, such as linear systems, can be found in [Die85]. More details on

[48] My translation using the assistance of DeepL. Original: Il faut pourtant se rappeler que ce n'est pas d'un seul point de vue qu'on peur embrasser un horizon si vaste. C'est au contraire par les efforts réunis de plusieus travailleurs suivant des voies différentes, c'est par l'application savante de toutes les ressources que la géometrie et l'analyse mettent aujourd'hui à notre disposition, qu'on peut espérer d'approfondir la théorie des surfaces, et de l'enrichir de nouvelles découvertes.

[49] My translation using the assistance of DeepL. Original: 'Par cette voie on arrive de manière aussi simple que rapide au cœur même de la théorie des surfaces.'

Italian geometry are given in [CE97] and in [Ba13]. To get into the mathemetics the Italian geometers used you may use the classical textbook [Za35] or [SR49]. More specific mathematics on projective geometry is described in [To65].

An outline of the developments from Poincaré onwards, including the Hirzebruch–Riemann–Roch theorem can be found in [Die89]. More specifically, the motivations for Poincaré to develop his analysis situs are described in the translator's introduction of [Po04] and in [JW20]. Lefschetz writes on his own contributions on topology and its applications in algebraic geometry in his autobiography [Lef68]. A great collection of articles on the history of the topology is [Jam99]. Atiyah wrote two biographical papers, on the developments of Hodge [At77], including Hodge's research on differential forms in algebraic geometry, and of Todd [At96], that we will briefly treat in §8.1. The history of first homology groups can be found in [Hi96]. A summary of the contributions of Steenrod on algebraic topology is given in [Pe70]. The main source for the investigations of the history of the Hirzebruch–Riemann–Roch theorem is the book of Hirzebruch [Hi56], which gives excellent references to the original works.

6 Sheaves and cohomology groups

The goal of the rest of this thesis is to analyse the generalisation of the concepts appearing in the Riemann–Roch theorem (Theorem 4.7). First of all we will generalise the space in which we work. In Riemann's theory we mainly worked with the Riemann surface algebraically described by an irreducible complex polynomial equation $F(s, z) = 0$. But Riemann's results hold more abstractly in the setting of a one-dimensional complex manifold. Both notions can be generalised. As we have seen in §5.1 the complex curve description of the Riemann surface can be reformulated via a projective curve. This notion generalises directly to higher dimensions. The manifold description also generalises directly to higher dimensions. However, as we will see for example in §9.5, it lacks some of the nice properties that we will need to generalise the Riemann–Roch theorem directly. It is possible, however, to generalise the Riemann–Roch theorem in this case too, as proved in [AH59]. These investigations to generalise the Riemann–Roch theorem to differentiable manifolds started the development towards the Atiyah–Singer index theorem [At88].

Other concepts we need to generalise are the divisor, the corresponding spaces $L(D)$ and $\Omega^1(D)$ and the topological quantity called the degree of the divisor. The generalisation of the divisor and its corresponding spaces will be executed in §7.2. The topological quantity of the degree of the divisor via the concept of Chern numbers in §8.2.

6.1 Sheaves and presheaves

In 1943, Steenrod developed a theory of homology with local coefficients in order to fill gaps in the old homology theory [St43], namely he was able to provide a full duality and intersection theory for non-orientable manifolds (see §14 of

[St43]). Steenrod stated that his theory of homology with local coefficients was a “natural and full generalization of the Withney notion of *locally isomorphic complexes* [Wh40]. Whitney, in turn, credits the source of his idea to de Rham's *homology of groups of the second kind in a non-orientable manifold* [dR32].” He also noted that independently Reidemeister came to a similar homology theory: the *Überdeckung* [Re].

In 1946, Leray published two Comptes Rendus notes [Ler46a][Ler46b] where he generalised Steenrod's concept of local coefficients and put it into one global structure: the sheaf and the corresponding sheaf cohomology, which we will introduce in this section. He developed his ideas while being a prisoner during the Second World War. The motivation for Leray to introduce the concepts of a sheaf and sheaf cohomology was to be able to relate the cohomologies of two topological spaces X and Y if there is a map $f : X \to Y$. Leray gave a list of applications in his second note [Ler46b]. This relative perspective predates Grothendiek and may originate from Leray's interest in fixed points, which are important in the study of partial differential equations [Mi00]. However, according to Dieudonné [Die89] “applications certainly went far beyond the wildest dreams of the inventor of these notions.” For more details on the history of the concept of a sheaf I will refer to [Mi00] or [Die89].

Now we will introduce the concept of a sheaf of abelian groups. The old terminology differs from the modern terminology. We will use the concept of a sheaf in the form that was used by Hirzebruch and created by H. Cartan [Car51] (Definition 6.1). However, this notion of a sheaf is what is now called an étalé space. The modern notion of a sheaf does not have an classical counterpart. The terminology of a presheaf was invented by Grothendiek [Mi00] (Definition 6.4).

Definition 6.1. A *sheaf* $\mathfrak{S}$ of (abelian) groups over a topological space X is a triple $\mathfrak{S} = (S, \pi, X)$ that satisfies the following properties:

1. S and X are topological spaces and $\pi : S \to X$ is a surjective continuous map.

2. For any point $\alpha \in S$ there exists an open neighbourhood $N \subset S$ of α and an open neighbourhood $M \subset X$ of $\pi(\alpha)$ such that $\pi|_N : N \to M$ is a homeomorphism.

3. For any $x \in X$ the preimage $\pi^{-1}(x)$ is an (abelian) group.

4. If we denote $S \oplus S \coloneqq \{(\alpha, \beta) \in S \times S| \ \pi(\alpha) = \pi(\beta)\}$ with the induced topology via the product and subset topology. The map $S \oplus S \to S$ given by $(\alpha, \beta) \mapsto \alpha - \beta$ is continuous.

(Abelian) groups can be replaced with any other algebraic structure. With the concept of a sheaf we introduce the following notation:

Definition 6.2. Let $\mathfrak{S} = (S, \pi, X)$ be a sheaf of (abelian) groups. Then we have the following definitions

- $S_x := \pi^{-1}(x)$ is called the *stalk over* x.
- A *section of* $\mathfrak{S}$ *over an open set* U is a continuous map $s : U \to S$ such that $\pi \circ s = id_U$.
- We define $\Gamma(U, \mathfrak{S})$ to be the group of sections of $\mathfrak{S}$ over U. If $U = \emptyset$, we set $\Gamma(U, \emptyset) = \{0\}$.

We should also define morphisms of between sheaves.

Definition 6.3. Let $\mathfrak{S} = (S, \pi, X)$ and $\tilde{\mathfrak{S}} = (\tilde{S}, \tilde{\pi}, \tilde{X})$ be sheaves over X. A homomorphism $h : \mathfrak{S} \to \tilde{\mathfrak{S}}$ is defined by the following properties:

1. h is continuous.
2. $\pi = \tilde{\pi} \circ h$, i.e. $h(S_x) \subset \tilde{S}_x$.
3. For any x, the map $h_x : S_x \to \tilde{S}_x$ is a group homomorphism.

Often a sheaf is defined using a presheaf.

Definition 6.4. A *presheaf over* X consists for every open set $U \subset X$ of an (abelian) group S_U and for each pair of open sets $V \subset U \subset X$ of a homomorphism $r_V^U : S_U \to S_V$, called the restriction, such that

1. If $U = \emptyset$, then $S_U = 0$.
2. $r_U^U = id_U$.
3. If $W \subset V \subset U$ are open sets of X, then $r_W^U = r_W^V \circ r_V^U$.

We also write $f|_V = r_V^U f$ with $f \in S_U$.

Any sheaf defines a *canonical presheaf* via $S_U = \Gamma(U, \mathfrak{S})$. However, different presheaves can define the same sheaf. We define the sheaf corresponding to a presheaf via the direct limit with respect to the restriction homomorphisms: i.e. each $f \in S_U$ defines an equivalence class, called a *germ* $f_x \in S_x$. Namely, we have for any neighbourhood U and V of x and for any $f \in S_U$ and $g \in S_V$ that $f_x = g_x$ if and only if there exists an open neighbourhood W of x contained in U and V such that $f|_W = g|_W$. Then we define $S = \bigsqcup_{x \in X} S_x$. There is a natural projection π that maps elements of S_x to x and the topology of the sheaf is defined by a basis given by $f(U)$ for each open set U in X and $f \in S_U$. In particular, if we take the direct limit of the canonical presheaf we recover the original sheaf.

We define the following notation for sheaves that we will use throughout the rest of this thesis.

Definition 6.5.

- Let X be a topological space and let A be a group. Define the *constant sheaf with stalk* A, also denoted by A, by the triple $(X \times A, \pi, X)$. We define $\pi : X \times A \to X$ to be the canonical projection and we give $X \times A$ the product topology, where A is considered to have a discrete topology. Addition is defined stalkwise: $(x, a) \pm (x, a') = (x, a \pm a')$, where $(x, a), (x, a') \in X \times A$.

- Let X be a topological space and let G be a topological group. Define G_c to be the sheaf such that for each open $U \subset X$, $\Gamma(U, G_c)$ consists of continuous functions $U \to G$.
- Let X be a differentiable manifold and let G be a real Lie group. Define G_d to be the sheaf such that for each open $U \subset X$, $\Gamma(U, G_d)$ consists of differentiable functions $U \to G$.
- Let X be a complex manifold and let G be a complex Lie group. Define G_ω to be the sheaf such that for each open $U \subset X$, $\Gamma(U, G_\omega)$ consists of holomorphic functions $U \to G$.

The constant sheaf in Definition 6.5 should not be confused with a locally constant sheaf that we will not define here, though it has the local nature of the sheaf. Namely, the sections of the constant sheaf are locally constant and therefore constant on the connected components. If G is a topological group endowed with the discrete topology we get that $G_c = G$ is a constant sheaf (and similarly for the differentiable and holomorphic case).

Let's consider a simple example of these concepts, which we introduce here since we will use it to define Chern classes (§8.2). Let X be a complex manifold and for any open set U define S_U to be the additive group of continuous complex valued functions on U. This defines a presheaf and therefore a sheaf $\mathbb{C}_c$. Similarly we define $\mathbb{C}_c^*$ as the sheaf of germs of continuous nowhere vanishing complex functions (the germs form a multiplicative group). Let $\mathbb{Z}$ denote the constant sheaf with stalk $\mathbb{Z}$. If we consider the map $h \mapsto e^{2\pi i h}$ we get the following exact sequence of sheaves:

Theorem 6.6. *There is an exact sequence of sheaves*

$$0 \to \mathbb{Z} \to \mathbb{C}_c \xrightarrow{h \mapsto e^{2\pi i h}} \mathbb{C}_c^* \to 0. \tag{6.1}$$

Proof. Exactness at $\mathbb{Z}$ and at $\mathbb{C}_c$ is clear. To see that h is surjective, take any f_x with $f \in \Gamma(U, \mathbb{C}_c^*)$. We can restrict f to a neighbourhood V such the logarithm is single-valued on the image $f(V)$ of f. Hence, there is a $g = \frac{1}{2\pi i} \ln f \in \Gamma(V, \mathbb{C}_c)$ that maps to f. Hence, the induced map $h_x : (\mathbb{C}_c)_x \to (\mathbb{C}_c^*)_x$ maps g_x to f_x. $\square$

Note that the surjectivity of $h \mapsto e^{2\pi i h}$ is in contrast to the non-surjectivity on the global sections through the map $\Gamma(X, \mathbb{C}_c) \to \Gamma(X, \mathbb{C}_c^*)$. For example, take $X = \mathbb{C} \setminus \{0\}$ and take $z \mapsto z$ as an element of $\Gamma(X, \mathbb{C}_c^*)$. The function $\frac{1}{2\pi i} \ln(z)$ is not an element of $\Gamma(X, \mathbb{C}_c)$, since it is not single-valued, but only locally single-valued. The local single-valuedness is expressed by the exact sequence of sheaves.

Another important example of a sequence in which the exactness does not hold for global sections but holds for sheaves is the sequence of differential forms with the exterior derivative as homomorphism.

Theorem 6.7 (Poincaré's Lemma in terms of sheaves)**.** *Let X be a real manifold and let $\mathfrak{A}^p$ be the sheaf of differential p-forms, then there is an exact sequence*

$$0 \to \mathbb{R} \xrightarrow{d} \mathfrak{A}^0 \xrightarrow{d} \mathfrak{A}^1 \xrightarrow{d} \dots. \tag{6.2}$$

Proof. Poincaré's lemma implies that if $\omega \in \Gamma(U, \mathfrak{A}^p)$ such that $d\omega = 0$, then for any $x \in U$ there is an open neighbourhood V of x such that $\omega = d\alpha$ with $\alpha \in \Gamma(V, \mathfrak{A}^{p-1})$. The statement follows from an argument analogous to the proof of 6.6. □

6.2 Cohomology classes with coefficients in sheaves

We now generalise the concept of cohomology groups with coefficients in abelian groups to cohomology groups with coefficients in sheaves of abelian groups. In this section, we will assume that the (pre)sheaves are (pre)sheaves of abelian groups. If $\mathfrak{S}$ is a sheaf of non-abelian groups however, the cohomology groups $H^0(X, \mathfrak{S})$ and $H^1(X, \mathfrak{S})$ can be defined analogously (see [Hi56]). We will define sheaf cohomology in two steps:

1. Define $H^q(\mathfrak{U}, \mathfrak{G})$ for an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of a topological space X with coefficients in a presheaf $\mathfrak{G}$.
2. Define $H^q(X, \mathfrak{S})$ as the direct limit of $H^q(\mathfrak{U}, \mathfrak{G})$ with respect to refinements of the cover, where $\mathfrak{G}$ is the canonical presheaf of $\mathfrak{S}$.

The notion of a direct limit was introduced by Steenrod in his PhD thesis (see [Pe70]). Let's start with the first step.

Definition 6.8. Let $\mathfrak{G}$ be a presheaf over a topological space X and $\mathfrak{U} = \{U_i\}_{i \in I}$ an open covering of X.

- A *q-cochain* is a function which maps an a tuple of $q+1$ indices $(i_0, \dots, i_q)$ to an element $f(i_0, \dots i_q) \in S_{(U_{i_0} \cap \dots \cap U_{i_q})}$.
- Define $C^q(\mathfrak{U}, \mathfrak{S})$ to be the group of q-cochains.
- Define the *coboundary* homomorphism $\delta : C^q(\mathfrak{U}, \mathfrak{S}) \to C^{q+1}(\mathfrak{U}, \mathfrak{S})$.

$$(\delta^q f)(i_0, \dots i_{q+1}) = \sum_{k=0}^{q+1} (-1)^k r^{U_{i_0} \cap \dots \cap \hat{U}_{i_k} \cap \dots \cap U_{i_{q+1}}}_{U_{i_0} \cap \dots \cap U_{i_{q+1}}} (f(i_0, \dots, \hat{i}_k, \dots, i_{q+1})). \tag{6.3}$$

You can prove $\delta^{q+1}\delta^q = 0$, so we can define the cohomology groups.

Definition 6.9. Let $\mathfrak{G}$ be a presheaf over a topological space X and $\mathfrak{U} = \{U_i\}_{i \in I}$ an open covering of X. We call

$$H^q(\mathfrak{U}, \mathfrak{G}) = \ker(\delta^q)/im(\delta^{q-1})$$

the *cohomology groups of an open cover* $\mathfrak{U}$ *with coefficients in* $\mathfrak{G}$.

To take the direct limit we have to show that the cohomology groups $H^q(\mathfrak{U}, \mathfrak{G})$ form a directed set with respect to refinement of the cover $\mathfrak{U} = \{U\}_{i \in I}$. Let

$\mathfrak{V} = \{V\}_{j\in J}$ be a refinement of $\mathfrak{U} = \{U\}_{i\in I}$ via a map $\tau : J \to I$ such that $V_j \subset U_{\tau(j)}$. This induces a map on cochains

$$\tau^* : C^q(\mathfrak{U}, \mathfrak{G}) \to C^q(\mathfrak{V}, \mathfrak{G}) \tag{6.4}$$

by the formula

$$(\tau^* f)(j_0, \dots, j_q) = f|_{V_{j_0},\dots,V_{j_q}}(\tau(j_0), \dots, \tau(j_q)). \tag{6.5}$$

The map τ^* commutes with δ^q, so it induces a homomorphism

$$\tau^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U}, \mathfrak{G}) \to H^q(\mathfrak{V}, \mathfrak{G}). \tag{6.6}$$

The map τ^* depends on the choice of τ. For the cohomology groups, however, $\tau^{\mathfrak{U}}_{\mathfrak{V}}$ is independent of the specific choice of map τ (see Lemma 2.6.1 in [Hi56]). Now it is clear that the maps $\tau^{\mathfrak{U}}_{\mathfrak{V}}$ make the cohomology groups of a cover a directed set with respect to refinements of the cover. Therefore we can create a definition of the cohomology groups with coefficients in a sheaf.

Definition 6.10. Let $\mathfrak{G}$ be a presheaf over a topological space X, then we define the *cohomology groups*

$$H^q(X, \mathfrak{G}) \coloneqq \varinjlim H^q(\mathfrak{U}, \mathfrak{G}) \tag{6.7}$$

as the direct limit with respect to refinement.

Let $\mathfrak{S}$ be a sheaf and let $\mathfrak{G}$ be its canonical presheaf. Then we define the cohomology groups with coefficients in $\mathfrak{S}$ to be

$$H^q(X, \mathfrak{S}) \coloneqq H^q(X, \mathfrak{G}). \tag{6.8}$$

As an example, we will explicitly determine $H^0(X, \mathfrak{S})$. Take any cover $\mathfrak{U} = \{U_i\}_{i\in I}$. Then an element $f \in H(\mathfrak{U}, \mathfrak{S})$ gives for each $i \in I$ a section $f_i \in \Gamma(U_i, \mathfrak{S})$ such that

$$0 = (\delta^0 f)(i, j) = f_j|_{U_i\cap U_j} - f_i|_{U_i\cap U_j}, \tag{6.9}$$

i.e., $f_j|_{U_i\cap U_j} = f_i|_{U_i\cap U_j}$. Therefore, $H^0(\mathfrak{U}, \mathfrak{S}) = \Gamma(X, \mathfrak{S})$, so

$$H^0(X, \mathfrak{S}) = \Gamma(X, \mathfrak{S}). \tag{6.10}$$

Two presheaves that define the same sheaf may define different cohomology groups on X. However, if the space is paracompact, the two presheaves define the same cohomology groups on X (see Lemma 2.9.1 in [Hi56]). This fact is important when we consider an exact sequence of sheaves

$$0 \to \mathfrak{S}' \to \mathfrak{S} \to \mathfrak{S}'' \to 0. \tag{6.11}$$

On each open set U of X we define S''_U via an exact sequence

$$0 \to \Gamma(U, \mathfrak{S}') \to \Gamma(U, \mathfrak{S}) \to S''_U \to 0. \tag{6.12}$$

We noted in the example of Theorem 6.6 that S''_U is not always equal to $\Gamma(U, \mathfrak{S}'')$. However, the corresponding presheaves, that assign S''_U and respectively $\Gamma(U, \mathfrak{S}'')$ to each open set U of X with the usual (induced) restrictions, define the same sheaf $\mathfrak{S}''$. Therefore, if X is paracompact, the presheaves define the same cohomology groups. The exact sequence of sheaves of equation (6.11) induces via the exact sequence of equation (6.12) a long exact sequence on the cohomology groups of a cover (like in the simplicial case, for details see [Hi56]). The direct limit preserves exact sequences so we get the following exact sequence:

Theorem 6.11. *Let X be paracompact and let*

$$0 \to \mathfrak{S}' \to \mathfrak{S} \to \mathfrak{S}'' \to 0 \tag{6.13}$$

be a short exact sequence of sheaves over X. Then we have a long exact sequence

$$\begin{aligned} 0 &\to H^0(X, \mathfrak{S}') \to H^0(X, \mathfrak{S}) \to H^0(X, \mathfrak{S}'') \\ &\to H^1(X, \mathfrak{S}') \to H^1(X, \mathfrak{S}) \to H^1(X, \mathfrak{S}'') \\ &\ddots \end{aligned} \tag{6.14}$$

6.3 Fine sheaves

We now introduce the notion of a fine sheaf, because if a sheaf is fine, many cohomology groups vanish (Theorem 6.13). The vanishing of cohomology groups can often be used to prove isomorphisms between cohomology groups (see for example equation 6.17 and Theorem 9.5).

Definition 6.12. Let $\mathfrak{S}$ be a sheaf over a paracompact space X. Then $\mathfrak{S}$ is a *fine sheaf* if for each locally finite covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, there exists a system $\{h_i\}_{i \in I}$ of homomorphisms $h_i : \mathfrak{S} \to \mathfrak{S}$ such that

1. For each $i \in I$ there is a closed set $A_i \subset X$ such that $A_i \subset U_i$ and for any $x \in X \setminus A_i$ we have $h_i(S_x) = 0$.
2. $\sum_{i \in I} h_i = id_{\mathfrak{S}}$.

Many sheaves are fine. For example, using a partition of unity we can prove that the sheaf $\mathbb{C}_c$ of continuous complex valued functions over a paracompact space X, but also the sheaf of real differential p-forms $\mathfrak{A}^p$ over a differentiable manifold is fine. On the other hand, constant sheaves, the sheaf $\mathbb{C}_c^*$ over a paracompact manifold or sheaves of holomorphic differential forms over a differentiable manifold are not fine in general.

Theorem 6.13. *If $\mathfrak{S}$ is a fine sheaf over a paracompact space X, then for any $q \geq 1$ we have $H^q(X, \mathfrak{S}) = 0$.*

Proof. See Theorem 2.11.1 in [Hi56]. □

We illustrate the usefulness of the notion of fine sheaves with an example that we will need in §8.2. In Theorem 6.6, we created the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{C}_c \to \mathbb{C}_c^* \to 0. \tag{6.15}$$

This induces the long exact sequence

$$\cdots \to H^1(X, \mathbb{C}_c) \to H^1(X, \mathbb{C}_c^*) \to H^2(X, \mathbb{Z}_c) \to H^2(X, \mathbb{C}_c) \to \dots. \tag{6.16}$$

But $\mathbb{C}_c$ is fine, so $H^1(X, \mathbb{C}_c) = H^2(X, \mathbb{C}_c) = 0$. Therefore we have an isomorphism

$$H^1(X, \mathbb{C}_c^*)) \cong H^2(X, \mathbb{Z}). \tag{6.17}$$

7 Fibre bundles and divisors

7.1 Fibre bundles

Although earlier mathematicians already used the ideas behind a fibre bundle (see [Jam99]), the formal development of the concept of a fibre bundle began with a paper by Whitney [Wh35]. In this paper, Withney introduced sphere bundles. For more details regarding the development of fibre bundles see [Jam99] or [Die89]. The development of fibre fundles was important since it evolved hand in hand with the concept of characteristic classes (see §8.2). Furthermore, the concept of a fibre bundle was used in a reinterpretation of the divisors that we will treat in the next paragraph (§7.2).

In the following subsection, let X and F be topological spaces and let G be a continuous group (written multiplicatively, since the group is no longer assumed to be abelian) with an effective continuous action $G \times F \to F$ ("effective" means that if the action of g is the identity on F, then g is the identity element of the group).

Definition 7.1. A topological space W with a continuous map $\pi : W \to X$ is called a *fibre bundle over X with structure group G and fiber F*, and π is called a projection, if there exist

1. an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of X,
2. homeomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times F$ that map $\pi^{-1}(U)$ onto $u \times F$ and
3. for all $i, j \in I$ elements $g_{ij} \in \Gamma(U_i \cap U_j, G_c)$ called transition maps such that for all $u \in U_i \cap U_j$ and $f \in F$ we have

$$h_i h_j^{-1}(u, f) = (u, g_{ij}(u) f). \tag{7.1}$$

Here, the transition maps g_{ij} are determined uniquely by h_i and h_j since the action is effective. The transition maps form a 1-cocycle on the covering $\mathfrak{U}$

(a 1-cochain g such that $\delta g = 1$, see also Definition 6.8), namely the equality $(\delta g)_{ijk} = g_{jk} g_{ik}^{-1} g_{ij} = 1$ is equivalent to $g_{ij} g_{jk} = g_{ik}$. On the other hand, a cocycle on an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X defines a fibre bundle with structure group G and fibre F if we identify $(u, f) \in U_j \times F$ and $(u, g_{ij}(u) f) \in U_i \times F$ on the disjoint union $\bigsqcup_{i \in I} U_i \times F$.

We will now define isomorphisms of fibre bundles. For this purpose we will introduce the concept of an admissible chart:

Definition 7.2. An *admissible chart* of a fibre bundle over X with coordinates given by a cocycle g is a homeomorphism $h_U : \pi^{-1}(U) \to U \times F$, with U open in X such that for any $i \in I$ there exist elements $g_{U,i} \in \Gamma(U \cap U_i, G_c)$ such that for all $u \in U \cap U_i$ and $f \in F$

$$h_U h_i^{-1}(u, f) = (u, g_{U,i}(u) f). \tag{7.2}$$

So we can define the following notion of an isomorphism between fibre bundles:

Definition 7.3. Let W be a fibre bundle with projection π and W' a fibre bundle with projection π', both over a topological space X with structure group G and fibre F. An isomorphism $k : W \to W'$ is a homeomorphism such that for each $x \in X$:

1. $k(\pi^{-1}(x)) = \pi^{-1}(x)$ and
2. there is an open neighbourhood U of x, an element $g_U \in \Gamma(U, G_c)$ and admissible charts $h_U : \pi^{-1}(U) \to U \times F$ for W and $h'_U : \pi^{-1}(U) \to U \times F$ for W' such that for all $u \in U$ and $f \in F$,

$$h'_U k h_U^{-1}(u, f) = (u, g_U(u) f). \tag{7.3}$$

2 is equivalent to saying that the cocycles g of W and g' of W' defined by the transition map define the same element in $H^1(X, G_c)$ (see [Hi56]). The cohomology groups $H^q(X, \mathfrak{S})$ are only well-defined for sheaves of abelian groups, however, the zeroth and the first cohomology groups are still well-defined in the non-abelian case (see [Hi56] for details). Therefore we have the following theorem.

Theorem 7.4. *There is a one-one correspondence between isomorphism classes of fibre bundles and with structure group G and fibre F and elements of $H^1(X, G_c)$.*

Definition 7.5. The elements of $H^1(X, G_c)$ are called *G-bundles*.

An example of a fibre bundle is a vector bundle. A vector bundle has as a vector space as its fibre and the corresponding general linear group as its structure group. For vector spaces we have a direct sum $\oplus$ and a tensor product $\otimes$. These operations can be extended fibrewise to vector bundles. We will also write $\oplus$ and $\otimes$ for the induced operations on GL-bundles. Analogously we extend the notion of the dual vector space and the exterior algebra.

For a complex vector bundle we can also define the conjugation operation. Let $\xi \in H^1(X, GL(n,\mathbb{C}))$ be represeted by g_{ij}, then we define $\overline{\xi}$ to be defined by $\overline{g_{ij}}$. Therefore, using Theorem 7.4, if W is a complex vector bundle, we can define $\overline{W}$ up to isomorphism. We define an *anti-isomorphism* to be an isomorphism except that it is anti-linear instead of linear. W and $\overline{W}$ are anti-isomorphic.

An important special case of a complex vector bundle is a complex line bundle. A complex line bundle has structure group $GL(1,\mathbb{C}) = \mathbb{C}^* = U(1)$. The group operation in $H^1(X, G_c)$ is given by the tensor product. If $\xi \in H^1(X, \mathbb{C}^*_c)$ is represented by $\{g_{ij}\}$, then the inverse ξ^{-1} is represented by $\{g_{ij}^{-1}\}$ such that $\xi \otimes \xi^{-1} = 1$, where 1 denotes the trivial bundle. The dual bundle ξ^* is also represented by g_{ij}^{-1}, so we have $\xi^* = \xi^{-1}$.

Let M be real differentiable manifold and $\{(U_i, (x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}))\}_{i\in I}$ be its atlas. The coordinate transformations

$$g_{ij} = \left(\frac{\partial x_r^{(i)}}{\partial x_s^{(j)}}\right) : U_i \cap U_j \to GL(n,\mathbb{R}) \tag{7.4}$$

define a vector bundle called the *tangent bundle* and is denoted by TM. Analogously we can associate to a complex differentiable manifold M with atlas $\{(U_i, (z_1^{(i)}, z_2^{(i)}, \ldots, z_n^{(i)}))\}_{i\in I}$ a holomorphic tangent bundle denoted by T_M. If we consider M as a real manifold by writing

$$(z_1^{(i)}, z_2^{(i)}, \ldots, z_n^{(i)}) = (x_1^{(i)} + iy_1^{(i)}, x_2^{(i)} + iy_2^{(i)}, \ldots, x_n^{(i)} + iy_n^{(i)})) \tag{7.5}$$

we get isomorphisms (see §4.7 of [Hi56])

$$TM \otimes \mathbb{C} \cong T_M \oplus \overline{T}_M \tag{7.6}$$

$$(TM \otimes \mathbb{C})^* \cong T_M^* \oplus \overline{T}_M^* \tag{7.7}$$

$$\lambda^r (TM \otimes \mathbb{C})^* \cong \bigoplus_{p+q=r} \lambda^p T_M^* \otimes \lambda^q \overline{T}_M^*, \tag{7.8}$$

where $\lambda^k W$ denotes the kth exterior power of a vector bundle W. The second isomorphism is locally given by $dz_r^{(i)} = dx_r^{(i)} + i\,dy_r^{(i)}$ and $d\overline{z}_r^{(i)} = dx_r^{(i)} - i\,dy_r^{(i)}$ with $r \in \{1, 2, \ldots, n\}$ and using the coordinates of equation (7.5).

7.2 Divisors

In this section, we consider a compact complex manifold V. In the case of a compact Riemann surface T, we defined divisors as finite formal sums of points (see §4.5)

$$\sum_{p\in T} a_p p. \tag{7.9}$$

This notion was generalised in the algebraic setting to higher dimensions, where the points are replaced by algebraic hyperplanes of codimension 1 (see [Die85] for the exact definition). There is a direct analog in the analytic setting. Here a divisor is defined by system of place functions:

Definition 7.6. $\{f_i\}$ is called a system of *meromorphic place functions* on a compact complex manifold V if $\mathfrak{U} = \{U_i\}_{i\in I}$ is an open covering of V and for each $i \in I$ the function f_i is meromorphic on U_i and the function f_i is not identically zero, and for each $i, j \in I$ the quotient $\frac{f_i}{f_j}$ has no poles or zeroes on $U_i \cap U_j$.

Two systems $\{f_i\}$ and $\{g_j\}$ of meromorphic place functions with $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ as respective covers are said to be equivalent if $\frac{f_i}{g_j}$ has no poles or zeroes on $U_i \cap V_j$, or in other words, if $U_i \cap V_j \neq 0$, then $\frac{f_i}{g_j} \in \Gamma(U_i \cap V_j, \mathbb{C}^*_\omega)$, where C^*_ω is the sheaf of germs of nowhere vanishing complex valued holomorphic functions on V. The equivalence classes of systems of meromorphic place functions are called *divisors*.

We will show this definition of divisors correspond with finite formal sums of points on a compact Riemann surface T. First, let $\{f_i\}$ be a system of meromorphic place functions on T corresponding to a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of T. Since T is compact, we can take a finite subcover of $\mathfrak{U}$ and f_i has a finite number of poles and zeroes (using a argument analogous to the one used to show the well-definedness of $\operatorname{div}(f)$ defined in equation (4.14)). If U_i and U_j overlap, f_i and f_j have poles and zeroes of equal order on $U_i \cap U_j$. So to any $p \in T$ we can associate a unique number $a_p = \operatorname{ord} f_i$, where i is chosen such that $p \in U_i$. This gives the finite formal sum $\sum_{p\in T} a_p p$. Any equivalent system of meromorphic place functions also have poles and zeroes of equal order. Conversely we can find for any finite formal sum a corresponding system of meromorphic place functions as follows: Denote the by $\{p_k\}_{k\in K}$ the set of points corresponding to the nonzero terms in a finite formal sum of points $\sum_{p\in T} a_p p = \sum_k a_{p_k} p_k$. Define non-overlapping neighbourhoods U_k corresponding to a chart centered around p_k for each $k \in K$. Define $U = V \setminus \{p_k\}_{k\in K}$. The functions $z \mapsto z^{a_{p_k}}$ on each U_k and the constant function 1 on U form a system of place functions. This establishes the correspondence.

Alternatively, the definition of divisors can be written in the language of sheaves.

Theorem 7.7. *Let $\mathfrak{G}$ be the sheaf of germs of local meromorphic not identically zero functions on a compact complex manifold V, where multiplication is considered the group operation. Then we define a sheaf $\mathfrak{D} = \mathfrak{G}/\mathbb{C}^*_\omega$ via the exact sequence*

$$0 \to \mathbb{C}^*_\omega \to \mathfrak{G} \to \mathfrak{D} \to 0. \tag{7.10}$$

Divisors correspond one-to-one to the elements of $H^0(V, \mathfrak{D})$.

Proof. Any system of meromorphic place functions defines an element of $H^0(V, \mathfrak{D})$ in the direct limit over refinements of the cover of the system. Being an element of $H^0(\mathfrak{U}, \mathfrak{D})$ exactly means that $\frac{f_i}{f_j}$ has no poles and zeroes on $U_i \cap U_j$. The equivalenve introduced by the direct limit is precisely the equivalence between two systems of meromorphic place functions. □

We will write the group operation on divisors induced by the sheaf definition of the divisors additively.

The function space $L(D)$ can now be defined as follows:

Definition 7.8. Define $L(0)$ to be the space of holomorphic functions on a compact complex manifold V. If a divisor D is represented by a system $\{f_i\}$, then

$$L(D) = \{g \in \mathcal{M}(V) | \forall i \in I \; gf_i \in \mathbb{C}_\omega(U_i)\}. \tag{7.11}$$

Here, $\mathbb{C}_\omega(U_i)$ denotes the space of holomorphic functions on U_i and $\mathcal{M}(V)$ the space of meromorphic functions on V.

In modern language, the Riemann–Roch problem is to determine the dimension of $L(D)$. In 1949, Weil introduced a new perspective on divisors in terms of line bundles, that makes it more natural to use tools of algebraic topology [Weil49]. The space $L(D)$ can also be written in terms of cohomology groups. First, we define a line bundle associated to a divisor.

Definition 7.9. Let a divisor D on V be represented by a system of place functions $\{f_i\}$ with open cover $\mathfrak{U} = \{U_i\}_{i \in I}$. We define the line bundle $\{D\}$ by identifying, in $\bigsqcup_{i \in I} U_i \times \mathbb{C}$, the elements $u \times k \in U_j \times \mathbb{C}$ and $u \times \frac{f_i(u)}{f_j(u)} k \in U_i \times \mathbb{C}$, when $u \in U_i \cap U_j$.

Let $\{D\}$ be the line bundle defined by a system of place functions $\{f_i\}$ with open cover $\mathfrak{U} = \{U_i\}_{i \in I}$. Note that sections $s \in H^0(V, \{D\})$ are given by holomorphic functions s_i on U_i such that on $U_i \cap U_j$

$$\frac{s_i}{f_j} = \frac{s_j}{f_j}. \tag{7.12}$$

This $\frac{s_i}{f_j} = \frac{s_j}{f_j}$ defines a global meromorphic function that we call $h(s)$. Therefore we have a map $h : H^0(V, \{D\}) \to L(D)$. This is an isomorphism.

Theorem 7.10. *Let D be a divisor on a complex manifold V, then $H^0(V, \{D\})$ and $L(D)$ are isomorphic via the mapping h just defined.*

This motivates a generalisation of the Riemann–Roch problem: to determine the dimension of $H^0(V, W)$, where W is a vector bundle. We focus however on the case of line bundles. This case includes the classical divisors since $\{D\}$ is a line bundle.

8 Characteristic classes and the Todd genus

8.1 Canonical classes

For a complex algebraic curve described by a complex polynomial equation $f(x, y) = 0$, and for a complex algebraic surface given by a complex polynomial equation $g(x, y, z) = 0$, the simple and respectively double integrals of the first kind are of the form (see equation (5.8) and [Se32]):

$$\int \frac{\varphi}{f_y} dx, \quad \int \frac{\psi}{g_z} dx\, dy, \tag{8.1}$$

where φ and ψ are polynomials that vanish in the singularities of the curve or respectively the surface. Severi said these polynomials have a "birational invariant meaning[50] [Se32]". I think he means that the dimension of the space of these polynomials remains unchanged. The simple integrals of the first kind on a surface can be expressed as (see [Pi84a])

$$u = \int \frac{Ady - Bdx}{g_z}, \tag{8.2}$$

where A and B are polynomials also defined by conditions set out in [Pi84a]. However, these polynomials depend on their coordinates x and y and it is therefore clear that they are not defining a space of polynomials with invariant dimension. Namely, you could take the birational transformation interchanging coordinates to vary the dimension of each space. Severi however, tried to give a birationally invariant meaning to a space corresponding to the simple Picard integrals of the first kind in [Se32]. He defined the Jacobian group to be the set of double points of the curves $u = const.$, which he says has invariant meaning. This observation of Severi initiated the development of new invariants: canonical systems (canonical systems are explained e.g. in [To57]).

In 1936, Todd and Eger independently generalised the canonical system of Severi into what is now called the Eger–Todd canonical classes [Eg37][To37a] (not to be confused with the Todd class $td(\xi)$ that we define in Definition 9.16). Todd associated to these classes arithmetical invariants (numbers), that enabled him to give a new expression for the arithmetical genus in terms of these arithmetical invariants, generalising an earlier theorem of Severi for three dimensions [To37b]. The expression of the genus in terms of the arithmetical characters is now called the *Todd genus*. This expression enables us to express the arithmetical genus in terms of local geometrical data. However, Todd's proof relied on an unproven lemma of Severi[51]. Hirzebruch proved this inequality as a special case of his Hirzebruch–Riemann–Roch theorem (Theorem 9.18).

The Eger–Todd classes are now commonly replaced by Chern classes, introduced by Chern in [Ch46]. Chern classes are dual, up to sign, to the homology

[50] My own translation. Originally: 'significato invariante per transformazioni birazionali.'

[51] I am unaware of its current status, however, this lemma was still unproven at the time Hirzebruch proved the equality of the arithmetic and the Todd genus [Hir54].

class of the Eger–Todd classes via Poincaré duality. Hodge proved this duality for a non-singular variety (i.e., an algebraic variety without singular points (see [Ho51a]), and the general case was later proven by Nakano [Na55]. Chern classes where originaly defined on complex manifolds. Consequently, Hodge asked the following question [Ho51a]:

> "Now a non-singular algebraic variety over the complex field is a complex variety, and hence Chern's results apply to it, and it is a natural question to ask whether the characteristic classes are related in any way to the known geometrical properties of the variety."

By answering this question, Hodge established the duality between the Chern classes and the homology classes of Eger–Todd classes.

8.2 Chern classes

Hirzebruch used the language of Chern classes. Their definition will be subject of this chapter.

We already saw that the coboundary homomorphism induces a isomorphism (equation (6.17))

$$H^1(X, \mathbb{C}_c^*) \cong H^2(X, \mathbb{Z}). \tag{8.3}$$

This isomorphism 6.17 associates a unique cohomology class $c_1(\xi)$ to each $\mathbb{C}^*$-bundle $\xi \in H^1(X, \mathbb{C}_c^*)$ (see definition 7.5). $U(1) = \mathbb{C}^*$, so any $\mathbb{C}^*$-bundle is a $U(1)$-bundle. We will now introduce a definition of the total Chern class of a $U(q)$-bundle. For a comparison of the many equivalent definitions see [BH58]. A topological space X will be called finite dimensional if every open covering $\mathfrak{U}$ of X has a refinement $\mathfrak{V}$ such that each point of X lies in at most $n+1$ open sets of $\mathfrak{V}$.

Definition 8.1. Let X be a locally compact and finite dimensional space X that is a countable union of compact sets, then the *Chern classes* can be defined by the following axioms:

1. For every $U(q)$-bundle ξ over X, there exists for any $i \geq 0$ a Chern class $c_i(\xi) \in H^{2i}(X, \mathbb{Z})$. And we will write

$$c(\xi) = \sum_{i=0}^{\infty} c_i(\xi). \tag{8.4}$$

2. For any $U(q)$-bundle ξ over X, we have $c_0(\xi) = 1$

3. For any $U(1)$-bundle ξ over X, we have an element $c_1(\xi) \in H^2(X, \mathbb{Z})$ given by the isomorphism $H^1(X, \mathbb{C}_c^*) \cong H^2(X, \mathbb{Z})$ of equation (6.17). The total Chern class of ξ is then defined to be $c(\xi) = 1 + c_1(\xi)$.

4. If $\xi_1, \ldots, \xi_q$ are continuous $U(1)$-bundles over X, then

$$c(\xi_1 \oplus \cdots \oplus \xi_q) = c(\xi_1) \ldots c(\xi_q). \tag{8.5}$$

   Here, multiplication is given by the cup product.

5. If $f : Y \to X$ is a continuous map and ξ a $U(q)$-bundle over X, then $c(f^*\xi) = f^*c(\xi)$.

It is clear that this definition is well-defined for $U(1)$-bundles. In order to show that this definition is well-defined for any q, we should make use of axioms 4 and 5. The proof of the well-definedness of the Chern classes by these axioms is given in §4.2 and Theorem 4.3.1 of [Hi56]. According to 4.1b of [Hi56], there is an isomorphism

$$H^1(X, U(q)_c) \cong H^1(X, GL(q, \mathbb{C})_c). \tag{8.6}$$

By Theorem 7.4, we can associate to a q-dimensional complex vectorbundle W an element $\xi \in H^1(X, GL(q, \mathbb{C})_c) \cong H^1(X, U(q)_c)$. So, using equation (8.6), we define the Chern class of the complex vector bundle W to be

$$c(W) := c(\xi). \tag{8.7}$$

We will now associate numbers to polynomials in Chern classes. Let M be a compact n-dimensional complex manifold. A complex manifold has a natural orientation. Namely, if $z_1, z_2, \ldots, z_n$ are local coordinates with $z_k = x_k + iy_k$, then the ordering

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n \tag{8.8}$$

defines the natural orientation. This orientation can also be expressed by an element of $H_{2n}(M, \mathbb{Z})$ called the fundamental cycle. Here is $H_{2n}(M, \mathbb{Z})$ the $2n$th singular homology group defined e.g. in §4.1 of [Sp66]. Our goal is to give a natural pairing

$$H^{2n}(M, \mathbb{Z}) \times H_{2n}(M, \mathbb{Z}) \to \mathbb{Z}, \tag{8.9}$$

in order to let Chern classes act on the fundamental cycle.

Since M can be considered a real differentiable manifold, there exists a triangulation (see [Cai35]). A triangulation enables us to consider the equivalent but more practical simplicial homology (see equation (8.12)). In the context of simplicial homology, we will define the fundamental cycle. First we define the simplicial complex

Definition 8.2. A *simplicial complex* K consists of a set V of *vertices* and a set of finite non-empty subsets of V called the set of *simplices* such that

1. any set of one vertex is a simplex
2. and any non-empty subset of a simplex is a simplex.

You can define a topological space associated to K what is called a *geometric realisation* $|K|$ of K (see p. 110 of [Sp66]). The geometric realisation $|K|$ of K is the disjoint union of *standard simplices*

$$\Delta^n := \{(t_1, \dots, t_d)|\ t_i \geq 0,\ \sum_{i=1}^{n} t_i \leq 1\}, \tag{8.10}$$

where d is the dimension of the simplex, glued along common faces, such that the vertices of the standard simplices in $|K|$ correspond to the vertices of simplices of K. A *triangulation* of M is then a homeomorphism

$$|K| \to M. \tag{8.11}$$

Theorem 4.6.8 of [Sp66] states that singular homology (p. 159 of [Sp66]) is isomorphic to the simplicial homology (p. 161 of [Sp66]), i.e. for any $0 \leq i \leq 2n$ we have

$$H_i^{\text{sing}}(M, \mathbb{Z}) \cong H_i^{\text{sing}}(|K|, \mathbb{Z}) \cong H_i^{\text{simp}}(K, \mathbb{Z}). \tag{8.12}$$

Therefore we are justified to use the simplicial homology of a simplicial complex K corresponding to a triangulation of M. We will now define the fundamental cycle in terms of simplicial homology using the orientation of the manifold M. Take an oriented atlas of M according to the ordering of equation (8.8). A chart $\varphi : U \to V \subset \mathbb{R}^n$ centered at x induces isomorphisms of relative singular homology groups via excision (see §3.3 of [Ha02]):

$$H_{2n}(M, M \setminus \{x\}) \cong H_{2n}(U, U \setminus \{x\}) \cong H_{2n}(V, V \setminus \{0\}) \cong H_{2n}(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \{0\}). \tag{8.13}$$

The orientation of $\mathbb{R}^{2n}$ at 0 is given by the homology class of the standard simplex translated such that 0 is in the interior. This induces an orientation on the $2n$-simplices of the triangulation as follows: Take an $2n$-simplex in K and consider the corresponding function $\sigma : \Delta^n \to M$ under the triangulation $|K| \to M$ (equation (8.11). σ determines the same homology group under that isomorphism $H_{2n}(M, M \setminus \{x\})$ as the standard simplex translated such that 0 is in its interior upto a sign. If the sign is positive, σ is called *positively oriented*. If the sign is negative, it should become positive by an interchange of vertices (see also [Ha02]).

We want to be able to let $H^{2n}(M, \mathbb{Z})$ act on the simplicial homology groups corresponding to a triangulation of M. Since $\mathbb{Z}$ is a constant sheaf, the sheaf cohomology defined in Definition 6.10 is the Čech cohomology (§6.7 of [Sp66]) that we will be able to pair with the simplicial cohomology. Namely, define

$$\mathfrak{U} := \{St(v_i) \subset M | v_i \text{ a vertex of } K \text{ on } M\}, \tag{8.14}$$

where $St(v_i)$ is called the *star* of v_i and is equal to the interior of the union of all simplices in K on M having v_i as a vertex. $\mathfrak{U}$ is an open cover of M. To each simplical p-cochain $f \in C^p(K, \mathbb{Z})$ corresponds a p-cochain $\check{f} \in C^p(\mathfrak{U}, \mathbb{Z})$ as

$$\check{f}_{U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_n}} = f(i_1, i_2, \dots, i_n), \tag{8.15}$$

defining an isomorphism

$$C^p(K,\mathbb{Z}) \cong C^p(\mathfrak{U},\mathbb{Z}) \tag{8.16}$$

of chain complexes (see p. 43 of [GH94] for more details). By barycentric subdivision we obtain a simplicial complex with simplicial homology groups isomorphic to those of K (see §2.1 of [Ha02]). Using such subdivision, we can make the cover $\mathbb{U}$ of open stars arbitrarily fine so that we get in the direct limit for each p an isomorphism

$$H^p(M,\mathbb{Z}) \cong H^p(K,\mathbb{Z}), \tag{8.17}$$

between the Čech and simplicial cohomology. This enables us to define the action of an $x \in H^{2n}(M,\mathbb{Z})$ on the fundamental class $[M]$ via the pairing

$$H^{2n}(K,\mathbb{Z}) \times H_{2n}(K,\mathbb{Z}) \to \mathbb{Z} \tag{8.18}$$

The action of a cohomology class $x \in H^{2n}(M,\mathbb{Z})$ on the fundamental cycle $[M]$ is denoted by $x[M] \in \mathbb{Z}$. Define the function $x_n : H^*(M,\mathbb{Z}) \to \mathbb{Z}$ to be the action of the $2n$th degree component cohomology class on the fundamental cycle $[M]$.

Let ξ be a $U(q)$-bundle. To every polynomial $p(c_1(\xi), c_2(\xi)\dots, c_n(\xi))$ of Chern classes of ξ we can now associate a number

$$x_n[p(c_1(\xi), c_2(\xi)\dots, c_n(\xi))] \tag{8.19}$$

Let's consider the example of the Chern class of a divisor on a Riemann surface.

Theorem 8.3. *Let T be a compact Riemann surface and D a divisor on T represented by a system of place functions $\{f_i\}$ with cover $\mathfrak{U} = U_i$. Then we have*

$$c_1(\{D\})[T] = \deg(D). \tag{8.20}$$

Proof. Label the zeroes and poles of D by $j \in J$. Let $K \to T$ be a triangulation of T subject to the cover $\mathfrak{U}$, with simplicial complex K (this is possible as can be seen in §3.3 of [Sp66]). Assume w.l.o.g. that each $j \in J$ lies on the interior of an image of a 2-simplex (a triangle). The standard 2-simplex has vertices ordered counterclockwise, so a positively ordered triangle has its vertices also ordered counterclockwise. Subdivide each triangle $A_jB_jC_j$ containing a $j \in J$, positively oriented, into three new triangles with vertex j. They have positive orientation jA_jB_j, jB_jC_j, jC_jA_j. Let j be a zero or pole of f_{i_j}. To the cover $\mathfrak{V}$ of open stars (equation (8.14)) we can associate place functions $g_k = f_{i_j}$ on each open star of $k \in J$ and $g_k = 1$ for any other open star. This defines an equivalent divisor since the systems have the identical poles and zeroes of equal order (see §7.2). The line bundle $\{D\}$ then has transition functions $g_{k\ell} = \frac{g_k}{g_\ell}$ equal to 1 except for $g_{jA_j} = g_{jB_j} = g_{jC_j} = -g_{A_jj} = -g_{B_jj} = -g_{C_jj} = g_j$, with $j \in J$.

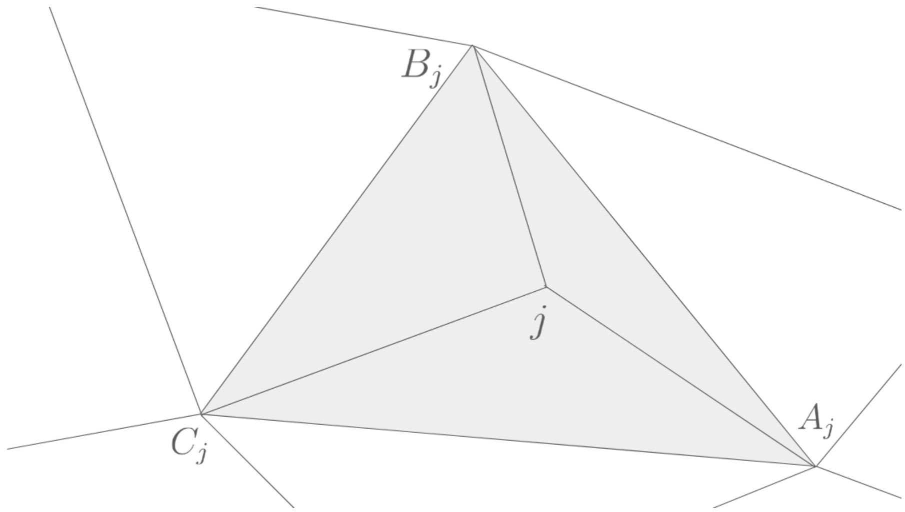

Figure 4: The subdivision of a triangle containing a point j into a three triangles having j as a vertext.

The functions $g_{k\ell}$ define an element of $H^1(T, \mathbb{C}^*)$. The first Chern class of $\{D\}$ is defined via the isomorphism

$$H^1(T, \mathbb{C}^*) \cong H^2(T, \mathbb{Z}) \tag{8.21}$$

arising from the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{C}_c \xrightarrow{\exp(2\pi i \cdot)} \mathbb{C}_c^* \to 0. \tag{8.22}$$

Let's investigate $g_{k\ell}$ under the mapping $H^1(X, \mathbb{C}^*) \cong H^2(X, \mathbb{Z})$. We have $g_{k\ell} \in H^1(\mathfrak{V}, \mathbb{C}^*)$ and the cover $\mathfrak{V}$ is fine enough such that $g_{k\ell}$ has an inverse under the mapping $\exp(2\pi i \cdot)$ on each open set of $\mathfrak{V}$:

$$h_{k\ell} := \frac{1}{2\pi i} \ln g_{k\ell} \in C^1(\mathfrak{V}, \mathbb{C}_c), \tag{8.23}$$

Applying the boundary operator on $h_{k\ell}$ we obtain an element

$$c_{k\ell m} := (\delta h)_{k\ell m} = \frac{1}{2\pi i}(\ln g_{\ell m} - \ln g_{km} + \ln g_{k\ell}) \in C^2(\mathfrak{V}, \mathbb{C}_c), \tag{8.24}$$

Theorem 6.11 shows that $c_{k\ell m} \in H^2(T, \mathbb{Z})$ and that the $c_{k\ell m}$ is independend of the choice of branch of ln when defining $h_{k\ell}$.

Let's go back to our situation. We have that $g_{k\ell}$ is equal to 1 except for $g_{jA_j} = g_{jB_j} = g_{jC_j} = -g_{A_j j} = -g_{B_j j} = -g_{C_j j} = g_j$, with $j \in J$. Therefore all $h_{k\ell} = 0$ except for permutations of $h_{jA_j} = h_{jB_j} = h_{jC_j}$. Denote the intersection of open stars of k and ℓ by $U_{k\ell}$. Let's choose the branch of $h_{jA_j} = \frac{1}{2\pi i} \ln g_j$ on U_{jA_j} and extend analytically to U_{jB_j} and then U_{jC_j}. On the overlap of U_{jC_j} and

U_{jA_j} the branches of the logarithm differ by $2\pi i \operatorname{ord}_j g_j$, since we extended the the logarithm once counterclockwise around te zero or pole. Therefore equation (8.24) gives

$$c_{jC_jA_j} = \frac{1}{2\pi i}(\ln g_{C_jA_j} - \ln g_{jA_j} + \ln g_{jC_j}) = \frac{1}{2\pi i}(0 + 2\pi i \operatorname{ord}_j g_j) = \operatorname{ord}_j g_j. \tag{8.25}$$

All other indices (except for permutations of jC_jA_j) will give zero. This is now an element of the cohomology $H^2(T, \mathbb{Z})$ and gives an element in in the simplicial cohomology $H^2(K, \mathbb{Z})$ that maps each triangle $k\ell m$ to $c_{k\ell m}$ (equation (8.15)). We can let this act on the fundamental cycle, defined as the sum of all triangles, positively oriented, of the triangulation. In the pairing of simplicial cohomology and homology, the only non-zero terms are the $c_{jC_jA_j}$ giving

$$c_1[T] = \sum_j c_{jC_jA_j} = \sum_j \operatorname{ord}_j g_j = \deg D. \tag{8.26}$$

□

9 The cohomology groups $H^{p,q}(V, W)$ and the arithmetic genus.

9.1 Kodaira's Riemann–Roch theorems

In 1950, Kodaira succeeded to prove a Riemann–Roch theorem for compact Kähler surfaces by applying the theory of harmonic integrals [Ko51]. The following year, Kodaira proved a similar Riemann–Roch theorem for complex manifolds of three dimensions [Ko52a]. The outline of Kodaira's incredible work on the Riemann–Roch theorems and other conjectures posed by the Italian geometers can be found in [Die89]. A great part of the research that preceded Kodaira's work on algebraic geometry was on harmonic integrals. Furthermore, Severi already wrote in [Se09] that he was not able to prove his conjecture on the arithmetic genus (equation (5.11)) because

> "the demonstration of this result will undoubtedly be achieved through serious difficulty, involving the introduction of many new elements and the development of others that currently exist only in embryonic form in the theory of algebraic functions of several variables.[52]"

The development of the theory of harmonic integrals consisted precisely of new elements in the theory of algebraic functions of several variables and indeed enabled Kodaira to make and prove Riemann–Roch theorems and to prove many

[52] Translated using the assistance of ChatGPT. Original: 'la dimostrazione di questo risultato si otterrà, indubbiamente attraverso a difficoltà gravissime, introducendo molti nuovi elementi e sviluppandone altri, che oggi esistono appena in germe nella teoria delle funzioni algebriche di più variabili.'

conjectures of the Italian geometers. For example, he was able to prove the conjecture of Severi on the arithmetic genus (equation (5.11)) in [Ko52b], and together with Spencer he proved another conjecture of Severi between two definitions of the arithmetic genus [KS53]. For these contributions, Kodaira was awarded the fields medal [IMU].

9.2 Dolbeault's note of 1953

The note of Dolbeault initiated what Dieudonne called the sprint towards the Hirzebruch–Riemann–Roch theorem [Die89]. In order to treat this paper we will introduce some notation and give some important preliminary results.

Let W be a complex analytic vector bundle over a complex manifold V, and let $\Omega(W)$ be the sheaf of germs of local holomorphic sections of W. We will introduce the following shorthand notation:

$$H^i(V, W) := H^i(V, \Omega(W)). \tag{9.1}$$

Let $\lambda^p T^* V$ denote the complex analytic vector bundle of covariant tangent p-vectors. Then we can define the following cohomology groups:

Definition 9.1.

$$H^{p,q}(V, W) = H^q(V, W \otimes \lambda^p T_V^*) \tag{9.2}$$

In particular we have $H^{0,q}(V, W) = H^q(V, W)$.

Definition 9.2. We call local sections of $\lambda^p T_V^* \wedge \overline{\lambda^q T_V^*}$ *local differentiable forms of type* (p, q) and define $\mathfrak{A}^{p,q}$ to be the sheaf of germs of local differentiable forms of type (p, q).

The operator d on differential forms splits into a sum

$$d = \partial + \overline{\partial}. \tag{9.3}$$

Here do ∂ and $\overline{\partial}$ denote differentiation with respect to the z and $\overline{z}$ variables respectively, the first increases the number p and the second operator increases the number q when acting on a differential form of type (p, q). Like $d^2 = 0$, we also have $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$.

9.2.1 Resolution of sheaves

In 1931, de Rham discovered a famous theorem now called de Rham's theorem [dR31]. Though de Rham's proof was accepted at the time of publication, it took for granted certain properties of differential manifolds. The first detailed proof is due to Weil and was communicated to Cartan in 1947 (see [Weil52] for Weil's publication):

Theorem 9.3. *If X is a differentiable manifold, we have an isomorphism*

$$H^p(X, \mathbb{R}) \cong H^p_{dR}(X), \tag{9.4}$$

where $H^p_{dR} = \ker(d^p)/im(d^{p-1})$ and d^p is the exterior derivative acting on differential forms of degree p.

It was found that de Rham's theorem 9.3 could be proved very neatly using Poincaré's lemma formulated in terms of an exact sequence of sheaves:

$$0 \to \mathbb{R} \xrightarrow{d} \mathfrak{A}^0 \xrightarrow{d} \mathfrak{A}^1 \xrightarrow{d} \dots, \tag{9.5}$$

where $\mathfrak{A}^p$ is the sheaf of germs of differentiable p-forms over a differentiable manifold X and d is the exterior derivative. The exterior derivatives also give maps on the sections of the sheaves:

$$0 \to \Gamma(X, \mathbb{R}) \xrightarrow{d} \Gamma(X, \mathfrak{A}^0) \xrightarrow{d} \Gamma(X, \mathfrak{A}^1) \xrightarrow{d} \dots, \tag{9.6}$$

whose quotients define the de Rham cohomology groups H^p_{dR}. The sheaves $\mathfrak{A}^p$ are fine (Definition 6.12). We have seen that the cohomology groups of non-zero degree of fine sheaves are zero (Theorem 6.13). This observation is key in the proof of de Rham's theorem. We will give a sequence of sheaves with this property a name and formulate the proof in general.

Definition 9.4. A sequence

$$0 \to \mathfrak{S} \xrightarrow{h} \mathfrak{S}_0 \xrightarrow{h^0} \mathfrak{S}_0 \xrightarrow{h^1} \mathfrak{S}_1 \xrightarrow{h^2} \dots \tag{9.7}$$

of sheaves over a paracompact space X is called a *resolution* of $\mathfrak{S}$ if the sequence is exact and for all $p \geq 0$ and $q \geq 1$ the cohomology groups $H^q(X, \mathfrak{S}_p) = 0$.

Theorem 9.5. *Let*

$$0 \to \mathfrak{S} \xrightarrow{h} \mathfrak{S}_0 \xrightarrow{h^0} \mathfrak{S}_0 \xrightarrow{h^1} \mathfrak{S}_1 \xrightarrow{h^2} \dots \tag{9.8}$$

be a resolution of $\mathfrak{S}$ over a paracompact space X. Then we have an induced sequence

$$0 \to \Gamma(X, \mathfrak{S}) \xrightarrow{h_*} \Gamma(X, \mathfrak{S}_0) \xrightarrow{h^0_*} \Gamma(X, \mathfrak{S}_0) \xrightarrow{h^1_*} \Gamma(X, \mathfrak{S}_1) \xrightarrow{h^2_*} \dots \tag{9.9}$$

such that for all $q \geq 0$

$$H^q(X, \mathfrak{S}) \cong \ker(h^q_*)/\operatorname{im}(h^{q-1}_*). \tag{9.10}$$

Here, h^{-1}_* is understood to be the zero mapping.

Proof. The case $q = 0$ is just equation (6.10). Now split the long exact sequence (9.8) into short exact sequences

$$0 \to \ker(h^p) \to \mathfrak{S}_p \xrightarrow{h^p} \ker(h^{p+1}_*) \to 0. \tag{9.11}$$

These induce for all $p \geq 1$ the long exact sequence that contains

$$0 = H^{q-1}(X, \mathfrak{S}_p) \to H^{q-1}(X, \ker(h^{p+1})) \to H^q(X, \ker(h^p)) \to H^q(X, \mathfrak{S}_p) = 0 \tag{9.12}$$

for each $q \geq 2$. Therefore by an inductive argument, using $\ker h^0 = im\ h = \mathfrak{S}$ we can show that for all $q \geq 1$

$$H^1(X, \ker(h^{q-1})) \cong H^q(X, \mathfrak{S}). \tag{9.13}$$

If p is replaced by $q-1$ in the short exact sequence (9.11), then we get the induced exact sequence

$$H^0(X, \mathfrak{S}_{q-1}) \xrightarrow{h_*^{q-1}} H^0(X, \ker(h^q) \to H^1(X, \ker(h^{q-1})) \to 0. \tag{9.14}$$

Therefore we have that $H^1(X, \ker(h^{q-1})) \cong H^0(X, \ker(h^q))/\operatorname{im}(h_*^{q-1})$. If we replace instead p with q we have an exact sequence

$$0 \to H^0(X, \ker(h^q)) \to H^0(X, \mathfrak{S}_q) \xrightarrow{h_*^q} \dots, \tag{9.15}$$

so $H^0(X, \ker(h^q)) = \ker(h_*^q)$. This proves $H^q(X, \mathfrak{S}) \cong \ker(h_*^q)/\operatorname{im}(h_*^q - 1)$. $\square$

De Rham's theorem is now a corollary of Theorem 9.5. Grothendieck proved an analog to Poincaré's lemma for the operator $\overline{\partial}$ [Do53]:

Lemma 9.6 (Grothendieck)**.** *Let $\Omega(\lambda^p T_V^*)$ denote the sheaf of germs of local holomorphic p-forms. The sequence*

$$0 \to \Omega(\lambda^p T_V^*) \xrightarrow{\overline{\partial}} \mathfrak{A}^{p,0} \xrightarrow{\overline{\partial}} \mathfrak{A}^{p,1} \xrightarrow{\overline{\partial}} \dots \tag{9.16}$$

is exact.

Proof. See [Do53] for a proof due to H. Cartan. At the time of publication of [Do53], Grothendieck's proof was still unpublished. $\square$

In [Do53], Dolbeault observed that the sequence (9.6) is a resolution, since the sheaves $\mathfrak{A}^{p,q}$ are fine. This led Dolbeault to prove an analog to de Rham's theorem for complex manifolds. This short note in the Comptes Rendus of Dolbeault [Do53] led to the discovery of theorems that year, which were crucial for the Hirzebruch–Riemann–Roch theorem. We will prove a slightly more general theorem then Dolbeault orignally treated by considering complex differential forms with coefficients in a complex analytic vector bundle W, i.e. sections of the differentiable vector bundle $W \otimes \lambda^p T_V^* \otimes \lambda^q \overline{T_V}^*$, since this enables us to introduce the divisors, as divisors can be considered line bundles (Definition 7.9). This was done by Serre [Se53] and independently by Spencer (according to Kodaira in [Ko53]).

Before we can generalize Dolbeault's theorem, we need to show that $\overline{\partial}$ induces a homomorphism on the sheaf of germs of complex differential forms with coefficients in W. This follows from the fact that the transition functions of W are matrix-valued functions with each component being a holomorphic function. But $\overline{\partial} = 0$ on holomorphic functions, so the action of $\overline{\partial}$ is independent on the local product structure $U \times \mathbb{C}$, where $U \subset V$ is open. So we get indeed an induced homomorphism $\overline{\partial}$ on the sheaf of germs of differential forms with coefficients in W.

Definition 9.7. Define $\mathfrak{A}^{p,q}(W)$ to be the sheaf of germs of differential forms of type (p,q) with coefficients in W and define $A^{p,q}(W) := \Gamma(V, \mathfrak{A}^{p,q}(W))$. Analogously define $A^{p,q} := \Gamma(V, \mathfrak{A}^{p,q})$.

Theorem 9.8. *The sequence*

$$0 \to \Omega(W \otimes \lambda^p T_V^*) \xrightarrow{\overline{\partial}} \mathfrak{A}^{p,0}(W) \xrightarrow{\overline{\partial}} \mathfrak{A}^{p,1}(W) \xrightarrow{\overline{\partial}} \dots \tag{9.17}$$

is exact.

Theorem 9.9 (Dolbeault)**.** *Let $\overline{\partial}^{p,q}$ be the restriction of $\overline{\partial}$ on the complex differential forms of type (p,q) with coefficients in a complex analytic vector bundle W, then we have*

$$H^{p,q}(V, W) \cong \ker \overline{\partial}^{p,q} / \operatorname{im}(\overline{\partial}^{p,q-1}) \tag{9.18}$$

A simple corollary of this theorem is that $H^{p,q}(V, W) = 0$ when p or q is larger then $\dim_{\mathbb{C}} V$. However, in order to apply the cohomology groups of $\Omega^p(W)$ to the theory of compact complex varieties, it was needed to prove that the dimension of these cohomology groups is finite. This step was made by Kodaira [Ko53]. But before we treat Kodaira's paper we treat Serre's letter to Borel [Se53].

9.3 Serre's letter

After Dolbeault published his note [Do53], Serre wrote a letter to Borel on his famous duality theorem (Theorem 9.10) and the Riemann–Roch theorem for higher dimensions. In his letter, Serre mentioned the work of Kodaira on the Riemann–Roch theorem for higher dimensions "that of course strongly inspired me [Serre][53]" [Se53]. Crucially, Serre considered the sheaf $\Omega(W \otimes \lambda^P T^* V)$, instead of just $\Omega(\lambda^p T^* V)$ (notation of Lemma 9.6). He derived the corresponding theorem (Theorem 9.9) analogous to that of Dolbeault.

The first result from the letter is Serre's duality theorem [Se53] and was published as Theorem 2 in [Se55]:

Theorem 9.10 (Serre duality)**.** *Let V be a compact complex manifold and let W be a complex analytic vector bundle over V, then we have an isomorphism*

$$H^{p,q}(V, W) \cong H^{n-p,n-q}(V, W^*). \tag{9.19}$$

[53] My own translation using the assistance of Google Translate. Original: 'dont je me suis bien etendu fortement inspiré.'

In particular, if $K \coloneqq \lambda^n T_V^*$ is the canonical line bundle, we have

$$H^p(V, W) \cong H^{n-p}(V, K \otimes W^*). \tag{9.20}$$

However, Serre relied on the assumption that the cohomology groups $H^{p,q}(V, W)$ were finite. This would be proven by Kodaira and independently by Cartan and Serre as we will see in the next section (Theorem 9.14). The idea of Serre's proof of his duality theorem (Theorem 9.10) was to show that there is a perfect pairing

$$H^{p,q}(V, W) \times H^{n-p,n-q}(V, W^*) \to \mathbb{C} \tag{9.21}$$

induced by the mapping

$$(\alpha, \beta) \mapsto \int_V \alpha \wedge \beta, \tag{9.22}$$

where α is a global (p, q)-form with coefficients in W and β is a $(n-p, n-q)$-current with coefficients in W^*. We will not delve into the theory of currents, which are to differential forms what distributions are to functions. Serre noted that the mapping of equation (9.22) was already studied to give a duality between forms and currents by Grothendieck and Schwartz.

Using this duality theorem Serre was able to relate the classical formulations of the Riemann–Roch theorems to the cohomology groups $H^{p,q}(V, W)$ and gave new proofs, about which he stated:

> "It is clear from the above that the general form of the Riemann–Roch theorem would be proven *if one could always express the arithmetic genus in terms of the Chern classes*[54]."

Indeed he was close to the general Hirzebruch–Riemann–Roch formula (Theorem 9.18). First let's introduce the following notation:

Definition 9.11. Let W be a complex analytic vector bundle over a compact complex manifold V of dimension n. Then we define the following notions (also relying on the finiteness of the dimension of $H^{p,q}(V, W)$):

- $h^{p,q}(V, W) \coloneqq \dim H^{p,q}(V, W)$
- $h^{p,q}(V) \coloneqq \dim H^{p,q}(V, 1)$, where 1 is the trivial vector bundle.
- $\chi(V, W) \coloneqq \sum_{q=0}^{n} (-1)^q h^{0,q}(V, W)$ is called the *Euler–Poincaré characteristic.*
- $\chi(V) \coloneqq \chi(V, 1) = \sum_{q=0}^{n} (-1)^q h^{0,q}(V)$ is called the *arithmetic genus.*

[54]My translation with the assistance of ChatGPT. Original: 'Il est clair d'après ce qui précède que la forme générale de Riemann–Roch serait démontrée *si l'on savait chaque fois exprimer le genre arithmétique à l'aide des classes de Chern.*'

The definition of the arithmetic genus is slightly different from that of Severi: it turns out that $\chi(V) = p_a + 1$, where p_a is one of the equivalent definitions of the arithmetic genus by Severi (see e.g. [Hi56]). This definition eliminates an commonly appearing -1 from the formulas of the Italian geometers.

Serre conjectured that, if D is a divisor with first Chern class x, and if the Chern classes of the tangent space of an n-dimensional variety X are given by $c_1, \dots, c_n$, then there exists a polynomial P only depending on n such that[55]

$$P(x, c_1, \dots, c_n)[X] = \chi(D). \tag{9.23}$$

This is indeed the right form of the Hirzebruch–Riemann–Roch formula (Theorem 9.18).

Let's take a look at how Serre was able to reformulate the classical Riemann–Roch formula (Theorem 4.7) in terms of the cohomology groups $H^q(V, W)$. We note

$$\Omega^1(-D) \cong H^0(V, \{D\}^{-1} \otimes \lambda^1 T_V^*) \cong H^0(V, \{D\}^* \otimes \lambda^1 T_V^*), \tag{9.24}$$

since $-D$ is represented by $\{D\}^{-1} \cong \{D\}^*$. Serre's duality (Theorem 9.10) gives

$$H^0(V, \{D\}^* \otimes \lambda^1 TV) \cong H^1(V, \{D\}). \tag{9.25}$$

Since we had shown $L(D) \cong H^0(V, \{D\})$ in Theorem 7.10, we have

$$L(D) - \Omega^1(-D) = H^0(V, \{D\}) - H^1(V, \{D\}) = \chi(V, \{D\}). \tag{9.26}$$

Therefore we can reformulate the Riemann–Roch theorem (Theorem 4.7) in the form

$$\chi(V, \{D\}) = \deg D + 1 - p. \tag{9.27}$$

In particular we have for the arithmetic genus

$$\chi(V) = \chi(V, \{0\}) = 1 - p, \tag{9.28}$$

Using these we can refine the Riemann–Roch theorem even further into:

Theorem 9.12. *Let V be a Riemann surface and let D be a divisor. Then*

$$\chi(V, \{D\}) = \deg D + \chi(V). \tag{9.29}$$

But remember, Serre formulated his general form of the Riemann–Roch theorem on the assumption that the cohomology groups $H^{p,q}(V, W)$ were finite. We will now look at Kodaira's proof of this assumption.

[55] Originally he wrote $\langle P(x, C_2, \dots, C_{2n}), X \rangle = \chi(D)$

9.4 Finiteness of the dimension of the cohomology groups, the arithmetic genus, and the theory of harmonic forms

In 1953, Kodaira published a paper [Ko53] on the finiteness of the dimension of $H^{p,q}(V, W)$ by applying the theory of harmonic forms with coefficients in a vector bundle[56]. At almost the same time, Cartan and Serre also published a paper on the finiteness of the Dolbeault cohomology [CS]. Cartan and Serre, however, considered the coefficients to consist of elements of coherent sheaves, a generalisation of vector bundles. We will not go into the theory of coherent sheaves and just treat the contents of Kodaira's paper.

In the real case an operator formally adjoint to d is constructed in order to define the Laplace–Beltrami operator Δ. The idea is to construct a Laplace–Beltrami operator analogous to the real case, by introducing a formal adjoint to the operator $\overline{\partial}$. In order to construct the formal adjoint to $\overline{\partial}$, we need an analogue to the Hodge star operator $\star$. The theory of real harmonic forms is described for example in [Ros97].

We set out to construct two anti-isomorphisms (maps that are isomorphisms except that they are anti-linear instead of linear)

$$\begin{aligned} \# &: W \otimes \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* \to W^* \otimes \lambda^{n-p} T_V^* \otimes \lambda^{n-q} \overline{T}_V^* \\ \tilde{\#} &: W^* \otimes \lambda^{n-p} T_V^* \otimes \lambda^{n-q} \overline{T}_V^* \to W \otimes \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* \end{aligned} \tag{9.30}$$

that induce anti-isomorphisms

$$\begin{aligned} \# &: \mathfrak{A}^{p,q}(W) \to \mathfrak{A}^{n-p,n-q}(W^*) \\ \tilde{\#} &: \mathfrak{A}^{n-p,n-q}(W^*) \to \mathfrak{A}^{p,q}(W). \end{aligned} \tag{9.31}$$

We start with the isomorphism $\overline{T}_V \to T_V^*$ via the pointwise map (defined in §15.3 of [Hi56]). Therefore we get

$$\begin{aligned} \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* &\cong \lambda^p T_V^* \otimes \lambda^q T_V \\ &\cong \lambda^{n-p} T_V \otimes \lambda^n T_V^* \otimes \lambda^n T_V \otimes \lambda^{n-q} T_V^* \\ &\cong \lambda^{n-p} \overline{T}_V^* \otimes \lambda^{n-q} T_V^*. \end{aligned} \tag{9.32}$$

Here we used the canonical isomorphisms of tensor and exterior products (chapter III of [Bo42]). If we compose the isomorphism of equation (9.32), that we call $\star : \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* \to \lambda^{n-p} \overline{T}_V^* \otimes \lambda^{n-q} T_V^*$, with complex conjugation c. We get an anti-isomorphism

$$c \circ \star : \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* \to \lambda^{n-p} T_V^* \otimes \lambda^{n-q} \overline{T}_V^*. \tag{9.33}$$

Together with the anti-isomorphisms $\psi : W \to W^*$ (defined in §15.3 of [Hi56]), this enables us to create the compositions $\# := \psi \otimes (c \circ \star)$ and $\# := \psi^{-1} \otimes (c \circ \star)$,

[56] In his paper he considered the particular case of a line bundle.

resulting in anti-isomorphisms

$$\begin{aligned}\# &: W \otimes \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* \to W^* \otimes \lambda^{n-p} T_V^* \otimes \lambda^{n-q} \overline{T}_V^*;\\ \tilde{\#} &: W^* \otimes \lambda^p T_V^* \otimes \lambda^q \overline{T}_V^* \to W \otimes \lambda^{n-p} T_V^* \otimes \lambda^{n-q} \overline{T}_V^*.\end{aligned} \tag{9.34}$$

These are the isomorphisms in equation (9.30) we set out to construct.

Now we can construct an inner product on the global (p,q)-forms $A^{p,q}(W)$. First we extend the wedge product in a natural way to a map

$$A^{p,q}(W) \times A^{r,s}(W^*) \to A^{p+r,q+s}(1) = A^{p+r,q+s}. \tag{9.35}$$

Namely, locally W and W^* are trivial. Let their trivialisations being given by $(w_1, \dots, w_k)$ and $(w_1^*, \dots w_k^*)$ respectively. We can then write $\alpha \in A^{p,q}(W)$ locally as $\alpha = \sum_i \alpha_i \otimes w_i$, with $\alpha_i \in A^{p,q}$, and we can write $\beta \in A^{r,s}(W^*)$ locally as $\beta = \sum_j \beta_j \otimes w_j^*$, with $\beta_i \in A^{r,s}$. Then there is a natural extension of the wedge product

$$\left(\sum_i \alpha_i \otimes w_i, \sum_j \beta_j \otimes w_j^* \right) \mapsto \sum_{i,j} w_i^*(w_j)(\alpha_i \wedge \beta_j) \tag{9.36}$$

that can be extended to a global product. Therefore we can define an inner product on the $A^{p,q}(W)$ as

$$\langle \alpha, \beta \rangle = \int_V \alpha \wedge \# \beta, \tag{9.37}$$

where $\alpha, \beta \in A^{p,q}(W)$. Checking that equation (9.37) indeed forms an inner product is analogous to the real case. We will now construct a formal adjoint with respect to this inner product,

$$\vartheta : \mathfrak{A}^{p,q}(W) \to \mathfrak{A}^{p,q-1}(W), \tag{9.38}$$

as the composition

$$\vartheta = -\tilde{\#} \overline{\partial} \#. \tag{9.39}$$

Namely, if we observe that[57] $\star\star = (-1)^{p+q}$, and so $\# \tilde{\#} = (-1)^{p+q}$, we get that for any $\alpha \in A^{p,q}(W)$ and $\beta \in A^{p,q-1}(W)$ we have

$$\begin{aligned}\langle \overline{\partial} \alpha, \beta \rangle - \langle \alpha, \vartheta \beta \rangle &= \int_V (\overline{\partial} \alpha \wedge \# \beta - \alpha \wedge \# \tilde{\#} \overline{\partial} \# \beta)\\ &= \int_V (\overline{\partial} \alpha \wedge \# \beta + (-1)^{p+q} \alpha \wedge \overline{\partial} \# \beta)\\ &= \int_V \overline{\partial}(\alpha \wedge \# \beta)\\ &= \int_V d(\alpha \wedge \# \beta)\\ &= 0.\end{aligned} \tag{9.40}$$

[57]Here we mean the operator given resulting in a multiplication by $(-1)^{p+q}$.

The fourth equality in equation (9.40) arises from the fact that $\partial = d - \overline{\partial}$ transforms the $(n, n-1)$-form $\alpha \wedge \#\beta$ into an $(n+1, n-1)$-form, which is zero. The last equality is due to Stokes' theorem. We define the corresponding Laplace–Beltrami operator to be

$$\Box := (\overline{\partial} + \vartheta)^2 = \overline{\partial}\vartheta + \vartheta\overline{\partial} : A^{p,q}(W) \to A^{p,q}(W). \tag{9.41}$$

Analogous to the real case (see [Ros97]) we have

$$A^{p,q}(W) = \overline{\partial}A^{p,q-1}(W) \oplus \vartheta A^{p,q+1}(W) \oplus B^{p,q}(V, W), \tag{9.42}$$

where $B^{p,q}(V, W)$ is the space of harmonic forms of type (p, q), i.e. the kernel of $\Box$. Therefore

$$\ker \overline{\partial}^{p,q} / \operatorname{im} \overline{\partial}^{p,q-1} \cong B^{p,q}(V, W), \tag{9.43}$$

and so by Theorem 9.9 we get the following theorem:

Theorem 9.13 (Kodaira)**.** *If V is a compact complex manifold and W is a complex analytic vector bundle over V, then*

$$H^{p,q}(V, W) \cong B^{p,q}(V, W). \tag{9.44}$$

Kodaira knew that the Laplace–Beltrami operator is an elliptic operator, and observed this was also the case in the complex case. An elliptic operator on a compact manifold has a finite dimensional kernel. So Kodaira obtains the following corollary:

Corollary 9.14 (Kodaira–Cartan–Serre)**.** *If V is a compact complex manifold and W is a complex analytic vector bundle over V, then*

$$\dim H^{p,q}(V, W) < \infty. \tag{9.45}$$

Severi conjectured that the arithmetic genus can be expressed as an alternating sum of the dimensions of the space of differential forms of the first kind (see equation (5.11))

$$\sum_{i=0}^{n} (-1)^i g_i, \tag{9.46}$$

where g_i is equal to the dimension of the space of differential i-forms of the first kind. The space of differential q-forms of the first kind, i.e., holomorphic differential forms, are exactly elements of $H^0(V, \lambda^q T_V) = H^{q,0}(V, 1)$. So $g_i = h^{q,0}$. Since we defined the arithmetic genus to be (Definition 9.11)

$$\chi(V) = \chi(V, 1) = \sum_{q=0}^{n} (-1)^q h^{0,q}, \tag{9.47}$$

we will have to prove $h^{0,q} = h^{q,0}$ to show that our definition of the arithmetic genus (Definition 9.11) is equivalent to the form conjectured by Severi in equation (5.11). But this is not true in general. It is however true in the case of a Kähler manifold, which we will treat in the next chapter.

9.5 Kähler manifolds

The notion of a Kähler metric was introduced by Kähler in his article[58] "On a remarkable Hermitian metric" [Ka33]. Kähler stated that in the study of invariants (with respect to the coordinate transformations of a complex manifold) corresponding to a Hermitian metric that can locally be written in the form

$$\sum_{\alpha,\overline{\beta}} g_{\alpha\overline{\beta}} dz^{\alpha} \otimes d\bar{z}^{\overline{\beta}}, \tag{9.48}$$

it is natural to use the alternating form

$$\omega \coloneqq \sum_{\alpha,\overline{\beta}} g_{\alpha\overline{\beta}} dz^{\alpha} \wedge d\bar{z}^{\overline{\beta}}. \tag{9.49}$$

Namely, we can use, in his words[59], "the elegant calculus of symbolic differential forms to produce invariants". As an example, Kähler gives the form $d\omega$. In his paper, he investigates the case where $d\omega = 0$. A metric for which $d\omega = 0$ is now called a *Kähler metric*. A *Kähler manifold* is a complex manifold that admits Kähler metric. It turns out that all algebraic varieties are Kähler (see §18.1 of [Hi56]), so Kähler manifolds include a wide variety of other manifolds.

In 1951, Hodge [Ho51b] and independently Garabedian and Spencer[60] published papers on the calculus of tensors and differential forms on Kähler manifolds. The most important result for the history of the Hirzebruch–Riemann–Roch theorem was proven in a subsequent paper [GS53]. In this work, Garabedian and Spencer showed that

$$\Box = \frac{\Delta}{2}, \tag{9.50}$$

where Δ is the real Laplace–Beltrami operator and $\Box$ the complex Laplace–Beltrami operator introduced in 9.41. This is now known as one of the Kähler identities. For the proof I will refer to the original source [GS53] or to proposition 3.1.12 of [Huy05]. The factor $\frac{1}{2}$ comes from the fact that Δ splits into two equal Laplace–Beltrami operators, one for ∂ and one for $\overline{\partial}$. Since the operator Δ commutes with conjugation, so does $\Box$. Therefore, we have an anti-isomorphism from $B^{p,q}$ to $B^{q,p}$, where $B^{p,q} \coloneqq B^{p,q}(V,1)$. So Theorem 9.13 gives

$$h^{p,q}(V) = \dim B^{p,q} = \dim B^{q,p} = h^{p,q}(V). \tag{9.51}$$

In particular this enables us to rewrite the arithmetic genus:

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[58] My translation. Original: "Über eine bemerkenswerte Hermitesche Metrik."

[59] My translation with the assistance of Google Translate. Orignal (whole sentence): "Diese invariant mit (1) [equation (9.48)] verknüpfte Form ω gibt Gelegenheit, den eleganten Kalkül der symbolischen Differentialformen zur Herstellung von Invarianten zu verwenden"

[60] I was not able to check the original source, but it was cited in [GS53]

Theorem 9.15. *Let V be a Kähler manifold, then the arithmetic genus (in the sense of definition 9.11) is equal to*

$$\chi(V) = \sum_{q=0}^{n} (-1)^q g_q, \tag{9.52}$$

where g_q is the dimension of the space of differential q-forms of the first kind.

Proof. Using equation (9.51) we get

$$\chi(V) = \sum_{q=0}^{n} (-1)^q h^{0,q} = \sum_{q=0}^{n} (-1)^q h^{q,0} = \sum_{q=0}^{n} (-1)^q g_q. \tag{9.53}$$

The last equality is arises from the definition of a differential q-forms of the first kind as elements of $\Gamma(V, \lambda^q T_V) = H^0(V, \lambda^q T_V)$. □

9.6 The Todd genus and the Hirzebruch–Riemann–Roch theorem

In §8.1, we saw that Hodge gave in [Ho51a] a duality between Chern classes and homology classes of the Eger–Todd classes for a non–singular variety. Motivated by this duality, Hirzebruch gave a definition of the Todd genus in terms of Chern classes in [Hir53]. Hirzebruch made use of a few tricks to simplify the many algebraic Lemmas used by Todd [To37b] in order to define his genus. This simplification provided Hirzebruch with much deeper insight into what was going on. The recent results of Kodaira and Spencer on the equality of various definitions of the arithmetic genus (see §9.1), and the work of Todd, which relied on an unproven Lemma of Severi (see §8.1), made it plausible to Hirzebruch that the equality between the arithmetic genus and the Todd genus indeed holds.

The first trick was the central topic of Hirzebruch's article [Hir53]: his invention of the multiplicative sequence. In [Hir53], Hirzebruch applied the concept of a multiplicative sequence to Steenrod's reduced powers, to the index of inertia and to the construction of the Todd genus. His expression of the index of inertia in terms of a multiplicative sequence of the Pontrjagin classes is now known as Hirzebruch's signature theorem and will be essential in Hirzebruch's proof of the Hirzebruch–Riemann–Roch theorem (Theorem 9.18) and is a direct result of combining the concept of a multiplicative sequence with the work of Thom on cobordism. Cobordism was developed in Thom's thesis [Th52] and announced in four comptes rendus notes [Th53]. I will not go deeper into the concept of a multiplicative sequence and how this was used to prove the Hirzebruch–Riemann–Roch theorem (Theorem 9.18). More on multiplicative sequences can be found in its publication [Hir53] or in [Hi56].

The second trick to ease his calculations was not his own invention. Namely, when Borel and Serre were calculating Steenrod's reduced powers of cohomology classes of certain Lie algebras, they noted that characteristic classes such as

Chern classes could be regarded as symmetric polynomials [BS51]. Hirzebruch called this the Borel–Serre method [Hir53]. Hirzebruch formulated this using generating functions. Namely, he factorised a formal polynomial of Chern classes as follows: Let X be a locally compact and finite-dimensional space X that is a countable union of compact sets and ξ a $U(q)$-bundle over X, then

$$\sum_{i=0}^{q} c_i(\xi)x^i = \prod_{j=0}^{q}(1+\gamma_j x), \tag{9.54}$$

where x and $\gamma_1, \dots, \gamma_q$ are formal variables. If we write the right hand side as a power series, we get

$$\begin{aligned}\prod_{j=0}^{q}(1+\gamma_j x) &= 1 + (\gamma_1 + \cdots + \gamma_q)x + \cdots + (\gamma_1 \cdots \gamma_q)x^q \\ &= e_0(\gamma_1, \dots, \gamma_q) + e_1(\gamma_1, \dots, \gamma_q)x + \cdots + e_q(\gamma_1, \dots, \gamma_q)x^q, \end{aligned} \tag{9.55}$$

where for each i the coeffcient of x^i is the ith order elementary symmetric polynomial $e_i(\gamma_1, \dots, \gamma_q)$. Equation (9.54) amounts to saying that for each i we have $c_i(\xi) = e_i(\gamma_1, \dots, \gamma_q)$. This factorisation is purely formal. However, using this factorisation Hirzebruch was able to express any polynomial, and therefore any power series symmetric in γ_j, in terms of the elementary symmetric polynomial of the γ_j, i.e. in terms of the Chern classes c_i. This enabled Hirzebruch to define the following notions [Hir54]:

Definition 9.16. Let X be a locally compact and finite-dimensional space X that is a countable union of compact sets and ξ a $U(q)$-bundle over X and let

$$\sum_{i=0}^{q} c_i(\xi)x^i = \prod_{j=0}^{q}(1+\gamma_j x). \tag{9.56}$$

be a formal factorisation with formal variables x and $\gamma_1, \dots, \gamma_q$, then we define:

- $ch(\xi) \coloneqq \sum_{i=1}^{q} e^{\gamma_i}$ is called the *Chern character*
- $td(\xi) \coloneqq \prod_{i=1}^{q} \frac{\gamma_i}{1-e^{-\gamma_i}}$ is called the *(total) Todd class* (not to be confused with the Eger-Todd classes).

In the definition of the Todd class we recognize the generating function

$$\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n, \tag{9.57}$$

where B_n are the Bernoulli numbers with the convention that $B_1 = \frac{1}{2}$ and x is again a formal variable.

Let V be a compact complex manifold. Recalling from §8.2 that we defined $x_n : H^*(V, \mathbb{Z}) \to \mathbb{Z}$ to be action the $2n$th cohomology class on the fundamental cycle, we can now introduce Hirzebruch's definitions of the the T-characteristic and the Todd genus [Hir54]:

Definition 9.17. Let V be a compact complex manifold and let W be a complex analytic vector bundle over V, then we define the *T-characteristic* to be

$$T(V, W) = x_n[ch(W)td(T_V)]. \tag{9.58}$$

The quantity $T(V, 1)$ is called the *Todd genus.*

The definition of $T(V, 1) = x_n[td(T_V)]$ is written in terms of Chern numbers, but is equivalent to Todd's original formulation via the duality discovered by Hodge between the Eger–Todd classes and the Chern classes (see §8.1). We are now able to formulate the Hirzebruch–Riemann–Roch theorem.

9.7 The Hirzebruch–Riemann–Roch theorem

In 1954 Hirzebruch published his famous theorem [Hir54] (Theorem 9.18). In his proof he had to assume that he worked over an *algebraic variety*, that is, a compact complex manifold that admits a complex analytic embedding as a submanifold of a complex projective space of some dimension (see 0.1 in [Hi56]). The Hirzebruch–Riemann–Roch theorem then states:

Theorem 9.18 (Hirzebruch–Riemann–Roch)**.** *Let V be an algebraic variety and let W be a complex analytic vector bundle over V. Then we have*

$$\chi(V, W) = T(V, W). \tag{9.59}$$

The case $\chi(V) = T(V)$, i.e. when $W = 1$ the trivial line bundle, is equivalent to Todd's original formula [To37b]. The Hirzebruch–Riemann–Roch theorem is indeed in the form conjectured by Serre (equation 9.23). We will first take a look how this theorem generalises the Riemann–Roch theorem (Theorem 9.12). The proof is beyond the scope of this thesis and I would like to refer to [Hir54] or Theorem 21.1.1 of [Hi56] for its details.

Theorem 9.19. *The Hirzebruch–Riemann–Roch theorem (Theorem 9.18) implies the Riemann–Roch theorem (Theorem 9.12).*

Proof. Let V be a Riemann surface and let D be a divisor. Let the Chern classes of $\{D\}$ and T_V be factored as in equation (9.54):

$$1 + c_1(\{D\})x = 1 + \gamma x, \tag{9.60}$$
$$1 + c_1(T_V)x = 1 + \gamma' x' \tag{9.61}$$

where x, γ, x' and γ' are formal variables. The Chern character corresponding to a divisor is

$$ch(\{D\}) = e^{\gamma} = 1 + \gamma + \cdots = 1 + c_1(\{D\}), \tag{9.62}$$

all higher Chern classes being 0 since V is two-dimensional and therefore all cohomology groups $H^i(V, \mathbb{Z})$ with $i > 2$ vanish. Similarly we have

$$td(T_V) = \frac{\gamma'}{1 - e^{-\gamma'}} = 1 + \frac{c_1(T_V)}{2}. \tag{9.63}$$

Therefore the T-characteristic corresponding to a divisor is

$$\begin{aligned} T(V,\{D\}) &= x_n \left[(1 + c_1(\{D\}) \left(1 + \frac{c_1(T_V)}{2} \right) \right] \\ &= x_n \left[1 + c_1(\{D\}) + \frac{c_1(T_V)}{2} \right] \\ &= c_1(\{D\})[V] + \frac{c_1(T_V)}{2}[V]. \end{aligned} \tag{9.64}$$

The pairing of the first Chern class of a divisor with the fundamental class of V is the degree of the divisor (Theorem 8.3). We will calculate the first Chern class of the tangent bundle in terms of the arithmetic genus using the Hirzebruch–Riemann–Roch Formula 9.18:

$$\chi(V) = T(V) = x_1[td(T_V)] = \frac{c_1(T_V)[V]}{2}. \tag{9.65}$$

Therefore the T-characteristic is

$$T(V,\{D\}) = c_1(\{D\})[V] + \frac{c_1(T_V)}{2}[V] = \deg(D) + \chi(V), \tag{9.66}$$

so Hirzebruch–Riemann–Roch implies

$$\chi(V,\{D\}) = T(V,\{D\}) = \deg(D) + \chi(V), \tag{9.67}$$

which is the Riemann–Roch theorem that we reformulated in Theorem 9.12. □

10 Outlook

Using the joint work of many mathematicians, who unified many areas in mathematics, Hirzebruch was able to close the problem of generalising the Riemann–Roch theorem to higher dimensions. However, his work also opened new areas of research. Most notably, in 1957, Grothendieck generalised the Hirzebruch–Riemann–Roch theorem by expressing the Hirzebruch–Riemann–Roch theorem as a relative theorem between two algebraic varieties called the Grothendieck–Riemann–Roch theorem. The theorem was first communicated in two letters to Serre, dated November 1 and November 17, 1957 [GS04], and was later published by Borel and Serre in [BS58]. Along the way Grothendieck created K-theory. The history of the Grothendieck–Riemann–Roch theorem is described in [Die89].

In 1957, Atiyah and Hirzebruch published a joint paper generalising the Hirzebruch–Riemann–Roch theorem to differentiable manifolds using the K-theoric methods of Grothendieck [AH59]. This was the start in a sequence of developments that created the Atiyah–Singer index theorem [At88]. In the commentary to volume 3 of his collected works, Atiyah outlines the immediate history of the Atiyah–Singer index theorem. The book [HP09] also contains much information in the biographical accounts of Atiyah and Singer. For a more mathematical account I would advise the article of Freed [Fr21]. A complete mathematical account of the original proof can be found in the lecture notes of Palais [Pa65].

When I was working on this thesis I stumbled upon a very interesting interview with Serre [CL85]. In this interview Serre called for more questions in papers:

> "papers should include more side remarks, open questions, and such. Very often, these are more interesting than the theorems actually proved. Alas, most people are afraid to admit that they don't know the answer to some question, and as a consequence they refrain from mentioning the question, even if it is a very natural one. What a pity! As for myself, I enjoy saying 'I do not know'."

Therefore I want to take this opportunity to make some remarks and ask some questions. First of all, we observed in §5.3 that great results arose when the mathematics attained a high degree of unity. This was also reconfirmed by the later sections. Nevertheless, there developed a strong tendency of the algebraic geometers after Zariski to reinterprete the theorems in a purely algebraic setting. This lead to the following quote of Lefschetz [Lef68]:

> "I cannot refrain, however, from mention of the following noteworthy activities: (...) II. The systematic algebraic attack on algebraic geometry by Oscar Zariski and his school, and beyond that of André Weil and Grothendieck. I do feel however that while we wrote algebraic GEOMETRY they make it ALGEBRAIC geometry with all that it implies."

However, this approach enabled the algebraic geometers to uncover deep analogies with number theory. Furthermore it may be misleading to consider the algebraic methods merely as a complement to the analytical methods. In many cases, algebraic methods can be extended to fields of arbitrary characteristic. For example the Grothendieck–Riemann–Roch theorem has been extended to fields of arbitrary characteristic (see [BS58]). But, by observing this historic pattern, I believe it's important that one should develop the algebraic side, the topological, and also the analytical side, search for equivalent definitions or approaches such as different definitions of invariants, but most importantly, not forget the underlying unity of mathematics.

The second question is a little bit more specific and I do not know if it has already been asked or answered. Riemann's inequality (Theorem 4.2) was a direct consequence of considering the space of functions as a subspace of the space of integrals. Roch's proof of the Riemann–Roch theorem used the same principle. My question is if it is possible to consider an analogous argument to in order to prove the Hirzebruch–Riemann–Roch theorem. I did not have the time to give it much thought.

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