QUANTIZATION AND SUPERSELECTION SECTORS II. DIRAC MONOPOLE AND AHARONOV-BOHM EFFECT

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The quantization procedure of the preceding paper is applied to study two generic topological quantum effects, viz. the charge quantization induced by (abelian) magnetic monopoles, and the Aharonov-Bohm effect. Prior to these applications, a general procedure is given for reducing unitary representations of a locally compact G which are induced by nontrivial unitary representations of $H \subset G$. This involves the use of spherical trace functions, and is useful in the determination of the eigenfunctions of the Hamiltonian of the particle in a given superselection sector. Such Hamiltonians, implementing the time-evolution on the given abstract C^*-algebra, are explicitly constructed and analyzed. The relevant quantum effects are found to be a consequence of the representation theory of the appropriate algebras of observables. In this way a group- and operator-theoretic elucidation of the mathematical structure of the given systems is attempted. This paper may be read independently of its predecessor.

1. Introduction

1.1. *Entree*

In this paper we apply some of the results and techniques developed in the preceding paper I [14] to an analysis of the two best-known and perhaps most surprising topological quantum effects, namely the magnetic monopole à la Dirac [7] and its associated charge quantization, and the Aharonov-Bohm effect [1]. To make this paper independently readable, we will briefly sum up the main points made in I, now employing a predominantly group-theoretic rather than operator-algebraic language. Sections, equations, and references in I will be referred to as I...., I.(...), and I.[...], respectively. Notational conventions are stated in I.1.4.

1. The homogeneous configuration space $Q = G/H$ is quantized by associating an abstract operator algebra $\mathscr{A}$ to it, whose self-adjoint elements are mapped into the physical observables of the system (i.e., the particle moving on Q). Here $\mathscr{A}$ is chosen to be the so-called transformation group C^*-algebra $\mathscr{A} = C^*(G, Q)$. An important feature of this particular algebra is that the set of equivalence classes of its irreducible representations π^χ (under unitary equivalence), which we identify with the superselection sectors of the system, as well as with its "inequivalent

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quantizations", equals $\hat{H}$, the dual of H. The unitary representation of G canonically associated to π^χ is just its representation induced by a unitary representation of H which is in the class χ.

2. The algebra $\mathscr{A}$ has a faithful representation on $\mathscr{H}_L = L^2(G)$ given by the pair I.(3.1), in which q_0 is a fixed H-invariant point on Q (corresponding to the coset $\{H\}$); this representation is used to construct a time-evolution on $\mathscr{A}$ in the following way (cf. I.3.3): any self-adjoint operator H_p, called the pre-Hamiltonian, whose bounded spectral projections of H_p must commute with all operators $\pi_R(h)$, $h \in H$, defines a unitary group $U_t = \exp(itH_p)$ with the property that it maps the algebra $\mathscr{A}$ into itself, i.e., $U_t \pi_L(\mathscr{A}) U_t^* = \pi_L(\mathscr{A})$. Thus, by the faithfulness of π_L, this uniquely defines an automorphism on $\mathscr{A}$. This automorphism is the time-evolution of the system at the algebraic level. The operator H_p is not the usual Hamiltonian of the system; a conventional Hamiltonian H^χ may be found (up to a c-number) in each superselection sector χ as the operator implementing the time-evolution α_t, that is, one demands that I.(3.17) holds.
3. In most practical examples one may construct H_p and the H^χ straightaway, by choosing a symmetric central element C in $\mathscr{U}(\mathscr{G})$ (the enveloping algebra of G). Then, assuming C to be an even polynomial, $H_p = \pi_R'(C)$ and $H^\chi = (\pi^\chi)'(C)$, where π' is the representation of $\mathscr{U}(\mathscr{G})$ derived from a unitary representation π of G [13, Chap. VI], and π_R is the right-regular representation on $L^2(G)$. More precisely, the respective Hamiltonians are obtained as the closure of the expressions given above, which are defined and essentially self-adjoint on the Gårding domain generated by $\pi(G)$. Moreover, in this case the Hamiltonians are G-invariant, so that their degenerate eigenspaces will form irreducible multiplets under G.

The simplest illustration of the whole scheme is provided by particle (without internal degrees of freedom) moving on $\mathbf{R}^3$. Firstly, take $G = Q$ to be the additive group $\mathbf{R}^3$ (hence $H = \{e\}$), and according to our prescription the algebra of observables is $\mathscr{A} = C^*(\mathbf{R}^3, \mathbf{R}^3)$. As this system, like all the others considered below, certainly satisfies the conditions in I.3.1, we can immediately conclude (e.g., from Theorem I.2) that $\mathscr{A} = \mathscr{K}(\mathscr{H})$. The C^*-algebra of compact operators is well-known (e.g. I.[2]) to have only one irreducible representation (up to unitary equivalence, of course), so that this choice of G only yields one "inequivalent quantization". Although the underlying mathematics is slightly different from what is used in its conventional formulation, this result is just another version of the Stone-von Neumann uniqueness theorem (cf. I.[34]). Here $\pi_L(\mathscr{A})$ is nothing but the usual Schrödinger representation of the CCR on $L^2(\mathbf{R}^3)$, and the simplest positive pre-Hamiltonian, which in this case coincides with the ordinary Hamiltonian, is evidently minus the Laplacian.

It is more interesting to take the covering of the Euclidean group $G = SU(2) \rhd \mathbf{R}^3$, which acts on Q in the conventional way ($SU(2)$ acts via its canonical epimorphism on $SO(3)$). Within the confines of "scalar quantum mechanics" (cf. I.1.2) this is the largest symmetry group of $\mathbf{R}^3$ respecting the (flat) Riemannian structure. It follows that $H = SU(2)$, so that the inequivalent representations of the quantum algebra $\mathscr{A} = C^*(G, Q)$ are labeled by (half-) integers j, to be interpreted as the spin of the particle.

Hence spin defines a superselection rule in the sense of labeling the inequivalent irreducible representations of the algebra of observables of a particle moving in three-space. This result is very close to Mackey's derivation of the concept of spin (cf. I.[27,39]) and is also in agreement with the original notion of a superselection sector I.[22,41].

As to the dynamics of the system, the simplest nontrivial choice of a positive pre-Hamiltonian is, as before, to select $C = T_1^2 + T_2^2 + T_3^2$, where the T_i are the generators of the Lie algebra of $\mathbf{R}^3$. A peculiar feature of this choice is that the Hamiltonians H^j in the irreducible representations $\pi^j(\mathscr{A})$ do not depend on j: they are (once again) just minus the Laplacian on $\mathbf{R}^3$. More general, spin-dependent interactions may be obtained by choosing C to be an element of $\mathscr{U}(\mathscr{G})$ which is just $SU(2)$-invariant (rather than invariant under the whole group G, as in the choice above).

1.2. *Plan of the paper*

The main goal of this paper is the actual construction and analysis of both the pre-Hamiltonian and the ordinary Hamiltonians in two concrete examples. One is usually interested in the (improper) eigenfunctions of the Hamiltonian, and it turns out that the study of our examples is facilitated by first giving a fairly abstract and general scheme to obtain these eigenfunctions. In case that the Hamiltonian is invariant under the group G, as in the construction given in ad 3 of the preceding section, this by and large amounts to decomposing the unitary representation π^χ of G on the Hilbert space $\mathscr{H}^\chi$ on which $\pi^\chi(\mathscr{A})$ is defined, into irreducible multiplets (subrepresentations) (note that $\mathscr{H}^\chi$ is irreducible as the carrier space of $\pi^\chi(\mathscr{A})$, but in general reducible as the carrier space of $\pi^\chi(G)$).

This problem has been extensively studied in the case that χ is the identity representation of H, so that $\pi^\chi = \pi^{\mathrm{id}}$ is the usual representation of G on $L^2(G/H)$ by left-translation of the cosets. The general case of nontrivial χ appears to have been studied very little in this explicit form, but is, under the hypotheses on G and H stated in I.3.1, in principle contained in the more ambitious task of reducing the left-regular representation π_L of G. The reason for this is that, as made explicit in I.3.2, $\mathscr{H}_L$ c.q. π_L can be decomposed as a direct sum/integral over (reducible) subspaces c.q. subrepresentations isomorphic to the $\mathscr{H}^\chi$ c.q. π^χ, so that the problem of completely decomposing $\mathscr{H}_L$ indeed amounts to the decomposition of *all* $\mathscr{H}^\chi$.

The full reduction of π_L has indeed been accomplished along these lines by Harish-Chandra (cf. [18] and refs. therein to the original literature) for G a semisimple Lie group, in which case $H = K$ is chosen to be a maximal compact subgroup of G. Some of the technical machinery introduced by Harish-Chandra (and certain other mathematicians), in particular the use of so-called spherical trace functions, is also available in a wider context, i.e. G locally compact, and $H = K$ compact but not necessarily maximal. It turns out, that a further restrictive assumption on the pair G, K (stated early in 2.1), which is satisfied in our application to the monopole problem, allows us to carry out the reduction process in a fashion which directly generalizes the reduction of the trivially induced representation on $L^2(G/K)$ in case that the latter is manageable, that is, when G, K is a so-called Gel'fand pair [6, 18, 19]. In this generalization, which is the principal subject of Chap. 2, the role played by spherical trace functions is analogous to the way zonal spherical functions enter in the trivially induced case.

It is hoped that these considerations will be of some mathematical interest, but in any case the main upshot of all this for the physics studied here is that explicit formulae for the energy eigenfunctions in terms of matrix elements of irreducible representations of G will be found. The discussion of so-called monopole harmonics, which are the eigenfunctions of the angular part of the Hamiltonian of a particle in a monopole field, in Chap. 3 is thereby rendered trivial.

Another special case which is both manageable and interesting for applications (e.g., the Aharonov-Bohm effect) obtains when H is discrete and G/H is compact. Here we have nothing to add to the literature, and at the end of Chap. 2 we just review some of the standard techniques (Selberg trace formula, duality theorem) that may be employed in the reduction of π^{χ}. An explicit determination of the eigenfunctions of the Hamiltonian in this representation amounts to a nontrivial exercise in the theory of automorphic functions, which becomes trivial in the case of the Aharonov-Bohm effect, where $G = \mathbf{R}$ and $H = \mathbf{Z}$, and the relevant automorphic functions are essentially simple exponentials.

In Chap. 3 we turn our attention to the Dirac magnetic monopole. The study of the real thing is preceded by a detailed analysis of a particle moving on the two-sphere (i.e., $Q = S^2$), which already contains all information on the superselection structure of the full model. The reason for this is that the relevant feature of a particle moving in the field of a monopole is not that it is moving in the field of a monopole, but that the location of the monopole (taken to be the origin) is excluded from its configuration space. Thus one has $Q = \mathbf{R}^3 - \{0\} \simeq \mathbf{R}^+ \times S^2$. The radial co-ordinate in some sense factorizes in the topological and representation-theoretic aspects of the problem (in particular, it does not affect the superselection structure of the model), and phenomena like the Dirac quantization condition will be seen (as is well-known, of course) to be a consequence of the S^2-part of Q. A more conventional way of putting this is to say that all "nontrivial topology" is contained in S^2.

The reduced problem $Q = S^2$ is much easier than the full one from the point of view of constructing the time-evolution and the (pre-) Hamiltonian, for here one is in the simple situation sketched in 1.1.3, where all Hamiltonians are representatives of a central element in the universal enveloping algebra of $G(= SO(3))$. This amounts to a drastic simplification of domain issues, and allows us to essentially blindly follow the algorithm given in I.

The inclusion of the radial co-ordinate destroys this simplicity, so that we have to do some of the analysis of domain problems of the pre-Hamiltonian by hand. Fortunately, these can be handled by completely standard techniques (leading to Theorem 1 in 3.2), so that the main conclusions of our analysis hopefully will not be obscured. These are, that the (Wess-Zumino-like) n-dependent terms in the Hamiltonian H^n (with n the quantized monopole charge), which are automatically induced by our quantization procedure, may be interpreted as terms describing the interaction of the particle with a magnetic monopole, that the Dirac quantization condition for electric and magnetic charge is a direct consequence of the "quantized" representation theory of the algebra of observables of a charged particle moving in a monopole field, and that the quantized charge labels a superselection rule for such a particle. As a technical corollary, it will be found that several apparently qualitatively different descriptions of the Dirac monopole, as the one in terms of a forbidden stringy region used by Dirac himself [7], the one in terms of a Hilbert space of sections introduced by Greub and Petry [10] and made popular by Wu and Yang [20], and the one in terms of induced representations used by Langlands [12], are unitarily equivalent from the point of view of the algebra of observables, and therefore equivalent from both a physical and a mathematical point of view, despite the fact that seemingly entirely different mathematical concepts are used in each of these descriptions.

Chap. 4 is devoted to the Aharonov-Bohm effect. As in the monopole case, it is the topology of the configuration space, here taken to be $Q = \mathbf{R}^3 - \{z\text{-axis}\} \equiv \mathbf{R}^+ \times \mathbf{R} \times S$ which determines most relevant features, notably the superselection rules, of the problem. As before, the factors $\mathbf{R}^{(+)}$ in Q are better omitted in a preliminary discussion of the superselective and topological aspects of the problem, so we start with the simpler case of a particle moving on a circle $Q = S$. (This situation is also of some independent interest, for the emergence of a θ-angle labeling the superselection sectors of this system is supposed to take place in quantum chromodynamics as well). Here our quantization method, including the construction of the pre-Hamiltonian and the Hamiltonian in each superselection sector can be carried out smoothly, yet, as in the monopole case, this smoothness is obscured by the inclusion of the radial coordinate. This again leads to certain domain problems, which can be handled in a very straightforward way.

In any case, the physical picture that emerges is analogous to that of the monopole example: the θ-dependent terms in the Hamiltonian H^θ in a given superselection sector θ may be interpreted as describing the interaction of the particle with the electromagnetic vector potential generated by an infinitely thin solenoid along the z-axis, so that, conversely, the magnetic flux Φ generated by such a solenoid determines a superselection rule for a particle moving in its field (with θ proportional to Φ).

2. Reduction Theory

Let $\mathscr{H}^\chi$ be the carrier space of the unitary representation π^χ of G, which is induced by an irreducible unitary representation π_χ (in the class χ) of H. A convenient realization of $\mathscr{H}^\chi$ is provided by the set of functions $\psi : G \to \mathscr{H}_\chi$ which satisfy $\psi(xh) = \pi_\chi(h)^* \psi(x)$, and which are square-integrable with respect to the inner product in $\mathscr{H}^\chi$, namely $(\psi_1, \psi_2) = \int_Q d\bar{x}(\psi_1(x), \psi_2(x))_\chi$, where $(\dots)_\chi$ is the inner product in $\mathscr{H}_\chi$ (cf. [8, VI.2], or I.2.3 ad 1). The representation π^χ then acts by $(\pi^\chi(y)\psi)(x) = \psi(y^{-1}x)$. Our task is to decompose $\pi^\chi(G)$ into irreducible constituents $\pi_\gamma(G)$.

One has to solve two problems:

1. the determination of $\hat{G}_\chi \subset \hat{G}$, that is, the set of irreducible representations of G which (weakly) occur in π^χ, as well as their multiplicities n^χ_γ;
2. the explicit form of the functions in $\mathscr{H}^\chi$ that transform irreducibly, or, in physical

terms, the eigenfunctions of the Hamiltonian in case it is G-invariant (or, if appropriate, to reduce the Schrödinger equation to a radial equation if the Hamiltonian is only partially G-invariant).

These problems are nontrivial even in case that $\chi = \mathrm{id}$, i.e. the identity representation of H. Take, for example, $G = SO(3)$ and $H = SO(2)$; then one has to find the explicit form and multiplicities of the spherical harmonics.

2.1. *Case 1: H compact*

As mentioned in the Introduction, we will be able to deal with two special cases, and here we start with the first one, in which $H = K$ is taken to be compact. In addition, for reasons to become clear shortly, we require that n_κ^γ is either 0 or 1 for all $\gamma \in \hat{G}$ and $\kappa \in \hat{K}$. Here the number n_κ^γ is defined as the multiplicity of κ in $\pi_\gamma(G \downarrow K)$ (that is, the representation π_γ restricted to K). This condition is quite strong, but is satisfied by the pair $G = SO(3)$, $K = SO(2)$, which corresponds to the Dirac monopole. A much weaker condition, sufficient for conventional harmonic analysis to work out nicely, consists in requiring the above just for $\kappa = \mathrm{id}$. A pair (G, K) meeting the latter demand is called a Gel'fand pair, and explicit conditions on G and K to meet it are known [6, 18, 19]. No such conditions are known (to the author) in the general case, but let us nevertheless call a pair G, K satisfying the strong condition a *generalized Gel'fand pair*. The statement above then amounts to saying that $(SO(3), SO(2))$ is a generalized Gel'fand pair.

Instead of reducing $\mathscr{H}^\kappa$, it is technically simpler to decompose $\mathscr{H}(\kappa) \simeq \oplus^{d_\kappa} \mathscr{H}^\kappa$ (cf. I.(3.10)). We realize $\mathscr{H}(\kappa)$ as the space of functions $\psi^\kappa : G \to \mathscr{M}_\kappa$ satisfying $\psi^\kappa(xk) = \pi_\kappa(k)^* \psi^\kappa(x)$, and being square-integrable with respect to the inner product in $\mathscr{H}(\kappa)$ defined by

$$(\psi_1^\kappa, \psi_2^\kappa) = d_\kappa \int_Q d\bar{x} \operatorname{Tr} \psi_1^\kappa(x)(\psi_2^\kappa(x))^* . \tag{2.1}$$

The representation $\tilde{\pi}^\kappa \simeq \oplus^{d_\kappa} \pi^\kappa$ of G on $\mathscr{H}(\kappa)$ is then defined as above, by

$$(\tilde{\pi}^\kappa(y)\psi^\kappa)(x) = \psi^\kappa(y^{-1}x). \tag{2.2}$$

For later use, we record how the representations $\tilde{\pi}^\kappa$ are embedded in the left-regular representation on $L^2(G)$. Specializing the results in I.3.2 to the compact case, we have, in the sense of the decomposition of unitary group representations,

$$L^2(G) \simeq \mathscr{H}_5 \equiv \oplus_{\kappa \in \hat{K}} \mathscr{H}(\kappa), \tag{2.3}$$

the unitary map $P : L^2(G) \to \mathscr{H}_5$ being given by

$$(P\psi_L)^\kappa(x) = \int dk\, \pi_\kappa(k) \psi_L(xk); \tag{2.4}$$

strictly speaking, this is defined on $L^1(G) \cap L^2(G)$, and extended by continuity. Now define a map $P^\kappa : L^2(G) \to \mathscr{H}(\kappa)$ by (2.4) as well, keeping κ fixed (and obviously replacing the left-hand side by $(P^\kappa \psi_L)(x)$). We are going to employ this map in the following considerations, which are inspired by a similar procedure for $\kappa = \mathrm{id}$ in [19, Chap. 16].

Distributions on G and unitary representations

Consider the Schwartz space $\mathscr{D}(G) \subset L^2(G)$, the inclusion being continuous [13, VI.5, Th.12(3)]. Define a sesquilinear form $\tilde{\omega}^\kappa$ on $\mathscr{D}(G) \times \mathscr{D}(G)$ by

$$\tilde{\omega}^\kappa(f_1, f_2) = (P^\kappa f_1, P^\kappa f_2), \tag{2.5}$$

the inner product being taken in $\mathscr{H}(\kappa)$. This form is separately continuous in each variable (in fact, by [13, VI.6, Th.16] it is jointly continuous because it is left-invariant, see below), so that, by the Bruhat-Maurin generalization [13, VI.6, Th.14] (to arbitrary locally compact groups) of the usual kernel theorem the form $\tilde{\omega}^\kappa$ is related to a functional ω^κ on $\mathscr{D}(G)$ (now considered a $*$-algebra with the standard convolution, and an involution given by $(f^*)(x) = \overline{f(x^{-1})}$) by

$$\tilde{\omega}^\kappa(f_1, f_2) = \omega^\kappa(f_1 * f_2^*). \tag{2.6}$$

Notice that $\tilde{\omega}^\kappa$ is left-invariant in the sense that $\tilde{\omega}^\kappa(\pi_L f_1, \pi_L f_2) = \tilde{\omega}^\kappa(f_1, f_2)$, and that, by the Schur orthogonality relations for compact groups, the projector P^κ satisfies $P^\kappa f = P^\kappa f *_K \overline{\theta_\kappa}$, where $\theta_\kappa = d_\kappa \chi_\kappa$ with $\chi_\kappa(k) = \mathrm{Tr}\, \pi_\kappa(k)$ (the convolution integral is over K only, see below). These facts imply that ω^κ has the following property: $\omega^\kappa(f) = \omega^\kappa(P_\kappa f)$, where P_κ projects $C_c(G)$ onto the well-known [18, I.4.5.1] (also cf. [8, 15]) function algebra $I_{c,\kappa}(G)$. Elements of $I_{c,\kappa}(G)$ are members of $C_c(G)$ which satisfy $f(kx) = f(xk)$ as well as $f *_K \overline{\theta_\kappa} = f$ (which taken together also imply that $\overline{\theta_\kappa} *_K f = f$), where the convolutions are taken over K only:

$$(f *_K \overline{\theta_\kappa})(x) = \int dk\, f(xk^{-1}) \overline{\theta_\kappa}(k);$$

$$(\overline{\theta_\kappa} *_K f)(x) = \int dk\, \overline{\theta_\kappa}(k) f(k^{-1}x). \tag{2.7}$$

Explicitly, the projector P_κ may be defined by

$$(P_\kappa f)(x) = \int dk (f *_K \overline{\theta_\kappa})(kxk^{-1}). \tag{2.8}$$

In addition, it is obvious that ω^κ is a positive definite distribution on $\mathscr{D}(G)$ in the sense that $\omega^\kappa(f * f^*) \geq 0$ for all f. It is well-known that positive definite functionals on the C^*-algebra $C^*(G)$ correspond to unitary representations of G. The construction given below requires a similar correspondence between distributions and unitary representations of G. In general, a distribution ω will not extend to a functional on $C^*(G)$, so that we cannot appeal to the standard C^*-algebra-result here. Fortunately enough, this would be quite unnecessary anyway, because, as shown in [13, VI.6], the GNS-like construction of a unitary representation from a positive definite functional can be carried out at the distributional level as well.

Thus one can associate a representation π_ω of the convolution algebra $\mathscr{D}(G)$ on a Hilbert space $\mathscr{H}_\omega$ to the distribution ω, in such a way that there is an identification of equivalence classes $\hat{f}$ of functions $f \in \mathscr{D}(G)$, and vectors in a pre-Hilbert space $\Phi_\omega \subset \mathscr{H}_\omega$, with the inner product in Φ_ω given by $(\hat{f}_1, \hat{f}_2) = \omega(f_1 * f_2^*)$, and the representation π_ω acting by $\pi_\omega(f)\hat{g} = \widehat{f * g}$. The equivalence relation is evidently given by $f \sim g$ if $f - g \sim 0$, and $f \sim 0$ if $\omega(f * f^*) = 0$. $\mathscr{H}_\omega$ is then defined as the Hilbert space closure of Φ_ω, and the above relations all extend to $\mathscr{H}_\omega$ by continuity. The associated representation of G (called π_ω as well), is then extracted by the connection $\pi_\omega(f) = \int dx\, f(x)\pi_\omega(x)$.

One even has more structure here (needed in the sequel) compared to the usual GNS construction: Φ_ω can be given a nuclear topology (derived form the usual Schwartz topology on $\mathscr{D}(G)$) which is finer than its Hilbert space topology, so that the embedding $\Phi_\omega \subset \mathscr{H}_\omega$ is continuous, and one has a Gel'fand triplet (rigged Hilbert space) $\Phi_\omega \subset \mathscr{H}_\omega \subset \Phi'_\omega$, where the last space is the dual of the first (with its nuclear topology).

It is now evident that the distribution ω^κ defined by (2.6) leads to the representation $\tilde{\pi}^\kappa$ on $\mathscr{H}(\kappa)$ in the above sense, i.e. $\mathscr{H}_{\omega^\kappa} = \mathscr{H}(\kappa)$ and $\pi_{\omega^\kappa} = \tilde{\pi}^\kappa$. As we see it as our task to reduce $\tilde{\pi}^\kappa$, we may now appeal to a decomposition theorem [13, VI.6, Th.20], stating that (as a consequence of the so-called complete von Neumann spectral theorem) the Hilbert space $\mathscr{H}(\kappa)$ and the distribution ω^κ can be decomposed in the sense of direct integrals [13] (also cf. I.3.2 for a very brief review of this notion):

$$\mathscr{H}(\kappa) \simeq \hat{\mathscr{H}}(\kappa) = \int_\Lambda^\oplus d\mu(\lambda)\mathscr{H}(\lambda);$$

$$\omega^\kappa = \int_\Lambda d\mu(\lambda)\omega_\lambda, \tag{2.9}$$

where each $\mathscr{H}(\lambda)$ carries the representation π_{ω_λ} of G. Here the set Λ may be taken to be $\hat{G}_\kappa$ (see the preamble to this chapter), and the measure μ is yet to be determined.

Connection with spherical trace functions

The above theorem concerns existence rather than explicit construction of the decomposition. In the present case it is possible to proceed, by relating the distributions ω_λ to so-called spherical trace functions. The following is a generalization of a computation for $\kappa = \mathrm{id}$, which establishes a similar connection with zonal spherical functions [19, Chap. 16]. The starting observation is that $\mathscr{H}(\kappa)$ carries a representation

ρ of $I_{c,\kappa}(G)$, defined by

$$\rho(f)\psi^\kappa = \psi^\kappa * f. \tag{2.10}$$

Since f is K-central, this is well-defined (i.e. the right-hand side satisfies the constraint mentioned prior to (2.1) that elements of $\mathscr{H}(\kappa)$ have to satisfy), and it is easily shown that the operators $\rho(f)$ are bounded (cf. [19]), and that ρ is nondegenerate (that is, no non-zero subspace is annihilated by all representatives $\rho(f)$). The same property of $I_{c,\kappa}(G)$ may be used to show, by an easy computation, that the algebra $\rho(I_{c,\kappa}(G))$ is in the commutant of $\tilde{\pi}^\kappa(G)$. Therefore, by standard reduction theory [13], decomposing an abelian subalgebra in the uniform closure I_κ^* of $\rho(I_{c,\kappa}(G))$ amounts to decomposing $\tilde{\pi}^\kappa$ (and vice versa). Now, under our assumption that the pair (G, K) satisfies the generalized Gel'fand condition the algebra $I_{c,\kappa}(G)$ can be shown to be abelian itself [18, 6.1.1.6]. Hence ρ is diagonal on $\hat{\mathscr{H}}(\kappa)$:

$$\rho \simeq \hat{\rho} = \int_\Lambda d\mu(\lambda)\rho_\lambda, \tag{2.11}$$

where $\rho_\lambda : I_{c,\kappa}(G) \to \mathbf{C}$ are representations of $I_{c,\kappa}(G)$. (The notation in (2.11) means that $(\hat{\rho}(f)\psi)(\lambda) = \rho_\lambda(f)\psi(\lambda)$.)

Now we are in a position to establish the connection between the ρ_λ and the ω_λ. Take $\psi_i^\kappa \in \mathscr{H}(\kappa)$, $i = 1, 2$. By the first member of (2.9), and (2.11) we have

$$(\rho(f)\psi_1^\kappa, \psi_2^\kappa) = \int_\Lambda d\mu(\lambda)\rho_\lambda(f)(\psi_1^\kappa(\lambda), \psi_2^\kappa(\lambda))_\lambda, \tag{2.12}$$

where $\psi^\kappa(\lambda)$ is the image of ψ^κ in $\mathscr{H}(\lambda)$ by the general spectral theorem [13, I.7], and $(\ldots)_\lambda$ is the inner product in $\mathscr{H}(\lambda)$. On the other hand, by (2.10) the same expression equals $\int d\mu(\lambda)((\psi_1^\kappa * f)(\lambda), \psi_2^\kappa(\lambda))_\lambda$.

Choose $f_i \in \mathscr{D}(G)$, in such a way that $P^\kappa f_i = \psi_i^\kappa$. The inner product in $\mathscr{H}(\lambda)$ is given by ω_λ, so that we can combine the information in this paragraph and the previous one to conclude that the following relation must hold almost everywhere (a.e.) with respect to μ:

$$\rho_\lambda(f)\omega_\lambda(f_1 * f_2^*) = \omega_\lambda(f_1 * f * f_2^*). \tag{2.13}$$

Now choose f and f_1 in $\mathscr{D}_\kappa(G) \equiv \mathscr{D}(G) \cap I_{c,\kappa}(G)$. Then $f * f_1 = f_1 * f$. Thus (2.13) implies, by the arbitrariness of f_2 (in $\mathscr{D}(G)$), a.e.

$$\rho_\lambda(f)\omega_\lambda(f_1) = \rho_\lambda(f_1)\omega_\lambda(f). \tag{2.14}$$

Hence $\omega_\lambda = \alpha(\lambda)\rho_\lambda$ as distributions on $\mathscr{D}_\kappa(G)$, where α is an a.e. strictly positive function. (Its positivity follows from the fact that $\alpha(\lambda) = \omega_\lambda(f)/\rho_\lambda(f)$ for arbitrary f, which may be taken an approximate unit for $\mathscr{D}_\kappa(G)$. ρ being a representation of $I_{c,\kappa}(G)$, $\rho_\lambda(f)$ must then converge to 1, whereas the ω_λ are a.e. positive definite by definition.)

The distributions ω_λ are actually defined on $\mathscr{D}(G)$, where they satisfy $\omega_\lambda(f) = \omega_\lambda(P_\kappa f)$, where P_κ is the projector on $I_{c,\kappa}(G)$ defined above (2.7). As we shall see below, the representations ρ_λ of $I_{c,\kappa}(G)$ may also trivially be extended to distributions on $\mathscr{D}(G)$, where they satisfy the same projection property. Thus we conclude that ω_λ and ρ_λ are proportional as distributions on G (for μ-almost all $\lambda \in \Lambda$). We may absorb the function α in the measure $d\mu$, defining $dv = \alpha\, d\mu$, and rewrite the first equation in (2.9) as

$$\hat{\mathscr{H}}(\kappa) = \int_\Lambda^\oplus dv(\lambda)\hat{\mathscr{H}}(\lambda), \tag{2.15}$$

where $\hat{\mathscr{H}}(\lambda)$ is the carrier space constructed from the distribution ρ_λ by the procedure sketched earlier in this section.

What do we know about the set Λ? The ρ_λ are irreducible representations of $I_\kappa^* \supset I_{c,\kappa}(G)$, so that $\hat{I}_\kappa^* \subset \hat{I}_{c,\kappa}$. More detailed information on the inclusion will be provided later on, but for the moment it appears to be relevant to investigate the set $\hat{I}_{c,\kappa}$ of irreducible representations of the algebra $I_{c,\kappa}(G)$.

Spherical trace functions

This representation theory can be reduced to the study of so-called spherical trace functions [18, II.6], [8, VII.3], [15]. Thus we proceed to give a very brief review of this theory, restricting ourselves to the case in which $I_{c,\kappa}(G)$ is commutative (although some of the results stated below are valid in general, cf. the references above).

Firstly, there is the trivial fact that all irreducible representations ρ ($\equiv \rho_\lambda$ for fixed λ) of $I_{c,\kappa}(G)$, being abelian, are one-dimensional. Less trivial is that each ρ is given by a spherical trace function $\hat{\rho}$ (which does, of course, depend on κ, but we suppress this dependence in the notation) by means of

$$\rho(f) = \int dx\, \hat{\rho}(x)f(x), \tag{2.16}$$

where $\hat{\rho}$ satisfies all properties of functions in $I_{c,\kappa}(G)$, *except* that it need not have compact support. Also, it is bounded, and normalized by $\hat{\rho}(e) = 1$. What makes the spherical trace functions amenable to explicit investigations is Godement's theorem: *For each non-zero spherical trace function $\hat{\rho}$ there exists a unitary representation $\tilde{\pi}_\rho$ of G on a Hilbert space $\mathscr{H}_\rho$, containing a representation π_κ of K in the class κ once, such that*

$$\hat{\rho}(x) = \frac{1}{d_\rho}\operatorname{Tr} P(\kappa)\tilde{\pi}_\rho(x)P(\kappa), \tag{2.17}$$

where $P(\kappa)$ is a projection operator on the subspace of $\mathscr{H}_\rho$ which carries $\pi_\kappa(K)$. Moreover, $\tilde{\pi}_\rho(G)$ is irreducible if and only if ρ is positive definite, and ρ characterizes $\tilde{\pi}_\rho$ up to unitary equivalence.

Now we have seen that the ρ_λ in (2.11) are indeed positive definite, but this does not mean that the carrier spaces $\hat{\mathscr{H}}(\lambda)$ in (2.15) are irreducible, because they correspond to ρ via the GNS construction, rather than by the connection (2.17). However, the relation between the $\hat{\mathscr{H}}(\lambda)$ and the $\mathscr{H}_{\rho_\lambda} \equiv \mathscr{H}_\lambda$ of Godement's theorem is trivial: decomposing the state ρ on $\mathscr{D}(G)$, one evidently has

$$\hat{\mathscr{H}}(\lambda) = \oplus^{d_\kappa} \mathscr{H}_\lambda; \qquad \pi_{\rho_\lambda} = \oplus^{d_\kappa} \tilde{\pi}_{\rho_\lambda}. \tag{2.18}$$

Hence (2.15) may be rewritten as

$$\hat{\mathscr{H}}(\kappa) = d_\kappa \cdot \int_\Lambda^\oplus d\nu(\lambda) \hat{\mathscr{H}}_\lambda, \tag{2.19}$$

where it is now understood that $\Lambda = \hat{I}_{c,\kappa}$, and that $\hat{\mathscr{H}}_\lambda$ is the carrier space of an irreducible representation $\pi_\lambda \equiv \tilde{\pi}_{\rho_\lambda}$ of G corresponding to the spherical trace function ρ_λ attached to the class κ (i.e. as representations of $I_{c,\kappa}(G)$) by means of (2.17).

Let us note, that some aspects of this results were to be expected on the basis of the (generalized) Frobenius reciprocity theorem [8, VI.11, Th.7]: the fact that each irreducible representation π_λ of G (weakly) occurring in $\mathscr{H}(\kappa)$, if subduced (restricted) to K contains the class κ once is an immediate corollary of this theorem, as $\mathscr{H}(\kappa)$ is the direct sum d_κ copies of $\mathscr{H}^\kappa$, which by definition is the carrier space of the representation $\pi^\kappa(G)$ induced from κ. This explains the multiplicity d_κ in (2.19) as well.

Finally, the measure ν on Λ is still unknown. In case that $\kappa = \mathrm{id}$ and G is a semisimple Lie group, an explicit expression exists in terms of Harish-Chandra's c-function [19], but in the general case discussed above this problem appears to be open (for partial results in the semisimple case cf. [15]). What we can say is that it follows from [15] that ν is supported by those points in Λ which correspond to the positive definite spherical trace functions.

A simple example

Let us now, as an illustration of the general formalism, consider the rather trivial yet illuminating case $G = K$. Note, that the pair (G, G) is a generalized Gel'fand pair, so that the theory developed above is indeed applicable. Then $\mathscr{H}(\kappa)$ by definition consists of those functions $\psi^\kappa : K \to M_\kappa$ which satisfy $\psi^\kappa(xy) = \pi_\kappa(y^{-1})\psi^\kappa(x)$. Hence $\mathscr{H}(\kappa)$ is isomorphic to the matrix algebra M_κ itself, by the correspondence $\mathscr{H}(\kappa) \ni \psi^\kappa_M \leftrightarrow M_{\psi^\kappa} \in M_\kappa$ given by $\psi^\kappa_M(x) = \pi_\kappa(x^{-1})M_{\psi^\kappa}$ and $M_{\psi^\kappa} = \psi^\kappa_M(e)$. The inner product in M_κ corresponding to the one in $\mathscr{H}_\kappa$ is evidently $(M, N) = d_\kappa \operatorname{Tr} MN^*$. G acts on M_κ by $\pi^\kappa(y)M = \pi_\kappa(y)M$, so that the commutant of $\pi^\kappa(G)$ is M_κ, acting on itself by right-multiplication. This leads to an explicit reduction of the carrier space M_κ, hence of $\mathscr{H}(\kappa)$, namely $\mathscr{H}(\kappa) = \oplus^{d_\kappa} \mathscr{H}_\kappa$, each copy of $\mathscr{H}_\kappa$ corresponding to a column vector in the matrix algebra M_κ.

Let us now try to reproduce this result by applying the formalism of spherical trace functions outlined above. Firstly, it is easily inferred, using the Schur orthogonality relations, that $I_{c,\kappa}(G)$ is spanned (as a vector space) by the single function $\overline{\theta_\kappa}$ (defined after (2.6)), so that $I_{c,\kappa}(G) \simeq \mathbf{C}$. Secondly, the representation ρ of $I_{c,\kappa}(G)$ on $\mathscr{H}(\kappa)$, constructed in (2.10), here comes out, in the realization on M_κ, to be $\rho(\overline{\theta_\kappa}) = 1$ (the unit matrix in M_κ). Hence ρ appears in fully reduced form, and we conclude, in conformity with the explicit discussion above, that M_κ is already the direct sum of irreducible carrier spaces. Thirdly, the spherical trace function corresponding to the irreducible subrepresentation of ρ of $I_{c,\kappa}(G)$ is easily seen to be $\hat{\rho} = \chi_\kappa/d_\kappa$ (the normalized character of κ), which is an explicit check of Godement's theorem above. Thus $\tilde{\pi}_\rho = \pi_\kappa$, as expected.

The monopole-like situation

A more general application of the formalism is G compact, but not equal to K. This is the situation relevant for the Dirac monopole, so let us specialize the abstract theory to this case (assuming, of course, that (G, K) is a generalized Gel'fand pair). The main simplification of the general theory is that $\hat{G}$ is now discrete, so that the direct integral machinery in unnecessary in favour of direct sums, and that Frobenius duality holds in its naive form. In particular, the measure ν as well as the set Λ may now be determined explicitly.

To find $I_{c,\kappa}(G)(=I_\kappa(G))$, embed it in $L^2(G)$ and map this into $\mathscr{H}(\kappa)$ by means of (2.4). It is easily seen (and follows from [18, 6.1.1]) that the image of $I_{c,\kappa}(G)$ under the map P^κ consists of those elements of $\mathscr{H}(\kappa)$ which are continuous functions, and satisfy $\psi^\kappa(kx) = \pi_\kappa(k^{-1})\psi^\kappa(x)$. Given the other constraints that ψ^κ has to satisfy in order to be in $\mathscr{H}(\kappa)$, this condition is seen to be met only by the linear span of functions of the type $\psi(x)_{\rho\sigma} = \overline{\pi_\gamma(x)_{\sigma\rho}}$, where ρ, $\sigma = 1, \dots d_\kappa$ are matrix indices, and $\pi_\gamma(G\downarrow K)$ is supposed to contain $\pi_\kappa(K)$ in its upper-left corner, with $\gamma \in \hat{G}_\kappa$. Now use the fact that the map P^κ, if restricted to $I_{c,\kappa}(G) \subset L^2(G)$ has an inverse, which is given by $((P^\kappa)^{-1}\psi^\kappa)(x) = d_\kappa \operatorname{Tr} \psi^\kappa(x)$, to conclude that $I_{c,\kappa}(G)$ is spanned by functions of the type $f_\gamma(x) = \overline{\operatorname{Tr} P(\kappa)\pi_\gamma(x)P(\kappa)}$, where the projector $P(\kappa)$ is defined after (2.17). Hence $I_{c,\kappa}(G)$ consists itself of spherical trace function which belong to the class $\bar{\kappa}$ (the class of representations conjugate to κ).

The spherical trace functions defining irreducible representations of $I_{c,\kappa}(G)$ are then (by the Schur orthogonality relations) obviously the functions $\hat{\rho}_\gamma(x) = \operatorname{Tr} P(\kappa)\pi_\gamma(x)P(\kappa)/d_\kappa$, as was to be expected from Godement's theorem (2.17). Hence $\Lambda = \hat{G}_\kappa$, and the representation theory of $I_{c,\kappa}(G)$ is completely determined by the equation $\rho_\gamma(f_{\gamma'}) = \delta_{\gamma\gamma'}$. It goes without saying that the spherical trace function ρ_γ corresponds to a representation $\pi_\gamma(G)$ in the class γ, so that one finally has

$$\mathscr{H}^\kappa \simeq \oplus_{\gamma \in \hat{G}_\kappa} \mathscr{H}_\gamma; \qquad \pi^\kappa \simeq \oplus_{\gamma \in \hat{G}_\kappa} \pi_\gamma. \tag{2.20}$$

Let $\mathscr{H}^\kappa$ be realized as sketched in the beginning of this chapter (with H and χ replaced by K and κ, respectively). It then follows from the one-but-last paragraph that the irreducible subspaces $\mathscr{H}_\gamma$ are embedded in $\mathscr{H}^\kappa$ as follows: basis vectors $e_{(\gamma)\mu}$, $\mu = 1 \dots d_\gamma$, are given by elements $\psi^\kappa_{(\gamma)\mu}$ in $\mathscr{H}^\kappa$, such that

$$\psi^\kappa_{(\gamma)\mu}(x)_\sigma = \overline{\pi_\gamma(x)_{\mu\sigma}}, \tag{2.21}$$

under the same conditions as stated above. This formula determines, among other things, the explicit form of monopole harmonics (cf. Chap. 3).

2.2. *Case 2: H discrete and Q compact*

Let us now assume that H is a discrete subgroup, which may be non-compact (for H compact and discrete the theory in the previous as well as the present section is applicable). The discussion is considerably simplified if we assume that $Q = G/H$ is compact, as in the application to the Aharonov-Bohm effect (although all of the results reviewed below may be generalized to the non-compact case [9, 13]). The following discussion is just a repetition of material in [9, Chap. 1]; also cf. [8, VII.6].

Let us start with the first problem mentioned in the beginning of this chapter. The main virtue of the compactness of Q is that $\mathscr{H}^\chi$ and π^χ can be discretely reduced, such that each irreducible representation π_γ occurring in π^χ has finite multiplicity n_γ^χ. Thus we have

$$\mathscr{H}^\chi \simeq \oplus_{\gamma \in \hat{G}} n_\gamma^\chi \cdot \mathscr{H}_\gamma; \qquad \pi^\chi \simeq \oplus_{\gamma \in \hat{G}} n_\gamma^\chi \cdot \pi_\gamma, \tag{2.22}$$

and the problem is to determine the multiplicities $n_\gamma^\chi < \infty$.

One of the tools available for solving this problem is the Selberg trace formula. Let $f \in L^1(G)$ be such that $\pi^\chi(f) = \int dx\, f(x)\pi^\chi(x)$ is of trace-class. By calculating $\mathrm{Tr}\,\pi^\chi(f)$ in two different ways one then finds

$$\int_F dx \sum_{h \in H} f(xhx^{-1})\,\mathrm{Tr}\,\pi_\chi(h) = \sum_{\gamma \in \hat{G}} n_\gamma^\chi \int_G dx\, f(x)\,\mathrm{Tr}\,\pi_\gamma(x). \tag{2.23}$$

Here F is a so-called fundamental domain in G, i.e., an open set such that the union of the closures of its translates under all $h \in H$ is G, whereas $h_1 F \cap h_2 F$ is empty if $h_1 \neq h_2$. For example, if $G = \mathbf{R}$ and $H = \mathbf{Z}$ then the set $]0, 1[$ is a fundamental domain. In case that $\mathscr{H}_\chi$ or $\mathscr{H}_\gamma$ are infinite-dimensional the traces have to be understood in the distributional sense, that is, "$\mathrm{Tr}\,\pi_\gamma$" is just the name of a distribution satisfying $(\mathrm{Tr}\,\pi_\gamma)(f) = \mathrm{Tr}(\pi_\gamma(f))$ for $f \in \mathscr{D}(G)$. The use of the Selberg trace formula in the reduction problem is brought about by choosing f in such a way that it determines the multiplicities n_γ^χ (see Chap. 4 for an example).

A second tool for determining the n_γ^χ is the generalized Frobenius reciprocity theorem (called the duality theorem in [9]) $n_\gamma^\chi = n_\chi^\gamma$, where n_χ^γ is the number of times the class χ occurs in $\pi_\gamma(G \downarrow H)$. For $d_\gamma = \infty$ this occurrence is understood to be in the weak sense, see below.

For $d_\chi = 1$ one may relate this application of the duality theorem, as well as the Selberg trace formula, to the theory of automorphic functions. Given a carrier space $\mathscr{H}_\gamma$ of $\pi_\gamma(G)$, construct the Gel'fand triplet $\Phi_\gamma \subset \mathscr{H}_\gamma \subset \Phi_\gamma'$ (this step is unnecessary if $d_\gamma < \infty$). Then the representation π_γ extends to Φ_γ' in the obvious way, that is, given a functional $\varphi \in \Phi_\gamma'$ sending $\psi \in \Phi$ to $\langle \varphi, \psi \rangle \in \mathbf{C}$ one defines $\pi_\gamma(x)\varphi$ by $\langle \pi_\gamma(x)\varphi, \psi \rangle = \langle \varphi, \pi_\gamma(x^{-1})\psi \rangle$. The duality theorem now says that n_χ^γ is equal to the number of linearly independent elements of Φ_γ' for which $\pi_\gamma(h)\varphi = \pi_\chi(h)\varphi$. Such distributions φ are called (generalized) automorphic functions, and one sees, that the Selberg trace formula is a statement about the dimensions of the respective spaces of automorphic functions corresponding to γ and χ. (Note that here we have followed [13, VII.9] rather than [9].) Once again, this procedure will be illustrated, in a rather trivial way, at the end of 4.1 below.

Finally, the second problem mentioned in the beginning of this chapter, that is, the explicit determination of the functions in $\mathscr{H}^\chi$ which transform irreducibly under π^χ, might be discussed. This problem is related to the theory of automorphic functions as well, and since its solution is absolutely trivial in our application to the Aharonov-Bohm effect, we will not comment on it any further.

3. Dirac Monopole

3.1. *Particle on a sphere*

As explained in the Introduction, the study of a particle moving on the two-sphere $Q = S^2$ is a prelude to the analysis of a charged particle whose configuration space is $\mathbf{R}^3$ minus the origin. We realize the two-sphere as $Q = SO(3)/SO(2) \equiv G/H$, so that the algebra of observables of the particle is $\mathscr{A} = C^*(SO(3), S^2)$. By the general theory (cf. 1.1.1) we know that the superselection sectors of this system correspond to $\widehat{SO}(2) = \mathbf{Z}$, that is, they are labeled by an integer n.

We label elements of $SO(3)$ by the Euler angles [3], so that we have

$$x = R(\alpha, \beta, \gamma) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}, \tag{3.1}$$

where the J_i are the standard generators of the Lie algebra $so(3)$, and $0 \le \alpha < 2\pi$, $0 \le \beta \le \pi$, $0 \le \gamma < 2\pi$. We identify H with the group of rotations around the z-axis, and realize Q as the set $\{Rq_0; R \in SO(3)\}$ with $\mathbf{R}^3 \ni q_0 = (0, 0, 1)$; its points are denoted in spherical co-ordinates by $q = (\phi, \theta)$. The representation π_n of $SO(2)$ is given by

$$\pi_n(R(0,0,\gamma)) \equiv \pi_n(R_z(\gamma)) = e^{-in\gamma}. \tag{3.2}$$

As we know from I (also cf. (1.1.1)), the irreducible representation π^n of the algebra $\mathscr{A}$ corresponds to the representation π^n of G which is induced by $\pi_n(H)$. In I.2.3 we gave three realizations of the carrier space $\mathscr{H}^\chi$ in the general case; let us now apply the abstract formulae given there to the concrete case studied here. Each realization will eventually correspond to a known description of the magnetic monopole.

Realization 1

The following realization has been used by Langlands in his study of the Dirac monopole [12]. Here π^n is realized by π_1^n acting on a Hilbert space $\mathscr{H}_1^n$, which is given by equivalence classes of complex-valued square-integrable functions on $SO(3)$ which satisfy the constraint

$$\psi_1^n(xR_z(\gamma)) = e^{in\gamma}\psi_1^n(x). \tag{3.3}$$

Hence the γ-dependence of ψ_1^n is given by $\psi_1^n(\alpha, \beta, \gamma) = \exp(in\gamma)\psi_1^n(\alpha, \beta, 0)$. The inner product in $\mathscr{H}_1^n$ is given by

$$(\psi_1^n, \varphi_1^n) = \frac{1}{4\pi} \int d\alpha\, d\beta \sin \beta \psi_1^n(\alpha, \beta, \gamma) \overline{\varphi_1^n(\alpha, \beta, \gamma)}. \tag{3.4}$$

The representation π_1^n of $SO(3)$ and $C(S^2)$ corresponding to $\pi_1^n(\mathscr{A})$ (cf. (1.1.1)) on $\mathscr{H}_1^n$ is now given by

$$\big(\pi_1^n(y)\psi_1^n\big)(x) = \psi_1^n(y^{-1}x);$$

$$\big(\pi_1^n(f)\psi_1^n\big)(\alpha, \beta, \gamma) = f(\alpha, \beta)\psi_1^n(\alpha, \beta, \gamma). \tag{3.5}$$

Realization 2

Next, consider $\pi_2^n(\mathscr{A})$ on $\mathscr{H}_2^n = L^2(S^2) = L^2([0, 2\pi] \times [0, \pi];\ d\phi\, d\theta \sin \theta)$. This realization is in some sense the most straightforward one, and, as we will see, it is the closest Hilbert space approximation to Dirac's description of the monopole [7]. To define $\pi_2^n(G)$ we need to choose a measurable section $s : S^2 \to SO(3)$; let us consider the two choices

$$s_{\pm}(\phi, \theta) = R(\phi, \theta, \mp\phi). \tag{3.6}$$

Evidently, s_+ is discontinuous at the south pole, whereas s_- is so at the north pole. This is irrelevant in the present Hilbert space context. To be definite, we adopt the choice $s = s_+$ in the formulae below. Writing $y = R(\alpha, \beta, \gamma)$, I.(2.13) is translated as

$$\big(\pi_2^n(y)\psi_2^n\big)(\phi, \theta) = \pi_n[R(\phi, -\theta, -\phi)R(\alpha, \beta, \gamma)R(\phi_y, \theta_y, -\phi_y)]\psi_2^n(\phi_y, \theta_y);$$

$$\big(\pi_2^n(f)\psi_2^n\big)(\phi, \theta) = f(\phi, \theta)\psi_2^n(\phi, \theta), \tag{3.7}$$

where (ϕ_y, θ_y) are the co-ordinates of the point (ϕ, θ) rotated by y^{-1}.

One may intertwine π_1^n and π_2^n by a unitary map $T_{12}^n : \mathscr{H}_1^n \to \mathscr{H}_2^n$, cf. I.(2.15), I.(2.16), given by

$$(T_{12}^n\psi_1^n)(\phi, \theta) = \psi_1^n(\phi, \theta, -\phi) = e^{-in\phi}\psi_1^n(\phi, \theta, 0), \tag{3.8}$$

that is, the right-hand side is not given by $\psi_1^n(\phi, \theta, 0)$, as one might naively expect on the basis of (3.4). The inverse is

$$\big((T_{12}^n)^*\psi_2^n\big)(\alpha, \beta, \gamma) = e^{in(\alpha+\gamma)}\psi_2^n(\alpha, \beta). \tag{3.9}$$

Realization 3

For reasons alluded to in Sec. 1, and to become clear shortly, in case that G is a Lie group, a realization of $\mathscr{H}^n$ inspired by differential geometry is convenient. In the context of a conventional quantum-mechanical description of the magnetic monopole, such a realization of the Hilbert space of states of a charged particle moving in a monopole field by means of "sectional wavefunctions" was first suggested in [10]; also cf. [20, 4].

The main goal here is to avoid the use of discontinuous sections in realization 2 above. Thus one covers S^2 with two (or more; this would lead to yet another unitarily equivalent realization) co-ordinate patches $U_{\pm}$, such that continuous sections $s_{\pm}: U_{\pm} \to SO(3)$ are possible. Choose $\varepsilon > 0$, and define U_+ and U_- to be the open sets of points on S^2 for which $\theta \in [0, \pi/2 + \varepsilon[$ and $\theta \in]\pi/2 - \varepsilon, \pi]$, respectively. Vectors ψ_3^n in the "Hilbert space of sections" $\mathscr{H}_3^n$ are pairs $(\psi_3^n)_{\pm}$, each of which is an equivalence class of L^2-functions defined on $U_{\pm}$, such that in the overlap region $U_+ \cap U_-$ the pair is related almost everywhere by

$$(\psi_3^n)_-(\phi, \theta) = e^{2in\phi}(\psi_3^n)_+(\phi, \theta). \tag{3.10}$$

The inner product in $\mathscr{H}_3^n$ is given by translating I.(2.18) with an appropriate choice of a partition of unity:

$$(\psi_3^n, \varphi_3^n) = \frac{1}{4\pi} \sum_{r=+,-} \int_{S_r^2} (\psi_3^n)_r \overline{(\varphi_3^n)_r} \tag{3.11}$$

where $S_+^2 \equiv S_N^2$ and $S_-^2 \equiv S_S^2$ are the northern and southern hemispheres, respectively.

The action of the representations π_3^n of $SO(3)$ and $C(S^2)$ follows from I.(2.19). Let r, s stand for the signs $+$ or $-$, r being $\pm$ if $(\phi, \theta) \in U_{\pm}$, and s being $\pm$ if $(\phi_y, \theta_y) \in U_{\pm}$ (cf. (3.7)). Then we have

$$\begin{aligned}(\pi_3^n(y)\psi_3^n)_{\pm}(\phi, \theta) &= \pi_n[R(r\phi, -\theta, -\phi)R(\alpha, \beta, \gamma)R(\phi_y, \theta_y, -s\phi_y)](\psi_3^n)_s(\phi_y, \theta_y);\\ (\pi_3^n(f)\psi_3^n)_r(\phi, \theta) &= f(\phi, \theta)(\psi_3^n)_r(\phi, \theta).\end{aligned} \tag{3.12}$$

The representations π_3^n and $\pi_{1,2}^n$ are, of course, unitarily equivalent, and their intertwining unitary maps $T_{13}^n: \mathscr{H}_1^n \to \mathscr{H}_3^n$ and $T_{23}^n: \mathscr{H}_2^n \to \mathscr{H}_3^n$ can be inferred from I.(2.20) and I.(2.22), respectively. This gives

$$(T_{13}^n \psi_1^n)_r(\phi, \theta) = \psi_1^n(\phi, \theta, -r\phi) = e^{-rin\phi}\psi_1^n(\phi, \theta, 0), \tag{3.13}$$

with inverse

$$((T_{13}^n)^* \psi_3^n)(\alpha, \beta, \gamma) = e^{in(\gamma + r\alpha)}(\psi_3^n)_r(\alpha, \beta), \tag{3.14}$$

where r relates to (α, β) in the same way as it does to (ϕ, θ). The connection between $\mathscr{H}_2^n$ and $\mathscr{H}_3^n$ is more direct: on the northern hemisphere the wavefunctions are identical, whereas on the southern hemisphere they are related by the gauge transformation (3.10). In the sequel we will only need the inverse transformation (also cf. I.(2.22))

$$\big((T^n_{23})^*\psi^n_3\big)(\phi,\theta) = (\psi^n_3)_+(\phi,\theta) \qquad \text{if } (\phi,\theta)\in U_+;$$

$$\big((T^n_{23})^*\psi^n_3\big)(\phi,\theta) = e^{-2in\phi}(\psi^n_3)_-(\phi,\theta) \qquad \text{if } (\phi,\theta)\in U_-. \tag{3.15}$$

Once again, in the overlap region either expression may be used.

Representation of the enveloping algebra

In order to construct the Hamiltonian later on, and as a matter of independent interest, we will now study the representation $(\pi^n)'$ of $\mathscr{U}(so(3))$, the enveloping algebra of $SO(3)$, which is derived from the unitary representation π^n constructed above in three different guises. The standard symmetric generators of $so(3)$ are denoted by J_i, $i = 1, 2, 3$. The representatives of elements in $\mathscr{U}(so(3))$ are unbounded operators, and it turns out that the related domain issues (which, as we shall see, have a direct physical content) are most easily handled on the sectional Hilbert space $\mathscr{H}^n_3$.

We write $\big((\pi^n_3)'(J_i)\psi^n_3\big)_r \equiv J^n_{i;r}(\psi^n_3)_r$, and find the following formal expressions from (3.12):

$$J^n_{1;r} = i\cos\phi\cot\theta\frac{\partial}{\partial\phi} + i\sin\phi\frac{\partial}{\partial\theta} + n\frac{\cos\phi}{\sin\theta}(1 - r\cos\theta);$$

$$J^n_{2;r} = i\sin\phi\cot\theta\frac{\partial}{\partial\phi} - i\cos\phi\frac{\partial}{\partial\theta} + n\frac{\sin\phi}{\sin\theta}(1 - r\cos\theta);$$

$$J^n_{3;r} = -i\frac{\partial}{\partial\phi} + rn. \tag{3.16}$$

(Choosing $r = +$ one obviously has formal expressions for the generators on $\mathscr{H}^n_2$.) We now have to find a domain in $\mathscr{H}^n_3$ where the operators in (3.16) are essentially self-adjoint. The standard procedure to follow in the representation theory of enveloping algebras is to find the Gårding domain [13], which consists of all vectors of the form $\pi(f)\psi$, with $f\in C^\infty_c(G)$ and $\psi\in\mathscr{H}$. Applying I.(2.23) to the present situation, it follows that the Gårding domain $D^3_G \subset \mathscr{H}^n_3$ is given by vectors

$$(\psi^n_f)_r(\phi,\theta) = \int_{SO(3)} d\mu(\alpha,\beta,\gamma) f\big(R(\phi,\theta,-r\phi)R(\alpha,\beta,\gamma)\big)e^{-in(\alpha+s\gamma)}(\psi^n_3)_s(-\gamma,-\beta), \tag{3.17}$$

with $f\in C^\infty(SO(3))$, and $d\mu$ the Haar measure on $SO(3)$.

To proceed, we use the following rather trivial argument: let an operator A' be a symmetric extension of an essentially self-adjoint operator A. Then A' is essentially self-adjoint. (This follows from the fact that $D(A'^*) \subset D(A^*)$, so that A' must have zero deficiency indices.) Now define the space $C^\infty_\pm(S^2)$ of pairs of functions $\psi_\pm$ which are in $C^\infty(U_\pm)$ and satisfy (3.10) on the overlap region. If we regard $C^\infty(U_\pm)$ as a subspace of $\mathscr{H}^n_3$, the conditions on ψ to be in this subspace are (apart from (3.10))

1. $\psi_\pm \in C^\infty([0,2\pi] \times [0,\pi] \cap U_\pm)$;
2. $\psi_\pm(0,\theta) = \psi_\pm(2\pi,\theta)$, and similar for all derivatives;
3. $\psi_\pm(\phi,0) = \psi_\pm(0,0)$; $(\partial^m \psi_\pm/\partial\phi^m)(\phi,0) = 0$ for all $m \geq 1$;
4. the previous condition, but now for $\theta = \pi$ rather than $\theta = 0$.

It is clear from (3.17) that $D_G^3 \subset C^\infty(U_\pm)$. The operators defined in (3.16) are still symmetric if extended to the domain $C^\infty(U_\pm)$. Hence they are essentially self-adjoint. A different way to arrive at this conclusion is to start at $\mathscr{H}_1^n$, and notice that the Gårding domain $D_G^1 = \mathscr{H}_c \cap \mathscr{H}_1^n$, where $\mathscr{H}_c$ is the space of functions which are convolutions $f * \psi$, with $f \in C^\infty(SO(3))$ and $\psi \in L^2(SO(3))$. Clearly $D_G^1 \subset C^\infty(SO(3)) \cap \mathscr{H}_1^n$. The generators $(\pi_1^n)'(J_i)$ (which are written down, e.g. in [3, Eq. (3.101)]) being symmetric also on this larger domain, it follows that they are essentially self-adjoint if extended to $C^\infty(SO(3)) \cap \mathscr{H}_1^n$. Now this domain is mapped onto $C^\infty(U_\pm)$ by the unitary intertwining map (3.13), and there we are.

It remains to find a domain of essential self-adjointness for the generators $(\pi_2^n)'(J_i) = J_{i;+}^n$ on $\mathscr{H}_2^n$ (see (3.16)). This trivially follows from (3.15); the result is that the appropriate domain is given by functions satisfying conditions analogous to 1, 2, and 3, above, whereas 4 is replaced by

4'. $\psi(\phi,\pi) = e^{-2in\phi}\psi(0,\pi)$.

As a check on this domain, one may notice that the singularity for $\theta = \pi$ in $(\pi_2^n)'(J_1)$ and $(\pi_2^n)'(J_2)$ is removed if these operators act on functions satisfying 4'. This condition is supposedly a relic of the Dirac string in the naive description of a magnetic monopole. Note, that the wavefunction by no means has to vanish at the south pole!

Finally, we should remark that the domain of the formal operators (3.16) has also been studied by Hurst [11], where no explicit conditions of the above type are given. However, the last section of this reference contains germinal ideas which, although stated in a non-C^*-algebraic language, resemble the way in which the dynamics of a particle moving in a monopole field is constructed below.

Dynamics

The construction of a time-evolution on the algebra of observables $\mathscr{A}$, as well as its implementation in its irreducible representations, has been reviewed in 1.1.2 and 1.1.3. Here we can literally follow the prescription in 1.1.3. We choose the Casimir operator $C = J_1^2 + J_2^2 + J_3^2$ in $\mathscr{U}(so(3))$, and according to I the Hamiltonian H^n in each irreducible representation $\pi^n(\mathscr{A})$ on $\mathscr{H}^n$ is given, up to a c-number, by $H^n = (\pi^n)'(C)$. For reasons to become apparent later, we choose the constant to be $-n^2$, so that in the realizations $\mathscr{H}_l^n$; $l = 1, 2, 3$ we have the Hamiltonians

$$H_l^n = \sum_{k=1}^{3} (\pi_l^n)'(J_k^2) - n^2 . \tag{3.18}$$

These operators, which are defined and essentially self-adjoint on the domains described earlier, will be interpreted at the end of the next section.

Monopole harmonics

The Hamiltonian H^n is $SO(3)$-invariant (its bounded spectral projections commute with the operators $\pi^n(x)$), so that in the realization H_1^n on $\mathscr{H}_1^n$ its eigenfunctions are given by (2.21). In this formula the index $\gamma \in \hat{G}$ should be replaced by $j \in \widehat{SO}(3) = \mathbf{N}$, the index $\kappa \in \hat{K}$ becomes $n \in \widehat{SO}(2) = \mathbf{Z}$, μ now runs from $-n$ to n, and σ takes the single value n. The condition for (2.21) to be meaningful is that $\pi_\kappa(K)$ occurs in $\pi_\gamma(G \downarrow K)$, which here means that $j \geq |n|$. Hence the eigenfunctions on $\mathscr{H}_1^n$ are given by

$$(\psi_1^n)_\mu^j(x) = \overline{D_{\mu n}^j(x)}, \tag{3.19}$$

with $x \in SO(3)$, and D the standard D-function [3, 3.6]. On $\mathscr{H}_2^n$ we find from (3.8) and the previous reference (where the d-functions appearing below are given as well) that

$$(\psi_2^n)_\mu^j(\phi, \theta) = e^{i\phi(\mu - n)} d_{\mu n}^j(\theta). \tag{3.20}$$

The "eigensections" in $\mathscr{H}_3^n$ are then apparent from (3.20) and (3.10). Note how the domain conditions 1–4 and 4' given earlier are satisfied as a consequence of the properties $d_{lm}^j(0) \sim \delta_{lm}$ and $d_{lm}^j(\pi) \sim \delta_{l,-m}$, with $l, m = -j, \ldots, j$.

3.2. *The Dirac Monopole*

We now include the radial co-ordinate in the problem, that is, we take the configuration space to be $Q = \mathbf{R}^3 - 0 \simeq \mathbf{R}^+ \times S^2$, where $\mathbf{R}^+$ is homeomorphic to $\mathbf{R}$ via the exponential map. A group acting transitively on Q is $G = \mathbf{R}^+ \times SO(3)$, where $\mathbf{R}^+$ (with generic element a), regarded as a group under ordinary multiplication, acts on the $\mathbf{R}^+$-part of Q by sending $r > 0$ to ar, whereas $SO(3)$ acts on S^2 as in the previous section. Since the former action is free, we still have $H = SO(2)$ as the little group. Taking the algebra of observables of a particle moving in Q to be $\mathscr{A} = C^*(G, Q)$, with the above choices of G and Q, we see that the superselection sectors are classified by $\hat{H} = \mathbf{Z}$, as in the case $Q = S^2$ studied in the previous section.

The radial co-ordinate not influencing the superselection structure of the model, the main technical difficulty its inclusion brings about lies in the construction of the pre-Hamiltonian (cf. 1.1.2). The problem is that the usual radial momentum operator is not the representative of an element in the enveloping algebra of $\mathbf{R}^+$, so that we cannot follow the algorithm stated in 1.1.3. To see this, consider the faithful representation of $\mathscr{A}$ given in I.(3.1). The Haar measure on $\mathbf{R}^+$ being dr/r, we have $\mathscr{H}_L = L^2(G) = L^2(\mathbf{R}^+; dr/r) \otimes L^2(SO(3))$, but in the present situation a more natural realization is provided by

$$\hat{\mathscr{H}}_L \equiv L^2(\mathbf{R}^3) \otimes L^2(S^1) = L^2(\mathbf{R}^+; r^2\, dr) \otimes L^2(SO(3)). \tag{3.21}$$

Vectors in $\mathscr{H}_L$ and $\hat{\mathscr{H}}_L$ are related by the unitary map $T: \mathscr{H}_L \to \hat{\mathscr{H}}_L$ defined by

$$(T\psi_L)(r,R) = (4\pi)^{-1/2} r^{-3/2} \psi_L(r,R), \tag{3.22}$$

and the corresponding representation $\hat{\pi}_L$ of $\mathscr{A}$ on $\hat{\mathscr{H}}_L$ follows from I.(3.1) and (3.22) to be

$$\begin{aligned} (\hat{\pi}_L(a,R)\hat{\psi}_L)(r,R') &= a^{-3/2}\hat{\psi}_L(r/a, R^{-1}R'); \\ (\hat{\pi}_L(f)\hat{\psi}_L)(r,R') &= f(r,\alpha,\beta)\hat{\psi}_L(r,\alpha,\beta,\gamma), \end{aligned} \tag{3.23}$$

where it is understood that R' corresponds to the Euler angles (α, β, γ).

In addition, $\hat{\mathscr{H}}_L$ carries the right-regular representation $\hat{\pi}_R(G)$, given by

$$(\hat{\pi}_R(a,R)\hat{\psi}_L)(r,R') = a^{3/2}\hat{\psi}_L(ar, R'R). \tag{3.24}$$

It follows that $\hat{\pi}_R(SO(2))$ is generated by

$$(\hat{\pi}_R)'(J_3) = -i\partial/\partial\gamma, \tag{3.25}$$

defined and essentially self-adjoint on functions in $\hat{\mathscr{H}}_L$ which are C^∞ and periodic in γ.

According to 1.1.2, the pre-Hamiltonian H_p must be an (essentially) self-adjoint operator that commutes with $\hat{\pi}_R(H) = \hat{\pi}_R(SO(2))$, where $SO(2)$ is embedded in SO(3) in the way explained after (3.1). The pre-Hamiltonian in any case should include a radial part p_r^2, where $p_r = -i(\partial/\partial r + 1/r)$ is the usual radial momentum on $L^2(\mathbf{R}^3)$. However, adopting the name p for the generator of $\mathbf{R}^+$, one has $(\pi_R)'(p) = -ir\partial/\partial r$ and hence $(\hat{\pi}_R)'(p) = -ir\partial/\partial r - 3i/2$, which comes nowhere near p_r. An additional problem comes from the $1/r^2$ factor multiplying the angular part of the usual free Hamiltonian expressed in polar co-ordinates. In view of this, there exists no element C of $\mathscr{U}(\mathscr{G})$ such that $H_p = (\hat{\pi}_R)'(C)$. Instead, we proceed "by hand". Consider the operator on $\hat{\mathscr{H}}_L$ defined by the formal expression

$$H_p = p_r^2 + \frac{1}{r^2}\sum_{k=1}^{2}(\hat{\pi}_R)'(J_k^2) = p_r^2 + \frac{1}{r^2}\big((\hat{\pi}_R)'(C) - (\hat{\pi}_R)'(J_3^2)\big), \tag{3.26}$$

where C is the usual quadratic Casimir operator in $\mathscr{U}(so(3))$. We define this on the domain $D = C_c^\infty(\mathbf{R}^+ \cup \{0\}) \otimes D_G^R(SO(3))_r$, where D_G^R is the Gårding domain for π_R on $L^2(SO(3))$, the suffix r meaning that one restricts this set to those functions ψ for which $H_p\psi \in \hat{\mathscr{H}}_L$, and the tensor product being the algebraic one. We then have

Theorem 1. *Define the pre-Hamiltonian* (3.26) *on D. Then*

1. H_p *is essentially self-adjoint;*
2. *the closure of* H_p *commutes with* $\hat{\pi}_R(SO(2))$;
3. *the closure of* H_p *commutes with* $\hat{\pi}_L(SO(3))$.

(To be precise, the bounded spectral projections of the closure of H_p *have the two last-mentioned properties.)*

Proof. As to item 1, the spherical part of this expression is essentially self-adjoint on the given domain, and it follows from (3.25) in conjunction with (3.3) that its closure has spectrum $\{l(l+1)-n^2\}$, where $l \in \mathbf{N}$, $n \in \mathbf{Z}$, and $|n| \leq l$. Hence the radial part of (3.26) can be examined from the partial pre-Hamiltonians $H_p^{l,n} = p_r^2 + (l(l+1) - n^2)/r^2$. It follows from [16, Th.X.11] (or by explicit computation) that these expressions are essentially self-adjoint for $l > 0$; for $l = n = 0$ the deficiency indices are easily shown by hand to be (0, 0), due to our specific choice of D, which enforces the boundary conditions $\lim_{r\to 0} r^2\psi(r) = \lim_{r\to 0} r^2\, d\psi/dr = 0$ on elements of $D((H_p^{l,n})^*)$. Note, that replacing $C_c^\infty(\mathbf{R}^+ \cup \{0\})$ in D by $C_c^\infty(\mathbf{R}^+)$ would spoil the essential self-adjointness, as it leads to arbitrary boundary conditions near 0 (also cf. [2]). Item 2 follows from the fact that C is central, so that $(\hat{\pi}_R)'(C) = (\hat{\pi}_L)'(C)$, and 3 follows from Segal's theorem stating that the von Neumann algebras generated by the left- and the right-regular representation are each other's commutant [13, VI.12]. □

By 1.1.2, this result means that the closure of H_p, which we will denote by H_p as well, defines a time-evolution on the algebra of observables $\mathscr{A}$, and this time-evolution is $SO(3)$-invariant.

Irreducible Hamiltonians

We now look at the Hamiltonians implementing the time-evolution on $\mathscr{A}$ in each of its irreducible representations. Most of the work has already been done in the previous section, because the inclusion of the radial co-ordinate does not affect the representation theory of $\mathscr{A}$. In particular, its representations π_n (we use the same notation as in the purely spherical case) are given by the tensor product of the representations constructed in 3.1 with a radial term that can be read from (3.23), on carrier spaces which are a tensor product of $L^2(\mathbf{R}^+; r^2\, dr)$ with the spaces $\mathscr{H}_i^n$ constructed in 3.1. Let us concentrate on the realization π_2^n on $\mathscr{H}_2^n = L^2(S^2)$ given in 3.1. Tensoring with $L^2(\mathbf{R}^+; r^2\, dr)$ then gives the carrier space $L^2(\mathbf{R}^3)$ expressed in spherical co-ordinates; this is the whole point of the transformation (3.22). The time-evolution defined by the pre-Hamiltonian (3.26) is then implemented by

$$\hat{H}_2^n = p_r^2 + \frac{1}{r^2} H_2^n, \tag{3.27}$$

where H_2^n is given in (3.18). The only part of this result that needs explanation is how the term $(\hat{\pi}_R)'(J_3^2)$ in (3.26) leads to the term n^2 in (3.18). This follows from the explicit reduction of the (abstract version of) the representation $\pi_L(\mathscr{A})$ (cf. I.3.2); in particular, the irreducible subspaces $\mathscr{H}_1^n$ are embedded in $\hat{\mathscr{H}}_L$ by means of the constraint (3.3) (with the radial co-ordinate added). As we see from (3.25), $(\hat{\pi}_R)'(J_3^2)$ is therefore diagonal on these subspaces with eigenvalue n^2. $\mathscr{H}_2^n$ being unitarily equivalent to $\mathscr{H}_1^n$ (cf. (3.8), (3.9)), the result therefore follows.

The domain of the formal expression (3.27) immediately follows from the appropriate domains for the spherical part and the radial dependence separately: the angular dependence of the function $\psi \subset D(\hat{H}_2^n)$, with argument (r, ϕ, θ), for fixed r is governed by the conditions 1–4' stated after (3.17), whereas the radial dependence is such that for fixed (ϕ, θ) it is in $C_c^\infty(\mathbf{R}^+ \cup \{0\})$, the complete function being further restricted by the demand that the operator (3.27) does not map it out of $L^2(\mathbf{R}^3)$. By our previous considerations, $\hat{H}_2^n$ is essentially self-adjoint on this domain, and its closure, which we denote by the same symbol, is a *bona fide* Hamiltonian of a particle moving in $\mathbf{R}^3 - 0$.

Punchline

$$\hat{H}_2^n = (\mathbf{p} - e\mathbf{A})^2, \tag{3.28}$$

where $\mathbf{A}$ is an electromagnetic potential given by

$$\mathbf{A}(r, \phi, \theta) = \frac{g(1 - \cos\theta)}{r \sin\theta} \mathbf{e}_\phi, \tag{3.29}$$

and e and g are related by $eg = -n$. In other words, the Hamiltonian $\hat{H}_2^n$, constructed by purely operator- and group-theoretic considerations, without any input from the theory of electromagnetism, is precisely the Hamiltonian of a charged particle moving in the field of a magnetic monopole with charge g sitting at the origin [7, 4]. The Dirac quantization condition for the charge is identically satisfied, and the quantized charge defines a superselection rule for a particle moving in its field. No Dirac string (or "veto") is necessary to describe this system; the apparent singularity along the negative z-axis is completely removed by the conditions on the wavefunctions in the domain of the Hamiltonian (3.27).

Alternatively, we could have adopted the Greub-Petry-Wu-Yang [10, 20] description of the monopole in terms of Hilbert spaces of sections, as in 3.1 (realization 3), and find it to be unitarily equivalent to the above description, cf. 3.1 (the radial co-ordinate does not affect these considerations) from the point of view of the representation theory of the C^*-algebra of observables of the particle.

Finally, we remark that half-integer monopole charges ($eg = n/2$) and, accordingly, half-integer monopole harmonics, can easily be incorporated in the present formalism by passing from $SO(3)$ to $SU(3)$, which acts on Q via its canonical epimorphism onto $SO(3)$. The stability group then picks up an extra factor $\mathbf{Z}_2$, which accounts for the possibility of having half-integer charges.

4. The Aharonov-Bohm Effect

4.1. *Particle on a circle*

We now choose the configuration space to be $Q = S = S^1 = U(1)$. We label points on S by the complex co-ordinate w, with $|w| = 1$ (the symbol z will be used later on for the z-axis). To find the desired structures, we take $G = \mathbf{R}$, which acts transitively on S as follows: $x \in \mathbf{R}$ sends $w \in S$ to $\exp(2\pi i x)w$. Obviously, the stability group is $H = \mathbf{Z}$. According to our program we take the C^*-algebra of observables to be

$C^*(\mathbf{R}, S)$, and find its set of superselection sectors (classes of unitarily equivalent representations) being $\hat{\mathscr{A}} = \hat{H} = U(1)$. Accordingly, the irreducible representations $\pi^\theta(\mathscr{A})$ are labeled by an angle $\theta \in [0, 2\pi[$, and the corresponding representation of $G = \mathbf{R}$ is the one induced by the character $\pi_\theta(n) = \exp(in\theta)$, $n \in \mathbf{Z}$.

As in the monopole case, it is illuminating to write out explicitly what the three realizations of π^θ, given in abstract form in I.2.3, look like.

Realization 1

Here the carrier space is the Hilbert space $\mathscr{H}_1^\theta$ of functions $\psi_1^\theta : \mathbf{R} \to \mathbf{C}$ which satisfy the constraint

$$\psi_1^\theta(x + n) = e^{-in\theta}\psi_1^\theta(x), \tag{4.1}$$

and have finite inner product

$$(\psi_1^\theta, \varphi_1^\theta) = \int_0^1 dx\, \psi_1^\theta(x)\overline{\varphi_1^\theta(x)}. \tag{4.2}$$

Taking the fixed point $q_0 \equiv w_0 = 1$, $\mathscr{A}$ is represented by (cf. I.(2.11))

$$\begin{aligned} (\pi_1^\theta(y)\psi_1^\theta)(x) &= \psi_1^\theta(x - y); \\ (\pi_1^\theta(f)\psi_1^\theta)(x) &= f(e^{2\pi i x})\psi_1^\theta(x). \end{aligned} \tag{4.3}$$

Realization 2

The following realization will turn out to be most appropriate if we wish to study the Aharonov-Bohm effect on $\mathbf{R}^3$. Now the carrier space is $\mathscr{H}_2^\theta = L^2(S; dw/2\pi i w)$, where, as above, S is supposed to be embedded in the complex plane (an alternative realization would be $L^2([0, 2\pi]; d\phi)$). To define $\pi_2^\theta(\mathscr{A})$ one must choose a measurable section $s : S \to \mathbf{R}$, and we take $s(w) = s_+(w)$, where $s_\pm(w)$ are defined by

$$s_\pm(w) = \frac{1}{2\pi i}\log_\pm w, \tag{4.4}$$

where $\log_+$ and $\log_-$ have their cuts on the positive and the negative real axis, respectively. Equivalently, the former is defined with respect to $\operatorname{Arg} w \in [0, 2\pi[$ whereas the latter has $\arg w \in [-\pi, \pi[$. Accordingly, $s_\pm$ is discontinuous in $w = \pm 1$. For later use, we record the relations

$$\log_- w = \log_+ w \text{ if } 0 < \operatorname{Arg} w < \pi;$$

$$\log_- w = \log_+ w - 2\pi i \text{ if } \pi < \operatorname{Arg} w < 2\pi;$$

$$\log_+ zw = \log_+ z + \log_+ w \text{ if } \operatorname{Arg} z + \operatorname{Arg} w < 2\pi;$$

$$\log_+ zw = \log_+ z + \log_+ w - 2\pi i \text{ if } \operatorname{Arg} z + \operatorname{Arg} w > 2\pi. \tag{4.5}$$

Computing the Wigner cocycle, we then find

$$(\pi_2^\theta(y = \beta + l)\psi_2^\theta)(w) = e^{il\theta}\psi_2^\theta(e^{-2\pi i\beta}w);$$

$$(\pi_2^\theta(f)\psi_2^\theta)(w) = f(w)\psi_2^\theta(w). \tag{4.6}$$

The first line only holds if $\operatorname{Arg} w > 2\pi\beta$; in the opposite case one should replace l by $l + 1$ on the right-hand side (here $l \in \mathbf{Z}$ and $\beta \in [0, 1[$). Thus the representation is strongly continuous in y, as it should be.

The unitary equivalence with π_1^θ is established by the unitary intertwiner $T_{12}^\theta : \mathscr{H}_1^\theta \to \mathscr{H}_2^\theta$ defined by

$$(T_{12}^\theta\psi_1^\theta)(w) = \psi_1^\theta((\log_+ w)/2\pi i), \tag{4.7}$$

with inverse

$$((T_{12}^\theta)^*\psi_2^\theta)(x = \beta + l) = e^{-il\theta}\psi_2^\theta(e^{2\pi i\beta}). \tag{4.8}$$

Note that $(T_{12}^\theta)^*$ does not map continuous functions in $\mathscr{H}_2^\theta$ into continuous functions in $\mathscr{H}_1^\theta$, unless the former satisfy $\lim_{w\downarrow 1}\psi_2^\theta(w) = e^{i\theta}\lim_{w\uparrow 1}\psi_2^\theta(w)$.

Realization 3

Now for $\mathscr{H}_3^\theta$, a "Hilbert space of sections". We choose $0 < \varepsilon < \pi/2$, and define two co-ordinate pathches on S as follows:

$$U_- = \left\{w \in S \,\middle|\, -\frac{\pi}{2} - \varepsilon < \arg w < \frac{\pi}{2} + \varepsilon\right\};$$

$$U_+ = \left\{w \in S \,\middle|\, \frac{\pi}{2} - \varepsilon < \operatorname{Arg} w < \frac{3\pi}{2} + \varepsilon\right\}. \tag{4.9}$$

The sections $s_\pm$ (4.4) are defined and analytic on $U_\pm$, and in the overlap regions they are related by a gauge transformation $s_+(w) = s_-(w) + g_{-+}(w)$. The overlap consists of two disjoint regions: region A, for which $\pi/2 - \varepsilon < \operatorname{Arg} w < \pi/2 + \varepsilon$, and region B with $3\pi/2 - \varepsilon < \operatorname{Arg} w < 3\pi/2 + \varepsilon$. Then $g_{-+}(w) = 0$ for $w \in A$ and $g_{-+}(w) = 1$ for $w \in B$.

Therefore, in view of I.(2.14), elements ψ_3^θ of $\mathscr{H}_3^\theta$ are given by pairs $(\psi_3^\theta)_\pm \in L^2(U_\pm)$ which satisfy

$$(\psi_3^\theta)_-(w) = (\psi_3^\theta)_+(w) \qquad \text{if } w \in A;$$

$$(\psi_3^\theta)_-(w) = e^{i\theta}(\psi_3^\theta)_+(w) \qquad \text{if } w \in B. \tag{4.10}$$

The inner product may be chosen as

$$(\psi_3^\theta, \varphi_3^\theta) = \lim_{\varepsilon\to 0} \sum_{r=+,-} \int_{U_r} \frac{dw}{2\pi i w} (\psi_3^\theta)_r(w)\overline{(\varphi_3^\theta)_r(w)}. \tag{4.11}$$

By I.(2.19), the representation $\pi_3^\theta(\mathscr{A})$ is given by

$$(\pi_3^\theta(y)\psi_3^\theta)_r(w) = \exp\frac{\theta}{2\pi}[-\log_r w + 2\pi i y + \log_s(e^{-2\pi i y}w)](\psi_3^\theta)_s(e^{-2\pi i y}w);$$

$$(\pi_3^\theta(f)\psi_3^\theta)_r(w) = f(w)(\psi_3^\theta)_r(w), \tag{4.12}$$

(cf. (3.12)) where $r = \pm$ if $w \in U_\pm$ and $s = \pm$ if $\exp(-2\pi i y)w \in U_\pm$. In the overlap region either expression may be used due to the compatibility condition (4.10). The expression in square brackets is an integer, which may be determined by using identities of the type (4.5).

The unitary equivalence with the other two realizations is assured by the intertwiners $T_{i3} : \mathscr{H}_i^\theta \to \mathscr{H}_3^\theta$, $i = 1, 2$. Firstly, from I.(2.20) we have

$$(T_{13}^\theta \psi_1^\theta)_r(w) = \psi_1^\theta\left(\frac{1}{2\pi i}\log_r w\right), \tag{4.13}$$

and note how this equation (with (4.5)) reproduces (4.10). We will neither need nor write down the inverse of T_{13}^θ, whereas we will only need the inverse of T_{23}^θ in the sequel, which according to (4.5) and I.(2.22) is

$$\begin{aligned}((T_{23}^\theta)^*\psi_3^\theta)(w) &= (\psi_3^\theta)_+(w) \text{ if } w \in U_+;\\ &= (\psi_3^\theta)_-(w) \text{ if } 0 < \operatorname{Arg} w < \frac{\pi}{2} + \varepsilon;\\ &= e^{-i\theta}(\psi_3^\theta)_-(w) \text{ if } \frac{3\pi}{2} - \varepsilon < \operatorname{Arg} w < 2\pi.\end{aligned} \tag{4.14}$$

Representation of the enveloping algebra

The enveloping algebra $\mathscr{U}(\mathbf{R})$ is generated by a single symmetric element, which we choose as $T = (2\pi i)^{-1} d/dx$ (acting on $C_c^\infty(\mathbf{R})$). We start with the easiest case, which is $(\pi_3^\theta)'$ on $\mathscr{H}_3^\theta$. From (4.12) we formally have

$$(\pi_3^\theta)'(T) = w\frac{d}{dw} \equiv p^\theta. \tag{4.15}$$

This object should exponentiate to the representation $\pi^\theta(\mathbf{R})$, which is explicitly θ-dependent. The θ-dependence of p^θ is obviously in its domain. By an argument similar to the one used in the previous section (in particular after (3.17)), which ultimately rests on I.(2.23), it may be shown that p^θ is essentially self-adjoint on the domain $C^\infty_\mp(S) \subset \mathscr{H}^\theta_3$ (and, once again, this result may also be derived from (4.13)). In analogy to $C^\infty_\mp(S^2)$, this domain consists of those ψ^θ_3 whose representatives $(\psi^\theta_3)_\pm$ are in $C^\infty(U_\pm)$. It goes without saying that this domain is θ-dependent because of the pasting condition (4.10).

The corresponding domain on $\mathscr{H}^\theta_2$, which is the realization with the most direct physical interpretation, may then be found by applying the unitary transformation (4.14). The result is that the appropriate domain of essential self-adjointness of p^θ on $\mathscr{H}^\theta_2$ is given by D^θ_2, which consists of those ψ^θ_2 which are in $C^\infty(S-1)$ (the circle with the point $w = 1$ removed) and satisfy the boundary condition

$$\lim_{w\uparrow 1} \psi^\theta_2(w) = e^{-i\theta} \lim_{w\downarrow 1} \psi^\theta_2(w). \tag{4.16}$$

This condition may be stated more naturally in the realization of $\mathscr{H}^\theta_2$ as $L^2([0, 2\pi])$, where $p^\theta = -id/d\phi$ is essentially self-adjoint on the equivalent domain consisting of those $\psi \in C^\infty([0, 2\pi])$ which satisfy $\psi(2\pi) = \exp(-i\theta)\psi(0)$. The exact domain on which the operator is self-adjoint may then be found in Example 1 of [16, X.1].

The θ-dependence of p^θ on $\mathscr{H}^\theta_2$ may be made more explicit by performing a unitary transformation $U : \mathscr{H}^\theta_2 \to \mathscr{H}^\theta_2$ defined by

$$(U\psi^\theta_2)(w) = \exp\left(\frac{\theta}{2\pi}\log_+ w\right)\psi^\theta_2(w) = w^{\theta/2\pi}\psi^\theta_2(w); \tag{4.17}$$

then UD^θ_2 is as before, but without the phase factor in (4.16), whereas the transform of p^θ as a formal differential operator is now given by

$$Up^\theta U^* = w\frac{d}{dw} - \frac{\theta}{2\pi}. \tag{4.18}$$

Time-evolution

We can now construct a reasonable time-evolution on $\mathscr{A}$ by following the prescription in 1.1.3. The faithful representation $\pi_L(\mathscr{A})$ on $\mathscr{H}_L = L^2(\mathbf{R})$ is given by (4.3) with ψ^θ_1 replaced by ψ_L, and the pre-Hamiltonian on $\mathscr{H}_L$ is taken to be

$$H_p = (\pi_L)'(4\pi^2 T^2) = -\frac{d^2}{dx^2}, \tag{4.19}$$

which is initially defined and essentially self-adjoint on the Gårding domain of functions of the form $f * \psi$, $f \in C^\infty_c(\mathbf{R})$ and $\psi \in L^2(\mathbf{R})$, and may be extended to its usual domain of self-adjointness [16].

According to 1.1.3, the Hamiltonian H^θ_i in the superselection sector labeled by θ is then given, in the realizations π^θ_2 and π^θ_3, by

$$H_i^\theta = 4\pi^2(\pi_i^\theta)'(T^2) = 4\pi^2\left(w\frac{d}{dw}\right)^2, \tag{4.20}$$

(with $i = 2, 3$) defined and essentially self-adjoint on the domain for the representation of the enveloping algebra constructed, in its various realizations, in the previous section. Put differently, given the formal differential operator (4.15) (defined, e.g. on functions vanishing near $w = 1$), inequivalent representations of the C^*-algebra $\mathscr{A}$ select different extensions of this operator in its role of implementing the time-evolution on the algebra in the given representation. In this way, the θ-dependence of these extensions will ultimately "explain" the Aharonov-Bohm effect.

Some very simple automorphic functions

Let us concentrate on $\mathscr{H}_2^\theta$, and study the eigenfunctions of the Hamiltonian H_2^θ (the corresponding functions in $\mathscr{H}_1^\theta$ and $\mathscr{H}_3^\theta$ may then be found by the inverse of (4.13) and by (4.14), respectively). Of course, this can easily be done by hand, but let us use the general methods reviewed in 2.2. As $S = \mathbf{R}/\mathbf{Z}$ is compact, we have a discrete decomposition

$$\mathscr{H}_2^\theta \simeq \oplus_{p \in \mathbf{R}}\, n_p^\theta \mathscr{H}_p, \tag{4.21}$$

where $\mathscr{H}_p = \mathbf{C}$, carrying the representation $\pi_p(x) = \exp(-ipx)$ of $\mathbf{R}$. To determine the multiplicities we first use the Frobenius reciprocity theorem $n_p^\theta = n_\theta^p$, where the right-hand side is the number of times that $\pi_\theta(\mathbf{Z})$ occurs in $\pi_p(\mathbf{R} \downarrow \mathbf{Z})$. This number is evidently either 0 or 1, and for it to be 1 it must be that $\exp(in\theta) = \exp(-inp)$ for all $n \in \mathbf{Z}$. This implies $p = -\theta + 2\pi n$, and therefore the multiplicities in (4.21) are as follows: $n_p^\theta = 1$ if $p = -\theta + 2\pi n$ for arbitrary $n \in \mathbf{Z}$, and $n_p^\theta = 0$ otherwise.

We may draw the same conclusion from the Selberg trace formula (2.23). The fundamental domain in $\mathbf{R}$ may be chosen as $F = \,]0, 1[$, and the formula becomes

$$\sum_{n \in Z} f(n)e^{in\theta} = \sum_{p \in \mathbf{R}} n_p^\theta \hat{f}(p), \tag{4.22}$$

for functions f on $\mathbf{R}$ for which both sides are finite. Here $\hat{f}$ is the Fourier transform of f (without any factor of 2π), and by the pre-compactness of F one knows that only a countable number of p's contribute to the right-hand side. Now choose $f(x) = \exp(iqx)/2\pi$, and use the identity $\sum_n \exp(in\phi) = 2\pi \sum_n \delta(\phi - 2\pi n)$. The trace formula (4.22) then demands that

$$\sum_{n \in Z} \delta(q + \theta - 2\pi n) = \sum_p n_p^\theta \delta(q - p)$$

for all $q \in \mathbf{R}$. This gives the same result for the n_p^θ as before. Substituting these multiplicities back into the Selberg trace formula, one finds a trivial generalization of the Poisson summation formula.

The above considerations have given the multiplicities of functions in $\mathscr{H}_2^\theta$ which transform irreducibly under $\pi_2^\theta(\mathbf{R})$ (hence the eigenfunctions of the Hamiltonian), but not their explicit form. This must be determined by other means, and here one finds, for $m \in \mathbf{Z}$,

$$(\psi_2^\theta)_m(w) = w^{m-\theta/2\pi}, \tag{4.23}$$

where the fractional power is defined with a cut along the positive real axis. Thus the domain condition (4.16) is satisfied. These eigenfunctions are special, and rather trivial cases of automorphic functions.

To end this section, we give some comments as to the interpretation of this result. The eigenvalues of the Hamiltonian H_2^θ are $E_m = (m - \theta/2\pi)^2$, so that for $\theta \in [0, \pi]$ $(\psi_2^\theta)_0$ is the ground state, whereas for $\theta \in [\pi, 2\pi[$ the ground state is $(\psi_2^\theta)_1$. For $\theta = \pi$ the ground state is evidently degenerate. Regarding these vectors as pure states on the C^*-algebra of observables $\mathscr{A}$, they may be identified with "θ-vacua" of the particle on a circle with the given time-evolution. Each θ-vacuum lies in, and defines, a superselection sector of the theory. A related model with an infinite numbers of degrees of freedom would presumably show a phase transition for $\theta = \pi$.

4.2. *The full configuration space*

To study the full Aharonov-Bohm effect we need to include a radial and an axial co-ordinate into the problem, that is, we now consider $Q = \mathbf{R}^3 - \{z\text{-axis}\} \simeq \mathbf{R}^+ \times \mathbf{R} \times S \simeq \mathbf{R}^2 \times S$. A group acting transitively on Q is $G = \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R} \simeq \mathbf{R}^3$ (the above identification of $\mathbf{R}^+$ and $\mathbf{R}$ proceeds via the exponential map), whose elements $x = (a, b, c)$ $(a > 0)$ send a point $q = (r, z, \phi)$ (expressed in cylindrical co-ordinates) to $xq = (ar, z + b, \phi + 2\pi c \bmod 2\pi)$. We see that the little group does not change in comparison with the case $Q = S$: we again have $H = \mathbf{Z}$, its elements corresponding to $(1, 0, n) \in G$, $n \in \mathbf{Z}$. Hence all remarks in the previous section on the superselection structure of the model still stand.

The following discussion parallels that of the monopole case in 3.2. To find a suitable time-evolution on $\mathscr{A} = C^*(G, Q)$ we investigate the faithful representation $\pi_L(\mathscr{A})$ (cf. I.(3.2)) on $\mathscr{H}_L = L^2(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R};\, dr/r\, dz\, dc)$. For later extraction of $L^2(\mathbf{R}^3)$ it is more convenient to realize π_L as $\hat{\pi}_L$ on $\hat{\mathscr{H}}_L = L^2(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R};\, r\, dr\, dz\, dc)$, the unitary equivalence between the two being given by the map $V: \mathscr{H}_L \to \hat{\mathscr{H}}_L$ defined by

$$(V\psi_L)(r, z, c) = \frac{1}{r}\psi_L(r, z, c). \tag{4.24}$$

The algebra $\mathscr{A}$ is represented by

$$(\hat{\pi}_L(a, b, c')\hat{\psi}_L)(r, z, c) = \frac{1}{a}\hat{\psi}_L(r/a, z - b, c - c');$$

$$(\hat{\pi}_L(f)\hat{\psi}_L)(r, z, c) = f(r, z, e^{2\pi i c})\hat{\psi}_L(r, z, c), \tag{4.25}$$

cf. (4.3). In addition, $\hat{\mathscr{H}}_L$ carries the right-regular representation $\hat{\pi}_R(G)$ defined by

$$\big(\hat{\pi}_R(a, b, c')\hat{\psi}_L\big)(r, z, c) = a\hat{\psi}_L(ar, z + b, c + c'). \tag{4.26}$$

For the same reasons as in the monopole case, the pre-Hamiltonian $\mathscr{H}_p$ we are after cannot be written as a representative of an element in the enveloping algebra of G, and we have to proceed in an indirect way. As in the monopole case, we are looking for a pre-Hamiltonian defining a time-evolution on $\mathscr{A}$ which in the "trivial" superselection sector $\theta = 0$ is implemented by the usual free Hamiltonian on $L^2(\mathbf{R}^3)$, i.e., (minus) the Laplacian. A suitable candidate for H_p is

$$-H_p = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial c^2}. \tag{4.27}$$

As in 3.2, only the radial terms cause problems; we may rewrite the expression as

$$-H_p = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + (\hat{\pi}_R)'(T_z^2) + \frac{1}{r^2}(\hat{\pi}_R)'(T_c^2), \tag{4.28}$$

where T_z and T_c are the generators of the subgroup $\mathbf{R} \times \mathbf{R} \subset G$ (hence $T_c = 2\pi T$, where T is the generator used in 4.1). An analysis similar to the one surrounding Theorem 1 in 3.2 (also cf. the treatment of the two-dimensional δ-function potential in [2]) shows that H_p is essentially self-adjoint on $D_2 = C_c^\infty(\mathbf{R}^+ \cup \{0\}) \otimes C_c^\infty(\mathbf{R}) \otimes C_c^\infty(\mathbf{R})_r$ (where the last factor refers to the z-coordinate), and its closure (called H_p as well) defines an acceptable pre-Hamiltonian. Namely, it obviously satisfies the basic demand (see I.3.3, or 1.1.3) of being affiliated to $\mathscr{M}_R(H)'$ (with $H = \mathbf{Z}$), a condition which in this case is trivial, because G is abelian. The time-evolution is not invariant under the full group $G = \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$, but just under $\mathbf{R} \times \mathbf{R}$, as the explicit r-dependence of H_p evidently breaks the $\mathbf{R}^+$-invariance.

The final step is to determine the "true" Hamiltonians $\hat{H}_i^\theta$ implementing the time-evolution in the irreducible representations $\pi_i^\theta(\mathscr{A})$, in their various guises $i = 1, 2, 3$. We confine our attention to $i = 2$, which leads to the most familiar realization of the Hilbert space in question. Most of the work has been done in 4.1; we now parametrize the complex number $w \in S$ used in realization 2 in 4.1 by $w = \exp(i\phi)$, and realize the $\mathscr{H}_2^\theta$ of the purely circular case as $L^2([0, 2\pi]; d\phi)$. We then use the same label $\mathscr{H}_2^\theta$ for the carrier space of $\pi_2^\theta(\mathscr{A})$ of the full algebra of observables, including those corresponding to the radial co-ordinate and the z-axis. Hence

$$\mathscr{H}_2^\theta = L^2(\mathbf{R}^+; r\,dr) \otimes L^2(\mathbf{R}; dz) \otimes L^2([0, 2\pi]; d\phi), \tag{4.29}$$

which is nothing but $L^2(\mathbf{R}^3)$ expressed in cylindrical co-ordinates (this is the point of the transformation (4.24)).

The irreducible representation π_2^θ of the algebra of observables $\mathscr{A}$ is then given by (cf. (4.6), (4.26))

$$(\pi_2^\theta(a,b,c=\beta+l)\psi_2^\theta)(r,z,\phi)=\frac{1}{a}e^{il\theta}\psi_2^\theta(r/a,z-b,\phi-2\pi\beta \bmod 2\pi);$$

$$(\pi_2^\theta(f)\psi_2^\theta)(r,z,\phi)=f(r,z,\phi)\psi_2^\theta(r,z,\phi). \tag{4.30}$$

Here we assumed that $\phi-2\pi\beta\in[0,2\pi[$; in the opposite case l should be replaced by $l+1$ in the exponent.

From (4.20), the remark after (4.16), and (4.27) in combination with the discussion on the domain, we finally infer that the Hamiltonian in the θ-superselection sector is

$$-\hat{H}_2^\theta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}, \tag{4.31}$$

which is defined and essentially self-adjoint on

$$D_\theta\equiv D(\hat{H}_2^\theta)=C_c^\infty(\mathbf{R}^+\cup\{0\})\otimes C_c^\infty(\mathbf{R})\otimes C_\theta^\infty([0,2\pi])_r, \tag{4.32}$$

where $C_\theta^\infty([0,2\pi])$ consists of those C^∞-functions which satisfy the twisted periodicity condition $\psi(2\pi)=\exp(-i\theta)\psi(0)$, and the suffix r means (as always) that the given domain is restricted to those functions ψ_2^θ for which $\hat{H}_2^\theta\psi_2^\theta$ is in $\mathscr{H}_2^\theta$; finally, the tensor product is evidently the algebraic one.

The main feature is, of course, the θ-dependence of this domain, hence of the Hamiltonian itself. As in (4.17), this dependence may be made more explicit by performing a unitary transformation $U':\mathscr{H}_2^\theta\to\mathscr{H}_2^\theta$ given by

$$(U'\psi_2^\theta)(r,z,\phi)=e^{i\theta\phi/2\pi}\psi_2^\theta(r,z,\phi), \tag{4.33}$$

which maps D_θ to D_0, and $\hat{H}_2^\theta$ into $U'\hat{H}_2^\theta(U')^*$, which is given by (4.31) upon replacement of $\partial/\partial\phi$ by $\partial/\partial\phi-i\theta/2\pi$, which is defined on D_0, cf. [5].

Another punchline

$$U'\hat{H}_2^\theta(U')^*=(\mathbf{p}-e\mathbf{A})^2, \tag{4.34}$$

where $\mathbf{A}$ is an electromagnetic potential given by

$$\mathbf{A}(r,z,\phi)=\frac{\Phi}{2\pi r}\mathbf{e}_\phi, \tag{4.35}$$

which is the field of an infinitely thin solenoid sitting along the z-axis. The magnetic flux is related to the θ-parameter by $\Phi=\theta/e$. This result is entirely analogous to (3.29) in the monopole case: once again, the main feature is that a given abstract time-evolution on the algebra of observables $\mathscr{A}$ is implemented by a Hamiltonian that explicitly depends on the superselection sector θ, and this θ-dependence, which in the present case is quite subtle, has clear physical consequences, viz. the Aharonov-Bohm effect.

The Aharonov-Bohm effect has two peculiar features. Firstly, the effect is periodic in Φ. This is explained in our description by the fact that everything is periodic in θ, which is a consequence of the representation theory of the algebra of observables of a particle moving in this particular configuration space, and as such is entirely analogous to the Dirac charge quantization. Secondly, the magnetic field itself vanishes off the z-axis, so that, in a conventional electrodynamical description, the particle feels a vector potential which is a pure gauge. It is occasionally argued (see, e.g., refs. in [17, 5]) that the Aharonov-Bohm effect arises because one cannot transform away the vector potential, for the required gauge transformation would be multivalued (in a single point on the circle, and on a plane in $\mathbf{R}^3 - \{z\text{-axis}\}$). This argument in itself is not quite sufficient to explain the effect, for one can construct a perfectly *bona fide* unitary operator implementing such a gauge transformation (namely the inverse of (4.33)); the real point, as stressed, e.g., in [5, 17], is that, as we have seen, such a unitary transformation affects not only the formal expression of the Hamiltonian (as a differential operator), but also its domain, and that by removing the flux-dependent term from the formal Hamiltonian one effectively re-introduces it by selecting a flux-dependent domain. The crucial importance of a correct specification of the domain of $\hat{H}^\theta$ is immediate in our approach to the Aharonov-Bohm effect as well: it follows from I.3.3 that, as reviewed in 1.1.3, the Hamiltonian implementing the abstract time-evolution on the algebra of observables is initially defined as an essentially self-adjoint operator on the Gårding domain corresponding to the representation π^θ of $G = \mathbf{R}$, which is evidently θ- (and thereby flux-) dependent, this unambiguously giving rise to the Aharonov-Bohm effect.

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