INDUCED REPRESENTATIONS, GAUGE FIELDS, AND QUANTIZATION ON HOMOGENEOUS SPACES

N. P. LANDSMAN*

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

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We study representations of the enveloping algebra of a Lie group G which are induced by a representation of a Lie subgroup H, assuming that G/H is reductive. Such representations describe the superselection sectors of a quantum particle moving on G/H. It is found that the representatives of both the generators and the quadratic Casimir operators of G have a natural geometric realization in terms of the canonical connection on the principal H-bundle G. The explicit expression for the generators can be understood from the point of view of conservation laws and moment maps in classical field theory and classical particle mechanics on G/H. The emergence of classical geometric structures in the quantum-mechanical situation is explained by a detailed study of the domain and possible self-adjointness properties of the relevant operators. A new and practical criterion for essential self-adjointness in general unitary representations is given.

1. Introduction

Finite-dimensional homogeneous configuration spaces $Q = G/H$, where G and $H \subset G$ are Lie groups, are worthy of study in their own right, classically as well as quantum-mechanically; moreover, such spaces may serve as caricatures of interesting field theories, some of whose (topological) features they may reflect in a tractable context, where one does not have to worry about the subtleties of an infinite number of degrees of freedom. For example, the configuration space of a (classical) gauge theory may be taken to be the space of orbits $\mathcal{Q} = \mathcal{G}/\mathcal{H}$, where $\mathcal{G}$ is the space of gauge fields on a manifold M, and $\mathcal{H}$ is the group of local gauge transformations, acting on $\mathcal{G}$ in the usual way: this situation is to some extent modeled by $Q = G/H$ (ignoring the fact that here G is a group). More directly, Q may be regarded as a one-dimensional σ-model with homogeneous target space.

A closely related goal is to understand the emergence of gauge fields in algebraic quantum mechanics and quantum field theory, where one starts with an algebra of observables $\mathcal{A}$, in which the gauge fields and the charged (matter) fields themselves are obviously invisible. The role of charged matter fields as intertwining operators between various representations of $\mathcal{A}$ is by now well understood in theories with global symmetries [7], but local gauge theories have defied a full understanding so far. Therefore, any example in which gauge fields emerge in connection with the representation theory of an algebra of observables should be welcome. As we shall see, particle motion on G/H provides a whole class of such examples.

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The quantization theory of a particle moving on an arbitrary homogeneous manifold $Q = G/H$ was initiated by Mackey [25], who replaced the canonical commutation relations as the basic object of study by systems of imprimitivity, and showed that such systems admit (unitarily) inequivalent representations, labeled by the dual $\hat{H}$ of H (that is, the set of equivalence classes of irreducible unitary representations of H). This work was extended in various directions in [5, 14]; it should be mentioned, that the idea that a given classical system may admit a family of inequivalent quantizations was independently arrived at in the context of geometric quantization [32].

In [21, 22] Mackey's approach was reformulated so as to fit into the algebraic theory of superselection sectors [13]. This reformulation is based on results of Glimm [9], who showed that Mackey's transitive system of imprimitivity over Q, which is formulated in terms of a unitary representation of G on a given Hilbert space $\mathscr{H}$, tied up with a projection-valued measure on Q, is equivalent to a (generically non-faithful) representation of the transformation group C^*-algebra $\mathscr{A} = C^*(G, Q)$. Taking $\mathscr{A}$ to be the algebra of observables of the system, which is appropriate since it contains all algebraic information about the position and momentum observables of the particle, one may then interpret the inequivalent representations of $\mathscr{A}$ as superselection sectors of the particle, or, equivalently, as inequivalent quantizations.

A given sector is, accordingly, labeled by a class $\chi \in \hat{H}$, and the corresponding representation π^χ of $\mathscr{A}$ is conveniently realized on a Hilbert space $\mathscr{H}^\chi$ of sections of the homogeneous vector bundle $E^\chi = G \times_H \mathscr{H}_\chi$ over Q (cf. subsection 2.1 for definitions and notation). This Hilbert space carries a unitary representation $\pi^\chi(G)$, closely related to $\pi^\chi(\mathscr{A})$, which is just the one induced by $\pi_\chi(H)$, where π_χ is an arbitrary element of the class χ (this geometric realization of induced representations of Lie groups is due to Bott, cf. [36, 37]). In addition, $\mathscr{H}^\chi$ carries a representation of $C_0(Q)$ (the C^*-algebra of continuous functions on Q which vanish at infinity) by multiplication operators, which together with $\pi^\chi(G)$ completely determines $\pi^\chi(\mathscr{A})$, and *vice versa*. This set-up, along with its quantum-mechanical significance, is explained in detail in [22], and in [23] it is shown through examples that several known topological quantum effects can be rigorously understood in these terms, topological quantum numbers and topologically quantized coupling constants generically corresponding to elements χ of $\hat{H}$. We note that the usual quantization on $L^2(Q)$ is just the one induced by the trivial representation of H.

The nicest feature of this approach to quantization is that it provides an explicit, geometric expression for the Hamiltonian in any superselection sector. One can look at the quantum Hamiltonian H^χ either as an operator implementing a given one-parameter automorphism group on $\mathscr{A}$ in the sector χ, or as the 'quantization' $d\pi^\chi(H)$ of some classical Hamiltonian H; the time-evolution that we will study is such that in the former description it is implemented in the trivially induced representation by (minus) the Laplacian on $L^2(Q)$, and in the latter representation corresponds to the symbol of the Laplacian. In any case, it was found in examples [23], and later in the more general case that G is compact [24], that H^χ is a gauge-covariant Laplacian on E^χ with respect to a background gauge field of a known type: the canonical (or reductive) connection on G [18, 19]. In other words, in the non-trivial superselection sectors the particle behaves as if it were moving in a fictitious external Yang-Mills field with gauge group H.

The aim of the present paper is to give a detailed and mathematically rigorous exposition of a circle of ideas generalizing this observation firstly to more general cosets (viz. reductive ones, with no compactness conditions), and secondly to a larger class of elements of the enveloping algebra of G (in the situation described above the Hamiltonian is the representative of the second-order Casimir operator of G). More specifically, we will analyze the role played by gauge fields in the theory of induced representations of Lie groups: as such, our results can be understood, without reference to quantum mechanics, as a possible contribution to the study of geometric structures arising in representation theory. However, we shall see that certain expressions, which from a purely mathematical point of view appear to be out of place, have a natural physical interpretation in terms of classical conservation laws in field theory as well as in particle mechanics.

Our starting point is the above-mentioned realization of a given induced representation $\pi^\chi(G)$ on a Hilbert space $\mathscr{H}^\chi$ of L^2-sections of a homogeneous vector bundle E^χ over Q. Our main concern is to study the associated representation $d\pi^\chi$ of the enveloping algebra $\mathscr{U}(\mathbf{g})$, concentrating on its first- and second-order elements. In any case, $\mathscr{U}(\mathbf{g})$ is represented by unbounded operators, so we have to address problems of domains and (essential) self-adjointness of the relevant operators. These problems are fairly straightforward, but since their treatment requires some functional-analytic tools (as opposed to the geometric ones used in the main body of the paper) we have deferred this discussion to the Appendix. For the benefit of those interested in the results only, and to keep matters transparent, we have written the Appendix in the lemma-theorem format.

The results are satisfactory, in that they justify the use of the differential-geometric machinery in the following sense: representatives $id\pi^\chi(X)$ of the Lie algebra $\mathbf{g}$, as well as of central (and several other) elements in $\mathscr{U}(\mathbf{g})$, are essentially self-adjoint on the domain Γ_c^χ of compactly supported smooth sections. More generally, the naive expression for a representative of $\mathscr{U}(\mathbf{g})$ as a differential operator on E^χ has the same closure as the corresponding operator defined on the Gårding domain, so that we can consistently define all unbounded operators of interest as naive differential operators. We also give an expression for their closure: this implies that we explicitly know the domain of self-adjointness of operators which are essentially self-adjoint on Γ_c^χ. These results eventually follow from a theorem of Thomas [35]. We also prove another useful result, which states that the (essential) self-adjointness of any $d\pi^\chi(X)$, $X \in \mathscr{U}(\mathbf{g})$, in a reducible unitary representation $\pi(G)$ of a type I Lie group G, is equivalent to its (essential) self-adjointness in all subrepresentations of π. This criterion is particularly convenient for induced representations, where the subrepresentations are easily determined from Frobenius reciprocity: a number of the known criteria guaranteeing essential self-adjointness on the Gårding domain [16] are recovered in this way, with the added benefit that we can work on the domain Γ_c^χ.

Feeling secure that we can work in a smooth (rather than an L^2) context, we start Sec. 2 with a quick summary of invariant metrics and connections on homogeneous spaces (a subject exhaustively treated in [19]), followed by an equally short review of induced representations realized on vector bundles. We then derive expressions for the representative $d\pi^\chi(Y)$ of an arbitrary element Y in the Lie algebra $\mathbf{g}$, and for generic second-order Casimir operators in $\mathscr{U}(\mathbf{g})$. One term in the former is easily understood (it is the covariant derivative, relative to the canonical connection, in the direction of the Killing vector field defined by Y), but there is a curious additional term, which has no straightforward geometric interpretation. To understand this extra term, we firstly analyze the classical field theory corresponding to the quantum particle (that is, we interpret the Schrödinger wave function, which here takes values in the carrier space $\mathscr{H}_\chi$ of $\pi_\chi(H)$, as a classical field), and show (inspired by the work of Jackiw and Manton [15]) that the extra term is precisely the additional contribution to the conserved charge associated to Y caused by the presence of the external gauge field.

Secondly, we explain how the extra term in $d\pi^\chi(Y)$ has a classical analogue in particle mechanics, where once again it can be understood as an additional term in a conserved quantity (here given by the momentum map of Y) necessitated by the gauge field. We use (and briefly summarize) the symplectic formalism developed by Sternberg et. al. [33, 38, 12] to construct the phase space of a classical particle with a Yang-Mills charge, and see explicitly how the additional contribution arises in the momentum map in the charged sector. These considerations lead to the intriguing idea, that a particle on G/H not only admits inequivalent quantizations, but has a family of distinct 'classicizations' as well. The parallel with the quantum case goes quite far, its main feature being the role played by Yang-Mills fields in either case. In this paper we will not go beyond drawing the analogies, leaving further work in this direction to the future.

We are grateful to a referee for pointing out an omission in the derivation of Eq. (2.18).

2. The Geometry of Induced Representations

2.1. *Homogeneous spaces*

Assumptions and notation

The results of this paper are derived under the following assumptions and notation:

1. G is a finite-dimensional Lie group of type I [37] (with Lie algebra $\mathbf{g}$), and H a closed Lie subgroup (with Lie algebra $\mathbf{h}$); $\mathbf{g}$ and $\mathbf{h}$ are generated by $\{T_a\}_{a=1,\ldots,d_G}$, and $\{T_i\}_{i=1,\ldots,d_H}$ (coinciding with the first d_H T_a's), respectively. The structure constants relative to this basis are called C_{ab}^c, etc.

2. $\mathbf{g}$ admits a reductive decomposition [19] $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$, where $\pi_{\mathrm{ad}}(h)X \in \mathbf{m}$ for all $h \in H$ and all $X \in \mathbf{m}$ (here π_{ad} denotes the adjoint representation). In particular, one must have $[\mathbf{h}, \mathbf{m}] \subset \mathbf{m}$. The dual $\mathbf{g}^*$ has a corresponding decomposition $\mathbf{g}^* = \mathbf{h}^* \oplus \mathbf{m}^*$. Hence one has representations $\pi_{\mathrm{is}}(H)$ on $\mathbf{m}$ and $\pi_{\mathrm{ci}}(H)$ on $\mathbf{m}^*$, which are the obvious restrictions of the adjoint representation π_{ad} and the co-adjoint representation π_{co}, respectively. We assume that π_{is} is faithful (the opposite extreme, in which $G = H \times M$ as a group, is easily handled as well; we leave this to the reader). The dimension of $\mathbf{m}$ is equal to d_Q, the dimension of $Q = G/H$. We choose a basis $\{T_\alpha\}_{\alpha=1,\ldots,d_Q}$ (coinciding with the last d_Q T_a's). Accordingly, doubly occurring indices α, β are to be summed from 1 to d_Q, a, b, c from 1 to d_G, and i, j, k from 1 to d_H.

Let $q_0 \in Q$ stand for either the coset $\{H\} \in G/H$, or for a given H-invariant point of any realization of Q, as an explicitly given manifold. We will identify $T_{q_0}Q$ (the tangent space of Q at q_0) with $\mathbf{m}$, so that $X \in \mathbf{m}$ corresponds to X_{q_0} defined by $(X_{q_0}f)(q_0) = d/dt\, f(\exp(tX)q_0)_{|t=0}$ for $f \in C^\infty(Q)$. More generally, any $Y \in \mathbf{g}$ defines a vector field Y_Q on Q whose value Y_q in T_qQ is given by $(Y_a f)(q) = d/dt\, f(\exp(tY)q)_{|t=0}$. All this can be dualized to $\mathbf{g}^*$, $\mathbf{m}^*$ and the cotangent bundle T^*Q: we denote the appropriate bases by $\{\theta_a\}$, $\{\theta_\alpha\}$, etc.

3. $\mathbf{m}$ has a $\pi_{\text{is}}(H)$-invariant nondegenerate inner product g_0 (not necessarily positive-definite). This gives rise to a G-invariant quasi-Riemannian metric g on Q, obtained from g_0 by left-translation [19]. Hence one has Killing vector fields $K(Y)$ for each $Y \in \mathbf{g}$, whose value at $q \in Q$ is $K(Y)_q = -Y_q$. We abbreviate $K(T_a) \equiv K_a$.

Let $p_{GQ}: G \to Q$ be the canonical projection (i.e., $p_{GQ}x = xq_0$ for $x \in G$), and let $s: Q \supset U \to G$ be a local section (that is, $p_{GQ} \circ s = \text{id}$), such that $q_0 \in U$. Assuming that the basis $\{T_\alpha\}$ has been chosen (quasi-) orthonormal with respect to g_0, we can define a vielbein $\{e_\alpha\}$ by giving its value at q as $(e_\alpha)_q = L'_{s(q)}(T_\alpha)_{q_0}$ (here L_x denotes the left-action of $x \in G$ on Q, and L' is the push-forward of any map L between two manifolds. We denote the pull-back by L^*). This implies $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$, where η is diagonal with entries ± 1, depending on the signature of g_0. The vielbein is related to the Killing vector fields by

$$(K_a)_q = -\pi_{\text{co}}(s(q))_a{}^\beta (e_\beta)_q; \qquad (e_\alpha)_q = -\pi_{\text{co}}(s(q)^{-1})_\alpha{}^b (K_b)_q. \tag{2.1}$$

The metric on Q defines a measure called dq, which is necessarily the unique (up to a constant scale) quasi-G-invariant measure on Q [37]. Also, π_{is} defines an embedding of H into the isometry group $SO(n, m)$ of g_0 (with $n + m = d_Q$).

This ends the list of 'essential' assumptions, i.e., conditions without which our approach would have to be seriously modified. The existence of a reductive decomposition and of an invariant metric is guaranteed if H is compact [19]. It should be pointed out that neither of these structures is necessarily unique. As will become clear shortly, the gauge field in the Hamiltonian and in the generalized momentum operators actually depends on the precise choice of the (arbitrary) reductive decomposition; moreover, the Hamiltonian depends on the metric as well. The point is that this dependence is irrelevant up to unitary equivalence: different choices of the various geometric structures (and this even goes as far as the signature of the metric on Q) give rise to different realizations of the induced representation $d\pi^\chi$ of $\mathcal{U}(\mathbf{g})$ (and, therefore of the full algebra of observables $C^*(G, Q)$) which are related by a unitary transformation. In this sense, the geometry is just an auxiliary tool, subordinate to the (analytic) representation-theoretic aspects of the problem.

In what follows, we make two further, inessential assumptions which simplify the formulae somewhat. Firstly, we will assume that H is compact (hence unimodular), and that G is unimodular. Hence we can choose g to be positive-definite, leading to an embedding of H into $SO(d_Q)$. Furthermore, the measure dq is now G-invariant. One can easily rewrite the formulae below so as to fit the general case, by inserting the appropriate Radon-Nikodym derivatives that appear in induced representations in the quasi-invariant case [37], and by replacing $SO(d_Q)$ by $SO(n, m)$. We will discuss representations $\pi^\chi(G)$ which are induced from an irreducible unitary representation $\pi_\chi(H)$. The latter is finite-dimensional for compact H; the generalization of the results below to non-compact H is non-trivial in case that π_χ is infinite-dimensional. One then has to restrict the space Γ_c^χ of compactly supported smooth sections of the vector bundle E^χ (cf. subsection 2.2 below) to those sections taking values in the domain of the representation $d\pi^\chi$ of the enveloping algebra $\mathscr{U}(\mathbf{h})$.

Secondly, the form g_0 defining the metric on Q is supposed to satisfy

$$g_0(X,[Y,Z]_{|\mathbf{m}}) + g_0(Z,[Y,X]_{|\mathbf{m}}) = 0 \tag{2.2}$$

for all X, Y, $Z \in \mathbf{m}$. The reason for this assumption will become apparent in the derivation of (2.18); it is always possible to choose g_0 so as to satisfy it in each of the following cases: G compact, G/H symmetric, G semi-simple, and more generally if g_0 is the restriction to $\mathbf{m}$ of a nondegenerate form on $\mathbf{g}$ which is invariant under $\pi_{\text{ad}}(G)$.

Invariant connections on homogeneous bundles over Q

We will employ two principal fibre bundles over Q. The general scheme is

$$\begin{array}{ccccc} G & \longrightarrow & P & \longleftarrow & K \\ & & \Big\downarrow{\scriptstyle p_{PQ}} & & \\ & & G/H & & \end{array}$$

That is, P is a bundle over $Q = G/H$ with gauge group K (acting from the right) and another group G acting on P from the left. (We use the symbol p_{PQ} for the projection of P onto Q, π generically denoting a representation.) We denote this situation by $P(Q, K, G)$. The two cases of interest to us are the bundle of orthonormal frames $O(Q, SO(d_Q), G)$, with the left-action of G on a frame given by the push-forward of the G-action on Q, and the H-structure $G(Q, H, G)$, with the natural left- and right-action of G and H on G, respectively [18]. The bundle G is a sub-bundle of O, a point $x \in G$ corresponding to the vielbein frame at $xq_0 \in Q$ defined in item 3 above, relative to a section satisfying $s(xq_0) = x$, i.e., $(e_\alpha)_{xq_0} = L'_x(T_\alpha)_{q_0}$.

Now take the fixed point $e \in P$ (here e is the identity of G); for $P = O$ this is defined via the embedding of G in O. Define a map $\lambda : H \to K$ by $\lambda = \text{id}$ for $P = G$, and $\lambda = \pi_{\text{is}}$ for $P = O$. The derived Lie algebra map is called $d\lambda : \mathbf{h} \to \mathbf{k}$. Invariant connections on P are in 1-1 correspondence with maps $\Lambda : \mathbf{g} \to \mathbf{k}$, which coincide with $d\lambda$ when restricted to $\mathbf{h}$, and satisfy the intertwiner relation

$$\Lambda(\pi_{\text{ad}}(h)Y) = \pi_{\text{ad}}(\lambda(h))\Lambda(Y) \tag{2.3}$$

for all $Y \in \mathbf{g}$ (on the right-hand side π_{ad} of course refers to K). With Y_e denoting the vector corresponding to $Y \in \mathbf{g}$ relative to the left-action of G on P, and Z_e the vertical vector defined by $Z \in \mathbf{k}$ via the right-action of K on P, (cf. item 2 above), the $\mathbf{k}$-valued one-form A defined in T_e^*P by $\langle A, Y_e\rangle = \Lambda(Y)$ and $\langle A, Z_e\rangle = Z$, and at other points of P either by vertical translation by K, or translation by G, is well-defined, and defines a connection because of (2.3) [19].

Of interest to us is the canonical connection (also called reductive, or H-connection, which name we shall adopt) A^H on G, defined by $\Lambda^H : \mathbf{g} \to \mathbf{h}$ given by

$$\Lambda^H(Y) = Y \quad (Y \in \mathbf{h}); \qquad \Lambda^H(Y) = 0 \quad (Y \in \mathbf{m}). \tag{2.4}$$

A^H is just the $\mathbf{h}$-component of the left-invariant Maurer-Cartan form on G. The space of horizontal vectors in T_xG is spanned by the left-invariant vectors $(T_a^L)_x$ (with T_a^L defined via the right (anti-) action $x \to xy$ of G on itself; similarly, the right-invariant vector fields T_a^R are defined via the left (anti-) action $x \to y^{-1}x$ of G on itself).

We will also use the Levi-Civita connection ω^{LC} on $P = O$, defined by $\Lambda^{LC} : \mathbf{g} \to \mathbf{so}(\mathbf{d}_Q)$. We take $\mathbf{so}(\mathbf{d}_Q)$ in its defining representation on $\mathbf{m}$, and define Λ^{LC} by its action on an arbitrary vector $X \in \mathbf{m}$ [19]:

$$\Lambda^{LC}(Y)X = [Y, X] \quad (Y \in \mathbf{h}); \qquad \Lambda^{LC}(Y)X = \frac{1}{2}[Y, X]_{|\mathbf{m}} \quad (Y \in \mathbf{m}). \tag{2.5}$$

Note that for symmetric spaces Q one has $[\mathbf{m}, \mathbf{m}] \subset \mathbf{h}$, so that (2.4) and (2.5) essentially coincide. Also, due to our assumption (2.2), Λ^{LC} indeed defines the Levi-Civita connection, cf. Sec. X.3 in [19].

2.2. *Induced representations and gauge fields*

Realization on sections of vector bundles

Given a representation $\pi_\chi(H)$ on a finite-dimensional Hilbert space $\mathscr{H}_\chi$, one can form the vector bundle $E^\chi = G \times_H \mathscr{H}_\chi$ associated to G; its base space is Q, and its fibers are isomorphic to $\mathscr{H}_\chi$, which is identified with the fiber $p_{EQ}^{-1}(q_0)$ (p_{EQ} being the projection of E^χ onto Q). Points of E^χ are equivalence classes $[x, \psi_\chi]$ ($\psi_\chi \in \mathscr{H}_\chi$) under the equivalence relation $(x, \psi_\chi) \sim (xh^{-1}, \pi_\chi(h)\psi_\chi)$ $(h \in H)$. Then G acts on E^χ by $y[x, \psi_\chi] = [yx, \psi_\chi]$, and therefore it also acts on Γ_c^χ (the space of smooth cross-sections of E^χ with compact support) by means of $(\pi^\chi(y)\Psi^\chi)(q) = y\Psi^\chi(y^{-1}q)$; one can close Γ_c^χ in the natural inner product (coming from the invariant measure dq on Q) [36, 37] to obtain a Hilbert space $\mathscr{H}^\chi$, on which the induced representation π^χ acts as above, extended to all of $\mathscr{H}^\chi$ by continuity. It is convenient to realize the cross-sections as functions $\psi^\chi : G \to \mathscr{H}^\chi$ which are H-equivariant, i.e., satisfy $\psi^\chi(xh) = \pi_\chi(h^{-1})\psi^\chi(x)$ for all $x \in G$ and $h \in H$. In this realization [36, 37]

$$(\pi^\chi(y)\psi^\chi)(x) = \psi^\chi(y^{-1}x). \tag{2.6}$$

Being in Γ_c^χ then means having compact support on G up to H-translations, that is, the projection of the support onto Q is compact; for H compact this is equivalent to compact support on G.

We wish to study the derived representation $d\pi^\chi$ of the enveloping algebra $\mathscr{U}(\mathbf{g})$ on $\mathscr{H}^\chi$. As shown in the Appendix, this is simply given on the domain $\Gamma_c^\chi \subset \mathscr{H}^\chi$ by

$$(d\pi^\chi(T_a)\psi^\chi)(x) = \frac{d}{dt}\psi^\chi(e^{-tT_a}x)_{|t=0}, \tag{2.7}$$

for all $T_a \in \mathbf{g}$, and extended to $\mathscr{U}(\mathbf{g})$ in the obvious way. Now take any invariant connection A on G, defined by a function Λ, as explained after (2.3) above. This defines a covariant derivative ∇^χ on the cross-sections Γ_c^χ of E^χ. The horizontal lift to T_xG of the Killing vector $(K_a)_{xq_0}$ in $T_{xq_0}Q$ is $(T_a^R)_x - \langle A_x, (T_a^R)_x \rangle$. The first term is just the right-hand side of (2.7), whereas the second term equals the vertical vector at x corresponding to $\Lambda(\pi_{\mathrm{ad}}(x^{-1})T_a) \in \mathbf{h}$. Hence from the H-equivariance of ψ^χ we find

$$(d\pi^\chi(T_a)\psi^\chi)(x) = (\nabla^\chi_{K_a}\psi^\chi)(x) + d\pi_\chi(\Lambda(\pi_{\mathrm{ad}}(x^{-1})T_a))\psi^\chi(x). \tag{2.8}$$

The first term is geometrically nice, but the second one looks odd. It will be interpreted in Sec. 3. The induced representation $\pi^\chi(G)$, hence the integrable representation $d\pi^\chi(\mathbf{g})$, in conjunction with the representation of $C_0(Q)$ by multiplication operators on $\mathscr{H}^\chi$, completely determines the associated representation π^χ of the algebra of observables $\mathscr{A} = C^*(G, Q)$ of the quantum particle moving on $Q = G/H$ [21, 22, 24].

The Hamiltonian (or second-order Casimir operator)

We wish to find an expression for $d\pi^\chi(C_2)$, where $C_2 \in \mathscr{U}(\mathbf{g})$ is a second-order Casimir invariant, which is more illuminating than the one obtained by just squaring (2.8). We start by recalling that $TQ = E^{\mathrm{is}} = G \times_H \mathbf{m}$ and $T^*Q = E^{\mathrm{ci}} = G \times_H \mathbf{m}^*$, these vector bundles being defined with respect to $\pi_{\mathrm{is}}(H)$ on $\mathscr{H}_{\mathrm{is}} = \mathbf{m}$ and $\pi_{\mathrm{ci}}(H)$ on $\mathscr{H}_{\mathrm{ci}} = \mathbf{m}^*$, respectively; an equivalence class $[x, X] \in E^{\mathrm{is}}$ corresponding to $L'_x X_{q_0} \in T_{xq_0}Q$, and $[x, \theta] \in E^{\mathrm{ci}}$ being identified with $L^*_{x^{-1}}\theta_{q_0} \in T^*_{xq_0}Q$. Hence for given $\psi^\chi \in \Gamma_c^\chi$ we can regard $\nabla^\chi\psi^\chi$ as a section of $E^{\mathrm{ci}\otimes\chi} \equiv E^{\mathrm{ci}} \otimes E^\chi$, that is, as an H-equivariant map $\nabla^\chi\psi^\chi : G \to \mathbf{m}^* \otimes \mathscr{H}_\chi$. But elements of $\mathbf{m}^*$ are (linear) functionals on $\mathbf{m}$, so that $\nabla^\chi\psi^\chi$ is, in fact, a map from $G \times \mathbf{m}$ into $\mathscr{H}_\chi$, whose value $\langle \nabla^\chi\psi^\chi | x, Y \rangle$ on $(x, Y) \in G \times \mathbf{m}$ is given by the evaluation of the covariant derivative $(\nabla^\chi\psi^\chi)(x)$ in the direction $[x, Y] \in T_{xq_0}Q$. Since $[x, Y] = L'_x Y_{q_0} = -\pi_{\mathrm{ad}}(x^{-1})K(Y)_{xq_0}$, we find from (2.8)

$$\langle \nabla^\chi\psi^\chi | x, Y \rangle = (d\pi_R(Y)\psi^\chi)(x) + d\pi_\chi(\Lambda(Y))\psi^\chi(x), \tag{2.9}$$

where we have written π^χ in terms of the right-regular representation (tensored with the trivial representation on $\mathscr{H}_\chi$) π_R of G, defined by $(\pi_R(y)\psi^\chi)(x) = \psi^\chi(xy)$. Note that π_R does not map $\mathscr{H}^\chi$ into itself.

Now choose a linear connection ω on the bundle O of orthonormal frames on Q, defined by a function $\hat{\Lambda} : \mathbf{g} \to \mathbf{so}(\mathbf{d}_Q)$, as explained after (2.3). This gives a covariant derivative $\hat{\nabla}$ on the cotangent bundle $T^*Q = O \times_{\mathbf{so}(\mathbf{d}_Q)} \mathbf{m}^*$. Since this bundle is identical to $E^{\mathrm{ci}} = G \times_H \mathbf{m}^*$, we can realize sections of T^*Q as H-equivariant functions $\psi^{\mathrm{ci}} : G \to \mathbf{m}^*$. Even if the connection ω on P is not reducible to a connection of $G \subset O$ (which would be the case if ω takes values in $\mathbf{h}$), we still have a formula analogous to (2.8), viz.

$$(d\pi^{\mathrm{ci}}(T_a)\psi^{\mathrm{ci}})(x) = (\hat{\nabla}_{K_a}\psi^{\mathrm{ci}})(x) + d\pi_d(\hat{\Lambda}(\pi_{\mathrm{ad}}(x^{-1})T_a))\psi^{\mathrm{ci}}(x), \tag{2.10}$$

where π_d is the defining representation of $SO(d_Q)$ on $\mathbf{m}$, identified with its dual on $\mathbf{m}^*$. To derive (2.10), one initially regards ψ^{ci} as an $SO(d_Q)$-equivariant function from O to $\mathbf{m}^*$, which, by equivariance, is determined by its values on G.

Note that, since $\psi^{\text{ci}}(x) \in \mathbf{m}^*$, one has for $Z \in \mathbf{so(d_Q)}$ and $X \in \mathbf{m}$

$$\langle d\pi_d(Z)\psi^{\text{ci}}(x), X\rangle = -\langle \psi^{\text{ci}}(x), d\pi_d(Z)X\rangle. \tag{2.11}$$

We now take the tensor product of the covariant derivatives ∇^χ on E^χ and $\hat{\nabla}$ on E^{ci} to obtain a connection $\widehat{\nabla^\chi}$ on $E^{\text{ci}} \otimes E^\chi$. For any section $\psi^{\text{ci}\otimes\chi}$ of $E^{\text{ci}} \otimes E^\chi$ we then have, analogously to (2.9),

$$\langle \widehat{\nabla^\chi}\psi^{\text{ci}\otimes\chi}|x, Y\rangle = (d\pi_R(Y)\psi^{\text{ci}\otimes\chi})(x) + [d\pi_d(\hat{\Lambda}(Y)) \otimes \mathbf{1}_\chi + \mathbf{1}_m \otimes d\pi_\chi(\Lambda(Y))]\psi^{\text{ci}\otimes\chi}(x), \tag{2.12}$$

where $\mathbf{1}_m$ and $\mathbf{1}_\chi$ are the unit matrices on $\mathbf{m}^*$ and $\mathscr{H}_\chi$, respectively. We apply this formula to the case $\psi^{\text{ci}\otimes\chi} = \nabla^\chi\psi^\chi$ (with $\psi^\chi \in \Gamma_c^\chi$); this is regarded as a map from $G \times \mathbf{m} \times \mathbf{m}$ into $\mathscr{H}_\chi$; the first factor $\mathbf{m}$ refers to the slot of $\widehat{\nabla^\chi}$, and the second one to that of ∇^χ. From (2.9), (2.11), and (2.12) we then have

$$\begin{aligned}\langle \widehat{\nabla^\chi}\nabla^\chi\psi^\chi|x, Y, X\rangle = \{&[d\pi_R(Y) + d\pi_\chi(\Lambda(Y))][d\pi_R(X) + d\pi_\chi(\Lambda(X))] \\ &- d\pi_R(d\pi_d(\hat{\Lambda}(Y))X) - d\pi_\chi(\Lambda[d\pi_d(\hat{\Lambda}(Y))X])\}\psi^\chi(x),\end{aligned} \tag{2.13}$$

with $x \in G$ and $X, Y \in \mathbf{m}$.

We define a second-order differential operator on E^χ by

$$(\Delta^\chi\psi^\chi)(x) = (\delta^{\alpha\beta}\widehat{\nabla^\chi_{e_\alpha}}\nabla^\chi_{e_\beta}\psi^\chi)(x), \tag{2.14}$$

in terms of the vielbein $\{e_\alpha\}_{\alpha=1,\ldots,d_Q}$. Using the second equation in (2.1) (with $q = xq_0$ and $s(q) = x$), it follows from (2.14) and the text preceding (2.9) (relating the Killing vectors to the $(T_a)_{q_0}$, noting that these vectors vanish if a is not in the range $1, \ldots, d_Q$) that

$$(\Delta^\chi\psi^\chi)(x) = \langle \widehat{\nabla^\chi}\nabla^\chi\psi^\chi|x, T_\alpha, T_\alpha\rangle. \tag{2.15}$$

This can be evaluated from (2.13), and it is clear that the resulting expression greatly simplifies if we take the connection A (defining ∇^χ) to be the H-connection A^H, cf. (2.4), and the linear connection ω on O to be the Levi-Civita connection defined by (2.5). Then Δ^χ becomes a gauge-covariant Laplacian, and we find

$$\Delta^\chi\psi^\chi = d\pi_R(T_\alpha)\, d\pi_R(T_\alpha)\psi^\chi. \tag{2.16}$$

Here and in what follows, the gauge-covariant Laplacian Δ^χ on Γ_c^χ is defined with respect to the H- and the Levi-Civita connection. Now assume that there exists a Casimir operator $C_2(H)$ in $\mathscr{U}(\mathbf{h})$, in such a way that

$$C_2(G) = -\sum_{\alpha=1}^{d_Q} T_\alpha^2 + C_2(H) \tag{2.17}$$

is a Casimir operator for G (i.e., a central element of $\mathscr{U}(\mathbf{g})$). The choice of $C_2(H)$ is not necessarily unique, and the following expression evidently holds for any such choice. Since $C_2(G)$ is invariant under (the extension from $\mathbf{g}$ to $\mathscr{U}(\mathbf{g})$ of) $\pi_{\mathrm{ad}}(G)$, one has $d\pi^\chi(C_2(G)) = d\pi_R(C_2(G))$, so that finally

$$H^\chi \equiv d\pi^\chi(C_2(G)) = -\Delta^\chi + d\pi_\chi(C_2(H)); \tag{2.18}$$

$\pi_\chi(H)$ being irreducible, the second term is a constant. The left-hand side is the Hamiltonian H^χ in the superselection sector χ (that is, the self-adjoint operator implementing time-evolution in the irreducible representation π^χ of the algebra of observables $\mathscr{A} = C^*(G, Q)$ of a particle moving on $Q = G/H$) [21, 22, 23, 24] so that we have found a geometric expression for the Hamiltonian. By the results of the Appendix, H^χ is essentially self-adjoint on the domain $\Gamma_c^\chi \subset \mathscr{H}^\chi$ of compactly supported smooth sections of E^χ, and, strictly speaking its closure defines the Hamiltonian. In the trivially induced sector (where χ is the identity representation of H) one evidently obtains the well-known result that the Hamiltonian is minus the ordinary Laplacian.

Given a family of local sections $s_\alpha : U \to G$ ($U_\alpha \subset Q$) one can represent ψ^χ by its local trivializations $\psi_\alpha^\chi : U_\alpha \to \mathscr{H}_\chi$, defined by $\psi_\alpha^\chi(q) = \psi^\chi(s_\alpha(q))$. The corresponding realization of the induced representation π^χ is reviewed in detail in [22] (also cf. [37] for the case of a single, generally discontinuous section $s : Q \to G$); for the Laplacian Δ^χ in (2.18) we find, in local co-ordinates q^μ on Q,

$$\Delta_\alpha^\chi = g^{\mu\nu}(\nabla_\mu + A_\mu^\chi)(\partial_\nu + A_\nu^\chi), \tag{2.19}$$

where ∇_μ is the ordinary covariant derivative in the Levi-Civita connection for the invariant metric g on Q, and A_μ^χ is short for $d\pi_\chi(\langle s_\alpha^* A^H, \partial_\mu \rangle)$. Similarly, we can rewrite (2.8) as an operator acting on the local trivializations ψ_α^χ. We abbreviate $(d\pi^\chi(T_a)\psi_\alpha^\chi)(q)$ as $-iJ_a^\chi(q)\psi_\alpha^\chi(q)$, and omit the $\psi_\alpha^\chi(q)$, as well as the α-dependence of J_a^χ and s_α. Using the H-connection, we find from (2.4)

$$-iJ_a^\chi(q) = \nabla_a^\chi(q) + \pi_{\mathrm{co}}(s(q))_a{}^i \, d\pi_\chi(T_i), \tag{2.20}$$

where the gauge-covariant derivative is evidently given by $\nabla^\chi = d + d\pi_\chi(s^* A^H)$; we write $\nabla_a^\chi \equiv \nabla_{K_a}^\chi$. The J_a^χ are generalized momentum operators in the sector χ, i.e., they implement infinitesimal G-translations on Q, in the representation $\pi^\chi(\mathscr{A})$ [22, 23]; they are defined and (by the Appendix) essentially self-adjoint on Γ_c^χ.

To close this section, we wish to make some bibliographical comments concerning (2.18) and its derivation. In the physics literature, this formula appears (in the form (2.19)) in [24] under the restriction that G (and H) are compact; the main steps in its original, non-geometric derivation, using harmonic analysis on $\mathscr{H}^\chi$, are due to Strathdee [34] in the context of Kaluza-Klein theories (also cf. [2]). This derivation cannot immediately be generalized to the non-compact case. In the mathematical literature, similar connections between the Casimir operator in an induced representation and invariant quadratic differential operators on vector bundles have been studied in the context of the Langlands program of realizing the discrete series of certain non-compact semi-simple Lie groups G in a particular, cohomological way; $H = K$ is then the maximial compact subgroup of G. Under the additional assumption that G/K is hermitian symmetric, Okamoto and Ozeki [27] relate the (unique) quadratic Casimir operator on $E^\chi \otimes E^{(0,q)}$ (where $E^{(0,q)}$ is the bundle of harmonic forms of type $(0, q)$ on the complex manifold G/K) to the Laplacian associated to the anti-holomorphic exterior derivative $\bar{\partial}$. This result does not (explicitly) involve a connection on G, and the relation to (2.18) is not clear to us. The canonical connection A^H on G appears in papers in which the discrete series is realized on spaces of sections of spin bundles over G/K, tensored with a given E^χ [28, 1, 31, 3]. Parthasarathy [28] relates the Casimir operator on such sections to the square of the Dirac operator in the background field A^H, using geometric techniques that partly inspired the derivation of (2.18) above; the intermediate step (2.13) appears (without derivation) in the work of Slebarski [31], which contains (2.8) as well. Also, the first term in (2.8) for $A = A^H$ was found in [6].

Finally, our derivation goes through essentially unchanged if we drop the assumption (2.2); however, in that case (2.5) defines the so-called natural torsion-free connection on G/H [19] rather than the Levi-Civita connection, and the differential operator Δ^χ may no longer be a gauge-covariant Laplacian on the bundle E^χ.

3. Induced Representations and Classical Conservation Laws

As we have seen (also cf. [22, 23, 24]), a particle on $Q = G/H$ quantized in the superselection sector χ acquires an internal degree of freedom, namely $\mathscr{H}_\chi$, and its conserved 'momenta' J_a^χ (cf. (2.20)) contain additional terms as compared to the usual trivially induced case $\chi = \mathrm{id}$. As we will now show, these extra terms can be understood classically in two quite different ways.

3.1. *Classical field theory*

Conserved charges

We can regard $\psi^\chi \in \mathscr{H}^\chi$ as a classical Schrödinger field. We will work in a fixed local trivialization, and drop both indices on ψ_α^χ; accordingly, the 'wave function' ψ is defined on Q, and takes values in $\mathscr{H}_\chi$. For simplicity we assume that ψ is smooth and has compact support. The Lagrangian corresponding to the Hamiltonian (2.18) is $L = \int \Omega_g \mathscr{L}$, where Ω_g is the volume-form on Q derived from the invariant metric g, and $\mathscr{L}$ is the Lagrangian density (omitting the constant)

$$\mathscr{L} = \frac{i}{2}(\langle \psi, \partial_t \psi \rangle - \langle \partial_t \psi, \psi \rangle) - g(\langle \nabla^\chi \psi, \nabla^\chi \psi \rangle), \tag{3.1}$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathscr{H}_\chi$. Here $\psi = \psi(q, t)$.

Applying the ideas of Jackiw and Manton [15] to the case at hand, we now ask how the gauge field A^χ (which we will simply call A in what follows) occurring in the covariant derivative ∇^χ in (3.1) modifies the conserved charges of the system. If A and the metric g were dynamical (that is, not given externally, but quantities to be varied in the action principle), then under a combined gauge transformation and diffeomorphism

$$\delta\psi = -\varepsilon\psi + L_X\psi;$$
$$\delta A = D\varepsilon + L_X A; \tag{3.2}$$
$$\delta g = L_X g,$$

where $\varepsilon = \varepsilon^i(q)\, d\pi_\chi(T_i)$, $D\varepsilon = d\varepsilon + [A, \varepsilon]$, and L_X is the Lie derivative in the direction of a vector field X on Q, the Lagrangian transforms as a total derivative, i.e., $\delta L = \int L_X(\Omega_g \mathscr{L})$. The Noether method then provides conserved currents and charges. However, we are not free to vary A and g because they are external and fixed, and one only obtains conserved charges if δA and δg vanish. This is the case if X is a Killing vector field, and if $L_X A = D\omega_X$ for some Lie-algebra valued function ω_X on Q: in that case, we can take $\varepsilon = -\omega_X$. Anticipating that this is indeed true, we take $X = K_a$, and obtain the Noether charges

$$Q_a^\chi = i\int dq \langle\psi(q), (K_a + \omega_a(q))\psi(q)\rangle, \tag{3.3}$$

with $\omega_a \equiv \omega_{K_a}$, and we have written $X\psi = L_X\psi$; ψ is a ($\mathscr{H}_\chi$-valued) scalar under diffeomorphisms. This can be written in terms of the inner product (,) on $\mathscr{H}^\chi$ as

$$Q_a^\chi = (\psi, J_a^\chi\psi), \tag{3.4}$$

with

$$-iJ_a^\chi = K_a + \omega_a = \nabla_a^\chi + \omega_a - \langle A, K_a\rangle, \tag{3.5}$$

the last term(s) being multiplication operator(s) on $\mathscr{H}^\chi$.

This should be compared with (2.20); to proceed, we have to show that one indeed has $L_{K_a}A = D\omega_a$, and identify ω_a. We recall that $A \equiv A^\chi = d\pi_\chi(s^*A^H)$ in terms of a fixed local section $s: U \to G$ ($U \subset Q$), and that $(K_a)_{xq_0} = p'_{GQ}(T_a^R)_x$ for the right-invariant vector field T_a^R on G (cf. the text after (2.7)). Since A^H is the $\mathbf{h}$-component of the left-invariant Maurer-Cartan form on G, and T_a^R generates left-translations on G, we clearly have

$$L_{T_a^R} A^H = 0. \tag{3.6}$$

Symmetric gauge fields

Defining a gauge field A on the base space of a principal bundle P to be symmetric under a diffeomorphism generated by a vector field X on Q if it is invariant up to a gauge transformation, that is, $L_X A = D\omega_X$, as above, Forgacs and Manton [8] derive equations constraining ω_X, and determining it in certain cases. Since we have more information, i.e., (3.6), we can use a more efficient procedure.

Let $\tilde{A}$ on P satisfy $L_Y \tilde{A} = 0$ for some vector field Y on P (in our case $\tilde{A} = A^H$, $P = G$, $Y = T_a^R$). Let $K = p'_{PQ} Y$ be the projection of Y on Q; for any (local) section $s: Q \to P$ it easily follows that

$$L_K s^* \tilde{A} = s^* L_{Y_v} \tilde{A}, \tag{3.7}$$

where $Y_v = (s' \circ p'_{PQ} - 1) Y$ is a vertical vector field on P. For any vector field Z on P one has [15]

$$L_Z \tilde{A} = d i_Z \tilde{A} + i_Z d\tilde{A} = D i_Z \tilde{A} + i_Z \tilde{F}, \tag{3.8}$$

where $\tilde{F} = D\tilde{A}$ is the curvature (and $D = d + [\tilde{A}, \cdot]$ as before). Taking $Z = Y_v$ and using the fact that $\tilde{F}$ vanishes on vertical vectors we find from (3.7) and (3.8)

$$L_K s^* \tilde{A} = D i_{Y_v} \tilde{A}. \tag{3.9}$$

Applying this to our situation above, we thus find

$$\omega_a(q) = d\pi_\chi(\langle A^H, (T_a^R)_v \rangle_{s(q)}); \tag{3.10}$$

note that we interchangeably use the notations $\langle A, X \rangle$ and $i_X A$ (for a one-form A evaluated at a vector X) according to typographical convenience. The right-hand side of (3.10) can easily be computed, for $(T_a^R)_x = -\pi_{co}(x)_a{}^b (T_b^L)_x$, and the left-invariant vectors T_α^L are annihilated by A^H (cf. the text after (2.4)). Being fundamental vertical vectors, the T_i^L are mapped into T_i, so that eventually

$$\omega_a(q) = \langle A, K_a \rangle_q + \pi_{co}(s(q))_a{}^i \, d\pi_\chi(T_i). \tag{3.11}$$

Substituting (3.11) into (3.5), we see that the Noether charge densities J_a^χ as defined in (3.4) coincide with the generalized momenta J_a^χ in the superselection sector χ, as determined by induced representation theory, leading to (2.20).

Physical interpretation

We started from the Lagrangian density (3.1), whose *classical* field ψ ($\equiv \psi^\chi$ for fixed χ) defined on $Q \times \mathbf{R}$, at any fixed time is taken to be a (locally trivialized) section of the bundle E^χ; recall that in the *quantum* theory of a particle moving on Q, the states in the sector χ are realized as sections of E^χ as well. In the absence of an external gauge field A^χ, the diffeomorphisms generated by the Killing vector fields K_a on Q lead to conserved Noether charges $Q_a^{id} = i \int dq \bar{\psi} K_a \psi$, which we recognize as $i(\psi, d\pi^{id}(T_a)\psi)$, that is, the expectation value of T_a in the state ψ in the trivially induced representation of G.

In a general static background gauge field A on Q no conserved charges of the classical field theory will exist, but in the presence of a symmetric field $A = A^\chi$ a modified charge can still be defined. As our computation shows, the modified charge is given by (3.4) with (2.20), which classically is the integral over a charge density corresponding to a particular field configuration ψ, and quantum-mechanically is the expectation value in the state ψ of the generator T_a, evaluated in the non-trivially induced representation $d\pi^\chi$ corresponding to the sector χ.

In particular, the peculiar second term in (2.20) has thus been accounted for: it is the generalization of the well-known Poincaré term in the angular momentum of a charged particle moving in the field of a magnetic monopole. As explained by Jackiw and Manton [15], this term can be thought of as the contribution to the charge generated by a small disturbance of the gauge field, caused by the particle moving in its field.

To conclude this subsection, we show how the Poincaré term is indeed a special case of the second term in (2.20). Since the radial co-ordinate is irrelevant here, we look at a charged particle moving on $S^2 = SO(3)/SO(2)$. As explained in [23, 24], representations π^n of $G = SO(3)$ which are induced by non-trivial representations π_n of $H = SO(2)$ (defined by $\pi_n(\phi) = \exp(in\phi)$), correspond to the particle moving in a magnetic monopole field of quantized charge $eg = n$. This is because the canonical connection A^H on $SO(3)$ is precisely the Dirac monopole field. The usual expression for this field on the base space S^2 is obtained by using the section (guage) $s: S^2 \to SO(3)$ defined by

$$s(\varphi, \theta) = e^{\varphi T_3} e^{\theta T_2} e^{-\varphi T_3}, \tag{3.12}$$

identifying H with rotations around the z-axis. Then

$$\pi_{\mathrm{co}}(s(\varphi, \theta))_a{}^3 = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta), \tag{3.13}$$

for $a = 1, 2, 3$. Since $d\pi_n(T_3) = in$, the second term in (2.20) is the multiplication operator $-n\hat{q}$, where $\hat{q}$ is the unit vector in the direction of $q \in S^2$. This is exactly the Poincaré term.

3.2. *Classical particle mechanics*

The phase space of a particle in a Yang-Mills field

Rather than looking at classical field theory, we now try to understand the extra term in (2.20) by finding its classical analogue at the level of particle mechanics. To do so, we use the formalism developed in [33, 38, 12], of which we now briefly review some aspects.

The starting point is a principal fibre bundle P with gauge group H over a configuration space Q (not necessarily homogeneous). The classical analogue of a unitary representation $\pi_\chi(H)$ is a co-adjoint orbit $\mathcal{O} \subset \mathbf{h}^*$. We say that the classical particle has charge $\mathcal{O}$. The space $T^*P \times \mathcal{O}$ has a right H-action ρ_h, given by the lift of R_h of H on

P to T^*P times the co-adjoint action of H on $\mathcal{O}$. We equip $T^*P \times \mathcal{O}$ with the symplectic form $\Omega = -d\alpha - \omega^K$, where α is the canonical 1-form (Liouville form) on T^*P, and ω^K is the canonical (Kirillov) symplectic form on $\mathcal{O}$. The H-action is symplectic with respect to Ω, and one has a moment map $\Phi : T^*P \times \mathcal{O} \to \mathbf{h}^*$. The phase space of a particle with charge $\mathcal{O}$ is then defined to be the Marsden-Weinstein reduced space $P_{\mathcal{O}} = \Phi^{-1}(0)/H$, equipped with the symplectic form ω provided by the M-W symplectic reduction procedure [12]; this phase space evidently depends on P as well as on Q and $\mathcal{O}$.

Now choose a connection $\tilde{A}$ on P. This defines a projection $p_{T^*PT^*Q} : T^*P \to T^*Q$ by

$$\langle p_{T^*PT^*Q}\theta, Y\rangle_{p_{PQ}(r)} = \langle \theta, \lambda_r(Y)\rangle_r, \tag{3.14}$$

where $\theta \in T_r^*P$, $Y \in T_{p_{PQ}(r)}Q$, and $\lambda_r(Y)$ is the horizontal lift of Y at $r \in P$. By acting trivially on $\mathcal{O}$, $p_{T^*PT^*Q}$ extends to a projection of $T^*P \times \mathcal{O}$ onto T^*Q, which is ρ_h-invariant, and therefore quotients to a well-defined projection $p_{\mathcal{O}} : P_{\mathcal{O}} \to T^*Q$ [38]. The point of this is that any function f on T^*Q pulls back to a function $p_{\mathcal{O}}^* f$ on $P_{\mathcal{O}}$.

Globally trivial example

To interpret these structures, take the case $P = Q \times H$, with connection $\tilde{A} = T_i A_\alpha^i dq^\alpha + \theta_{MC}$; here θ_{MC} is the left-invariant Maurer-Cartan form on H, and we use local co-ordinates q^α on Q. The functions A_α^i satisfy $A_\alpha^i(q, h_1 h) = \pi_{\mathrm{ad}}(h^{-1})^i{}_j A_\alpha^j(q, h_1)$ in order that $\tilde{A}$ indeed transforms as a connection.

The cotangent bundle $T^*P \simeq T^*Q \times H \times \mathbf{h}^*$, relative to a left-invariant trivialization of T^*H. We use canonical (Darboux) co-ordinates (q, p) on T^*Q, and denote points in T^*H by $(h, \beta) \equiv \beta_i \theta^i(h)$, with $\{\theta^i\}$ being a basis of $\mathbf{h}^*$, realized as left-invariant one-forms on T^*H dual to the basis $\{T_i\}$ of $\mathbf{h}$. Also, we use co-ordinates γ_i on $\mathcal{O}$, which are adapted to the embedding $\mathcal{O} \subset \mathbf{h}^*$ (that is, γ stands for $\gamma_i \theta^i \in \mathbf{h}^*$). Accordingly denoting points in $T^*P \times \mathcal{O}$ by (q, p, h, β, γ), the H-action is given by

$$\rho_h(q, p, h_1, \beta, \gamma) = (q, p, h_1 h, \pi_{\mathrm{co}}(h^{-1})\beta, \pi_{\mathrm{co}}(h^{-1})\gamma), \tag{3.15}$$

with $(\pi_{\mathrm{co}}(h^{-1})\beta)_i = \pi_{\mathrm{co}}(h^{-1})_i{}^j \beta_j$, etc. The moment map is

$$\Phi(q, p, h, \beta, \gamma) = -(\beta_i + \gamma_i)\theta^i \in \mathbf{h}^*, \tag{3.16}$$

so that the reduced space $P_{\mathcal{O}}$ is isomorphic to $T^*Q \times \mathcal{O}$, a point (q, p, γ) corresponding to the equivalence class $[q, p, e, \gamma, -\gamma] \in \Phi^{-1}(0)/H$. In terms of these co-ordinates, the projection $p_{\mathcal{O}} : P_{\mathcal{O}} \to T^*Q$ is given by

$$p_{\mathcal{O}}(q, p, \gamma) = (p_\alpha - \gamma_i A_\alpha^i(q, e))\, dq^\alpha. \tag{3.17}$$

Finally, the symplectic form on the reduced space is $\omega = dq^\alpha dp_\alpha + \omega^K$; note the sign change of ω^K relative to Ω on $T^*P \times \mathcal{O}$, caused by the minus sign in $[\ldots \gamma, -\gamma]$ above.

Let $Y(q) = Y^\alpha(q)\partial/\partial q^\alpha$ be a vector field on Q. Its symbol $f_Y \in C^\infty(T^*Q)$ is given by $f_Y(q,p) = Y^\alpha(q)p_\alpha$. Clearly, from (3.17)

$$(p_{\mathcal{O}}^* f_Y)(q,p,\gamma) = Y^\alpha(p_\alpha - \gamma_i A_\alpha^i(q,e)). \tag{3.18}$$

Interpreting f_Y as the phase-space representative of the derivative in the direction of Y, we see that its pull-back to $P_{\mathcal{O}}$ is the covariant derivative in the gauge field $\tilde{A}$. This conclusion evidently holds in the globally non-trivial case as well, as we can reproduce this result in local trivializations.

Application to homogeneous spaces

We now go back to our pet situation $Q = G/H$, and take P to be the bundle $G(Q,H,G)$ already used in subsection 2.1. Then $T^*P = T^*G \simeq G \times \mathbf{g}^*$ under the left-trivialization, and we denote $p_a\theta^a(x) \in T_x^*G$ by (x,p) (here θ^a are canonical left-invariant 1-forms on G). When convenient, we use a separate notation $\beta_i \equiv p_i$ for the co-ordinates on $\mathbf{h}^*$, and $\mu_\alpha \equiv p_\alpha$ for those on $\mathbf{m}^*$. As in the example above, we use co-ordinates γ_i on $\mathcal{O}$. The canonical 1-form on T^*G is $\alpha(x,p) = p_a\theta^a(x)$ (in which the p_a are regarded as functions on $\mathbf{g}^*$, and the θ^a are elements of the T^*G-factor in $T^*(T^*G) \simeq T^*G \times T^*\mathbf{g}^*$), and the Maurer-Cartan equations provide the symplectic form on $T^*G \times \mathcal{O}$

$$\Omega(x,p,\gamma) = -dp_a\theta^a(x) + \frac{1}{2}C_{ab}^c p_c\theta^a(x)\theta^b(x) - \omega^K(\gamma). \tag{3.19}$$

The H-action on $T^*G \times \mathcal{O}$ is given by

$$\rho_h(x,p,\gamma) = (xh, \pi_{\mathrm{co}}(h^{-1})p, \pi_{\mathrm{co}}(h^{-1})\gamma), \tag{3.20}$$

and the moment map $\Phi : T^*G \times \mathcal{O} \to \mathbf{h}^*$ is

$$\Phi(x,p,\gamma) = -(\beta_i + \gamma_i)\theta^i, \tag{3.21}$$

cf. (3.16). Hence $\Phi^{-1}(0) \simeq G \times \mathbf{m}^* \times \mathcal{O}$ under the correspondence $\Phi^{-1}(0) \ni (x,\mu,\gamma,-\gamma) \leftrightarrow (x,\mu,\gamma)$. Since $\mathcal{O} \subset \mathbf{h}^*$ so that $\mathbf{m}^* \times \mathcal{O} \subset \mathbf{g}^*$, we have an embedding $i : \Phi^{-1}(0) \to T^*G$. We identify $\Phi^{-1}(0)$ with the corresponding subset of T^*G, and take i to be the embedding of this subset into T^*G. The pre-symplectic form $\tilde{\omega}$ on $\Phi^{-1}(0)$ is

$$\tilde{\omega} = -i^*\,d\alpha + \omega^K; \tag{3.22}$$

as in the previous example, note the sign change of ω^K. This form has null vector-fields $T_i^L + C_{i\alpha}^\beta \mu_\beta \partial/\partial\mu_\alpha + C_{ij}^k\gamma_k\partial/\partial\gamma_j$ (which are indeed tangent to $\Phi^{-1}(0)$, as the last two terms generate the co-adjoint action of H on $\mathbf{m}^* \times \mathcal{O}$), which generate the foliation of $\Phi^{-1}(0)$ defined by the H-action

$$\rho_h(x,p) = (xh, \pi_{\mathrm{co}}(h^{-1})p), \tag{3.23}$$

cf. (3.20); for notational convenience we have re-assembled $(\mu, \gamma) \in \mathbf{m}^* \times \mathcal{O}$ into a single vector p in $\mathbf{m}^* \times \mathcal{O} \subset \mathbf{g}^*$. The quotient space $\Phi^{-1}(0)/H \equiv G_{\mathcal{O}}$ is the Marsden-Weinstein reduced space with respect to Ω and Φ, taken to be the phase space of a particle on Q with charge $\mathcal{O}$. If pr is the canonical projection of $\Phi^{-1}(0)$ onto $G_{\mathcal{O}}$, then $G_{\mathcal{O}}$ has a uniquely determined symplectic form ω such that $\tilde{\omega} = pr^*\omega$.

The simplest example of this construction is obtained by choosing $\mathcal{O} = \{0\}$; this yields $G_0 = G \times_H \mathbf{m}^* = T^*Q$, cf. subsection 2.2 (text prior to (2.9)). As before, we denote elements of T^*Q by H-equivalence classes $[x, \theta] = L^*_{x^{-1}}\theta_{q_0}$ (cf. subsection 2.1, item 2 for the notation used here). Similarly, we denote points in $G_{\mathcal{O}}$ by classes $[x, \mu, \gamma] = [xh, \pi_{co}(h^{-1})\mu, \pi_{co}(h^{-1})\gamma]$. We now choose the canonical connection A^H on the bundle $P = G$, and easily find that the projection $p_{\mathcal{O}} : G_{\mathcal{O}} \to T^*Q$ is simply given by

$$p_{\mathcal{O}}[x, \mu, \gamma] = [x, \mu]. \tag{3.24}$$

This is clearly independent of the particular representatives of the equivalence classes involved.

G-action and moment map

Consider the left-action of G on itself ($L_y x = yx$); this lifts to a G-action on T^*G ($L^*_y(x, p) = (yx, p)$), extended to an action on $T^*G \times \mathcal{O}$ by a trivial action on $\mathcal{O}$. This action commutes with the right H-action defined previously, so that we eventually obtain a G-action λ on the phase space $G_{\mathcal{O}}$, given by

$$\lambda_y[x, \mu, \gamma] = [yx, \mu, \gamma]. \tag{3.25}$$

This action is symplectic with respect to the symplectic form ω on $G_{\mathcal{O}}$, and there is a moment map $J^{\mathcal{O}} : G_{\mathcal{O}} \to \mathbf{g}^*$ with respect to this action. One then has generalized momenta $J^{\mathcal{O}}_a$ defined by $J^{\mathcal{O}}_a(m) = \langle J^{\mathcal{O}}(m), T_a \rangle$ (where $m \in G_{\mathcal{O}}$), whose Poisson bracket relative to ω coincides with (minus) the Lie bracket in $\mathbf{g}^*$ (cf. the general theory of the moment map [12]). One finds from (3.22) and (3.25)

$$J^{\mathcal{O}}_a([x, p]) = \pi_{co}(x)_a{}^b p_b, \tag{3.26}$$

which is a well-defined function on the quotient space, cf. (3.23) (as before, $p \equiv (\mu, \gamma)$).

We now compare $p^*_{\mathcal{O}} J^0_a$ with $J^{\mathcal{O}}_a$, each function being defined on $G_{\mathcal{O}}$. Relative to a local section s, we choose local co-ordinates (q, p) on $G_{\mathcal{O}}$, such that (q, p) are the co-ordinates of $[s(q), p]$. Since on G_0 the p have no components γ, one has

$$J^0_a(q, \mu) = \pi_{co}(s(q))_a{}^\beta \mu_\beta. \tag{3.27}$$

Finally, (3.24) and (3.26) give

$$J^{\mathcal{O}}_a(q, \mu, \gamma) = (p^*_{\mathcal{O}} J^0_a)(q, \mu) + \pi_{co}(s(q))_a{}^i \gamma_i. \tag{3.28}$$

Interpretation

We compare the classical expression (3.28) for the conserved quantity $J^{\mathcal{O}}_a$ with its quantum-mechanical analogue (2.20).

CLASSICAL	QUANTUM
1. Charge $\mathcal{O} \in \mathbf{h}^*$	Unitary representation $\pi_\chi(H)$
2. Phase space $G_\mathcal{O}$	Hilbert space of sections $\mathcal{H}^\chi$
3. Moment map $J^\mathcal{O}$	Representation $d\pi^\chi$
4. Function $J_a^\mathcal{O}$	Operator $d\pi^\chi(T_a) = -iJ_a^\chi$, $T_a \in \mathbf{g}$
5. $C^\infty(Q) \subset C^\infty(G_\mathcal{O})$	$C^\infty(Q)$ as multiplication operators on $\mathcal{H}^\chi$
6. Momenta $\mu_\alpha \in C^\infty(G_0)$	Vielbein e_α (operator on $\mathcal{H}^{\text{id}}$)
7. $J_a^0 \in C^\infty(G_0)$	Killing vector field K_a (on $\mathcal{H}^{\text{id}}$)
8. $p_\mathcal{O}^* J_a^0 \in C^\infty(G_\mathcal{O})$	Covariant derivative ∇_a^χ on $\mathcal{H}^\chi$
9. $\pi_{\text{co}}(s(q))_a{}^i \gamma_i$ in $J_a^\mathcal{O}$	$\pi_{\text{co}}(s(q))_a{}^i \, d\pi_\chi(T_i)$ in J_a^χ.

Most of these correspondence are self-evident. We wish to stress that the first one is by no means one-to-one: only 'quantizable' orbits in $\mathbf{h}^*$ in the sense of geometric quantization are associated to (irreducible) unitary representations of H, and conversely, many groups H have unitary irreducible representations which do not correspond to a coadjoint orbit. In item 5, $C^\infty(Q)$ is understood to be embedded in $C^\infty(G_\mathcal{O})$ via the pull-back of the projection of $G_\mathcal{O}$ onto Q. As to 6 and 7, cf. (2.1) with (3.27), and note that $\mathcal{H}^{\text{id}} = L^2(Q)$ (we have omitted possible minus signs in 7 and 8; they depend on the sign convention for the Poisson bracket). The differential operators featuring in 6–8 are, us usual, defined on the domain $\Gamma_c^\chi \subset \mathcal{H}^\chi$ of compactly supported smooth sections of E^χ (with $\chi = \text{id}$ in 6 and 7).

The picture that emerges is that the algebra of quantum observables $\mathcal{A} = C^*(G, Q)$ has inequivalent representations π^χ, which have a geometric realization in terms of the canonical connection A^H on G (evaluated in various representations $d\pi^\chi(\mathbf{h})$), and that, in a very similar way, the Poisson algebra generated by $C^\infty(Q)$ and the functions J_a^0, has inequivalent representations as Poisson algebras on the various symplectic manifolds $G_\mathcal{O}$. Surprisingly enough, these representations can be realized by means of the connection A^H as well. We may, therefore, speak of inequivalent quantizations as well as of inequivalent 'classicizations': the former refer to representations as concretely given C^*-algebras on Hilbert spaces, and the latter refer to representations as Poisson algebras on symplectic manifolds. The underlying algebraic structure that is being represented here in two different categories is, loosely speaking, a crossed product of G and $C^\infty(Q)$.

Part of this structure was independently studied heuristically by Koch [20], motivated differently (this work became available to us only while finishing this paper). While not discussing the role played by the gauge field, his work essentially contains the correspondences 1–5 above, as well as the idea that the symplectic leaves $G_\mathcal{O}$ of $(T^*G)/H$ carry representations of the Poisson algebra mentioned above. Also, [20] contains a discussion of the classical analogue of the algebra of quantum observables $C^*(G, Q)$.

A. Representations of the Enveloping Algebra

We are going to discuss how to pass from a representation of a Lie group to a representation of its enveloping algebra. The latter will be represented by unbounded operators, and we have to discuss the domain and possible self-adjointness properties of these operators. References for the part of this Appendix preceding and following the theorems are Warner [37], Jorgensen [16], and Schmüdgen [30]. Theorems 1 and 2 follow directly from Thomas [35]. Theorem 3 is a generalization of a theorem in [35], proved by different techniques; the version in [35] holds for bi-invariant Hilbert subspaces of $\mathscr{D}(G)'$, and does not apply to induced representations. Detailed information on the analysis of direct integral decompositions of unitary group representations of the type (A.7) below, which are central to our Theorem 3, may be found in Goodman [10] and Penney [29]. Chapter I of the latter contains relevant background material for Theorem 1 below as well.

Let π be a continuous unitary representation of a Lie group G on a separable Hilbert space $\mathscr{H}$. This defines a representation of the convolution algebra $\mathscr{D}(G)$, called π as well, by

$$\pi(f) = \int_G dx\, f(x)\pi(x); \tag{A.1}$$

for convenience we assume that G is unimodular, so that dx is a left- and right-invariant Haar measure. We realize the Lie algebra $\mathbf{g}$ as right-invariant vector fields acting on $\mathscr{D}(G)$, i.e.,

$$(Yf)(x) = \frac{d}{dt} f(e^{-tY}x)_{|t=0} \tag{A.2}$$

for $Y \in \mathbf{g}$. This can be extended to an action of any element X of the enveloping algebra $\mathscr{U}(\mathbf{g})$ in the obvious way. More generally, we use the notation Xf for the action of $X \in \mathscr{U}(\mathbf{g})$ on smooth functions f taking values in a Hilbert space $\mathscr{H}_\chi$. This is defined as in the scalar case, the limit $t \to 0$ now being taken in the topology of $\mathscr{H}_\chi$.

The Gårding domain $\mathscr{H}_G \subset \mathscr{H}$ consists of all vectors of the form $\pi(f)\psi$ for $f \in \mathscr{D}(G)$ and $\psi \in \mathscr{H}$. This domain is dense, and carries a representation $d\pi$ of $\mathscr{U}(\mathbf{g})$, given by

$$d\pi(X)\pi(f)\psi = \pi(Xf)\psi. \tag{A.3}$$

It can be shown that $\mathscr{H}_G$ exactly consists of the C^∞-vectors for π, that is, $\psi \in \mathscr{H}_G$ if and only if the function $f_\psi : G \to \mathscr{H}$ defined by $f_\psi(x) = \pi(x)\psi$ is in $C^\infty(G, \mathscr{H})$. An equivalent definition of $d\pi$ is then given by $d\pi(X)\psi = -(Xf_\psi)(e)$.

Given a unitary representation $\pi_\chi(H)$, we realize the induced representation $\pi^\chi(G)$ by H-equivariant functions $\psi^\chi : G \to \mathscr{H}_\chi$ whose $\mathscr{H}_\chi$-norm is square-integrable on $Q = G/H$ (cf. [37] for more details; also cf. subsection 2.2 above). For any $\psi^\chi \in \mathscr{H}_G^\chi$ one then simply has $d\pi^\chi(X)\psi^\chi = X\psi^\chi$. However, we wish to work on the domain Γ_c^χ of compactly supported smooth cross-sections of the vector bundle E^χ (cf. subsection 2.2). It is trivial to show that Γ_c^χ consists of C^∞-vectors for π^χ, so that $\Gamma_c^\chi \subset \mathscr{H}_G^\chi$, and that Γ_c^κ is dense in $\mathscr{H}^\chi$ (which, indeed, can be defined as the closure of Γ_c^χ). The following theorem shows that it entails no loss of generality to define $d\pi^\chi(\mathscr{U}(\mathbf{g}))$ on Γ_c^χ rather than on $\mathscr{H}_G^\chi$.

Theorem 1. *The space Γ_c^χ of compactly supported smooth cross-sections of the vector bundle E^χ over Q is a common dense invariant domain for $d\pi^\chi(\mathscr{U}(\mathbf{g}))$, with the property that, for all $X \in \mathscr{U}(\mathbf{g})$, $d\pi^\chi(X)$ defined on Γ_c^χ has the same closure as $d\pi^\chi(X)$ defined on the Gårding domain $\mathscr{H}_G^\chi$.*

Proof. The invariance of Γ_c^χ is obvious, since the $d\pi^\chi(X)$ are realized as differential operators. Another way of showing that Γ_c^χ is dense and invariant is to define a map $j : \mathscr{H}^\chi \to \mathscr{D}(G)'$, depending on the choice of a fixed vector $u \in \mathscr{H}_\chi$, by giving the action of $j\psi^\chi$ on an arbitrary $\phi \in \mathscr{D}(G)$,

$$\langle j\psi^\chi, \phi\rangle = \int_G dx \langle \psi^\chi(x), u\rangle \phi(x), \tag{A.4}$$

where $\langle\ ,\ \rangle$ is the inner product in $\mathscr{H}_\chi$. In other words, we identify the locally integrable function $j\psi^\chi = \langle \psi^\chi, u\rangle$ with a distribution on G. It is easily checked that $j\psi^\chi$ is indeed in $\mathscr{D}(G)'$ (it is, in fact, a distribution of order 0), and that j is a continuous injection (the injective property follows from the H-equivariance of ψ^χ, combined with the fact that $u \in \mathscr{H}_\chi$ is cyclic for $\pi_\chi(H)$). The dual $j^* : \mathscr{D}(G) \to \mathscr{H}^\chi$ is given (for unimodular H, with Haar measure dh) by

$$(j^*\phi)(x) = \int_H dh\, \pi_\chi(h)\phi(xh). \tag{A.5}$$

It follows from Lemma 5.1.1.4 in [37], proving the surjectivity of j^* onto Γ_c^χ, and the fact that the projection of the support of ϕ onto Q is compact, that $j^*\mathscr{D}(G) = \Gamma_c^\chi$. This proves a nice characterization of Γ_c^χ, with the intertwining property $d\pi^\chi(X) \circ j^* = j^* \circ X$, showing that Γ_c^χ, like $\mathscr{D}(G)$ must be invariant under the action of $\mathscr{U}(\mathbf{g})$. The density of Γ_c^χ in $\mathscr{H}^\chi$ is a consequence of the continuity of the inclusion j^*, cf. [35] (where analogous inclusion maps j are studied in a more abstract context).

The last part of the theorem is a special case of Proposition 1b in [35]. We reproduce the proof, which is short and elementary. Let $\psi^\chi = \pi^\chi(f)\phi$ be an arbitrary element of $\mathscr{H}_G^\chi$. Then there exists a sequence $\{\phi_n\} \to \phi$, with all $\phi_n \in \Gamma_c^\chi$ (as Γ_c^χ is dense). By the invariance of Γ_c^χ, $\psi_n^\chi \equiv \pi(f)\phi_n$ is in Γ_c^χ, and $d\pi^\chi(X)\psi_n^\chi = \pi(Xf)\phi_n$ converges to $\pi(Xf)\phi = d\pi(X)\psi^\chi$, as $\pi(f)$ is bounded for all $f \in \mathscr{D}(G)$. Hence for every $\psi^\chi \in \mathscr{H}_G^\chi$ there is a sequence $\{\psi_n^\chi\}$ in Γ_c^χ such that $\psi_n^\chi \to \psi^\chi$ and $d\pi^\chi(X)\psi_n^\chi \to d\pi^\chi(X)\psi^\chi$, for all $X \in \mathscr{U}(\mathbf{g})$. The claim follows. □

In the following theorem, X^+ is the transpose of $X \in \mathscr{U}(\mathbf{g})$, defined by linear extension of $(Y_1 \ldots Y_n)^+ = (-1)^n Y_n \cdots Y_1$, $Y_i \in \mathbf{g}$. Let $\mathscr{H}_X^\chi$ consist of those elements ψ^χ of $\mathscr{H}^\chi$ for which $X^+ j\psi^\chi \in j\mathscr{H}^\chi$; here X^+ is regarded as a weak differential operator acting on $\mathscr{D}(G)'$, i.e., $\langle X^+F, \phi\rangle = \langle F, X\phi\rangle$ for $F \in \mathscr{D}(G)'$ and $\phi \in \mathscr{D}(G)$. Since j is injective, it can be inverted on $j\mathscr{H}^\chi$, so that we can extend the action of $X^+ \in \mathscr{U}(\mathbf{g})$ from $\mathscr{H}_G^\chi$ to $\mathscr{H}_X^\chi$ by defining

$$X^*\psi^\chi = j^{-1}X^+ j\psi^\chi. \tag{A.6}$$

Theorem 2. *The domain of the adjoint $d\pi^\chi(X)^*$ of $d\pi^\chi(X)$ (defined on either Γ_c^χ or $\mathscr{H}_G^\chi$) is $\mathscr{H}_X^\chi$, and on this domain $d\pi^\chi(X)^* = X^*$, as defined in* (A.6) *above. In particular, X^* is the self-adjoint closure of $d\pi^\chi(X)$ in case the latter is essentially self-adjoint (on Γ_c^χ).*

Proof. This is a special case of Proposition 1c in [35], upon identification of $\mathscr{H}^\chi$ with the arbitrary right-invariant Hilbert space $\mathscr{H}$ in $\mathscr{D}(G)$ studied there (interchanging left and right), and the general injection j in [35] given by (A.4) above. □

A more explicit description of $\mathscr{H}_X^\chi$ may be given in case that X is ellipic, cf. [11].

Although Theorem 1 in conjunction with the well-known (Nelson-Stinespring) criteria for essential self-adjointness on the Gårding domain [37, 16] is sufficient to establish that the Casimir operators studied in the main text are essentially self-adjoint on Γ_c^χ, we wish to provide a more general result, that applies to any unitary representation of G, and to arbitrary elements of $\mathscr{U}(\mathbf{g})$. The notion of a direct integral decomposition of Hilbert spaces, bounded operators, and representations is explained in [4], and in [26, 30] for unbounded operators. For this theorem we assume that G is a type I Lie group [37], a condition satisfied by all examples of interest.

Theorem 3. *Let π be a continuous unitary representation of a type I Lie group G on a separable Hilbert space $\mathscr{H}$. Let*

$$\mathscr{H} = \int_{\hat{G}}^{\oplus} d\mu(\gamma)\mathscr{H}_\gamma \otimes \mathscr{H}^{n_\gamma}; \qquad \pi = \int_{\hat{G}}^{\oplus} d\mu(\gamma)\pi_\gamma \otimes \mathbf{1}_{n_\gamma} \tag{A.7}$$

be the primary decomposition of $\mathscr{H}$ and π over the dual $\hat{G}$; the π_γ are irreducible representations of G on Hilbert spaces $\mathscr{H}_\gamma$, and the $\mathscr{H}^{n_\gamma}$ are multiplicity spaces (with unit operator $\mathbf{1}_{n_\gamma}$). Then, for every $X \in \mathscr{U}(\mathbf{g})$, $d\pi(X)$ is essentially self-adjoint if and only if $d\pi_\gamma(X)$ is essentially self-adjoint for μ- almosrt every γ (these operators being defined on their respective Gårding domains).

The proof will follow from three lemma's. In what follows, $\bar{A}$ denotes the closure of an operator A, $D(A)$ stands for its domain, and X is an arbitrary element of $\mathscr{U}(\mathbf{g})$. Also $\mathscr{A}'$ is the commutant of $\mathscr{A}$.

Lemma 1. *$\overline{d\pi(X)}$ is affiliated to $\pi(G)''$ (the von Neumann algebra generated by $\pi(G)$).*

Proof. We must show that, for all unitary operators $U \in \pi(G)'$, $UD(\overline{d\pi(X)}) \subset D(\overline{d\pi(X)})$ and $U^*\overline{d\pi(X)}U = \overline{d\pi(X)}$ on its domain. By 5.6.3 in [17], it suffices that these properties hold on a core D_0 of $\overline{d\pi(X)}$. We take $D_0 = \mathscr{H}_G$. For $\psi = \pi(f)\phi \in \mathscr{H}_G$ one has $U\psi = \pi(f)U\phi \in \mathscr{H}_G$, and $U^*d\pi(X)U\pi(f)\phi = U^*\pi(Xf)U\phi = d\pi(X)\pi(f)\phi$. □

Lemma 2. *Relative to the decomposition* (A.7) *of $\mathscr{H}$, $\overline{d\pi(X)}$ can be decomposed as*

$$\overline{d\pi(X)} = \int_{\hat{G}}^{\oplus} d\mu(\gamma)X_\gamma \otimes \mathbf{1}_{n_\gamma}, \tag{A.8}$$

where the field $\{X_\gamma\}_{\gamma\in\hat{G}}$ is μ-measurable, and the X_γ are certain (μ-a.e.) closed operators defined on a dense domain $D(X_\gamma) \subset \mathscr{H}_\gamma$. The domain of $\overline{d\pi(X)}$ consists of those measurable vector fields $\psi \in \mathscr{H}$ (defined by their components $\psi(\gamma) \in \mathscr{H}_\gamma \otimes \mathscr{H}^{n_\gamma}$) for which $\psi(\gamma) \in D(X_\gamma) \otimes \mathscr{H}^{n_\gamma}$ and $\overline{d\pi(X)}\psi \in \mathscr{H}$ (where $(\overline{d\pi(X)}\psi)(\gamma) = X_\gamma \otimes \mathbf{1}_{n_\gamma}\psi(\gamma)$).

Proof. The preceding lemma, in combination with Theorem 2 of Nussbaum [26] (where the definition of a measurable field of unbounded operators may be found) implies the existence of a decomposition $\overline{d\pi(X)} = \int_{\hat{G}}^{\oplus} d\mu(\gamma) X(\gamma)$ for some field $\{X(\gamma)\}$ of closed operators, with similar properties as in the statement of the lemma above. We show that $X(\gamma)$ is affiliated to $\pi_\gamma(G)''$ (a.e.). Since the decomposition (A.7) is primary, any unitary $U \in \pi_\gamma(G)'$ can be decomposed as $U = \int_{\hat{G}}^{\oplus} d\mu(\gamma) U(\gamma)$, with all $U(\gamma)$ unitary (up to a null set, where they may be redefined so as to be unitary), and of the form $U(\gamma) = \mathbf{1}_\gamma \otimes U_\gamma$. Conversely, any field $\{U(\gamma)\}$ of this form defines a unitary operator in $\pi(G)'$. By Lemma 1, $U^*\overline{d\pi(X)}U\psi = \overline{d\pi(X)}\psi = \int_{\hat{G}}^{\oplus} d\mu(\gamma) X(\gamma)\psi(\gamma)$ for $\psi \in D(\overline{d\pi(X)})$, but the left-hand side is also equal to $\int_{\hat{G}}^{\oplus} d\mu(\gamma) U^*(\gamma) X(\gamma) U(\gamma)\psi(\gamma)$. The existence of a non-null set $\Gamma \in \hat{G}$ on which $U^*(\gamma)X(\gamma)U(\gamma) \neq X(\gamma)$ on $D(X(\gamma))$ would then lead to a contradiction. Hence $X(\gamma)$ is affiliated to $\pi_\gamma(G)''$ a.e., and it follows from the factorization of the $U(\gamma)$ that $X(\gamma) = X_\gamma \otimes \mathbf{1}_{n_\gamma}$ for some X_γ. Since $X(\gamma)$ is closed, its domain must be of the form $D(X_\gamma) \otimes S$, with $D(X_\gamma) \subset \mathscr{H}_\gamma$ such that X_γ on this domain is a closed operator on $\mathscr{H}_\gamma$, and S a closed subspace of $\mathscr{H}^{n_\gamma}$. But $D(\overline{d\pi(X)})$ is dense in $\mathscr{H}$, so that $D(X_\gamma)$ must be dense in $\mathscr{H}_\gamma$ and $S = \mathscr{H}^{n_\gamma}$. □

Lemma 3. *The operators X_γ in* (A.8) *are equal to $\overline{d\pi_\gamma(X)}$, the closure of $d\pi_\gamma(X)$ (defined on the Gårding domain $\mathscr{H}_\gamma^G$ in $\mathscr{H}_\gamma$).*

Proof. We divide the proof into 4 parts. We omit the qualification 'μ-a.e.' at various appropriate places.

1. We first show that $\mathscr{H}_\gamma^G \subset D(X_\gamma)$. Take $\psi = \pi(f)\phi \in \mathscr{H}_G \subset D(\overline{d\pi(X)})$. Then by the previous lemma $\psi = \int_{\hat{G}}^{\oplus} d\mu(\gamma)\psi(\gamma)$, with $\psi(\gamma) \in D(X_\gamma) \otimes \mathscr{H}^{n_\gamma}$. Since ϕ has a similar decomposition (without the domain restriction) it follows from (A.7) that $\psi(\gamma) = (\pi_\gamma(f) \otimes \mathbf{1}_{n_\gamma})\phi(\gamma)$, hence $\pi_\gamma(f)\phi(\gamma) \otimes \mathbf{1}_{n_\gamma} \in D(X_\gamma) \otimes \mathscr{H}^{n_\gamma}$. Since f and ϕ are arbitrary, the claim follows.

2. Moreover, X_γ restricted to $\mathscr{H}_\gamma^G$ coincides with $d\pi_\gamma(X)$. This follows similarly: on the one hand, by definition of $d\pi(X)$ one has $d\pi(X)\psi = \pi(Xf)\phi = \int_{\hat{G}}^{\oplus} d\mu(\gamma)(\pi_\gamma(Xf) \otimes \mathbf{1}_{n_\gamma})\phi(\gamma)$, while on the other hand by (A.7) this must equal $\int_{\hat{G}}^{\oplus} d\mu(\gamma)(X_\gamma \pi_\gamma(f) \otimes \mathbf{1}_{n_\gamma})\phi(\gamma)$.

3. We now define the unbounded operator X' on $\mathscr{H}$ as follows: the domain $D(X')$ consists of those vectors $\psi = \int_{\hat{G}}^{\oplus} d\mu(\gamma)\psi(\gamma)$ in $\mathscr{H}$ for which $\psi(\gamma) \in \mathscr{H}_\gamma^G \otimes \mathscr{H}^{n_\gamma}$, and $X'\psi = \int_{\hat{G}}^{\oplus} d\mu(\gamma)(d\pi_\gamma(X) \otimes \mathbf{1}_{n_\gamma})\psi(\gamma)$ is in $\mathscr{H}$. Then $\overline{X'} = \int_{\hat{G}}^{\oplus} d\mu(\gamma)\overline{d\pi_\gamma(X)} \otimes \mathbf{1}_{n_\gamma}$, with domain as described in Lemma 2 (with X_γ replaced by $\overline{d\pi_\gamma(X)}$). This follows from the fact that if $\psi_n \to \psi$ in $\mathscr{H}$ then $\{\psi_n\}$ contains a subsequence $\{\psi_{n_k}\}$ such that $\psi_{n_k}(\gamma) \to \psi(\gamma)$ in $\mathscr{H}_\gamma \otimes \mathscr{H}^{n_\gamma}$, cf. the proof of Proposition 7 in [26], and Lemma 14.1.3 in [17].

4. Finally, $\overline{X'} = \overline{d\pi(X)}$. This follows from the inclusions $d\pi(X) \subseteq X' \subseteq \overline{d\pi(X)}$. Then first inclusion follows as in the proof of item 1 above, and the second one is equivalent to the claim in item 1. Together with item 3, this proves the lemma. □

We note that, rather than proving part 1 and 2 of this lemma directly, we could have appealed to a result of Goodman (Lemma 3.1 in [11]), who shows that C^∞ vectors for π on $\mathscr{H}$ decompose (in the sense of (A.7)) into direct integrals of C^∞ vectors for π_γ on $\mathscr{H}_\gamma$, and that $d\pi(X)$ decomposes as in (A.8), with X_γ replaced by $d\pi_\gamma(X)$. Also cf. [29].

Proof of Theorem 3. Combining Lemma 2 and Lemma 3, we have

$$\overline{d\pi(X)} = \int_{\hat{G}}^{\oplus} d\mu(\gamma)\, \overline{d\pi_\gamma(X)} \otimes \mathbf{1}_{n_\gamma}. \tag{A.9}$$

Theorem 3 of Nussbaum [26] states that in a decomposition of the type $X = \int_{\hat{G}}^{\oplus} d\mu(\gamma) X(\gamma)$, with X and the $X(\gamma)$ closed, X is self-adjoint if and only if almost every $X(\gamma)$ is. □

Applied to induced representations ($\pi = \pi^\chi$), and using Theorem 1, Theorem 3 can be used to prove essential self-adjointness of many operators $d\pi^\chi(X)$ on the domain Γ_c^χ of compactly supported sections of the vector bundle E^χ. The conditions can often be checked, since by Frobenius reciprocity [37] we know which subrepresentations π_γ occur in π^χ.

More generally, some well-known theorems stating essential self-adjointness of representatives $d\pi(X)$ on $\mathscr{H}_G$ are immediate corollaries of Theorem 3. If G is compact then the π_γ are finite-dimensional, so that any $d\pi(X)$ is essentially self-adjoint if it is symmetric (i.e., $X^+ = X$). Also, for any G, central elements $Z \in \mathscr{U}(\mathbf{g})$ are scalars in each irreducible representation π_γ, so that $d\pi(Z)$ is essentially self-adjoint if $Z^+ = Z$ (Nelson-Stinespring). Finally, it easily follows that $id\pi_\gamma(Y)$, where $Y \in \mathbf{g}$, must be essentially self-adjoint in irreducible representations (because by irreducibility the range $R(\overline{d\pi_\gamma(Y)} + i)$ must be $\mathscr{H}_\gamma$), so that Theorem 3 implies that $id\pi(Y)$ is essentially self-adjoint in any representation (I. E. Segal).

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