History of Mathematical Definitions

Erik Mulders

January 2025

Bachelor Thesis in Mathematics

Supervisor:
prof. dr. N.P. Landsman

Second reader:
C.C. Kooloos

In this paper we will be going over the definition used in mathematics through out history. First we will go over what exactly we consider a definition and what sorts of definitions there are in mathematics. Then we will go over the mathematical definitions that can be found in ancient Mesopotamia, ancient Greece, and ancient China. Finally we will look at the *Foundations of Arithmetic* by Gottlob Frege to see what the idea of mathematical definitions has become in modern times and what sorts of problems mathematicians still face.

Contents

1 Introduction

"*Οροι* begins the first book of Euclid's *Elements*[12]. It means *definitions* and afterwards Euclid indeed proceeds to give 23 definitions, followed by 5 postulates and those in turn are followed by 5 common notions. Euclid's *Elements* is one of the best-known and influential mathematical works in history. There are numerous ways Euclid could have started this work but he decided that this was how he would do it. Even today it is very common to see a mathematics book start with a definition, if not multiple. This is hardly surprising since if an author wants to write a book about a mathematical subject it makes sense to first describe that subject. A book on trees would hardly be useful for someone who has no concept of what a tree is. In mathematics this description is usually given as a definition.

Since mathematics deals with abstract objects the only real way we can get an understanding of these objects is through their definitions. However, definitions are not cast in stone. They can be changed, new ones can be made, and old ones can be discarded. Sometimes two definitions can describe the same thing and at other times some definition can be interpreted differently by different people.

With definitions playing such an important role in mathematics, taking a close look at how people defined mathematical structures and concepts in the past can help us understand how mathematics evolved and changed throughout history. In this paper we will go through several major time periods in history and try to see what kinds of definitions were used, how they were used, and compare those definitions with common definitions used today. Of course, since we are looking at general time periods which lasted hundreds of years it is impossible to pick a single definition for a mathematical concept that was used throughout that entire time period, so instead we will look at the definitions used in the most influential works from that time. When looking at historical works it is generally also useful to look at the context in which they were created so we will also give a general recap of the important historical events and mathematical developments that were happening at the time each work was written. We will also look at the work as a whole, see how the definitions in it were used, and, if we can, determine whether or not it succeeds, given the goal of the author.

However, before we can look at any definitions in historical works we should have a clear idea of what exactly a definition is. When can we say something is a definition? What do we want from a definition? If we have two definitions of the same thing, can we determine that one definition is strictly "better" than the other? To answer these and other questions we will start off with a chapter on definitions themselves and give ourselves a basis to work from when analyzing the definitions we find throughout history.

After that we will look at the the use of mathematics in ancient Mesopotamia.

Here we find the some of the oldest records of mathematics in human history. Then we go over the use of definitions in ancient Greece and and ancient China were we find the oldest use of definitions in mathematics. Finally we look at the *Foundations of Arithmetic* by Gottlob Frege to see how the use of definitions in mathematics has changed in the modern era.

2 Definitions

What is a definition? Though this question might seem simple at first, actually trying to find a conclusive answer might prove difficult. Someone might say that a definition is a sort of “description” of a certain “something”. That something can quite literally be anything. Its definition in this case describes that something, it tells us what that something looks like or what it can be used for. Someone else might describe a definition as a sort of ’classification’. Its definition gives us a certain number of conditions that an object must adhere to in order to be classified as the thing defined.“If A fulfills conditions one and two then we can say that it is a [definition]”. Now these are two definitions of a definition that one might reasonably agree or disagree with. Perhaps you could even consider them to be equivalent or maybe they are two ideas of different types of definitions. After all, there is no universal law that states that all definitions must adhere to the same rules.

In essence, what we try to do when we make a definition is to communicate an idea or concept to someone else. There are multiple ways you could go about doing that. Ideally, the two of you end up with the same idea and understanding of what it is you are trying to define, but that can be quite difficult since there will inevitably be some way in which the two ideas each of you has about that object do not line up. Now this does not need to be a problem so long as for the purposes you intend to use the definitions for each of your ideas overlap. So a definition does not need to be perfect; but just good enough. This of course leads to the question of when a definition is good enough. Luckily, there are plenty of people who have also pondered this question, especially in the mathematical field, so we can look at their works for inspiration.

We start by looking at an article written by Carlo Cellucci [5]. Cellucci states the following:

> According to the axiomatic conception of mathematics, the axioms with which the mathematician starts are about certain terms that are left undefined. All other terms are defined from preciously introduced terms. The nature of the definition is expressed by the stipulative conception according to which: ([5]p.606)
>
> 1. A definition merely stipulates the meaning of a term;
> 2. A definition is an abbreviation;
> 3. A definition is always correct;

4. A definition can be eliminated;
5. A definition says nothing about the existence of the thing defined.

To give us a clear understanding of each of these requirements Cellucci gives us the explanations for each given by Pascal and Frege. The explanation Celluci gives us for Frege is as follows: ([5]p.607)

> 1. A definition merely stipulates the meaning of a term. Indeed, 'a definition is an arbitrary stipulation by which a new sign' is 'introduced to take the place of a complex expression whose sense we know' (Frege 1979, 211).
> 2. A definition is an abbreviation. Definitions 'merely introduce abbreviative notations (names)' (Frege 2013, VI). Their use 'is to bring about an extrinsic simplification by stipulating an abbreviation' (Frege 1967, 55). This 'and this alone is the use of definitions in mathematics' (Frege 1984, 274).
> 3. A definition is always correct. For, 'a definition does not assert anything,' thus, a definition is not 'something that is in need of proof or of some other confirmation of its truth' (Frege 1980, 36).
> 4. A definition can always be eliminated. A definition is 'something wholly inessential and dispensable' since, 'if the definiens occurs in a sentence and we replace it by the definiendum, this does not affect the thought at all' (Frege 1979, 208). Therefore, 'nothing follows from' a definition 'that could not also be inferred without it' (Frege 1967, 55).
> 5. A definition says nothing about the existence of the thing defined. The requirement on the definition of a concept that there be objects falling under the concept 'encounters great difficulties,' because 'the only apparent way to show' that 'a concept has this property, is to cite an object falling under the concept. But to do that, you already need the concept' (Frege 1979, 179). Therefore, 'the admissibility of a concept is entirely independent of the question whether objects fall under it, and if so which, or in other words, whether there be objects, and if so which, of which it can be truly asserted' (ibid.).

Cellucci then quotes the works of Pascal and Frege to give us a better understanding what each of these conditions entails. For the first point, this gives us that a definition only really does one thing, it gives us the conditions to which an object must adhere to be labeled something. The second is fairly straightforward, a definition is an abbreviation. So in theory one should be able to replace for example something defined as A with the definition of what A is and nothing would change besides the length of the text. In this sense, the only real purpose of definitions is to make mathematical texts easier to read. The third statement is a bit misleading in the sense that it sort of implies that a definition states a fact when this is not what it is supposed to do. A definition should not assert something that is either correct or false. In that same sense, one does not need to provide any sort of proof for a definition since it is not something that can be proven. For the fourth, it is important to remember that we are working with the axiomatic conception of mathematics: starting from a set of axioms we use logical reasoning to conclude. In this sense introducing a definition should not introduce any new knowledge. So all the conclusions we can get from the set of axioms are the same ones we can get from the set of axioms and a definition. The last point is again very straightforward, if for example, I want to define an "ultimate number" as "a prime number that is divisible by two other numbers" it would not be hard to show that there are no "ultimate numbers". However, this does not mean that this definition is not a definition. It is simply a definition of something that does not exist.

Despite the fact that the stipulative conception of a definition has been popular for a long time, Cellucci argues that each point has some serious shortcomings.

- By claiming that a definition merely stipulates the meaning of a term, the stipulative conception cannot account for the fact that finding a suitable definition can make the difference in discovering a solution to a problem. ([5])p.610)

To support this claim Celluci uses the definitions of Plato and Euclid for a sphere as an example. Plato's defines a sphere using distance to the center in the same way someone would define a circle but then in three dimensions. Euclid on the other hand uses motion in his definition by looking at the path of a semicircle as it rotates around its axis. Both definitions are definitions of a sphere but the one used by Euclid is essential in helping him solve the problem of the five platonic figures.

- By claiming that a definition is an abbreviation, the stipulative conception cannot account for the fact that mathematicians often use concepts for a long time, even centuries, before they can find a suitable definition for them. ([5]p.611)

Here Celluci gives the development of the derivative as example. It was already being used by Newton and Leibniz but it was only properly defined by Cauchy and Weierstrass.

- By claiming that a definition is always correct, the stipulative conception cannot account for the fact that mathematicians often give definitions which afterwards turn out to be incorrect. ([5])p.612)

Here we get the example of Jordan's definition of a curve. Jordan made his definition in 1887 and it was widely used but in 1973 Peano defined a curve that according to Jordan's definition should be a curve but at the same time goes through every point of a square and thus is a "two dimensional curve".

- By claiming that a definition can always be eliminated, the stipulative conception cannot account for the fact that it is not generally the case that a definition can be eliminated.([5]p.613)

To demonstrate this Celluci gives us the definition *If* $a \neq 0$ *then* $\frac{a}{b} = q$ *is an abbreviation for* $b = qa$. Which he argues cannot be removed from the of $\frac{1}{0} \neq 2$. To make it possible for the definition of fractions to be eliminated Celluci adds the part *and if* $a = 0$ *then* $q = 0$. But adding this part would lead to contradictions for example $\frac{1}{1} + \frac{1}{0} = 1 + 0 = 1 \neq 0 = \frac{1}{0} = \frac{1\times0+1\times1}{1\times0} = \frac{1}{1} + \frac{1}{0}$. Thus the definition of fractions can either not be eliminated or leads to contradictions.

- By claiming that a definition says nothing about the existence of the thing defined, the stipulative conception cannot account for the fact that in mathematics it is assumed that the thing defined exists.

Here Celluci uses the definition of the derivative of a function. He gives the definition as $f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$ *provided the limit exists*. This definition he argues introduces the derivative of a function as an existing thing thus contradicting the idea that a definition should not say anything about the existance of the thing it is defining.

Whether these shortcomings raised by Cellucci are enough to turn away from the stipulative model is up for debate. Personally, I feel that some of the arguments made by Cellucci are not that strong, for example, the arguments he makes against the fifth point. The concepts mathematicians work with in a sense do not really "exist", at least not in a physical sense. Or let us go back to my definition of an "ultimate number" on page 6 (A prime number divisible by itself, one, and another third number). By their definition these numbers cannot exist but that does not mean that the definition is "wrong" or is not a definition. It is simply a definition of something that does not exist and for that reason, it is not a definition worth using. Studying things that we know do not exist is by its very nature pointless so arguing that mathematicians assume that the thing defined exists seems to me like a weak argument. You could argue that definitions of things that do not exist are "bad" or "pointless" definitions, but to me those arguments are not a reason to say that the stipulative form of definitions has "serious shortcomings".

Of course, the stipulative conception of a definition is not the only possible way to look at a definition. According to Cellucci the correct way to get a conception of a definition is through the heuristic conception of mathematics. Recall that for the stipulative conception, we thought of mathematics in an axiomatic sense. In the heuristic conception of mathematics, you do not start from a set of axioms. Instead, when you encounter a problem you determine a hypothesis from which you can deduce a solution to the problem. As Cellucci explains it:

> to solve a problem, one looks for some hypothesis that is a sufficient condition for solving the problem, that is, such that a solution to the problem can be deduced from the hypothesis. The hypothesis is obtained from the problem, and possibly other data already available, by some non-deductive rule – induction, analogy, metaphor, etc. – and must be plausible, that is, such that the arguments for the hypothesis are stronger than the arguments against it, on the basis of the existing knowledge. ([5]p.616)

Comparing the axiomatic conception and the heuristic conception they seem to almost pull in opposite directions. Both of these conceptions have their strengths and weaknesses and of course their own concepts of definitions. In the case of heuristic mathematics all one does is turn hypotheses into logical definitions. *Specifically, a definition is the hypothesis that there exists something which satisfies the condition stated in the definiens* ([5]p.618). This gives a clear contrast with the axiomatic concept of a definition where the thing defined does not need to exist. Cellucci earlier gave five arguments against the five rules for axiomatic definitions; now he gives us five arguments for the use of heuristic definitions to demonstrate why this conception of definition would be the better version. ([5]p.619)

1. The heuristic conception can account for the fact that finding a suitable definition can make the difference in discovering a solution to a problem. For, a definition is a hypothesis, and finding a suitable hypothesis may be a crucial step towards finding a solution to a problem.
2. The heuristic conception can account for the fact that mathematicians often use concepts for a long time, even centuries, before they can find a suitable definition for them. For, a definition is not merely the expression of a volition, finding it may require as much effort and time as finding any hypothesis capable of leading to a solution to a problem.
3. The heuristic conception can account for the fact that mathematicians often give definitions which afterwards turn out to be incorrect. For, a definition is a hypothesis, and a hypothesis which is plausible at one stage may become implausible at a later stage, whenever the arguments against the theory become stronger than for against it.
4. The heuristic conception can account for the fact that it is not generally the case that a definition can be eliminated. For, a definition is a hypothesis and, when a hypothesis is a sufficient condition for solving a problem and is plausible, it plays an essential role in solving the problem, therefore it cannot be eliminated. Of course, different definitions may lead to different solutions to the same problem and play an essential role in their respective solutions, but each of them cannot be eliminated from its respective solution to the problem.
5. The heuristic conception can account for the fact that, in mathematics, it is assumed that the thing defined exists. For, a definition is the hypothesis that there exists something which satisfies the condition stated in the definiens, and the hypothesis must be plausible

These are the five arguments that Cellucci puts forward for the use of the Heuristic use of definitions. You can decide for yourself if these arguments form a strong enough reasoning to choose the heuristic conception over the axiomatic conception. Personally, I find these arguments together to not be enough of a reason to choose the heuristic over the axiomatic conception. The reason for that being that I can only really see one key difference between the core idea of a definition between the heuristic and axiomatic conception, namely, that the heuristic conception requires a definition to be "useful". Useful, in the sense that it helps to solve some form of problem. Of course, it is great if a definition is useful but in return for getting a definition that is always useful in that sense, we have to accept a very broad concept of a definition. *Specifically, a definition is the hypothesis that there exists something which satisfies the condition stated in the definiens.* ([5]p.618) What this leaves us with is that a definition is simply a description of something that exists. This does not leave us with a framework to assess definitions by themselves and only really in the context of the problem they were created to solve. The axiomatic conception by contrast gives us a set of rules to which a definition must adhere allowing us to evaluate a definition by itself. Now this does mean that any error in the construction of these rules could in turn lead to an error in our evaluation of a definition. In this respect the heuristic conception has a leg up in my opinion since it puts a lot more emphasis on the fact that mathematical rules and ideas are made by humans and humans make mistakes. In the axiomatic conception, we can get a tendency to think of certain theorems as unshakable facts when in reality these "facts" can change.

Now let's turn our attention towards a different work related to definitions, specifically 'On Rigorous Definitions' by Nuel Belnap [2]. Here we find what Belnap refers to as the standard theory for definitions. Belnap attributes the creation of this theory to Lésniewski, who worked from the works Frege. In this theory there are two criteria to which a definition must adhere. Those are the criteria of eliminability and of conservativeness. According to Belnap, these two criteria arise naturally from the two things we want from a definition: ([2]p.119)

> Under the concept of a definition as explanatory, (1) a definition of a word should explain all the meaning that a word has, and (2) it should do only this and nothing more.

I think most of us would agree that these two things are what we want from a definition. A definition is a description or explanation of something so we want it to fully describe everything about the thing it is defining and nothing more than that. Any additional information given by a definition that is not about the object it is defining is needless baggage and serves no purpose for the use of the definition. With these two requirements of a definition the first criterion that arises from "a definition of a word should explain all the meaning that a word has" is that of eliminability. We have seen this criterion before in the axiomatic conception of a definition as *A definition can always be eliminated.* The second requirement leads to the criterion of conservativeness.

As we have already seen the criterion of eliminability, let's start with that one. We can simply use the explanation given by Frege if we have a sentence containing a defined object A that we can simply replace A with the full definition of A and the sentence would still mean the same. For example, we could define 'ball' as 'a spherical object'. Then the sentences *I kick a ball* and *I kick a spherical object* mean the same. This criterion is one that fundamentally should be required of each definition since if there is a sentence in which this is not possible then there is a context in which the thing defined and its definition are not the same. This would mean that they are not equivalent so the given definition does not define the thing we want it to define.

The criterion of conservativeness does not have a direct equivalent in the axiomatic conception of definitions. However, you could say that it is in a way similar to the first or second requirements of the axiomatic conception. A short explanation of the criterion of conservativeness would be that a definition should never introduce new information. If you could not draw a certain conclusion before you knew a definition then only learning the definition should not change that. Belnap himself describes it as that a definition *does not permit the proof of anything we couldn't prove before* ([2]p.122). We should note that the idea of not introducing new information depends on the information already available. This means that a definition can satisfy conservativeness in one context but not in another. Ideally we would want this to not be the case but this is in a sense impossible. Take for example a definition A in a context where A is simply an abbreviation of an already known object B with property C. Then we can conclude $A = A$. However if we go to a context in which the object B does not exist then in a way the definition A *is* introducing new information into the context and this violates conservativeness.

With this, we now have three ways of looking at a definition:

1. The axiomatic conception, where we use five criterion to see if a definition is truly a definition.

2. The heuristic conception, where we consider a definition a hypothesis that there exists something which satisfies the definiens. In this conception we determine the validity of a definition by seeing how useful it is in helping us solve mathematical problems.

3. The standard conception, where we use two criteria, the criterion of eliminability and the criterion of conservitaveness.

There are of course many more ways to evaluate a definition but I believe these three together form a good general platform to determine if a definition is "good" or "bad". The axiomatic conception is perhaps what most people would expect from a mathematical idea of a definition. It provides five clear conditions to which a definition must adhere and if it does we can say that it is indeed a definition. The standard conception (or standard theory) does not have as clearly defined conditions as the axiomatic conception but the two conditions it has mean that a definition that adheres to them does what we expect from a definition and nothing more. The heuristic conception is the least restrictive, but in contrast to the other two it looks towards the results that we gained by using the definition. This is also a very interesting way to look at definitions, since a lot of definitions changed through history and comparing the old versions to the modern one we are always more inclined to pick the modern version as the better version. However, if the old version was used in some groundbreaking mathematical discoveries it would be at the very least a bit shortsighted to simply write it off as a bad definition.

Now that we have these three conceptions of a definition we can start looking at the way mathematics and mathematical definitions have evolved throughout history, starting in 2000 BC. in Mesopotamia.

3 Mesopotamia

Talking about Mesopotamia we are talking about the valley between and around the Euphrates and Tigris rivers during the time period of roughly 2000 to 600 BC ([28]p.21). These two rivers provided the region with ample fertile land and water, making for an ideal plot of land for early civilisation. It is no surprise then that this area saw the rise of what is possibly the earliest human civilisation of Sumer. This civilisation was not unified in the way that a country like Germany is today. The Sumerians where divided in multiple different city-states and the valley itself was open to invasions, which happened a lot during this time period of more then a millennium. The political power of the time shifted frequently but there was a sufficiently high degree of cultural unity throughout the valley. ([28]p.22)

This unity can among other things be attributed to the writing system that was used at the time. Namely, the Cuneiform writing system. The earliest accounts of this writing system were found at the location of the ancient city of Uruk and date from roughly 5000 years ago. This writing system used stylized representations to depict specific words. These images were pressed into clay tablets which were then heated so they hardened. These clay tablets were extremely durable and as a result a great number of them have survived even to this day. Because of this we have a great deal of information about the daily lives of the Mesopotamian people and the developments that happened during this time period. ([28]p.22)

Figure 1: A clay tablet from the city of Uruk

Among these tablets we also find a great number of mathematical texts. From these texts we learn that the Mesopotamians used a sexagesimal system. That is, a system with base 60 ([28]p.23). So for example, if we take the number 372, (that is 3:7:2 or 3 times 100, 7 times 10 and 2 times 1) in a sexagesimal system that would be 6:12 (6 times 60 for 360 and 12 times 1 for 372). The way they would depict this is by using a short vertical stripe with a thicker indent at the top to denote single units. They would use up to 9 of these stripes to denote the numbers 1 through 9. Then they would add up to 5 wedges in front of these single units to depict the numbers 10, 20, 30, 40 or 50. So 3 wedges 4 single units would be equal to 34. With this they could depict any number from 1 to 59. Then for the number 60 they would simply write a 1 again ([13]). In the texts from around 2000 BC we see that there was no symbol for 0, which means that 1 and 60 or 3600 were written the exact same way. At the time it seems that the readers themselves would simply have to determine the magnitude of the number by relying on context clues from the text. However, in later tablets starting from roughly 300 BC we see that they started using two diagonal units as a placeholder for 0. However these were not used at the start of a number. This symbol seems to only have been meant to distinguish for example 3601 (1:0:1) from 61 (1:1). Both of these number could also mean 216060 or 3660 respectively. ([28]p.24)

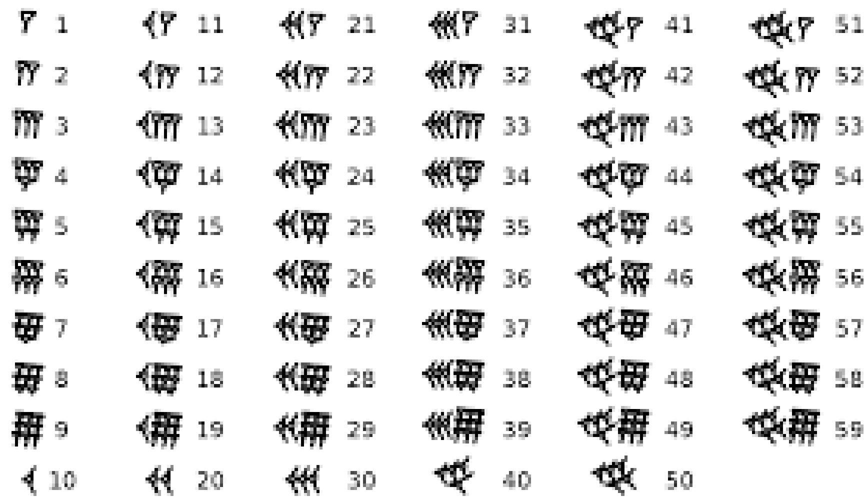

Figure 2: The Mesopotamian number system

While this is a sexagesimal system, we can also clearly see the semblance to the decimal system we use today. Why the Mesopotamians used this system and not a decimal one is not known. It could be that because they didn't distinguish between orders of magnitude that they preferred this system because it allowed for easy division and multiplication when working with 2,3,4,5,6,10,12,15,20 and 30 ([28]p.23). This is to be compared favorably to the 2 and 5 for which this holds in the decimal system. Nowadays this might seem like a trivial reason until we remember that for all our technology we use a binary system because it works with only two different numbers. The main reason why we don't use binary today ourselves is simply because it is not very intuitive for people to read. 110101 might seem like a huge number, but is really just 53, and 1101011 might seem similar but is really 107, more than its double. With this in mind it would make sense for the Mesopotamians to choose a number system that was easy to read and represent on clay tablets while also allowing for easy calculating when working with multiplication and division.

Now that we know what number system the Mesopotamians used we can look at what they used this system for. To do this we must first look at who was practicing mathematics at the time. During this time literacy was something that was reserved for special classes within the Mesopotamian society. The main places where literacy was practiced would be within the palace, temples, and households of wealthy families. Within these places scribes would make their living by producing and reading documents. The primary use of these documents would have been administrative, like keeping track of food storage, taxes, or recording land ownership. These scribes would have had to undergo training for some time which involves hypothetical mathematical problems like calculating the area of land and its yield in grain. We know this because of a multitude of clay tablets that survived which seem to contain what could be described as homework assignments for scribes in training. We know this due to the fact that a lot them seem to present a problem in a very factual way with only the required information giving while having suspiciously “round” answers. ([34])

These homework assignments can give us an idea of what kind of mathematical knowledge was expected from scribes at the time. The oldest known exercise for scribes was found on a tablet now known as W 19408,76 and seems to contain an exercise about calculating the area of 2 fields. ([34]p.73)

Figure 3: W19408,76 found at the Warka site in Iraq and currently kept at the German archilogical institute

The front and back of this tablet contain 2 different calculations containing multiplication, division and addition. When fully calculated both sides yield the same result. It is likely that the scribe was required to show that both fields had the same area. W19408,76 might be the first preserved tablet containing an exercise on calculating the area of a give surface, but it is far from the only one. Another tablet, BM 15285, contains a multitude of figures with a short description pertaining to each figure describing how the figure was constructed. It seems that the scribe was required to find the surface area of a part of the figure using the given information. Most of the figures are constructed using straight lines but some include circles as well. This indicates that the Mesopotamians already had an idea of how to calculate the area of a circle. ([34]p.92)

Figure 4: BM15285 (British museum)

These sorts of geometry problems appear on a large number of tablets and seem to have been common practice for the Mesopotamians. The Mesopotamians seemed to have a great deal of understading about geometry and even used it to solve quadratic and cubic equations. Another tablet, BM13901, contains a list of 24 quadratic problems and solutions for each problem ([34]p.104). Within each problem the questions seem to have been to find an unknown value given limited information. These sort of mathematical equations are similar to our modern version of algebra, where one is required to find the value of x given a relation such as $x^2 - \frac{1}{2}x = 1$.

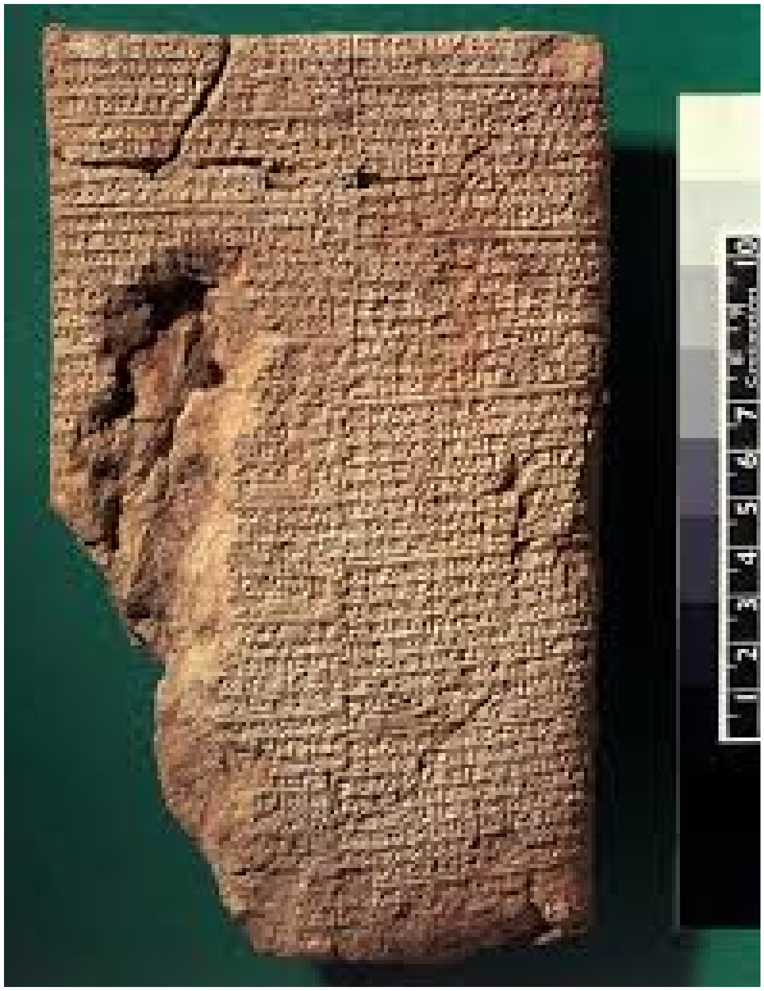

Figure 5: BM13901 (British museum)

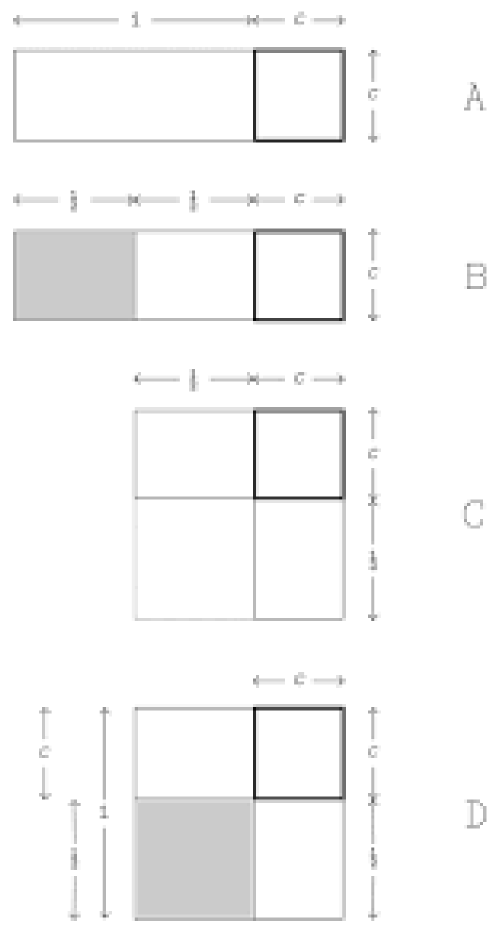

Figure 6: A depiction of the first problem described in BM 13901

For this problem the assignment was to find the value of c. The student was given that the total surface of the area at step A was equal to 45 ($\frac{3}{4}$ to us), which means the problem would be equivalent to solving $x^2 + x = \frac{3}{4}$. The way this was intended to be solved is by first splitting the area of $c \times 1$ into two halves and placing one half below the square c^2 as depicted in step B and C. Now we can look at it as a square with sides $\frac{1}{2} + c$, which has a surface area equal to that of the original rectangle and the grey square. We know that the sides of the grey square are $\frac{1}{2}$ which means it has a surface area of $\frac{1}{4}$ so the total surface area of the square is $\frac{1}{4} + \frac{3}{4} = 1$. For a square to have a surface area of 1 its sides must be equal to one, so $\frac{1}{2} + c = 1$, which gives us $c = \frac{1}{2}$. ([19])

A problem like this would be easy enough to solve for a scribe without any form of assistance, but when working with more complex problem this might get difficult. Scribes would have been able to do complex multiplication and division by hand but when working with problems that required repeated uses of multiplication and division it would become a bit more difficult. The chance of error and the time required would increase exponentially the more complex a problem became. Because of these sorts of reasons scribes seemed to have made ample use of mathematical tables. These tables had several rows of number depicted on them, which had a certain relationship to one another. For example, a table that has been found numerous times is a multiplication table, which list 2 rows of numbers which when multiplied together all equal 60. Since the number system used allowed for easy multiplication by magnitudes of 60 this table seems to have been used to easily find inverses for specific numbers. But these are far from the only sorts of tables found. The most interesting such table is Plimpton 322. This specific tablet contains 3 columns that have a very interesting relation to each other in that they appear to be Pythagorean triples, or, in other words, they have the relation of $a^2 + b^2 = c^2$. This is not the exact way in which the numbers a written on the tablet, but the relationship they have is equivalent to that of Pythagorean triples. This tablet is a great testament of the capabilities of the mathematicians of ancient Mesopotamia. One of the numbers on the table appears to contain no less then nine different entries which means it would be the equivalent of working with a number greater then 167 trillion. Some of the numbers on Plimpton 322 appear to have errors in them but this tablet is proof that even in ancient Mesopotamia mathematics was already being practiced simply for the purpose of mathematics.([4][24])

Figure 7: Plimpton 322, G.A. Plimpton collection, number 322, Columbia university

3.1 Definitions in Mesopotamia

Trying to analyse the mathematical definitions used in Mesopotamia is a challenging if not impossible task. Whereas the later time periods all provide us with several influential mathematical works that have been preserved to modern times, the Mesopotamian time period does not afford us such a luxury. Although a great number of mathematical clay tablets have been found none of them appear to contain something what we could interpret as a definition. The tablets we do have contain mathematical tables, geometrical figures, and even some training exercises for aspiring scribes. These give us insight into the mathematics they were capable of, but they don't provide us with in insight into what they thought of mathematics itself or of the mathematical structures they were working with. We can try to extrapolate some sort of plausible definition that could have been used in Mesopotamia, but this would not really serve much of a point. To give a comparison, say you are a sculptor and you want to compare the tools you use with those used by sculptors in some ancient civilisation. However, the only thing you have is the statues they left behind. Those statues could give you some insight into what techniques they used and what they were capable of sculpting. Perhaps you could even see some of the markings left by the tools they used. However, when it comes to what the handles of those tools looked like, what color they were, and what sort of materials they were made of, you really have nothing to go from. Worse, the problem is not only that, but the tools used throughout the existence of this civilisation probably changed dozens of times and probably even varied between different sculptors. Trying to reconstruct some version of the tools they used does not really have any use since you have no way of verifying if you were correct.

With that in mind, trying to get an understanding of the definitions used in ancient Mesopotamia is not really possible. This Chapter merely serves to give an idea of the starting point of human mathematical knowledge. The Mesopotamian mathematical sources we have are the oldest in history and show just how long humans have been interested in mathematics, and the things they were able to do with it. Over four thousand years ago people were already using mathematics for incredible building projects, to gain an understanding of the movements of the stars, and to solve complex geometrical and arithemtical problems. For anyone wanting to read more about Mesopotamian mathematics I would recommend *Mesopotamian Mathematics* by Eleanor Robson ([34]).

4 Ancient Greece

From the Mesopotamian era, we have a good understanding of what kind of mathematics they practiced. We know what kind of techniques they were capable of and for what purpose they used them. However, we do not have any idea of what they thought of mathematics as a subject. We do not have the answers to more philosophical mathematical questions like "What do they consider a number?". We know for example that they tried to find the square root of 2 and that they had a decently accurate approximation for it. However, they did not seem to have a specific sort of representation for the exact number of $\sqrt{2}$. Whether they did consider it a number existed or something that simply did not exist we do not know. Even if these sorts of questions were discussed in Mesopotamia no one bothered to write this discussion down or none of these tablets survived to this day.

This changes when we switch our focus to the Hellenistic world. From roughly 800 BC to 800 AD in the area around the Mediterranean Sea we can find a great deal of discussion not just about mathematical problems but about the nature of mathematics itself. In this period, we find great Philosophers such as Pythagoras, Zeno, Aristotle and of course Euclid and many others. These men had exchanges with their peers who were alive during the same time and of those who weren't they commented on the work they left behind. While from most of them we do not have direct accounts or mathematical works, we are still left with an abundance of sources, though the validity of some of these sources can be questioned because they were written sometimes centuries after the passing of their main character. However, we should still consider ourselves fortunate due to the sheer number of works we still have today. ([28])

Greek mathematics started when tribes from the north of Greece moved south toward the Mediterranean Sea. Here they used trade routes to other civilizations around the Mediterranean. These trade routes did not only serve as a way for goods to be exchanged around the Mediterranean, but also knowledge. The Greeks were very eager to learn, and absorbed a great deal of knowledge from other cultures into their own and improved upon it. Mathematics was no exception and many scholars from all over Greece even traveled to Egypt and Babylonia to study mathematics. These scholars didn't simply learn what had already been discovered by the Babylonians and Egyptians, but were determined to master everything these cultures had developed and transform it into something even better.([28]p.41)

During the 6th century BC we find the first records of Thales and Pythagoras. These two men are the first Greeks to whom definite mathematical discoveries are attributed. However, we should note that the sources that attribute these discoveries to them are not from this period but rather later and certainly not from either of their times. For example, one source that attributes Thales with certain mathematical achievements is Eudemus of Rhodes in roughly 320 BC.

This is three centuries after Thales and even then we are not working with Eudemus' own original text but with others describing his work. ([28]p.43) Most sources seemed to be based on tradition rather than on any form of historical document. Thales and Pythagoras are the main representatives of the two earliest Greek schools of mathematical thought. Thales founded the Ionian School of Mathematics, named after the settlement of Ionia in which he lived. Pythagoras on the other hand founded the Pythagorean school, named after the man himself, like his followers the Pythagoreans.

4.1 Thales

Thales was born in the city of Miletus in the Greek region of Ionia in 620 BC. Although we have very little in the way of direct information about Thales' life, ancient sources refer to him as an incredibly clever man and the first philosopher. During his life, Thales is said to have traveled to Babylonia and Egypt, where he learned about mathematics and geometry. While Thales himself most likely did not write any mathematical works, or if he did, they did not survive, he is credited with proving some mathematical theorems. Now the proofs Thales used were most likely nothing like the mathematical proofs of today, they were most likely some form of demonstration, like a drawing of a circle and then constructing the diameter in a certain way which and then showing that the circle is bisected. ([16]p.130)

There are five theorems Thales is said to have proven:

1. A circle is bisected by its diameter
2. The base angles in a isosceles triangle are equal
3. Two intersecting straight lines create 2 equal angles
4. If there are two triangles with two angles and one side equal to one another, then both triangles are equal
5. The angle in a semicircle is a right angle

Of these five theorems, the first one is more akin to a definition, whereas the other four are more clearly provable statements. A diameter was defined as a straight line through the center of a circle that ends at its edge. This might seem trivial to almost anyone, but for this definition of diameter to work we already need to have assumed multiple things. For one, we need to assume that every circle has a center. Now we can define a circle as all points with a certain distance to a point which we then call the center. But could it then be possible for a circle to have 2 centers? Or is it possible to find a straight line through the center that doesn't intersect with the edge of the circle? The answer to these questions might seem obvious when simply looking at a circle, but giving formal proofs of them can sometimes be tricky.

It is very easy to forget that some of the things we simply accept as facts might not be so clear to someone else. Suppose we have to explain the idea of a circle to someone who has never seen one. The easiest way and the way most of us would probably do it would be to draw a circle. We could simply point to it and say "This is a circle". However, this method would prove difficult when working with more abstract shapes or mathematical constructs. If we simply build our mathematical proofs on things we assume everyone agrees on instead of on clear rules and definitions, then we could run into issues when going into the more abstract notions of mathematics. Thales himself most likely had a specific definition of a circle that would have sufficed for this definition of a diameter and for his proof that the diameter indeed bisected the circle. Sadly we do not have the proofs that Thales supposedly used to prove the theorems he is credited with, if he had these, so it is impossible to try to infer what kind of definition of a circle he worked with.

4.1.1 Definitions used by Thales

Since we don't have any works from Thales himself it is not possible to know any of the definitions he used, or what kind of thoughts he had on them. Nonetheless, as with his five theorems, there are certain definitions also attributed to Thales. For one, Thales is said to have given the first definition of a Number (p.70 [16]) which he gives as *a collection of units.* When we try to look at this definition in the heuristic conception from chapter 2 we run into the problem that without any direct works from Thales evaluating if the effectiveness of this definition becomes impossible. The best we can say is that since this is one of the few things attributed to Thales, it is possible that he made use of this concept in the proofs of his theorems, or that he (or his followers) decided it was important enough to consider it one of his achievements. Of course, a glaring problem with this definition is that it excludes all fractions as being numbers. Negative and complex numbers have the same problem, but since they were not known to the Greeks at the time it wouldn't be fair to expect from Thales that he takes those into account. We can get around this by simply considering it to be a definition of a natural number instead of a definition suitable for all numbers. If we do this, then in the standard conception we can consider this a fine definition, provided we a) consider a collection of one to also be a collection and b) do not consider 0 a number. When we do this, we can safely say that this definition fulfills the eliminability criterion since any natural number x can be replaced with the sentence *A collection of (x) units.* As for conservativeness, this definition does not contain any new or unnecessary information, so it would fulfill this one as well. Lastly, we then have the axiomatic conception:

- Does the definition merely stipulate the meaning of a term?

Yes, it stipulate the requirement that any (natural) number must be made up of a certain number of units

- Is the definition an abbreviation?
  Yes, it abbreviates the stipulation ‘a collection of units’ into ‘number’.
- Is the definition always correct?
  Yes, the definition by itself does not assert any claim that can be either true or false. It can however clash with one’s own interpretation of a number, so in that sense it could potentially be considered false.
- Can the definition always be eliminated?
  Yes, any use of the term ’number’ could be replaced with the sentence *a collection of units*
- Does the definition say nothing about the existence of the thing defined?
  Yes, the definition itself does not in any way imply that a *number* as defined does or doesn’t exist.

When we go step by step we can certainly argue that in the axiomatic conception and the standard conception this definition can be considered a *good* or *proper* definition. However, for any modern mathematician this definition certainly leaves a lot to be desired. As already mentioned using this definition fractions are not considered to be numbers, and negative and complex numbers are even less so. We can work around this by limiting this definition to natural numbers but even then this definition is vague with regard to 1 or 0 being numbers. Now 0 not being included in the natural numbers is not that unusual, but excluding 1 is something that is rarely done. To the Greeks 1 seemed to be special in comparison to the other numbers. From Aristotle’s Metaphysics, we get

> The essence of what is one is to be some kind of beginning of number; for the first measure is the beginning, since that by which we first know each class is the first measure of the class; the one, then, is the beginning of the knowable regarding each class (p.45 [36]).

Besides 1 possibly not being a number, this definition can also include infinity as being a possible number: since a collection can potentially have an infinite number on units, infinity according to this definition would be considered a number. With all that in mind, this definition of numbers would definitely not be usable in modern mathematics.

4.2 Pythagoras

Pythagoras was born around 570 BC on the island of Samos. During his life he is said to have traveled to Egypt and Babylon before settling in Crotos when he was around 40. Pythagoras himself did not leave any written works, nor did any of his contemporaries write down any of his direct statements or ideas. Many of the accounts describing Pythagoras come from much later in history.

This means that anything attributed to Pythagoras should be subject to some scrutiny. With that in mind, while Pythagoras is nowadays mostly thought of a mathematician, he was a far cry from what we now consider one. Much like Thales, the theorem he is most famous for, $a^2 + b^2 = c^2$, was known well before his time but Pythagoras seems to have been the first person to have given a formal sort of proof ([28]p.45). However, this proof (if it existed) has been lost. While it is true that Pythagoras did practice a lot of mathematics he also seemed to have been an expert on religious affairs. Perhaps because of this, Pythagoras took a more philosophical approach to mathematics. In fact, some even attribute to Pythagoras the creation of the words Philosophy (love of knowledge) and mathematics (that which is learned). Where most before him seemed to have thought of mathematics mostly as a tool for things like measuring land or keeping track of food supplies, Pythagoras took a more mystical approach to numbers. Since we have no direct sources of Pythagoras his own ideas are impossible to know; however, the Pythagorean school of thought he created and its followers produced many works. So rather than looking towards the man himself for his ideas on mathematics it is best to look at the ideas and practices of the Pythagorean school. ([28]p.43; [21]; [16]p.65)

4.2.1 Pythagoreanism

The people who practiced Pythagoreanism or the Pythagoreans for short, seemed to have believed in a couple of ideas central to the school of thought. They believed that the human soul was immortal and could reincarnate into animals after death. But more importantly for us, they seemed to have held the belief that can be described by the quote *all is number* ([28]p.45). This quote attributed to Pythagoras was at the heart of the Pythagorean school of thought. It relates to their fundamental idea that all things in the universe could be described using numbers.

Several concepts are credited to the Pythagoreans. One example is the table of opposites, credited to them by Aristotle in his book Metaphysics([36]). The idea was that everything in the universe fell into one of each of these opposite categories. So for example an object would be either light or dark but not both and also not neither. Some of these concepts like good or bad would certainly not be found in a mathematical discussion nowadays, but this nonetheless shows a clear shift in what was considered to be mathematics. The Pythagoreans tried to use mathematical concepts to explain the world around them and in doing so they started to drag mathematics in a more abstract direction, away from simple geometry towards what could be considered an early form of mathematical physics. These kinds of shifts are what ultimately force people to innovate in a field such as mathematics. The maths they had available at the time would be sufficient to serve all their daily needs but when trying to use it to explain complex natural phenomena it would be insufficient. Therefore if people would have to expand and improve it if they ever hoped to gain a deeper understanding of the world around them. ([16]p.65)

male	–	female
limit	–	unlimit
light	–	dark
good	–	bad
one	–	many
right	–	left
rest	–	motion
straight	–	curved
odd	–	even
square	–	oblong

Figure 8: The Pythagorean table of opposites [21]

While it is true that the Pythagoreans expanded the mathematical discussion to subjects beyond geometry, they most certainly did not abandon the discussion of geometrical problems. One of the most fascinating geometrical figures to the Pythagoreans was the pentagram, and even more specifically, its relationship to the golden ratio. The golden ratio divides a line at a specific point such that the ratio of the whole line to the larger segment is equal to the ratio of the larger segment to the smaller segment. ([28]p.46)

$$\varphi = \frac{p}{q} = \frac{p+q}{p} = 1.6180339887 \text{ (approx.)}$$

Figure 9: Golden ratio

Now if we look at the pentagram we can see this golden ratio appear several times. In the figure below we see that the ratio the the red line to the green line is equal to the ratio of the green line to the blue line which in turn is again equal to the ratio of the blue line to the pink line.

$$\frac{red}{green} = \frac{green}{blue} = \frac{blue}{pink} = \frac{1+\sqrt{5}}{2} \cong 1.6180339887 \quad (1)$$

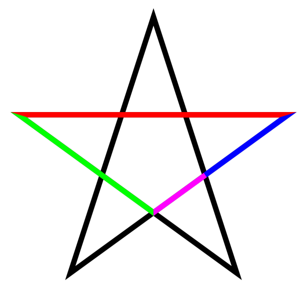

Figure 10: The golden section in a pentagram

4.2.2 Definitions in Pythagoreanism

The Pyhtagoreans wanted to understand mathematics to its fullest extent and definitions were of course no exception. One of these definitions we find in Proclus's commentary on Euclid's *Elements*. The Pythagoreans defined a point as *A unit that has position*(p.96 [30]). We can split this definition into two parts, firstly a point is a unit and secondly what differentiates it from other units as being a point is that it has a position. A point being a unit appears to be a direct contradiction to the definition used by Euclid in the *Elements* namely: *A point is what has no parts*([12]p.1). If we consider a point a unit then we can take half of it, which in turn would mean that it has parts. Another problem with thinking of a point as a unit is that a line can have an infinite number of points on it and therefore would consist of an infinite number of units and thus be infinite. The part about a unit having a position is interesting because it is not included in the definition by Euclid. Since all points are units, all their properties would be the same and the one thing that differentiates two different points is their location. Being able to have a clear way to distinguish points is something that is lacking in the definition used by Euclid. If we consider two points to be different, then we could consider that as indicating that some *part* of them is different, which would be impossible if a point has no parts. Being able to use position as a way to differentiate would give us a clear way to say why for example one point is part of a certain circle whereas another is not.

Now we shall look at these definitions in the axiomatic, heuristic, and standard conceptions. Starting with the axiomatic conception, for the Pythagorean definition we can argue that the idea that a point is a unit clashes with the second and fourth axioms of a definition. Since units by themselves are not related to geometry, introducing the notion that they are the same is not compatible with a definition merely being an abbreviation or being eliminable. The definition of Euclid does not have this problem, since it does not relate a point to something outside of geometry. The Pythagorean definition conflicting with the eliminability axiom also means that it conflicts with the standard conception of a definition. As for the heuristic conception we could argue that the

Pythagorean conception of a point is more effective since it relates a point to the position it is in. Today, when we work with geometry we often simply see a point as coordinates. This means that the only relevance a point has to us is its location and the actual thing it is. Considering a point as a unit can lead to some problems, but if we look at it in a less direct way and see a point as having the same or a similar role in geometry as the unit does in number theory, then this would make it more acceptable. A unit in number theory, is the starting point when trying to construct other numbers. We can add units to get any other number we want, and in a similar way we can add points to a geometrical structure to create more complex forms and shapes.

Regrettably, Proclus does not give us some more definitions that were used by the Pythagoreans. The closest thing we get to another definition is the following:

> But let us recall the more Pythagorean doctrine that posits the point as analogous to the monad, the line to the dyad, the surface to the triad, and the solid to the tetrad. (p.98 [30])

While this is certainly an interesting way to look at the connection between geometrical objects it does not provide much in terms of mathematical explanation. Thus, we move on to the life of Euclid and his best-known work, *Elements*.

4.3 Euclid

Euclid's *Elements* is the most successful mathematical work in human history. Despite this, remarkably little is known about its author. We do not know when or where exactly Euclid was born. All we know about the man himself is that around roughly 300 BC he was a scholar in the city of Alexandria in Egypt. Because of this, with Euclid we have sort of the opposite problem that we had with Thales and Pythagoras. Where there were plenty of stories about the life of Thales and Pythagoras, we have no works which can be attributed to them, but with Euclid we only have his works. Even though more than half of the works of Euclid have been lost to time five of them have survived. Those are *Data*, *Division of Figures*, *Phaenomena*, *Optics*, and of course *Elements*.

In *Optics* Euclid gives a mathematical approach to understanding vision, using geometry to explain how human vision works. *Phenomena* concerns the motion of the stars and the sun. *Division of Figures* as the name suggests explains how to divide plane configurations. *Data* describes the relationship between magnitudes and plane figures. And lastly, *Elements* is a textbook covering all elementary mathematics, that is to say, arithmetic, geometry, and algebra (although there are similarities to our modern notions of these subjects they still differ significantly in some aspects). What is perhaps even more important for mathematical history is not the theorems that are contained in the books, but the way they are constructed. With *Elements* is the clearest example of this.

4.3.1 Elements

Elements[1] is not so much one book rather than a collection of thirteen books. These books can generally be characterized by the subject matter they focus on. The first six books focus on plane geometry, the next three books concern number theory, the tenth book is about incommensurables and the final three focus on solid geometry. The contents of *Elements* were not all invented by Euclid himself. Euclid drew a lot on the works of his predecessors but that is not to say that he didn't add his own proofs wherever he could. The reason for this is *Elements* was never intended to be a collection of works solely by Euclid. Rather, it was meant to be a textbook that could be used by scribes in training to familiarize themselves with the subjects included in it. *Elements* was not the first such textbook to be produced, but none of the textbooks that were made before *Elements* have survived. The reason for this seems to be that, compared to its contemporaries, *Elements* was simply the superior textbook. One reason for this appears to be the way Euclid decided to structure *Elements* with each of the thirteen books following a very similar structure.

After Euclid gives his twenty-three definitions in book I, which we will discuss in more detail in 4.3.2, he proceeds to give five postulates:

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight line) that the (sum of the internal angles) is less than right-angles (and do not meet on the other side)

The first three postulates seem to be about things that are possible to do in a plane. We can draw a line between any two points or draw a circle of any size anywhere. But the fourth seems to be more akin to an axiom. We define all right angles to be equal. And the fifth, seems to resemble more of a theorem. In other words when two lines cross a different line they will either meet on one side of the line and not on the other, or they will be parallel to each other. As much as the fifth postulate seems like a theorem that could be proven no such proof has been found even to this day.

[1] It should be noted that we work with the English translation made by Richard Fitzpatrick of the Greek text of J.L.Heiberg. We will consider this text as *Elements* by Euclid himself, though it should be noted that throughout the centuries many parts of the text could have been altered or added by other scholars

After this Euclid gives five more common notions:

1. Things equal to the same things are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole is greater than the part.

The purpose of the common notions appears to be to give the conditions under which one can say that two things are equal. However, the last common notion differs from the others, in that it does not say when two things are equal but rather when things are not equal. If the common notions about equality is what makes them different from the postulates, that does raise the question as to why the fourth postulate is not included in the common notions. Perhaps this is because it is not about the equality between two "things" but about all arbitrary right angles. However, if we replaced all right angles with any two right-angles then wouldn't that make it a common notion? Euclid didn't provide any commentary in *Elements* as to why exactly he chose these postulates and common notions so we will most likely never know the answer.

Now we will look at the first theorem Euclid gives in *Elements*. "To construct an equilateral triangle on a given finite straight line." This would most likely be something that most mathematicians at the time knew how to do but Euclid's goal with *Elements* was to teach even the most elementary mathematics, so this was included as well.
Let AB be the line that we want to construct the equilateral triangle on. First, draw a circle with center A and radius AB and draw a circle with center B and radius AB. [post. 3] Then take point C as the intersection between these two circles and draw lines AC and BC. [post. 1] Since C is on a circle with center A and radius AB, $AC = AB$, the same holds for $BC = AB$. [Def 15] Then by common notion 1 we have $AC = AB = BC$, so the triangle is equilateral. (Def 15 here being the definition of the circle where all points on the circumference of the circle have the same distance to its center).

In this example, we start with a given line of unknown length. The goal is to construct a triangle with even sides with one of the sides being the given line. According to Euclid's third definition, the extremities of this line are points which he names A and B. For the next step, he uses his third postulate stating that we can draw a circle with any center and radius to draw two circles: one with point A as its center and one with point B as its center. According to this postulate any length can be chosen for the radius of the circle, so Euclid chooses the length of the initial line as the radius. Since the distance from point A to point B is the length of the initial line, this results in point B being on

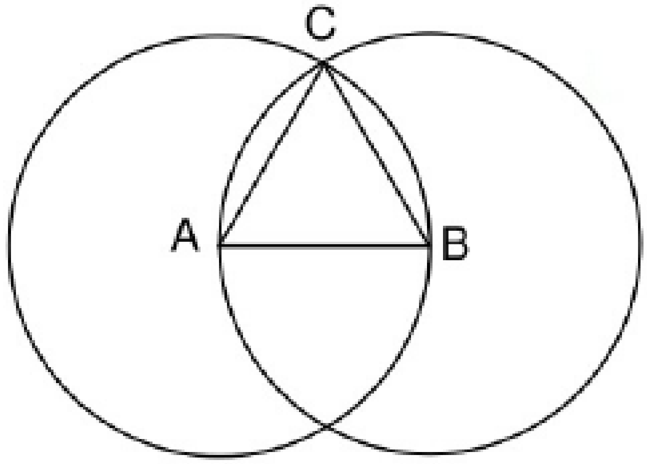

Figure 11: Euclid's first proposition

the circle with center A, and vice versa. Euclid then names a new point C as one of the intersection points of these two circles. Euclid never states in his definitions, postulates or common notions that the intersection of two circles is a point. One would expect that since most of his postulates rely on points to construct geometrical figures Euclid would have included a postulate on how to construct points. Euclid also never proves that the two circles intersect each other. Perhaps this is due to his reliance on visual demonstrations. If you follow the steps Euclid gives, then you'll see that the circles intersect and that by itself might have been proof enough for Euclid. Regardless, now that Euclid has three points A, B, and C he uses his first postulate to draw lines between them and then he uses his fifteenth definition to show that the distance between A and C is the same as the distance between A and B which is in turn equal to the distance between B and C. So then by the first common notion all of these distances are equal and as a result, we have a triangle with even sides.

Euclid's proof is not perfect, but in terms of structure we can already see a lot of similarities with modern mathematics. We start with some given information and a goal we want to reach. To get to that goal we take several steps, and for each of those steps we can use the given definitions, postulates, and common notions. Today we do mathematics similarly but instead of postulates and common notions we have lemmas and theorems that have rigorous proofs behind them, and aren't just given as facts. But the general structure is already there: we work from an established pool of knowledge to take steps that we justify using that knowledge to eventually get to a conclusion. The major thing that changes throughout time is that people notice flaws or discrepancies in the knowledge that at that time was regarded as correct, and cause it to change.

4.3.2 Definitions in Elements

Because *Elements* had such a long-lasting impact on the definitions and postulates that would be used in the coming eras of mathematics, it would make sense to familiarise us with the definitions that Euclid decided to use. However, as we know *Elements* consists of 13 books, most of which open with a list of definitions and postulates. Throughout all the books of *Elements*, Euclid provides us with 124 definitions. To go through all of them would require a considerable amount of time, paper and effort for what would probably be very little payoff. However, that doesn't mean it is not worth our time looking at least at some of the specific definitions used. We will look at whether these definitions still hold up to this day or whether they lack in some critical aspect, and if so, how.

We already know the first definition given in *Elements*: the definition of a point. Following this, Euclid gives us the definitions of a line and a surface. Since these concepts form the very basis of geometry it makes sense to see how exactly Euclid decided to define them:

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines.
7. A plane surface is (any) one which lies evenly with straight-lines on itself.

Since we have already looked at the first definition we shall skip this one. Looking at the second definition, we see that Euclid defines a line as a length without breadth. While this definition at first glance makes sense, we see a similar problem to the first definition of a point, in that Euclid defines a line by the concepts of length and breadth. A line possesses the property of length whilst simultaneously not having the property of breadth. This could be considered contradictory in a way, since when something does not have a breadth, is there "something" that could have a length? Indeed, in a way there is nothing to have a length. This is a sort of philosophical problem that arises from a more practical problem. We live in a three-dimensional world. For anything to exist in this world, it needs to possess a non-zero property in each of these three dimensions. In other words, for something to exist it needs a height, length and width (breadth). So when we talk about a line we can see, we talk about a *line* that exists in our world so by that idea of existing in our world it needs to have a height and breadth. However small they may be. This sort of problem can be solved by simply considering mathematical objects as things that do not need to exist in our *real* world. They exist in a more abstract place that we can only really observe in our thoughts and when we make a representation of these objects in our world we simply adjust them slightly for the sake of representation.

In his commentary, Proclus says:

> He taught us what the point is through negations only, since it is the principle of all magnitudes; but the line he explains partly by affirmation and partly by negation. ([30]p.97)

This is an interesting observation, because when we look at it this way the definitions of Euclid build themselves up like the dimensions of the geometrical object. A point has no dimensions; therefore, it would be logical to define only in terms of what it does not have. A line has one dimension, therefore, we say that it includes that one dimension *length* but that it also does not include further dimensions such as *breadth.* According to Proclus, stating that something is without breath it also implies that it also does not have any other dimensions, so no further clarification would be needed.

> For everything that is without breadth is also without depth, but the converse is not true. Thus in denying breadth of it he has also taken away depth, and this is why he does not add "without depth," since this is implied in the absence of breadth. ([30]p.97)

In this way, it would be logical then to define a surface by adding the next dimension "breadth" to a line and that is indeed what Euclid does. "And a surface is that which has length and breadth only." ([12]p.1) This only leaves us with the last dimension, depth, which we can find if we go to the start of the eleventh book of *Elements* where we find: *A solid is (figure) having length and breadth and depth.* ([12]p.424) This shows just how much preparation Euclid put into his work. The definitions are not just chosen for their own sake but also to fit properly with all the definitions following them. According to Proclus, *The line has also been defined in other ways. Some define it as the "flowing of a point",* ([30]p.97) which for modern mathematicians who are familiar with functions might seem like a better or more intuitive definition but if it had been used as a replacement for Euclid's definition, then it would feel very much out of place. When we use this conception of a point to build towards a surface we get that a surface would be considered a flowing line. The thing is that while this definition also builds upon the definition of a point, it is dependent on it, whereas the definition of Euclid is not. This advantage of the definitions working both in and outside of their context of *Elements* contributes to the effectiveness of *Elements* as an educational text. To fully use it you can either read the books from the beginning and they will naturally guide you through all the concepts, or you can look for specific information in the books and use it without having to read any other context from the book.

Looking at Euclid's "definitions" in terms of our three conceptions of definitions we can adjust our viewpoint for the heuristic definition slightly to get an interesting new perspective. We can consider the effectiveness of Euclid's definitions not in terms of how successful they were in solving mathematical problems, but how effective they were in teaching new mathematicians. In this case, the long-time popularity of *Elements* among mathematicians leaves no doubt that in this regard Euclid fully succeeded. There might have been definitions at the time that better helped mathematicians in solving problems but the definitions of Euclid served as a great gateway into mathematics for aspiring mathematicians for a very long time. The main point of *Elements* was never to be a groundbreaking addition to the mathematical knowledge at the time but to be a collection of the knowledge that was already out there. In this sense, we can consider most if not all definitions used in *Elements* as good definitions in the heuristic conception. For the standard and axiomatic conception, we can argue that the definitions cannot be eliminated since they use the concepts of length and breadth, which by themselves are not defined in geometry. In this case, I would argue that the heuristic conception is a better way to evaluate this definition, since even if the definitions Euclid used are not proper as we see them now, they were still very successful in their role as an educational tool.

Lastly, we look at the third definition given by Euclid: *The Extremities of a line are points.* The third criterion of the axiomatic conception is that a definition is always correct. You could look at this definition and say that it is indeed correct, but that is taking this criterion too literally. A definition is always correct because a definition is not an assertion of something that can be proven true or false. This indicates that this is not a definition in the axiomatic conception. Even if we were to evaluate this definition in the heuristic or standard conception, a definition not asserting something is generally agreed upon as being key to something being a definition. To Euclid, this might not have been the case but a definition not being an assertion is what differentiates it from theorems, lemmas and other mathematical statements like axioms. It is also important to consider why Euclid decided to include this definition, since he already defined both a point and a line. In the commentary of Heath on *Elements* we find that Euclid believed it was necessary to give a connection between a point and a line. ([17]p.165) But if this was his main reason for including this definition, Heath points out that this definition in that regard is not sufficient.

> We miss a statement of the facts, equally requiring to be known, that a "division" of a line, no less than its "beginning" of "end" is a point and that the intersection of two lines is also a point.([17]p.178)

So this definition is not good for our understanding of the concept, and even if we consider it a necessary addition of information that should be known, it is still insufficient.

Recalling that the thirteen books can be subdivided into several subjects, let us now switch focus to the seventh book in *Elements*. Book 7 is the first book that primarily focuses on number theory instead of geometry. Because of this large shift in subject matter, book 7 again opens with quite a large list of definitions. This list falls only one short of the list of definitions at the start of book 1, being a total of twenty-two definitions long. Unlike in book one, these definitions are not followed by any postulates or common notions. Instead, they are directly followed by the first proposition.

To start this list of definitions Euclid first provides us with the definition of a unit. Quite a logical place to start, but the definition Euclid decided to use is certainly lacking when compared to modern standards. They are as follows:

1. A unit is (that) according to which each existing (thing) is said (to be) one.

2. And a number (is) a multitude composed of units.

3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.

4. But (the lesser is) parts (of the greater) when it does not measure it.

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

The unit is the most fundamental of fundamental part when working with numbers. It is the starting point from which we can generate all other numbers. Giving a good definition of a unit is certainly not easy, but the way Euclid decided to define it certainly leaves a lot to be desired. Trying to define something by referring to every other thing in existence is quite a bold move. But in a way, it does make some form of sense. Everyone is familiar with the concept of *one thing*, it doesn't matter whether we are talking about trees or stones; we can always tell when we have exactly one of those things. In the work of Heath, we find the definition used by the Pythagoreans *According to "some of the Pythagoreans", "a unit is the boundary between numbers and parts."* ([18]p.279) One could argue that this is a better definition, since it does not refer to as broad a concept as everything in existence. However, what it does refer to are the numbers and parts, which in turn is not ideal since usually numbers are in turn defined by the unit, which creates a sort of circular defintion. The problem here is that the unit can be seen as the starting point from which all numbers and parts originate but then when defining the unit we essentially have nothing else to refer to. By referring to everything in existence Euclid essentially gets around the reader's need to have any specific prerequisite knowledge. The reader only needs to know "something" that exists. Looking at it in this way it is quite a clever way to get around the difficulty of making a definition without having any previously established notions to work from.

Looking at this definition in the context of the standard and the axiomatic conceptions, the biggest criterion that we would expect this definition to clash with is that of eliminability. Since the unit is our starting point, removing that starting point could cause all preceding definitions to in turn fall apart. However, since Euclid refers to everything, we can argue that we can take any other starting point where *something* exists. Since we have that something we also have *one* of that something, which in turn we could use to get our concept of a unit. So we see that we can quite easily remove this definition, since we can in turn acquire it from anywhere else. This in turn makes it also a good definition in the heuristic conception. The definition being so universally accessible for any reader makes it an excellent starting point for introducing the reader to the definitions that follow. Euclid's goal of *Elements* was to teach mathematics, so ideally it would be easily accessible for as many people as possible. When trying to teach something it can be easy to omit some information that you know because you see it as something not worth mentioning, however, the person you are teaching might not have this knowledge. By starting with the broadest concept possible, namely everything in existence, and working from there, it becomes in a way impossible to require any prerequisite knowledge.

Nonetheless, Euclid's definition of the unit is not perfect. One person who went into great detail as to why was Gottlob Frege. In his *Foundations of Arithmetic* [43] Frege goes into great detail as to why exactly the definition of the unit given by Euclid is mathematically insufficient. In his introduction, Frege says:

> then we shall very likely be invited to select something for ourselves - anything we please- to call one. Yet if everyone had the right to understand by this name whatever he pleased, then the same proposition about one would mean different things for different people,-such propositions have no common content. ([43]p.12)

This points to a fatal flaw in Euclid's definition of the unit. By trying to create a definition that is easily understandable for all readers, he leaves room for each reader's personal interpretation. This by itself in a way undermines the goal of *Elements* and the idea of a definition. Definitions are made so that each mathematician who works with them has the same structure in mind. This way we can debate its properties, what it can be used for, and what conclusion we can draw from it. This by itself would be impossible or meaningless if the structure we are analyzing is different for each individual. At this point, Frege does not mention Euclid by name but one can infer that this section is referencing the definition given in *Elements*.

Frege's counterargument to Euclid's definition is informal but in turn it gives rise to another question. Are different units identical to one another? We have tried to find a suitable definition of a unit, but is there one definition that can fully encapsulate every *one* that we recognize? One apple and one pear are similar in a lot of ways but also different in many others. Both of them are one and together they can be considered two pieces of fruit, but at the same time, they are not two apples or two pears. We can add them together in some way but to do that we have to change our understanding from one apple to one piece of fruit. In doing this the object we are working with has not changed but our identification of it has. So there are situations in which we cannot add certain units together; would this then not mean that the units that we are working with are different units? If they are, then that would imply that there would be different definition for each of them and thus no one definition that works for every unit. Frege also analyses this question in the *Foundations of Arithmetic* ([43]). Frege points to the solution given by Thomae:

> Disregard, in considering separate things, those characteristics which serve to distinguish them ([43]p.45)

By doing this all things we are considering should become equal in the sense that each of them is a unit, and together they form a number of the things considered. Frege, however, rejects this notion on two grounds. The first is the same as the one I used before. Even if we consider an apple and a pear as pieces of fruit our shift in perspective has not changed anything about the fact that they are an apple and a pear. The differences between the two objects don't disappear simply because we refuse to acknowledge them. For his second point Frege quotes Descartes *The number (or better, the plurality) in things arises from their diversity.* ([43]p.46) In other words, if I have two things that are equal in every way, then they would be the same thing so then I would only have one thing. To better illustrate this, Frege quotes Jevons:

> It has often been said that units are units in respect of being perfectly similar to each other; but though they may be perfectly similar in some respects, they must be different in at least one point, otherwise they would be incapable of plurality. If three coins were so similar that they occupied the same place at the same time, they would not be three coins, but one. ([43]p.46)

This creates an interesting dilemma: we are looking for a universal unit but if there is such a thing then that would imply that there is no plurality since there would be nothing to add to this unit. As Frege puts it:

> If we try to produce the number by putting together different distinct objects the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not the number, But if we try to do it in the other way, by putting together identicals, the result runs perpetually together into one and we never reach a plurality. ([43]p.50)

The main problem here is finding a definition of the unit that allows us to go form the unit into other numbers. Perhaps then it would be beneficial to switch our focus to the numbers as a whole to get a broader view of the problem at hand. To do this, we first look at the way Euclid defined a number.

With his definition of the unit established, Euclid then defines a number. *A number is a multitude composed of units.* From this, we can immediately tell that Euclid only considers natural numbers to be numbers. What is less clear is whether the unit itself is considered a number. In Heath's commentary on the unit, we find Iamblichus saying that the school of Chrysippus defined it in a confused manner as *multitude one* ([18]p.279). Going from this reasoning we could consider the unit a number; however, Heath's use of the word "confusing" could imply that this viewpoint was not one commonly held among the Greeks. However in Heath's commentary on the number we find the definition given by Moderatus:

> A number is a collection of units, or a progression of multitude beginning from an unit and a retrogression ceasing at an unit ([18]p.280).

Moderatus starts with the same definition as Euclid but then expands on it. By stating that a progression of multitudes starts at a unit, Moderatus implies that the unit is itself included in the progression of multitudes and is thus also a multitude. The same is implied by stating that a retrogression ends at a unit. Euclid did not include this analysis in *Elements*, however, so whether or not he agreed with them is hard to say. However, Heath cites the work of Iamblichus, which says that the definition used by Euclid lacked the words *Even though it be collective* ([18]p.279), which according to him was generally included in the definition of a unit. Euclid specifically not including this would in turn imply that he did not see the unit as a number. Whatever is the case, Euclid himself did not provide any further comments on the matter in *Elements*, so perhaps he simply wanted the reader to draw their own conclusion.

When we look at this definition in terms of our three conceptions, we can be very straightforward in terms of the axiomatic and standard conception. This definition can be simplified to the form $A = X \times B$, where X is the unit, B the amount of units, and A the number (always equal to B). This is by all accounts a good mathematical definition, if A, B, and X are properly defined. As for the heuristic conception, in a way, the definition used by Euclid lines up reasonably well with our modern-day definition of natural numbers. The exceptions are the unit and zero. Excluding zero from the natural numbers is certainly not uncommon, but the unit being excluded is. Euclid himself does not specifically exclude one as a number but his definition being vague on the matter does not help. In either case, Euclid confined his idea of numbers only to the natural ones for which this definition could be seen as sufficient.

Going back to Frege's *Foundations of Arithmetic*, we see Frege raise the same argument as the one I made before.

> Some writers define numbers as a set or multitude or plurality. All these views suffer from the drawback that the concept will not then cover the numbers 0 and 1. ([43]p.38)

He criticizes these sorts of definitions as being vague. This would certainly be the case when working with the definitions given by Euclid, since he defined the unit in terms of everything in existence. So by that logic, we can add one moon, one cat, and one meter to get a total of three. It is not surprising that Frege rejects these sorts of definitions outright.

> No analysis of the concept of number, therefore, is to be found in a definition of this kind. ([43]p.38)

Another critique from Frege towards this definition of natural numbers we find when he quotes Schröder's definition of the natural numbers as *A natural number is sum of ones.* ([43]p.55) According to Frege, in doing this Schröder is not thinking of the meaning of numbers but only of their application in mathematics.

> This passage shows that for Schröder number is a symbol. · · · Even by the word "one" he understands the symbol 1, not its meaning. ([43]p.55)

Euclid also provides a multitude of definitions for specific sorts of numbers. Some of them we are still very familiar with, such as the even, odd, and prime numbers. In addition to these, we also get the definition of squared and cubed numbers. All of these would still be familiar to any modern reader but besides these, we also get some more unusual definitions. For example, that of a plane number:

> And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another. ([12]p.194)

If we exclude one from the numbers then any nonprime number would be a plane number. A solid number is defined in a way similar to a plane number being the product of three numbers instead of two. The purpose of these definitions seems to be to draw a connection from the numbers as concepts to geometrical figures. This is not surprising, since in a lot of his proofs Euclid relies heavily on geometrical demonstrations.

There are plenty more definitions to be found in Euclid's *Elements* but at least we have looked at some of the most interesting ones. When you understand how a point is defined and how you go from a point to a line, you can extrapolate how to get to a plane and a solid. The most interesting definitions to study are generally the ones at the very start of a subject. Once you have a decent number of definitions to work from, making new ones tends to become quite easy, since you already have an established frame to work from. The definitions Euclid chose for the unit and a point might not be perfect in some regards, but they lay a good foundation for Euclid to work from. Euclid also uses his choices for these definitions to great effect, since the definitions and theorems themselves are perfectly chosen for the definitions that follow them. The choices Euclid made for which definitions to include show how much thought he put into even these decisions when writing *Elements*. As a result, *Elements* became one of the most influential mathematical works in history and became a gateway for generations of aspiring mathematicians into the subject.

5 Ancient China

Most Western mathematicians will at least be somewhat familiar with the mathematical developments that happened in ancient Greece. They might not know the exact details like specific dates, but they will at least know the broad strokes of Greek mathematical development and can tell you some of the big names and what they did for mathematics. But when asked about the historical development of mathematics in ancient China their knowledge tends to be a lot more limited. Perhaps this is because when people look at history they look at it from the viewpoint of their own predecessors. A good example of this would be when people are asked about the numbers we use (or more specifically the symbols we use to depict our numbers). A fair number of people would probably give an answer that includes *Arabic numerals*. However, the origins of this number system lie in India. But in our European historical context, they were introduced via the Arabic world so in that context it does make sense to refer to them as Arabic. By this logic, it makes sense that most people wouldn't know about historical Chinese mathematical developments. Any idea that originated in China would have had to make its way through India and the Middle East before finally arriving in Europe, and when it finally did any context regarding its origin in China would have been gone.

To start we shall look at the oldest two surviving works of mathematics from China. These are the *Zhoubi Suanjing* and the *Juizhang Suanshu*. The latter of these two is generally translated as *the Nine Chapters on the Mathematical Art* which is then shortened to *the Nine Chapters*. The translation of the former however is a bit more difficult. The reason for this is that *Zhou* could either refer to *the path of the heavens* or the Zhou dynasty ([26] p.13). The exact period in which these works were made is unknown, as are their authors. A likely reason for this is that in 213 BC the emperor of China ordered almost all books in China to be burned with few exceptions. Mathematics however was not of of those exceptions, however not all mathematical knowledge was lost and both the *Zhoubi* and *the Nine Chapters* survived this event but did lose a lot of context because of it ([28] p.176). Out of these two works *the Nine Chapters* was the one with the biggest influence on Chinese mathematics. Similar to *Elements* in the West, *the Nine Chapters* was something that almost any Chinese mathematician would have studied at some point. As Jean-Claude puts it:

> *The Jiuzhang suanshu* became, in the Chinese "tradition", the mandatory reference, the classic of classics. ([26] p.14)

5.1 The Nine Chapters

It might surprise some of you but *the Nine Chapters* consists of nine chapters. In ascending order, they are called:

> Field Measurements, Millet and Rice, Proportional Distribution, Short width, Construction consultations, Fair levies, Excess and deficit, Rectangular arrays, Right-Angled triangles ([31] p.39).

From *the Nine Chapters*, we learn that the Chinese at this point in time already knew about Pythagoras' theorem. They also knew how to find the root of a number, used 3 as a substitute for π, and knew how to solve linear systems of equations. Most interestingly, we also learn that the Chinese were already using negative numbers to solve equations ([26] p.14). We will return to negative numbers in due time but for now, let's compare the structure of *the Nine Chapters* to that of *Elements*. The first thing to note is that while *Elements* consists of theorems, *the Nine Chapters* consists of problems. Now at first, this might seem like a small detail. If we look at the first theorem in *Elements*; *To construct an equilateral triangle on a given straight line*, we could easily rewrite this as a problem. But if we compare it to one of the first problems of the Nine chapters we can see the difference more clearly. The fifth problem of the Nine Chapters goes:

> Now given a fraction $\frac{12}{18}$. Tell: reducing it (to its lowest terms), what is obtained? ([31] p.41).

This is a "specific" problem, in that it contains certain given constants that when changed also change the answer to the question. In *Elements*, we are given variables. In the first theorem the length of the given line is never specified because the answer is the same for any line of any length. Now if we compare the answers in *Elements* we get a step by step proof showing how to construct the triangle which then also proves (sort of) that the sides of the triangle are equal. The answer given to problem five in *the Nine Chapters* is *Answer:* $\frac{2}{3}$ ([31] p.41). Now while this answer is correct, this specific knowledge is not really useful to have. Knowing only that $\frac{12}{18}$ is equal to $\frac{2}{3}$ won't help you with other fractions. For this particular problem *the Nine Chapters* doesn't show a step by step solution, but a a little further in the book it does give the general method for finding the solution:

> The Rule for Reduction of Fractions. If (the denominator and numerator) can be halved, halve them. If not, lay down the denominator and numerator, subtract the smaller from the greater. Repeat the provess to obtain the Greatest Common Divider (GCD). reduce them by the GCD. ([31] p.41)

Applying this to problem five we see that both 12 and 18 can be halved so we get $\frac{6}{9}$. 9 can't be halved so we subtract 6 from 9 to get $\frac{3}{3}$. This would then be the GCD so we divide 6 and 9 by 3 to get $\frac{2}{3}$. Now *the Nine Chapters* doesn't specify when to stop repeating the process to get the GCD, but I assume that it intends to stop when both the numerator and denominator are equal. This rule for finding the GCD is also found in *Elements*, book 7, theorem 2:

> To find the greatest common measure of two numbers not prime to one another. ([12] p.196)

Here Euclid uses a similar method of continuously subtracting the smaller number from the larger until eventually we find a number that divides (measures) both numbers. Now Euclid's proof is quite a lot longer than the instructions given in *the Nine Chapters* since Euclid explains why every logical step he makes is mathematically correct. The main conclusion I draw from this difference is that *Elements* as a book is more concerned with mathematical correctness and laying the foundation for mathematical reasoning to the fullest extent possible, whereas *the Nine Chapters* is more focused on practical applications and teaching mathematics that would be useful in daily life. Both of these approaches have their merits with the more theoretical approach of *Elements* being useful for developing mathematics as a whole, the more practical approach of *the Nine Chapters* makes it more accessible and useful for people like bureaucrats, merchants, land surveyors etc.

For our purposes, like studying the use of definitions, it would have been nice if *the Nine Chapters* took an approach more similar to *Elements.Elements* gave a clear list of definitions for us to examine, but *the Nine Chapters* doesn't. The only way to get any form of definition from *the Nine Chapters* is to construct some that would best fit the context. For this, we can turn to *The History of Chinese Mathematics* by Jean-Claude Martzloff ([26]). In chapter twelve Martzloff analyses the use of numbers and numeration in the context of Chinese mathematics itself. Especially useful for us are the parts here about the use of positive and negative numbers, and the use of zero. With the help of these sections, we can try to form some definitions that we can compare with the definitions used by Euclid in *Elements*.

5.2 Negative numbers

Negative numbers are introduced in *the Nine Chapters* in chapter eight with the sign rule:

> Like signs subtract. Opposite signs add. Positive without extra, make negative; negative without extra makes positive. Opposite signs subtract; same signs add; positive without extra, make positive; negative without extra, make negative. ([31] p.46)

At first glance, this explanation seems confusing but if we break it down we see that it is indeed correct. The rule distinguishes four possibilities depending on whether you are trying to add or subtract and on whether the two numbers you are using the *same sign* (both positive or both negative) or *different sign* (one positive and one negative). So *Like signs subtract* would mean you are trying to subtract a from b and they are either both positive or both negative. Then those four total possible cases are split into two pairs: *like signs subtract. Opposite signs add* and *opposite signs subtract; same signs add.* Now for each pair of possible scenarios, the rules are explained. For *like signs subtract. Opposite signs add.* we are given the rules: *positive without extra, make negative.* and *Negative without extra makes positive.* So say we want to calculate $a - b$ with a and b having the same sign. If they are both positive then by the minus sign b would become negative in the calculation. Now if $a < b$ then the positive a would be *without extra* so the result of the calculation would be negative. And if $a > b$ then the negative b would be *without extra* so the result would be positive. The same rules apply to adding two numbers of opposite signs together. Now in these two cases, you always end up with one positive and one negative number in the calculation. In the other two cases, both of the numbers in the calculation will be either positive or negative. In this case *positive without extra* would seem to imply that but numbers are positive which is a bit confusing.

As one might expect from *the Nine Chapters*, negative numbers are not introduced in their own right but are explained alongside the rules for using them in calculations. The explanation itself is very short and doesn't give us much to go on in terms of trying to make a suitable definition of negative numbers. To start we turn again to *the History of Chinese mathematics* [26], Martzloff makes three main observations about the ways negative numbers are used in *the Nine Chapters* and in Chinese mathematics more generally. Firstly, he says:

> In the *Jiuzhang suanshu*, as in all other Chinese mathematical works, negative numbers are never found in the statements of problems and therefore cannot be considered as relative numbers in any situation whatsoever (they only exist via computational procedures).([26] p.201)

Thinking of negative numbers purely with regard to computational problems does make defining them a lot easier. If we have a definition we can simply try if it is properly stated in terms of addition, multiplication etc. And if it is, then we can conclude that it is a proper definition. Now from an analytical perspective, this is a lot less interesting. Martzloff believes the reason that negative numbers were only seen as a computational tool was:

> We may assume that this has to do with the fact that for Chinese mathematicians numbers were essentially born of concrete situations (even if, most often, it was a fictitious concrete) ([26]p.201)

Personally, I agree with Martzloff and I think most people would. This is not surprising since the first mathematical problems always arise out of some real world scenario which we then try to represent in some abstract way to then be able to draw a conclusion that is then applicable to the initial real world scenario. When we then try to start to look more closely at the possibilities of that abstract representation we see things like negative numbers arise. In the case of negative numbers we see that in some ways they are purely an abstract concept, we can't find negative three apples for example, while in some ways they can apply to real-world scenarios, you can owe someone twenty euros and that could be considered a negative amount of money. Now negative numbers being not always applicable to the real-world situations leads us to the question of whether we accept negative numbers as valid answers to mathematical questions. For the Chinese, the general answer appears to have been no.

> Similarly, none of the problems ever have answers which are negative numbers. ([26]p.201)

The second observation Martzloff makes about negative numbers in Chinese mathematics is:

> Positive and negative numbers only occur as computational intermediates during the execution of highly particular algorithms including, for example, square-array algorithms (fangcheng) in the Jzuzhang suanshu and algorithms for extracting the roots of polynomials in the Suanxue qimeng (1299) and the Shushu jzuzhan, (1247) ([26]p.201)

What this would seem to imply is that in Chinese mathematics the rules surrounding the use of negative numbers were a lot more strict than the ones we have today. Perhaps this is because these sorts of algorithms are where negative numbers originated and there was never a real reason for Chinese mathematicians to work on more general uses. If a problem was simple enough that it didn't require the algorithm they could solve it without the use of negative numbers. In that case, going out of your way to define rules for using negative numbers outside of those algorithms makes no sense. The last thing Martzloff notes about negative numbers in China is:

> Chinese negative numbers were given a concrete embodiment by specific objects, namely counting-rods (suan). ([26]p.202)

Having a physical representation for negative numbers could have contributed to them being seen as just a tool used in calculation rather than as a subject of study in and of itself. Whatever the case might be, the usage of negative numbers in Chinese mathematics appears to have been for practical reasons like computations and any exploration of negative numbers in a philosophical sense therefore was probably rarely present if it was present at all.

Now we can try to forge a definition for negative numbers that would fit with these three observations. Note that the third observation doesn't influence the definition, since it only concerns the physical representation of negative numbers and not the concept. From Martzloff's first two observations we know that the most important thing is that a definition should work for algorithmic computations: the philosophical reasoning behind the definition was not important in ancient Chinese mathematics. With that in mind any definition of negative numbers in the form of them being the inverse of positive numbers with regards to addition would suffice. Other than that there isn't much else we can say about a definition of negative numbers that would have been used in ancient China. It would have been nice to be able to make a more complex definition that would have been more interesting to explore further but sadly we really can't make one with the information we have about negative numbers in ancient China.

5.3 Zero

When we look at the use of zero in Chinese mathematics we see a similar situation to the negative numbers. In some specific ways, we see zero but we don't see it used as a concept by itself. Martzloff identifies three main ways in which zero was used in Chinese mathematics:

> When one speaks of zero, this may mean (for simplicity), zero as a number with the same status as any other number. It may also mean that symbol which when written immediately after the final units of a number enables us to multiply this number by the base (10 in base ten). Yet again, it may mean the special symbol which shows us that certain orders of units are absent. ([26]p.204)

Of these three, the first form of zero is the most interesting one, with the other two forms being the only way aiding the notation of certain numbers. Sadly, the first form is absent from ancient Chinese mathematics:

> Chinese problems never have zero for a solution and Chinese mathematics never involves a number zero which is freely subjected to operations like the other numbers. ([26]p.204)

The only form that was present in ancient Chinese mathematics is the third form, but a symbol for zero does show up around the seventh century. The reason that zero is not found in the use of mathematical operations is probably because for addition, subtraction, multiplication, and division, using zero is rarely useful. Adding or subtracting zero changes nothing, you cannot divide by zero, and if you multiply anything by zero the result will always be zero. An operation that either does nothing or has a constant outcome is not something that would have been of interest to the Chinese mathematicians. Negative numbers certainly had their uses so we see them used constantly to solve all sorts of problems. But even for negative numbers, the Chinese seemed to focus solely on their practical use and zero not having any practical use was thus mostly ignored.

5.4 Geometry

Ancient Chinese mathematics was not particularly known for its proficiency in geometry. Of course, they had a decent understanding of it since they used it for things like land surveying but again the focus is on practical application rather than on mathematical exploration. However, it was certainly practiced even if not to the degree of the Greeks. As Martzloff puts it:

> A priori, it is difficult to conceive of geometry without definitions, axioms, postulates, theorems and proofs. ([26]p.273)

This means that Chinese geometry could provide us with some actual concrete definitions used in ancient Chinese mathematics. Martzloff points out two schools of disciplines that we could potentially look towards for a more detailed understanding of Chinese reasoning and by extension geometry, namely the Sophist school and the Mohist school. About the first one, Martzloff states:

> In the case of the sophists, the connection with later mathematical developments is not particularly obvious and remains to be precisely assessed.([26]p.273)

So diving deeper into the sophist school would most likely not yield any results. However, for the Mohist school, Martzloff says:

> however it has long been noted that the Mohists defined certain geometrical object. ([26]p.273)

Now this does seem promising since it could allow us to directly compare definitions used in China with those used in Greece at around the same time. Martzloff himself goes into some of the definitions used but most importantly he refers us to the work *Later Mohist Logic Ethics and Science* by A.C. Graham [14]. In this work, we find the Canons used in the Mohist school. The Canons consist of two lists of ninety-eight and eighty-two items long (referred to as A 1-98 and B 1-82). For us, only a couple of these Canons are of interest namely A 52-69. Why we pick these two can be best explained in two quotes from Graham: *All Canons in A 1-75 really are intended to be definitions.*([14]p.261) and the section title *The sciences: (Geometry) (A)52-69)* ([14]p.301). So the first seventy-five Canons are definitions and fifty-two through sixty-nine are specifically definitions with regard to geometry.

5.4.1 Defintions in Chinese Geometry

Although Graham says that all Canons A 52-69 are about geometry one could argue that for some that is kind of a stretch. We can already see this in the first Canon: ***P'ing*** *(level/flat) is of the same height.* ([14]p.305) Height is something that is used in three-dimensional geometry but when comparing geometrical structures we hardly ever refer to them as flat or level. When we use these terms today we generally refer to a real-world surface that has no noticeable rise in elevation anywhere on it. In geometry, these terms are hardly used or even necessary since when we talk about a plane we assume that the plane is perfectly flat unless specified otherwise. Graham does give a reason as to why this definition might have been included in the Canons:

> The first definition may be taken as also a declaration that from now on we can forget the third dimension, all measurements will be on a flat surface. ([14]p.305)

This could be seen as a bit of a leap from the definition itself, which mentions height and never implies that we should disregard it from that point on. However, we must also remember that this is a translation of a very old Chinese text and a lot of context can be lost on us modern English readers. Graham does have an argument to support his claim:

> The very first Canon is the only one which introduces the third dimension. ([14]p.305)

With this in mind his claim does make some sense, especially if we notice that, height itself is not defined but only things of the same height. We could even go as far as to say that the next Canon also implies this:

> (A 53) **T'ung ch'ang** (of the same length) is each when laid straight exhausting the other. ([14]p.305)

Now the reason we say this also implies Graham's claim is not because of the definition itself but because of its position. We are only given a way to compare length after we have been given a definition of things of the same height. This method of course works for all dimensions of height, width and length so why are we only given it after the definition of things being the same height? If we were to consider height it would make more sense to explain how to compare lengths and after that give a name to two things of the same height. We might say: *this is how we compare the dimensions of two objects and if this dimension (height) is the same we call them level*. But the Canons don't do this so I think Graham's claim that the third dimension wasn't considered in Chinese geometry is a reasonable conclusion to draw.

Now the first two Canons give us some interesting insight into the geometry the Chinese practiced but as an account of definitions by themselves, they aren't really noteworthy. The third Canon:

> (A 54)**The chung** (centre) is (the place from which (?)) they are the same in length.([14]p.305)

This one is a bit more interesting but the only geometrical object in two dimensions that has a true center in this sense is a circle. The definition of a circle comes a bit later A 58 so we will revisit this definition then. Canon A 55 is certainly interesting and the example noted with it may be even more so:

> (A 55) **Hou** (having bulk/thickness/dimension) is having something than which it is bigger. ([14]p.305)

If we exclude A 52, what we see with the other three Canons is that they do not define a specific geometrical structure, but rather define properties that a geometrical structure can have. A structure can be *the same length as another one* or *be at the center of another one* or *have a dimension*. Having these sorts of definitions can be useful but having them before even knowing what sort of structures you are working with seems pointless. The definition for having dimension works in the sense of 'if something has a certain dimension (length or width) then it has a certain length (or breadth) which means we can halve the length (or breadth) and thus have something which is smaller.' But having good rules does nothing if you don't know what those rules apply to. However, some Canons have an example to help give a bit more clarity as to what they mean and the example we get for A 55 is:

> Only (the starting-point (?)) has nothing than which it is bigger. ([14]p.305)

The translation *the starting point* might not be entirely accurate, but getting a perfect one-to-one translation is almost always impossible since some implication and context are easily lost. So for the sake of comparison to *Elements* let us take this as a definition of a point. Graham even informs us that this conception of a point also existed outside of the Mohist Canons.

> **Wu hou** (the dimensionless, the point) was a central concept of the sophists, as in Hui Shih's paradox: -...-*The dimensionless cannot be accumulated, yet its size is a thousand miles.* ([14]p.306)

We can connect these definitions with the simple reasoning that since a point has nothing which is bigger than it cannot have dimension thus it is dimensionless. Now we already saw that even in Ancient China there were people raising arguments about this sort of definition of a point:

> Hui Shih's paradox is that points cannot be accumulated yet a length is the sum of the units into which it is ultimately divisible, which are points. ([14]p.306)

Now in this explanation, we see that units and points are considered the same thing. A unit is something that can be halved (specifically if we know of fractions then we know that $\frac{1}{2}$ is half of the unit) and a point cannot so this already gives a contradiction in the explanation. Even going a a bit deeper into the idea of dividing a length we see some contradiction. If we consider a length as made up of an infinite amount of points then dividing it we again get two lengths which in turn consist of an infinite amount of points. Comparing two infinite things against each other is different from just comparing two numbers. Take for example all the natural numbers. We can split them into even and odd numbers, then in a way, all the even numbers are a smaller group than all the natural numbers since the natural numbers contain all the even numbers plus an additional infinite odd numbers. However, for every natural number its double is contained in the even numbers so with this we can make a one-to-one comparison to the even numbers which would imply that they are the same size. Similarly, for every length we can make a one-to-one comparison to any other length so they would essentially contain the same number of points. In that sense if we divide a length into two smaller lengths the number of points in each length is the same so have we really divided the number of points?

Now if we compare the Mohist definition to the one given in *Elements* we see that they are fundamentally very similar. They might not be exactly identical but there are several ways in which we could try to argue that they are equivalent. For example, you could say that a point having no part would imply that it has no dimensions since if it had a dimension we could halve that dimension and thus get a part of a point. Going the other way a point having nothing which is bigger than it would imply that we cannot take a part of it since a part of a point would have to be smaller than it and thus the point would have something than which it is bigger. We could also try a different approach in which we use the context from *Elements*. Since Euclid defined a point before defining length, breadth, and height we can assume that a point does not possess any of these and is thus dimensionless. This approach doesn't work as well the other way around since we get the definitions for the same height and length before we get the definition of a point. It is also interesting to see that the position of a point is not brought up in the Canons, so the definition of a point to the Mohists is fairly different from the one used by the Pythagoreans.

Despite the Canons A 52-69 being geometrical definitions the idea of geometry that was present in ancient China was quite different from what we see today and even very different from what we see in ancient Greece. As such the definitions we get from the Canons are about things that would never have been considered in Greek geometry. For example, A 56 gives us ***Jih Chung*** *(the sun at the centre/noon) is the sun being due South.* ([14]p.307) This is not something that would be relevant to anyone practicing geometry. Because of this, the number of definitions from the Canons that we can compare to our modern standards and to the definitions from *Elements* is highly limited. Therefore, the last two definitions we will look at are that of the square and the circle. ([14]p.309)

> C. "*Yüan* (circular) is having the same lengths from one centre.
> E. The compasses draw it *in the rough* (?).
> C. *Fang* (square) is *circuiting* in four from a *right angle* (?).
> E. The carpenter's square shows it *in the rough* (?)"

This definition of a circle is almost identical to the one Euclid gives in *Elements*. The only real differences are that Euclid gives the definition of the circle and the center of the circle as two different definitions, and that is the precise about what it means to have the same length from the center. This is not that surprising since the example given even mentions a compass, which works by marking all points at a certain distance from another point. The *in the rough* in both examples appears to be a difficult translation since it can either refer to the material on which the figures were drawn (*The compasses draw on rough wood* [14]p.309) or it can be a reference to the idea that the circles and squares drawn are never "perfect"(*compasses and carpenter's square draw only rough approximations to the true circle and square.* ([14]p.309)). The definition of a square is a bit more questionable since for one it mentions a *right angle* which normally would not be an issue except that the Canons themselves never define a right angle. So the definition relies on something that has not been defined. At the time there probably was a general understanding of what a right angle was but leaving something like this open to interpretation is not something you want from a definition. Another thing is that the sides of the square are never explicitly said to be equal. There are two possible explanations for this. The first lies with the word *circuiting* used in the example. Here this word is used for lack of a better translation so it is possible that the word used in Chinese had more context to it which implies that all the sides would be of equal length. This context could either have been lost when it was translated to English when the Chinese language changed. The second explanation is given by Graham:

> Although fang is used characteristically of the true square it can be applied to all rectangles. ([14]p.309).

The square is a rectangle of course and it is possible that the Mohists did not deem it necessary to differentiate the square from all the other rectangles. The Canons do not contain a specific definition for a rectangle which would support this idea. Graham leans more towards the second explanation that the square and the rectangle were used interchangeably: *the definition as we understand it does not specify that the sides are equal in length* ([14]p.309). In contrast to the Canons in *Elements*, Euclid identifies “Four main quadrilateral figures” and gives a name to all the remaining ones: ([12]p.7)

> 22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

The Mohist Canons provide an interesting insight in what the ancient Chinese considered geometry. Although the Canons can hardly be considered representative for the whole of ancient China in the way that one could consider *Elements* a representation of Greek geometry since their influence was far less, it still provides us with an idea of things that in some way were considered geometry in ancient China.

6 Frege

The definitions we have seen from ancient Greece and ancient China often leave a lot to be desired when it comes to mathematical correctness. This is not surprising since they are some of the oldest mathematical definitions in the world. Making something for the first time is always the hardest but over time we get better at it. The same goes for making definitions. As mathematics progresses old definitions get revisited and, if deemed necessary, changed. This is something that is still going on even now and will most certainly continue to happen in the future. In chapter four we already looked at parts of Gottlob Frege's *The Foundations of Arithmetic* [43]. In chapter four we mostly looked at how parts of *the Foundations of Arithmetic* related to *Elements*. Now we will look at *the Foundations of Arithmetic* itself.

6.1 The Foundations of Arithmetic

> When we ask someone what the number one is, or what the symbol 1 means, we get as a rule the answer "Why, a thing". and if we go on to point out that the proposition "the number is one thing"is not a definition, because it has the definite article on one side and the indefinite on the other, or that it only assigns the number to the class of things, without stating which thing it is, then we shall very likely be invited to select something for ourselves-anything we please-to call one. ([43]p.13)

This is the way Frege decided to start *the Foundation of Arithmetic*. The point he is making in this section is that, despite everyone having an understanding of the number one, that understanding tends to be limited and when we ask them to elaborate on it their answer will often lead to contradictions. In this example we are told that we can pick anything we please to call one but, as we already saw in Chapter 4, this does not work when trying to make a proper definition of one.

> Yet if everyone had the right to understand by this name whatever he pleased than the same proposition about one would mean different things for different people ([43]p.13)

Leaving something up to individual interpretation goes against one of the core goals we have when making definitions. We use definitions so that everyone has the same understanding of an object or concept. That way when we discuss the thing defined there is no room for misunderstandings. For that reason, a good definition leaves no room for individual interpretation. So then a definition that lets everyone pick their own object to use a one would of course not suffice as a proper definition. However, is this really such a big problem? Even if we don't all have the exact same understanding of the number one, if we all understand the rules of arithmetic and agree on things like $1+1=2$ then everything should be fine.

> Many people will be sure to think this not worth the trouble. Naturally, they suppose, this concept is adequately dealt with in the elementary textbooks, where the subject is settled once and for all. Who can believe that he has anything still to learn on so simple a matter? ([43]p.15)

This is a question that will always be asked when trying to convince other people to show interest in any subject. Of course, from an individual perspective, we can always answer "personal interest". However, this doesn't help when we try to convince others. Now for most people, not knowing a proper definition of "one" will have no impact on their lives. Even for mathematicians, the subject will rarely come up. This is although trying to give a proper definition of one might seem like a purely mathematical problem, it is actually closer to a philosophical; one and since mathematicians understand how the symbol 1 works in subjects like calculus they don't see the point in going out of their comfort zone in exploring this problem. While finding a proper definition of the number one might not be necessary for mathematicians to make progress in mathematics as a whole, Frege argues that this sort of mindset itself can be a problem:

> It is sad and discouraging to observe how discoveries once made are always threatening to be lost again in this way, and how much work promises to have been done in vain, because we fancy ourselves so well off that we need not bother to assimilate its results. ([43]p.15)

So even if there is no personal interest or direct benefit from going back to the fundamentals, if we are unwilling to even look back we might cause mathematical knowledge that was already discovered to become lost.

Now let's first get a general overview of *the Foundations of Arithmetic*. The book is divided into five parts:

1. Views of certain writers on the nature of arithmetical propositions.
2. Views of certain writers on the concept of Number.
3. Views on unity and one.
4. The concept of Number.
5. Conclusion.

Each of these parts is again divided into smaller subsections. What's most notable is that in the first three chapters, most of these subsections are formulated as questions. This is not that surprising since at the core of these parts are not Frege's own ideas but those of other mathematicians. He is trying to find answers to certain questions as *Are numerical formulae provable?* in the works of others. The goal of these chapters is to provide context for the problems these sorts of questions pose while at the same time giving the ideas of other mathematicians and showing how those contradict or have flaws in them. It is only in the last two parts is that Frege really gets into his own ideas and conclusions.

6.1.1 Views of certain writers on the nature of arithmetical propositions

In this section Frege tries to find the answers to three questions:

1. Are numerical formulae provable?
2. Are the laws of arithmetic inductive truths?
3. Are the laws of arithmetic synthetic a priori or analytic?[2]

Starting with the first one Frege immediately clarifies that we must distinguish between numerical formulae such as $2 + 3 = 5$ and general laws of arithmetic. Here we are dealing with the first kind. Frege starts with the ideas held by Kant, who holds that these formulae are unprovable and synthetic. Kant tries to draw upon our intuition of fingers to imply that these formulae are self evident but Frege rejects this by saying that this might be the case for small numbers (like 2+3) but not for large numbers (like 7472+89147). This in turn means we must distinguish between small and large numbers, which also provides a problem. ([43]p.6)

After Kant, Frege discusses the ideas of Leibniz, who holds that these formulae are provable. Leibniz does this by arguing that every natural number is defined in terms of its predecessor +1. Then any formula $a+b$, b can be substituted by b times +1 to get $a + 1 \cdots +1$. Now we can substitute $a + 1$ for a's successor and repeat this process to eventually get to the answer to the formula. Frege points out that Leibniz makes a slight oversight in that by substituting b for its predecessor +1 we actually get $a + (c + 1)$. This is not a big problem since we only need to add the property $a + (b + c) = (a + b) + c$ to make it complete. Frege does agree with Leibniz (as do I) that defining the natural numbers this way is the most apt way to define them. It starts from the very first natural number and then proceeds to define every other natural number in the same way without relying on something like intuition to define the natural numbers. Grassmann and Hankel have a similar opinion to Leibniz but they try to state the law:

$$a + (b + 1) = (a + b) + 1 \tag{2}$$

Frege criticizes this approach but not so much the idea behind it, as this method but more the work of Grassmann itself. Frege points out certain points at which Grassmann's proof falls short but these do not distract from the whole idea behind this method. ([43]p.7-8)

The last viewpoint Frege explores for the question *are numerical formulae provable?* is that of Mill. Mill starts off with the same definitions as Leibniz but then

[2]Here Frege questions what kind of knowledge the laws of arithmetic are. Differentiating between a priori (knowledge independent of experience) or a posteriori (knowledge gained through emperical evidence) and between analytic (something is true solely based of its meaning) or synthetic (something is true based on how it relates to the world).

deviates by working from the premise that all knowledge is empirical, which also influences Mill's conception of a definition:

> He informs us, in fact, that these definitions are not definitions in the logical sense; not only do they fix the meaning of a term, but they also assert along with it an observed matter of fact. ([43]p.9)

Frege disagrees with this idea and argues that there is no observed fact about, for example, the definition of the number 777864. Mill gives an example of his view by using the number three. Mill holds that if we have a collection of three objects we can split it into a set of one object and another set of two objects. Mill states that with our senses we can differentiate between the sets of one, two, and three and thus empirically prove that $1 + 2 = 3$. Frege counters Mill's reasoning by stating that this idea will not work on things that aren't physical objects such as flavors. Things that don't impress the senses in the same way as three physical objects are still sets of three. Frege then proceeds to point out that if all equations are proven empirically then $999999 + 1 = 1000000$ would only be true if someone actually witnessed an object being added to a set of 9999999. Needless to say, Frege rejects Mill's idea of empirically proving formulae. ([43]p.9-12)

Now we get to the question *Are the laws of arithmetic inductive truths?* At the start of the previous section on *Are numerical formulae provable?* Frege differentiates between formulae in the form of $2+3=5$ and general laws. Frege then addressed the first of the two and here he addresses the other. Starting with Mill who states: *Whatever is made up of parts, is made up of parts of those parts.* ([43]p.12), Frege believes that Mill is trying to give a law that is comparable to Leibniz's axiom: *If equals be substituted for equals, the equality remains.* ([43]p.13) However, the implication I get from Mill's law is that we can break a number down into smaller parts but it doesn't seem to imply that we can substitute a sum of numbers with the number their sum adds up to. Mill refers to his law as *an inductive truth, and a law of nature of the highest order* ([43]p.12), but Frege argues:

> But in order to be able to call arithmetical truths laws of nature, Mill attributes to them a sense which they do not bear. For example, he holds that the identity $1 = 1$ could be false, on the ground that one pound weight does not always weigh precisely the same as another. But the proposition $1 = 1$ is not intended in the least to state that it does. ([43]p.13)

Mill's insistence upon using empirical proofs means he takes physical representations of arithmetical sums to be equivalent to the sum itself. Since it is almost impossible to have a perfect representation of, for example, $3 + 7 = 10$ Mill believes that those cannot be proven to always be correct. Frege puts it best when he says that physical representations of arithmetical formulae are not the meaning of those formulae but just an application of them.([43]p.13)

Finding or determining the laws of arithmetic is not an easy task. In trying to do so we face problems that are found nowhere else and to demonstrate this Frege uses Leibniz response to the assertion that all numbers only differentiate themselves from other numbers by being either more or less than them. Leibniz responds to this with:

> That can be said of time and of the straight line, but certainly not of the figures and still less of the numbers, which are not merely different in magnitude but also dissimilar. An even number can be divided into two equal parts, an odd number cannot; three and six are triangular numbers, four and nine are squares, eight is a cube, and so on. And this is even more the case with the numbers than with the figures; for two unequal figures can be perfectly similar to each other, but never two numbers. ([43]p.14)

Frege illustrates this further by comparing the numbers to a species of animals. When studying a species of animal usually you work with a sample size, rarely will you ever be able to observe every single member of a species, and even if you could there will be future members of that species that you have not yet seen. Within this sample size, you can determine patterns or common traits but there is no guarantee that those will hold when the sample size changes. So when trying to make a statement about a member of that species that you have never seen before at best the statement is an educated guess. With numbers this is not the case. Even if we see a number we have never seen before we can say for sure which number proceeds it and which one follows it. We can see if it is an even or odd number. Even if we have never actually seen every number we know all of them, and as Leibniz states, each number has different properties from every other number. Where Mill holds the belief that the laws of arithmetic are empirical, Leibniz believes them to be innate: ([43]p.15)

> The truths of number are in us and yet we still learn them ([43]p.17)

Now we get to the last question of this part: *Are the laws of arithmetic synthetic a priori or analytic?* Frege only considers synthetic a priori or analytic here but initially there are actually four options, since we have the option between analytic or synthetic and the option between a priori or a posteriori. However, analytic truths are inherently a priori so analytic a posteriori need not be considered. This in turn means that any a posteriori truth has to be synthetic which would coincide with the views of Mill which Frege has already rejected. Then only two option remain: synthetic a priori or analytic. ([43]p.17)

Kant, and following him, Baumann, and Lipschitz believe synthetic a priori to be the correct choice, holding that this truth is self evident from pure intuition. Hankel mostly agrees with them and bases his theory of real numbers on three common notions, which he defends as follows:

> Once expounded they are perfectly self-evident; they are valid for magnitudes in every field, as vouched for by our pure intuition of magnitude;

> and they can without losing their character be transformed into definitions simply by defining the addition of magnitudes as an operation which satisfies them. ([43]p.18)

Frege, however, points out that Hankel here uses magnitudes instead of numbers. Magnitudes are not inherently the same as numbers and a lot of different things can be ascribed as magnitudes. Numbers, angles, and masses can all be described as magnitudes, but they are far from the same thing and out intuition of they can vary wildly. Thus Frege rejects the idea of calling upon *pure intuition of magnitude* when trying to define numbers:

> We are all too ready to invoke inner intuition, whenever we cannot produce any other ground of knowledge. But we have no business, in doing so, to lose sight altogether of the sense of the word "intuition". ([43]p.19)

The mathematicians believing synthetic a priori to be correct all rely upon intuition to justify this. However, what exactly do they mean by "intuition". Kant defines it as: *An intuition is an individual idea, a concept is a general idea or an idea of reflexion.* ([43]p.19) Frege criticizes this definition by saying that it lacks a connection to sensibility, which he believes to be essential when using intuition as an argument for synthetic a priori knowledge. Frege then proceeds to compare arithmetic to geometry. Frege says that while there are similarities there are also most certainly differences. The reason Frege does this is to show that while geometrical knowledge may be synthetic a priori, arithmetical knowledge is not.

> For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self contradictions when we proceed to our deductions, despite the conflict between our assumptions and our intuition. The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and consequently synthetic. Can the same be said of the fundamental propositions of the science of number? ([43]p.21)

Frege certainly doesn't think that this is the case. Which then only leaves us with the possibility that the laws of arithmetic are analytic. Leibniz agrees with Frege about this mostly because of his belief that all a priori knowledge is analytical. However, Leibniz also believes that every truth *has its proof a priori derived from the concept of the terms, notwithstanding it does not always lie in our power to achieve this analysis.* ([43]p.21); which Frege disagrees with.

This now leads us to Frege's own conclusion. Despite believing the laws of arithmetic to be analytical and thus a priori, Frege does acknowledge the following:

> However much we may disparage deduction, it cannot be denied that the laws established by induction are not enough. New propositions must

> be derived from the which are not contained in any one of them by itself.([43]p.23)

To prevent the clash between the laws of arithmetic being a priori knowledge yet requiring deduction Frege proposes an alternative way of finding them. Instead of trying to reason from known facts we can work from conditions we establish based on the known facts. This way we can determine which results depend on which conditions. The way I understand this is that instead of trying to solve the whole problem of the laws of arithmetic all at once we instead take a small section of it and examine it closely. If we accept that all laws of arithmetic are indeed analytic then using deduction would seem counterintuitive but since we are only looking at a small section of the problem we can use deduction to find out which conditions are required for a certain outcome to be true. Then later we can try to make all the conditions we have found to be required fit together as one general set of facts that govern the laws of arithmetic. Frege states:

> This would make them analytic judgments, despite the fact that they would not normally be discovered by thought alone; for we are concerned here not with the way in which they are discovered but with the kind of ground on which their proof rests. ([43]p.23)

6.1.2 Views of certain writers on the concept of Number

This section is very similar to the first one and is divided into three parts as well, though not all of them are questions this time.

1. Is Number a property of external things?
2. Is number something subjective?
3. The set theory of Number.

And again just like the first section Frege starts by making an important distinction. When talking about numbers we can either refer to individual numbers like 3 and 4, or we can be talking about the concept of Number in general. In the case of individual numbers, we have already seen the way Leibniz chooses to define them, in terms of their predecessor +1. This is a fine way to define individual numbers provided that we have a proper definition of the number 1 and the concept of adding 1. Though this gives us a starting point for defining the individual numbers Frege points out that from these individual definitions alone it is not possible to extract general arithmetic laws that govern all numbers. To extract these laws Frege examines the general concept of Numbers and in doing so also tries to find definitions for 1 and for addition with 1, which then in turn helps us to define the individual numbers. ([43]p.24)

> At this point I shoud like straight away to oppose the attempt to think of number geometrically, as a ration between lengths or surfaces. Obviously, the thought behind this was to facilitate the numerous applications of

> arithmetic to geometry by putting the rudiments of both in the closest connexion from the outset. ([43]p.25)

As we can see Frege rejects the thought that the concept of numbers can be found with the help of geometry. For geometry itself, he doesn't give any direct counterarguments but we can find his reasoning when Frege argues against the concept of numbers used by Newton. Newton thinks of numbers as a relationship between magnitudes. To Newton, a number is a representation of the relationship between a magnitude and a magnitude of the same kind which is taken as the unit. Frege argues against this by referring to Euclid's use of equimultiples when identifying the ratio between two magnitudes. Using equimultiples we can determine the ratio between two magnitudes without ever addressing the concept of numbers. So, according to Frege, there is fundamentally no direct relationship between magnitudes and numbers. ([43]p.26)

> The first question to be faced then, is whether number is definable. Hankel declares that it is not, in these words; "What we mean by thinking or putting a thing once, twice, three times, and so on, cannot be defined, because of the simplicity in principle of the concept of putting."([43]p.26)

Hankel gives a fair argument. The idea behind his argument is that the concept of number is so broad and all-encompassing that any attempt to define it would surely fall short in some sense of the idea of number. Frege however, does not believe this to be the case:

> If the general inclination is, on the whole, to hold that Number is indefinable, that is more because attempts to define it have failed than because anything has been discovered in the nature of the case to show that it must be so. ([43]p.27)

Now Frege turns to the first question of this section, *Is number a property of external things?* Frege notes that in language numbers are generally used to describe a property of something else, in the same way that words like hard, heavy, or red are used. Two mathematicians who seem to agree with this idea are Cantor and Schröder. Although their views are not identical they both believe that the concept of Number originates when abstracting properties of real-world objects, which in turn would put it in a similar category as color. To counter this idea Frege puts forth an idea of Baumann: ([43]p.27)

> Baumann rejects the view that numbers are concepts extracted from external things: "The reason being that external things do not present us with any strict units; they present us with isolated groups or sensible points, but we are at liberty to treat each one of these itself again as a many." ([43]p.28)

Frege agrees with this and notes it as an important distinction between the concept of color and the concept of number. If we observe a book, for example, we can see that the pages are white and the letters are black. No matter how we change our perspective of the book, those colors will not change. But we can change our perspective from one book to one hundred pages. In this case, we are still observing the same object but the number we associate with it is completely different. ([43]p.29)

> To the question: What is it that the number belongs to as a property? Mill replies as follows: the name of a number connotes, "of course, some property belonging to the agglomeration of things which we call by the name: and that property is the characteristic manner in which the agglomeration is made up of, and may be separated into parts." ([43]p.30)

So far Frege has not agreed with Mill on anything and this point is no different. Frege criticizes his use of *characteristic manner* since it implies that there is a definite way to split up every group of objects while this is almost never the case. For example, take a group of ten children. You can split them into ten individuals, five pairs, a group of boys and a group of girls, or several groups of siblings. Mill's argument is based on his belief that everything that is made up of parts is made up of parts of those parts. But Frege points out that you can just keep deconstructing or reordering everything to get any number you want. Furthermore, Mill's viewpoint implies that only physical objects can be numbered which, Frege also rejects. Locke and Leibniz are of the same opinion and express their beliefs as follows: ([43]p.30)

> Locke: "Number applies itself to men, angels, actions, thoughts-everything that either doth exist or can be imagined." ([43]p.31)

> Leibniz: "Some things cannot be weighed, as having no force and power; some things cannot be measured, by reason of having no parts: but there is nothing which cannot be numbered. Thus number is, as it were a kind of metaphysical figure." ([43]p.31)

Frege concludes that the property of Number is not something that is merely a property of external things. Physical objects and groups of objects can have the property of any one number that someone wishes to ascribe to it and are things that can be applied to far more things than just physical objects. In his own words, *It does not make sense that what is by nature sensible should occur in what is non-sensible.* ([43]p.31) This of course doesn't mean that numbers are entirely detached from physical objects but rather that their nature is something far broader than something that is extracted from physical objects. Frege ends with a quote from Berkeley:

> It ought to be considered that number... is nothing fixed and settled, really existing in things themselves. It is entirely the creature of the mind, considering, either an idea by itself, or any combination of ideas to which it gives one name, and so makes it pass for a unit. According as the mind variously combines its ideas, the unit varies; and as the unit, so the number, which is only a collection of units, doth also vary. We call

> a window one, a chimney one, and yet a house in which there are many windows, and many chimneys, hath an equal right to be called one, and many houses go to the making of one city. ([43]p.33)

But if we agree with Berkeley that the number is not something that is fixed or settled we run into a new question: *Is Number something subjective?* If it is, then trying to define it would be far to complicated since the definition would vary depending on each person, context, and the thing numbered. This means that every time we talk about numbers the first thing we should do is define what the unit is and from there every other number would follow. Yet in plenty of cases we don't even bother defining what the unit is, we are working with and when trying to answer $3 + 4$ there is nothing subjective in the answer 7. Frege agrees with this and explains his own thoughts as such:

> I distinguish what I call objective from what is handleable or spatial or actual. The axis of the earth is objective, so is the centre of mass of the solar system, but I should not call them actual in the way the earth itself is so. We often speak of the equator as an *imaginary* line; but it would be wrong to call it an imaginary line in the dyslogistic sense; it is not a creature of thought, the product off a psychological process, but is only recognized or apprehended by thought. ([43]p.35)

Frege goes on to explain that even if there was something subjective about the idea of Number then that subjective part would remain entirely inside someone's own intuition and purely intuitive ideas cannot be communicated. Using the example of geometry again since every geometric axiom has a dual counterpart we could have two people, where one interpretation of a plane is the same interpretation the other has of a point and vice versa. At no point would these two people ever run into a contradiction when discussing geometry with each other. The only difference would be their mental pictures of the problems they discuss, but that is merely an aesthetic change. Another example would be color. A white object can appear in many different colors depending on the light that we shine on it but that doesn't change the fact that the object itself is white. Even colorblind people can distinguish between certain colors that they cannot perceive based on the fact that other people distinguish between those colors. The subjective part about numbers would thus merely be aesthetic in nature and not a fundamental part of the idea of number. ([43]p.35-36)

Then we are left with the last part of this section: *The set theory of Number.* This is by far the shortest part of this section, consisting of no more than one page of text. Here Frege contemplates the idea of using set theory to find the concept of number but rejects it quite quickly in the following words:

> Some writers define number as a set or multitude or plurality. All of these views suffer from the drawback that the concept will not then cover the numbers 0 and 1. Moreover, these terms are utterly vague: sometimes they approximate in meaning to "heap" or "group" or "agglomeration',

referring to a juxtaposition in space, sometimes they are so used as to be practically equivalent to "Number", only vaguer. ([43]p.38)

6.1.3 Views on unity and one.

This section is divided into four parts:

- Does the number word "one" express a property of objects?
- Are units identical with one another?
- Attempts to overcome the difficulty
- Solution of the difficulty

The first question seems almost identical to the first question from the second section but there is a slight difference. In this case, instead of looking at the property of numbers themselves, we are looking at the properties of the words we use to describe numbers. This might seem like an insignificant difference, but when we look at the question "*Is number something subjective?*" we saw that any subjective part of the concept of Number would be intuitive and could therefore not be communicated. This would imply that the communication used for the concept of Number would be completely objective. So it might actually make sense to more closely examine the language used around numbers to find some objective truths about them.

> In the definitions which Euclid gives at the beginning of book VII of the Elements, he seems to mean by the word "*μονάs*" sometimes an object to be numbered, sometimes a property of such an object, and sometimes the number one. We can translate it consistently by the German "Einheit", but only because that word itself shifts over the same variety of meanings. ([43]p.39)

As Frege demonstrates here, one word can have different meanings depending on the context in which it is used. In the case of the words "one" and "unit," this is helpful since it allows us to use these concepts in a broad variety of situations without having to explain in detail what exactly we mean by them. The person we are communicating with will use their own intuitive understanding of the word and potentially fill in any gaps in the information we left out. While this flexibility is very useful in everyday conversation it poses a problem when trying to find a proper definition, since it leaves so much information up to context or someone's own interpretation. We have already rejected the idea that numbers are a property inherent to physical objects but Frege remarks that the word "one" is used in the same way as words like "wise" or "heavy". However, Frege also notes that:

> It must strike us immediately as remarkable that every single thing should possess this property. It would be incomprehensible why we should still ascribe it expressly to a thing at all. ([43]p.40)

The words "wise" and "heavy" provide context to a person or object. Providing context implies that it differentiates the object we are talking about from other objects that do not have that property. Thus there must be objects to which the word cannot be applied. But the word "one" can be applied to any object we can conceive of. Therefore the word itself does not provide any context to the object we are talking about.

> The content of a concept diminishes as its extension increases; if its extension becomes all-embracing, its content must vanish altogether. ([43]p.40)

According to Frege this is further supported by the fact that no one has been able to give a proper definition of "one" as a property. Any conceivable thing can both be regarded as "one" and not as "one" depending on the way we choose to classify "one" in a given context. Baumann says:

> Whatever we take as a point, or refuse to take as further subdivided into parts, that we regard as one; but every one of outer intuition, whether empirical or pure, can also be regarded as many. Every idea is one when isolated in contrast with another; but in itself it can again be distinguished into many.. ([43]p.41)

Baumann here uses the word *intuition* but at the same time describes "one" as something that cannot be divided into parts but also admits that any "one" can also be many. This goes against the idea of "one" being intuitive. Furthermore, Frege holds that if the concept of one is intuitive it should also be present in animals to a certain extent. Frege states that a dog is perhaps capable of differentiating between one dog and many dogs but he believes that it would be unlikely that a dog is aware of a common concept between one stone and one human. ([43]p.42)

Baumann says that a unit is something that cannot be divided but then contradicts himself soon after. Some others hold an even stricter view of the unit. Köpp holds that a unit is something that is *self-contained and incapable of dissection*([43]p.43). By applying such strict criteria to a unit Köpp hopes to give a definition that is internally coherent and free from any arbitrary interpretation. But Frege points out that:

> This attempt collapses because we are then left with practically nothing fit to be called a unit and to be numbered. ([43]p.43)

With this Frege concludes that there is no point in defining one as a property. Properties distinguish between different things and because they distinguish they are inherently limited, in some way, in their use. One as a property applies to everything and thus does not distinguish between objects. Frege goes on to state that trying to force any notions on the property one can actually be disadvantageous since it can lead us to reach false conclusions based on our notions of the property "one". In turn, we lose nothing by having no direct criteria for one as a property since its use is all-encompassing.

> If it does no harm, and in fact is actually necessary, to take our strict units none too strictly, what was the point of being strict? ([43]p.44)

If we cannot even define "one" as a property to something and it doesn't help us differentiate between different things then why do we still use it in the same way as a word like "heavy"? Schröder says that we do it to ascribe an identity to things that we want to number. However, Frege counters that the word "one" is not our only option in this case. Words like "object" and "thing" would suffice just as well. Furthermore, Frege asks whether it is really necessary to ascribe a unit when we want to number a group of things and that designating one out of a group as the unit also leads to other problems.

> In any case, no two objects are *ever* completely identical. On the other hand, of course, we can practically always engineer some respect in which any two objects whatever agree. ([43]p.44)

Frege goes on to say that many different philosophers and mathematicians, like Hobbes, Hume, and Thomae, declare all units to be equal without proper justification. Thomae holds that every member of a set is a unit and that all units are identical. This would imply that every member of the set is identical to every other member, but if this is the case how can we possibly distinguish between different members of the set? Thomae wants us to *disregard, in considering separate things, those characteristics which serve to distinguish them* ([43]p.45). However, even if we disregard those things that differentiate them they will still remain. Us not considering them does not make them magically disappear and turn every member of a group into an exact copy of one another.

> We cannot succeed in making different things identical simply by dint of operations with concepts. But even if we did, we should then no longer have things in the plural, but only one thing; for, as Descartes says, the number (or better, the plurality) in things arises from their diversity. ([43]p.46)

This then would lead us to the conclusion that there must be different units. Any two objects must be different in at least one aspect since otherwise they would not be two objects but one object. There must be some way to distinguish different objects otherwise plurality would be impossible. Thus designating one thing as "the" unit would imply that every other thing that is different in at least one aspect is not a unit and thus not "one". However, Frege goes on to demonstrate that the idea of there being multiple units also has its own problems. To do this Frege uses the view of Jevons who holds that:

> Whenever I use the symbol 5 I really mean $1 + 1 + 1 + 1 + 1$ and it is perfectly understood that each of these units is distinct from each other. If requisite I might mark them thus $1' + 1'' + 1''' + 1'''' + 1'''''$. ([43]p.47)

If we were to use the method used by Jevons then we would have to always distinguish each number in terms of the units that were used for it. If we want to write $3 + 5$ we first have to identify eight different ones and clarify which belong to 3 and which belong to 5. At the same time, we could never just write $1 = 1$ since that equation would be lacking in context. This way of thinking is pointless according to Frege:

> The symbols $1', 1'', 1'''$ tell the tale of our embarrassment. We must have identity - hence the 1; but we must have difference - hence the strokes; only unfortunately, the latter undo the work of the former. ([43]p.48)

Then it would seem we have reached a dead end. We cannot take all units as identical since then we would not be able to make a plurality but at the same time, we cannot differentiate between each of them since that would defeat the whole purpose of having a unit in the first place. To overcome this problem Frege proposes we do one crucial thing. Of all of the mathematicians whose viewpoints we have seen not one of them distinguished between the unit and one. All of them considered them to be the same idea and used them interchangeably. Frege suggests that we should instead differentiate between them.

> However, if confusion is not to become worse confounded, it is advisable to observe a strict distinction between unit and one. When we speak of "the number one", we indicate by means of the definite article a definite and unique object of scientific study. There are not divers numbers one but only one.([43]p.49)

In doing this we can think of all 1 in arithmetic as being equal thus we would not need to write $1' + 1'' + 1'''$, as Jevons believes. This would also mean that numbers are not agglomerations of things but individual things by themselves.

> Only concept words can form a plural. If, therefore, we speak of "units', we must be using the word not as equivalent to the proper name "one", but as a concept word. ([43]p.50)

Then we are left to define the unit different from the number one which is not an easy task. Though we now have a definition of the number one which we can use for arithmetic, for the definition of unit we are stuck in a similar place as before. If we define it using objects we cannot use the unit when defining numbers but if we define it using the number one we can never get a plurality. This brings us to the next part of this section, which Frege calls *Attempts to overcome the difficulty.*

> The first suggestion is to all for assistance on a certain property of time and space, as follows. One point of space, considered by itself, is absolutely indistinguishable from another, and so are a straight line, or a plane, or one of a number of congruent bodies or areas or line-segments: they are distinguishable only when conjoined as elements in a single total intuition. ([43]p.51)

This seems like a promising place to start since this approach seems to have the two things we want from a unit, being identical and being distinguishable. Hobbes and Thomae believe that this is the best possible solution but Leibniz, Baumann, and Jevons disagree. Thomae says that when we number different points in space then we automatically discriminate between them based on the order in which we number them. Individually they may not possess distinguishable features but nonetheless we can distinguish between them based on their location or the time when we number them. Jevons on the other hand says that: *Three coins are three coins, whether we count them successively or regard them all simultaneously* ([43]p.52). Frege leans towards the side of Jevons and states:

> I would add that, if the objects numbered do not follow one after another in actual fact, but it is only that they are numbered one after another, then time cannot be ground of discrimination between them. ([43]p.52)

Frege then goes on the point out that even if we use time and space as a differentiating factor we still have objects that are not truly identical to one another. Therefore we are still not able to identify a unit. Changing the way we distinguish between objects does not help us get closer to a definition for a unit. For a similar reason we cannot use a generalized concept of series to differentiate between objects. For there to be a generalized series there must already be a distinction between them. Frege then moves on to the concept of number. providing us with the definition of Schröder that *A natural number is a sum of ones* ([43]p.55). But rejecting it as merely giving an explanation for the representation of $A = 1 + ... + 1$, Frege then quotes Jevons:

> When I speak of *three men* I need not at once specify the marks by which each may be known from each. Those marks must exist if they are really three men and not one and the same, and in speaking of them as many I imply the existence of the requisite differences. Abstract number, then, is *the empty form of difference.* ([43]p.56)

Frege also rejects this idea for several reasons. First, he states that this method can never get us to numbers such as 10000 since we are incapable of grasping so many differences. Secondly, the idea behind *the empty form of difference* is very vague and unclear. Thirdly, Frege argues that by simply abstracting from objects we can never get to the numbers 1 and 0. Since speaking of 1 or 0 men does not by itself imply the existence of differentiating marks between the objects we are speaking of since there are no other objects to differentiate between. And lastly, Frege states that this idea does not remove the problems with distinguishing between units in the way of $1' + 1'' + 1'''$.

We then reach the last part of this section: *Solution to the difficulty.* Frege starts this part by summarizing the question we have yet to answer. Firstly we have yet to determine what exactly we mean when we use numbers to assert something about an object. We have already determined that number is not a physical property that something has like color or weight but at the same time it is also not something that is subjective to the mind of each person. And although number is not subjective, if we change our perspective of an object we can change the number we associate with it. This leads to the second unanswered question:

> How are we to curb the arbitrariness of our ways of regarding things, which threatens to obliterate every distinction between one and many? ([43]p.58)

We have already made the distinction between the “one”, which is an object of mathematical study. The one we defined for this does not make a plural when added together with another one, which then means that the mathematical notation $1 + 1 = 2$ does not mean putting together two ones. The 2 in this case not a collection of ones but an object of mathematical study in its own right.

As for the first unanswered question of this part, Frege states that when we use numbers to describe something we are in fact not describing something about the objects themselves. The difference between *four companies* and *500 men* is not any individual object or any agglomeration of objects. Rather the terminology we use has changed, we have substituted the concept of *companies* for that of *men*. Therefore numbers do not ascribe something to an object but rather to a concept.

> This is perhaps clearest with the number 0. If I say “Venus has 0 moons”, there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the *concept* “moon of Venus”, namely that of including nothing under it. ([43]p.59)

Frege proceeds to defend his viewpoint against certain critiques someone might give against them. Someone might say that concepts can change over time and thus the numbers associated with them are also not rigid. Frege says that objects themselves also change constantly but this does not prevent us from recognizing two objects as the same concept. Further, he argues that when we use numbers to assert something about a concept we are asserting about the concept as it is at the time of the assertion. Therefore, we can always add the time at which an assertion is made to it to make it unchangeable for all eternity. For example, if we state *Germany has X inhabitants* then the number X can vary over time but *Germany has X inhabitants at new year 1900* is the same for all of time. ([43]p.59)

Another argument one could make against Frege’s idea is that concepts are subjective. Therefore, anything that asserts something about a concept must also be subjective, but we already established that numbers are not subjective. Frege in turn argues that the view that concepts are subjective itself is mistaken.

> If, for example, we bring the concept of body under that of what has weight, or the concept of whale under that of mammal, we are asserting something objective; but if the concept themselves were subjective, the the subordination of one to the other, being a relation between them, would be subjective too, just as the relation between ideas is. ([43]p.60)

Frege also says that in seeing Number as something attached to concepts rather than objects we also avoid the problem of being able to assign multiple numbers to one object. When we change our perspective from “book” to “pages” we can change the number assigned to it without changing the object. However, when we change the number we assign to the object we also change the concept we assign to it. We are assigning the number 1 to “book” and assigning the number 100 to “pages”. This also explains why numbers can assigned to anything we can conceive and anything we can observe, and also why any group of things can be assigned a number regardless of what those things are. Anything can be classified under an infinite number of concepts which we can define however we please. ([43]p.61)

> Corroboration for our view is to be found in Spinoza, where he writes: “I answer that a thing is called one or single simply with respect to its existence, and not with respect to its essence; for we only think of things in terms of number after they have first be reduced to a common genus.” ([43]p.62)

With this established we are now also able to find a definition of the unit. To start Frege quotes Schröder:

> This generic name or concept will be called the denomination of the number formed by the method given, and constitutes, in effect, what is meant by its unit. ([43]p.66)

Frege finds this a very apt suggestion and resolves to do just that. Frege defines a concept as a unit relative to the number which belongs to it. This works because the concept in general will isolate the things that fall under it from other things. *The concept “letters in the word three” isolates the t from the h, the h from the r, and so on* ([43]p.66). Frege himself also acknowledges that this does not always work. As an example, Frege says that things falling under the term “red” can be indefinitely split up whilst still remaining “red”. However, Frege argues that a concept as broad as “red” could never have a finite number ascribed to it in the first place and it would therefore never have been possible to find a unit for it in the first place. Frege summirizes it as:

> Only a concept which isolates what falls under it in a definite manner, and which does not permit any arbitrary division of it into parts, can be a unit relative to a finite Number. ([43]p.66)

> In the Proposition “Jupiter has four moons”, the unit is “moon of Jupiter’. Under this concept falls moon I, and likewise also moon II, and moon III too, and finally moon IV. Thus we can say: the unit to which I relates is identical with the unit to which II relates, and so on. This gives us our identity. But when we assert the distinguishability of units, we mean that the things numbered are distinguishable. ([43]p.67)

6.1.4 The concept of Number.

This section is divided into:

- Every individual number is a self-subsistent object.
- To obtain the concept of Number, we must fix the sense of a numerical identity.
- Our definition completed and its worth proved.
- Infinite Numbers.

We have established that numbers assert things about concepts, but this only tells us what numbers do and not what they are. This is what Frege wants to address in this section. If we want to define every number in its own right it is only logical that we start with the numbers 0 and 1. Starting with 0, Frege proposes to define it as:

> the number 0 belongs to a concept, if the proposition that *a* does not fall under that concept is true universally, whatever *a* may be. ([43]p.67)

Frege defines 0 in a way that is similar to how someone might define a property that could belong to an object. It is similar to how one might define a word like "heavy" or "yellow", which is not surprising. So mathematicians have tried to define numbers as a sort of immaterial object which has never fully worked because that is not what numbers are. We have a good understanding of the word "heavy" despite not being able to form a mental image of heavy by itself because it is not something that can exist by itself. For "yellow" it is a bit different since we can picture the color yellow in our minds but this would be the same as if we were staring at a big yellow wall in front of us. It is not truly yellow as an object by itself. The same holds for numbers. When we try to find an image of the number 3 we either think of the symbol 3 we use to represent it or of three objects, in other words, a concept with the property 3. Therefore the only logical way forward is to define numbers as properties of concepts by stating what a concept has to adhere to have have the property of that number.

> Similarly we could say: the number 1 belongs to *F*, if the proposition that *a* does not fall under *F* is not true universally, whatever *a* may be, and if from the propositions "*a* falls under *F*" and "*b* falls under *F*" it follows that *a* and *b* are the same. ([43]p.67)

Despite these two definitions seeming like logical conclusions to the steps we have taken so far, Frege still does not believe these definitions to be satisfactory to our ultimate goal. For as much as it seems that these are two definitions of 0 and 1 Frege declares that they are not exactly:

> In reality we have only fixed the sense of the phrases "the number 0 belongs to" and "the number 1 belongs to"; but we have no authority to pick out the 0 and 1 here as self-subsistent objects that can be recognized as the same again. ([43]p.68)

What Frege seems to imply here is that if we, for example, have two concepts that both have no objects that fall under them, thus having the number 0, we cannot say that those 0 that belong to each concept are the same. I think that a good example of this would be if you had two flowers, both red, but one has a slightly lighter color than the other. Both of the objects would fall under the concept of "red flower" and are thus "red", but at the same time if you asked someone if they are the same color they could very reasonably say no.

> It is time to get a clearer view of what we mean by our expression "the content of a statement of number is an assertion about a concept". ([43]p.68)

Frege goes on to state that thinking of numbers as properties is not the correct way to go about defining them. Because in doing so we can't talk about numbers as objects by themselves. Despite the way we use numbers in everyday language making it seem like they are properties of other things this is merely because of the way we word things. Frege demonstrates this by taking the sentence *Jupiter has four moons* ([43]p.69) and fully deconstructing it into the sentence: *The number of Jupiter's moons is the number four, or* 4 ([43]p.69). Both sentences have the same meaning but in one sentence *four* appears to be a mere property but in the other one *the number four* is more clearly an object itself.

> A possible criticism is, that we are not able to form of this object which are calling Four or the Number of Jupiter's moons any sort of idea at all which would make it something self-subsistent. ([43]p.69)

Frege says that while it seems we can use external things to form ideas about numbers, like the four dots on a die giving us insight into the number four, this is not the case.

> We can form no idea of the number either as a self-subsistent object or as a property in an external thing, because it is not in fact either anything sensible or a property of an external thing. ([43]p.70)

Frege also states that not being able to form an exact mental image of something does not mean our understanding of it is wrong. Frege uses examples like the earth and the sun which are so big no one can form a proper mental picture of them. At best we get two balls with the same aproximate ratio to one another. Yet at the same time, we are able to calculate the distance between them without a perfect mental image. Those calculations are not wrong and we can use them to gain a further understanding of the earth and sun. But the sun and the earth are physical objects, even if we can't fully imagine them we can see them, which is not the case for numbers. Frege says that despite numbers not being spatial objects this does not mean that they are not objects in their right.

> Yet even granted that what is subjective has no position in space, how is it possible for the number 4, which is objective, not to be anywhere? Now I contend that there is no contradiction in this whatever. It is in fact that the number 4 is exactly the same for everyone who deals with it; but that has nothing to do with being spatial. Not evey objective object has a place. ([43]p.72)

With this we have reached the next part *To obtain the concept of Number, we must fix the sense of numerical identity.*

> How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. ([43]p.73)

By doing this Frege wants to get around the problem with defining a number in the same way one would define a property. If we don't define it as a property but define in which context, it can be used in a similar way to a property. That way the numbers themselves are still objects in their own right and we can declare that the "four" used in "four apples" and "four houses" is the same since we have used "four" in two different situations in which it can be used instead of finding two situations in which the word "four" is used. Frege describes it as that we want to be able to say "*The number which belongs to the concept F is the same as the which belongs to the concept G*" ([43]p.73); and to be able to do that we must avoid the expression *the Number which belongs to the concept F* ([43]p.73). But this also raises the question of how we can be certain that the number belonging to two concepts is in fact the same number.

> Hume long ago mentioned such a means: "When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal." ([43]p.73)

Frege is not entirely convinced by this method, mostly because numbers are not the only things that have the relationship of identity. Other things besides numbers can also be identical to each other, so why is there a need to give a specific definition for numbers? Frege proposes to first find the criteria that determine *the concept of identity* and together with the *concept of Number* we can find the way to determine when two numbers are identical. Frege then puts forth the definition of identity as given by Leibniz:

> Things are the same as each other, of which one can be substituted for the other without loss of truth. ([43]p.76)

Frege takes this as his own definition of identity. Although Leibniz's definition uses the words *the same* instead of *identical*, they are interchangeable. According to Frege universal substitutability contains all the laws of identity. This makes sense, if two objects are the same in every possible context then there is no possible way to distinguish between them, therefore they have to be identical. However, Frege still has some doubts about this definition that he wants to clear up. As an example, Frege gives us two parallel lines a and b. We can substitute *the direction of* a everywhere for *the direction of* b because they are parallel. In this case, since we already knew that their directions were identical showing that we can substitute them is quite easy. But what if we want to compare *the direction of* a with something that is not a direction, say q? How can we say for sure *the direction of line* a *is identical to* q? Frege tries the idea that we can say that q is a direction if there exists a line b with direction q, but then we already have to know that *q is identical to the direction of line b*, which leads us in a circle. Another idea is that we could only consider the question if q is introduced as the direction of line b, but then we would only be considering q as a property and not as an object. If this definition does not work for lines it will surely not work for numbers, so Frege proposes to try a different approach: ([43]p.77-79)

> If line a is parallel to line b, then the extension of the concept "line parallel to line a" is identical with the extension of the concept "line parallel to line b"; and conversely, the the extensions of the two concepts just named are identical, then a is parallel to b. ([43]p.79)

Frege then says that if we substitute the lines used here for concepts and parallelism the one to one correlation of the objects that fall under those concepts we can apply this method to Number. Frege clarifies that he will use the word *equal* when this condition is satisfied but makes sure that we understand that this is merely an arbitrarily selected symbol. Then we get Frege's definition for Number:

> The number which belongs to the concept F is the extension of the concept "equal to the concept F".([43]p.80)

That numbers can be thought of as extensions of concepts might not be immediately obvious. Frege shows this by tackling two conceptions we have about numbers: *that they are identical* and *that one is wider than the other* ([43]p.80). Frege says that:

> the proposition: the extension of the concept "equal to the concept F" is identical with the extension of the concept "equal to the concept G", is only true if and only if the proposition: "the same number belongs to the concept F as to the concept G" is also true. ([43]p.80)

The *if and only if* relation between these two statements implies that they can be substituted for one another without loss of truth and thus they are equivalent. As for the idea that *one is wider than the other*, Frege says that while we say that one extension of a concept might be wider then another extension we do not say this about numbers. However, Frege also says that it is impossible for there to be a case where if, *the same number belongs to the concept F as to the concept G*, that the extension of the concept "equal to the concept G" would be wider than the extension of concept "equal to the concept F".

> when all concepts equal to G are also equal to F, then conversely also all concepts equal to F are equal to G. ([43]p.81)

This then brings us to the next part of this section: *Our definition completed and its worth proved*. In this part, Frege wants to show us that we can use his definition of numbers to find well-known properties of numbers. To start Frege wants to give a more precise meaning of the term *equality*. We have already described it as a one-one correlation but Frege wants to give a more detailed account of what we are to understand with this relation.

> In the proposition "a lies immediately to the right of A" we conceive first one and then another object inserted in place of a and again of A, then that part of the content which remains unaltered throughout this process constitutes the essence of the relation. ([43]p.82)

Frege refers to the proposition *a lies immediately to the right of* A as a *judgment-content*. In this judgment-content we have the objects a and A, if we remove a and A then what we are left with is what Frege calls a *relation-concept*. A relation-concept by itself doesn't assert anything but can be completed in multiple different ways to form a judgment-content that does assert something. Using these ideas Frege wants to find the logical form the the equality relation:

> Just as "a fall under the concept F"is the general form of a judgment-content which deals with an object a, so we can take "a stands in the relationship ϕ to b" as the general form of a judgment-content which deals an object a and an object b. ([43]p.83)

> If now every object which falls under the concept F stands in the relation ϕ to an object falling under the concept G, and if to every object which falls under G there stands in the relation ϕ an object falling under F, then the objects falling under F and under G are correlated with each other by the relation ϕ. ([43]p.83)

But what if there is no object falling under F? Frege explains that every a under F having relation ϕ to all objects under G can be deconstructed as there being no a for which *a falls under* F and *a does not stand in the relation* ϕ *to any object falling under* G are true. Now with this, we can try to substitute the one-one relation for ϕ to get a definition for equality. To do this we must first get an idea of what we want the one-one relation to do. Frege believes it should do two things: if in the relation ϕ d stands to a and d stands to e then a should be the same as e, and vice versa, if d stands to a and b stands to a then d and b should be the same. All regardless of what a, b, d, e are.

With this Frege has now established that *the concept* F *is equal to the concept* G is to mean:

> There exists a relation ϕ which correlates one to one the objects falling under the concept F with the objects falling under the concept G. ([43]p.85)

And he takes n *is a Number* to mean:

> There exists a concept such that n is the Number which belongs to it. ([43]p.85)

Here the number that belongs to a concept is defined as *The extension of the concept "equal to the concept* F*"*. With this, we have finally defined a Number and we can use this definition to show that if the numbers that belong to the concepts F and G are identical then F and G are equal. To do this we must show that if we have three concepts F, G, and H, then if H is equal to F it is also equal to G, and of course vice versa.

We already had ϕ as a one-to-one relationship between F and G; now let ψ do the same from H to F. Then what we want to find is the judgment-content for $H\psi F\phi G$. Frege says that this judgment-content can be given as:

> There exists an object to which c stands in the relation ψ and which stands to b in the relation ϕ. ([43]p.86)

If we take this as a relation-concept we can see that it is one-one and we can give a similar proof for the vice versa account. With this established Frege moves on to the definitions of individual numbers. 0 being the most logical place to start, Frede defines it as:

> 0 is the Number which belongs to the concept "not identical with itself".([43]p.87)

Because every object is identical to itself there is no object which falls under this concept. The reason that Frege chose to use this concept for the definition of 0 is because it can be shown that no object falls under it purely by using logic. Frege then shows that any concept under which no object falls is equivalent to any other concept under which no object falls and that the 0 he defined is thus the number belonging to all of those concepts. Now if F and G are such concepts then we want to find a relation ϕ that all objects under F in relation to an object under G and the other way around. But since F and G are both empty, any relation ϕ will do that and thus also the one-one relation.

Now that we have a proper definition of 0 it makes sense to move on to 1. But there are two ways to do this. We can either try to find a concept that has the number 1 belonging to it, or we can try to find a way to define the action +1 and then apply this to 0. If we choose the latter then when we move from 1 to 2 we already have an easy method of doing so. It is no surprise then that this is also what Frege does, at least he sort of does that. What Frege wants to do is define the relation between two adjacent natural numbers:

> There exists a concept F, and an object falling under it x, such that the Number which belongs to the concept F is n and the Number which belongs to the concept 'falling under F but not identical with x'is m.([43]p.89)

Frege takes this proposition to mean the same as *n follows in the series of natural numbers directly after m.* Now Frege explains how we use this proposition to get from 0 to 1. First, we start with the concept *identical to* 0, which of course has the number 0 falling under it. However, because it has the number 0 falling under it the number belonging to this concept is not 0. Now if we then add this concept as the F in our proposition and the number 0 as our x we get: *identical with* 0 *but not identical with* 0. Because the first and second requirements contradict each other there is no object that falls under this concept and thus the number belonging to it is 0. Now with our proposition, we know that the number belonging to *identical with* 0 follows in the natural numbers directly after 0 so if we define 1 as the number belonging to the concept *identical to* 0 we can conclude that: 1 *follows in the series of natural number directly after* 0.

Frege then states several truths that follow easily from this definition of 1. For example, if we take a as the number after 0 it always follows that $a = 1$. Also, if 1 is the number belonging to F then there is an object x that falls under F. If there is also an y that falls under F it follows $x = y$, similarly if for all x and y under F we have $x = y$ then the number belonging to F is 1.

Now one might expect that with this we can fully define all the other natural numbers but there is one question we still have to answer. While we know what it means for n to come after m how do we know that such an n exists? We need to prove that after every natural number there follows another number. The way to do this is to find the concept F to which this number n belongs since if we know that there exists a concept to which n belongs then n is a number. To do this Frege first puts forward the concept: *member of the series of natural numbers ending with* n. Frege then gives us the definition of *following in a series*:

> If every object to which x stands in the relation ϕ falls under the concept F, and if from the proposition that d falls under the concept F it follows universally, whatever d may be, that every object to which d stands in the relation ϕ falls under the concept F, then y falls under the concept F, whatever F may be. ([43]p.92)

Frege uses this to mean y *follows in the* ϕ*-series after* x and x *comes in the* ϕ*-series after* y. The way Frege chose to word this might make it a bit hard to follow exactly what he means so I will break it down to make it easier to follow. In total there are two things that must be true, the first is: *every object to which* x *stands in* ϕ *falls under* F. This one is easy to understand but it is the second one that is a bit more tricky. We need to have that the proposition d *falls under* F implies universally that (*every object to which* d *stands in* ϕ *falls under* F implies that y *falls under* F). So we have a statement that implies an implication, having two implications directly following each other can make it confusing but writing it down this way makes it a lot clearer to follow. Frege admits that this is not the most intuitive way of defining that an object follows another in a series but argues that defining it this way makes it a matter of fact. He gives the example of a definition in which we *transfer our attention continually from one object to another* ([43]p.92). Frege says that while this definition might be more natural it also relies on our own senses and therefore would not work unless we observe to result or perform the action.

Now Frege uses this definition of *following in the phi-series* on the proposition of n *follows in the series of natural numbers directly after* m. To do this Frege proposes a further definition of y *follows in the* ϕ*-series after* x *or* y *is the same as* x ([43]p.94). This means the same as both y *is a member of the* ϕ*-series beginning with* x and x *is a member of the* ϕ*-series ending with* y. We can now use these to show that the number belonging to the concept *member of the series of natural numbers ending with* n follows directly after n. However, Frege does not provide us with the full proof of this since according to him *To give the proof in full here would take us too far afield.* ([43]p.94)

To conclude this section we now arrive at the part *infinite numbers*. The section itself is no more than three pages long and it mostly consists of Frege giving his thoughts on the recent works of Cantor regarding infinite numbers. Frege himself does define an infinite number ∞_1 as the number belonging to the concept *finite Number*.

> About the infinite number ∞_1 so defined there is nothing mysterious or wonderful. "The number which belongs to the concept of F is ∞_1" means no more and no less than this: that there exists a relation which correlates one to one the objects falling under the concept F with the finite numbers. ([43]p.96)

From this, it becomes quite clear that Frege himself has very little interest in exploring the arithmetical laws that govern the infinite numbers. However, Frege does speak highly of the recent work of Cantor and says: *I hearthily share his contempt for the view that in principle only finite numbers ought to be admitted as actual* ([43]p.97). While the works of Cantor are certainly also interesting and I would like to analyze them at some point I will not do so now.

6.1.5 Conclusion.

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgments and consequently a priori. Arithmetic thus becomes simply the development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one. ([43]p.99)

Frege tried his best to define numbers in a logical way and I believe that he supports his views very well. However when the Russell's paradox was put forth by Bertrand Russell it would show that Frege's ideas did not work. Despite this, I still believe *Foundations of Arithmetic* is still a very interesting read. The way Frege breaks down why other definitions of numbers don't work and the way he supports his own viewpoints makes you think about your own idea of what a number really is. Even if we was eventually proven to be incorrect I still believe *Foundations of Arithmetic* is a good book for any mathematician willing to delve deeper into the philosophy behind mathematics.

7 Conclusion

We started by looking at the origins of mathematics in ancient Mesopotamia. While we didn't find any definitions as they would have been used by the people of Mesopotamia we still saw that they already had a remarkable knowledge of mathematics. They were able to do calculations with numbers of any magnitude and had a great understanding of geometry. They were even aware of things like the theorem of Pythagoras and had entire tables dedicated to Pythagorean triples.

In ancient Greece, we saw mathematics continue to develop with the help of great mathematicians like Thales, Pythagoras, and Euclid. The last of which also provided us with probably the most influential mathematical work of all time: *Elements*. These great mathematicians also gave us our first concrete definition to examine. Even if these definitions do not hold up to today's standards they still give us insight into the thought processes of mathematicians thousands of years ago and allow us to see how mathematics has changed since then.

In ancient China, we find another great mathematical work in *the Nine Chapters*. This also gives us one of the first instances of negative numbers being used in mathematics. While in ancient China we do not find definitions to the same extent as we did in ancient Greece it was still interesting to see how mathematics was practiced differently in these two regions of the world.

And finally in the modern developments, we see Frege's *Foundations of Arithmetic*. In this work we see Frege explore what numbers truly are and not just try to give a definition of them that we can work with. It was really interesting to see how even definitions of "a number" that seemed perfectly fine at first glance run into problems when analyzed fully. Most interesting of all was the definition of Number he eventually arrived at.

Analyzing definitions like this really gave me a new perspective on mathematics as a whole. We are used to thinking about what definitions imply and what we can conclude from them given certain information. When doing mathematics I always believed we were looking forward, trying to find new answers to questions that arose when we found other answers. But in this instance, we are looking backward, examining the conclusions we have already drawn and looking at our initial starting points. And when we get to something at the very heart of mathematics like numbers and just ask the question: *What are they?*, we also get to a discussion that involves a great deal of philosophy. When we have nothing to go on but our own pure reasoning we really put our understanding of mathematics as a whole to the test and I hope to be able to examine this subject further in the future.

References

[1] Madiha Baharuddin. *The book Al-Jabr*. University of Malaya, 2014.

[2] Nuel Belnap. "On Rigorous Definitions". In: *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition 72(2)* (1993).

[3] Ian Bruce. *Euler's Introductio in Analysin Infinitorum*. https://www.17centurymaths.com/contents/introductiontoanalysisvol1.htm. 2013.

[4] Bill Casselman. *The Babylonian tablet Plimpton 322*. https://personal.math.ubc.ca/~cass/courses/m446-03/pl322/pl322.html.

[5] Carlo Celluci. "Definition in mathematics". In: *philosophy of mathematics* (2018).

[6] Karine Chemla. *Reading instructions of the past, classifying them, and reclassifying them: commentaries on the canon The Nine Chapters on Mathematical Procedures from the third to the thirteenth centuries*. Published online by the Cambridge University press. 2020.

[7] Arthur A. Clarke. *Disquisitiones Arithmeticae*. Yale University, 1966.

[8] Jeremy Cripps. *Summa De Arithmetica, Geometria, Proportioni Et Proportionalita*. https://archive.org/details/summa-de-arithmetica-geometria-proportioni-et-proportionalita/mode/1up. 1994.

[9] Christopher Cullen. *Astronomy and Mathematics in Ancient China: The "Zhou bi suan jing"*. Cambridge University, 1999.

[10] Nicolas de Cusa. *Doctissimi Viri et Mathe*. https://archive.org/details/ARes352131/page/n9/mode/2up. 1533.

[11] Mary Domski. *Descartes' Mathematics*. https://plato.stanford.edu/entries/descartes-mathematics/#BooOneDesGeoAna. 2011.

[12] Richard Fitzpatrick. *Euclid's Elements of Geometry*. Richard Fitzpatrick, 2008.

[13] Carlos Gonçalves. *Cuneiform Mathematics: Outstanding Techniques in a System of Texts*. https://www.asor.org/anetoday/2023/02/cuneiform-mathematics. 2023.

[14] A.C. Graham. *Later Mohist Logic Ethics and Science*. Chinese University Press, 1978.

[15] Jeremy John Gray. *Carl Friedrich Gauss*. https://www.britannica.com/biography/Carl-Friedrich-Gauss. 2024.

[16] Thomas Heath. *A History of Greek Mathematics*. Humphrey Milford, 1921.

[17] Thomas Heath. *The Thirteen Books of Euclid's Elements. Vol I*. Cambridge University Press, 1908.

[18] Thomas Heath. *The Thirteen Books of Euclid's Elements. Vol II*. Cambridge University Press, 1908.

[19] Jens hoyrup. *BM 13901 1.* https://math.libretexts.org/Bookshelves/Applied_Mathematics.

[20] Carl Huffman. *Pythagoras.* https://plato.stanford.edu/entries/pythagoras/. 2005.

[21] Carl Huffman. *Pythagoreanism.* https://plato.stanford.edu/entries/pythagoreanism/. 2006.

[22] Nikolaos L. Kechris. *Book of Lemmas.* https://archive.org/details/enbibliolimmaton/page/n1/mode/2up. 2018.

[23] David Eugene Smith Marcia L. Latham. *The Geometry of René Descartes.* The Open Court Publishing Company, 1925.

[24] Daniel F. Mansfield. *Plimpton 322: A Study of Rectangles.* Foundations of Science, 2021.

[25] Robin Wilson Marlox Anderson Victor Katz. *Sherlock Holmes in Babylon.* The Mathematical Association of America, 2004.

[26] Jean-Claude Martzloff. *A History of Chinese Mathematics.* Masson, 1987.

[27] Bahman Mehri. *From Al-Khwarizmi to Algorithm.* Sharif Univeristy of Technology, 2017.

[28] Uta C. Merzbach and Carl B. Boyer. *A History of Mathematics 3rded.* John Wiley & Sons, Inc., Hokoben, New Jersey, 2011.

[29] *Mesopotamian Tablets search site.* https://www.spurlock.illinois.edu/collections/search-collection/index.php?advsf=1&n=cuneiform+tablet&a=&d=1&g=All&cl=&mt=&cr=&g1=All&g2=&g3=&gs=&g4=&g5=&c1=All&c2=All&c3=All&Search=Search.

[30] Glenn R. Morrow. *Proclus, A Commentray on the First Book of Euclid's Elements.* Princeton University Press, 1970.

[31] Rik Brandenburg Keimpe Nevenzeel. *The Nine Chapters on the History of Chinese Mathematics.* University of Groningen, 2007.

[32] Patricia O'Grady. *Thales of Miletus.* https://iep.utm.edu/thales/.

[33] Luca Pacioli. *Summa De Arithmetica, Geometria, Proportioni Et Proportionalita.* https://books.google.nl/books?id=iqgPe49fhrsC&pg=RA1-PA30&hl=nl&source=gbs_selected_pages&cad=1#v=onepage&q&f=false. 1494.

[34] Eleanor Robson. *Mesopotamian Mathematics.*

[35] Frederic Rosen. *The Algebra of Mohammed Ben Musa.* The Oriental Translation Fund, 1831.

[36] W.D. Ross. *Aristotle's Metaphysics.* NuVision Publications Publisher, 2009.

[37] Laurence Sigler. *Fibonacci's Liber Abaci.* Springer Science + Business Media, 2003.

[38] A.J.E.M. Smeur. *De Thiende Simon Stevin.* De Graaf, 1965.

[39] Ira Spar. *The Origins of Writing*. https://www.metmuseum.org/toah/hd/wrtg/hd_wrtg.htm. 2004.

[40] Michael Stifel. *Arithemtica Integra*. https://archive.org/details/bub_gb_fndPsRv08R0C/page/n15/mode/2up. 1544.

[41] Heinz Klaus Strick. *Abu'l-Wafa Al-Buzjani*. Spektrum Der Wissenschaft Verlagsgesellschaft Heidelberg, 2014.

[42] Heinz Klaus Strick. *Mohammed Al Khwarizmi*. Spektrum der Wissenschaft Verlagsgesellschaft Heidelberg, 2009.

[43] Gottlob Frege Translation J.L.Austin. *The Foundations of Arithemtic*. Harper Torchbooks, 1960.

[44] T. Richard Witmer. *Ars Magna or the Rules of Algebra*. Dover Publications, 1968.