QUANTIZATION AND SUPERSELECTION SECTORS I. TRANSFORMATION GROUP C^*-ALGEBRAS

N. P. LANDSMAN

Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, NL-1018 XE Amsterdam, The Netherlands and *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom*[1]

Received 5 March 1990
Revised 12 April 1990

Quantization is defined as the act of assigning an appropriate C^*-algebra $\mathscr{A}$ to a given configuration space Q, along with a prescription mapping self-adjoint elements of $\mathscr{A}$ into physically interpretable observables. This procedure is adopted to solve the problem of quantizing a particle moving on a homogeneous locally compact configuration space $Q = G/H$. Here $\mathscr{A}$ is chosen to be the transformation group C^*-algebra corresponding to the canonical action of G on Q. The structure of these algebras and their representations are examined in some detail. Inequivalent quantizations are identified with inequivalent irreducible representations of the C^*-algebra corresponding to the system, hence with its superselection sectors.

Introducing the concept of a pre-Hamiltonian, we construct a large class of G-invariant time-evolutions on these algebras, and find the Hamiltonians implementing these time-evolutions in each irreducible representation of $\mathscr{A}$. "Topological" terms in the Hamiltonian (or the corresponding action) turn out to be representation-dependent, and are automatically induced by the quantization procedure. Known "topological" charge quantization or periodicity conditions are then identically satisfied as a consequence of the representation theory of $\mathscr{A}$.

1. Introduction

1.1. *The stage*

The problem considered in this paper is the quantization of a particle moving on a homogeneous configuration space $Q = G/H$. For example, Q might be a line, a circle, a sphere, three-dimensional Euclidean space, etc. Physical systems of this kind are very interesting, because, in their simplicity, they may exhibit so-called "topological quantum effects" in case that Q and G are suitably chosen. Well-known examples of such effects, which will be studied in detail in the sequel to this paper, are the Aharonov-Bohm effect and the Dirac magnetic monopole charge quantization, which arise if one deletes a line or a point from $\mathbf{R}^3$, respectively. Phenomena of this kind are usually associated with what physicists call "topologically non-trivial" configuration spaces (which means that Q is not homeomorphic to some Euclidean space), and the examples mentioned fit well into this idea, but actually such phenomena may occur for any configuration space. For example, the existence of spin is an example of an effect whose

[1] Present address; supported by SERC

Reviews in Mathematical Physics Volume 2 No 1 (1990) 45–72

mathematical description (or, at least the one given here) is completely analogous to that of the two effects mentioned earlier, while its arena is just $\mathbf{R}^3$ with its usual topology. As we shall see, such phenomena are bound to occur whenever the little group H is non-trivial.

Our restriction to locally compact homogeneous configuration spaces is, unfortunately, necessitated by the fact that the quantization scheme we are going to offer in its present formulation only works for such spaces; this restriction excludes field theories from the present treatment as well. Both restrictions may eventually be removed by employing the formalism of continuous tensor products of C^*-algebras, but we leave this development to a future paper. We will actually impose still further restrictions on the pair G, Q, to be detailed in 3.1 below, but these are of a very technical nature, and are easily met by all obvious physical examples.

Another limitation is that we start from the formulation of a physical system in terms of its configuration space Q; its phase space T^*Q (in case that Q is a manifold) does not enter the discussion. Yet physical systems exist which are directly formulated in terms of their phase-space variables, and which do not admit an underlying configuration space. In spite of these limitations to the applicability of the method described here, a wide range of realistic systems remains for which the technique *is* applicable, some of which are difficult to handle with other methods (e.g., the case in which Q is neither a manifold nor a vector space).

1.2. *Quantization and C^*-algebras*

Origins

Our present approach to the quantization problem has been inspired by, and, indeed, is mathematically modeled on the foundational work in quantum (field) theory due to Segal [36], Haag-Kastler [22], and Araki [1]. Among other things, these authors drew attention to some limitations of the conventional Hilbert space formalism, useful as it may be in practical calculations. For one thing, the phenomenon of superselection sectors [41] is incorporated in an *ad hoc* way, and another point is that quite often the set of physically realizable states of a physical system is simply not exhausted by the set of density matrices in a given Hilbert space representation. Instead, these authors proposed that all relevant physical information ought to be contained in certain abstract operator algebras $\mathscr{A}$ generated by the observables of the system. This approach is vindicated *a priori* by a deep study of the algebraic structure of quantum mechanics; it is remarkable that, the main emphasis in the eventual C^*-algebraic formulation of quantum mechanics being on abstract operator algebras, its mathematical origin is actually tied to the algebraic structure of the state space. For a comprehensive discussion of these issues, we refer to [5, 21] and refs. therein, the first of which also contains extensive historical notes.

For physical as well as mathematical reasons, which are explained in [36, 22, 5, 21], it is at least highly convenient, and possibly mandatory, to assume that $\mathscr{A}$ is a C^*-algebra. There is an intrinsic axiomatic way to define C^*-algebras [8, 31, 5] as special instances of Banach algebras with involution. A Banach algebra is a Banach space over **C**, which is also an associative algebra, and in which the norm satisfies $\|AB\| \leq \|A\| \|B\|$. An involution is an antilinear operation $*$, such that $(AB)^* = B^*A^*$ and $A^{**} = A$. What defines a C^*-algebra is the extra demand $\|AA^*\| = \|A\|^2$. These axioms may be motivated by noting that they are identically satisfied by the ordinary operator norm on a norm-closed algebra of bounded operators on a Hilbert space. Conversely, particularly in reference to quantum mechanics, the upshot of especially the last requirement is that the usual Hilbert-space formalism can be recovered from, and is a special case of the C^*-algebraic setup. In other words, the axioms characterizing a C^*-algebra allow one to prove that every C^*-algebra is isomorphic with a norm-closed subalgebra of $\mathscr{B}(\mathscr{H})$ (cf. 1.4.7 for notation).

It so happens, that in the past C^*-algebraic techniques have almost exclusively been employed in the study of infinite systems, in particular quantum field theories and non-relativistic thermodynamic systems (cf. [24] and [6], respectively, for a review). The reason for this might have been the belief that, in view of the Stone-von Neumann uniqueness theorem, the application of these techniques to finite systems (by which one apparently implicitly referred to systems consisting of a finite number of particles moving in Euclidean space) would not lead to results which could not also have been obtained in a more elementary way by conventional Hilbert space (or other) techniques. However, the moment one studies the sort of systems considered in this paper, rather than the special case mentioned above, the distinct features of the C^*-algebra approach, and the way it is an extension of the usual Hilbert space formalism, emerge.

One of the main advantages of studying the simple systems considered here lies in the fact that they lead to C^*-algebras which are far more tractable than the ones considered in field theory and quantum statistical mechanics. Namely, C^*-algebras come in two sorts: they can be either "extremely well behaved" ($\equiv$type I $\equiv$ postliminary $\equiv$ GCR [8, 2, 31]; in the context of the present paper, this means in particular that their representation theory is well under control), or "totally misbehaved (antiliminary)" [31][a]. Now the C^*-algebras we are going to use to quantize a particle moving on a locally compact homogeneous space turn out, under mild extra assumptions, to be of the former type (cf. 3.1), whereas the ones employed in infinite systems (at least those of the type mentioned) invariably belong to the latter sort.

The sort of C^*-algebras employed in algebraic quantum field theory or quantum statistical mechanics admit a tremendous number of unclassifiable irreducible representations, most of which presumably have no physical interpretation whatsoever. Indeed, much of the effort in algebraic field theory goes into the procedure of selecting "admissible" states (or, equivalently, representations) giving rise to physically meaningful structures. Thus it would be meaningless to call any equivalence class of

[a] Actually, the totally misbehaved C^*-algebras are in a certain sense better behaved than the well-behaved ones we employ, since the ones used in field theory, etc. which belong to the former category are simple, hence admit faithful irreducible representations, whereas the latter usually are not, as exemplified in Chap. 3 below.

(irreducible) representations a superselection sector, so that in the context of quantum field theory this notion usually refers to a very special class of representations, namely the ones which in some well-defined sense are close to a vacuum representation.

Instead, the physical systems we are presently dealing with are so elementary that it should be possible to associate a well-behaved (type I) C^*-algebra $\mathscr{A}$ to the configuration space Q, so that, in particular, *all* (irreducible) representations of $\mathscr{A}$ are physically meaningful, and correspond to admissable superselection sectors, or admissable "inequivalent quantizations", which are the same.

Quantization by systems of imprimitivity (Mackey)

Let us introduce some of the relevant concepts by discussing a very elementary example, which also provides a simple-minded motivation for concentrating on abstract algebraic relations between operators, rather than on their concrete representations, at least in the initial stage of the quantization process.

Consider a particle moving on the real line $\mathbf{R}$. According to canonical quantization, this system is quantized by the following postulates:

1. the Hilbert space of states of the system is $\mathscr{H} = L^2(\mathbf{R})$;
2. the position and momentum variables are represented by (unbounded) operators q and p satisfying the canonical commutation relations (CCR) $[q, p] = i$.

The Stone-von Neumann uniqueness theorem is then usually invoked to demonstrate that the CCR are necessarily represented on $\mathscr{H}$ by $q = q$ (multiplication operator) and $p = -id/dq$.

Now item 1 should be considered in the light of the fact that there is only one infinite-dimensional separable Hilbert space. Hence 1 is a statement about a convenient concrete realization of this Hilbert space, which in a sense corresponds to a "choice of co-ordinates", and which can at best be a matter of convenience. Therefore, all essential, "co-ordinate free" information must reside in the second item.

However, the fact that the operators involved are unbounded makes it difficult to directly formulate the essential contents of the CCR in a representation-independent way. It is therefore convenient, and involves no loss of information whatsoever, to reformulate the CCR in terms of bounded operators [27, 34][b]. Firstly, note that the multiplication operator q is completely characterized by its spectral projections, which also characterize all bounded multiplication operators f, which act on $\mathscr{H}$ by $(f\psi)(q) = f(q)\psi(q)$, where $f \in C_0(\mathbf{R})$ (cf. 1.4.4 below for notation). Secondly, the additive group $G = \mathbf{R}$ is represented on $\mathscr{H}$ by unitaries π such that $(\pi(x)\psi)(q) = \psi(q - x)$, and this representation may equivalently be characterized by the operator relation (which is essentially a system of imprimitivity)

$$\pi(x) f \pi(x)^* = \alpha_x[f], \tag{1.1}$$

where the automorphism α_x acts on $f \in C_0(\mathbf{R})$ by

[b] The reformulation of the CCR in terms of Weyl operators [36, 6] differs significantly from the one sketched here (e.g. it leads to an inequivalent C^*-algebra), although it serves the same goal. The Weyl-Segal formulation of the CCR appears to be possible only if the underlying phase space is linear, as in the present example.

$$\alpha_x[f] = f^x; \qquad f^x(q) = f(x^{-1}q), \tag{1.2}$$

(in which $x^{-1}q \equiv q - x$) in the sense of multiplication operators. However, one may consider $C_0(\mathbf{R})$ as an abstract algebra of functions in its own right (with pointwise multiplication), and $G = \mathbf{R}$ as an abstract group acting on $C_0(\mathbf{R})$ via the α_x, so that (1.1) and (1.2) actually no longer contain any reference to the realization $\mathscr{H} = L^2(\mathbf{R})$ of the Hilbert space on which the operators had originally been defined. In a word, these operator relations are the desired abstract reformulation of the CCR. Their physical content resides, of course, in the identification of elements of the abstract function algebra $C_0(Q)$ with functions of the actual physical position variable q describing the localization of the particle; the physical interpretation of the operators $\pi(x)$ (or rather their generators, in case that G is a Lie group) as momentum operators then automatically follows, in view of the fact that they generate translations of the position variable. Thus, according to the above quantization method, which is due to Mackey [27] (also cf. [39] for an exhaustive review), all one has to do is find the (irreducible) representations of the system of imprimitivity given above, and it is easily shown that, up to unitary equivalence, there is only one such irreducible representation [27, 39, 4, 15].

A powerful feature of this approach is that it can trivially be generalized to deal with particles moving on arbitrary (separable locally compact) homogeneous spaces [27, 39, 11]. For let a group G act transitively on the configuration space Q (so that, in self-evident notation, $x \in G$ sends $q \in Q$ to $xq \in Q$). The position variable is then fully described by the algebra $C_0(Q)$ (generalizing $C_0(\mathbf{R})$ above), and the action of G on Q is contained in the same formulae (1.1), (1.2); in a word, the particle is quantized by the above system of imprimitivity. Finding the possible "quantizations" of the system is then equivalent to classifying the irreducible representations of this system of imprimitivity, which is a problem admitting a complete and straightforward solution [27, 39, 4, 15] (also cf. Chap. 2 below).

Before proceeding, it should be remarked that the representation $Q = G/H$ is highly non-unique, so that the choice of G should be restricted by demanding that it respects certain additional structures, e.g., a metric. Even so, one may form arbitrary nontrivial extensions E of G by K ($G = E/K$), and let E act on Q via the canonical epimorphism $p: E \to G$. Such extensions can be classified by cohomological methods [35], and, as will rapidly become clear, inequivalent extensions will lead to different quantizations in the present method. (This is true for trivial extensions of G as well, but these just correspond to the incorporation of internal degrees of freedom.) In ordinary quantum mechanics only central extensions are taken into account, whereas the proposed method is more general[c], so that one is faced with an *embarras du choix* which may be a blessing or a curse. In any case, it will turn out that *topological quantum effects are caused by a non-minimal choice of the group G*.

[c] Note in this connection that all projective representations of a given group G_0 can be obtained from the unitary representations of a single group extension $\overline{G_0}$ (the splitting group of G_0 [7]). If we choose $G = \overline{G_0}$, then the confines of scalar quantum mechanics in the context of the quantization method used in the present paper require that we restrict ourselves to those states on the C^*-algebra $C^*(G, Q)$ which give rise to a representation of G for which the kernel of p is mapped into multiples of the unit operator, cf. 2.2 and 2.3.

C^*-algebraization

It is just a tiny step to embed the above procedure in the general C^*-algebraic setting of quantum mechanics, that is, to "code" the information carried by the generalized CCR (1.1), (1.2) into a given C^*-algebra. Before attempting to do so, we would like to make a number of general comments concerning the quantization procedure. The above example has been included to motivate the reader in accepting that the act of quantizing a given physical system consists of the following three steps:

1. associating an operator algebra $\mathscr{A}$ with the system (in the above case, some algebra generated by the system of imprimitivity);
2. giving a rule leading to the interpretation of the self-adjoint elements of $\mathscr{A}$ as observables;
3. finding the unitarily inequivalent (irreducible) representations of $\mathscr{A}$ by an algebra of operators on a Hilbert space, thus entering the conventional description of quantum mechanics.

The basic postulate from which this paper starts is that, for the present class of systems, the so-called transformation group C^*-algebra $\mathscr{A} = C^*(G, Q)$ (to be defined in 2.2 below) is the correct choice [28, 26]. Our reason for this choice is that, as will be explained in the sequel, together with an appropriate implementation of item 2 in the quantization procedure (identifying the observables), this algebra actually does encode the correct system of imprimitivity describing the quantum particle. Now it would be nice if this algebra in some sense would generate its own physical interpretation, i.e., if it would uniquely characterize Q as well as the group action of G, and its associated set of observables, thus minimizing the role of step 2 above. However, while $\mathscr{A}$ is now given explicitly, it is still highly "uninformative" as to the nature of the quantized configuration space it is supposed to describe: there is by no means a one-to-one correspondence between the abstract C^*-algebra $C^*(G, Q)$ and the pair G, Q; as we shall see later, $C^*(G_1, Q_1)$ is isomorphic to $C^*(G_2, Q_2)$ if $\hat{H}_1$ is homeomorphic to $\hat{H}_2$ and $L^2(Q_1)$ is isomorphic to $L^2(Q_2)$. Hence step 2 is indispensable. We will see in 2.2 and 2.3 below, that the algebra $C^*(G, G)$ contains $C_0(Q)$ in an improper way, and the latter may be extracted out of the former in any representation. However, the same goes for any other $C_0(Q')$. What rule 2 in the present case allows us to do is extract the correct $C_0(Q)$, and identify its abstract elements $f, g, \ldots$ with functions of the physical position variable q. The other $C_0(Q')$'s are still imporperly embedded in $\mathscr{A}$, but their elements simply are not to be associated with any position variable. In similar vein, all relevant information on the group G may be extracted from $\mathscr{A}$, and if G is a Lie group then rule 2 says its generators are to be identified with physical momentum observables.

Although the momentum operators (apart from being unbounded, so that they are not properly contained in $\mathscr{A}$) do not admit any local structure, the position observables do, and this allows some analogy with algebraic quantum field theory to be drawn here. If E is a compact subset of Q then $C(E)$ is a C^*-algebra which is properly embedded in $C_0(Q)$, and improperly contained in $\mathscr{A}$. Thus we have a local structure $E \to C(E) \to C_0(Q)$, and $C_0(Q)$ is actually the C^*-inductive limit of all $C(E)$'s embedded in it. The difference with the situation in algebraic field theory [22, 24] (where one puts all physical information in the "net" $\mathcal{O} \to \mathscr{A}(\mathcal{O}) \to \mathscr{A}$, which assigns an algebra of localized observables to a region $\mathcal{O}$ in space-time) is that there the local structure refers to all observables, not just to the position variable; furthermore, the meaning of the notion of locality is different in the two cases: in the relativistic context of algebraic quantum field theory this refers to the causal structure of Minkowski space-time, whereas in the present setting it just has its naive, spacelike meaning. Indeed, the elements of the algebras $C(E)$ are *not* local operators in the sense of [22], cf. [23].

As to step 3 of the quantization procedure, the problem of finding the inequivalent representations of the systems is actually at the heart of the whole process of quantization. We suggest that *inequivalent irreducible representations of a given C^*-algebra describing a physical system are to be identified with its "inequivalent quantizations"*, and we adopt the terminology that these are the *superselection sectors* of the theory; in case that the existence of these superselection sectors is a consequence of the non-Euclidean topology of the configuration space, as in most of the examples studied here, we will call them *topological superselection sectors*[d]. The above-mentioned problem, then, is a purely mathematical one, which can be solved completely for the class of systems considered in this paper (cf. Chap. 2).

1.3. *Plan of the paper*

Having introduced some notation in 1.4 below, we take off in Chap. 2 by describing some relevant mathematics, namely those aspects of the theory of crossed products and transformation group C^*-algebras that we are going to employ. We need only a very small class of the latter objects (the transitive ones), whose complete representation theory is derived in 2.2. Three convenient realizations of their irreducible representations are given in 2.3, and we find that the idea of a Hilbert space of sections enters very naturally at this stage; here the differential geometry of fibre bundles enters the stage in an otherwise purely functional-analytic setting. Chap. 2 is little more than a collection of known facts, amended by a sequence of fairly trivial derivations of some results following form these.

The fact that Mackey's approach to the quantization of particles on homogeneous spaces can be reformulated in terms of C^*-algebras does not, in itself, add anything to the practical aspects of the quantization problem. The C^*-algebraic description really comes to its own where the description of time-evolution is concerned. The abstract algebraic formulation of the time evolution of a quantum system in terms of a one-parameter automorphism group on its C^*-algebra [25, 37, 5] is a very powerful generalization of the conventional Hamiltonian description, and its use in the present context lies at the basis of understanding the appearance of what are usually called "topological" terms in the Hamiltonian, and their connection with the quantization procedure.

The ultimate goal of Chap. 3 is to describe time evolutions of particles moving on homogeneous spaces by constructing one-parameter automorphism groups (G-

[d] Topological superselection sectors have previously been identified in the context of (two-dimensional) quantum field theory [38, 17], also cf. [16].

invariant or not) on transitive transformation group C^*-algebras. To do so, we need a detailed structural analysis of these algebras, which is given in 3.2. The precise conditions under which our approach works are stated and motivated in 3.1. Note that an essential part of the analysis in this chapter, and thereby the possibility of defining a time-evolution in the way we do, ultimately relies on a theorem of Williams [42] concerning the structure of a large class of transformation group C^*-algebras, which includes the transitive ones relevant to this paper.

The connection between the techniques proposed here, and other quantization methods is a delicate and involved issue, which will be taken up in a future paper. Thus the lack of references in this paper to other quantization techniques is not to be interpreted in any negative sense.

1.4. *Definitions and notation*

The following conventions will be used throughout this paper and its sequel, and any potentially obscure or ambiguous notation used in the main text without explanation is given below.

1. G is a locally compact group, whose elements are denoted by x or y. We assume G to be unimodular, type I (postliminary), and amenable (cf. 3.1), and its Haar measure is written as dx. The integral $\int dx$ stands for $\int_G dx$ (the same holds with x replaced by y). The Hilbert space $L^2(G)$ is defined with respect to the Haar measure. The dual of G is denoted by $\hat{G}$; this is the space of equivalence classes of irreducible unitary representations of G (equipped with the hull-kernel topology [14]).

2. H is a closed unimodular subgroup of G with generic element h, and Haar measure dh (etc.); once again, $\int dh$ stands for $\int_H dh$, and $L^2(H)$ is constructed on the basis of the Haar measure on H. Elements of its dual $\hat{H}$ are generically called χ, and their dimension is d_χ. The Plancherel measure on $\hat{H}$ is denoted by $d\nu(\chi)$. If H is compact, we use the symbols K, k, $\hat{K}$, and κ instead of H, h, $\hat{H}$, and χ, respectively.

3. Elements of the locally compact Hausdorff configuration space Q are just called q or q'. G acts on Q in such a way that x sends q to xq. The action is assumed to be transitive, so that we have $Q = G/H$, where H is the little group (stability group) of an arbitrarily chosen point $q_0 \in Q$. Hence yq_0 may be identified with the coset $yH \equiv \bar{y}$; thus $x\bar{y} = \overline{xy}$. Since we will assume both G and H to be unimodular, Q carries a natural G-invariant measure called dq. The Hilbert space $L^2(Q)$ is obviously constructed in terms of this measure.

4. For any locally compact Hausdorff space X, $C_0(X)$ stands for the space of continuous complex-valued functions on X which vanish at infinity (that is, the set $\{x \in X : |f(x)| > \varepsilon\}$ is compact for any $\varepsilon > 0$ and $f \in C_0(X)$). This space can be normed by setting $\|f\| = \sup_{x\in X}|f(x)|$, and is complete in this norm; it is even a commutative C^*-algebra under pointwise multiplication of functions, and with an involution * sending f to its complex conjugate [8, 31, 5][e]. The space $C_c(X)$ of continuous functions

[e] And, conversely, all commutative C^*-algebras arise in this way. The fact that the underlying space X is necessarily locally compact follows inescapably from the Banach-Alaoglu theorem of functional analysis, and is the major barrier preventing an extension of the present quantization method to field theories.

on X with compact support is uniformly (i.e., in the above norm) dense in $C_0(X)$. Functions in $C_0(Q)$ or $C_0(G)$ will be denoted by f or g, whereas upper-case letters F, G correspond to elements of $C_c(G \times Q)$ (confusion with the group G is impossible). For reasons explained in Chap. 2, elements of the transformation group C^*-algebra $\mathscr{A} = C^*(G, Q)$ are called F or G as well. More generally, C^*-algebras are denoted by calligraphic letters $\mathscr{A}$, $\mathscr{C}$ and their elements are written as upper-case roman letters, unless $\mathscr{A}$ is commutative, in which case we also use lower-case letters f, g as above. The dual of $\mathscr{A}$ (etc.) is the space of equivalence classes of its irreducible representations (carrying the hull-kernel topology), and is called $\hat{\mathscr{A}}$.

5. *-Automorphisms of a C^*-algebra $\mathscr{A}$ are denoted by α; these are linear surjective and injective maps of $\mathscr{A}$ into itself which respect the *-algebraic structure, i.e. $\alpha[AB] = \alpha[A]\alpha[B]$ and $\alpha[A^*] = (\alpha[A])^*$. It follows [5, 2.3.4] that α is norm-preserving. If α depends on $x \in G$ in such a way that $\alpha_x \circ \alpha_y = \alpha_{xy}$, so that the α_x's form a representation of G on $\mathscr{A}$ (regarded as a Banach space), then G is said to be an automorphism group of $\mathscr{A}$ (and the triple $(\mathscr{A}, G, \alpha)$ is called a C^*-dynamical system) [31, 5]. Throughout this paper, this representation of G will be required to be strongly continuous, that is, the maps $x \to \alpha_x[A]$ are continuous for all $A \in \mathscr{A}$. The C^*-dynamical systems of interest for us will be those in which $\mathscr{A} = C_0(Q)$, $Q = G/H$, and α_x acts on $\mathscr{A}$ in the way described by (1.2). These systems will simply be referred to as (G, Q).

6. A concrete representation of any algebraic object will be denoted by the symbol π, usually with some index attached to it. For example, an irreducible representation of H corresponding to an element $\chi \in \hat{H}$ is called π_χ, and the same symbol can be used to denote a representation of a group algebra of H, such as $L^1(H)$ or $C^*(H)$. The d_χ-dimensional carrier space of π_χ is called $\mathscr{H}_\chi$. The representation of G induced by the representation π_χ of H is called π^χ, and its carrier space is $\mathscr{H}^\chi$. As we shall see, irreducible representations of $C^*(G, Q)$ uniquely correspond to elements $\chi \in \hat{H}$, and, more particularly, to induced representations π^χ of G; for this reason the symbol π^χ will stand for an irreducible representation of $C^*(G, Q)$, too.

7. The unique abstract separable infinite-dimensional Hilbert space will be called $\mathscr{H}$, and the C^*-algebra of all compact operators on this space is denoted by $\mathscr{K}(\mathscr{H})$. The C^*-algebra of all bounded operators on $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$. Concretely given Hilbert spaces will generally be called $\mathscr{H}_i$ for some index i. The carrier space $\mathscr{H}_\kappa$ is finite-dimensional, and $M_\kappa \equiv \mathscr{B}(\mathscr{H}_\kappa)$. Finally, we will simply say that two Hilbert spaces are isomorphic if they are isometrically isomorphic (unitarily equivalent).

Finally, a non-integer appearing in a reference is a reference to a paragraph or theorem in the work preceding that non-integer (e.g. [31, 7.7.1] refers to a result stated in Sec. 7.7.1 of [31]).

2. Crossed Products and Transformation Group C^*-Algebras

2.1. *Crossed products*

Crossed products, of which the transformation group C^*-algebras on which the mathematics in this paper is based are special cases, were introduced in [10]. We will give some more information on their structure than is actually required to understand the rest of this paper, because we believe that the approach due to Fell [15, Chap. VIII] sketched below invites useful generalizations of the quantization procedure advertised here. An alternative discussion may be found in [31]. These references also provide an exhaustive bibliography on the subject.

We start by defining a so-called semidirect product bundle $\mathscr{S}$ (which in itself is a special case of the more general concept of a C^*-algebraic bundle [14, 15]) in terms of a C^*-dynamical system $(\mathscr{C}, G, \alpha)$ (which later on we take to be (G, Q)). $\mathscr{S}$ is a space whose elements are pairs $\langle A, x\rangle$, $A \in \mathscr{C}$, $x \in G$. In order to construct the crossed product algebra later on, it is necessary to define a multiplication in $\mathscr{S}$ according to the semi-direct product rule

$$\langle A, x\rangle \langle B, y\rangle = \langle A\alpha_x[B], xy\rangle. \tag{2.1}$$

Also, we will need an involution and a norm on $\mathscr{S}$, given by

$$\langle A, x\rangle^* = \langle \alpha_{x^{-1}}[A^*], x^{-1}\rangle; \tag{2.2}$$

$$\|\langle A, x\rangle\|_b = \|A\|, \tag{2.3}$$

respectively. The bundle structure is defined by the projection $p : \mathscr{S} \to G$, given by $p\langle A, x\rangle = x$. The fibers are isomorphic to $\mathscr{C}$, and the base space is just G; $\mathscr{S}$ is a trivial fibre bundle (that is, it has the topology of the product $G \times \mathscr{C}$). Note that $\mathscr{S}$ is not a Banach algebra: addition is only defined for elements in the same fiber.

Let $C_c(\mathscr{S})$ be the space of continuous cross-sections (i.e. functions $F : G \to \mathscr{S}$ such that $p \circ F$ is the identity map) of $\mathscr{S}$ with compact support. (Since the bundle topology is trivial, this space may be identified with the space $C_c(G, \mathscr{C})$ of compactly supported continuous functions from G to $\mathscr{C}$, by identifying F and $\tilde{F}$ in the relation $F(x) = \langle \tilde{F}, x\rangle$.) This space can be normed by putting $\|F\|_1 = \int dx\, \|F(x)\|$, and the completion of $C_c(\mathscr{S})$ in this norm may be made into a Banach algebra $L^1(\mathscr{S})$ upon defining a product and an involution by (assuming G to be unimodular)

$$(FG)(x) = \int dy\, F(xy^{-1})G(y); \tag{2.4}$$

$$(F^*)(x) = (F(x^{-1}))^*. \tag{2.5}$$

Thus (2.4) is an operator-valued convolution product, which may be defined as it stands (whereas (2.5) is defined by extension of the continuous involution in $C_c(\mathscr{S})$).

$L^1(\mathscr{S})$ is not a C^*-algebra because the axiom $\|AA^*\|_1 = \|A\|_1^2$ is not satisfied, but it can easily be made into one by the canonical procedure [8, 2.7]: define $\|F\| = \sup_\pi \|\pi(F)\|$, where the norm in the right-hand side is the ordinary norm of the bounded operator $\pi(F)$ on the Hilbert space carrying the representation $\pi(L^1(\mathscr{S}))$, and the supremum is taken over all representations (or, equivalently, over all irreducible representations) of $L^1(\mathscr{S})$. Completing $L^1(\mathscr{S})$ in the norm thus defined yields a C^*-algebra, which we denote by $C^*(\mathscr{S})$. This, then is the *crossed product* of $\mathscr{C}$ and G with respect to the automorphism α, and to stress the role of α one also uses the notation $\mathscr{C} \times_\alpha G$ instead of $C^*(\mathscr{S})$ [31]. Note that, because the supremum is taken in defining the norm, this is the smallest possible C^*-completion of $L^1(\mathscr{S})$. Also observe that (as always) $\|\cdot\|_1 \geq \|\cdot\|$, so that $C_c(\mathscr{S})$ is uniformly dense in $C^*(\mathscr{S})$.

The crucial property of $C^*(\mathscr{S})$ (or $L^1(\mathscr{S})$) is that its nondegenerate[f] representations π are in one-to-one correspondence with the nondegenerate representations of $\mathscr{S}$ itself, which in turn are in one-to-one correspondence with the so-called covariant representations of the C^*-dynamical system $(\mathscr{C}, G, \alpha)$ [10], [31, 7.6], [15, VIII.13]. A representation π of $\mathscr{C}$ (occurring in $(\mathscr{C}, G, \alpha)$) is said to be covariant with respect to G if there is a unitary representation (also called π) of G on the same carrier space on which $\pi(\mathscr{C})$ is defined, such that $\pi(G)$ implements α, in other words, if

$$\pi(x)\pi(A)\pi(x)^* = \pi(\alpha_x[A]). \tag{2.6}$$

The explicit correspondence between the representations (all called π) of $\mathscr{S}$, $(\mathscr{C}, G, \alpha)$, and $C^*(\mathscr{S})$ is given by the relations

$$\pi(\langle A, x\rangle) = \pi(A)\pi(x);$$

$$\pi(F) = \int dx\, \pi(\tilde{F}(x))\pi(x), \tag{2.7}$$

where $\tilde{F}$ is related to F in the way explained prior to (2.4). More constructively, $\pi(\mathscr{C})$ and $\pi(G)$ may be extracted from $\pi(C^*(\mathscr{S}))$ by exploiting the existence of an approximate unit in $C^*(\mathscr{S})$ [31, 7.6.4], [15, VIII.11.8 & 13.8].

2.2. *Transformation group C^*-algebras*

The simplest nontrivial case of the above construction arises by choosing $\mathscr{C} = \mathbf{C}$ (the 1×1 matrix algebra over the complex numbers), with G acting trivially on $\mathscr{C}$. The crossed product corresponding to this C^*-dynamical system is just the group C^*-algebra $C^*(G)$, which is equipped with the conventional convolution product [31], and the results quoted in the last section just boil down to the well-known fact that unitary representations of G are in one-to-one correspondence with nondegenerate representations of its group algebra (e.g., [31]).

A more involved special case of the general crossed product construction, which is of great relevance for us, follows by taking the dynamical system $(\mathscr{C}, G, \alpha)$ to be equal

[f] A representation is nondegenerate if no nonzero subspace of the carrier space $\mathscr{H}$ is annihilated by all representatives. In particular, irreducible representations are nondegenerate.

to (G, Q) (cf. 1.4.5); the C^*-algebra $C^*(\mathscr{S})$ is then denoted by $C^*(G, Q)$, and is called the *transformation group C^*-algebra*[g] corresponding to the pair G, Q. Thus we have $\mathscr{C} = C_0(Q)$, and G acting on this algebra as an automorphism group as in (1.2).

By the general theory, nondegenerate representations π of $C^*(G, Q)$ correspond to covariant representations π of the system (G, Q); in other words, given $\pi(C^*(G, Q))$ one can extract representations $\pi(C_0(Q))$ and $\pi(G)$, the latter being unitary, in such a way that (2.6), which in this case boils down to (1.1) (with f replaced by $\pi(f), f \in C_0(Q)$) is satisfied. If $\pi(C^*(G, Q))$ is faithful, this allows us to extract the algebras $C_0(Q)$ and $C^*(G)$ from their crossed product. However, these algebras are not properly contained in $C^*(G, Q)$, unless Q is compact and G is discrete; in the general case, we will still say that they are embedded in $C^*(G, Q)$. Conversely, covariant representations of (G, Q) give rise to nondegenerate representations of $C^*(G, Q)$ by (2.7).

The observation in the previous section that $C_c(\mathscr{S})$ is dense in $C^*(\mathscr{S})$ allows us to be more concrete in the present case: here $C_c(G, Q)$ consists of the compactly supported continuous functions from G to $C_0(Q)$, or, equivalently, of the complex-valued functions on $G \times Q$ which are C_c on G and C_0 on Q. The norm of such a function F (identified with $\tilde{F}$ defined previously) is majorized by its norm in $L^1(G, Q)$ given by $\|F\|_1 = \int dx \sup_Q |F(x, q)|$, which, in combination with the fact that $C_c(Q)$ is uniformly dense in $C_0(Q)$ leads to the conclusion that $C_c(G \times Q)$ is dense in $C^*(G, Q)$. Thus identifying $C_c(G \times Q)$ with a dense subalgebra of $C^*(G, Q)$ allows us to state the contents of the rules (2.4), (2.5) in very simple terms for elements of this subalgebra, viz.

$$(FG)(x, q) = \int dy\, F(xy^{-1}, q) G(y, yx^{-1}q); \tag{2.8}$$

$$(F^*)(x, q) = \overline{F(x^{-1}, x^{-1}q)}. \tag{2.9}$$

Furthermore, the assumption that G and Q are separable now allows us to conclude that $C^*(G, Q)$ is separable.

It is a simple matter to derive all irreducible representations of $C^*(G, Q)$. The general theory in [15, VIII.18] is evidently applicable here, but the fact that G acts transitively on Q allows us to take a shortcut here [26]. Namely, (2.6) (or (1.1)) is a transitive system of imprimitivity for the representation π of G, and Mackey's imprimitivity theorem [27, 39, 4, 15][h] allows us to conclude that such systems of imprimitivity, hence the nondegenerate representations π of $C^*(G, Q)$, are in one-to-one correspondence with the unitary representations of the little group H. In particular, the unitary representation $\pi(G)$ associated to $\pi(C^*(G, Q))$ is just the one induced by the given representation of H. Specializing to the irreducible representations of $C^*(G, Q)$, it

[g] These were introduced by Glimm [19] in the context of induced group representations. To define them, it is not necessary to assume that the action of G on Q is transitive, as we do.

[h] Mackey's derivation of this theorem is based on the assumption that G is separable and 2nd countable, but these conditions were later proven to be unnecessary, cf. [15] and refs. therein. Nevertheless, we will need to impose them, for different reasons, at a later stage of the analysis (cf. Chap. 3). Also, note that Mackey, etc. use the term "system of imprimitivity" in a slightly different, yet completely equivalent sense.

follows that

$$[C^*(G, G/H)]\hat{} = \hat{H}. \tag{2.10}$$

This justifies the use of the notation π^χ for the irreducible representations of $C^*(G, Q)$, as well as for the representations of G induced by $\chi \in \hat{H}$, cf. 1.4.6. As we shall see later (Sec. 3.2), for nontrivial H all irreducible representations are unfaithful (they have large kernels), but we will first proceed by studying their explicit forms.

2.3. *Irreducible representations of $C^*(G, G/H)$*

We have seen that certain covariant representations of the C^*-dynamical system (G, Q) give rise to irreducible representations of the transformation group C^*-algebra $C^*(G, Q)$. These representations are parametrized by $\chi \in \hat{H}$, and by the imprimitivity theorem we know that that the corresponding unitary representation π^χ of G is the one induced by a particular representative π_χ of χ. The associated representation $\pi^\chi(C_0(Q))$ is also given by the imprimitivity theorem, as it is determined by the spectral projections of each $\pi^\chi(f)$, $f \in C_0(Q)$ [27]. Given these data, the corresponding representation $\pi^\chi(C^*(G, Q))$ follows by the second member of (2.7). We will now give three useful (unitarily equivalent) realizations of π^χ (or $\mathscr{H}^\chi$), which stand out for their analytic, physical, and differential-geometric convenience, respectively. The Hilbert spaces used in the first two realizations are well-known in the context of group representation theory.

Realization 1

Let us first consider $\mathscr{H}^\chi$ to be realized as $\mathscr{H}_1^\chi$, which is the space of equivalence classes (under the inner product defined in 3 below) of functions[i] $\psi_1 : G \to \mathscr{H}_\chi$ satisfying (cf. e.g. [18, VI.2], [4, 16.1]):

1. $(\psi_1(x), \psi)$ is a measurable function of x for all $\psi \in \mathscr{H}_\chi$;
2. $\psi_1(xh) = \pi_\chi(h)^*\psi_1(x)$ for all $h \in H$ a.e. with respect to the Haar measure on G;
3. $(\psi_1, \psi_1)_1 \equiv \int_Q d\bar{x}(\psi_1(x), \psi_1(x))_\chi < \infty$, where, as explained in 1.4.3., $d\bar{x}$ is the invariant measure on $Q = G/H$, and $(\cdot, \cdot)_\chi$ is the inner product in $\mathscr{H}_\chi$.

The realization π_1^χ of π^χ of G and $C_0(Q)$ is then given by

$$(\pi_1^\chi(y)\psi_1)(x) = \psi_1(y^{-1}x);$$

$$(\pi_1^\chi(f)\psi_1)(x) = f(xq_0)\psi_1(x). \tag{2.11}$$

Thus $F \in C_c(G \times Q)$ is represented by

$$(\pi_1^\chi(F)\psi_1)(q) = \int dy\, F(xy^{-1}, xq_0)\psi_1(y). \tag{2.12}$$

[i] Here and in the following we will omit the appropriate upper index χ on the ψ's.

Realization 2

The constraint in 2 above obscures the physical interpretation of the state vectors ψ_1. Therefore, we now give another realization on a Hilbert space $\mathscr{H}_2^\chi$ consisting of equivalence classes of functions $\psi_2 : Q \to \mathscr{H}_\chi$, which satisfy the analogues of 1 and 3 above, but are unconstrained otherwise. To specify π_2^χ we need to choose a measurable section[j] $s : Q \to G$ (that is, $s(q)q_0 = q$), in terms of which

$$(\pi_2^\chi(y)\psi_2)(q) = \pi_\chi(\gamma(y,q))\psi_2(y^{-1}q);$$

$$(\pi_2^\chi(f)\psi_2)(q) = f(q)\psi_2(q), \tag{2.13}$$

where $\gamma(y,q) = (s(q))^{-1}ys(y^{-1}q)$ is the Wigner cocycle. A formula analogous to (2.12) is then easily derived, and will be omitted here. We see, that elements of $\mathscr{H}_2^\chi$ are essentially (vector-valued) wave-functions in the usual sense.

Different choices of the section s correspond, of course, to unitarily equivalent representations. Namely, let sections s_α and s_β be almost everywhere related by a "gauge transformation" $s_\beta(q) = s_\alpha(q)g_{\alpha\beta}(q)$, with $g_{\alpha\beta}(q) \in H$. To emphasize that $\mathscr{H}_2^\chi$, π_2^χ and ψ_2 depend on the choice of s, we will affix a label α or β to these symbols. Then the unitary transformation $T_{\alpha\beta} : \mathscr{H}_{2;\alpha}^\chi \to \mathscr{H}_{2;\beta}^\chi$ defined by

$$(T_{\alpha\beta}\psi_2^\alpha)(q) = \pi_\chi(g_{\beta\alpha}(q))\psi_2^\alpha(q) \tag{2.14}$$

intertwines $\pi_{2;\alpha}^\chi$ and $\pi_{2;\beta}^\chi$ (to be precise, one needs to define the above transformation on compactly supported continuous wave-functions, and extend it to the whole Hilbert space by continuity).

The intertwiner of π_1^χ and π_2^χ is easily established as well (we now take $s = s_\alpha$ fixed): it is the unitary map $T_{12} : \mathscr{H}_1^\chi \to \mathscr{H}_2^\chi$ defined (on compactly supported continuous functions, etc.) by

$$(T_{12}\psi_1)(q) = \psi_1(s(q)), \tag{2.15}$$

its inverse being given by

$$(T_{12}^*\psi_2)(x) = \pi_\chi(x^{-1}s(xq_0))\psi_2(xq_0). \tag{2.16}$$

Let now G and H be Lie groups. The induced representations π_1^χ of G constructed above then have a geometric meaning [40] in terms of fiber bundles. Namely, G is the total space of a principal fiber bundle with base manifold $Q = G/H$ and group H, the projection $p : G \to Q$ being the canonical one (cf. 1.4.3) $px = \bar{x} \equiv xH$. One then forms the vector bundle E_χ associated to G; its fibers are isomorphic to $\mathscr{H}_\chi$, and its points are equivalence classes $[x, v]$ $(v \in \mathscr{H}_\chi)$ under the equivalence relation $(x, v) \equiv (xy^{-1}, \pi_\chi(y)v)$. Then G acts on E_χ by $y[x,v] = [yx, v]$, and therefore it also acts on $\Gamma(E_\chi)$ (the space of C_c^∞-cross-sections of E_χ) by means of

[j] This is equivalent to choosing a measurable Mackey decomposition of G with respect to H, as in [4, 16.1].

$$(\pi^{\chi}(y)\psi)(q) = y\psi(y^{-1}q), \tag{2.17}$$

where $\psi \in E_\chi$. If we realize the cross-section ψ in terms of ψ_1 given by $\psi(q) = [x, \psi_1(x)]$, where x satisfies $px = q$, then consistency with the definition of the equivalence class $[x, v]$ requires that $\psi_1 : G \to \mathscr{H}_\chi$ satisfies $\psi_1(xh) = \pi_\chi(h)^* \psi_1(x)$ for all $h \in H$. Defining the obvious inner product in $\Gamma(E_\chi)$ and completing it into a Hilbert space then evidently reproduces the construction leading to the representation π_1^χ given previously.

Realization 3

We would now like to incorporate the passage of π_1^χ to π_2^χ, or, in other words, the passage to wave-functions having a local interpretation, into the differential-geometric setting of induced representations of Lie groups reviewed above. However, this goal seems to be frustrated by the fact that the section s used in constructing $\mathscr{H}_2^\chi$ cannot, in general, be chosen so as to be continuous (as we mentioned, it only needs to be measurable in the general Hilbert space context). The obvious way out is to construct a "Hilbert space of sections". This leads to a third realization of π^χ, which we call π_3^χ.

We cover Q with open sets $\{U_\alpha\}_{\alpha \in I}$ for some index set I, so that the U_α are homeomorphic to a Euclidean space, and do this in such a way that we can locally define smooth sections $s_\alpha : U_\alpha \to G$. For $q \in U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$ the sections s_α and s_β are then related by the gauge transformation given previously, where now $g_{\alpha\beta} : Q \to H$ is required to be smooth, and to be such that it satisfies the consistency (cocycle) condition $g_{\alpha\beta}(q) g_{\beta\gamma}(q) = g_{\alpha\gamma}(q)$ on $q \in U_\alpha \cap U_\beta \cap U_\gamma$.

Elements of $\mathscr{H}_3^\chi$ are sectional wave-functions ψ_3, which are concretely represented by their local trivializations $\psi_3^\alpha : U_\alpha \to \mathscr{H}_\chi$ for all $\alpha \in I$. In overlap points $q \in U_\alpha \cap U_\beta$ different trivializations of the single section ψ_3 are required to be related a.e. by the analogue of (2.14), viz. $\psi_3^\beta(q) = \pi_\chi(g_{\beta\alpha}(q))\psi_3^\alpha(q)$. Furthermore, the obvious translations of the conditions 1 and 3 for $\mathscr{H}_1^\chi$ should be satisfied; as far as the inner product in $\mathscr{H}_3^\chi$ is concerned, one has

$$(\psi_3, \varphi_3)_3 = \sum_{\alpha \in I} \int_{U_\alpha} dq\, \chi_\alpha(q) (\psi_3^\alpha(q), \varphi_3^\alpha(q))_\chi < \infty, \tag{2.18}$$

in terms of a partition of unity[k] χ.

The representation π_3^χ on $\mathscr{H}_3^\chi$, then, is defined as follows:

$$(\pi_3^\chi(y)\psi_3)^\alpha(q) = \pi_\chi((s_\alpha(q))^{-1} y s_\beta(y^{-1}q))\psi_3^\beta(y^{-1}q);$$

$$(\pi_3^\chi(f)\psi_3)^\alpha(q) = f(q)\psi_3^\alpha(q). \tag{2.19}$$

Here it has been supposed that $q \in U_\alpha$ and $y^{-1}q \in U_\beta$; which particular U_β is selected

[k] This is a collection of functions $\chi_\alpha : Q \to \mathbf{R}$ such that the support of each χ_α is in U_α, and $\sum_{\alpha \in I} \chi_\alpha(q) = 1$ for all q. The value of the inner product is not affected by the precise choice of χ in view of the compatibility between different trivializations of ψ_3 in overlap regions, and the unitarity of π_χ.

in overlap regions is immaterial due to the compatibility conditions on the trivializations of ψ_3.

It is an easy matter to intertwine π_3^χ and the previously constructed realizations: the unitary equivalence with π_1^χ is established by the map $T_{13}: \mathscr{H}_1^\chi \to \mathscr{H}_3^\chi$, defined in complete analogy with (2.15) by

$$(T_{13}\psi_1)^\alpha(q) = \psi_1(s_\alpha(q)), \tag{2.20}$$

with the inverse

$$(T_{13}^*\psi_3)(x) = \pi_\chi(x^{-1}s_\alpha(xq_0))\psi_3^\alpha(xq_0), \tag{2.21}$$

where it is assumed that $q \in U_\alpha$ (once again, this particular choice is immaterial). The unitary equivalence with π_2^χ is inherent in (2.14): let $s \equiv s_\gamma$ be the fixed (discontinuous, in general) section employed in the construction of $\mathscr{H}_2^\chi$. Then $T_{23}: \mathscr{H}_2^\chi \to \mathscr{H}_3^\chi$ defined by

$$(T_{23}\psi_2)^\alpha(q) = \pi_\chi(g_{\alpha\gamma}(q))\psi_2(q), \tag{2.22}$$

in combination with its self-evident inverse, intertwines π_2^χ and π_3^χ. Note that $g_{\alpha\gamma}$ need not be smooth, if s is not.

Gårding domain

When both modes of description are available, it may, in general, be a matter of taste whether one prefers a functional-analytic or a differential-geometric setting; in the present case, however, the third (fibre bundle) realization of π^χ turns out to be particularly useful in the problem of constructing the Gårding domain [29, 4] for $\pi^\chi(G)$. Doing so is essential for the analysis of those unbounded operators on the representation space $\mathscr{H}^\chi$ which are representatives of elements of the universal enveloping algebra $\mathscr{U}(\mathscr{G})$ of (the Lie algebra of) G; as we shall see in 3.3 below, the Hamiltonians implementing the time-evolution in (irreducible) representations of $C^*(G, Q)$ are of this kind.

The Gårding domain D_G for a representation $\pi(G)$ of a Lie group is by definition the linear subspace of the carrier space $\mathscr{H}$ of π spanned by the vectors $\pi(f)\psi \equiv \int dx\, f(x)\pi(x)\psi$, where f and ψ run through $C_c^\infty(G)$ and $\mathscr{H}$, respectively. D_G is dense in $\mathscr{H}$, and its importance derives from the fact [29, 4] that representatives of symmetric elements of $\mathscr{U}(\mathscr{G})$ are essentially self-adjoint on D_G.

In the present case, simple manipulations show that on $\mathscr{H}_3^\chi$ one has

$$(\pi_3^\chi(f)\psi_3)^\alpha(q) = \int dx\, f(s_\alpha(q)x)\pi_\chi(xs_\beta(x^{-1}q_0))\psi^\beta(x^{-1}q_0), \tag{2.23}$$

where f and ψ_3 are as indicated above, and $q \in U_\alpha$, $x^{-1}q_0 \in U_\beta$. The q-dependence of the right-hand side is particularly straightforward: it allows one to conclude that sections in $D_G \subset \mathcal{H}_3^\chi$ are represented by local trivializations ψ_3^α which are C^∞ on U_α for each $\alpha \in I$. If $\mathcal{H}_\chi$ is finite-dimensional (as in all our applications), so that the factor $\pi_\chi(\ldots)$ in (2.23) is innocent, then more detailed information on D_G is easily extracted, cf. Chap. 4 below, and also the sequel to this paper. The corresponding Gårding domains on $\mathcal{H}_2^\chi$ and $\mathcal{H}_1^\chi$ may subsequently be found by performing the unitary transformations T_{23}^* and T_{13}^*, respectively, on $D_G \subset \mathcal{H}_3^\chi$.

3. The Structure of $C^*(G, Q)$

3.1. *Conditions on G and Q*

The only essential technical assumption made so far on the pair (G, Q), with $Q = G/H$, is that G is locally compact, H is closed, and Q is Hausdorff. This sufficed to prove the key result (2.10) concerning the representation theory of the crossed product $C^*(G, Q)$. Although most of the formulae in Chap. 2 also require the unimodularity of G and H, this condition has not been essential, and just enhanced the clarity of the expressions by saving some writing (cf. [31, 15] and [18, 4] for the incorporation of the modular function in the context of crossed products and induced representations, respectively).

Further progress can be made if we impose stronger conditions on G and H. Let us note already at this stage that these are all going to be met in our examples (and, indeed, in most realistic examples one can think of). Firstly, assume that H is a type I ($\equiv$ postliminary $\equiv$ GCR [2, 8, 18, 31, 14, 15]) group. This condition means that any factorial representation of H properly contains an irreducible representation (so that it can be discretely reduced), and is met by all compact groups, all locally compact abelian groups, all semisimple Lie groups, etc. By (2.10), this allows us to conclude that the crossed product $C^*(G, Q)$ is itself postliminary, which is in agreement with, and a special case of a considerably deeper result of Gootman [20].

What is more, the further assumption that H is amenable and CCR, and that the pair (G, Q) is 2nd countable, implies, by a theorem of Williams [42][1] that $\mathcal{A} = C^*(G, Q)$ is even a CCR algebra[m]. This means, by definition [2, 8, 31], that any irreducible representation of $\mathcal{A}$ consists of $\mathcal{K}(\mathcal{H})$ for some $\mathcal{H}$ (which is allowed to be finite-dimensional). A group H is said to be CCR if its group C^*-algebra $C^*(H)$ is CCR. A sufficient condition, which can easily be checked in practical cases, for a group H to be CCR is [18, VII.5] that the following three demands are met: it is locally compact, it has an Iwasawa decomposition, and the set of its finite-dimensional representations separates points on H. *Necessary* and sufficient conditions of a more complicated sort are given in [30]. In any case, practically all (finite-dimensional) groups encountered in theoretical physics are CCR.

[1] For H compact this theorem is easily checked by looking at the explicit realizations of the irreducible representations of $C^*(G, Q)$ constructed in the preceding section, from which it may be inferred that $\pi^\chi(F)$ is Hilbert-Schmidt for $F \in C_c(G \times Q)$. The general proof follows quite different lines, and is highly involved.

[m] Here CCR means Completely Continuous Representation (theory); in older literature a compact operator is called completely continuous. What physicists call a CCR algebra (for Canonical Commutation Relations) is, in the infinite-dimensional case, not a CCR algebra in the above sense.

The condition of amenability [32] on H is less straightforwardly explained (for a brief summary, cf. the notes to Secs. 4.3.1, 4.3.2 in [5]); here we just mention that all locally compact abelian and solvable groups, all compact groups, and all factor groups and semidirect products of these three classes are amenable. Noncompact semisimple Lie groups, while type I, are not amenable. For us, one of the important consequences of amenability is that all irreducible representations of H occur in the direct integral decomposition of its left-regular representation (cf. 3.2 below). More generally, the fact that $C^*(G, Q)$ is CCR will be decisive in our construction of time evolutions on this algebra in Sec. 3.3.

Furthermore, the proofs of the first two Theorems below require G and H to be separable[n] (so that $C^*(G, Q)$ is separable, and the carrier spaces of its (irreducible) representations are separable); this allows us, in particular, to carry out direct integral decompositions whenever required. Moreover, we demand that G is amenable as well; this is necessary for the regular representation of $C^*(G, Q)$ to be faithful [31, 7.7.5], which we use in Theorem 1, and in the construction of time evolutions later on. However, the whole construction also works for non-amenable groups G, provided that we slightly modify our quantization prescription: instead of quantizing the configuration space Q with its associated group action by means of the crossed product $C^*(G, Q)$, we rather have to work with the so-called reduced crossed product [31, 7.7]. A large number of "inequivalent quantizations" of Q will still be covered by this modified procedure.

We will maintain our assumption that G and H are unimodular: as to the former, this remains just a matter of convenience, whereas the unimodularity of H will actually be crucial in the harmonic analysis part of the reasoning in the next section. However, this assumption is not essential for the quantization procedure in itself. Finally, Theorem 2, which may be interesting, but which will not really be employed in the sequel, requires the space $\hat{H}$ (equipped with the hull-kernel topology [8, 15]) to be Hausdorff. We make this assumption because it is satisfied by all our examples; in general, there exists a "folk-conjecture", which has been proved in [3] for the locally compact separable case, that a group dual $\hat{H}$ is Hausdorff if and only if H is type I, and is an extension of an abelian group by a compact group. (The more general question whether the dual of a transformation group C^*-algebra is Hausdorff is addressed in [43].)

3.2. *Reduction of a regular representation*

In this section we will exhibit a very simple faithful representation π_L of $C^*(G, Q)$. This is reducible, and its decomposition over irreducible representations will completely elucidate the structure of the transitive transformation group C^*-algebras we are interested in.

Consider the representations $\pi_L(G)$ and $\pi_L(C_0(Q))$ defined on the Hilbert space $\mathscr{H}_L = L^2(G)$ by

[n] This is automatically true if G and Q are 2nd countable.

$$(\pi_L(y)\psi_L)(x) = \psi_L(y^{-1}x);$$

$$(\pi_L(f)\psi_L)(x) = f(xq_0)\psi_L(x), \tag{3.1}$$

respectively, or, equivalently, the corresponding representation of $F \in C_c(G \times Q)$ given by

$$(\pi_L(F)\psi_L)(x) = \int dy\, F(xy^{-1}, xq_0)\psi_L(y). \tag{3.2}$$

A representation of $C^*(G, Q)$ then follows by passing to the uniform closure of the set of operators on $\mathscr{H}_L$ defined by (3.2). The restriction of π_L to G is evidently the left-regular representation. We have

Theorem 1. *The representation $\pi_L(C^*(G, Q))$ is faithful if G is amenable.*

We precede the proof of this theorem by an intermezzo on direct integrals, a machinery which is heavily used below as well as in the next paper.

Direct integrals of Hilbert spaces and representations

Direct integrals are particular realizations of a Hilbert space. We will consider only the simplest form of this construction, and our treatment will be very sketchy. For more detail and rigour we refer to [29, 9].

Let Λ be a measure space with a Borel measure μ, and let each $\lambda \in \Lambda$ correspond to a Hilbert space $\mathscr{H}(\lambda)$ with inner product $(\ ,\ )_\lambda$. By definition, elements of the direct integral $\hat{\mathscr{H}} = \int_\Lambda^\oplus d\mu(\lambda)\mathscr{H}(\lambda)$ are measurable vector-valued functions v on Λ, such that $v(\lambda) \in \mathscr{H}(\lambda)$, and $(v, v) = \int_\Lambda d\mu(\lambda)(v(\lambda), v(\lambda))_\lambda < \infty$; this also defines the inner product in $\hat{\mathscr{H}}$.

This realization singles out two special classes of operators on $\hat{\mathscr{H}}$. Let a bounded operator A act on v by $(Av)(\lambda) = a(\lambda)v(\lambda)$. If a is a (measurable) complex-valued function on Λ then A is called diagonal, whereas A is said to be decomposable if $a(\lambda) \in \mathscr{B}(\mathscr{H}(\lambda))$. The connection between these classes, which lies at the basis of reduction theory, is that the algebra of all diagonal operators is the commutant of the algebra of all decomposable operators.

Let now $\mathscr{M}$ be a commutative von Neumann algebra° of bounded operators acting on a Hilbert space $\mathscr{H}$. The spectral theorem then states that there exists a unitary map $U: \mathscr{H} \to \hat{\mathscr{H}}$, for some choice of Λ (identified with the spectrum of $\mathscr{M}$) and μ, such that $U\mathscr{M}U^*$ consists of diagonal operators on $\hat{\mathscr{H}}$.

Furthermore, let $\mathscr{A}$ be a C^*-algebra with nondegenerate representation π on $\mathscr{H}$, such that $\mathscr{M}$ is in the commutant of $\pi(\mathscr{A})$. Then $U\pi(\mathscr{A})U^*$ is a C^*-algebra of decomposable operators on $\hat{\mathscr{H}}$. Thus one may reduce a representation π of a C^*-algebra $\mathscr{A}$ by finding a commutative von Neumann algebra $\mathscr{M}$ in the commutant of $\pi(\mathscr{A})$, and

° A von Neumann algebra is a concretely given C^*-algebra which is closed in the weak operator topology; equivalently, it is its own bi-commutant [9].

diagonalizing it. Moreover, the decomposition is extremal (i.e., into irreducible subrepresentations) if and only if $\mathcal{M}$ is maximal. One symbolically writes $\mathcal{H} \simeq \hat{\mathcal{H}}$, with $\hat{\mathcal{H}}$ defined as above, and $\pi \simeq \int_\Lambda^\oplus d\mu(\lambda)\pi_\lambda$, where the subrepresentative $\pi_\lambda(A)$ coincides with the operator $a(\lambda)$ defined above.

Proof of Theorem 1. Take the canonical faithful representation π_c of $C_0(Q)$ by multiplication operators on $\mathcal{H}_c = L^2(Q)$. This gives rise to a so-called regular representation[p] $\pi_r(C^*(G,Q))$ on $\mathcal{H}_r = L^2(G,\mathcal{H}_c) \simeq L^2(G \times Q)$ defined by

$$(\pi_r(F)\psi_r)(x,q) = \int dy\, F(xy^{-1}, xq)\psi_r(y,q), \tag{3.3}$$

which is equivalent to the pair

$$\begin{aligned}(\pi_r(y)\psi_r)(x,q) &= \psi_r(y^{-1}x,q);\\ (\pi_r(f)\psi_r)(x,q) &= f(xq)\psi_r(x,q).\end{aligned} \tag{3.4}$$

Now the von Neumann algebra $L^\infty(Q)$ acts on $\mathcal{H}_r$ in the obvious way (as an algebra of multiplication operators, leaving the argument x of ψ_r unaffected), and it is easily seen to commute with $\pi_r(C^*(G,Q))$. Thus we may follow the general strategy for decomposing π_r sketched above; note that the present decomposition is neither extremal nor central. This gives

$$L^2(G \times Q) \simeq \int^\oplus dq\, \mathcal{H}(q); \qquad \pi_r \simeq \int^\oplus dq\, \pi_q, \tag{3.5}$$

with $\mathcal{H}(q) = L^2(G)$, and π_q given by

$$\begin{aligned}(\pi_q(y)\psi_q)(x) &= \psi_q(y^{-1}x);\\ (\pi_q(f)\psi_q)(x) &= f(xq)\psi_q(x).\end{aligned} \tag{3.6}$$

It is evident that $\mathcal{H}_L = \mathcal{H}_{q_0}$, $\pi_L = \pi_{q_0}$; moreover, all representations π_q are unitarily equivalent to π_L by the unitary intertwiner $T_{qq_0}: \mathcal{H}_q \to \mathcal{H}_{q_0}$ defined by $(T_{qq_0}\psi_q)(x) = \psi_q(x(s(q))^{-1})$ for an arbitrary measurable section $s: Q \to G$. Hence the regular representation (3.3), which for G amenable is faithful by a well-known theorem [31, 7.7.5], is decomposed as a direct integral of identical copies of π_L. It follows that π_L itself must be faithful. □

Apart from carrying π_L, $\mathcal{H}_L$ also carries the right-regular representation of G, defined by $(\pi_R(y)\psi_L)(x) = \psi_L(xy)$. Bounded operators on $\mathcal{H}_L$ commuting with the von Neumann algebra $\mathcal{M}_R(H)$ generated by $\pi_R(H)$ will be called ad(H)-invariant.

[p] It is easy to check that the procedure given here is a special case of the general construction of a regular representation of a crossed product [31, 7.7.1].

Corollary 1. *Let G be compact. Then $C^*(G, G/H)$ is isomorphic to the algebra $\mathscr{K}(\mathscr{H})_H$ of ad(H)-invariant compact operators on $L^2(G)$.*

Proof. A Hilbert-Schmidt operator K on $L^2(G)$ defined by a kernel $k(x,y)$ is evidently ad(H)-invariant if $k(xh, yh) = k(x,y)$ a.e. It then follows from (3.2) that operators of the form $\pi_L(F)$, $F \in C_c(G \times Q)$ are in $\mathscr{K}(\mathscr{H})_H$, and by passing to the uniform closure we infer that $\pi_L(C^*(G,Q)) \subset \mathscr{K}(\mathscr{H})_H$.

We now prove the converse. To do so, it suffices to show that any ad(H)-invariant Hilbert-Schmidt operator K represented by a kernel $k \in C_c(G \times G)$ is of the form $K = \pi_L(F)$ for some $F \in C_c(G \times Q)$. The sufficiency of this condition follows from the fact that the set of operators of the former type is uniformly dense in $\mathscr{K}(L^2(G))$, and this in turn is implied by the observation that $C_c(G \times G)$ is L^2-dense in $L^2(G \times G)$, combined with the inequality [33, VI.6] $\|K\| \leq \|k\|_{L^2}$. Suppose, then, that $k(xh, yh) = k(x,y)$. Define a function f on $G \times G$ by $f(x,y) = k(x, y^{-1}x)$. Then $f(xh, y) = f(x,y)$, so that f must have the form $f(x,y) = F(xq_0, y)$ for some $F \in C_c(G \times Q)$. It follows that $K = \pi_L(F)$, where the function F is identified with the corresponding element of $C^*(G,Q)$. □

This Corollary is a special case of a theorem of Evans [12], which was proved by quite different means.

Decomposition of $\pi_L(C^*(G,Q))$

The representation π_L is reducible, because, as in the compact case discussed in the Corollary, $\pi_L(C^*(G,Q))$ commutes with $\mathscr{M}_R(H)$. (An argument similar to the one used in proving the Corollary may be employed to show that $\pi_L(C^*(G,Q))' = \mathscr{M}_R(H)$, but we will not need this fact.) We are going to decompose π_L by reducing $\pi_R(H)$ on $\mathscr{H}_L$, that is, by diagonalizing an arbitrary maximal abelian subalgebra of $\mathscr{M}_R(H)$. It will follow by inspection that the corresponding decomposition of $\pi_L(C^*(G,Q))$ is extremal (i.e., it is decomposed as a direct integral of irreducible representations), and it is unique by the type I property of H.

The first step is to perform a unitary transformation $T_{L4}: \mathscr{H}_L = L^2(G) \to \mathscr{H}_4 \equiv L^2(Q \times H) \simeq L^2(Q) \otimes L^2(H)$. Given a section s, this is defined by

$$(T_{L4}\psi_L)(q,h) = \psi_L(s(q)h), \tag{3.7}$$

with inverse

$$(T_{L4}^*\psi_4)(x) = \psi_4(xq_0, (s(xq_0))^{-1}x). \tag{3.8}$$

This transformation is unitary with the obvious inner product on $\mathscr{H}_4$. With $\pi_4 = T_{L4}\pi_L T_{L4}^*$ it follows that

$$(\pi_4(y)\psi_4)(q,h) = \psi_4(y^{-1}q, (\gamma(y,q))^{-1}h);$$

$$(\pi_4(f)\psi_4)(q,h) = f(q)\psi_4(q,h), \tag{3.9}$$

where γ is the Wigner cocycle (cf. (2.13)); the result follows by noting that $s(q)q_0 = q$.

The next step consists in reducing the left-regular representation of H on the $L^2(H)$-factor in $\mathscr{H}_4$ by the (Fourier-) Plancherel transform [8, 18.8]. Thus define

$$\mathscr{H}_5 = \int_{\hat{H}}^{\oplus} d\nu(\chi)\mathscr{H}(\chi); \qquad \mathscr{H}(\chi) = L^2(Q) \otimes HS[\mathscr{H}_\chi], \tag{3.10}$$

where $HS[\mathscr{H}]$ is the Hilbert space of Hilbert-Schmidt operators on $\mathscr{H}$ [8, A66] (with inner product $(A, B) = \operatorname{Tr} AB^*$). Hence the inner product in H_5 is

$$(\psi_5, \psi_5)_5 = \int dq \int_{\hat{H}} d\nu(\chi) \operatorname{Tr}[\psi_5(q,\chi)\psi_5(q,\chi)^*]. \tag{3.11}$$

If H is abelian then $\mathscr{H}_\chi = \mathbf{C}$, whereas $\mathscr{H}_\chi$ is always finite-dimensional if H is compact; in that case $HS[\mathscr{H}_\chi]$ is just $M_{d_\chi}(\mathbf{C})$ (the algebra of complex d_χ-dimensional matrices), $\hat{H}$ is discrete, and the Plancherel measure assigns the measure d_χ to the point $\chi \in \hat{H}$.

In the general case, the Plancherel transform is effected by the unitary transformation $T_{45}: \mathscr{H}_4 \to \mathscr{H}_5$ defined by

$$(T_{45}\psi_4)(q,\chi) = \int dh\, \pi_\chi(h)\psi_4(q,h), \tag{3.12}$$

which is well-defined as it stands for $\psi_4 \in L^2(Q) \otimes (L^1(H) \cap L^2(H))$, and is extended to all of $\mathscr{H}_4$ by continuity. The inverse follows from (3.11) and (3.12) as

$$(T_{45}^*\psi_5)(q,h) = \int_{\hat{H}} d\nu(\chi) \operatorname{Tr}[\pi_\chi(h)^*\psi_5(q,\chi)]. \tag{3.13}$$

The point of the unitary transformation $T_{L5} \equiv T_{45}T_{L4}: \mathscr{H}_L \to \mathscr{H}_5$ is that it decomposes $\pi_R(H)$ on $\mathscr{H}_L$, and thereby also decomposes $\pi_L(C^*(G,Q))$. Indeed, with $\pi_5 = T_{L5}\pi_L T_{L5}^*$ it is easily inferred from the above formulae that

$$\begin{aligned}(\pi_5(y)\psi_5)(q,\chi) &= \pi_\chi(\gamma(y,q))\psi_5(y^{-1}q,\chi);\\ (\pi_5(f)\psi_5)(q,\chi) &= f(q)\psi_5(q,\chi).\end{aligned} \tag{3.14}$$

To interpret this result, let us recall [8, A66] the canonical isomorphism $HS[\mathscr{H}_\chi] \simeq \mathscr{H}_\chi \otimes \mathscr{H}_{\bar{\chi}}$, where $\bar{\chi}$ is a representation conjugate to χ. Thus regarding $\psi_5(q,\chi)$ as a vector in $\mathscr{H}_\chi \otimes \mathscr{H}_{\bar{\chi}}$, (3.14) shows that only its components in $\mathscr{H}_\chi$ transform. Inspecting (2.13), the reader will then notice that

$$\mathscr{H}_5 = \int_{\hat{H}}^{\oplus} d\nu(\chi)\, d_\chi \cdot \mathscr{H}_2^\chi; \qquad \pi_5 = \int_{\hat{H}}^{\oplus} d\nu(\chi)\, d_\chi \cdot \pi_2^\chi, \tag{3.15}$$

so that π_5 is indeed a direct integral over irreducible representations of $C^*(G,Q)$, as advertised before.

The structure of $C^*(G,Q)$

Since π_L is a faithful representation of $C^*(G,Q)$, and π_5 above is unitarily equivalent to it, the latter is faithful as well. Now *-isomorphisms of algebras of decomposable operators on Hilbert spaces represented as direct integrals do not distinguish between different nonzero (possibly infinite) multiplicities, and they do not see the difference between equivalent measures (defining the direct integral) either. Hence (3.15) allows us to identify

$$C^*(G,Q) = \int_{\hat{H}}^{\oplus} d\chi\, \pi^\chi(C^*(G,Q)), \tag{3.16}$$

where $d\chi$ is any measure on $\hat{H}$ which is supported on all of $\hat{H}$ (recall that we assumed H to be amenable, so that all irreducible representations are weakly contained in the regular representation), and the right-hand side is obviously defined on the Hilbert space $\mathscr{H} = \int^{\oplus} d\chi\, \mathscr{H}^\chi$. To completely define this direct integral, we need to specify a fundamental family of vector fields; this is trivial in the present case, as each π^χ occurs with multiplicity one, so that by [8, 8.6] the direct integral is a constant field (in the sense of [9] or [8, A71]). In other words, the Hilbert space defined by the direct integral consists of equivalence classes of all functions ψ on $\hat{H}$ such that $\psi(\chi) \in \mathscr{H}^\chi$, and its norm is $d\chi$-square-integrable.

Furthermore, under the assumptions stated in the previous section, we know [42] that $\pi^\chi(C^*(G,Q))$ is the algebra $\mathscr{K}(\mathscr{H})$ of compact operators on an infinite-dimensional Hilbert space[q], so that we may identify $C^*(G,Q)$ with a certain space of functions from $\hat{H}$ to $\mathscr{K}(\mathscr{H})$. It then follows that the kernel of the irreducible representation π^χ consists of those functions which vanish on χ. We can completely determine the nature of this function space in case that $\hat{H}$ is Hausdorff. In that case, [8, 3.3.9] says that the function $\chi \to \|\pi^\chi(A)\|$ is continuous on $\hat{H}$ for each $A \in C^*(G,Q)$, whereas [14, VII.6.7] implies that the same function vanishes at infinity. Hence we have proved

Theorem 2. *Let G and H be locally compact, amenable, separable, type I, and 2nd countable, and let H also be CCR, unimodular, and such that $\hat{H}$ is Hausdorff. Then* $C^*(G,G/H) = \mathscr{C}_0(\hat{H} \times \mathscr{K}(\mathscr{H}))$, *the algebra of continuous cross-sections vanishing at infinity of the trivial bundle with base space $\hat{H}$ and fibers $\mathscr{K}(\mathscr{H})$.*

Note that multiplication of two cross-sections is defined by the product of the operators in their image, that is, pointwisely. Also note that the triviality of the bundle could have been derived from [13, Th. 1.3]. The above theorem confirms a very general theorem [13], [14, VII.8.14] stating that any separable C^*-algebra whose spectrum is Hausdorff is of the form $\mathscr{C}_0(\mathscr{B})$ for some bundle $\mathscr{B}$ whose fibers are isomorphic to $\mathscr{K}(\mathscr{H})$ (where $\mathscr{H}$ may vary from fiber to fiber, and may be finite-dimensional).

[q] We exclude the case where Q consists of a finite number of points.

3.3. *Dynamics*

The aim of this section is to apply Theorem 1 to construct a reasonable family of time-evolutions on the transitive transformation group C^*-algebras describing the physical systems under consideration. Before doing so, we give a brief review of some standard facts on the C^*-algebraic description of dynamics.

Algebraic time-evolution

In the abstract algebraic approach to quantum mechanics, time evolution is not *a priori* described by a Hamiltonian on a fixed Hilbert space (which is not yet there), but rather by a one-parameter *-automorphism group (cf. 1.4.5) on the algebra of observables $\mathscr{A}$ [25, 37, 5, 6].

A general strategy for constructing such a one-parameter group consists in determining a faithful representation π of $\mathscr{A}$ on some $\mathscr{H}$, and finding a group of unitaries U_t (or, equivalently, a self-adjoint operator H_p such that $U_t = \exp(itH_p)$) for which $A_t \equiv U_t A U_t^* \in \pi(\mathscr{A})$ for all $A \in \pi(\mathscr{A})$. Since π is faithful, this gives rise to a unique automorphism group α satisfying $\pi(\alpha_t[A]) = U_t \pi(A) U_t^*$ for all $A \in \mathscr{A}$.

Given a group of *-automorphisms α_t on $\mathscr{A}$ one may still, under favourable circumstances, arrive at a conventional Hamiltonian description within each representation of $\mathscr{A}$: let π^χ be such a representation[r]. The Hamiltonian H^χ on the carrier space $\mathscr{H}^\chi$ is defined, if it exists, as the operator implementing α_t in the following sense:

$$e^{itH^\chi} \pi^\chi(A) e^{-itH^\chi} = \pi^\chi(\alpha_t[A]) \tag{3.17}$$

for all $A \in \mathscr{A}$. One sees that H^χ is only defined up to a c-number. In the models studied here, the existence of H^χ is guaranteed (for we will explicitly construct it).

It is clear that the Hamiltonian thus defined is, in general, a representation-dependent object: it derives its structure both from the automorphism group α_t and from the explicit form of π^χ. Now the latter carries global (topological) information about the algebra $\mathscr{A}$, hence on the configuration space Q. In particular, so-called "topological terms" in the Hamiltonian may be expected to arise without any further ado, if there is reason to expect them at all. An exceptional situation arises if the α_t are inner, that is, if unitary elements $U_t \in \mathscr{A}$ exist implementing the time-evolution within the algebra itself (rather than in a given representation). In that case, we evidently have $\exp(itH^\chi) = \pi^\chi(U_t)$, so that the Hamiltonians in all representations are essentially the same. As in quantum field theory, this rather trivial situation will not arise in the present context.

Application to $C^*(G, Q)$

It is reasonable to demand that the time-evolution α_t is G-invariant; this means in concrete terms that the Hamiltonian in any representation $\pi^\chi(\mathscr{A})$ commutes with $\pi^\chi(G)$. This requirement may be translated into a condition on the α_t. To do so, note

[r] We use the symbol π^χ because in this paper we concentrate on irreducible representations, but the definition (3.17) applies in any representation.

that G acts as an automorphism group not only on $C_0(Q)$ (this action was used in the definition of the transformation group C^*-algebra), but also on $C^*(G, Q)$ itself. We denote the latter action by β, and define it on $C_c(G \times Q)$ by[s]

$$(\beta_y[F])(x, q) = F(y^{-1}xy, y^{-1}q), \tag{3.18}$$

to be extended to the whole algebra by continuity. Note that β is implemented in any representation π of $C^*(G, Q)$: the unitary implementing β_x is simply $\pi(x)$. The G-invariance of the time-evolution on $\mathscr{A}$ is now simply expressed by the requirement $\alpha_t \circ \beta_x = \beta_x \circ \alpha_t$ for all $t \in \mathbf{R}, x \in G$. This is guaranteed if the unitary group U_t mentioned above commutes with $\pi(G)$.

After the work in the previous section, the actual construction of a large number of G-invariant time-evolutions on $\mathscr{A}$ is now a piece of cake. We take the faithful representation $\pi = \pi_L$ given in (3.1), (3.2). Let $\mathscr{M}_L$ and $\mathscr{M}_R$ be the von Neumann algebras generated by the left- and right- regular representations of G on $\mathscr{H}_L = L^2(G)$, respectively. A well-known theorem [29, VI.12] asserts that these algebras are each other's commutant, i.e. $\mathscr{M}'_L = \mathscr{M}_R$. Hence any unitary group in $\mathscr{M}_R$ is G-invariant in the above sense. The nontrivial part of proving that such a unitary group defines an automorphism of $\mathscr{A}$ is to show that it maps $\mathscr{A}$ into itself, cf. the one-but-last paragraph. This issue is completely settled by the following theorem, which is stated on the same assumptions as Theorem 2, except that we do not have to assume that $\hat{H}$ is Hausdorff, and that, for simplicity, we assume that G is a Lie group (this assumption may be avoided, at the expense of a more involved formulation of the theorem, by noting that the universal enveloping algebra may be constructed for any connected locally compact group [29, VI.6]). We will use the notation π' for the representation of the universal enveloping algebra $\mathscr{U}(\mathscr{G})$ of G which is derived from a unitary representation π of G [29, 4]. The adjoint representation of G on its Lie algebra $\mathscr{G}$ (with generators T_i) extends to the whole $\mathscr{U}(\mathscr{G})$; this leads to an obvious notion of ad-invariance of elements of $\mathscr{U}(\mathscr{G})$, which is compatible with our earlier use of this term.

Theorem 3. *Let C be a symmetric and ad(H)-invariant element of $\mathscr{U}(\mathscr{G})$. Then the "pre-Hamiltonian" $H_p = \pi'_R(C)$ is essentially self-adjoint on the Gårding domain for $\pi'_R(\mathscr{U}(\mathscr{G}))$ on $L^2(G)$, and is affiliated[t] to $\mathscr{M}'_L$. The unitary group $U_t = \exp(itH_p)$ defines a G-invariant one-parameter *-automorphism group on $C^*(G, Q)$ satisfying $\pi_L(\alpha_t[A]) = U_t \pi_L(A) U_t^*$ for all $A \in C^*(G, Q)$, where π_L is the representation defined by (3.2). If, in addition, $C = C(T_i)$ is central, then the Hamiltonian H^χ in an irreducible representation π^χ is given by the closure of $(\pi^\chi)'(C(-T_i))$ (defined on the appropriate Gårding domain).*

Proof. The essential self-adjointness of H_p follows from standard theorems on representations of enveloping algebras [29, 4]. The second claim, as well as the G-invariance of the time-evolution, follows immediately from the remarks in the previous paragraph.

[s] This is a special case of the definition $(\beta_y[\tilde{F}])(x) = \alpha_y[\tilde{F}(y^{-1}xy)]$, where $\tilde{F}: G \to \mathscr{C}$ is an element of a general crossed product, cf. Sec. 2.1.

[t] In the present case, this means that the bounded spectral projections of its closure commute with all elements of $\mathscr{M}_L$. For the general definition cf. [5, 2.5.7].

The built-in ad(H)-invariance of the unitaries U_t implies that the operators $U_{t;5} \equiv T_{L5} U_t T_{L5}^*$ are decomposable on $\mathscr{H}_5$ (cf. (3.10), etc.). Then (3.14), (3.15) in combination with Williams' theorem [42] quoted in the previous two sections, which guarantees the compactness of the operators $\pi^\chi(A)$, show that $U_{t;5}\pi_5(A)U_{t;5}^* \in \pi_5(C^*(G,Q))$ for all $A \in C^*(G,Q)$ (because the compact operators form a two-sided ideal in $\mathscr{B}(\mathscr{H})$). Therefore, the α_t as defined in the theorem indeed define a *-automorphism group of $C^*(G,Q)$. The final statement follows from the explicit decomposition of π_L in the previous section as well, for $\pi'_L(X(-T_i)) = \pi'_R(X(T_i))$ if X is central. □

Note that the construction of the appropriate Gårding domains will be facilitated by the considerations at the end of Sec. 2.3.

Leaving trivial instances apart, one sees from the construction above that the automorphism groups thus constructed are outer, i.e. they are not implemented within the algebra itself. This fact accounts for the possibility of that topological terms in the Hamiltonian may be "induced" in given representations.

Let us also remark that the construction of arbitrary (i.e., not necessarily G-invariant) time-evolutions on $\mathscr{A}$ follows from a trivial generalization of the above theorem: any ad(H)-invariant[u] group of unitaries on $L^2(G)$ defines such a time-evolution. However, even the class of G-invariant time-evolutions is already very large (for example, any central element of $\mathscr{U}(\mathscr{G})$ will do). One may restrict this class by demanding that, for example, the spectrum of each Hamiltonian H^χ is positive. For G compact this is automatically satisfied if we choose X to be a Casimir operator; for noncompact G one has to be more careful. Also, one may require that the pre-Hamiltonian forces the particle to move on geodesics (in case that Q is a symmetric space), at least in the classical approximation.

Finally, we wish to point out that it should be possible to realize the time-evolutions constructed above as modular automorphism groups, as in the Tomita-Takesaki theory (cf. [5], and refs. therein to the original literature). This could possibly give rise to a proof of Theorem 3 which is independent of the explicit decomposition theory in the previous section; in addition, it would hint at a deep connection between states and time-evolutions in quantum mechanics. The rationale for this belief is that the von Neumann algebras $\mathscr{M}_L$ and $\mathscr{M}_R$ on $L^2(G)$ are connected by a modular conjugation J, defined by $(J\psi_L)(x) = \overline{\psi_L(x^{-1})}$ (which intertwines $\pi_L(G)$ and $\pi_R(G)$). Any vector Ω which is cyclic and separating for $\mathscr{M}_R$ defines a modular group, which is implemented by unitaries $U_t \in \mathscr{M}_R$ (because G, and therefore $\mathscr{M}_{L,R}$ are assumed to be type I). The further requirement of H-invariance on Ω then produces a time-evolution satisfying all our requirements.

References

[1] H. Araki, "On the algebra of all local observables", *Prog. Theor. Phys.* **32** (1964) 844–854.

[2] W. Arveson, *An Invitation to C*-Algebras*, Springer, Berlin, Heidelberg, New York, 1976.

[3] L. Baggett and T. Sund, "The Hausdorff dual for connected groups", *J. Funct. Anal.* **43** (1981) 60–68.

[u] This notion is now used in the sense defined prior to the Corollary in the previous section.

[4] A. O. Barut and R. Raçka, *Theory of Group Representations and Applications*, Panstwowe Wydawnictwo Naukowe, Warszawa, 1977.

[5] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol.* 1 *(2nd revised edition)* Springer, Berlin, Heidelberg, New York, 1987.

[6] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol.* 2, Springer, Berlin, Heidelberg, New York, 1981.

[7] J. F. Cariñena and M. Santander, "Projective covering groups versus representation groups", *J. Math. Phys.* **21** (1980) 440–443.

[8] J. Dixmier, *C*-Algebras*, North-Holland, Amsterdam, 1977.

[9] J. Dixmier, *Von Neumann Algebras*, North-Holland, Amsterdam, 1981.

[10] S. Doplicher, D. Kastler and D. W. Robinson, "Covariance algebras in field theory and statistical mechanics", *Commun. Math. Phys.* **3** (1966) 1–28.

[11] G. G. Emch, "Quantum and classical mechanics on homogeneous Riemannian manifolds", *J. Math. Phys.* **23** (1982) 1785–1791.

[12] B. D. Evans, "C^*-bundles and compact transformation groups", *Mem. AMS* **39**, No. 269 (1982).

[13] J. M. G. Fell, "The structure of algebras of operator fields", *Act. Math.* **106** (1962) 233–280.

[14] J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles, Vol.* 1, Academic Press, San Diego, 1988.

[15] J. M. G. Fell and R. S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundels, Vol.* 2, Academic Press, San Diego, 1988.

[16] K. Fredenhagen, K. H. Rehren and B. Schroer, "Superselection sectors with braid group statistics and exchange algebras, I: General theory", *Commun. Math. Phys.* **125** (1989) 201–226.

[17] J. Fröhlich, "New super-selection sectors ("soliton-states") in two-dimensional Bose quantum field models", *Commun. Math. Phys.* **47** (1976) 269–310.

[18] S. A. Gaal, *Linear Analysis and Representation Theory*, Springer, Berlin, Heidelberg, New York, 1973.

[19] J. Glimm, "Families of induced representations", *Pacific J. Math.* **12** (1962) 855–911.

[20] E. C. Gootman, "The type of some C^*- and W^*-algebras associated with transformation groups", *Pacific J. Math.* **48** (1973) 93–106.

[21] R. Haag, "Quantum physics and C^*-algebras," in *Fields and Geometry*, ed. A. Jadczyk, World Scientific, Singapore, 1986.

[22] R. Haag and D. Kastler, "An algebraic approach to quantum field theory", *J. Math. Phys.* **5** (1964) 848–861.

[23] G. C. Hegerfeldt, "Difficulties with causality in particle localization", *Nucl. Phys.* **B** (Proc. Suppl.) **6** (1989) 231–237.

[24] S. S. Horuzhy, *Introduction to Algebraic Quantum Field Theory*, Reidel, Dordrecht, 1988. D. Kastler (ed.), *The Algebraic Theory of Superselection Sectors: Introduction and Recent Results*, World Scientific, Singapore, 1990.

[25] R. V. Kadison, "Transformation of states in operator theory and dynamics", *Topology* **3**, Suppl. 2 (1965) 177–198.

[26] N. P. Landsman, "C^*-algebraic quantization and the origin of topological quantum effects", *Lett. Math. Phys.* **20** (1990) 11–18.

[27] G. W. Mackey, *Induced Representations*, Benjamin, New York, 1968.

[28] S. Majid, "Hopf algebras for physics at the Planck scale", *Classical and Quantum Gravity* **5** (1988) 1587–1606, and Ph.D. thesis, Harvard Univ. 1988.

[29] K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups*, Panstwowe Wydawnictwo Naukowe, Warszawa, 1968.

[30] C. C. Moore and J. Rosenberg, "Groups with T_1 primitive ideal spaces", *J. Funct. Anal.* **22** (1976) 204–222.

[31] G. K. Pedersen, *C*-Algebras and Their Automorphism Groups*, Academic Press, New York, 1979.

[32] J. P. Pier, *Amenable Groups*, Wiley, New York, 1984.

[33] M. Reed and B. Simon, *Functional Analysis*; *Methods of Modern Mathematical Physics, Vol. I* (*2nd revised edition*), Academic Press, New York, 1980.

[34] M. A. Rieffel, "On the uniqueness of the Heisenberg commutation relations", *Duke Math. J.* **39** (1972) 745–752.

[35] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer, Berlin, Heidelberg, New York, 1982.

[36] I. E. Segal, *Mathematical Problems of Relativistic Physics*, Amer. Math. Soc., Providence, 1963.

[37] B. Simon, "Quantum dynamics, from automorphism to Hamiltonian", in *Studies in Mathematical Physics*, eds. E. H. Lieb et. al., Princeton Univ. Press, Princeton, 1976.

[38] R. F. Streater and I. F. Wilde, "Fermion states of a Bose field", *Nucl. Phys.* **B24** (1970) 561–575.

[39] V. S. Varadarajan, *Geometry of Quantum Theory*, *Vol.* 2, Van Nostrand Reinhold, New York, 1970.

[40] N. R. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Dekker, New York, 1973.

[41] G. C. Wick, A. S. Wightman and E. P. Wigner, "The intrinsic parity of elementary particles", *Phys. Rev.* **88** (1952) 101–105.

[42] D. P. Williams, "The topology of the primitive ideal space of transformation group C^*-algebras", *Trans. AMS* **266** (1981) 335–355.

[43] D. P. Williams, "Transformation group C^*-algebras with Hausdorff spectrum", *Illinois J. Math.* **26** (1982) 317–321.