Algebraic K-theory, periodic cyclic homology, and the Connes-Moscovici Index Theorem

Master's thesis by
Bram Mesland
under supervision of
prof dr. N.P. Landsman

University of Amsterdam, Faculty of Science
Korteweg-de Vries Institute for Mathematics
Plantage Muidergracht 24, 1018 TV Amsterdam
June 20th, 2005

Does it matter that this waste of time is what makes a life for you?

-Frank Zappa

Algebraic K-theory, cyclic homology, and the Connes-Moscovici Index Theorem

Abstract. We develop algebraic K-theory and cyclic homology from scratch. The boundary map in periodic cyclic cohomology is shown to be well-behaved with respect to the external product. Then we prove that the Chern-Connes character induces a natural transformation from the exact sequence in lower algebraic K-theory to the exact sequence in periodic cyclic homology. Using this, the Gohberg-Krein index theorem is easily derived. Finally, we prove the Connes-Moscovici index theorem, closely following Nistor in [20].

Keywords: Algebraic K-theory, cyclic homology, Chern-Connes character, index theorem, noncommutative geometry.

Contents

Introduction

Index theory and noncommutative geometry

This paper contains an exposition of two of the basic tools of what is commonly called "Noncommutative Geometry". This branch of mathematics studies algebras (usually over $\mathbb{C}$, but even over $\mathbb{Z}$) using tools and ideas originating in geometry. The philosophy of noncommutative geometry is close to that of algebraic geometry, in which the interplay between algebra and geometry can be illustrated by the equivalence of categories

$$\text{affine algebraic varieties over } \mathbb{C} \Longleftrightarrow \text{finitely generated domains over } \mathbb{C},$$

implemented by associating to such a variety its algebra of complex valued regular functions. This basically states that all the information in a certain geometric structure is encoded in an algebraic structure associated to it. Of course, in proving such a statement, the direction $\Leftarrow$ is the more interesting one. In topology, there is a similar theorem, due to Gel'fand and Naimark, stating the equivalence of categories

$$\text{locally compact Hausdorff spaces} \Longleftrightarrow \text{commutative } C^*\text{-algebras over } \mathbb{C},$$

implemented by associating to such a space its algebra of complex valued continuous functions vanishing at infinity.
The important thing about the above equivalences is that the algebras involved are commutative. Apparently, noncommutative algebras do not correspond to any concrete geometric structure. However, the study of spaces involves such constructions and objects as vector bundles, connections, differential forms and integration and the above equivalences allow one to carry out these constructions using the function algebra, without any reference to the space itself. For some of these constructions, the commutativity of the function algebra does not have any particular significance. It is at this point that noncommutative geometry starts to deviate from algebraic geometry. For in this case, one can use geometric constructions and intuition, although there is no geometric object to refer to.
Enlarging the scope from commutative to noncommutative algebras at times greatly simplifies and clarifies results in geometry and topology. Two nice examples of this are provided by the extension of Atiyah-Hirzebruch topological K-theory from the category of spaces (or, equivalently, commutative C^*-algebras)

to that of Banach algebras. This theory associates to a space X a sequence of abelian groups $K^i(X)$, $i \in \mathbb{Z}$, such that for closed subspaces $Y \subset X$, there is a long exact sequence

$$... \longrightarrow K^i(X/Y) \longrightarrow K^i(X) \longrightarrow K^i(Y) \longrightarrow K^{i-1}(X/Y) \longrightarrow ...$$

The central result in topological K-theory is Bott periodicity, stating that $K^i(X) \cong K^{i+2}(X)$ for all i. Bott's original proof of this remarkable fact is far more complicated then the short and elegant proof given by Cuntz in [8]. This proof takes place almost entirely in the noncommutative category.
The second example is the C^*-algebraic proof of the Atiyah-Singer index theorem. This celebrated theorem, for which Atiyah and Singer received the Abel Prize in 2004, states that the analytic index of an elliptic pseudo differential operator P on a compact manifold M of dimension n can be computed from the topological formula

$$\mathrm{Ind}(P) = (-1)^n \int_{T^*M} \mathrm{ch}(\sigma(P)) \wedge \mathrm{Td}(M).$$

Elliptic pseudo differential operators are Fredholm operators when viewed as operators in the Hilbert space $L^2(M)$, meaning that both $\dim\ker P$ and $\dim\ker P^*$ are finite, where P^* is the formal adjoint of P. The analytic index of P is just the Fredholm index

$$\mathrm{Ind}(P) = \dim\ker P - \dim\ker P^*.$$

The right hand side of the Atiyah-Singer formula, as an integral of differential forms over a manifold, is a topological invariant. The form $\mathrm{Td}(M)$ is an invariant of the manifold M, whereas $\mathrm{ch}(\sigma(P))$, the Chern character of the principal symbol of P, can vary with P. Classically, the Chern character on a manifold M is a homomorphism

$$\mathrm{ch} : K^i(M) \to \bigoplus_{j=0}^{\infty} H_{DR}^{2j+i}(M),$$

from the topological K-theory of M to its DeRham cohomology made 2-periodic. The current paper treats a generalization of this homomorphism. One modern proof of the index theorem is related to the existence of a short exact sequence

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi^0(M) \xrightarrow{\sigma} C(S^*M) \longrightarrow 0$$

of C^*-algebras. Here $\mathcal{K}(L^2(M))$ is the ideal of compact operators on the separable Hilbert space $L^2(M)$, $\Psi^0(M)$ is the completion of the algebra of order at most 0 pseudodifferential operators on M when viewed as an algebra of operators on $L^2(M)$, and σ is the principal symbol map.
An operator P is elliptic if its principal symbol $\sigma(P)$ is invertible, and this implies that $\sigma(P)$ defines a class in the group $K_1^{top}(C(S^*M))$. The superscript

'top' stresses that this is the C^*-algebraic K_1-group, as opposed to the purely algebraic one we will discuss later and mainly work with in this paper. The boundary map

$$K_1^{top}(C(S^*M)) \to K_0^{top}(\mathcal{K}) \cong \mathbb{Z}$$

maps the K_1 class of the symbol $\sigma(P)$ to its Fredholm index. This map is therefore denoted Ind. This shows that the index is actually a K-theoretic quantity, and is starting point for a more general, algebraically flavoured index theory, which is the subject of this paper.

Non-compact index theory

The name most commonly associated with noncommutative geometry is that of Alain Connes. He initiated the subject in the late 1970's. In 1990, Henri Moscovici and Connes published the paper [7], in which they proved an index theorem for a class of elliptic operators on noncompact manifolds $\tilde{M}$, equipped with a free action of a discrete group Γ, such that the quotient $\tilde{M}/\Gamma$ is compact. Such spaces are just normal covering spaces of M with Γ the group of covering transformations. Normal covering spaces are classified by homotopy classes of maps $\psi : M \to \mathbb{B}\Gamma$, where $\mathbb{B}\Gamma$ is the classifying space of Γ.
In the noncompact setting, problems arise when one tries to define the index of an elliptic operator. These are in general not Fredholm, so the Fredholm index does not make sense. One could try to define the index as a K-theory element. This is impossible however, since the topological K-theory of a non-compact manifold $\tilde{M}$ is the K-theory of the C^*-algebra $C_0(\tilde{M})$ of functions on M vanishing at infinity. This is a nonunital algebra, and therefore does not posses any invertible elements. The symbol $\sigma(P)$ is an invertible element in the algebra $C(M)$, so we cannot associate an element in $K^1(\tilde{M})$ to it.
It is possible to define a K-theoretic index for more restrictive classes of elliptic operators. If the operator P is Γ-invariant, then it has a principal symbol $\sigma(P) \in C^\infty(S^*M)$ and an index

$$\mathrm{Ind}[\sigma(P)] \in K_0(\hat{\mathcal{K}} \otimes \mathbb{C}[\Gamma]).$$

Here $\hat{\mathcal{K}}$ is the ideal of smooth compact operators on L^2M and $\mathbb{C}[\Gamma]$ is the group algebra of Γ. These are not C^*-algebras, and the K-group in which the index lives, is defined purely algebraically.
To associate numerical invariants to a K-theoretic index, we need algebraic analogues of DeRham cohomology, the Chern character and the integration of differential forms. These tools are provided by (continuous) periodic cyclic homology, the Chern character on algebraic K-theory which for a locally convex algebra A is a homomorphism

$$\mathrm{Ch} : K_i(A) \to HP_i^c(A),$$

and the pairing between periodic cyclic homology and cohomology, respectively. This gives a pairing

$$K_i(A) \otimes HP^i_c(A) \to \mathbb{C},$$

between periodic cyclic cohomology and algebraic K-theory. Periodic cyclic homology is a generalization of DeRham cohomology in the sense that

$$HP_i^c(C^\infty(M)) \cong \bigoplus_{j=0}^{\infty} H_{DR}^{2j+i}(M).$$

For a discrete group Γ, we have

$$\bigoplus_{j=0}^{\infty} H^{2j+i}(\Gamma, \mathbb{C}) \subset HP^i(\mathbb{C}[\Gamma]).$$

The theorem of Connes and Moscovici now reads as follows: Let $\tilde{M} \to M$ be a covering given by $\psi : M \to \mathbb{B}\Gamma$. Let P be a matrix of Γ-invariant elliptic pseudo-differential operators on $\tilde{M}$. Then, for $\xi \in H^*(\Gamma, \mathbb{C}) = H^*(\mathbb{B}\Gamma)$, we have

$$(\mathrm{Tr} \otimes \xi)_* \mathrm{Ind}[\sigma(P)] = (-1)^n \int_{T^*M} \mathrm{ch}[\sigma(P)] \wedge \mathrm{Td}(M) \wedge \psi^*(\xi).$$

Two conjectures

The main motivation for proving this result was the Novikov conjecture, which states the following: Let M be a compact oriented manifold and Γ a group that can be defined by a finite number of generators and relations. Suppose a map $\psi : M \to \mathbb{B}\Gamma$ is given. Then the number

$$\mathrm{Sg}_\xi(M) := \int_M L(M) \wedge \psi^*(\xi)$$

is a homotopy invariant of the pair (M, ψ). That is, given a homotopy equivalence of manifolds $f : N \to M$, then $\mathrm{Sg}_\xi(M, \psi)$=$\mathrm{Sg}_\xi(N, \psi \circ f)$. Here $L(M)$ is a characteristic class called the Hirzebruch L-genus.
It turns out that, for any covering space there exits a Γ-invariant operator, called the signature operator D_s, which is elliptic, and has the property that

$$[\mathrm{ch}(\sigma(D_s)) \wedge \mathrm{Td}(M)] = [L(M)],$$

as cohomology classes. Thus if we knew that $\mathrm{Ind}(D_s)$ was a homotopy invariant of the pair (M, ψ), then the Connes-Moscovici theorem would imply the Novikov conjecture, since T^*M and M are homotopic.
Unfortunately, homotopy invariance of $\mathrm{Ind}(D_s)$ is only known at the level of the K-theory of the larger algebra $\mathcal{K} \otimes C^*(\Gamma)$, which is a C^*-algebra. The group C^*-algebra $C^*(\Gamma)$ is obtained form $\mathbb{C}[\Gamma]$ by taking the norm-closure in $B(\ell^2(\Gamma))$ for the regular representation of Γ on the Hilbert space $\ell^2(\Gamma)$. The Novikov is thus reduced to the extension of the cyclic cocycle $Tr \otimes \xi$ from $\hat{\mathcal{K}} \otimes \mathbb{C}[\Gamma]$ to $\mathcal{K} \otimes C^*(\Gamma)$. The problem is that periodic cyclic cohomology is not well behaved for C^*-algebras in the sense that HP^0 is often absurdly large, whereas HP^1 vanishes in most cases of interest.

Another conjecture related to index theory on non-compact manifolds is the Baum-Connes conjecture [1]. It roughly states that for a locally compact Hausdorff topological group G, acting properly on a space X, such that X/G is compact, the K-theory of the reduced group C^*-algebra $C^*_r(G)$ is generated by indices of G-equivariant elliptic differential operators on X. The reduced group C^*-algebra is the completion of the convolution algebra of G, viewed as an algebra of operators on $L^2(G)$. The article [1] gives a clear exposition of the ideas related to the Baum-Connes conjecture.

Structure of the paper

We will discuss the proof of the Connes-Moscovici theorem given by Nistor in [20]. This proof differs drastically from the one given in the original paper [7] of Connes and Moscovici. They used estimates with heat kernels and Alexander-Spanier cohomology. The advantage of Nistor's proof is that all the analysis is eliminated, at the cost of the use of the Atiyah-Singer index theorem (so, actually, the analysis has been moved to the proof of that theorem).
The proof uses the naturality of the Chern character as a natural transformation of the exact sequence in lower algebraic K-theory, to the exact sequence in periodic cyclic homology. In this way we can reduce the computation of the boundary map Ind in algebraic K-theory, to the computation of the boundary map ∂ in periodic cyclic homology. For the last one we can use homological methods, and it is therefore easier to compute.
Cyclic homology comes in two flavours, discrete and continuous, the difference lying in the choice of tensor product in the definition of the complexes computing them. The theorem we want to prove is a statement in continuous periodic cyclic homology, but we will use the discrete variant to achieve our goal. The interplay between the two is crucial, and we shall put some emphasis on it in the last chapter.
The first three chapters of this paper are devoted to a discussion of lower algebraic K-theory, cyclic homology, and the Chern character relating the two. The theorem at the end of chapter 3 is particularly aesthetic and we worked it out in detail. From this result, one immediately derives the Gohberg-Krein index theorem for Toeplitz operators.
The writing process of this thesis was mostly a thorough study of cyclic homology. I chose an axiomatic approach, to be able to derive Connes' interpretation of cyclic homology as derived functors. The second chapter is therefore the longest, and the main body of work of this project.
In the fourth chapter we return to the theorem we want to prove, and use the tools devoloped to come to the final result.

Prerequisites

Although at first I intended to produce a self contained paper, I'm aware that there are some gaps in the exposition now. I tried to give some background in the appendices. Some familiarity with homological algebra surely helps a great deal in understanding what is happening, especially in chapter 2. Knowledge of algebraic topology, algebraic geometry and C^*-algebras are not required, but the reader familiar with (one of) these subjects, will benefit from this.

Acknowledgements

This project grew out of my study of algebraic K-theory at the Universidad de Barcelona, under supervision of Jose Ignacio Burgos. I thank him for the time he took to introduce me to the world of K-theory and homological algebra. I thank Klaas Landsman for stimulating and inspiring me to produce the present paper. I thank Maarten Solleveld for the useful discussions we had and for taking the time to read the whole work.Iris Hettelingh has been an indispensable support during the six years I spent at UvA. I'm indebted to Jonathan Rosenberg and Victor Nistor for some very useful remarks. To Bram, Vincent, Michel, Ruben and Bert I would like to say that I really enjoyed sharing a room with you and I hope you do well during the rest of your careers. René, thank you for keeping the door open all these years!

Dankwoord

Omdat jullie er zijn:
Martijn, Susana, Vera, Bob, Annemieke, Lot, Joost, Bram, Sjors, Annafloor, Jelle, Friso, Kim, Dunya, Carolien, Femke, Barbara, en wie ik verder nog vergeten ben.

Chapter 1

Lower algebraic K-theory

Since the algebras involved in the Connes-Moscovici theorem are not Banach algebras, it is more convenient to work with algebraic K-theory, than with, for example, Cuntz's K-theory for locally convex algebras. The first part of the paper consists of an overview of lower algebraic K-theory. This part of the theory is classical in the following sense. Grothendieck defined the functor K_0 on the category of rings, when he was working on a Riemann-Roch type problem in algebraic geometry in the 1950's. Atiyah and Hirzebruch then picked up his ideas and developed their topological K-theory. Actually, if X is a topological space, and $C(X)$ the ring of continuous functions on X, then $K^0(X) \cong K_0(C(X))$. This is an important corollary of the Serre-Swan theorem that relates algebraic and topological K-theory. In topological K-theory, in turned out to be easy to define the higher K-functors K_n for $n \in \mathbb{Z}$. They are just the composition of K_0 with some functor in the category of spaces. For rings, the definition of the negative K-groups is difficult but in a sense straightforward, and analogous to the procedure in topology. During the 1960's and 1970's, Whitehead, Bass and Milnor defined the algebraic K-groups K_1 and K_2, by purely algebraic methods. Everyone felt that there must be higher K-functors as they exist in topological K-theory, and this feeling was justified by the work of Daniel Quillen in the mid 1970's, for which he was awarded the Fields medal. Quillen defined the higher algebraic K-groups using homotopy theory, which in a sense revealed the true nature of algebraic K-theory.
We will only give an overview of the classical functors K_0 and K_1. The exposition given here is well-known, and most of it can be found in Rosenberg's book [23].

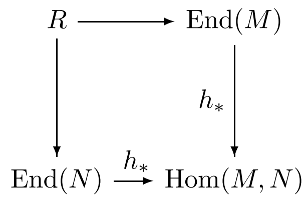
1.1 Projective modules

Throughout this paper, the word *ring* will mean unital ring, and the word *ring homomorphism* will mean unital ring homomorphism, unless otherwise specified. Let R be a ring. An *R-module* is an abelian group M together with a ring

homomorphism $m : R \to \text{End}M$. We usually do not mention m explicitly and write rx for $m(r)x$ ($x \in M$). Technically, we should distinguish left and right modules, but we will always work with left modules unless otherwise specified. A right module is an abelian group M with a homomorphism $R^{op} \to \text{End}(M)$. If R is abelian, the two notions coincide.
Let M and N be R-modules and $h : M \to N$ a group homomorphism. The map h induces maps

$$\begin{aligned} h_* : \text{End}(N) &\to \text{Hom}(M, N) \\ \phi &\mapsto \phi \circ f \\ h^* : \text{End}(M) &\to \text{Hom}(M, N) \\ \phi &\mapsto f \circ \phi. \end{aligned}$$

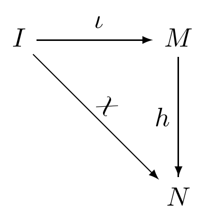
We say that h is an *R-module morphism* if the diagram



commutes. Given a ring R, the modules over this ring together with R-module morphisms form a category which we will denote by $\boldsymbol{M}_R$.
Vector spaces are modules over fields, and they have several nice properties, such as the existence of a basis, and the fact that surjective morphisms of vector spaces (i.e. linear maps) are *split*. This means that for a surjective linear map $f : V \to W$, there is a map $g : W \to V$ with $f \circ g = \text{id}_W$. This amounts to the isomorphism $V \cong \ker f \oplus \text{im } f$ and the dimension theorem. For modules over a general rings, this need not at all be the case, for example consider the canonical surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, which is clearly not split (since $\mathbb{Z}$ does not have torsion).

Definition 1.1. Let R be a ring, M an R-module. An R-module P is *projective* if any surjective R-module homomorphism $M \to P$ splits.

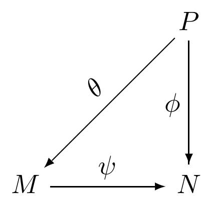
There is a special kind of modules that deserves our attention, in order to be able to give a characterization of projective modules . Let I be a set. An R-module M is called *free on the set I* if there is an injective set map $\iota : I \to M$ such that for any R-module N and set map $\chi : I \to N$, there is a unique R-module morphism $h : M \to N$ such that the diagram



commutes. It is clear that any two free modules on the same set I are isomorphic. One checks that the module $\bigoplus_{i\in I} R$ with coordinatewise multiplication as module structure is a free module on the set I.
Every R-module M is the image of a free module, since one can choose a set of generators for M over R, that is, a subset $I \subset M$ such that any $m \in M$ can be written as $M = \sum_{i\in I} r_i i$ with $r_i \neq 0$ for only finitely many i. Then one considers the free module on I and the map $h : \bigoplus_{i\in I} R \to M$ induced by the inclusion $I \hookrightarrow M$.
There are several characterizations of projective modules which we will summarize now.

Proposition 1.2. *Let P be a module over the ring R.The following are equivalent:*

1. *P is projective.*
2. *There exists an R-module Q and a free R-module F, such that $P \oplus Q \cong F$.*
3. *For any pair of R modules N, M, and morphisms $\phi : P \to N$ and $\psi : M \to N$, with ψ surjective, there exists $\theta : P \to M$ such that the diagram*



   commutes.
4. *The functor*

$$\begin{aligned} \mathrm{Hom}_R(P, -) : \boldsymbol{M}_R &\to \boldsymbol{Ab} \\ M &\mapsto \mathrm{Hom}_R(P, M) \end{aligned}$$

   is exact.

Proof. 1⇒ 2: Let F be a free module and $\psi : F \to P$ a surjective morphism. The splitting $\phi : P \to F$ gives an isomorphism $F \cong P \oplus \ker \psi$.
2⇒ 3: Choose a module Q such that $F := P \oplus Q$ is free. We replace $\psi : M \to N$ by

$$\psi \oplus \mathrm{id}_Q : M \oplus Q \to N \oplus Q$$

and $\phi : P \to N$ by

$$\phi \oplus \mathrm{id}_Q : P \oplus Q \to N \oplus Q.$$

Since F is free and ψ is surjective, we can choose for each generator f_i of F an element $e_i \in \psi^{-1}(\phi(f_i))$ and this defines a morphism $\theta : F \to N \oplus Q$ with the property that $\psi \circ \theta = \phi$. Moreover, by definition of θ, its restriction to $P \subset F$

completes the original diagram.
3⇒ 4: Let

$$0 \longrightarrow K \xrightarrow{\ i\ } L \xrightarrow{\ \pi\ } M \longrightarrow 0$$

be a short exact sequence of R-modules. Applying $\mathrm{Hom}_R(P, -)$ yields a sequence

$$0 \longrightarrow \mathrm{Hom}_R(P,K) \xrightarrow{i_*} \mathrm{Hom}_R(P,L) \xrightarrow{\pi_*} \mathrm{Hom}_R(P,M) \longrightarrow 0,$$

which we show to be exact. Let $f \in \mathrm{Hom}_R(P,K)$. If $i \circ f = 0$, then by injectivity of i, $f = 0$, so we have exactness on the left.
Let $g \in \mathrm{Hom}_R(P,L)$ be such that $\pi \circ g$=0. Then im $g \subset \ker \pi = \mathrm{im}\ i$. Therefore $f : P \to K$ defined by $f := i^{-1}g$ is a well defined morphism and $g = i \circ f$. Moreover it is clear that $\pi \circ i \circ g = 0$, so we have exactness in the middle. Note that this part of the argument does not depend on any special property of P.
To prove exactness on the right, let $f \in \mathrm{Hom}_R(P,M)$. Since $\pi : L \to M$ is surjective, there exists $\theta : P \to L$ such that the diagram

$$\begin{array}{ccc} & & P \\ & \swarrow^{\theta} & \downarrow f \\ L & \xrightarrow{\ \pi\ } & K \end{array}$$

is commutative. Thus $f = \pi \circ \theta$ and we have exactness on the right.
4⇒ 1. Let $\phi : M \to P$ be a surjective morphism, and let $N := \ker f$. By 4. there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(P,N) \longrightarrow \mathrm{Hom}_R(P,M) \xrightarrow{\phi_*} \mathrm{Hom}_R(P,P) \longrightarrow 0.$$

Thus there is a morphism $s : P \to M$ such that $\psi \circ s = \mathrm{id}_P$. □

The preceeding lemma shows how one can generalize the notion of projectivity to arbitrary abelian categories. One calls an object A in such a category $\mathcal{C}$ projective when the functor $\mathrm{Mor}_{\mathcal{C}}(A, -)$ is exact. The lemma also motivates the definition of an *injective* module (or object), namely as a module M for which the functor $\mathrm{Hom}_k(-, M)$ is exact.

1.2 Grothendieck's K_0

From proposition 1.2 we see that a module P is projective if and only if it is isomorphic to a direct summand in a free module. Therefore, the direct sum of two projective modules is again projective.
Recall that a module is called *finitely generated* if it has a finite set of generators. We saw above that then it is the image of a free module on a finite set. The

direct sum of two finitely generated modules is again finitely generated. Thus the class $\mathcal{P}R$ of finitely generated projective R-modules is closed under direct sums. Now let us look at the following definition.

Definition 1.3. Let S be a set. S is a *semigroup* if it admits a binary associative composition operation $S \times S \to S$ denoted $(r, s) \mapsto rs$. A semigroup S is a *monoid* if there is a element $e \in S$ such that $es = se$ for all $s \in S$.

It seems that $\mathcal{P}R$ is almost a monoid under the operation

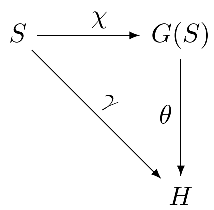
$$P + Q := P \oplus Q.$$

The identity element would be the zero module. Unfortunately $\mathcal{P}R$ is not a set, and the direct sum operation is not associative. These two problems can be solved by passing to Proj R, the set of isomorphism classes of elements of $\mathcal{P}R$. This is a set since it can be defined as the set of isomorphism classes of direct summands in R^n, $n \in \mathbb{N}$. We have associativity of the direct sum because there is an obvious isomorphism

$$(P \oplus Q) \oplus S \to P \oplus (Q \oplus S).$$

Our monoid is even commutative, for $P \oplus Q \cong Q \oplus P$. This allows us to construct a group our of Proj R, by the following theorem.

Theorem 1.4. *Let S be an abelian semigroup. Then there exists a group $G(S)$ and a homomorphism of semigroups $\chi : S \to G(S)$, with the following properties: The image of χ generates G and if H is any abelian group and $\gamma : S \to H$ a semigroup homomorphism, then there is a unique group homomorphism $\theta : G(S) \to H$ such that the diagram*



commutes. If $G'(S)$ and $\chi' : S \to G'(S)$ is another such pair, then there is an isomorphism $\alpha : G'(S) \to G(S)$ such that $\alpha \circ \chi' = \chi$.

Proof. Define an equivalence relation $\sim$ on $S \times S$ by

$$(x, y) \sim (u, v) \Longleftrightarrow \exists t \in S \quad x + v + t = y + u + t.$$

Denote by $[(x, y)]$ the equivalence class of (x, y) and set

$$G(S) := \{[(x, y)] : x, y \in S\}.$$

There is a well-defined associative addition on S

$$[(x, y)] + [(u, v)] := [(x + y, u + v)]$$

and the element $[(x,x)]$ (which is equal to $[(y,y)]$ for any $y \in S$) is the identity element of $G(S)$. Since

$$[(x,y)] + [(y,x)] = [(x+y, x+y)],$$

$G(S)$ is a group.
Define $\chi : S \to G(S)$ by $x \mapsto [(x+x, x)]$. We have

$$[(x,y)] = [(x+x,x)] + [y, y+y)] = [(x+x,x)] - [(y+y,y)] \quad (1),$$

which shows that $\chi(S)$ generates $G(S)$. Now let H be any abelian group and $\gamma : S \to H$ a morphism of semigroups. Define $\theta : G(S) \to H$ by

$$[(x,y)] \mapsto \gamma(x) - \gamma(y).$$

Then θ is a homomorphism and $\theta \circ \chi(x) = \gamma(x)$. From (1) it is clear that θ is unique.
Now let $G'(S)$ and χ' be another such pair. From their universal properties we obtain maps $\alpha' : G(S) \to G'(S)$ and $\alpha : G'(S) \to G(S)$ associated to the maps χ' and χ ,respectively. They satisfy $\alpha' \circ \chi = \chi'$ and $\alpha \circ \chi' = \chi$. It follows that $\alpha' \circ \alpha \circ \chi = \chi'$ and $\alpha \circ \alpha' \circ \chi' = \chi$. Since the images of the $\chi's$ generate the groups, it follows that α and α' are each others inverses, hence isomorphisms. □

The group $G(S)$ is called the *Grothendieck group* of S.

Definition 1.5. Let R be a ring. We define $K_0(R)$ as the Grothendieck group of the abelian monoid Proj R.

Note that K_0 is a covariant functor from the category of unital rings to that of abelian groups. A ring homomorphism $\psi : R \to T$ defines a map $\psi_* : \text{Proj } R \to \text{Proj } T$ by considering T as a right R-module via ψ and defining $P \mapsto T \otimes_\psi P$. This is a well defined homomorphism of semigroups since it is additve and if $P \oplus Q \cong R^n$ for some n, then

$$(T \otimes_\psi P) \oplus (T \otimes_\psi Q) \cong T \otimes_\psi (P \oplus Q) \cong T \otimes_\psi R^n \cong T^n$$

so $\psi_*(P) \in \text{Proj } T$. Composing this with the Grothendieck group construction yields a homomorphism $K_0(R) \to K_0(T)$.

1.3 Idempotents

For a finitely generated R-module P, we can choose a surjection $\pi : R^n \to P$ with splitting $s : P \to R^n$. The composite $s \circ \pi \in \text{End}R^n$ is an idempotent endomorphism, for

$$s \circ \pi \circ s \circ \pi = s \circ \text{id}_P \circ \pi = s \circ \pi.$$

It is clear that different isomorphism classes define different idempotents. We can identify EndR^n with the matrix ring $M_n(R)$. An idempotent $e \in M_n(R)$ defines a projective R-module $P := R^n e$ by multiplying from the right (since the module action comes from the left). This module is projective since

$$R^n e \oplus R^n(1-e) \cong R^n.$$

Thus to each class $[P] \in$ Proj R we can associate an idempotent $e \in M_n(R)$ for some n. However, different idempotents (for possibly different n's) can give rise to isomorphic projective modules. In order to describe exactly when this happens, we need some definitions.

Definition 1.6. For $n \in \mathbb{N}$, define

$$\begin{aligned} i_n : M_n(R) &\hookrightarrow M_{n+1}(R) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \\ j_n : GL(n,R) &\hookrightarrow GL(n+1,R) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Note that the i_n are non-unital ring homomorphisms and the j_n are group homomorphisms. With these maps (and their compositions) $(M_n(R))_{n\in\mathbb{N}}$ is a directed system of rings and $(GL(n,R))_{n\in\mathbb{N}}$ a directed system of groups. Denote their direct limits by $M(R)$ and $GL(R)$, respectively. Then let Idem R be the set of idempotent matrices in $M(R)$.

Note that $M_n(R)$ injects in $M(R)$ and $GL(n,R)$ in $GL(R)$ for each n and that $GL(R)$ acts on Idem (R) by conjugation. For matrices p and q, their *block sum* is the the matrix

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Idem R is closed under the block sum operation. Moreover if e is conjugate to p (by g) and f is conjugate to q (by h), then

$$e \oplus f = (g \oplus h)(p \oplus q)(g \oplus h)^{-1},$$

such that the block sum is well defined on the orbit space of Idem R under $GL(R)$. It so becomes an abelian monoid, since $e \oplus f$ is conjugate to $f \oplus e$ and the zero matrix serves as the identity. We show that this monoid is essentially Proj R.

Proposition 1.7. *Let $e \in M_n(R)$ and $f \in M_k(R)$ be idempotents. Then the projective modules $R^n e$ and $R^k f$ are isomorphic if and only if e and f are in the same $GL(R)$- orbit for its action on* Idem R.

Proof. $\Leftarrow$. By adding zeroes we may assume that e and f are of the same size $n \times n$ and that there is a matrix $g \in GL(n, R)$ such that $e = gfg^{-1}$. Since conjugation is an automorphism of R^n, we see that $R^n e$ and $R^n f$ are isomorphic. $\Rightarrow$ Suppose $\alpha : R^n e \to R^k f$ is an isomorphism. Then α extends to a morphism $a : R^n \to R^k$ of modules by setting $a(x) = j \circ \alpha(xe)$, where $j : R^k f \hookrightarrow R^k$ is the natural inclusion. Similarly α^{-1} extends to a morphism $b : R^k \to R^n$. Clearly $ab = e$ and $ba = f$, such that the matrix

$$g := \begin{pmatrix} 1-e & a \\ b & 1-f \end{pmatrix}$$

satisfies $g^2 = 1_{n+k}$. Moreover $g(e \oplus 0_k)g^{-1} = 0_n \oplus f$ and we are done, since $0_n \oplus q$ is conjugate to q by a permutation matrix. □

Corollary 1.8. *$K_0(R)$ is isomorphic to the Grothendieck group of the orbit space of the action of $GL(R)$ on* Idem R.

Proof. The previous proposition shows that Proj R is isomorphic to the orbit space (Idem $R)/GL(R)$ with the block sum operation. The statement then follows by functoriality of the Grothendieck group construction. □

When do two idempotents e and f define the same element in $K_0(R)$? Well, $[e] = [f]$ means that

$$e \oplus e \oplus f \oplus t = g(f \oplus f \oplus e \oplus t)g^{-1}$$

for some $t \in$ Idem R and $g \in GL(R)$. Now we can choose an idempotent q such that $e \oplus f \oplus t \oplus q$ is conjugate to 1_n for some n. It follows that $e \oplus 1_n$ is conjugate to $f \oplus 1_n$, and this condition is also sufficient.
In particular we see that for any idempotent $p \in M_n(R)$ there exists an idempotent $q \in M_k(R)$ such that $p \oplus q$ is conjugate to 1_{n+k}. In this description of K_0, the functoriality takes a more concrete form, since for a ring homomorphism $\phi : R \to T$, the induced map $\phi_* : K_0(R) \to K_0(T)$ is given by $\phi_*([e]) = [\phi(e)]$. Another advantage is that we can immediately deduce the following result.

Theorem 1.9 (Morita invariance for K_0). *There is a natural isomorphism $K_0(R) \cong K_0(M_n(R))$.*

Proof. It is clear that Idem $M_n(R) =$ Idem R and $GL(M_n(R)) = GL(R)$. □

Proposition 1.10. *Let R_1 and R_2 be rings. There is a natural isomorphism $K_0(R_1 \times R_2) \cong K_0(R_1) \oplus K_0(R_2)$.*

Proof. This is immediate, since $M(R_1 \times R_2) \cong M(R_1) \times M(R_2)$ by mapping a matrix (r_{ij}) over $R_1 \times R_2$ to the matrix $(p_1(r_{ij}), p_2(r_{ij}))$ and vice versa. It is clear that this maps $GL(R_1 \times R_2)$ to $GL(R_1) \times GL(R_2)$ and Idem $R_1 \times R_2$ to Idem $R_1 \times$ Idem R_2 and thus it induces an isomorphism $K_0(R_1 \times R_2) \to K_0(R_1) \oplus K_0(R_2)$. □

1.4 Whitehead's K_1

In the construction of the functor K_0, we encountered the group $GL(R)$ of invertible matrices over R. We will use this group to construct the functor K_1. Informally, K_0 can be viewed as a classifier of projective modules over R, which are the analogues of vector spaces. K_1 will classify the linear automorphisms between projective modules.

Definition 1.11. Let R be a ring, and $GL(n,R)$ the group of invertible $n \times n$ matrices over R. For $0 \leq i,j \leq n$, $i \neq j$, define the matrix $e_{ij}(a) \in GL(n,R)$ as the matrix having 1's on the diagonal and 0's elsewhere, except for an a in the (i,j)-slot. Such a matrix is called *elementary.* Denote by $E(n,R)$ the group generated by the elementary $n \times n$ matrices, and by $E(R) \subset GL(R)$ the direct limit of the groups $E(n,R)$.

The elementary matrices encode the row- and column-operations from linear algebra. Multiplication from the left by $e_{ij}(a)$ adds a times the i-th row to the the j-th row. Multiplication on the left corresponds to the column operations.

Lemma 1.12. *The elementary matrices over a ring R satisfy the relations*

1.) $e_{ij}(a)e_{ij}(b) = e_{ij}(a+b)$

2.) $e_{ij}(a)e_{kl}(b) = e_{kl}(b)e_{ij}(a) \qquad j \neq k \quad i \neq l$

3.) $e_{ij}(a)e_{jk}(b)e_{ij}(a)^{-1}e_{jk}(b)^{-1} = e_{ik}(ab) \qquad i,j,k$ distinct

4.) $e_{ij}(a)e_{ki}(b)e_{ij}(a)^{-1}e_{ki}(b)^{-1} = e_{kj}(-ab) \qquad i,j,k$ distinct

Furthermore, each upper and lower triangular matrix with 1's on the diagonal belongs to $E(R)$.

Proof. The relations are checked by calculation. Furthermore, we know from linear algebra that an upper or lower triangular matrix can be reduced to the identity by elementary row and column operations, that is, by multiplication with elementary matrices. □

Lemma 1.13. *For $A \in GL(n,R)$, the matrix*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

is in $E(2n,R)$.

Proof.

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The first three factors are in $E(2n,R)$ by lemma 1.12 and the last factor is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

so it is also in $E(2n, R)$ by 1.12 $\square$.

Recall that, given a group G, we denote its *commutator subgroup*

$$< ghg^{-1}h^{-1} : g, h \in G >,$$

the group generated by all commutators, by $[G, G]$. $[G, G]$ always is a normal subgroup of G.

Proposition 1.14 (Whitehead's lemma).

$$[GL(R), GL(R)] = [E(R), E(R)] = E(R)$$

Proof. The second equality follows immediately from relation 3 of lemma 1.12. For the first one we compute

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

Thus by lemma 1.13 and the second equality we have

$$[GL(R), GL(R)] \subset E(R) = [E(R), E(R)] \subset [GL(R), GL(R)]. \quad \square$$

A group G satisfying $[G, G] = G$ is called a *perfect* group. Thus for any ring R, $E(R)$ is perfect.

Definition 1.15. Let R be a ring. We define

$$K_1(R) := GL(R)/E(R).$$

Thus $K_1(R)$ is the maximal abelian quotient of $GL(R)$. As with K_0, K_1 is a functor, since a ring homomorphism $\phi : R \to T$ induces a map $\phi_* : GL(R) \to GL(T)$ by coordinatewise applying ϕ. Moreover, it is clear that $\phi(E(R)) \subset \phi(E(T))$, such that we have an induced map $\phi_* : K_1(R) \to K_1(R)$.
The product in K_1 may be described in two different ways. Since it is a quotient of $GL(R)$, we have $[A].[B] = [AB]$. But since $B^{-1} \oplus B \in E(R)$ and $AB = AB \oplus 1$ in $GL(R)$, we have

$$[AB] = [(AB \oplus 1)(B^{-1} \oplus B)] = [A \oplus B]$$

in $K_1(R)$. Thus we may also take the block sum. As with K_0, we have the following result.

Theorem 1.16 (Morita invariance for K_1). *There is a natural isomorphism $K_1(R) \cong K_1(M_n(R))$.*

Proof. We saw that $GL(R) = GL(M_n(R))$. It remains to show that under this identification, $E(M_n(R))$ is mapped to $E(R)$. Since an elementary matrix over $M_n(R)$ regarded as a matrix over R is upper triangular, we have $E(M_n(R)) \subset E(R)$ by lemma 1.12. Conversely, the image of the generators of $E(M_n(R))$

generates $E(R)$ because it contains all elementary matrices, except the ones with an entry in some slot of an $n \times n$ identitity matrix on the diagonal. But if $e_{ij}(a)$ is such a matrix, then $e_{(i+n)j}(1)$ and $e_{i(i+n)}(a)$ are not and we have the relation

$$e_{ij}(ab) = e_{ik}(a)e_{kj}(b)e_{ik}(a)^{-1}e_{kj}(b)^{-1}.$$

So by taking $k = i+n$ and $b = 1$, we see that $E(R) \subset E(M_n(R))$. So we obtain an isomorphism

$$GL(M_n(R))/E(M_n(R)) \to GL(R)/E(R). \quad \square$$

Proposition 1.17. *Let R_1 and R_2 be rings. There is a natural isomorphism $K_1(R_1 \times R_2) \cong K_1(R_1) \oplus K_1(R_2)$.*

This is immediate, since $GL(R_1 \times R_2) \cong GL(R_1) \times GL(R_2)$ and this isomorphism maps $E(R_1 \times R_2)$ to $E(R_1) \times E(R_2)$. $\square$

1.5 Relative K-theory

We defined K_0 and K_1 for unital rings R. Our aim is to construct a long exact sequence in K-theory, associated to a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

of rings. Since a non-trivial ideal I is in general non-unital (and if it is unital, its unit does not coincide with the unit of R) we do not yet have the means to associate K-groups to I. We will show how to do this in a convenient way in this section.

Definition 1.18. Let R be a ring and $I \subset R$ an ideal. Define

$$D(R, I) := \{(r, s) \in R \times R : r - s \in I\},$$

the *double of R along I*.

$D(R, I)$ is a ring under pointwise mulitplication, since

$$r_1r_2 - s_1s_2 = (r_1 - s_1)r_2 + s_1(r_2 - s_2),$$

and it is clearly unital for this multiplication.The projection $p_2 : D(R, I) \to R$ is a homomorphism with kernel isomorphic to I. This motivates the following

Definition 1.19. Let R be a ring and $I \subset R$ an ideal. Define

$$K_i(R, I) := \ker(p_{2*} : K_i(D(R, I)) \to K_i(R)), \quad i = 0, 1.$$

It is called the *relative K-theory of I with respect to R.*

The definition of $K_i(R,I)$ depends on R. It will turn out that this dependence is superficial for K_0, but essential for K_1. This will be the topic of the next section. Now, we will discuss the central results concerning relative K-groups. Denote by γ the inclusion $I \hookrightarrow D(R,I)$ in the first coordinate. The inclusion $i : I \to R$ induces a map $i_* : K_0(R,I) \to K_0(R)$ since the diagram

$$\begin{array}{ccc} D(R,I) & \xrightarrow{p_1} & R \\ \uparrow & \nearrow_{i} & \\ I & & \end{array}$$

commutes. Hence i_* is essentially p_{1*}.

Proposition 1.20 (Half exactness of K_0). *Let*

$$0 \longrightarrow I \xrightarrow{\ i\ } R \xrightarrow{\ \pi\ } R/I \longrightarrow 0$$

be a short exact sequence of rings. Then the induced sequence

$$K_0(R,I) \xrightarrow{i_*} K_0(R) \xrightarrow{\pi_*} K_0(R/I)$$

on K_0 is exact.

Proof. We need to show $\operatorname{im} i_* = \ker \pi_*$. To this end, let $[e]-[f] \in K_0(R)$ be such that $\pi_*([e]-[f]) = 0$, with e and f idempotents in some matrix ring over R. Then we have that $\pi(e) \oplus 1_n$ is conjugate to $\pi(f) \oplus 1_n$ for some n. Since π is unital, we may replace e and f by $e \oplus 1_n$ and $f \oplus 1_n$. Thus, for some $g \in GL(R/I)$, $\pi(e) = g\pi(f)g^{-1}$. However, g need not lift through π to a matrix in $GL(R)$. But by lemma 1.13, $h := g \oplus g^{-1} \in E(R/I)$, and this clearly lifts to some $\hat{h} \in E(R) \subset GL(R)$. Moreover, $h(\pi(e) \oplus 0_k)h^{-1} = \pi(f) \oplus 0_k$, for some k. Thus, replacing e by $\hat{h}(e \oplus 0_k)\hat{h}^{-1}$ and f by $f \oplus 0_k$, we may assume $\pi(e) = \pi(f)$. But this means that $(e,f) \in \operatorname{Idem} D(R,I)$ and the class $[(e,f)] - [(f,f)] \in K_0(R,I)$ maps to $[e]-[f]$ under p_{1*}. Thus $\ker \pi_* \subset \operatorname{im} i_*$.
Now assume $[e]-[f] \in \operatorname{im} p_{1*}$. Let $[(e_1,e_2)] - [(f_1,f_2)] \in p_{1*}^{-1} \subset K_0(R,I)$. Then, using that $K_0(R \times R) \cong K_0(R) \oplus K_0(R)$, we have $[e_2]-[f_2] = 0$ and $[e]-[f] = [e_1]-[f_1]$ in $K_0(R)$. Since $(e_1,e_2),(f_1,f_2) \in \operatorname{Idem} D(R,I)$ we have $\pi(e_1) = \pi(e_2)$ and $\pi(f_1) = \pi(f_2)$. Therefore

$$\pi_*([e]-[f]) = \pi_*([e_1]-[f_1]) = [\pi(e_1]-[\pi(f_1)] = 0,$$

and this completes the proof of exactness. □

For K_1 we now prove the analogue of the previous proposition. Again we write i_* for p_{1*}

Proposition 1.21 (Half exactness of K_1). *Let*

$$0 \longrightarrow I \xrightarrow{\ i\ } R \xrightarrow{\ \pi\ } R/I \longrightarrow 0$$

be a short exact sequence of rings. Then the induced sequence

$$K_1(R,I) \xrightarrow{i_*} K_1(R) \xrightarrow{\pi_*} K_1(R/I)$$

on K_1 is exact.

Proof. Let $(A,B) \in GL(D(R,I))$ be such that $[(A,B)] \in \ker p_{2*}$. Then $B \in E(R)$ since $\pi(E(R)) = E(R/I)$. Multiplying by $(B,B)^{-1}$ brings (A,B) in the form $(A',1)$, without changing its class in $K_1(R,I)$. It follows that $\pi(A') = 1$, thus $\operatorname{im} i_* \subset \ker \pi_*$.
If $B \in GL(R)$ is such that $\pi_*([B]) = 1$, then there exists $B' \in E(R)$ with $\pi(B') = \pi(B)$, since $\pi(E(R)) = E(R/I)$. Then $\pi(BB'^{-1}) = 1$, and therefore $(BB'^{-1}, 1) \in GL(D(R,I))$. We have $[B] = p_{1*}([(BB'^{-1},1)])$ in $K_1(R)$. □

Theorem 1.22. *Let*

$$0 \longrightarrow I \xrightarrow{\ i\ } R \xrightarrow{\ \pi\ } R/I \longrightarrow 0$$

be a short exact sequence of rings. There exists a natural homomorphism

$$\mathrm{Ind} : K_1(R/I) \to K_0(R,I),$$

such that the sequence

$$K_1(R,I) \xrightarrow{i_*} K_1(R) \xrightarrow{\pi_*} K_1(R/I) \xrightarrow{\mathrm{Ind}} K_0(R,I) \xrightarrow{i_*} K_0(R) \xrightarrow{\pi_*} K_0(R/I)$$

is exact.

Proof. First we construct Ind. Let $[A] \in K_1(R/I)$, $A \in GL(n, R/I)$. Using A, we will construct a projective module over $D(R,I)$. Define

$$M_A = R^n \times_A R^n := \{(x,y) \in R^n \times R^n : \pi(x) = \pi(y)A\}.$$

This is a $D(R,I)$-module, as

$$(r_1, r_2)(x,y) := (r_1 x, r_2 y),$$

which is well defined since $\pi(r_1) = \pi(r_2)$. The map is additive in the following sense: Let $A_1 \in GL(n, R/I)$, $A_2 \in GL(m, R/I)$.

$$\begin{aligned} M_{A_1 \oplus A_2} &= \{(x,y) \in R^{n+m} \times R^{n+m} : \pi(x) = \pi(y)(A_1 \oplus A_2)\} \\ &\cong \{((x_1,x_2),(y_1,y_2)) \in R^n \oplus R^m \times R^n \oplus R^m : \pi(x_i) = \pi(y_i)A_i\} \\ &\cong M_{A_1} \oplus M_{A_2} \end{aligned}$$

To show that M_A is finitely generated projective, we observe that for $A \in E(R/I)$, we can choose a lift $\hat{A} \in E(R)$ and then define a map

$$\begin{aligned} \phi_{\hat{A}} : M_A &\to D(R,I)^n \\ (x,y) &\mapsto (x, y\hat{A}). \end{aligned}$$

ϕ is an isomorphism since $\hat{A}$ is invertible. Since for any matrix $A \in GL(n, R/I)$, the matrix $A \oplus A^{-1}$ is in $E(2n, R/I)$, we see that M_A is a direct summand in $D(R,I)^{2n}$, hence finitely generated projective. These observations motivate us to define

$$\mathrm{Ind}([A]) := [M_A] - [D(R,I)^n]$$

for $A \in GL(n, R/I)$. This is well defined, as we saw that for $A \in E(n, R/I)$, $M_A \cong D(R,I)^n$. It is a homomorphism since

$$M_{A \oplus B} \cong M_A \oplus M_B \text{ and } D(R,I)^{n+m} \cong D(R,I)^n \oplus D(R,I)^m.$$

It remains to show $\ker \mathrm{Ind} = \mathrm{im}\ \pi_*$ and $\mathrm{im}\ \mathrm{Ind} = \ker i_*$. For the first equality, note that for $A \in \mathrm{im}\ \pi_*, A = \pi(\hat{A})$, we can define $\phi_{\hat{A}} : M_A \to D(R,I)^n$ as above. Thus $\mathrm{im}\ \pi_* \subset \ker \mathrm{Ind}$. For the other inclusion, $\mathrm{Ind}([A]) = 0$ means that there exists an $m \in \mathbb{N}$ with

$$M_A \oplus D(R,I)^m \cong D(R,I)^{n+m}.$$

Thus, replacing A by $A \oplus 1_m$ we may assume $M_A \cong D(R,I)^n$. Let

$$\begin{aligned} \phi : D(R,I)^n &\to M_A \\ (x,y) &\mapsto (\phi_1(x,y), \phi_2(x,y)) \end{aligned}$$

be an isomorphism. Let $e_j, j = 1, ..., n$ be the standard basis of R^n. Define matrices $B_i \in GL(n,R)$ $i = 1,2$ by

$$e_j B_i := \phi_i(e_j, e_j).$$

The B_i are invertible since their inverses are the matrices $C_i, i = 1,2$ defined by

$$e_j C_i := \phi_i^{-1}((e_j B_1, e_j B_2)).$$

By definition of $D(R,I)$ the B_i satisfy $\pi(B_1) = \pi(B_2)A$ and therefore $A = \pi(B_1 B_2^{-1})$ and we are done.
From the definition of Ind, it is clear that

$$p_{1*}([M_A] - [D(R,I)^n]) = [R^n] - [R^n] = 0,$$

so $\mathrm{im}\ \mathrm{Ind} \subset \ker i_*$. On the other hand, let $p_{1*}([P] - [D(R,I)^n]) = 0$. Then, for some m, $p_1(P) \oplus R^m \cong R^{n+m}$ and we may assume $p_1(P) \cong R^n$. Since $[P] \in K_0(R,I)$ we have $p_2(P) \cong R^n$ as well. We will construct a matrix $A \in GL(n, R/I)$ such that $P \cong R^n \times_A R^n$. We may view P as a submodule of $D(R,I)^k$ for some k. Choose isomorphisms $\phi_i : p_i(P) \to R^n$. The maps $\psi_i :=$

$\phi_i \circ p_i$ are surjective R-module homomorphisms, when we view P as an R-module via the diagonal inclusion $R \hookrightarrow D(R,I)$. Since R^n is projective, they admit splittings $s_i : R^n \to P$. This allows us to define matrices $\hat{A}, \hat{B} \in M_n(R)$ by

$$x\hat{A} := \psi_1 s_2(x) \quad x\hat{B} := \psi_2 s_1(x)$$

which is well defined because $\psi_2 \circ s_1$ and $\psi_1 \circ s_2$ are endomorphisms of R^n. Set

$$A := \pi(\hat{A}) \quad B := \pi(\hat{B}).$$

We claim that $B = A^{-1}$, which we will prove by showing $\pi_*\psi_2 s_1$ and $\pi_*\psi_1 s_2$ are inverse to each other. Since $P \subset D(R,I)^k$ and $\ker \psi_i = \ker p_i$, we have

$$p_2(x - s_1\psi_1(x)) \in I^k, \quad p_1(x - s_2\psi_2(x)) \in I^k$$

for all $x \in P$. Therefore

$$\begin{aligned} \pi_*(\psi_2 s_1 \psi_1 s_2) &= \pi_*(\psi_2 s_2) = \pi_*(\mathrm{id}) = \mathrm{id} \\ \pi_*(\psi_1 s_2 \psi_2 s_1) &= \pi_*(\psi_1 s_1) = \pi_*(\mathrm{id}) = \mathrm{id}. \end{aligned}$$

Next we show that the map

$$\begin{aligned} \psi : P &\to R^n \times_A R^n \\ x &\mapsto (\psi_1(x), \psi_2(x)) \end{aligned}$$

is an isomorphism. First of all, note that ψ is well defined since

$$\pi(\psi_1(x)) = \pi(\psi_1 s_2 \psi_2(x)) = \pi(\psi_2(x))A$$

and it is clearly a $D(R,I)$-module morphism. It is injective since

$$\ker \psi_1 \cap \ker \psi_2 = \ker p_1 \cap \ker p_2 = \{0\} \times I^k \cap I^k \times \{0\} = 0.$$

For the surjectivity, let $(r_1, r_2) \in R^n \times_A R^n$. Then $r_1 - \psi_1 s_2 r_2 \in I^n$. Therefore

$$r_1 - \psi_1 s_2 r_2 = \sum_{j=1}^{n} i_j e_j,$$

where e_j is the standard basis of R^n and $i_j \in I$. The element

$$y := \sum_{j=1}^{n} (i_j, 0) s_1 e_j \in P$$

then satisfies $\psi_2(y) = 0$ and $\psi_1(y) = r_1 - \psi_1 s_2 r_2$, because we may view R^n as a $D(R,I)$-module in two different ways via $(a,b)r := p_i(a,b)r$ and then ψ_i is a $D(R,I)$-module map. The element

$$x := y + s_2 r_2 \in P$$

then satisfies $\psi_1(x) = r_1$ and $\psi_2(x) = r_2$. $\square$

Corollary 1.23 (Explicit formula for Ind**).** *Let R be a ring and $I \subset R$ an ideal. Let $u \in GL(n, R/I)$ for some n and let $[u]$ denote its class in $K_1(R/I)$. Then*

$$\mathrm{Ind}([u]) = [\begin{pmatrix} (2ab-(ab)^2, 1_n) & (a(2_n - ba)(1_n - ba), 0_n) \\ ((1_n - ba)b, 0_n) & ((1_n - ba)^2, 0_n)] \end{pmatrix}] - [(1_n, 1_n) \oplus (0_n, 0_n)]$$

where $a, b \in M_n(R)$ and $\pi(a) = u$ and $\pi(b) = u^{-1}$.

Proof. Applying π to first coordinates of the above matrix shows that it is a matrix over $D(R, I)$, and applying p_{1*} shows that it is in $K_0(R, I)$. Recall from the proof of theorem 1.22 that the projective $D(R, I)$-module associated to u is $R^n \times_u R^n$, which is a direct summand in $D(R, I)^{2n} \cong R^{2n} \times_{u \oplus u^{-1}} R^{2n}$. The idempotent corresponding to M_u is $(v, 1_{2n})(1_n, 1_n) \oplus (0_n, 0_n)(v^{-1}, 1_{2n})$,where v is a lifting of $u \oplus u^{-1}$. If a lifts u and b lifts u^{-1}, then

$$v := \begin{pmatrix} 2a - aba & ab - 1_n \\ 1_n - ba & b \end{pmatrix}$$

lifts $u \oplus u^{-1}$ and the formula then follows by computation. □

1.6 Excision

In the previous section, we defined relative K-theory of ideals $I \subset R$, where R is a unital ring. We could however construct K-groups for ideals, or more generally, non-unital rings, without using the embedding $I \hookrightarrow R$. In this section we discuss how this is done.

Definition 1.24. Let k be a commutative and unital ring, and I a k-algebra. The *unitization of I (over k)* is the ring

$$I_+ := I \oplus k$$

(as an abelian group) with the multiplication

$$(x, n)(y, m) := (xy + mx + ny, nm).$$

The unit in I_+ is $(0, 1)$. Note that every ring is a $\mathbb{Z}$-algebra, so the definition applies. This definition is convenient, because the unitization of a k-algebra can be chosen to be a k-algebra.
Calculations in matrix algebras over I_+ are done with the same multiplication rule, since one easily checks that for matrices $(A, A'), (B, B') \in M_n(I_+)$ the product is calculated as

$$(A, A')(B, B') = (AB + A'B + AB', A'B'),$$

where the products on the right hand side are just the ordinary matrix multiplications in I and k. A homomorphism of $\phi : A \to B$ non-unital k-algebras extends to a unital homomorphism

$$\begin{aligned} \phi_+ : A_+ &\to B_+ \\ (a,n) &\mapsto (\phi(a), n). \end{aligned}$$

If I itself is already unital, then $I_+ \cong I \times k$ by the isomorphism $\phi : (x,n) \mapsto (x + n.1, n.1)$, as is checked by calculation. Note that the projection $\rho : I_+ \to k$ induces a split extension

$$0 \longrightarrow I \longrightarrow I_+ \overset{\rho}{\rightleftarrows} k \longrightarrow 0.$$

Definition 1.25. Let I be a (not necessarily unital) ring. We define

$$K_i(I) := \ker(\rho_* : K_i(I_+) \to K_i(k)) \quad i = 0, 1.$$

This definition coincides with the usual one when I is unital, since then $I_+ \cong I \times k$, so $K_i(I_+) \cong K_i(I) \oplus K_i(k)$ and $\ker \rho_* = K_i(I)$. When I is non-unital, and I is an ideal in some ring R it is not at all clear whether $K_i(R,I) \cong K_i(I)$. However, there is a map

$$\begin{aligned} \gamma : I_+ &\to D(R,I) \\ (x,n) &\mapsto (x + n.1, n.1), \end{aligned}$$

and thus a map $\gamma_* : K_i(I) \to K_i(R,I)$ for which the diagram

$$\begin{array}{ccc} I_+ & \xrightarrow{\gamma} & D(R,I) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle p_2} \\ k & \longrightarrow & R \end{array}$$

commutes.

Theorem 1.26 (Excision for K_0). *Let R be a k-algebra and $I \subset R$ an ideal. The map $\gamma_* : K_0(I) \to K_0(R,I)$ is a natural isomorphism.*

Proof. First we show γ_* is injective. Let $[e] - [f] \in K_0(I)$. Then

$$e = (e_1, e_2), \quad f = (f_1, f_2), \quad \rho_*([e]) = \rho_*([f]),$$

and first of all, by taking direct sums with $(0, 1_s) - e$ for suitable s, we may assume $e = (0, 1_s)$. Then, for suitable n, $1_s \oplus 1_n = g(f_2 \oplus 1_n)g^{-1}$ for some matrix $g \in GL(k)$. Thus by taking direct sums with 1_n if necessary, we have $1_s = gf_2g^{-1}$. Since we may view g as a matrix over I_+, we may replace f_2 by gf_2g^{-1} and we have $f_2 = 1_s$ as well. Now $\gamma_*([(0,1_s)] - [(f_1, 1_s)]) = 0$, means that $[(1_s, 1_s)] = [(f_1 + 1_s, 1_s)]$. Thus, again by taking direct sums with 1_n for suitable n, we assume that

$$g_1(f_1 + 1_s)g_1^{-1} = 1_s \qquad g_2 1_s g_2^{-1} = 1_s,$$

where $(g_1, g_2) \in GL(D(R,I))$. Since $(g_2^{-1}, g_2^{-1}) \in GL(D(R,I))$ it follows that $(g_2^{-1}g_1, 1) \in GL(D(R,I))$ and since $g_2^{-1}g_1 - 1 \in M(I)$ we actually have

$$(g_2^{-1}g_1, 1) = \gamma(g_2^{-1}g_1 - 1, 1).$$

One checks by calculation that

$$(g_2g_1^{-1} - 1, 1)^{-1} = (g_1g_2^{-1} - 1, 1)$$

and that

$$(g_2g_1^{-1} - 1, 1)(f_1 + 1_s, 1_s)(g_1g_2^{-1} - 1, 1) = (0, 1_s),$$

using

$$g_2^{-1}g_1(1_s + f_1)g_1^{-1}g_2 = g_2^{-1}1_sg_2 = 1_s.$$

Thus we have $\ker \gamma_* = 0$.
For surjectivity, let $[e] - [f] = [(e_1, e_2)] - [(f_1, f_2)] \in K_0(R, I)$ be arbitrary. As above, we may assume $(e_1, e_2) = (1_s, 1_s)$. Since $[e] - [f] \in \ker p_{2*}$, $gf_2g^{-1} = 1_s$ for some $g \in GL(R)$, thus by conjugating with $(g, g) \in GL(D(R, I))$, we have $f_2 = 1_s$. But then, since $(f_1 - 1_s, 1_s) \in \text{Idem } I_+$, we have $(f_1, 1_s) = \gamma(f_1 - 1_s, 1_s)$. Therefore

$$[e] - [f] = \gamma_*([(0, 1_s)] - [(f_1 - 1_s, 1_s)]),$$

and this completes the proof. □

Corollary 1.27 (Excision and the exact sequence). *For any exact sequence*

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

of rings there is a natural exact sequence

$$K_1(R, I) \xrightarrow{i_*} K_1(R) \xrightarrow{\pi_*} K_1(R/I) \xrightarrow{\text{Ind}} K_0(I) \xrightarrow{i_*} K_0(R) \xrightarrow{\pi_*} K_0(R/I)$$

where have written Ind *for* $\gamma_*^{-1}\text{Ind}$ *and* i_* *for* $i_*\gamma$. *Explicitly, for* $[u] \in K_1(R/I)$ *we have*

$$\text{Ind}([u]) = [\begin{pmatrix} (-(1_n - ab)^2, 1_n) & (a(2_n - ba)(1_n - ba), 0_n) \\ ((1_n - ba)b, 0_n) & ((1_n - ba)^2, 0_n)) \end{pmatrix}] - [(0_n, 1_n) \oplus (0_n, 0_n)]$$

Proof. Immediate by applying excision to theorem 1.22 and applying γ_* to the above formula yields the formula for Ind from corollary 1.23. □

We now have extended the domain of K_0 to the category of non unital rings. Since non-unital homomorphisms $\phi : A \to B$ extend to unital homomorphisms and because of the way in which this extension is defined, we see that K_0 is functorial for nonunital rings. We would like to do the same for K_1, but unfortunately it turned out that this is not possible, as the simple counterexample in appendix C shows.

1.7 Topological K-theory

So far our exposition has been purely algebraic. Grothendieck defined K_0 the way we did it here and his ideas were picked up by Atiyah and Hirzebruch, who approached Grothendieck's from a topological point of view. The functor K^0 (*contravariant*, hence the upper index), arose in problems concerning holomorphic vector bundles over a complex manifold. Working together, topological K-theory for locally compact Hausdorff spaces was developed. We will follow a different path, by proving certain properties of the functor K_0 with respect to Banach algebras and certain well behaved subalgebras of these. Definitions and basic properties can be found in appendix A. Specializing the results of this section to commutative C^*-algebras, which, by the Gelfand-Naimark theorem are rings of continuous functions on a locally compact Hausdorff space, yields the topological K-theory of Atiyah and Hirzebruch, see appendix D. The ground ring k will be $\mathbb{R}$ or $\mathbb{C}$ in this section, and we will refer to it as $\mathbb{F}$.

Lemma 1.28. *Let A be a unital Banach algebra. If $x \in A$ is such that $\|x-1\| < 1$, then $x^\alpha \in A$ is defined for any $\alpha \in \mathbb{R}$. In particular, x is invertible in A.*

Proof. Define
$$x^\alpha := \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n}(\alpha - j)}{n!}(x-1)^n.$$
By the hypothesis, this series is a Cauchy sequence, since we have
$$\begin{aligned}\|\sum_{n=k}^{m} \frac{\prod_{j=0}^{n}(\alpha - j)}{n!}(x-1)^n\| &\leq \sum_{n=k}^{m} |\frac{\prod_{j=0}^{n}(\alpha - j)}{n!}|\|(x-1)^n\| \\ &\leq \sum_{n=k}^{m} |\frac{\prod_{j=0}^{n}(\alpha - j)}{n!}|\|x-1\|^n \to 0, k \to \infty.\end{aligned}$$
So by completeness it is convergent. Since $x = (1+(x-1))$, this series is just the usual power series of x^α and the element thus defined has the desired algebraic properties. ☐

Proposition 1.29. *Let A be a Banach algebra. If $e, f \in A$ are idempotents such that $\|e-f\| < \min\{\|e\|^{-2}, \|f\|^{-2}\}$, then the projective modules Ae and Af are isomorphic.*

Proof. Consider the unital algebras $P := eAe$ and $Q := fAf$. In Q we have
$$\|fef - f\| \leq \|f\|\|e-f\|\|f\| < 1$$
and similarly $\|efe-e\| < 1$, by hypothesis. Thus, by lemma 1.28, $x := (efe)^{-\frac{1}{2}}$ is defined in P. x commutes with efe and has the following properties:
$$\begin{gathered}(xf)(fx) = xefex = x^2efe = e \\ e(xf) = xf = (xf)f, \quad (fx)e = fx = f(fx) \\ (fx^2f)(fef) = fx^2fef = fx^2(efe)f = fef.\end{gathered}$$

Thus $(fx)(xf) = f$ since fef is invertible in Q. But then the map

$$\begin{aligned} \phi : Ae &\to Af \\ ae &\mapsto aexf \end{aligned}$$

sets up an isomorphism, with inverse $af \mapsto affx$. $\square$

The preceeding results allow us to prove homotopy invariance for K_0. First some terminology. Note that for a Banach algebra B, the algebra

$$C([0,1], B) := \{f : [0,1] \to B : f \text{ continuous}\}$$

is a Banach algebra in the norm

$$\|f\| := \sup_{x\in[0,1]} \|f(x)\|.$$

The map $e_t : C([0,1], B)$ defined by $f \mapsto f(t)$ is a continuous homomorphism.

Definition 1.30. Let A and B be Banach algebras. A *homotopy of homomorphisms* from A to B is a homomorphism

$$\psi : A \to C([0,1], B).$$

We denote by the composition $e_t \circ \psi$ by ψ_t.

Theorem 1.31 (Homotopy invariance for K_0). *Let A and B be unital Banach algebras and ψ a homotopy of homomorphisms from A to B. Then the maps $\psi_i : A \to B$ induce the same map $\psi_{i*} : K_0(A) \to K_0(B)$ for $i = 0, 1$.*

Proof. Let e be an idempotent in $M_n(A)$ for some n. The function

$$t \mapsto \psi_t(e)$$

defines a continuous path of idempotents in $M_n(B)$ from $\psi_0(e)$ to $\psi_1(e)$. There exist finitely many $t_i \in [0,1], i = 0, ..., n$,

$$0 = t_0 \leq t_1 \leq ... \leq t_{n-1} \leq t_n = 1,$$

such that $\|\psi_t(e) - \psi_s(e)\| < 1$ for all $s, t \in [t_i, t_{i+1}]$. Thus the class $[\psi_t(e)] \in K_0(B)$ does not change in these intervals. By connectivity of $[0,1]$, it doesn't change in the whole interval and the result follows. $\square$

For a non-unital Banach algebra, note that its unitization A_+ as an $\mathbb{F}$-algebra is a Banach algebra in the norm

$$\|(x, n)\| := \|x\| + \|n\|.$$

A homotopy ψ of nonunital homomorphisms from A to B, extends to a unital homotopy of homomorphisms $\psi_+ : C([0,1], A_+) \to B_+$ by $\psi_+(a, n) := (\psi(a), n)$.

Corollary 1.32. *Let A and B be Banach algebras and $\psi : A \to C([0,1], B)$ a homomorphism. Then the maps $\psi_i : A \to B$ induce the same map $\psi_{i*} : K_0(A) \to K_0(B)$ for $i = 0, 1$.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A_+) & \longrightarrow & K_0(\mathbb{F}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{t*}} & & \downarrow{\scriptstyle \psi_{+t*}} & & \downarrow{\scriptstyle \mathrm{id}_*} & & \\
0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(B_+) & \longrightarrow & K_0(\mathbb{F}) & \longrightarrow & 0
\end{array}$$

which commutes for $t \in [0, 1]$. The map in the middle is the same for all t. Therefore, the map on the left doesn't change either. □

We now come to the discussion of higher topological K-theory. Recall the definition of $\mathrm{Ind} : K_1(A/I) \to K_0(I)$ for an ideal I in a Banach algebra A. If $GL_0(A)$ denotes the connected component of the identity in the locally path connected topological group $GL(A)$, then one can show that $\pi_* GL(A)_0 = GL(A/I)_0$, where $\pi : A \to A/I$ is the quotient map. See for example [22]. This implies that if $u, v \in GL(A)$ are path connected, then so are the idempotents defining $A^n \times_u A^n$ and $A^n \times_v A^n$ in Idem $D(A, I)$. Therefore their classes in $K_0(D(A, I))$ agree and so do the elements $\mathrm{Ind}([u])$ and $\mathrm{Ind}([v])$ in $K_0(R, I) \cong K_0(I)$. Thus, $\mathrm{Ind}(GL_0(A/I)) = 0$. Since $E(A/I) \subset GL_0(A/I)$, Ind descends to a homomorphism

$$GL(A/I)/GL_0(A/I) \to K_0(I).$$

This motivates the following definition.

Definition 1.33. Let A be a unital Banach algebra. We define

$$K_1^{top}(A) := GL(A)/GL_0(A),$$

and denote by $S_A : K_1(A) \to K_1^{top}(A)$ the canonical surjection induced by the inclusion $E(A) \subset GL_0(A)$. For an ideal $I \subset A$ we define

$$K_1^{top}(A, I) := \ker(p_{2*} : K_1^{top}(D(A, I)) \to K_1^{top}(A)),$$

and denote by $S_{A,I} : K_1(A, I) \to K_1^{top}(A, I)$ the canonical surjection induced by the inclusion $E(D(A, I)) \subset GL_0(D(A, I))$.

K_1^{top} is a functor from the category of Banach algebras (and continuous algebra homomorphisms) to that of abelian groups. A continuous homomorphism $f : A \to B$, induces a group homomorphism $\hat{f} : GL(A) \to GL(B)$, and $\hat{f}(GL_0(A)) \subset GL_0(B)$ by continuity. Therefore it induces a well defined map

$f_*^{top} : K_1^{top}(A) \to K_1^{top}(B)$. It is clear that the diagram

$$\begin{CD} K_1(A) @>{f_*}>> K_1(B) \\ @V{S_A}VV @VV{S_B}V \\ K_1^{top}(A) @>{f_*^{top}}>> K_1^{top}(B) \end{CD}$$

commutes, and this implies half-exactness of K_1^{top}.

Theorem 1.34. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

be an exact sequence of Banach algebras. Then there is a map

$$\mathrm{Ind}_1 : K_1^{top}(A/I) \to K_0(I),$$

such that the sequence

$$K_1^{top}(A,I) \xrightarrow{i_*^{top}} K_1^{top}(A) \xrightarrow{\pi_*^{top}} K_1^{top}(A/I) \xrightarrow{\mathrm{Ind}_1} K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I)$$

is exact.

Proof. We show that the exact sequence from theorem 1.22 factors through K_1^{top}. We already saw that we may define Ind_1 as $\mathrm{Ind} \circ S_{A/I}^{-1}$. Then $\mathrm{im}\ \mathrm{Ind}_1 = \mathrm{im}\ \mathrm{Ind} = \ker i_*$, so have exactness at $K_0(I)$. Also,

$$\ker \mathrm{Ind}_1 = S_{A/I}(\ker \mathrm{Ind}) = S_{A/I}(\mathrm{im}\ \pi_*) = \mathrm{im}\ \pi_*^{top} \circ S_A = \mathrm{im}\ \pi_*^{top},$$

so we have exactness at $K_1^{top}(A/I)$. Exactness at $K_1^{top}(A)$ is a consequence of the fact that $\pi(GL_0(A)) = GL_0(A/I)$ such that the proof is identical to the proof for K_1. □

We can extend topological K-theory from Banach algebras to a class of dense subalgebras of those, that is stated in the following theorem.

Theorem 1.35 (Karoubi). *Let B a Banach $\mathbb{C}$-algebra and $A \subset B$ a dense subalgebra which is stable, that is, with the property that $i(M) \in GL(n,B)$ implies $M \in GL(n,A)$ for all n. Endowing A with the relative topology, we have an isometric inclusion $i : A \to B$, and $i_* : K_i^{top}(A) \to K_i^{top}(B)$ is an isomorphism for i=0,1 .*

It shows that for a compact manifold M, $C^\infty(M)$ and $C(M)$ have the same topological K-theory.
Topological K-theory has some favourable properties which makes it easier to compute then algebraic K-theory. One of those is that there is essentially only

the functor K_0 which one needs to consider. For a Banach algebra A, one defines its *cone* CA as

$$CA := \{f \in C([0,1], A) : f(0) = 0\}$$

and its *suspension* SA as

$$SA := \{f \in C([0,1], A) : f(0) = f(1) = 0\} \cong \{f : S^1 \to A : f(1) = 0\}$$

where S^1 denotes the unit circle in $\mathbb{C}$. Cone and suspension are functors in the category of Banach algebras, and they are exact. That is, they map short exact sequences to short exact sequences. We will omit the proof, it can be found in [22].

It is not hard to see that there is an exact sequence

$$0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0.$$

Also id : $CA \to CA$ is homotopic to the zero map, by the homotopy of homomorphisms

$$\psi : CA \to C([0,1], CA)$$

given by $\psi(f)(s,t) := f(st)$. It follows that $K_0(CA) = 0$. The same argument shows that any element in $GL(CA_+)$ is path connected to the identity, and hence that $K_1^{top}(CA_+) = 0$. Unitizing SA and CA gives us an exact sequence

$$0 \longrightarrow SA_+ \longrightarrow CA_+ \longrightarrow A \longrightarrow 0 \quad (*).$$

Since $K_0(CA_+) \cong K_0(CA) \oplus K_0(\mathbb{C}) \cong \mathbb{Z}$, examining the exact sequence of theorem 1.34 associated to (*), we see that Ind_1 is injective and its image is $K_0(SA)$. So we have a natural isomorphism $K_1^{top}(A) \cong K_0(SA)$. Thus K_1^{top} is homotopy invariant and excisive. This suggests that we may define $K_i^{top}(A) := K_0(S^iA)$, the i-th iterated suspension of A, since by exactness of S and theorem 1.34, we then have an exact sequence

$$\ldots \xrightarrow{\text{Ind}_{i+1}} K_i^{top}(I) \longrightarrow K_i^{top}(A) \longrightarrow K_i^{top}(A/I) \xrightarrow{\text{Ind}_i} K_{i-1}^{top}(I) \longrightarrow \ldots$$

for any extension of Banach algebras.

The fundamental property of topological K-theory is *Bott periodicity*, which we will state without proof.

Theorem 1.36 (The Bott isomorphism). *Let A be a Banach algebra. There is a natural isomorphism $K_0(A) \to K_1^{top}(SA)$, induced by*

$$e \mapsto ze + (1_n - e), \quad z \in S^1, e \in \text{Idem}\ (n, A)$$

when A is unital. For non-unital A, we define the Bott isomorphism on $K_0(A_+)$ and then restrict it to $K_0(A)$.

One easily checks well-definedness in the non-unital case. An immediate corollary to this theorem is that the long exact sequence above collapses to a periodic six term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/I) \\ \mathrm{Ind}_1 \uparrow & & & & \downarrow \mathrm{Ind}_0 \\ K_1^{top}(A/I) & \xleftarrow{\pi_*^{top}} & K_1^{top}(A) & \xleftarrow{i_*^{top}} & K_1^{top}(I) \end{array}$$

which is indeed very nice to have at one's disposal for concrete calculations of these groups. Since there are only two groups in topological K-theory, the density theorem 1.35 shows that we have a six term exact sequence for dense stable subalgebras of Banach algebras. We saw that to obtain this sequence, it suffices to have homotopy invariance of K_0 and the property $\pi_* GL_0(A) = GL(A/I)_0$ for surjective continuous homomorphisms. These properties hold for more general classes of topological algebras, but are much harder to prove already for Frechet algebras. This is done by Phillips in [21]. Recently Cuntz defined bivariant topological K-theory for locally convex algebras, with the desired properties, see [8]. Bott periodicity also motivates the study of periodic cyclic (co)homology in the next chapter, where we have such a sequence for any extension of algebras. Cyclic homology is related to algebraic K-theory by a natural transformation called the Chern character, which is the subject of chapter 3 .

1.8 C^*-algebras and index theorems

As mentioned in the the introduction, topological K-theory for C^*-algebras has been particularly fruitful in index theory. The connecting morphism in K-theory carries the name Ind because of this fact. We will discuss two simple index theorems, but postpone their proofs until chapter 3. With the machinery developed there, we will be able to give very short proofs. These proofs use cyclic homology, which is discussed in the next chapter, and to be able to do so, it is necessary to pass from C^*-algebras to dense subalgebras with a finer topology. In doing so, certain subtleties arise, and it is useful to be aware of them, because in chapter 4 we need to reformulate the Atiyah-Singer index theorem, in the presence of similar subtleties.
Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and denote by $B(\mathcal{H})$ the algebra of bounded operators. The algebra $\mathcal{K} \subset B(H)$ of compact operators is a closed two sided ideal and the quotient $\mathcal{Q} := B(\mathcal{H})/\mathcal{K}$ is called the *Calkin algebra*. Thus by construction, there is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0$$

of C^*-algebras. The following notion is particularly important in index theory.

Definition 1.37. An operator $T \in B(\mathcal{H})$ is called a *Fredholm operator* if both $\dim\ker T$ and $\dim\ker T^*$ are finite.

The following theorem gives a characterization of Fredholm operators, which links them to the above exact sequence.

Theorem 1.38 (Atkinson). *An operator $T \in B(\mathcal{H})$ is Fredholm if and only if its image $\pi(T) \in \mathcal{Q}$ is invertible.*

We call $\pi(T)$ the *symbol* of T. For Fredholm operators, there exists a lifting $S \in B(\mathcal{H})$ of $\pi(T)^{-1}$, such that $1-ST$ and $1-TS$ are the orthogonal projections on $\ker T$ and $\ker T^*$, respectively. Such an S is called a *parametrix* for T. Recall that a projection is a self-adjoint idempotent. Since S lifts $\pi(T)^{-1}$, $1-ST$ and $1-TS$ are compact operators. They are also projections, so their ranges must be finite dimensional. The quantity

$$\dim\ker T - \dim\ker T^*$$

is called the *Fredholm index* of T.
The be able to state the link with K-theory, we need to identify the K_0 group of $\mathcal{K}$. Since this is a nonunital algebra, its K-groups are defined using the unitization construction. An idempotent e in a nonunital ring I defines an idempotent $(e,0)$ in I_+, and $K_0(\mathcal{K})$ is exhausted in this way by idempotents in $\mathcal{K}$. The range of these idempotents is finite dimensional and there is an isomorphism $K_0(\mathcal{K}) \to \mathbb{Z}$ given by mapping an idempotent to its trace. We have the following result.

Proposition 1.39. *Under the isomorphism $K_0(\mathcal{K}) \cong \mathbb{Z}$, the boundary map $\mathrm{Ind}_0 : K_1^{top}(\mathcal{Q}) \to K_0(\mathcal{K})$ maps the symbol $\pi(T)$ of a Fredholm operator T to its index.*

A proof will be given in section 3.1. Because of this result, the construction of the boundary map in K-theory is sometimes called the *parametrix construction.*
A more interesting index theorem is the Gohberg-Krein index theorem for Toeplitz operators. The Toeplitz C^*-algebra $\tilde{\mathcal{T}}$ can be constructed as follows. Identify $\ell^2(\mathbb{Z})$ with $C(S^1)$ via Fourier transformation. Under this isomorphism, $\ell^2(\mathbb{N})$ corresponds to the functions whose Fourier coefficients vanish in negative degrees. A function $f \in C(S^1)$ acts a linear operator F on $\ell^2(\mathbb{Z})$ by multiplication : $F(g) = fg$. If $p : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ is the orthogonal projection, then $f \in C(S^1)$ acts on $\ell^2(\mathbb{N})$ by

$$T_f g := p \circ F(g) = p(fg).$$

T_f is called the *Toeplitz operator with symbol f*. We define $\tilde{\mathcal{T}}$ as the subalgebra of $B(\ell^2(\mathbb{N}))$ generated by all Toeplitz operators. It is quite easy to see that $\tilde{\mathcal{T}}$ is in fact generated as a C^*-algebra the operator T_z, which is commonly denoted S, because it acts as a shift on the basis $\{z^n\}$ of $\ell^2(\mathbb{N})$. We have $T_z^* = T_{\overline{z}}$ and the relation $S^*S = 1$ holds trivially. Adding the relation $SS^* = 1$ as well, gives the universal C^*-algebra generated by one unitary, and this is isomorphic to $C(S^1)$, the continuous functions on the circle. By construction, there is a surjection $\tilde{\mathcal{T}} \to C(S^1)$, and the kernel of this map turns out to be isomorphic to $\mathcal{K}$. We are thus dealing with a subextension

$$0 \longrightarrow \mathcal{K} \longrightarrow \tilde{\mathcal{T}} \longrightarrow C(S^1) \longrightarrow 0$$

of the Calkin extension, called the *Toeplitz extension*. The purely algebraic version of this extension plays a crucial role in chapter 3. There we give a more detailed construction of this extension and obtain its C^*-algebraic and smooth version as certain completions of the algebraic one.
Clearly, the Toeplitz extension has a continuous linear section given by $f \mapsto T_f$. If the symbol of a Toeplitz operator is invertible, then it is Fredholm, and it has an index. From theorem 1.39 and the fact that the Toeplitz extension is a subextension of the Calkin extension, that $\mathrm{Ind}_0 : K_1^{top}(C(S^1) \to K_0(\mathcal{K}) \cong \mathbb{Z}$ maps the K_1^{top}-class of a Fredholm Toeplitz operator to its index. We have the following relation between the symbol and the index.

Theorem 1.40 (Gohberg-Krein). *Let $f \in C(S^1)$ be an invertible element. Then the index of the Toeplitz operator T_f is equal to minus the winding number of f. In other words, under the isomorphism $K_0(\mathcal{K}) \cong \mathbb{Z}$, we have*

$$\mathrm{Ind}_0([f]) = -\frac{1}{2\pi i}\int_{S^1} \frac{df}{f}.$$

Another description of the winding number uses the fundamental group $\pi_1(S^1) \cong \mathbb{Z}$. An invertible element $f \in C(S^1)$ is a non vanishing function $f : S^1 \to \mathbb{C}$. The function $\frac{f}{\|f\|} : S^1 \to S^1$ defines an element in $\pi_1(S^1)$ and the winding number of f is just the image of this class in $\mathbb{Z}$. Theorem 1.40 can be derived as an immediate consequence of theorem 3.14. All this should be sufficient to motivate the abstract discussion of cyclic homology, given in the next chapter.

Chapter 2

Cyclic homology

Cyclic homology was discovered by Alain Connes, Joachim Cuntz, Daniel Quillen and Boris Tsygan in the early 1980's. Connes was looking for a target for the Chern character in the non-commutative setting, while Tsygan was motivated by the wish to have an additive version of algebraic K-theory. The (co)homology theories for algebras that we will need are known as cyclic type homologies. For our purposes, it will be convenient to define these homology theories in a more abstract setting, using cyclic modules. The theory for algebras is obtained by assigning to a given algebra a cyclic module. Our main goal in this chapter is to describe the cyclic type homologies as Tor and Ext functors on the category of cyclic modules, which will allow us to define certain product structures in a very elegant way. The nature of the present chapter will therefore be very abstract. In the next chapter we will relate the discussion to algebraic K-theory. Most of the results in this chapter can be found in Loday's book [17].

2.1 The simplicial and cyclic categories

In algebraic topology, one studies various homology theories associated to spaces. An important notion is that of a simplicial object, which is most conveniently described as a functor from a certain small category to some other category (which in topology is the category of spaces). Cyclic homology, which is a generalization of the ordinary homology theories in topology, can be described in very much the same way, by adding some structure to the source category.

Definition 2.1. The *simplicial category* Δ is the small category having objects

$$[n] := \{0, 1, \ldots, n-1, n\} \quad \text{(ordered sets)}$$

and the morphism sets $\mathrm{Mor}_\Delta([n], [m])$ consist of maps

$$\phi : [n] \to [m] \quad \text{such that } i \leq j \Rightarrow f(i) \leq f(j).$$

The morphisms of Δ are called *increasing maps*. The *faces* $\delta_i^n : [n-1] \to [n]$ $(0 \leq i \leq n)$ and *degeneracies* $\sigma_j^n : [n+1] \to [n], (0 \leq j \leq n)$ defined by

$$\delta_i^n(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$
$$\sigma_j^n(k) = \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } k > j, \end{cases}$$

deserve special attention, because of the lemma below. The faces and degeneracies satisfy the following relations:

$$\begin{array}{ll} (1) & \delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \quad \text{for } i < j \\ (2) & \sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n \quad \text{for } i \leq j \\ (3) & \sigma_j^{n-1} \circ \delta_i^n = \begin{cases} \delta_i^{n+1} \circ \sigma_{j-1}^n & \text{for } i < j, \\ \mathrm{id}_{[n]} & \text{for } i = j, i = j+1, \\ \delta_{i-1}^{n+1} \circ \sigma_j^n & \text{for } i > j+1 \end{cases} \end{array}$$

as is checked by calculation. The upper indices will often be suppressed in calculations.

Lemma 2.2. *For any morphism $\phi : [n] \to [m]$, there is a unique decomposition*

$$\phi = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_r}\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_s}$$

such that $i_1 \leq i_2 \leq \cdots \leq i_r$ and $j_1 < j_2 < \cdots < j_s$ with $m = n - s + r$. If the index set is empty, then ϕ is understood to be $\mathrm{id}_{[n]}$.

Proof. First assume f is injective. We perform an induction on m. If $m = 0$, then $n = 0$ and $f = \mathrm{id}$. Now assume that we have the desired decomposition for $k < m$. Since ϕ is injective, $m \geq n$ and if $m = n$ then $\phi = \mathrm{id}$ so we may assume $m > n$. Then there is a $k \in [m]$ such that $k \notin \mathrm{im}\ \phi$. But then $\phi = \delta_k\sigma_k\phi$ and $\sigma_k\phi : [n] \to [m-1]$ has a decomposition

$$\sigma_k\phi = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_r},$$

with $m - 1 = n + r$. No σ's occur since ϕ is injective. Thus

$$f = \delta_k\delta_{i_1}\delta_{i_2}\ldots\delta_{i_r},$$

and we can move δ_k in its right place using relation (1) and we have $m = n+r+1$. Next assume that ϕ is not injective. We do an induction on n. If $n = 0$, then ϕ is injective, a contradiction. For $n = 1$, ϕ has to be constant, so $\phi(0) = k$. Then $\phi = \delta_m \ldots \delta_{k+1}\delta_k \ldots \delta_2\delta_1\sigma_0$. Now suppose that for $k < n$ we have the desired decomposition. Since ϕ is not injective, there exist $j, k \in [n], j \neq k$ such that $\phi(i) = \phi(j)$ for $i \leq j \leq k$. But then $\phi = \phi\delta_i\sigma_i$ and $f\delta_i : [n-1] \to [m]$ has a decomposition

$$\phi\delta_i = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_r}\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_s},$$

with $m = n - 1 - s + r$. Hence

$$\phi = \delta_{i_1}\delta_{i_2}\dots\delta_{i_r}\sigma_{j_1}\sigma_{j_2}\dots\sigma_{j_s}\sigma_i,$$

and we can move σ_i in its right place by relation (2). Moreover, $m = n - 1 - s + r = n - (s+1) + r$.
Having established existence of the decomposition, it remains to show uniqueness. So suppose we have two distinct expressions

$$\delta_{i_1}\delta_{i_2}\dots\delta_{i_r}\sigma_{j_1}\sigma_{j_2}\dots\sigma_{j_s}, \quad \delta_{k_1}\delta_{k_2}\dots\delta_{k_t}\sigma_{l_1}\sigma_{l_2}\dots\sigma_{l_u}$$

satisfying the above hypotheses. By multiplying from the left with σ's and from the right with δ's, using relation (3), we may assume $i_1 < k_1$ or $j_s > l_u$. In the first case, we have

$$\delta_{i_1}\delta_{i_2}\dots\delta_{i_r}(i_1) \geq i_1 + 1 \neq i_1 = \delta_{k_1}\delta_{k_2}\dots\delta_{k_t}(i_1).$$

Since the σ's are surjective, this proves distinctness of the two expressions.
In the second case, we may assume there are no δ's left, and hence $u = s$. But since $j_s > l_u > \dots > l_1$, we have

$$\sigma_{j_1}\sigma_{j_2}\dots\sigma_{j_s}(j_s) = j_s - s + 1 \neq j_s - s = \sigma_{l_1}\sigma_{l_2}\dots\sigma_{l_s}(j_s).$$

This proves distinctness of the two functions and hence uniqueness. □

Corollary 2.3. Δ *can be defined as the category with objects* $[n]$*, for* $n \in \mathbb{N}$*, and morphisms generated by*

$$\begin{aligned} \delta_i^n &: [n-1] \to [n] \quad 0 \leq i \leq n \\ \sigma_j^n &: [n+1] \to [n] \quad 0 \leq j \leq n, \end{aligned}$$

subject to the relations (1), (2) and (3).

Proof. By lemma 2.2 (and its proof), the category defined in this way will be equivalent to Δ. □

Definition 2.4. Let $\mathcal{C}$ be a category. A *simplicial object in* $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. A *cosimplicial object in* $\mathcal{C}$ is a covariant functor $\Delta \to \mathcal{C}$.

In the next section we will encounter examples of simplicial objects. If $\mathcal{C}$ is the category of sets, X a simplicial and Y a cosimplicial object, then we can form their *product over* Δ as follows:

$$X \otimes_\Delta Y := \bigcup_{n \in \mathbb{N}} X_n \times Y_n / \equiv,$$

where $\equiv$ is the equivalence relation generated by

$$(x_1, y_1) \equiv (x_2, y_2) \Leftrightarrow (x_1, f_* y_1) = (f^* x_2, y_2), f \in \mathrm{Mor}\Delta.$$

This allows us to associate a topological space to any simplicial set X. Define the *geometric n-simplex* by

$$\Delta^n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1, \quad \sum x_i = 1\},$$

and subsequently the cosimplicial set $|\Delta|$ by

$$\begin{aligned} |\Delta|_n &:= \Delta^n, \\ \delta^n_{i*}(x_0, ..., x_{n-1}) &:= (x_0, ..., x_{i-1}, 0, x_i, ..., x_{n-1}), \\ \sigma^n_{j*}(x_0, ..., x_{n+1}) &:= (x_0, ..., x_j + x_{j+1}, ..., x_{n+1}). \end{aligned}$$

The *geometric realization* of X is defined as $|X| := X \otimes_\Delta |\Delta|$.

Definition 2.5. Let ${}^{pre}\Delta$ denote the small category with objects $[n]$, $n \in \mathbb{N}$, and morphisms generated by

$$\delta^n_i : [n-1] \to [n] \quad 0 \leq i \leq n$$

subject to the relation (1).${}^{pre}\Delta$ is called the *pre-simplicial category.* If $\mathcal{C}$ is an arbitrary category, then a *pre-simplicial object* in $\mathcal{C}$ is a contravariant functor ${}^{pre}\Delta \to \mathcal{C}$.

We will now describe the cyclic category Λ. In order to do this, we recall that the *degree* of a continuous map $f : S^1 \to S^1$ (S^1 denotes the circle in the complex plane) is the image of its class $[f] \in \pi_1(S^1)$ under the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$. It tells us how often it wraps S^1 around itself.
Endowed with its usual trigonometric orientation, two elements $x, y \in S^1$ determine an *interval* $[x, y]$. A continuous map $f : S^1 \to S^1$ is called *increasing* if

$$f([x, y]) \subset [f(x), f(y)]$$

for all $x, y \in S^1$.
The finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ can be embedded in S^1 by $\overline{k} \mapsto e^{\frac{2\pi ik}{n}}$. If f_0 and f_1 are two degree 1 maps from S^1 to itself, such that

$$f_i(\mathbb{Z}/n\mathbb{Z}) \subset \mathbb{Z}/m\mathbb{Z},$$

then we call them *λ-homotopic* if there exists a homotopy $F : S^1 \times [0, 1] \to S^1$ between them such that

$$F(\mathbb{Z}/n\mathbb{Z}, t) \subset \mathbb{Z}/m\mathbb{Z} \quad \forall t \in [0, 1].$$

Definition 2.6. The *cyclic category* Λ has objects $[n]$ (the objects of Δ) and morphism sets $\mathrm{Mor}_\Lambda([n], [m])$ consisting of the λ-homotopy classes of degree 1 increasing maps $f : S^1 \to S^1$ such that $f(\mathbb{Z}/(n+1)\mathbb{Z}) \subset \mathbb{Z}/(m+1)\mathbb{Z}$.

It is immediate from this definition that the image of $\mathbb{Z}/(n+1)\mathbb{Z}$ under a map $f \in [f] \in \mathrm{Mor}_\Lambda([n], [m])$ only depends on $[f]$. Therefore it induces a map

$\phi_f : [n] \to [m]$ of sets. If $\phi : [n] \to [m]$ is a non-constant increasing map, one obtains a degree 1 map $f_\phi : S^1 \to S^1$ by mapping the line segment

$$[e^{\frac{2\pi ij}{n+1}}, e^{\frac{2\pi i(j+1)}{n+1}}] \mapsto [\phi(e^{\frac{2\pi ij}{n+1}}), \phi(e^{\frac{2\pi i(j+1)}{n+1}})]$$

("by multiplication"), for $0 \leq j \leq n-1$ and mapping

$$[e^{\frac{2\pi in}{n+1}}, 1] \mapsto [\phi(e^{\frac{2\pi in}{n+1}}), \phi(0)].$$

If ϕ is constant, $\phi(i) = k$ then we map $[e^{\frac{2\pi in}{n+1}}, 1]$ around S^1. Since $\phi_{f_\phi} = \phi$, we may identify Δ with a subcategory of Λ in this way.

Lemma 2.7. *Each $f \in \mathrm{Mor}_\Lambda([n],[m])$ admits a unique decomposition $f = \phi \circ \tau_n^k$, where $\phi \in \mathrm{Mor}_\Delta([n],[m])$, $0 \leq k \leq n$, and τ_n is the invertible morphism defined by $\tau_n(x) := e^{\frac{2\pi i}{n+1}}x$.*

Proof. The maps f_ϕ map the interval $[e^{\frac{2\pi in}{n+1}}, 1]$ onto

$$[e^{\frac{2\pi if(n)}{m+1}}, e^{\frac{2\pi if(0)}{m+1}}] \supset [e^{\frac{2\pi im}{m+1}}, 1].$$

So if we choose for each n an element $\theta_n \in [e^{\frac{2\pi in}{n+1}}, 1]$, then we can choose a representative f of f_ϕ such that $f(\theta_n) \in [e^{\frac{2\pi im}{m+1}}, 1]$. On the other hand, if a morphism $g \in \mathrm{Mor}_\Lambda([n],[m])$ has a representative f with this property, then it is in the image of Δ, because it defines an increasing map $\phi_f : [n] \to [m]$ with $f_{\phi_f} = f$. Now for arbitrary $f \in \mathrm{Mor}_\Lambda([n],[m])$ there is a unique interval

$$I := [e^{\frac{2\pi ij}{n+1}}, e^{\frac{2\pi i(j+1)}{n+1}}]$$

such that $\theta_m \in f(I)$. If we define $\tau : S^1 \to S^1$ by $\tau(x) = e^{\frac{2\pi i}{n+1}}x$, then

$$\theta_m \in f \circ \tau^{j-n}([e^{\frac{2\pi in}{n+1}}, 1]),$$

thus $f \circ \tau^{j-n} \in \Delta$. This clearly proves existence and uniqueness of the decomposition. □

Lemma 2.8. $|\mathrm{Mor}_\Lambda([n],[0])| = n+1$

Proof. For each k, $0 \leq k \leq n$, we have a morphism ϕ_k defined by sending everything to 1 except the interval $[e^{\frac{2\pi ik}{n+1}}, e^{\frac{2\pi i(k+1)}{n+1}}]$, which is wrapped around S^1. This gives us $n+1$ non λ-homotopic maps and it is clear that there are no others. □

Corollary 2.9. $\mathrm{Aut}_\Lambda([n]) = \{[\tau^k] : 0 \leq k \leq n\}$*, where $\tau : S^1 \to S^1$ is the map from lemma 2.7.*

Proof. It is clear that the powers of τ define distinct automorphisms of $[n]$. There can be no others, since if $f \in \mathrm{Mor}_\Lambda([n],[0])$ and g, h are distinct automorphisms of $[n]$, then $f \circ g, f \circ h \in \mathrm{Mor}_\Lambda([n],[0])$ will be distinct morphisms. $\square$

These results suggest examining the relations between τ and the generators of Δ. By calculation one finds that they are

$$\begin{aligned}
&(4) \quad \tau_n \circ \delta_i^n = \delta_{i-1}^n \circ \tau_{n-1} \quad \text{for } 1 \le i \le n, \quad \tau_n \circ \delta_0^n = \delta_n^n \\
&(5) \quad \tau_n \circ \sigma_i^n = \sigma_{i-1}^n \circ \tau_{n+1}, \quad \text{for } 1 \le i \le n, \quad \tau_n \circ \sigma_0^n = \sigma_n^n \circ \tau_{n+1}^2, \\
&(6) \qquad \tau_n^{n+1} = \mathrm{id}_{[n]}.
\end{aligned}$$

This immediately translates into

Corollary 2.10. *Λ can be defined as the category with objects $[n]$, $n \in \mathbb{N}$, and morphisms the morphisms and relations of Δ and in addition the morphisms $\tau_n : [n] \to [n]$ subject to the relations (4),(5) and (6).*

Proof. The relations 4,5 and 6 enable us to write any morphism in the category $\mathcal{C}$ defined by them as a product of an element in Δ with some power of τ. Since all relations hold in Λ we have a functor from $\mathcal{C} \to \Lambda$, which is surjective. Because we have the unique decomposition from lemma 2.7 there is a functor in the other direction which in inverts the previous one, which hence must be an equivalence. $\square$

Definition 2.11. Let $\mathcal{C}$ be a category. A *cyclic object* in $\mathcal{C}$ is a contravariant functor $\Lambda \to \mathcal{C}$.

In the next section we will encounter examples of cyclic objects. The presentation from corollary 2.10 can be used to prove a remarkable property of Λ. Recall that, given a category $\mathcal{C}$, we can form is *opposite* $\mathcal{C}^{op}$ by taking as elements the elements of $\mathcal{C}$ and defining $\mathrm{Mor}_{\mathcal{C}^{op}}(A,B) := \mathrm{Mor}_{\mathcal{C}}(B,A)$.

Proposition 2.12. *The category Λ is isomorphic to its opposite category Λ^{op}.*

Proof. We will define a contravariant functor $F : \Lambda \to \Lambda$, such that $F \circ F$ is an inner automorphism of Λ. It then follows that F induces an equivalence $\Lambda \cong \Lambda^{op}$. By the presentation of corollary 2.10, it suffices to define F on the generators of Λ.

$$\begin{aligned}
F(\delta_i^n) &:= \sigma_i^{n-1} \quad 0 \le i \le n-1 \\
F(\delta_n^n) &:= \sigma_0^{n-1}\tau_n^n = \tau_{n-1}^{n-1} \circ \sigma_{n-1}^{n-1} \circ \tau_n \\
F(\sigma_i^n) &:= \delta_{i+1}^{n+1} \quad 0 \le i \le n \\
F(\tau_n) &:= \tau_n^n = \tau_n^{-1}
\end{aligned}$$

We simply *define* F to be contravariant. To check that we have a functor, one checks that the above definition respects all the relations in Λ, which is just calculation. Also, one calculates that for $F : [n] \to [m]$, $F \circ F(f) = \tau_m^m \circ f \circ \tau_n$ $\square$.

Corollary 2.13. *Let $\mathcal{C}$ be any category. A cyclic object in $\mathcal{C}$ is given by a covariant functor $\Lambda \to \mathcal{C}$.*

Proof. Just compose the given cyclic object with F. □

Corollary 2.14. *Each $f \in \mathrm{Mor}_\Lambda([n],[m])$ admits a unique decomposition $f = \tau_n^\ell \circ \psi$, where $\psi \in F(\mathrm{Mor}_\Delta([n],[m]))$, $0 \le \ell \le n$ and τ_n is the invertible morphism defined by $\tau_n(x) := e^{\frac{2\pi i}{n+1}} x$.*

By proposition 2.12, $f = F(g)$ for some $g \in \mathrm{Mor}_\Lambda([m],[n])$. By lemma 2.7 $g = \phi \circ \tau_n^k$, with $\phi \in \Delta$. Applying F yields the desired result with $\ell = n - k$. □

Of course $F(\Delta) \cong \Delta^{op}$, so we have an embedding $\Delta^{op} \hookrightarrow \Lambda$, and we can decompose each $f \in \Lambda$ as $f = \tau_n^\ell \circ \psi$ with $\psi \in \Delta^{op}$. For the study of cyclic homology, we need one more definition. It involves the category most relevant to us.

Definition 2.15. The *pre cyclic* category ${}^{pre}\Lambda$ is the category with objects $[n], n \in \mathbb{N}$ and morphisms generated by

$$\delta_i^n : [n-1] \to [n] \quad 0 \le i \le n$$
$$\tau_n : [n] \to [n]$$

subject to the relations (1), (4) and (6).
If $\mathcal{C}$ is an arbitrary category, then a *pre cyclic object* in $\mathcal{C}$ is a contravariant functor ${}^{pre}\Lambda \to \mathcal{C}$.

In the sequel we will refer to the (pre) simplicial and (pre) cyclic objects in a category as *cyclic type* objects. If $\mathcal{C}$ is any category, then the each class of cyclic type objects in $\mathcal{C}$ forms a category in which the morphisms are natural transformations of functors. We will see what this means in more concrete terms in the next section.

2.2 Cyclic modules

Let k be a commutative ring with unit and denote by $\boldsymbol{M}_k$ the category of k modules.This is an abelian category. We will discuss cyclic type objects in $\boldsymbol{M}_k$, and look at the most important examples. From the presentations of the categories in the previous section, we deduce a more concrete description of cyclic type objects in $\boldsymbol{M}_k$. A pre-simplicial k-module is a graded k-module

$$X = \bigoplus_{n=0}^{\infty} X_n,$$

together with maps

$$d_i^n : X_n \to X_{n-1} \quad \text{for } 0 \le i \le n$$

subject to the following relation:

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n \quad \text{for } i < j.$$

A pre-cyclic k-module is a pre-simplicial k-module together with maps

$$t_n : X_n \to X_n$$

such that

$$\begin{aligned} d_i^n \circ t_n &= t_{n-1} \circ d_{i-1}^n \quad \text{for } i \neq 0 \\ t_n^{n+1} &= \mathrm{id}_{X_n}. \end{aligned}$$

This is all immediate, by dualizing the relations in the relevant category. A morphism $f : X \to Y$ of pre-cyclic or -simplicial modules, is a graded k-module homomorphism that commutes with the operators d_i^n and t_n.

Let A be a k-algebra. Set

$$C_n(A) := A^{\otimes n+1},$$

the $n+1$-fold tensor product (over k) of A with itself. Then

$$A^\sharp := \bigoplus_{n=0}^{\infty} C_n(A)$$

is a graded k-module. We make it into a pre-cyclic k-module by defining

$$\begin{aligned} d_i^n(a_0 \otimes ... \otimes a_n) &:= a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n \quad \text{for } 0 \leq i \leq n-1 \\ d_n^n(a_0 \otimes ... \otimes a_n) &:= a_n a_0 \otimes a_2 \otimes ... \otimes a_{n-1} \\ t_n(a_0 \otimes ... \otimes a_n) &:= a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \end{aligned} .$$

A simplicial k-module is a pre-simplicial k-module X together with maps

$$s_i^n : X_n \to X_{n+1} \quad \text{for } 0 \leq i \leq n$$

subject to the relation

$$s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n \quad \text{for } i \leq j$$

and such that

$$d_i^{n+1} \circ s_j^n = \begin{cases} s_{j-1}^{n-1} \circ d_i^n & \text{for } i < j \\ \mathrm{id}_{X_n} & \text{for } i \in \{j, j+1\} \\ s_j^{n-1} \circ d_{i-1}^n & \text{for } i > j+1. \end{cases}$$

A cyclic k-module is a pre-cyclic k-module which is a simplicial k-module, such that

$$s_i^n \circ t_n = \begin{cases} t_{n+1} \circ s_{i-1}^n & \text{for } i \neq 0 \\ t_{n+1}^2 \circ s_n^n & \text{for } i = 0. \end{cases}$$

This is again immediate by dualization. As above, a morphism $f : X \to Y$ of cyclic or simplicial modules is given by a graded k-module homomorphism that commutes with the operators s_i^n, d_j^n and t_n.

If A is a unital k-algebra, then $A^\sharp$ becomes a cyclic k-module by defining

$$s_i^n(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n,$$

for $0 \leq i \leq n$.
If the algebra A carries a locally convex topology that is compatible with the algebra structure, then we replace the algebraic tensor product in the above definitions by the projective tensor product $\hat{\otimes}$. The operations d_i^n, s_j^n and t_n are continuous for the projective seminorms and thus extend to the completion of the algebraic tensor product in these seminorms. This cyclic module is denoted $A_c^\sharp$.
Given a cyclic module X, we can construct a "dual" cyclic module Y by setting $Y_n := \mathrm{Hom}_k(X_n, k)$. Λ now acts in a covariant way by

$$(f_*\phi_m)(x_n) := \phi_m(f^*x_n), \quad f \in \mathrm{Mor}_\Lambda([n],[m]), \quad \phi_m \in \mathrm{Hom}_k(X_m, k), \quad x_n \in X_n.$$

Since $\Lambda \cong \Lambda^{op}$, this induces a cyclic module structure on Y. In the locally convex case, we take the topological dual, consisiting of continuous homomorphisms $X_n \to k$.

2.3 Hochschild homology

This section deals with the simplest of the cyclic type homologies. Given a pre-cyclic k-module X, we can define maps

$$\begin{aligned} b_n &:= \sum_{i=0}^{n} (-1)^i d_i^n : X_n \to X_{n-1} \\ b'_n &:= \sum_{i=0}^{n-1} (-1)^i d_i^n : X_n \to X_{n-1}. \\ \lambda_n &:= (-1)^n t_n : X_n \to X_n \end{aligned}$$

Lemma 2.16. *We have*

$$b_{n-1} \circ b_n = b'_{n-1} \circ b'_n = 0$$

and

$$b_n \circ (1 - \lambda_n) = (1 - \lambda_{n-1}) \circ b'_{n-1}.$$

That is, the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_n & \xrightarrow{b_n} & X_{n-1} & \xrightarrow{b_{n-1}} & X_{n-2} & \longrightarrow & \cdots \\ & & \uparrow{\scriptstyle 1-\lambda_n} & & \uparrow{\scriptstyle 1-\lambda_{n-1}} & & \uparrow{\scriptstyle 1-\lambda_{n-2}} & & \\ \cdots & \longrightarrow & X_n & \xrightarrow{-b'_n} & X_{n-1} & \xrightarrow{-b'_{n-1}} & X_{n-2} & \longrightarrow & \cdots \end{array}$$

is a double complex.

Proof. Since X is a pre-cyclic k-module, we have for $i < j$

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n,$$

and we can compute:

$$\begin{aligned} b_{n-1} \circ b_n &= \Big(\sum_{i=0}^{n-1}(-1)^i d_i^{n-1}\Big) \circ \Big(\sum_{j=0}^{n}(-1)^j d_j^n\Big) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j} d_i^{n-1} \circ d_j^n \\ &= \sum_{j=0}^{n}\Big(\sum_{i<j}(-1)^{i+j} d_i^{n-1} \circ d_j^n + \sum_{j\le i}(-1)^{i+j} d_i^{n-1} \circ d_j^n\Big) \\ &= \sum_{j=0}^{n}\Big(\sum_{i<j}(-1)^{i+j} d_{j-1}^{n-1} \circ d_i^n + \sum_{j\le i}(-1)^{i+j} d_i^{n-1} \circ d_j^n\Big) \\ &= \sum_{j=0}^{n}\Big(\sum_{j\le i}(-1)^{i+j+1} d_i^{n-1} \circ d_j^n + \sum_{j\le i}(-1)^{i+j} d_i^{n-1} \circ d_j^n\Big) \\ &= 0. \end{aligned}$$

Examining this computation shows that $b'_{n-1} \circ b'_n = 0$ as well. For the last equality, observe that $b_n - b'_n = (-1)^n d_n^n$ and that

$$\begin{aligned} &d_i^n \circ \lambda_n = -\lambda_{n-1} \circ d_{i-1}^n, \quad 1 \le i \le n \\ &d_0^n \circ \lambda_n = (-1)^n d_n^n. \end{aligned}$$

Then compute

$$\begin{aligned} b_n \circ \lambda_n - \lambda_{n-1} \circ b'_n &= \sum_{i=0}^{n}(-1)^i d_i^n \circ \lambda_n - \sum_{i=0}^{n-1}(-1)^i \lambda_{n-1} \circ d_i^n \\ &= (-1)^n d_n^n + \sum_{i=1}^{n}(-1)^{i-1}\lambda_{n-1} \circ d_{i-1}^n - \sum_{i=0}^{n-1}(-1)^i \lambda_{n-1} \circ d_i^n \\ &= (-1)^n d_n^n. \end{aligned}$$

This proves the desired equality. □

Definition 2.17. Let X be a pre-cyclic k-module. The double complex of lemma 2.16 is called the *Hochschild complex* of X. The *Hochschild homology* of X is the homology of the total complex associated to its Hochschild complex. It is denoted $HH_*(X)$.

It is clear that that the HH_i are functors on the category of pre-cyclic k-modules, by definition of the morphisms in $\boldsymbol{M}_k({}^{pre}\Lambda)$. If A is a k-algebra, then we will refer to $HH_*(A^\sharp)$ as the Hochschild homology of A, and denote it by $HH_*(A)$. HH_i is a functor on the category of k-algebras, since an algebra morphism $f : A \to B$ induces a morphism of pre-cyclic modules $f_* : C_*(A) \to C_*(B)$ by

$$f_*(a_0 \otimes ... \otimes a_n) := f(a_0) \otimes ... \otimes f(a_n)$$

which is a chain map. Therefore it induces a k-module map on Hochschild homology.
When X is a cyclic k-module, we kan compute its Hochschild homology using only the top row of the Hochschild complex. This is expressed in the following proposition.

Proposition 2.18. *If X is a cyclic k-module, then $HH_*(X)$ is isomorphic to the homology of the complex (X_*, b).*

Proof. We will show that the bottom row of the Hochschild complex is acyclic, so the proposition follows by general homological algebra (cf. Appendix B). Define $s_{-1}^n : X_n \to X_{n+1}$ by $s_{-1}^n = t_{n+1} \circ s_n^n$. We will show

$$s_{-1}^{n-1} \circ b'_n + b_{n+1} \circ s_{-1}^n = id_{X_n},$$

such that s is a contracting homotopy.

$$\begin{aligned}
s_{-1}^{n-1} \circ b'_n + b'_{n+1} \circ s_{-1}^n &= \sum_{i=0}^{n-1} (-1)^i t_n \circ s_{n-1}^{n-1} \circ d_i^n + \sum_{j=0}^{n} -(1)^j d_j^{n+1} \circ t_{n+1} \circ s_n^n \\
&= \sum_{i=0}^{n-1} (-1)^i t_n \circ d_i^n \circ s_n^n + \sum_{j=0}^{n} -(1)^j d_j^{n+1} \circ t_{n+1} \circ s_n^n \\
&= \sum_{i=0}^{n-1} (-1)^i d_{i+1}^{n+1} \circ t_{n+1} \circ s_n^n + \sum_{j=0}^{n} -(1)^j d_j^{n+1} \circ t_{n+1} \circ s_n^n \\
&= d_0^{n+1} \circ t_{n+1} \circ s_n^n \\
&= t_n^n \circ d_1^{n+1} \circ t_{n+1}^2 \circ s_n^n \\
&= t_n^n \circ d_1^n \circ s_0^n \circ t_n \\
&= t_n^{n+1} \\
&= \mathrm{id}_{X_n}. \quad \square
\end{aligned}$$

The operators s_{-1}^n are called the *extra degeneracies* of the cyclic module X.
In case we have a unital algebra, we can describe its Hochschild homology as

a Tor functor. This has the advantage that one can use a resolution for its computation. Let A^{op} be the algebra A with the multiplication reversed. That is $a * b := ba$ in A^{op}. Moreover let A^e denote the algebra $A \otimes_k A^{op}$. Then an A^e-module is just an A-bimodule.

Proposition 2.19. *Let A be a unital algebra. There are canonical isomorphisms $HH_n(A) \cong \mathrm{Tor}_n^{A^e}(A, A)$.*

Proof. It suffices to provide a resolution of A by A^e-modules, which, when tensored over A^e with A gives the complex $(A^\sharp, b)$. We saw that the complex $(A^\sharp, b')$ is a resolution of A, and b' is an A^e-module map. The tensor powers of A are free A^e modules for $n \geq 2$, so the b'-complex of A is indeed a projective resolution of A by A^e-modules. Tensoring with A over A^e gives us the b-complex of A, so we are done. ☐

Note that that we can study the complex (X, b) for any pre-simplicial module X. In particular, if we have a simplicial set Y, then we can form the simplicial module $k[Y]$, which in degree n is the free k-module on Y_n. There is a map ν from the the complex $(k[Y], b)$ to the singular complex $C(|Y|, d)$ of the geometric realization $|Y|$ with coefficients in k. It is defined by associating to $y \in Y_n$ the n-simplex $f_y : \Delta^n \to |Y|$ given by $x \mapsto (y, x)$.

Theorem 2.20. *Let Y be a simplicial set. The map ν induces an isomorphism*

$$\nu_* : H_*(k[Y], b) \to H_*(|Y|, k).$$

A proof can be found in.

We will now define Hochschild cohomology. We can dualize the Hochschild complex by setting $X^n := \mathrm{Hom}_k(X_n, k)$ and

$$b^n(f) = f \circ b_n, \quad b'^n(f) = f \circ b'_n, \quad \lambda^n(f) = f \circ \lambda_n,$$

for $f \in X^n$. We then obtain a complex

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & X^n & \xleftarrow{b^n} & X^{n-1} & \xleftarrow{b^{n-1}} & X^{n-2} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle 1-\lambda^n} & & \downarrow{\scriptstyle 1-\lambda^{n-1}} & & \downarrow{\scriptstyle 1-\lambda^{n-2}} & & \\
\cdots & \longleftarrow & X^n & \xleftarrow{-b'^n} & X^{n-1} & \xleftarrow{-b'^{n-1}} & X^{n-2} & \longleftarrow & \cdots
\end{array}$$

Its cohomology is denoted by $HH^*(X)$ and in the case of an algebra by $HH^*(A)$, it is called the *Hochschild cohomology* of X (resp. A). If X is a cyclic k-module, then its Hochschild cohomology is isomorphic to the cohomology of the cocomplex (X^*, b^*). For a unital k-algebra A, we can interpret elements of $C^n(A)$ as $n+1$ linear functionals on A^{n+1}. The boundary maps can be expressed as

$$b^n f(a_0, ..., a_n) = (-1)^n f(a_n a_0, a_1, ..., a_{n-1}) + \sum_{i=0}^{n-1} (-1)^i f(a_0, ..., a_i a_{i+1}, ..., a_n).$$

Thus, a Hochschild 0-cocycle $f : A \to k$, is a k-linear map satisfying

$$b^1 f(a_0, a_1) = f(a_0 a_1) - f(a_1 a_0) = f([a_0, a_1]) = 0,$$

that is, a trace on A. Since $b^0 = 0$, there are no Hochschild 0-coboundaries. Therefore $HH^0(A)$ consists exactly of the traces on A.
Let $q : X \to Y$ be a surjective morphism of pre-cyclic modules. Since the category $\boldsymbol{M}_k({}^{pre}\Lambda)$ is abelian, $W := \ker f \subset X$ is a cyclic module and the inclusion $i : W \hookrightarrow X$ is a morphism. By general homological algebra there are long exact sequences

$$\ldots \longrightarrow HH_n(W) \xrightarrow{i_*} HH_n(X) \xrightarrow{q_*} HH_n(Y) \xrightarrow{\partial} HH_{n-1}(W) \longrightarrow \ldots$$

$$\ldots \longrightarrow HH^n(W) \xrightarrow{i^*} HH^n(X) \xrightarrow{q^*} HH^n(Y) \xrightarrow{\partial} HH^{n+1}(W) \longrightarrow \ldots$$

From proposition 2.19 it is immediate that, when A is unital, there are canonical isomorphisms $HH^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$, since we can use the same resolution to compute these groups.
Hochschild homology shares an important property with K-theory, namely that of Morita invariance.

Definition 2.21. Let A be an algebra, and $M_r(A)$ the algebra of $r \times r$ matrices over A. Then any $M \in M_r(A)$ can be written as a sum $\sum M_i a_i$ with $M_i \in M_r(k)$ and $a_i \in A$. The *generalized trace map* Tr: $M_n(A)^{\otimes n+1} \to A^{\otimes n+1}$ is defined by

$$\mathrm{Tr}(M_0 a_0 \otimes \ldots \otimes M_n a_n) := \mathrm{Tr}(M_0 \ldots M_n) a_0 \otimes \ldots \otimes a_n.$$

Since the trace $\mathrm{Tr} : M_r(k) \to k$ is permutation invariant, and $M_i a_i = a_i M_i$, the generalized trace commutes with b and thus defines a map $\mathrm{Tr}_* : HH_*(M_r(A)) \to HH_*(A)$.

Theorem 2.22 (Morita invariance of Hochschild homology). *Let A be a unital k-algebra. Then $\mathrm{Tr}_* : HH_*(M_r(A)) \to HH_*(A)$ is an isomorphism.*

We will not provide a proof since it is lengthy and not very illuminating. See [11] or [17]. The inverse to Tr is induced by the chain map given by the inclusion $A \hookrightarrow M_r(A)$ in the upper left corner or in any other diagonal entry.

Corollary 2.23. *Let A be a unital k-algebra and $g \in A$ an invertible element. Then the conjugation map $\phi_g : a \mapsto gag^{-1}$ induces the identity map in Hochschild homology.*

Proof. Let $h := \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$.There is a commutative diagram of cyclic modules

$$\begin{array}{ccccc} A^\sharp & \xrightarrow{i_1} & M_2(A)^\sharp & \xleftarrow{i_2} & A^\sharp \\ \| & & \downarrow \phi_h & & \downarrow \phi_g \\ A^\sharp & \xrightarrow[i_1]{} & M_2(A)^\sharp & \xleftarrow{i_2} & A^\sharp, \end{array}$$

to which we apply HH. Since i_{1*} and i_{2*} are both inverses of Tr_*, they are isomorphisms and we have $i_{1*}^{-1}\phi_h i_{1*} = \mathrm{id}$ so $\phi_{h*} = \mathrm{id}$ and therefore $i_{2*}^{-1}\phi_{g*}i_{2*} = \mathrm{id}$ and $\phi_{g*} = \mathrm{id}$. $\square$

2.4 Cyclic homology

To define cyclic homology, we need to enlarge the Hochschild complex to a bigger double complex. In order to do this, we need one more operator. It is defined as

$$\mathcal{N}_n := \sum_{i=0}^{\infty} \lambda_n^i : X_n \to X_n.$$

Clearly, $\mathcal{N}_n$ has the property that $\mathcal{N}_n \circ \lambda_n = \mathcal{N}_n$, or, equivalently $(1-\lambda_n)\circ\mathcal{N}_n = \mathcal{N}_n(1-\lambda_n) = 0$. Even more holds:

Lemma 2.24. *$b'_n \circ \mathcal{N}_n = \mathcal{N}_{n-1} \circ b_n$, that is, the diagram*

$$\begin{array}{ccccccccc}
\downarrow{\scriptstyle b_3} & & \downarrow{\scriptstyle -b'_3} & & \downarrow{\scriptstyle b_3} & & \downarrow{\scriptstyle -b'_3} & & \\
X_2 & \xleftarrow{1-\lambda_2} & X_2 & \xleftarrow{\mathcal{N}_2} & X_2 & \xleftarrow{1-\lambda_2} & X_2 & \xleftarrow{\mathcal{N}_2} & \ldots \\
\downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle -b'_2} & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle -b'_2} & & \\
X_1 & \xleftarrow{1-\lambda_1} & X_1 & \xleftarrow{\mathcal{N}_1} & X_1 & \xleftarrow{1-\lambda_1} & X_1 & \xleftarrow{\mathcal{N}_1} & \ldots \\
\downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle -b'_1} & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle -b'_1} & & \\
X_0 & \xleftarrow{1-\lambda_0} & X_0 & \xleftarrow{\mathcal{N}_0} & X_0 & \xleftarrow{1-\lambda_0} & X_0 & \xleftarrow{\mathcal{N}_0} & \ldots
\end{array}$$

is a double complex.

Proof. This is just computation. First, observe that

$$\begin{aligned}
d_i^n \circ \lambda_n^j &= (d_i^n \circ \lambda_n) \circ \lambda_n^{j-1} \\
&= \begin{cases} (\lambda_{n-1} \circ d_{i-1}^n) \circ \lambda_n^{j-1}, & 1 \le i \le n-1, \\ (-1)^n d_n^n \circ \lambda_n^{j-1}, & i = 0, \end{cases} \\
&= \begin{cases} (-1)^j \lambda_{n-1}^j \circ d_{i-j}^n, & j \le i, \\ (-1)^i \lambda_{n-1}^i \circ d_0^n \circ \lambda_n^{j-i}, & j > i, \end{cases} \\
&= \begin{cases} (-1)^j \lambda_{n-1}^j \circ d_{i-j}^n, & j \le i, \\ (-1)^{i+n} \lambda_{n-1}^i \circ d_n^n \circ \lambda_n^{j-i-1}, & j > i, \end{cases} \\
&= \begin{cases} (-1)^j \lambda_{n-1}^j \circ d_{i-j}^n, & j \le i, \\ (-1)^{n+j+1} \lambda_{n-1}^{j-1} \circ d_{i+n-j+1}^n, & j > i. \end{cases}
\end{aligned}$$

Then compute

$$\begin{aligned}
b'_n \circ \mathcal{N}_n &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(-1)^i d_i^n \circ \lambda_n^j \\
&= \sum_{0\le j\le i\le n-1} (-1)^{i+j}\lambda_{n-1}^j \circ d_{i-j}^n + \sum_{0\le i<j\le n-1} (-1)^{n+i+j-1}\lambda_{n-1}^{j-1} \circ d_{i+n-j+1}^n \\
&= \big(\sum_{j=0}^{n-1}\lambda_{n-1}^j\big) \circ \big(\sum_{i\le j}(-1)^{n+i+j} d_{i+n-j}^n + \sum_{i\ge j}(-1)^{i+j} d_{i-j}^n\big) \\
&= \mathcal{N}_{n-1}\circ b_n. \qquad \square
\end{aligned}$$

The complex of lemma 2.24 is called the *cyclic double complex* of X and is denoted $CC_{**}(X)$. The cyclic double complex is an extension of the Hochschild double complex, which consists of its first two columns.

Definition 2.25. Let X be a pre-cyclic k-module. The *cyclic homology* of X is the homology of the total complex associated to $CC_*(X)$, whose term in degree n is

$$\bigoplus_{i+j=n,i,j\ge 0} CC_{ij}(X) = \bigoplus_{j=0}^{n} X_j.$$

It is denoted $HC_*(X)$.

We dualize the cyclic double complex in exactly the same way as we did with the Hochschild double complex to define the cyclic cohomology groups $HC^*(X)$. Also, in the case $X = A^\sharp$ for an algebra A, we write $HC_*(A), HC^*(A)$. To a short exact sequence

$$0 \longrightarrow W \xrightarrow{\ i\ } X \xrightarrow{\ q\ } Y \longrightarrow 0$$

of pre cyclic modules correspond long exact (co)homology sequences

$$\dots \longrightarrow HC_n(W) \xrightarrow{i_*} HC_n(X) \xrightarrow{q_*} HC_n(Y) \xrightarrow{\partial} HC_{n-1}(W) \longrightarrow \dots$$

$$\dots \longrightarrow HC^n(W) \xrightarrow{i^*} HC^n(X) \xrightarrow{q^*} HC^n(Y) \xrightarrow{\partial} HC^{n+1}(W) \longrightarrow \dots$$

There is a beautiful relation between HH_* and HC_*. Consider the self map S of the cyclic double complex that shifts everything two columns to the left. The kernel of this map is exactly the Hochschild complex, and the image is the cyclic double complex with a degree shift. Therefore we have the following theorem.

Theorem 2.26 (Connes). *Let X be a pre-cyclic k-module. There is a natural long exact sequence*

$$\dots \xrightarrow{\ S\ } HC_{n-1}(X) \xrightarrow{B} HH_n(X) \xrightarrow{I} HC_n(X) \xrightarrow{S} HC_{n-2}(X) \xrightarrow{\ B\ } \dots$$

Here B denotes the boundary map associated to short exact sequence of chain complexes, and I is the map induced by the inclusion of the Hochschild complex into the cyclic double complex. In cohomology we get a similar sequence.

Corollary 2.27. *Let $\phi : X \to Y$ be a morphism of pre-cyclic modules. ϕ induces an isomorphism in Hochschild homology if and only if it induces an isomorphism in cyclic homology.*

Suppose $\phi_* : HH_n(X) \to HH_n(Y)$ is an isomorphism for all n. Since HC vanishes in negative degrees, examining the beginning of the SBI sequence shows that $\phi_* : HC_i(X) \to HC_i(Y)$ is an isomorphism for $i = 0, 1$. The result now follows using the five lemma in each degree. The other implication is immediate. □

Corollary 2.28 (Morita invariance of cyclic homology). *Let A a unital k-algebra. The generalized trace induces an isomorphism in cyclic homology.*

Proof. It is obvious that Tr commutes with the cyclic operators, and thus is a map of pre-cyclic modules. Since it induces an isomorphism in Hochschild homology, it induces an isomorphism in cyclic homology. □.

Corollary 2.29. *Let A be a unital k-algebra and g an invertible element. The conjugation map ϕ_g induces the identity map in cylic homology.*

Proof. Immediate from corollary 2.23 and the SBI sequence. □

For cyclic modules, we saw that the odd numbered columns of the cyclic double complex have vanishing homology. This led to a simpler recipe to compute its Hochschild homology. For cyclic homology, the situation simplifies as well, when dealing with a cyclic module.

Definition 2.30. Let X be a cyclic module with extra degeneracies $s^n_{-1} : X_n \to X_{n+1}$. Define the *Connes B-operator* $B_n : X_n \to X_{n+1}$ by $B := (1 - \lambda_{n+1}) \circ s^n_{-1} \circ \mathcal{N}_n$.

Proposition 2.31. *Let X be a cyclic module. Then $b_{n+1} \circ B_n + B_{n-1} \circ b_n = B_{n+1} \circ B_n = 0$ and the cyclic homology of X is isomorphic to the homology of* *the double complex*

$$\begin{array}{ccccc}
\downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 \\
X_2 & \xleftarrow{B_1} & X_1 & \xleftarrow{B_0} & X_0 \\
\downarrow b_2 & & \downarrow b_1 & & \\
X_1 & \xleftarrow{B_0} & X_0 & & \\
\downarrow b_1 & & & & \\
X_0. & & & &
\end{array}$$

Proof. That B is a differential is immediate from the relations $(1-\lambda_n)\circ\mathcal{N}_n = \mathcal{N}_n\circ(1-\lambda_n) = 0$. For the second relation we compute

$$\begin{aligned}
b_{n+1}\circ B_n + B_{n-1}\circ b_n &= b_{n+1}\circ(1-\lambda_{n+1})\circ s_{-1}^n\circ\mathcal{N}_n + (1-\lambda_n)\circ s_{-1}^{n-1}\circ\mathcal{N}_{n-1}\circ b_n \\
&= (1-\lambda_n)\circ b'_{n+1}\circ s_{-1}^n\circ\mathcal{N}_n + (1-\lambda_n)\circ s_{-1}^{n-1}\circ b'_n\circ\mathcal{N}_n \\
&= (1-\lambda_n)(b'_{n+1}\circ s_{-1}^n + s_{-1}^{n-1}\circ b'_n)\circ\mathcal{N}_n \\
&= (1-\lambda_n)\circ\mathcal{N}_n \\
&= 0.
\end{aligned}$$

So the indicated diagram is indeed a double complex, living in nonnegative degrees. It is called the *mixed complex of* X and is denoted $MC_{**}(X)$. To show that its homology is isomorphic to $HC_*(X)$, we construct a map

$$\Phi : \mathrm{Tot}(MC_{**}(X)) \to \mathrm{Tot}CC_{**}(X),$$

which induces an isomorphism in homology. We define Φ as a map

$$MC(X)_{i,j} = X_{j-i} \to X_{j-i}\oplus X_{j-i+1} = CC_{2i,j-i}\oplus CC_{2i-1,j-i+1}$$

by $x\mapsto(x, s_{-1}^n\mathcal{N}_n(x))$. This defines a map of total complexes since it preserves the bidegree and commutes with the differentials:

$$\begin{aligned}
\Phi\circ(b_n+B_n)(x) &= \Phi(b_n(x), (1-\lambda_n)\circ s_{-1}^n\mathcal{N}_n(x)) \\
&= (b_n(x), s_{-1}^{n-1}\mathcal{N}_{n-1}b_n(x), (1-\lambda_n)\circ s_{-1}^n\mathcal{N}_n(x), 0) \\
&= (b_n(x), s_{-1}^{n-1}b'_n\mathcal{N}_n(x), (1-\lambda_n)s_{-1}^n\mathcal{N}_n(x), 0) \\
&= (b_n(x), \mathcal{N}_n(x) - b'_{n+1}s_{-1}^n\mathcal{N}_n(x), (1-\lambda_n)s_{-1}^n\mathcal{N}_n(x), 0) \\
&= \partial(x, s_{-1}^n\mathcal{N}_n(x)) \\
&= \partial\Phi(x),
\end{aligned}$$

where ∂ denotes the differential of $\mathrm{Tot}CC_{**}(X)$. Since Φ is obviously injective and its cokernel is the total complex of the b'-columns of $CC_{**}(X)$, which is acyclic, it follows that Φ induces and isomorphism in homology. $\square$

From the cyclic double complex we can construct yet another complex, denoted $C^{\lambda}_{*}(X)$, whose term in degree n is

$$C^{\lambda}_{n}(X) := X_n/\text{im }(1-\lambda).$$

The map $b_n : X_n \to X_{n-1}$ descends to a differential on this complex, since $b(1-\lambda) = (1-\lambda)b'$. Therefore there is a chain map $q : CC_{**}(X) \to C^{\lambda}(X)$ defined by sending everything to zero except the first column, on which it is the quotient map.We denote the homology of $C^{\lambda}_{*}(X)$ by $H^{\lambda}_{*}(X)$.

Proposition 2.32. *If k contains $\mathbb{Q}$ and X is a pre-cyclic k-module, then the quotient map $q : CC_{**}(X) \to C^{\lambda}_{*}(X)$ induces an isomorphism on homology. That is, $HC_n(X) \cong H^{\lambda}_{n}(X)$.*

Proof. Since k contains $\mathbb{Q}$, we have that $(1-\lambda_n)(x) = 0$ implies

$$x = \frac{1}{n+1}(\sum_{i=0}^{n} \lambda_n^i(x)) = \mathcal{N}_n(\frac{1}{n+1}x).$$

On the other hand $\mathcal{N}_n(x) = 0$ implies

$$(1-\lambda_n)(\sum_{i=1}^{n} \lambda_n^i(x)) = \sum_{i=1}^{n} \lambda_n^i(x) - nx = -(n+1)x,$$

thus

$$x = (1-\lambda_n)(\frac{-1}{n+1}\sum_{i=1}^{n} i\lambda_n^i(x)).$$

This shows that the rows of the cyclic double complex are acyclic, so by general homological algebra (cf. appendix B) q induces an isomorphism on homology. $\square$

Again , one can dualize the statement and proof to obtain a result in cohomology.

2.5 Periodic and negative cyclic homology

A priori there seems to be no reason why the cyclic double complex is chosen to live in the first quadrant of the plane. The two homologies discussed in this section deal with the complex chosen to live in the upper half plane and the second quadrant. In the first case the *periodic double complex*, denoted $CC^{per}_{**}(X)$, becomes

$$\begin{array}{ccccccccc}
 & & \downarrow{\scriptstyle -b'_3} & & \downarrow{\scriptstyle b_3} & & \downarrow{\scriptstyle -b'_3} & & \\
\cdots & \xleftarrow{1-\lambda_2} & X_2 & \xleftarrow{\mathcal{N}_2} & X_2 & \xleftarrow{1-\lambda_2} & X_2 & \xleftarrow{\mathcal{N}_2} & \cdots \\
 & & \downarrow{\scriptstyle -b'_2} & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle -b'_2} & & \\
\cdots & \xleftarrow{1-\lambda_1} & X_1 & \xleftarrow{\mathcal{N}_1} & X_1 & \xleftarrow{1-\lambda_1} & X_1 & \xleftarrow{\mathcal{N}_1} & \cdots \\
 & & \downarrow{\scriptstyle -b'_1} & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle -b'_1} & & \\
\cdots & \xleftarrow{1-\lambda_0} & X_0 & \xleftarrow{\mathcal{N}_0} & X_0 & \xleftarrow{1-\lambda_0} & X_0 & \xleftarrow{\mathcal{N}_0} & \cdots
\end{array}$$

If we take the total complex $\mathrm{Tot}(CC^{per}_{**}(X))$ then its term in degree n, which is

$$\bigoplus_{i+j=n, j\geq 0} CC_{ij}(X) = \bigoplus_{j=0}^{\infty} X_j,$$

is an infinite direct sum. If k contains $\mathbb{Q}$, the rows of $CC^{per}_{**}(X)$ are acyclic by the proof of proposition 2.32. We will show that the homology of $\mathrm{Tot}(CC^{per}_{**}(X))$ vanishes. Let

$$x = \sum_{j=0}^{n} x_j \in \bigoplus_{j=0}^{\infty} X_j,$$

with $x_j \in X_j$. If $n = 0$, then x being a cycle means that either $(1-\lambda_0)x = 0$ or $\mathcal{N}_0 x = 0$. In the first case x is an arbitrary element of X_0, since $1 - \lambda_0 = 0$. Since $\mathcal{N}_0 = 1$, we have $x = \mathcal{N}_0 x$, so x is a boundary. In the second case we have $x = 0$, so it is also a boundary. We proceed by induction on n. Suppose all cycles $x \in \bigoplus_{j=0}^{\infty} X_j$ with $x_i = 0$ for $i \geq n$ are boundaries, and let x be as above with $x_n \neq 0$. Since x is a cycle, we have either $(1-\lambda_n)x_n = 0$ or $\mathcal{N}_n x_n = 0$. Since the rows of $CC^{per}_{**}(X)$ are exact, we have either $x_n = \mathcal{N}_n y_n$ or $x_n = (1-\lambda_n)y_n$ for some $y_n \in X_n$. Define $y := (1-\lambda_n)y_n - b'_n y_n$ in the first case, $y := \mathcal{N}_n y_n + b_n y_n$ in the second case. In both cases y is a boundary, so we may replace x by $x - y$ without changing its homology class. But then $x_n = 0$, so by the inductive hypothesis x is a boundary and the homology of $\mathrm{Tot}CC^{per}_{**}(X)$ vanishes. For this reason we look at the complex whose term in degree n is

$$\prod_{i+j=n, j\geq 0} CC_{ij}(X) = \prod_{j=0}^{\infty} X_j.$$

It is denoted $\mathrm{ToT}(CC^{per}_{**}(X))$.

Definition 2.33. Let X be a pre-cyclic k-module. The *periodic cyclic homology* of X is the homology of the complex $\mathrm{ToT}(CC^{per}_{**}(X))$. It is denoted $HP_*(X)$.

It is immediate that $HP_n(X) \cong HP_{n+2}(X)$, since the complex computing HP is two-periodic. The isomorphism is given by the periodicity operator S, and is therefore natural in X.So we actually have two functors HP_0 and HP_1. A short exact sequence of pre-cyclic k-modules induces a long exact sequence in periodic cyclic homology. Using the natural isomorphism S, we obtain a periodic six term exact sequence

$$\begin{array}{ccccc} HP_0(W) & \longrightarrow & HP_0(X) & \longrightarrow & HP_0(Y) \\ \partial_0 \uparrow & & & & \downarrow \partial_1 \\ HP_1(Y) & \longleftarrow & HP_1(X) & \longleftarrow & HP_1(W) \end{array}$$

which will play a key role in this paper.
The negative theory is obtained by deleting from the periodic double complex the columns CC_{i*} for $i > 1$. This is called the *negative double complex* and denoted $CC^-_{**}(X)$.

Definition 2.34. Let X be a pre-cyclic k-module. The *negative cyclic homology* of X is the homology of the complex $\mathrm{ToT}(CC^-_{**}(X))$, whoxe term in degree n is

$$\prod_{i+j=n, i\leq 1, j\geq 0} CC_{ij}(X) = \prod_{j=\max 0, n-1}^{\infty} X_j.$$

It is denoted $HC^-(X)$.

Again the periodicity operator S is a self map of the negative complex and induces a homomorphism $S : HC^-_n(X) \to HC^-_{n-2}(X)$. S is injective on$CC^-_{**}(X)$, and its cokernel is the Hochschild complex. This gives an "SBI" sequence

$$\ldots \xrightarrow{S} HC^-_{n-1}(X) \longrightarrow HH_{n-1}(X) \longrightarrow HC^-_n(X) \xrightarrow{S} HC^-_{n-2}(X) \longrightarrow \ldots$$

relating Hochschild to negative cyclic homology.
To define periodic and negative cyclic cohomology, we dualize the complexes above in the same way as we did with Hochschild and ordinary cyclic homology. Since there is an isomorphism

$$\mathrm{Hom}_k(\bigoplus_{j\in J} X_j, k) \cong \prod_{j\in J} \mathrm{Hom}_k(X_j, k),$$

we see that if k contains $\mathbb{Q}$, the homology of the complex $\mathrm{ToT}(CC^{**}_{per}(X))$ (using direct *products*) will vanish, by dualizing the argument given for the cyclic double complex. Therefore we have the following definition.

Definition 2.35. Let X be a pre-cyclic k-module. The *periodic cyclic cohomology* of X is the homology of the complex $\mathrm{Tot} CC^{**}_{per}(X)$, whose term in degree n is

$$\bigoplus_{j\geq 0} CC^{ij}(X) = \bigoplus_{j=0}^{\infty} \mathrm{Hom}_k(X_j, k).$$

The *negative cyclic cohomology* of X is the homology of the complex $\mathrm{Tot}(CC_-^{**}(X))$, whose term in degree n is

$$\bigoplus_{i\leq 1, j\geq 0} CC^{ij}(X) = \bigoplus_{j=\max 0, n-1}^{\infty} \mathrm{Hom}_k(X_j, k).$$

One derives a six term exact sequence in periodic cyclic cohomology in the standard way.

Proposition 2.36. *Let X be a pre-cyclic k-module. There are natural isomorphisms*

$$HP_n(X) \cong \lim_{\rightarrow} HC^-_{n-2j}(X), \quad HP^n(X) \cong \lim_{\rightarrow} HC^{n+2j}(X),$$

the direct limits being taken with respect to the periodicity operator S.

Proof. We have to directed systems of abelian groups

$$\ldots \xrightarrow{S} HC_n^-(X) \xrightarrow{S} HC_{n-2}^-(X) \xrightarrow{S} \ldots$$

$$\ldots \xrightarrow{S} HC^n(X) \xrightarrow{S} HC^{n+2}(X) \xrightarrow{S} \ldots$$

We may describe their direct limits as the disjoint union of terms of the system modulo the equivalence relation generated by $y = Sx$. We saw that any cycle in $CC_*^{per}(X)$ differs by a boundary from a cycle having zeroes in the first n coordinates, for any n. Thus it is of the form $S^n(x)$ for some cycle $x \in CC^-_{**}(X)$. For the cohomology system the argument is exactly dual. □

Corollary 2.37. *Let $\phi : X \to Y$ be a map of cyclic modules. If ϕ induces an isomorphism in Hochschild (co)homology, then it induces an isomorphism in periodic cyclic (co)homology.*

Proof. ϕ induces an isomorphism in cyclic cohomology so this follows immediately the above direct limit representation. For cyclic homology there is a functorial exact sequence

$$0 \longrightarrow \lim^1 HC_n(X) \longrightarrow HP_*(X) \longrightarrow \lim_{\leftarrow} HC_n(X) \longrightarrow 0,$$

so the result follows from the fact that ϕ induces an isomrophism in cyclic homology. □

Corollary 2.38 (Morita invariance of negative and periodic cyclic homology). *Let A be a unital k-algebra. Then the generalized trace induces an isomorphism in periodic and negative cyclic (co)homology.*

Proof. The generalized trace induces an isomorphism in Hochschild homology. □

Definition 2.39. A map $\phi : X \to Y$ of cyclic modules is called a *quasi isomorphism* if it induces an isomorphism in Hochschild homology.

The construction in this chapter have their obvious analogues using $MC_{**}(X)$ for a cyclic module X, yielding isomorphic groups.

2.6 Normalization and excision

We will now turn to the question of excision. To discuss this, we must first overcome the apparent ambiguity in the definition of the cyclic type homologies for non-unital algebras. Since these homologies are defined for pre-cyclic modules, the presence of a unit is not necessary in their construction. For a non-unital algebra I, we can use the pre-cyclic module $I^\sharp$ to compute them. On the other hand, the standard way of extending a homology theory from unital to non-unital algebras is by using *relative* homology groups.

Definition 2.40. Suppose we have a non-unital algebra I that is embedded in a unital algebra A as an ideal. Consider the cyclic module $(A, I)^\sharp$, associated to the pair (A, I), as the kernel of the map $\pi_* : A^\sharp \to (A/I)^\sharp$. The *relative cyclic type homologies of I with respect to A* are the cyclic type homologies of $(A, I)^\sharp$.

This module fits into exact sequences in the cyclic type homologies, for

$$0 \longrightarrow (A, I)^\sharp \xrightarrow{i_*} A^\sharp \xrightarrow{\pi_*} (A/I)^\sharp \longrightarrow 0$$

is by construction an exact sequence of cyclic modules. Now to define the cyclic type homologies for I, we consider the homologies of the cyclic module $(I_+, I)^\sharp$, where I_+ is the unitization of I as a k-algebra. Fortunately, these definitions coincide and this is in fact a consequence of the following general facts about simplicial modules and the mixed complex $MC_{**}(X)$ associated to a cyclic module.

Definition 2.41. Let X be a simplicial module. The complex of *degenerate elements* $D(X)$ is given by

$$D_n(X) := \operatorname{span}_k\{\operatorname{im} s_i^{n-1} : 0 \leq i \leq n-1\}.$$

Its *normalization* is the complex $N(X) := X/D(X)$.

Theorem 2.42. *Let X be a simplicial module. The complex $D(X)$ is b-contractible, that is, the quotient map*

$$X \to N(X)$$

induces an isomorphism in homology.

Proof. We will construct a filtration F_i, $i \in \mathbb{N}$,

$$0 = F_0 \subset F_1 \subset ... \subset D(X),$$

by subcomplexes with the properties that

- $F_p \cap D_n(X) = D_n(X)$ for $n \leq p$,
- the quotients F_i/F_{i-1} are acyclic.

Since $F_0 = 0$ is acyclic and for each i there is an exact sequence

$$0 \longrightarrow F_{i-1} \longrightarrow F_i \longrightarrow F_i/F_{i-1} \longrightarrow 0$$

it follows that each F_i is acyclic. By the first property of the filtration it then follows that $D(X)$ is acyclic. Define

$$(F_p)_n : \mathrm{span}_k\{\mathrm{im}\ s_i^{n-1} : 0 \leq i \leq \min\{p, n\}\}.$$

This is a subcomplex because of the simplicial identities and it clearly satisfies the first condition. To prove the second one we define a contracting homotopy

$$\begin{aligned} H : (F_i/F_{i-1})_n &\to (F_i/F_{i-1})_{n+1} \\ [x] &\mapsto (-1)^i[s_i^n(x)] \end{aligned}.$$

To prove this, we first compute

$$\begin{aligned} bs_i + s_ib &= (\sum_{j=0}^{n+1}(-1)^j d_j^{n+1})s_i^n + s_i^{n-1}(\sum_{j=0}^{n}(-1)^j d_j^n) \\ &= \sum_{j=0}^{i}(-1)^j s_{i-1}^{n-1} d_j^n + \sum_{j=i+1}^{n}(-1)^j s_i^{n-1} d_{j+1}^n + s_i^{n-1}(\sum_{j=0}^{n}(-1)^j d_j^n) \\ &= \sum_{j=0}^{i}(-1)^j (s_{i-1}^{n-1} + s_i^{n-1}) d_j^n. \end{aligned}$$

This last expression equals $\sum_{j=0}^{i}(-1)^j s_i^{n-1} d_j^n$ modulo F_{i-1}. An element $[x] \in (F_i/F_{i-1})_n$ can be represented by $[s_i^{n-1}(y)]$ for some $y \in (F_i/F_{i-1})_{n-1}$. Combining these observations gives

$$\begin{aligned} (Hb + bH)[x] &= [(-1)^i(\sum_{j=0}^{i}(-1)^j s_i^{n-1} d_j^n) s_i^{n-1}(y)] \\ &= [(-1)^i(\sum_{j=0}^{i-1}(-1)^j s_i^{n-1} s_{i-1}^{n-2} d_j^{n-1}(y)) + s_i^{n-1}(y)] \\ &= [(-1)^i(\sum_{j=0}^{i-1}(-1)^j s_{i-1}^{n-1} s_{i-1}^{n-2} d_j^{n-1}(y)) + x] \\ &= [x]. \qquad \square \end{aligned}$$

Note that $N(X)$ is not a cyclic k-module. The cyclic operators t_n do not descend to $N(X)$, for $t_n s_n \notin D(X)$ (think of the algebra case!). The mixed complex of X, however, does behave well with respect to normalization.

Proposition 2.43. *For any cyclic k-module X, the Connes B-operator descends to $N(X)$, such that the cyclic homology of X is isomorphic to the homology of the the double complex*

$$
\begin{array}{ccccc}
\downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 \\
N(X_2) & \xleftarrow{B_1} & N(X_1) & \xleftarrow{B_0} & N(X_0) \\
\downarrow b_2 & & \downarrow b_1 & & \\
N(X_1) & \xleftarrow{B_0} & N(X_0) & & \\
\downarrow b_1 & & & & \\
N(X_0). & & & &
\end{array}
$$

Proof. It suffices to show that B descends, since then the normalization map is a map of double complexes that induces an isomorphism on the columns, and hence by general homological algebra, it induces an isomorphism on the total complexes.
Since $\lambda_n s^n_{-1} = \lambda^2_n s^n_n = s^n_0 \lambda_n$, whose image lies in $D(X)$, we only have to show that $D(X)$ is stable for $s^n_{-1}\mathcal{N}_n$. Thus for $0 \le i \le n-1$ we compute

$$
\begin{aligned}
s^n_{-1}\mathcal{N}_n s^{n-1}_i &= s^n_{-1}(\sum_{j=0}^{n} \lambda^j_n) s^{n-1}_i \\
&= s^n_{-1}(\sum_{j=0}^{n-1-i} s^{n-1}_{i+j}\lambda^j_{n-1} + \sum_{j=1}^{i+1} \lambda^j_n s^{n-1}_{n-1}\lambda^{n-1-i}_{n-1}) \\
&= \sum_{j=0}^{n-1-i} \lambda_{n+1} s^n_n s^{n-1}_{i+j}\lambda^j_{n-1} + \sum_{j=1}^{i+1} \lambda_{n+1} s^n_n \lambda^j_n s^{n-1}_{n-1}\lambda^{n-1-i}_{n-1} \\
&= \sum_{j=0}^{n-1-i} \lambda_{n+1} s^n_{i+j} s^{n-1}_{n-1}\lambda^j_{n-1} + \sum_{j=1}^{i+1} \lambda^{j+1}_{n+1} s^n_{n-j} s^{n-1}_{n-1}\lambda^{n-1-i}_{n-1} \\
&= \sum_{j=0}^{n-1-i} s^n_{i+j+1}\lambda_n s^{n-1}_{n-1}\lambda^j_{n-1} + \sum_{j=1}^{i+1} \lambda^{j+1}_{n+1} s^n_n s^{n-1}_{n-j}\lambda^{n-1-i}_{n-1} \\
&= \sum_{j=0}^{n-1-i} s^n_{i+j+1}\lambda_n s^{n-1}_{n-1}\lambda^j_{n-1} + \sum_{j=1}^{i+1} s^n_{n-j}\lambda^{j-1}_n s^{n-1}_{n-j}\lambda^{n-1-i}_{n-1},
\end{aligned}
$$

and the image of the last expression is in $D(X)$. □

Corollary 2.44. *For any not-necessarily unital algebra I, the cyclic type (co)homologies of the pre-cyclic modules $(I_+, I)^\sharp$ and $I^\sharp$ are naturally isomorphic.*

Proof. Since $(I_+, I)^\sharp$ is a cyclic module, its cyclic homology is the homology of $MC_{**}(N(I_+, I)^\sharp)$ by proposition 2.43 and its Hochschild homology is the b-homology of $N(I_+, I)^\sharp$ by theorem 2.42. It is not difficult to see that $N(I_+, I)^\sharp$ isomorphic to $I_+ \otimes I^{\otimes n}$ in degree $n > 0$ and to I in degree 0. Define a map $\gamma : I_* \oplus I_{*-1} \to N(I_+, I)_*$ by

$$(i_0 \otimes ... \otimes i_n, j_1 \otimes ... \otimes j_n) \mapsto (i_0, 0) \otimes i_1 \otimes ... \otimes i_n + (0, 1) \otimes j_1 \otimes ... \otimes j_n.$$

It is straightforward to check that this induces a chain map from the total complex of $CC_{**}(I)$ to the total complex of $MC_{**}(N(I_+, I)^\sharp)$. It induces an isomorphism in homology because there is an inverse chain map $s : N(I_+, I)_* \to I_* \oplus I_{*-1}$ defined by

$$(i_0, x_0) \otimes i_1 \otimes ... \otimes i_n \mapsto ((i_0 \otimes ... \otimes i_n), x_0 i_1 \otimes ... \otimes i_n).$$

This is also checked by straightforward calculation. This gives isomorphisms for HH and HC, and since the same procedure can be applied to the complexes computing HP and HC^- and the cohomology groups, the statement follows. $\square$.

The value of corollary 2.44 should not be underestimated. It tells us that the cyclic theory for algebras can be carried out entirely using cyclic (instead of pre-cyclic) modules. This will become apparent in the next section.
The canonical map $j : I_+ \to A$, given by $(i, x) \mapsto i + x$ is an algebra map, and thus induces a map $j_* : I_+^\sharp \to A^\sharp$ of cyclic modules. It is clear that it maps $(I_+, I)^\sharp$ to $(A, I)^\sharp$. The question is, when is j_* a quasi-isomorphism? The following class of algebras deserves our attention.

Definition 2.45. Let I be a (not necessarily) unital k-algebra. I is called *H-unital* if the complex $(I^\sharp, b')$ is acyclic.

Without proof we state the main result on H-unitality.

Theorem 2.46 (Wodzicki). *Let I be a not necessarily unital algebra. The following are equivalent*

- *I is H-unital,*
- *For any algebra A, such there is an embedding $I \hookrightarrow A$ as a two sided ideal, the map $\phi_* : (I_+, I)^\sharp \to (A, I)^\sharp$ is a quasi-isomorphism.*

This is proved in [26]. A lot of algebras occurring in applications are H-unital, although not all of them. The following result, obtained by Cuntz and Quillen in their monumental paper [10], tells us that for the periodic theory, the situation is much more favourable.

Theorem 2.47 (Cuntz-Quillen). *Let A be an algebra over a field k of characteristic zero and $I \subset A$ a two-sided ideal. The map $j_* : (I_+, I)^\sharp \to (A, I)^\sharp$ induces an isomorphism in periodic cyclic (co)homology.*

The Cuntz-Quillen theorem is in fact far more general, for they proof excision in both variables for a certain bivariant theory $HP^*(A,B)$, with the properties that $HP^*(A,k) \cong HP^*(A)$ and $HP^*(k,B) \cong HP_*(B)$. The excision theorem implies that the periodic theory is "truly" Morita invariant.

Corollary 2.48. *Let I be a not-necessarily unital algebra. Then the generalized trace induces an isomorphism in periodic cyclic (co)homology.*

Proof. This follows by examining six-term exact sequence associated to the split exact sequence

$$0 \longrightarrow I \longrightarrow I_+ \longrightarrow k \longrightarrow 0$$

of algebras and then using excision and Morita invariance of I_+ and k, which are unital. □

Excision holds only in a limited sense for continuous periodic cyclic homology HP_c. Cuntz adapted the proof of theorem 2.47 in [9], to establish excision for HP_c on the category of m-algebras (see appendix A).

Theorem 2.49 (Cuntz). *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

be an extension of m-algebras that admits a continuous linear splitting. Then the inclusion $j_* : (I_+, I)^\sharp \to (A, I)^\sharp$ induces an isomorphism in continuous periodic cyclic (co)homology.

For future reference, we also include the following theorem by Goodwillie, on which the excision theorem depends in a crucial way. Recall that an ideal $I \subset A$ in an algebra is called *nilpotent* if there exists $n \in \mathbb{N}$ for which $I^n = 0$.

Theorem 2.50 (Goodwillie). *Let I be a nilpotent ideal in an algebra A over a field of characteristic zero. Then the quotient map $\pi : A \to A/I$ induces an isomorphism in periodic cyclic (co)homology.*

This is proved in [13]. We do not include a proof since we will only need to refer to this theorem sideways.

2.7 Differential forms

We are now ready to relate the developed cyclic theory to ordinary topology and geometry. To be able to state the main results on this relation, we need a generalization of the concept of differentiable forms on a manifold or algebraic variety.

Definition 2.51. Let A be a k-algebra. A *differential graded algebra over A* is a graded k-algebra $\Omega := \bigoplus_{n=0}^{\infty} \Omega^n$, together with a degree 1 map $d : \Omega^* \to \Omega^{*+1}$, satisfying $d^2 = 0$ and $d(xy) = d(x)y + (-1)^{deg x} x d(y)$. A *differential calculus over A* is a differential graded algebra Ω^* together with an algebra map $\rho : A \to \Omega^0$.

For a manifold M, the algebra $\Omega^*(M)$ of ordinary differential forms on M, is a differential calculus over $C^\infty(M) = \Omega^0(M)$. Similarly, for a smooth algebraic variety V over $\mathbb{C}$, the algebra $\Omega(V)$ of *algebraic* forms over V is a differential calculus over the ring $\mathcal{O}(V)$ of regular functions on V. Note that if we consider V as a smooth manifold, these two calculi need not coincide.

Proposition 2.52. *Every algebra A has a canonical differential calculus $(\Omega^*(A), d, \rho)$, with the property that for any other differential calculus (Ω'^*, d', ρ') over A, there is a unique map $f : \Omega^*(A) \to \Omega'^*$ of graded algebras satisfying $f \circ d = d' \circ f$ and $f \circ \rho = \rho'$.*

Proof. First assume that A is unital. Denote by $\overline{A}$ the quotient (as a linear space)$A/k.1$ and define

$$\Omega_u^n(A) : A \otimes \overline{A}^{\otimes n}, \qquad d(a_0 \otimes \overline{a_1} \otimes ... \otimes \overline{a_n}) := 1 \otimes \overline{a_0} \otimes ... \otimes \overline{a_n},$$

and we write

$$a_0 \otimes \overline{a_1} \otimes ... \otimes \overline{a_n} = a_0 da_1 ... da_n.$$

To make d satisfy the Leibniz rule the product in $\Omega_u^n(A)$ must be defined by

$$\begin{aligned}
&(a_0 da_1 ... da_n)(b_0 db_1 ... db_m) := \\
&= a_0 da_1 ... d(a_n b_0) db_1 ... db_m \\
&\quad + \sum_{i=1}^{n-1} a_0 da_1 ... d(a_{n-i} a_{n-i+1}) ... da_n db_0 ... db_m, \\
&\quad + (-1)^n a_0 a_1 da_2 ... da_n db_0 ... db_m.
\end{aligned}$$

which is consistent with the notation. Since $\Omega_u^0(A) = A$, we take $\rho = \mathrm{id}$, and given another calculus (Ω', d', ρ'), f is completely determined by the conditions $f = \rho'$ in degree 0 and $f \circ d = d' \circ f$. If A does not have a unit, we define

$$\Omega^*(A) := \ker(\Omega_u^*(A_+) \to \Omega_u^*(k)).$$

Then since $\Omega_u(k) = k$, concentrated in degree 0, we have $\Omega^0(A) = A$ and $\Omega^n(A) \cong A_+ \otimes A^{\otimes n}$. Then given a calculus Ω' over A, Ω'_+ (where the adjunction is in degree 0) is a calculus over A_+ and we get a map $f : \Omega_u^*(A_+) \to \Omega'_+$, whose restriction to $\Omega^*(A)$ lands in Ω' and is the desired map. □

Note that for a unital algebra, we have defined two 'canonical' differential calculi. The difference is that $\Omega_u^*(A)$ is universal in the unital category, while $\Omega^*(A)$ is universal in the non-unital category. We will work mainly with $\Omega^*(A)$. Viewing $(\Omega^*(A), d)$ as a complex, one could be interested in the homology of this complex, but it is quite easy to see that this vanishes. There is however a way to construct an interesting homology out of this complex. First of all, if $\omega \in \Omega^n(A)$ and $\eta \in \Omega^m(A)$, their *graded commutator* is the form

$$[\omega, \eta] := \omega\eta - (-1)^{nm} \eta\omega \in \Omega^{n+m}(A).$$

The linear space spanned by all graded commutators is denoted $[\Omega^*(A), \Omega^*(A)]$. Since

$$\begin{aligned} d[\omega, \eta] &= d(\omega)\eta + (-1)^n \omega d(\eta) - (-1)^{nm} d(\eta)\omega - (-1)^{(n+1)m} \eta d(\omega) \\ &= [d(\omega), \eta] + (-1)^n [\omega, d\eta], \end{aligned}$$

d descends to $\Omega^*_\sharp(A) := \Omega^*(A)/[\Omega^*(A), \Omega^*(A)]$.

Definition 2.53. Let A be a k-algebra. The *noncommutative De Rham homology* of A is the homology of the complex $(\Omega^*_\sharp(A), d)$.

The isomorphisms $\Omega^n(A) \cong A_+ \otimes A^{\otimes n} \cong A^{\otimes n+1} \oplus A^{\otimes n}$ (as k-modules) for $n \geq 1$ and $\Omega^0(A) = A$ tell us that $\Omega^*(A) = N(A_+, A)^\sharp$. Therefore we have operators B and b on $\Omega^*(A)$. To describe these operators on forms, we need to define an operator $\kappa : \Omega^*(A) \to \Omega^*(A)$, known as *Karoubi's operator*. It is given by $\kappa(\omega da) = (-1)^{deg \omega} da\omega$.

Lemma 2.54. *For $\omega \in \Omega^n(A)$ and $a \in A$, we have the following equalities:*

$$B(\omega) = \sum_{i=0}^{n} \kappa^i d\omega, \qquad b(\omega da) = (-1)^n [\omega, a].$$

Proof. We have an isomorphism of total complexes $\text{Tot}MC_{**}(A) \cong \text{Tot}N(A_+, A)$, which shows that b and B have the matrix representation

$$b = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \mathcal{N}_n & 0 \end{pmatrix}$$

with respect to the decomposition $\Omega^n(A) \cong A^{\otimes n+1} \oplus A^{\otimes n}$. It is clear that for forms $da_0...da_n$, the action of $\sum_{i=0}^n \kappa^i$ corresponds to the action of $\mathcal{N}_n$ on $A^{\otimes n}$. Since d has matrix representation $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we have

$$B = \mathcal{N}_n \circ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and thus $B = \sum_{i=0}^n \kappa^i d$. For b, we have

$$\begin{aligned} b(a_0 da_1 ... da_n da) &= a_0 a_1 da_2 ... da_n da + \sum_{i=1}^{n} (-1)^i a_0 da_1 ... d(a_i a_{i+1}) da_{i+2} ... da_n \\ &\quad + (-1)^{n+1} a a_0 da_1 ... da_{n-1} \\ &= (-1)^n ((a_0 da_1 ... da_n) a - (-1)^n a a_0 da_1 ... da_n \end{aligned},$$

by applying the Leibniz rule repeatedly. For forms $da_1...da_n da$ a similar calculation can be made. □

Any complex can be regarded as a mixed complex with $b = 0$. In particular, we can consider $(\Omega^*_\sharp(A), d, 0)$ as a mixed complex. Define a map

$$\begin{aligned} f_A : \Omega^*(A) &\to \Omega^*_\sharp(A) \\ a_0 da_1 ... da_n &\mapsto \frac{1}{n!}\rho(a_0) d\rho(a_1) ... d\rho(a_n) \end{aligned}.$$

We claim that this map commutes with the differentials. Since κ acts as the identity modulo graded commutators, we see that B becomes $(n+1)d$ and that $f \circ B = (n+1) f \circ d = d \circ f$. For b, it is immediate that $f \circ b = 0$.
Now let V be a smooth complex affine algebraic variety and $\Omega^*(V)$ its module of algebraic forms. This is the exterior algebra $\bigwedge_{\mathcal{O}(V)} \Omega^1_{\mathbb{C}}(V)$, where $\Omega^1_{\mathbb{C}}(V)$ is the module generated by the expresions f and df, $f \in \mathcal{O}(V)$, subject to the usual Leibniz rules. There is a canonical map $\chi_V : \Omega^*(\mathcal{O}(V)) \to \Omega^*(V)$, by universality of $\Omega^*(\mathcal{O}(V))$. This map factors through the quotient map $\Omega^*(\mathcal{O}(V)) \to \Omega^*_\sharp(\mathcal{O}(V))$, since $\Omega^*(V)$ is graded commutative. So we have a canonical map $\chi'_V : \Omega^*_\sharp(\mathcal{O}(V)) \to \Omega^*(V)$. Let $\chi := \chi'_V \circ f_{\mathcal{O}(V)} : \Omega^*(\mathcal{O}(V)) \to \Omega^*(V)$.

Theorem 2.55 (Hochschild-Konstant-Rosenberg). *The map*

$$\chi : (\Omega^*(\mathcal{O}(V)), B, b) \to (\Omega^*(V), d, 0)$$

is a quasi isomorphism of mixed complexes.

From this it is immediate that $HP_*(\mathcal{O}(V)) \cong \bigoplus_{i=0}^{\infty} H^{*+2i}_{DR}(V)$, the algebraic DeRham cohomology of V. Actually, the theorem asserts that χ induces an isomorphism on the homology of the columns, thus $HH_n(\mathcal{O}(V)) \cong \Omega^n(V)$, but by general homological algebra, it also induces an isomorphism on the total complexes. Since the complex $\ker \chi$ has acyclic columns and $\mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ is an exact functor it follows that $\chi^* : \mathrm{Hom}_{\mathbb{C}}(\Omega^*(V), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(\Omega^*(\mathcal{O}(V), \mathbb{C})$ is also a quasi isomorphism of mixed complexes.
For a smooth manifold M, there is an analogous result, using continuous forms and homology. This means that in the construction of $\Omega^*(C^\infty(M))$, we replace the algebraic by the complete projective tensor product, and we obtain in the same way a map $\chi : \Omega^*_c(C^\infty(M)) \to \Omega^*(M)$. Recall that $\Omega^*(M) = \bigwedge_{C^\infty(M)} \Omega^1(M)$, and $\Omega^1(M)$ is the module of sections of the cotangent bundle of M.

Theorem 2.56 (Connes). *The map*

$$\chi : (\Omega^*_c(C^\infty(M)), B, b) \to (\Omega^*(M), d, 0)$$

is a quasi isomorphism of mixed complexes.

A proof can be found in [4]. For complex manifolds N, the algebra of of holomorphic functions $\mathcal{O}(N)$ is a locally convex algebra, and there is a map $\chi : \Omega^*_c(\mathcal{O}(N)) \to \Omega^*(N)$ to the differential calculus of holomorphic forms. We have (cf.[8])

Theorem 2.57. *The map*

$$\chi : (\Omega^*_{top}(\mathcal{O}(N)), B, b) \to (\Omega^*(N), d, 0)$$

is a quasi-isomorphism of mixed complexes.

These theorems provide us with a powerful tool to actually compute the cyclic type homologies for various algebras. It is immediate that for a smooth manifold M, $HH_*(C^\infty(M)_c)^\sharp \cong \Omega^*(M)$ and $HP_*(C^\infty(M)_c^\sharp) \cong \bigoplus_{i=0}^\infty H_{DR}^{*+2i}(M)$. These are important motivations for studying cyclic homology, since we produced the De Rham complex and -cohomology without using commutativity of $C^\infty(M)$. We may thus regard periodic cyclic homology as an extension of De Rham cohomology to the category of $\mathbb{C}$-algebras.
For future reference, we'll now have a closer look at the periodic cyclic (co)homology of the algebra $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials with complex coefficients. This is the coordinate ring of the variety

$$V := \{(z, w) \in \mathbb{C}^2 : f(z, w) := zw - 1 = 0\},$$

which is smooth because $\frac{\partial f}{\partial z} = w, \frac{\partial f}{\partial w} = z$ and these do only vanish simultaneously at $(0, 0)$, which is not in V. Since $\Omega^k(V) = 0$ for $k \geq 2$, $HP_*(\mathbb{C}[z, z^{-1}])$ is computed by the complex

$$\ldots \xleftarrow{0} \Omega^1(V) \xleftarrow{d} \Omega^0(V) \xleftarrow{0} \Omega^1(V) \xleftarrow{d} \Omega^0(V) \xleftarrow{0} \ldots$$

where $\Omega^0(V)$ is in the even degrees. Of course $\Omega^0(V) = \mathbb{C}[z, z^{-1}]$ and the kernel of d are precisely the constants. Since evidently the image of 0 is 0, it follows that $HP_0(\mathbb{C}[z, z^{-1}]) \cong \mathbb{C}$. In odd degrees, we have $\Omega^1(V) \cong \mathbb{C}[z, z^{-1}]dz \cong \mathbb{C}[z, z^{-1}]$ (as linear spaces), and the kernel is everything. The image of d under this isomorphism is $\text{span}_{\mathbb{C}}\{z^n : n \in \mathbb{Z} \setminus \{-1\}\}$. Thus in this case the residue map $\sum a_i z^i \mapsto a_{-1}$ induces an isomorphism $HP_1(\mathbb{C}[z.z^{-1}]) \cong \mathbb{C}$. Again, dualization yields the same result in cohomology.

2.8 Cyclic cohomology and Ext^n_Λ

When the ground ring k is a field, one can interpret the cyclic type homologies as derived functors in the category of cyclic k-modules. This insight is due to Alain Connes, and the discussion in this paragraph can be found in [6].
Thus from now on, k is a field. Viewing k as a k-algebra, we obtain the trivial cyclic module $k^\sharp$ cf. paragraph 1.2, since k is unital. We will construct a biresolution of this cyclic module by injective cyclic k-modules. Then applying $\text{Hom}_\Lambda(X, -)$ to this resolution and taking homology of the resulting complex will give us $\text{Ext}^n_\Lambda(X, k^\sharp)$. Let

$$P^m := \bigoplus_{j=0}^{\infty} k[\text{Mor}_\Lambda([m], [j])],$$

the graded k-module which in degree j is the free k-module on the set $\mathrm{Mor}_\Lambda([m],[j])$. P^m becomes a cyclic k-module by defining

$$\phi_*(f) := \phi \circ f$$

for $\phi \in \mathrm{Mor}_\Lambda([j],[n])$ and $f \in \mathrm{Mor}_\Lambda([m],[j])$ and extending by linearity. Denote by P_m its dual, given by $P_m^j := \mathrm{Hom}_k(P_j^m, k)$.

Lemma 2.58. *P^m is a projective and P_m an injective cyclic k-module.*

Proof. Cf. the remark after proposition 1.2, to show that P_m is injective, we must show that the functor $\mathrm{Mor}_\Lambda(-, P_m)$ is exact. To prove projectivity of P^m, one dualizes the following argument. Let E be an arbitrary cyclic k-module, $e_j \in E_j$, $\phi \in \mathrm{Mor}(E, P_m)$ and $f \in P_j^m$ a generator. We have

$$\phi(e_j)(f) = \phi(e_j)(f \circ \mathrm{id}_m) = (f^*\phi(e_j))(\mathrm{id}_{[m]}) = \phi(f^*e_j)(\mathrm{id}_{[m]}),$$

so ϕ is completely determined by the functional $\chi_\phi : e_m \mapsto \phi(e_m)(\mathrm{id}_{[m]})$ on E_m. The map

$$\begin{aligned} \mathrm{Mor}_\Lambda(E, P_m) &\to \mathrm{Hom}_k(E_m, k) \\ \phi &\mapsto \chi_\phi \end{aligned}$$

is an isomorphism (of groups). Its inverse is $\chi \mapsto \phi_\chi$, where ϕ_χ is defined by

$$\phi_\chi(e_j)(f) := \chi(f^*e_j).$$

Now if

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

is an exact sequence of cyclic modules, then applying $\mathrm{Mor}_\Lambda(-, P_m)$ and this isomorphism yields

$$0 \longrightarrow \mathrm{Hom}_k(G_m, k) \to \mathrm{Hom}_k(F_m, k) \to \mathrm{Hom}_k(E_m, k) \longrightarrow 0.$$

Since the maps are just the maps induced by the restriction of the original maps to degree m, and $\mathrm{Hom}_k(-, k)$ is exact, this sequence is exact and we are done. $\square$

Note that k being a field was only needed to assure that $\mathrm{Hom}_k(-, k)$ is an exact functor. The cyclic modules P^m are projective for any commutative and unital ground ring k.

Next we construct a *projective* biresolution of $k^\sharp$ using the P^m's. This construction resembles the cyclic double complex in a strong way. Therefore we will use the same notation for its maps. Define $b_m, b'_m : P^m \to P^{m-1}$ by

$$\begin{aligned} b_m : f &\mapsto \sum_{i=0}^{m} (-1)^i f \circ \delta_i^m \\ b'_m : f &\mapsto \sum_{i=0}^{m-1} (-1)^i f \circ \delta_i^m \end{aligned}$$

and $\lambda_m : P^m \to P^m$ by

$$\lambda_m : f \mapsto (-1)^m f \circ \tau_m.$$

Then, as expected $\mathcal{N}_m := \sum_{i=0}^m (-1)^i t_m^i$.

Proposition 2.59. *The homology of the double complex*

$$\begin{array}{ccccccccc}
\downarrow b_3 & & \downarrow -b'_3 & & \downarrow b_3 & & \downarrow -b'_3 & & \\
P^2 & \xleftarrow{1-\lambda_2} & P^2 & \xleftarrow{\mathcal{N}_2} & P^2 & \xleftarrow{1-\lambda_2} & P^2 & \xleftarrow{\mathcal{N}_2} & \cdots \\
\downarrow b_2 & & \downarrow -b'_2 & & \downarrow b_2 & & \downarrow -b'_2 & & \\
P^1 & \xleftarrow{1-\lambda_1} & P^1 & \xleftarrow{\mathcal{N}_1} & P^1 & \xleftarrow{1-\lambda_1} & P^1 & \xleftarrow{\mathcal{N}_1} & \cdots \\
\downarrow b_1 & & \downarrow -b'_1 & & \downarrow b_1 & & \downarrow -b'_1 & & \\
P^0 & \xleftarrow{1-\lambda_0} & P^0 & \xleftarrow{\mathcal{N}_0} & P^0 & \xleftarrow{1-\lambda_0} & P^0 & \xleftarrow{\mathcal{N}_0} & \cdots
\end{array}$$

vanishes in positive degrees and equals $k^\sharp$ in degree zero.

Proof. The fact that this is a double complex follows by the same reasoning as for the cyclic double complex. To show that it has vanishing homology, it suffices to show that the rows are exact and the b-columns are exact modulo im $(1-\lambda_m)$ (cf. the proof of proposition 2.32). By lemma 2.2, each $f \in \mathrm{Mor}_\Lambda([m],[n])$ can be written as $f = \psi \circ \tau_m^\ell$. Since $\mathrm{Aut}_\Lambda([m]) \cong \mathbb{Z}/(m+1)\mathbb{Z}$, we see that P^m is a free $k(\mathbb{Z}/(m+1)\mathbb{Z})$ module with basis Δ. Thus each $x \in P^m$ can be written

$$x = \sum_{i=0}^n \phi_i (\sum_{j=0}^m a_{ij} \tau_m^j).$$

Now $(1-\lambda_m)(x) = 0$ if and only if for all i

$$\phi_i(a_{i0} - (-1)^k a_{im})\tau_k^0 + \sum_{j=1}^m \phi_i(a_{ij} - (-1)^m a_{i(j-1)})\tau_m^j = 0$$

and since $k(\mathbb{Z}/(m+1)\mathbb{Z})$ is free over k, this holds if and only if for all i

$$a_{i0} = (-1)^m a_{im} \quad \text{and } a_{ij} = (-1)^m a_{i(j-1)}.$$

It follows that

$$\sum_{j=0}^m a_{ij}\tau_m^j = \mathcal{N}_m(a_{i0}\tau_m^0)$$

and

$$x = \mathcal{N}_m(\sum_{i=0}^m \phi_i a_{i0} \tau_m^0).$$

Thus $\ker(1-\lambda_m) \subset \mathrm{im}\,(\mathcal{N}_m)$. One uses a similar argument to show that the other inclusion holds as well. Thus the rows are exact. We are left with the complex

$$\dots \xrightarrow{b_{m+1}} P^m/(1-\lambda_m) \xrightarrow{b_m} P^{m-1}/(1-\lambda_{m-1}) \xrightarrow{b_{m-1}} \dots$$

Using lemma 2.2 again, it follows that

$$P^m/(1-\lambda_m) \cong \bigoplus_{j=0}^{\infty} k[\mathrm{Mor}_\Delta([m],[j])]$$

and the complex above is in fact a family of complexes

$$\dots \xrightarrow{b_{m+1}} k[\mathrm{Mor}_\Delta([m],[j])] \xrightarrow{b_m} k[\mathrm{Mor}_\Delta([m-1],[j])] \xrightarrow{b_{m-1}} \dots$$

indexed by the $\mathbb{N}$-grading of the P^m's. This is exactly the complex computing the homology of the simplicial module $k[\mathrm{Mor}_\Delta(-,[j])]$, whose geometric realization is the j-simplex. The homology of the complex is isomorphic to the topological homology of j-simplex, and this vanishes in positive degrees for all j. In degree 0 it equals k, so the homology of the total complex is $k^\sharp$. □

Applying the exact functor $\mathrm{Hom}_k(-,k)$ to this projective resolution yields an injective resolution of $k^\sharp$, since $\mathrm{Hom}_k(k^\sharp,k) \cong k^\sharp$. Then, for a given cyclic k-module X, the groups $\mathrm{Ext}^n_\Lambda(X,k^\sharp)$ are computed as the homology of the complex obtained by applying $\mathrm{Mor}_\Lambda(X,-)$ to the above injective resolution.

Theorem 2.60. *For any cyclic k-module X the groups $HC^n(X)$ and $\mathrm{Ext}^n_\Lambda(X,k^\sharp)$ are isomorphic.*

Proof. We show that the complexes computing these groups are isomorphic, whence the result follows. We saw that

$$\mathrm{Mor}_\Lambda(X,P_n) \cong \mathrm{Hom}_k(X_n,k).$$

Since the maps on the cyclic double complex are defined using the (covariant) cyclic module structure on $\mathrm{Hom}(X,k)$, it suffices to show that for $\chi : X_n \to k$, $f:[n]\to[m]$, $x_n \in X_n$ and $g \in P^m_n$ we have

$$\phi_{f_*\chi}(x_n)(g) = \phi_\chi(x_n)(gf).$$

We calculate

$$\begin{aligned}\phi_{f_*\chi}(x_n)(g) &= (f_*\chi)(g^*x_n)\\ &= \chi(f^*g^*x_n)\\ &= \phi_\chi(x_n)(gf). \quad \square\end{aligned}$$

One can of course also show that $HC_n(X) \cong \mathrm{Tor}^\Lambda_n(k^\sharp,X)$, and even that $HH^n(X) \cong \mathrm{Ext}^n_\Delta(X,k^\sharp)$ and $HH_n(X) \cong \mathrm{Tor}^\Delta_n(k^\sharp,X)$, but since we will not

use these isomorphisms, we will not discuss them here. For the Tor-groups, no extra assumption on k is needed.
All proves in this section extend to the case of locally convex cyclic objects, by replacing the duals by topological duals.

2.9 Products

With the Ext interpretation of cyclic cohomology, we can introduce two products and we will use these to give a canonical description of the shift operator S on the Ext-groups. These products allow us to prove homotopy invariance of periodic cyclic homology. This homotopy invariance is more restrictive than it is in topological K-theory, for HP is only invariant under differentiable homotopies.
The external product is the analogue ofthe wedge product in DeRham cohomology, and we will prove that it behaves well with respect to the boudnary map. This will be useful in the proof of the index theorem.
Cf. Appendix B, the group $\mathrm{Ext}^n_\Lambda(X,Y)$ is isomorphic to the group of equivalence classes of n-extensions $(X_i)_{i=0}^{n-1}$

$$0 \longrightarrow Y \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_0 \longrightarrow X \longrightarrow 0$$

of X by Y. If we consider an element $[y] = (Y_i)_{i=0}^{m-1}$ of $\mathrm{Ext}^m_\Lambda(Y,Z)$, and an element $[x] = (X_j)_{j=0}^{n-1}$ of $\mathrm{Ext}^n_\Lambda(X,Y)$, we can form an $m+n$ extension $[y \circ x]$ of X by Z, by splicing the extensions together: Since $\psi_0 : Y_0 \to Z$ is surjective and Z embeds in X_0, we can regard ψ_0 as a map from Y_0 to X_{n-1}, without disturbing exactness. This construction defines an associative product

$$\begin{array}{c} \mathrm{Ext}^n_\Lambda(X,Y) \otimes_{\mathbb{Z}} \mathrm{Ext}^m_\Lambda(Y,Z) \to \mathrm{Ext}^{n+m}_\Lambda(X,Z) \\ x \otimes y \mapsto y \circ x \end{array},$$

called the *Yoneda* product. In particular, $\mathrm{Ext}^*_\Lambda(k^\sharp,k^\sharp)$ is a $\mathbb{Z}$-graded ring with this product and for any X, $\mathrm{Ext}^*_\Lambda(X,k^\sharp)$ is a $\mathbb{Z}$-graded left $\mathrm{Ext}^*_\Lambda(k^\sharp,k^\sharp)$-module. There is a good reason to find this module structure particularly interesting. To describe the action of S on the groups $\mathrm{Ext}^n_\Lambda(X,k^\sharp)$, we need an injective resolution of $k^\sharp$ by cyclic modules. This resolution is provided by proposition 2.59 and taking the total complex. We will denote this resolution by

$$k^\sharp \xrightarrow{\ i\ } \mathcal{I}_0 \xrightarrow{\ d_0\ } \mathcal{I}_1 \xrightarrow{\ d_1\ } \mathcal{I}_2 \xrightarrow{\ d_2\ } \ldots$$

Lemma 2.61. *The $\mathbb{Z}$-graded rings $\mathrm{Ext}^*_\Lambda(k^\sharp,k^\sharp)$ and $k[\sigma]$, with σ in degree 2, are isomorphic. Moreover, for any cyclic vector space X, the Yoneda product with σ defines a map*

$$\mathrm{Ext}^n_\Lambda(X,k^\sharp) \to \mathrm{Ext}^{n+2}_\Lambda(X,k^\sharp),$$

which under the isomorphism of theorem 2.60 corresponds to the operator $S : HC^n(X) \to HC^{n+2}(X)$.

Proof. The complex computing $HC^*(k)$ is

$$
\begin{array}{ccccccccc}
0 & & -1 & & 0 & & -1 & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
k & \xrightarrow{0} & k & \xrightarrow{3} & k & \xrightarrow{0} & k & \xrightarrow{3} & \ldots \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 0} & & \\
k & \xrightarrow{2} & k & \xrightarrow{0} & k & \xrightarrow{2} & k & \xrightarrow{0} & \ldots \\
\uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle -1} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle -1} & & \\
k & \xrightarrow{0} & k & \xrightarrow{1} & k & \xrightarrow{0} & k & \xrightarrow{1} & \ldots
\end{array}
$$

where we used the isomorphism $\mathrm{Hom}_k(k,k) \cong k$. From this it is immediate that $HC^n(k) = 0$ in odd degrees and $\cong k$ in even degrees. By theorem 2.60, this complex is isomorphic to the one computing $\mathrm{Ext}^*_\Lambda(k^\sharp, k^\sharp)$, and we see that the we can choose any $f \in \mathrm{Mor}_\Lambda(k^\sharp, \mathcal{I}_2)$ as a generator of $\mathrm{Ext}^2_\Lambda(k^\sharp, k^\sharp)$. The Ext-complex also comes with a shift operator $\mathbb{S}$ and it is clear that this operator coincides with the usual one under the isomorphism of theorem 2.60. Let $i : k^\sharp \to \mathcal{I}_0$ be the embedding of $k^\sharp$ in $\mathcal{I}_0$ as $\ker d_0 : \mathcal{I}_0 \to \mathcal{I}_1$. Define $\sigma = \mathbb{S}i$. The 2-extension corresponding to σ is

$$E: \quad 0 \longrightarrow k^\sharp \xrightarrow{\;i\;} \mathcal{I}_0 \xrightarrow{\;\psi\;} E_0 \xrightarrow{\;\pi\;} k^\sharp \longrightarrow 0$$

With

$$E_0 := \{(i_1, x) \in \mathcal{I}_1 \oplus k^\sharp : d_1(i_1) = \sigma(x)\},$$

$\psi(i_0) = (d_0 i_0, 0)$ and $\pi(i_1, x) = x$.
We will show that for an n-extension $F := (F_i)_{i=0}^{n-1}$ of some cyclic vector space X by $k^\sharp$, corresponding to an n-cocycle $\chi \in \mathrm{Mor}_\Lambda(X, \mathcal{I}_n)$, the Yoneda product $E \circ F$ is equivalent to the extension $G := (G_i)_{i=0}^{n+1}$ corresponding to the $n+2$-cocycle $S\chi$. In degree 2 to $n-2$ we have $(E \circ F)_i = \mathcal{I}_{n-i+2}$, while $G_i = \mathcal{I}_{n-i}$. In these degrees we take $S : \mathcal{I}_{n-i} \to \mathcal{I}_{n-i+2}$ as morphisms. It is clear that these commute with the $d_i' s$. Since

$$(E \circ F)_0 = \{(i_{n+1}, x) \in \mathcal{I}_{n+1} \oplus k^\sharp : d_{n+1} i_{n+1} = \chi(x)\}$$

and

$$G_0 := \{(i_{n+3}, x) \in \mathcal{I}_{n+3} \oplus k^\sharp : d_{n+3} i_{n+3} = S\chi(x)\},$$

we can take the map $(i_{n+1}, x) \mapsto (S i_{n+1}, x)$. Checking commutativity is straightforward. So much for the tails of the extensions. At the heads, we have

$(E \circ F)_{n+1} = G_{n+1} = \mathcal{I}_0$ so here we take equality as a morphism. In degree n we take projection on the first factor $E_1 \to \mathcal{I}_1$. This is the crucial point of the equivalence, for the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{i \circ \pi} & \mathcal{I}_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \mathbb{S}} \\ \mathcal{I}_1 & \xrightarrow{d_1} & \mathcal{I}_2 \end{array}$$

commutes because of the definition of σ, as one easily checks. $\square$

Definition 2.62. Let X and Y be cyclic vector spaces. The *external product* of X and Y is the cyclic vector space $X \times Y$ obtained by

$$(X \times Y)_n := X_n \otimes_k Y_n,$$

with structural morphisms defined by the diagonal action of the structural morphisms of X and Y. In the category of locally convex cyclic vector spaces, this product is defined using the complete projective tensor product.

Observe that $X \otimes k^\sharp \cong X$ for any cyclic vector space and that for unital algebras A and B, $(A^\sharp \times B^\sharp) \cong (A \otimes B)^\sharp$.

Definition 2.63. Let $x = (X_i, \phi_i)_{i=0}^n$, with $X_n = X$, be a resolution of X by X', and $y = (Y_j, \psi_j)_{j=0}^m$, with $Y_m = Y$, a resolution of Y by Y'. Their external product is the resolution

$$x \times y := (\bigoplus_{k=0}^{l} Y_k \times X_{l-k})_{l=0}^{n+m},$$

with maps

$$\sum_{k=0}^{l} (-1)^k \mathrm{id} \times \phi_{l-k} + (-1)^{k+1} \psi_k \times \mathrm{id} : (x \times y)_l \to (x \times y)_{l-1}.$$

This construction defines an associative product

$$\mathrm{Ext}^n_\Lambda(X, X') \otimes_{\mathbb{Z}} \mathrm{Ext}^m_\Lambda(Y, Y') \to \mathrm{Ext}^{n+m}_\Lambda(X \times Y, X' \times Y'),$$

called the *external product*.

In this context it is useful to consider

$$\tau : \mathrm{Ext}^{n+m}_\Lambda(X \times Y, X' \times Y') \to \mathrm{Ext}^{n+m}_\Lambda(Y \times X, Y' \times X'),$$

the natural transformation that interchanges the factors. It maps the class of an extension

$$0 \longrightarrow X' \times Y' \xrightarrow{\phi_{n+m}} \ldots \xrightarrow{\phi_0} X \times Y \longrightarrow 0$$

to the class of

$$0 \longrightarrow Y' \times X' \xrightarrow{\phi_{n+m} \circ \tau^{-1}} ... \xrightarrow{\tau \circ \phi_0} Y \times X \longrightarrow 0.$$

Lemma 2.64. *Let $x \in \mathrm{Ext}^n_\Lambda(X, X')$, $y \in \mathrm{Ext}^m_\Lambda(Y, Y')$. The Yoneda product $\circ$ and the external product $\times$ satisfy the following identities:*

$$x \times y = (\mathrm{id}_{X'} \times y) \circ (x \times \mathrm{id}_Y) = (-1)^{mn}(x \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \times y),$$
$$\mathrm{id}_X \times (y \circ x) = (\mathrm{id}_X \times y) \circ (\mathrm{id}_X \times x)$$
$$x \times y = (-1)^{mn}\tau(y \times x)$$
$$x \times \mathrm{id}_{k^\sharp} = x = \mathrm{id}_{k^\sharp} \times x.$$

Proof. The second and last identities are trivial. We will prove the first one by providing a morphism from the defining representative of $x \times y$ to the defining representatives of $a := (\mathrm{id}_{X'} \times y) \circ (x \times \mathrm{id}_Y)$ and $b := (-1)^{mn}(x \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \times y)$, respectively. We have

$$a_l = Y \times X_l, \quad l < n, \quad a_l = Y_{l-n} \times X', \quad l \geq n$$

so for $l < n$ we define the map $(x \times y)_l \to a_l$ to be the composite

$$\bigoplus_{k=0}^{l} Y_k \times X_{l-k} \longrightarrow Y_0 \times X_l \xrightarrow{\psi_0 \times \mathrm{id}} Y \times X_l.$$

For $l \geq n$, we just take the canonical projection. It is straightforward to check that this defines a morphism $(x \times y) \to a$. Recall from the module structure on extensions (appendix B), that $(-1)(E_i, \chi_i)$ is equivalent to the extension (E_i, g_i) with $g_i = \chi_i$ for $i > 0$ and $g_0 = -\chi_0$. As with a, we have

$$b_l = Y_l \times X, \quad l < m, \quad b_l = Y' \times X_{l-m}, \quad l \geq m.$$

For $l < m$, we define the map $(x \times y)_l \to b_l$ to be the composite

$$\bigoplus_{k=0}^{l} Y_k \times X_{l-k} \longrightarrow Y_l \times X_0 \xrightarrow{(-1)^{nm}\mathrm{id} \times \phi_0} Y_l \times X.$$

For $l \geq m$, we define it to be

$$(-1)^{n(n+m-l)}p : (x \times y)_l \to Y' \times X_{l-m},$$

with p the canonical projection. Again it is diagram drawing to verify that this defines a morphism.
The third equality also follows by providing a morphism, we omit the details. □

Corollary 2.65. *Let $x \in \mathrm{Ext}^n_\Lambda(X, k^\sharp)$ and $y \in \mathrm{Ext}^m_\Lambda(k^\sharp, Y)$.*

- $\sigma \circ x = x \times \sigma$,

- $y \circ \sigma = y \times \sigma$.

Proof. Immediate, by calculation:

$$\sigma \circ x = (\mathrm{id}_{k^\sharp} \times \sigma) \circ (x \times \mathrm{id}_{k^\sharp}) = x \times \sigma,$$

and

$$y \circ \sigma = (\sigma \times \mathrm{id}_{k^\sharp}) \circ (\mathrm{id}_{k^\sharp} \times y) = y \times \sigma. \quad \square$$

The following extends the shift operator on cyclic cohomology to the bivariant Ext-groups.

Definition 2.66. The *periodicity operator* $\mathbb{S} : \mathrm{Ext}^n_\Lambda(X, Y) \to \mathrm{Ext}^{n+2}_\Lambda(X, Y)$ is defined as $\mathbb{S}x = x \times \sigma$.

Lemma 2.67. *Let* $x \in \mathrm{Ext}^n_\Lambda(X, X')$, $y \in \mathrm{Ext}^m_\Lambda(Y, Y')$.

- $\mathbb{S}x = \sigma \times x$
- $\mathbb{S}x \times y = \mathbb{S}(x \times y) = x \times \mathbb{S}y$, .

Proof. Again just calculation:

$$\mathbb{S}x = x \times \sigma = (-1)^{2n}\tau(\sigma \times x) = \sigma \times x,$$

by examining the action of τ. The other equalities now follow from this and the associativity of $\times$. $\square$

From this corollary it follows that we can define the external product on periodic cyclic cohomology by passing to the direct limit

$$HP^i(X) \cong \varinjlim \mathrm{Ext}^{i+2n}(X, k^\sharp).$$

In particular, for unital algebras we obtain a product

$$HP^i(A) \otimes HP^j(B) \xrightarrow{\times} HP^{i+j}(A^\sharp \times B^\sharp) \xrightarrow{\sim} HP^{i+j}(A \otimes B),$$

denoted

$$HP^i(A) \otimes HP^j(B) \xrightarrow{\otimes} HP^{i+j}(A \otimes B).$$

A first important application of these products is the following. Recall that, for a locally convex algebra A, $C^\infty([0,1], A) := C^\infty([0,1])\hat{\otimes}A$.

Theorem 2.68 (Algebraic homotopy and diffotopy invariance for HP). *Let A be a complex unital algebra and $A[t]$ the algebra of polynomials with coefficients in A. There is a natural isomorphism $HP^*(A[t]) \cong HP^*(A)$. If A is locally convex, then $HP^*_c(C^\infty([0,1], A)) \cong HP^*_c(A)$.*

Proof. Using differential forms, on sees that the inclusions $\mathbb{C} \hookrightarrow \mathbb{C}[t]$ and $\mathbb{C} \hookrightarrow C^\infty([0,1])$ induce isomorphisms in HP. We will prove the statement in the discrete case, the continuous case is exactly similar. Since

$$HP^*(\mathbb{C}[t]) \otimes HP^*(A) \cong \mathbb{C} \otimes HP^*(A) \cong HP^*(A),$$

and there is a split injection $A \to A[t]$, it suffices to show that the product

$$HP^*(A) \otimes HP^*(\mathbb{C}[t]) \to HP^*(A[t]),$$

is surjective. To this end, let $x \in HP^*(A[t])$ be represented by the n-extension

$$x : 0 \longrightarrow \mathbb{C}^\sharp \longrightarrow ... \xrightarrow{\phi} X_0 \longrightarrow A[t]^\sharp \longrightarrow 0.$$

Using the inclusion $A \hookrightarrow A[t]$, we define

$$G_0 := \{(x_0, a) \in X_0 \oplus A : \phi(x_0) = a\}$$

and obtain a n-extension

$$0 \longrightarrow \mathbb{C}^\sharp \longrightarrow ... \longrightarrow G_0 \longrightarrow A^\sharp \longrightarrow 0$$

of $A^\sharp$ by $\mathbb{C}^\sharp$, defining an element x'. Next recall the periodicity extension

$$\sigma : \quad 0 \longrightarrow \mathbb{C}^\sharp \xrightarrow{i} \mathcal{I}_0 \xrightarrow{\psi} E_0 \xrightarrow{\pi} \mathbb{C}^\sharp \longrightarrow 0.$$

Using the evaluation map $e_0 : \mathbb{C}[t] \to \mathbb{C}$ sending f to $f(0)$, we define

$$F_0 : \{(i_1, x, f) \in E_0 \oplus \mathbb{C}[t] : f(0) = x\}$$

and obtain an extension

$$0 \longrightarrow \mathbb{C}^\sharp \longrightarrow \mathcal{I}_0 \longrightarrow F_0 \longrightarrow \mathbb{C}[t]^\sharp \longrightarrow 0,$$

which we will denote σ_t. By the relation $x' \times \sigma_t = \sigma_t \circ (x' \times \mathrm{id}_{\mathbb{C}[t]})$, we find that $x' \times \sigma_t$ is equivalent to the extension

$$0 \longrightarrow \mathbb{C}^\sharp \longrightarrow \mathcal{I}_0 \longrightarrow F_0 \longrightarrow ...$$

$$... \longrightarrow G_0 \otimes \mathbb{C}[t]^\sharp \longrightarrow A[t]^\sharp \longrightarrow 0.$$

Using the projections $F_0 \to E_0$ and $G_0 \to X_0$ and the evaluation $e_0 : \mathbb{C}[t] \to \mathbb{C}$, one sees that this is equivalent to the extension $\sigma \circ x$. Since the shift acts trivially on HP^*, x and $\sigma \circ x$ represent the same element and the product is surjective. □

For non-unital algebras, things are a bit more complicated. Since we will use the excision theorem 2.47 in these constructions, our ground field will be $\mathbb{C}$. It will be useful to introduce the notation $I^\flat := (I_+, I)^\sharp$, for a non-unital algebra I. The excision theorem then says that for any extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

of $\mathbb{C}$-algebras, the inclusion $j_{I,A} : I^\flat \to (A, I)^\sharp$ of cyclic vector spaces induces an isomorphism in periodic cyclic (co)homology. Now let B be an arbitrary unital algebra. The map

$$(i \otimes b, z) \mapsto (i, 0) \otimes b + (0, z) \otimes 1,$$

is an inclusion $(I \otimes B)_+ \to I_+ \otimes B$ of unital algebras. It determines a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I \otimes B)^\flat & \longrightarrow & (I \otimes B)^\sharp_+ & \longrightarrow & \mathbb{C}^\sharp & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_{I,B}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\flat \times B^\sharp & \longrightarrow & (I_+ \otimes B)^\sharp & \longrightarrow & B^\sharp & \longrightarrow & 0 \end{array}$$

with exact rows. We have used the identification $I^\flat \times B^\sharp$, which follows from the isomorphism $(I_+ \otimes B)^\sharp \cong I_+^\sharp \times B^\sharp$ of cyclic vector spaces. Thus, $\eta_{I,B}$ is essentially $j_{I \otimes B, I_+ \otimes B}$ and induces an isomorphism in periodic cyclic (co)homology. Therefore we can extend the external product by

$$HP^i(I) \otimes HP^j(B) \xrightarrow{\otimes} HP^{i+j}(I^\flat \times B^\sharp) \xrightarrow{\eta^*_{I,B}} HP^{i+j}(I \otimes B)$$

to non-unital algebras.
Given an extension of algebras, we denote the corresponding extension

$$0 \longrightarrow (A, I)^\sharp \longrightarrow A^\sharp \longrightarrow (A/I)^\sharp \longrightarrow 0$$

of cyclic vector spaces by $[A, I]$. It defines an element of $\mathrm{Ext}^1_\Lambda((A/I)^\sharp, (A, I)^\sharp)$, for which we will use the same notation. Since we work over a field, tensoring an extension by a unital algebra B yields an exact sequence

$$0 \longrightarrow I \otimes B \longrightarrow A \otimes B \longrightarrow (A/I) \otimes B \longrightarrow 0.$$

Lemma 2.69. *Let A and B be unital $\mathbb{C}$-algebras and $I \subset A$ a two-sided ideal. Then the cyclic vector spaces $(A, I)^\sharp \times B^\sharp$ and $(A \otimes B, I \otimes B)^\sharp$ are isomorphic and we have*

$$[A \otimes B, I \otimes B] = [A, I] \times \mathrm{id}_{B^\sharp} \in \mathrm{Ext}^1_\Lambda((A/I \otimes B)^\sharp, (A \otimes B, I \otimes B)^\sharp).$$

Proof. Since the functor $X \mapsto X \times B^\sharp$ is exact and we have identifications $(A \otimes B)^\sharp \cong A^\sharp \times B^\sharp$ and $((A/I) \otimes B)^\sharp \cong (A/I)^\sharp \times B^\sharp$, it follows that $(A, I)^\sharp \times B^\sharp \cong (A \otimes B, I \otimes B)^\sharp$. $\square$

By the excision theorem every element $\xi \in HP^*(I)$ is of the form $\xi = j_{I,A*}(\xi_0) = \xi_0 \circ j$ for some $\xi_0 \in HP^*(A,I)$. Furthermore, in the long exact Ext-sequence obtained by Yoneda, the boundary map

$$\partial_{I,A} : \mathrm{Ext}^n_\Lambda((A,I)^\sharp, \mathbb{C}^\sharp) \to \mathrm{Ext}^{n+1}_\Lambda((A/I)^\sharp, \mathbb{C}^\sharp)$$

is given by $\xi_0 \mapsto \xi_0 \circ [A,I]$. Since the HP-groups are direct limits of the Ext-groups, we can choose representatives of ξ and ξ_0 in $\mathrm{Ext}^n_\Lambda(I^\flat, \mathbb{C}^\sharp)$ and $\mathrm{Ext}^n_\Lambda((A,I)^\sharp, \mathbb{C}^\sharp)$, for the same n. The long exact Ext sequence maps naturally to the to the long exact HP-sequence, so we have

$$\partial_{I,A}(\xi) = \xi_0 \circ [A,I].$$

For tensoring by a unital algebra B, the Cuntz-Quillen map

$$j_{I\otimes B, A\otimes B} : ((I\otimes B)_+, I\otimes B)^\sharp \to (A\otimes B, I\otimes B)^\sharp$$

satisfies $j_{I\otimes B,A\otimes B} = (j_{I,A} \times \mathrm{id}_{B^\sharp}) \circ \eta_{I,A}$.

Theorem 2.70 (Nistor). *Let A and B be unital $\mathbb{C}$-algebras and $I \subset A$ a two-sided ideal. Then the boundary maps*

$$\partial_{I,A} : HP^i(I) \to HP^{i+1}(A/I)$$

and

$$\partial_{I\otimes B,A\otimes B} : HP^i(I\otimes B) \to HP^{i+1}(A\otimes B)$$

satisfy

$$\partial_{I\otimes B,A\otimes B}(\xi\otimes\zeta) = \partial_{I,A}(\xi)\otimes\zeta$$

for all $\xi \in HP^*(I)$ and $\zeta \in HP^*(B)$.

Proof. Choosing representatives for ξ, ξ_0 and ζ in the relevant Ext-groups, we may calculate

$$\begin{aligned}
\partial_{I,A}(\xi)\otimes\zeta &= (\xi_0 \circ [A,I]) \times \zeta \\
&= (\mathrm{id}_{\mathbb{C}^\sharp} \times \zeta) \circ ((\xi_0 \circ [A,I]) \times \mathrm{id}_{B^\sharp}) \\
&= (\mathrm{id}_{\mathbb{C}^\sharp} \times \zeta) \circ (\xi_0 \times \mathrm{id}_{B^\sharp}) \circ ([A,I] \times \mathrm{id}_{B^\sharp}) \\
&= (\xi_0 \times \zeta) \circ [A\otimes B, I\otimes B] \\
&= \partial_{I\otimes B,A\otimes B}((\xi_0\times\zeta)\circ j_{I\otimes B,A\otimes B}) \\
&= \partial_{I\otimes B,A\otimes B}((\xi_0\times\zeta)\circ (j_{I,A}\times \mathrm{id}_{B^\sharp})\circ\eta_{I,A}) \\
&= \partial_{I\otimes B,A\otimes B}((\xi\times\zeta)\circ\eta_{I,A}) \\
&= \partial_{I\otimes B,A\otimes B}(\xi\otimes\zeta). \quad \square
\end{aligned}$$

2.10 Discrete groups

An interesting class of algebras are those associated with discrete groups. Recall that the adjective "discrete" stresses the fact that we consider the group given with the discrete topology. The theory below therefore applies to any group although there are important modifications of it for topological groups (i.e. groups with a non-discrete topology). We discuss only the the disrete case, because the group of covering transformations of a normal covering space is usually treated as a discrete group.

Definition 2.71. Let $\mathcal{C}$ be a small category. The *nerve* $\mathbb{B}_*\mathcal{C}$ of $\mathcal{C}$ is the simplicial set determined by

$$\begin{aligned}
&\mathbb{B}_0(\mathcal{C}) := \text{obj } \mathcal{C},\\
&\mathbb{B}_n(\mathcal{C}) := \{(f_0, ..., f_{n-1}) : f_i \in \text{Mor}_{\mathcal{C}}(\mathcal{C}_i, \mathcal{C}_{i+1}), \mathcal{C}_i \in \text{obj } \mathcal{C}\}, \quad n \geq 1,
\end{aligned}$$

and

$$\begin{aligned}
&d_n^0(f_0, ..., f_{n-1}) := (f_1, ..., f_{n-1})\\
&d_n^i(f_0, ..., f_{n-1}) := (f_0, ..., f_{i-1} \circ f_i, ..., f_{n-1}), \quad 1 \leq i \leq n-1\\
&d_n^n(f_0, ..., f_{n-1}) := (f_0, ..., f_{n-2}), \quad n \geq 2\\
&s_n^i(f_0, ..., f_{n-1}) := (f_0, ..., f_i, \text{id}_{\mathcal{C}_{i+1}}, f_{i+1}, ..., f_{n-1}), \quad n \geq 1
\end{aligned}$$

while

$$d_1^0(f) = \mathcal{C}_0, \quad d_1^1(f) = \mathcal{C}_1, \quad \text{for } f : \mathcal{C}_0 \to \mathcal{C}_1$$

and

$$s_0^0(\mathcal{C}_0) = \text{id}_{\mathcal{C}_0}.$$

The *classifying space of* $\mathcal{C}$ is the geometric realization of its nerve and is denoted $\mathbb{B}\mathcal{C}$.
A group Γ can be considered a small category with one object (Γ itself) and morphism set $\text{Mor}_\Gamma(\Gamma, \Gamma) = \Gamma$ with composition given by multiplication. The *nerve* of Γ is the nerve of this category, and the *classifying space* of Γ is the classifying space of this category, denoted $\mathbb{B}\Gamma$.
The *(co)homology of* Γ *with coefficients in* k is the (co)homology of the simplicial module $k[\mathbb{B}_*\Gamma]$ and is denoted $H_*(\Gamma, k)$.

It is immediate that $H_0(\Gamma, k) \cong k$, while $H_1(\Gamma, k) \cong \Gamma_{ab} \otimes_{\mathbb{Z}} k$. This can be seen as follows: $b_1 = 0$, so $\ker b_1 = k[G]$, while the image of $b_2 : k[\Gamma^2] \to k[\Gamma^1]$ consists of expressions

$$\sum k_i(g_{0i} - g_{0i}g_{1i} + g_{1i}),$$

so im b_2 is the k-submodule of $k[G]$ generated by the elements $g_0 + g_1 - g_0g_1$. Moreover there is a homomorphism of groups

$$\begin{aligned}
&\xi : k[\Gamma] \to k \otimes_{\mathbb{Z}} \Gamma_{ab}\\
&\sum k_i g_i \mapsto \sum k_i \otimes [g_i],
\end{aligned}$$

and $\ker b_2 \subset ker\xi$, so it gives a homomorphism $\xi : H_1(G,k) \to k \otimes_{\mathbb{Z}} G_{ab}$. This map is surjective since we can lift $a \otimes [g]$ to ag. For injectivity, it suffices to show that $a \otimes g = a \otimes gvuv^{-1}u^{-1}$, since then our lift actually defines an inverse to ξ. But in $H_1(G,k)$ we have $[g_0g_1] = [g_0] + [g_1]$, which in particular implies $[1] = [0]$ and $[h^{-1}] = -[h]$. Using these relations the desired equality follows easily. So ξ is injective and hence an isomorphism.

Proposition 2.72. *For any ring R, $K_1(R) \cong H_1(GL(R), \mathbb{Z})$.*

Proof. This is immediate since $\mathbb{Z} \otimes_{\mathbb{Z}} GL(R)_{ab} = GL(R)_{ab} = K_1(R)$. □

We want to relate the (co)homology of the group Γ to the periodic cyclic (co)homology of the group ring $k[\Gamma]$. To be able to do this, we need to make $k[\mathbb{B}_*\Gamma]$ into a cyclic module, which is done by defining

$$t_n(g_0, ..., g_{n-1}) := ((g_0 ... g_{n-1})^{-1}, g_2, ..., g_{n-2}).$$

Subsequently we define the map $\iota : k[\mathbb{B}_*\Gamma] \to k[\Gamma]^\sharp$ by

$$\iota(g_0, ..., g_{n-1}) = g_0 ... g_{n-1}^{-1} \otimes g_0 \otimes ... \otimes g_{n-1},$$

which is a map of cyclic modules. Let $k[(\Gamma, 1)]$ be the graded k-module generated by the sets

$$(\Gamma_n, 1) := \{(g_0 \otimes ... \otimes g_n) \in \Gamma^{\otimes n+1} : g_0 ... g_n = 1\}.$$

ι embeds $k[\mathbb{B}_*\Gamma]$ as the cyclic submodule $k[(\Gamma, 1)] \subset k[\Gamma]^\sharp$. This leads us to defining $k[(\Gamma, z)]$ as the graded k-module generated by

$$(\Gamma_n, z) := \{(g_0 \otimes ... \otimes g_n) \in \Gamma^{n+1} : \exists g \in \Gamma \quad g(g_0 ... g_n)g^{-1} = z\},$$

for any $z \in \Gamma$. Since the structural maps d_n^i, s_n^i and t_n on $k[\Gamma]^\sharp$ do not change the conjugation class of the product of the factors of an elementary tensor, we get cyclic submodules $k[(\Gamma, z)] \subset k[\Gamma]^\sharp$. It is plain that $k[(\Gamma, z)] = k[(\Gamma, w)]$ whenever z and w are conjugate in Γ. Thus, if $\overline{\Gamma}$ denotes the set of conjugation orbits of Γ, there is a direct sum decomposition

$$k[\Gamma]^\sharp \cong \bigoplus_{\overline{z} \in \overline{\Gamma}} k[(\Gamma, z)],$$

of cyclic modules. In particular, we see that ι is split injective and it gives us split injections in (co)homology. For our purposes, the image of these split injections is the most interesting part of the periodic cyclic (co)homology of $k[\Gamma]$, and we will now determine it.

Definition 2.73. Let Γ be a discrete group. Define a small category $E\Gamma$ by

$$\text{obj } E\Gamma := \Gamma, \quad \text{Mor}_{E\Gamma}(g_0, g_1) = \{g_0^{-1}g_1\},$$

while composition is group multiplication in Γ. We denote the nerve $\mathbb{B}_*E\Gamma$ by $\mathbb{E}_*\Gamma$ and the classifying space by $\mathbb{E}\Gamma$.

Since $E\Gamma$ has a unique morphism connecting any two of its objects, there is a bijection between $\mathbb{E}_n\Gamma \to \Gamma^{n+1}$. Using this bijection, we can define the structure of a cyclic set by

$$t_n(g_0, ..., g_n) = (g_n, g_0, ..., g_{n-1}),$$

and so obtain a cyclic k-module $k[\mathbb{E}_*\Gamma]$. This is also a free left $k[\Gamma]$ module by

$$g(g_0, ..., g_n) = (gg_0, ..., gg_n),$$

with basis

$$\{(1, g_1, ..., g_n) : g_i \in \Gamma\}.$$

The b-complex $k[\mathbb{E}_*\Gamma]$ is contractible, for

$$(g_0, ..., g_n) \mapsto (1, g_0, ..., g_n)$$

defines a contracting homotopy. Therefore

$$HH_0(k[\mathbb{E}_*\Gamma]) = k, \quad HH_n(k[\mathbb{E}_*\Gamma]) = 0, \quad n \geq 1.$$

If we define the trivial $k[\Gamma]$ module structure $g \circ x = x$ on k, then by general homological algebra, the double complex $CC_{**}^{per}(k[\mathbb{E}_*\Gamma])$ is a $k[\Gamma]$-free resolution of the $\mathbb{Z}$-graded complex of $k[\Gamma]$-modules

$$\mathcal{K}^{per} : ... \longrightarrow k \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow ...$$

consisting of k in even degrees and 0 in odd degrees. This follows from the fact that the maps d_n^i and t_n are $k[\Gamma]$-module homomorphisms. The double complex consisting of $C_*(k[\mathbb{E}_*\Gamma])$ in the even columns, and 0 in the odd ones, is also a free resolution of $\mathcal{K}^{per}$. The following result is due to Burghelea [3].

Proposition 2.74.

$$HP_n(k[\mathbb{B}_*\Gamma]) \cong \prod_{i\in\mathbb{Z}} H_{2i+n}(\Gamma, k), \quad HP^n \cong \bigoplus_{i\in\mathbb{Z}} H^{2i+n}(\Gamma, k),$$
$$HC_n^-(k[\mathbb{B}_*\Gamma]) \cong \prod_{i=0}^{\infty} H_{2i+n}(\Gamma, k), \quad HC_-^n \cong \bigoplus_{i=0}^{\infty} H^{2i+n}(\Gamma, k),$$
$$HC_n(k[\mathbb{B}_*\Gamma]) \cong \bigoplus_{i=0}^{\infty} H_{n-2i}(\Gamma, k), \quad HC^n \cong \bigoplus_{i=0}^{\infty} H^{n-2i}(\Gamma, k).$$

Proof. We can use both resolutions to compute the groups $\mathrm{Tor}_n^{k[\Gamma]}(\mathcal{K}^{per}, k)$. To prove the statements on HC and HC^-, one considers the complexes $\mathcal{K}^+$ and $\mathcal{K}^-$, which are just the positive and negative parts of $\mathcal{K}^{per}$ and to prove the results in cohomology, one computes the relevant Ext-groups.
There is an isomorphism of cyclic modules $k[\mathbb{E}_*\Gamma] \otimes_{k[\Gamma]} k \cong k[\mathbb{B}_*\Gamma]$, given by

$$(g_0, ..., g_n) \mapsto (g_0^{-1}g_1, ..., g_{n-1}^{-1}g_n)$$

and using this isomorphism and the first resolution of $\mathcal{K}^{per}$, we see that

$$\mathrm{Tor}_n^{k[\Gamma]}(\mathcal{K}^{per}, k) \cong HP_n(k[\mathbb{B}_*\Gamma]).$$

On the other hand, using the second resolution, we see that

$$\mathrm{Tor}_n^{k[\Gamma]}(\mathcal{K}^{per}, k) \cong \prod_{i\in\mathbb{Z}} HH_{n+2i}(k[\mathbb{B}_*\Gamma]) = \prod_{i\in\mathbb{Z}} H_{n+2i}(\Gamma, k),$$

whence the assertion. □

Thus, the group homology $H(\Gamma, \mathbb{C})$ can be embedded in the periodic cyclic homology of $\mathbb{C}[\Gamma]$. This fact will be useful in chapter 4.
Now we consider a discrete group Γ acting on a complex algebra A by homomorphisms, that is, a homomorphism

$$\begin{aligned} \Gamma &\to \mathrm{Aut}_{\mathbb{C}} A \\ \gamma &\mapsto \alpha_\gamma. \end{aligned}$$

The *algebraic crossed product* $A \rtimes \Gamma$ is the free A-module on Γ. It consists of formal sums $\sum_{i=0}^n a_i\gamma_i$ and the product is given by

$$(a\gamma)(b\mu) := (a\alpha_\gamma(b))\gamma\mu.$$

An ideal $I \subset A$ is called Γ-*invariant* if $\alpha_\gamma(x) \in I$ for all $x \in I$ and $\gamma \in \Gamma$. For such an ideal $I \rtimes \Gamma$ is an ideal in $A \rtimes \Gamma$ and the sequence

$$0 \longrightarrow I \rtimes \Gamma \longrightarrow A \rtimes \Gamma \longrightarrow A/I \rtimes \Gamma \longrightarrow 0$$

is exact.
Define a homomorphism

$$\begin{aligned} \delta : A \rtimes \Gamma &\to (A \rtimes \Gamma) \otimes \mathbb{C}[\Gamma] \\ a\gamma &\mapsto a\gamma \otimes \gamma. \end{aligned}$$

Using δ, we can make $HP^*(A \rtimes \Gamma)$ into a $HP^*(\mathbb{C}[\Gamma])$ module by

$$HP^*(A \rtimes \Gamma) \otimes HP^*(\mathbb{C}[\Gamma]) \xrightarrow{\otimes} HP^*(A \rtimes \Gamma \otimes \mathbb{C}[\Gamma]) \xrightarrow{\delta^*} HP^*(A \rtimes \Gamma).$$

Theorem 2.75 (Nistor). *Let Γ be a discrete group acting on the unital $\mathbb{C}$-algebra A. Then the boundary map*

$$\partial_{I\rtimes\Gamma, A\rtimes\Gamma} : HP^*(I \rtimes \Gamma) \to HP^{*+1}(A/I \rtimes \Gamma)$$

is $HP^*(\mathbb{C}[\Gamma])$-linear.

Proof. δ fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \rtimes \Gamma & \longrightarrow & A \rtimes \Gamma & \longrightarrow & A/I \rtimes \Gamma & \longrightarrow & 0 \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ 0 & \longrightarrow & (I \rtimes \Gamma) \otimes \mathbb{C}[\Gamma] & \longrightarrow & (A \rtimes \Gamma) \otimes \mathbb{C}[\Gamma] & \longrightarrow & (A/I \rtimes \Gamma) \otimes \mathbb{C}[\Gamma] & \longrightarrow & 0 \end{array}$$

with exact rows. Therefore $\delta^* \circ \partial = \partial \circ \delta^*$, and for $x \in HP^*(\mathbb{C}[\Gamma])$ and $\xi \in HP^*(I \rtimes \Gamma)$ we compute

$$\partial(\xi x) = \partial(\delta^* \xi \otimes x) = \delta^* \partial(\xi \otimes x) = \delta^*(\partial(\xi) \otimes x) = \partial(\xi) x,$$

using the definition of the $HP^*(\mathbb{C}[\Gamma])$-module structure and theorem 2.70. □

Chapter 3

The Index Formula

We have now come to describe the relationship that exists between K-theory and cyclic homology. This relationship has its origin in topology, in particular the study of characteristic classes. We will hear more about those in the next chapter. With the results in this chapter we will prove the index theorems discussed in section 1.8. The proofs of these theorems become very simple with the machinery developed here. The central result, theorem 3.14, will be of great importance in chapter 4. This theorem is proved in [20], and we follow the line of proof presented there.

Most of what is done from now on depends on the Cuntz-Quillen excision theorem, which is known only in characteristic zero. Therefore we will fix our ground field to be $\mathbb{C}$, except in the construction of the Chern character and the pairings related to it, which we will do for an arbitrary commutative and unital ground ring.

3.1 Pairings and the Fredholm index

Recall that a trace on an algebra A is a linear map $\tau : A \to k$, such that $[A, A] \subset \ker \tau$. The group $HH^0(A)$ consists of the traces on A, and we will construct a pairing

$$K_0(A) \otimes_{\mathbb{Z}} HH^0(A) \to k.$$

Assume for a moment that A is unital. We extend τ to a trace $\tau : M_\infty(A) \to k$ by composing it with $\mathrm{Tr} : M_\infty(A) \to A$. τ defines a homomorphism

$$\begin{aligned} \tau_* : K_0(A) &\to k \\ [e] - [f] &\mapsto \tau(e) - \tau(f), \end{aligned}$$

since it is additive on block sums and for $g \in GL(A)$ we have

$$\tau(geg^{-1} - e) = \tau((ge)g^{-1} - g^{-1}(ge)) = \tau([ge, g^{-1}]) = 0.$$

It is clear from this definition that it is compatible with the Morita invariance isomorphisms on K_0 and HH^0. If A is not unital, then we first extend τ to

A_+ by making it zero on 1, and then obtain, as above, a map $\tau : K_0(A_+) \to k$, which we then restrict to $K_0(A)$. If A is unital, the two constructions coincide. An ideal $I \subset A$ is never unital, and $HH^0(I)$ is not Morita invariant in general. There is however a split injective map $\mathcal{M} : HH^0(I) \to HH^0(M_n(I))$ for each n, given by the above extension construction. The splitting is given by associating to a trace $\eta : M_n(I) \to k$ the trace $\eta' : I \to k$ defined as $\eta'(i) = \eta(i \oplus 0_{n-1})$. The pairing defined here is compatible with $\mathcal{M}$ in the sense that $\tau_*([e]) = (\mathcal{M}\tau)_*([e])$, where on the left hand side $[e]$ is considered as an element of $K_0(I)$, whereas on the right hand side it is an element of $K_0(M_n(I))$. Note that in the other direction compatibility need not hold.
For unital algebras A, there is a pairing

$$K_1(A) \otimes_{\mathbb{Z}} HH^1(A) \to k,$$

which we will now construct. Let $u \in GL(n, A)$ for some n and write $u = (a_{ij})$, $u^{-1} = (b_{ij})$. Subsequently, if $\phi : A \otimes A \to k$ is a Hochschild 1-cocycle, we define a map $\phi_* : GL(A) \to k$ by

$$u \mapsto \sum_{i,j=1}^{n} \phi(a_{ij}, b_{ji}),$$

and claim that this descends to a morphism $\phi_* : K_1(A) \to k$. To check this, it suffices to show that $\phi_*(uv) = \phi_*(vu) = \phi_*(u)$ whenever $v \in E(A)$ and $\phi_*(u \oplus v) = \phi_*(u) + \phi_*(v)$, but this is obvious. We will show that

$$\phi_*(e_{ij}(a)u) = \phi_*(u) \qquad (1)$$

for each elementary matrix $e_{ij}(a)$. This suffices because $e_{ij}(a)^{-1} = e_{ij}(-a)$ and thus

$$\phi_*(e_{ij}(a)) = \sum_{k=0}^{n} \phi(1,1) = 0,$$

since ϕ is a Hochschild 1-cocycle and $\phi(1,1) = \phi \circ b(1,1,1)$. So by induction,(1) implies that ϕ_* vanishes on $E(A)$ and that $\phi_*(uv) = \phi_*(vu) = \phi_*(u)$, since $E(A) = [E(A), E(A)]$. We compute:

$$\begin{aligned}
\phi_*(e_{ij}(a)u) &= \sum_{k,l=1}^{n} \phi((e_{ij}(a)u)_{kl}, (u^{-1}e_{ij}(-a))_{lk}) \\
&= \sum_{l=1}^{n} \sum_{k \neq i,j} \phi(u_{kl}, u^{-1}_{lk}) + \sum_{l=1}^{n} \phi(u_{il} + au_{jl}, u^{-1}_{li}) + \phi(u_{jl}, -u^{-1}_{li}a + u^{-1}_{lj}) \\
&= \sum_{k,l=1}^{n} \phi(u_{kl}, u^{-1}_{lk}) + \sum_{l=1}^{n} \phi(au_{jl}, u^{-1}_{li}) - \phi(u_{jl}, u^{-1}_{li}a) \\
&= \phi_*(u) + \sum_{l=1}^{n} \phi \circ b(u_{jl}, u^{-1}_{li}, a) \\
&= \phi_*(u),
\end{aligned}$$

because $j \neq i$ and therefore $\sum_{l=1}^n u_{jl} u_{li}^{-1} = 0$. This pairing is by definition compatible with Morita invariance.

Lemma 3.1. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

be an exact sequence of $\mathbb{C}$*-algebras and* $\tau : I \to \mathbb{C}$ *a trace on* I. *Let* $\ell : A/I^2 \to A$ *be an arbitrary linear lifting of the quotient map* $A \to A/I^2$ *, which if* A *is locally convex we assume to be continuous. Then the expression*

$$\phi_\tau(a, b) := \tau([\ell(a), \ell(b)] - \ell([a, b])),$$

defines a Hochschild 1*-cocycle on* A/I^2*, whose class in* $HH^1(A/I^2)$ *is independent of* ℓ*. If* $\tau[A, I] = 0$*, then we may everywhere replace* I^2 *by* I.

Proof. From the identity $[a, xy] = [ax, y] + [ya, x]$ it follows that $\tau[A, I^2] = 0$. To check that ϕ_τ is a cocycle we compute

$$\begin{aligned}
\phi_\tau \circ b(x, y, z) &= \phi_\tau((xy, z) - (x, yz) + (zx, y)) \\
&= \tau([\ell(xy), \ell(z)] - \ell[xy, z]) - \tau([\ell(x), \ell(yz)] - \ell[x, yz]) \\
&\quad + \tau([\ell(zx), \ell(y)] - \ell[zx, y]) \\
&= \tau([\ell(xy), \ell(z)]) - \tau([\ell(x), \ell(yz)]) + \tau([\ell(zx), \ell(y)]) \\
&= \tau([\ell(x), \ell(y)\ell(z) - \ell(yz)]) + \tau([\ell(y), \ell(z)\ell(x) - \ell(zx)]) \\
&\quad + \tau([\ell(z), \ell(x)\ell(y) - \ell(xy)]) \\
&= 0,
\end{aligned}$$

because the arguments in the last expression are in $[A, I^2]$.
Let $m : A/I^2 \to A$ be another linear lifting and denote the associated morphisms by ϕ_τ^ℓ and ϕ_τ^m. We want to show that $\phi := \phi_\tau^\ell - \phi_\tau^m$ is a coboundary, since then we have $[\phi_\tau^\ell] = [\phi_\tau^m]$ in $HH^1(A/I^2)$.

$$\begin{aligned}
\phi(x, y) &= \tau([\ell(x), \ell(y)] - \ell[x, y] - [m(x), m(y)] + m[x, y]) \\
&= \tau([\ell(x), \ell(y) - m(y)] - [m(y), \ell(x) - m(x)] - \ell[x, y] + m[x, y]) \\
&= \tau(m[x, y] - \ell[x, y]) \\
&= \tau \circ (m - \ell) \circ b(x, y). \quad \square
\end{aligned}$$

Lemma 3.2 (Nistor). *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

be an exact sequence of $\mathbb{C}$*-algebras and* τ *a trace on* I*. Then we have*

$$\tau_* \circ \mathrm{Ind} = \phi_{\tau *} : K_1(A/I^2) \to \mathbb{C},$$

where $\mathrm{Ind} : K_1(A/I^2) \to K_0(I^2)$ *is the connecting morphism in* K*-theory associated to the extension*

$$0 \longrightarrow I^2 \longrightarrow A \longrightarrow A/I^2 \longrightarrow 0.$$

If $\tau[A, I] = 0$*, then we may everywhere replace* I^2 *by* I.

Proof. Let $[u] \in K_1(A/I^2)$ and apply τ_* to the formula from corollary 1.27,

$$\mathrm{Ind}([u]) = [\begin{pmatrix} (-(1_n - ab)^2, 1_n) & (a(2_n - ba)(1_n - ba), 0_n) \\ ((1_n - ba)b, 0_n) & ((1_n - ba)^2, 0_n)) \end{pmatrix}] - [(0_n, 1_n) \oplus (0_n, 0_n)],$$

for $u \in GL(n, A)$. Extending $\ell : A/I^2 \to A$ to $M_n(A/I^2)$, we can choose $a = \ell(u)$ and $b = \ell(u^{-1})$. Then

$$\tau_* \circ \mathrm{Ind}[u] = \tau((1 - ba)^2) - \tau((1 - ab)^2) = 2\tau([a, b]) - \tau([a, bab])$$

and

$$\tau([a, bab] - [a, b]) = \tau([a, b(ab - 1)]) = 0,$$

because $ab - 1 \in M_n(I^2) = M_n(I)^2$ and $\tau([M_n(A), M_n(I)^2]) = 0$. Thus $\tau_* \circ \mathrm{Ind}([u]) = \tau([a, b])$. Computing $\phi_{\tau*}([u])$ gives

$$\phi_{\tau*}([u]) = \tau([\ell(u), \ell(u^{-1})] - \ell([u, u^{-1}])) = \tau([\ell(u), \ell(u^{-1})]) = \tau([a, b]),$$

and we are done. □

We will now give a proof of proposition 1.39. To be able to use the results in this section, we need an another, equivalent formulation of the theorem. Recall from section 1.8 exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0.$$

Although the K-theory isomorphism $K_0(\mathcal{K}) \cong \mathbb{Z}$ is given by mapping an idempotent to its trace, the trace Tr is not defined on $\mathcal{K}$, because this algebra is in a sense too big. We pass to a smaller algebra, on which Tr is a Hochschild 0-cocycle.
Recall that the *p-th Schatten ideal* $\mathcal{L}_p(\mathcal{H})$ consists of the *p-summable operators* on $\mathcal{H}$. We make this precise in a definition.

Definition 3.3. Let $\{e_i\}$ be a basis of $\mathcal{H}$. For any operator $T \in B(\mathcal{H})$, TT^* is a positive operator, and thus $|T| := (TT^*)^{\frac{1}{2}}$ is defined using functional calculus. Define

$$\mathcal{L}_p := \{T \in B(\mathcal{H}) : \mathrm{Tr}(|T|^p) := \sum_{i=0}^{\infty} \langle |T|^p e_i, e_i \rangle < \infty\}.$$

For each p, there are inclusions $M_\infty(\mathbb{C}) \subset \mathcal{L}_p \subset \mathcal{L}_{p+1} \subset \mathcal{K}$, and these are K_0-equivalences. For all these algebras, we have $M_\infty(\mathcal{L}_p) \cong \mathcal{L}_p$. As with $\mathcal{K}$, the K_0-groups of the Schatten ideals are exhausted by idempotents in the algebras itself, and these idempotent are in turn elements of $\mathcal{L}_1$. Thus, sending an idempotent to its trace gives an isomorphism $K_0(\mathcal{L}_p) \cong \mathbb{Z}$. The map on K_0 induced by these inclusions is clearly the identity. On $\mathcal{L}_1$, the trace Tr is well defined, and the isomorphism $K_0(\mathcal{L}_1) \cong \mathbb{Z}$ is realized by the pairing with this Hochschild 0-cocycle.
Recall that for a Fredholm operator $T \in B(\mathcal{H})$, there is a parametrix S, such

that $1 - ST$ and $1 - TS$ are compact projections. Since these have finite trace, they are actually in $\mathcal{L}_1$, and it follows that T is invertible modulo $\mathcal{L}_1$. Since invertibility modulo $\mathcal{L}_1$ clearly implies invertibility module $\mathcal{K}$, we find that an operator is Fredholm if and only if it is invertible modulo $\mathcal{L}_1$.
The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & B(\mathcal{H}) & \xrightarrow{q} & B(\mathcal{H})/\mathcal{L}_1 & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & B(\mathcal{H}) & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

implies that $\mathrm{Ind}([q(T)]) = \mathrm{Ind}([\pi(T)])$, for Fredholm operators T. Since $1 - ST$ and $1 - TS$ are projections on the kernels of T and T^*, respectively, we see from the proof of the lemma that for the trace $\mathrm{Tr} : \mathcal{L}_1 \to \mathbb{C}$ we get

$$\mathrm{Tr} \circ \mathrm{Ind}([q(T)]) = \mathrm{Tr}(1 - ST) - \mathrm{Tr}(1 - TS) = \dim\ker T - \dim\ker T^*,$$

which is just the Fredholm index of T. This proves proposition 1.39.

3.2 The boundary map

We now further investigate the result of section 3.1. A trace $\tau : I \to \mathbb{C}$ satisfies $\tau[A, I^2] = 0$. Recall that the relative cyclic cohomology group $HC^0(A, I^2)$ was defined as the cohomology of the complex

$$C^*_\lambda(A, I^2) = \mathrm{coker}\ (\pi^* : C^*_\lambda(A/I^2) \to C^*_\lambda(A)).$$

Therefore, $HC^0(A, I^2)$ consists of the traces $\sigma : I^2 \to \mathbb{C}$ satisfying $\sigma[A, I^2] = 0$. By restriction we thus obtain a map $HC^0(I) \to HC^0(A, I^2)$.

Lemma 3.4. *Let $\partial : HC^0(A, I^2) \to HC^1(A/I^2)$ be the boundary map in cyclic cohomology associated to the extension*

$$0 \longrightarrow I^2 \longrightarrow A \xrightarrow{\pi} A/I^2 \longrightarrow 0.$$

If $[\tau] \in HC^0(A, I^2)$ is the class of a trace $\tau : I \to \mathbb{C}$, then $\partial[\tau] = [\phi_\tau]$.

Proof. First note that ϕ_τ is by definition a cyclically invariant Hochschild cocycle on A/I^2, so it defines a class in $HC^1(A/I^2)$. The lemma follows by tracing back the definition of the boundary map associated to the exact sequence of complexes

$$0 \longrightarrow C^*_\lambda(A/I^2) \longrightarrow C^*_\lambda(A) \longrightarrow C^*_\lambda(A, I^2) \longrightarrow 0.$$

In degrees 0 and 1 the diagram is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^1_\lambda(A/I^2) & \longrightarrow & C^1_\lambda(A) & \longrightarrow & C^1_\lambda(A,I^2) & \longrightarrow & 0 \\
 & & \uparrow b & & \uparrow b & & \uparrow b & & \\
0 & \longrightarrow & C^0_\lambda(A/I^2) & \longrightarrow & C^0_\lambda(A) & \longrightarrow & C^0_\lambda(A,I^2) & \longrightarrow & 0.
\end{array}$$

The cocycle $\partial[\tau]$ is then defined as follows. Choose an element $\sigma \in C^0_\lambda(A)$ that maps to τ. Then $b\sigma \in C^1_\lambda(A)$. Since the diagram commutes, the rows are exact and $b\tau = 0$, we can pull back $b\sigma$ to an element $\phi \in C^1_\lambda(A/I^2)$. One defines $\partial[\tau] = [\phi]$.It is basic to check that this is well defined.
In our case, let $\ell : A/I^2 \to A$ be a linear lifting of the quotient map. Then as a $\mathbb{C}$-vector space, A decomposes as $A/I^2 \oplus I^2$ by the isomorphism $a \mapsto (\pi(a), a - \ell\pi(a))$. We can define σ by $\sigma(a) = \tau(a - \ell\pi(a))$. Using the decomposition $A/I^2 \oplus I^2$ it follows that it maps to τ. Thus $b\sigma \in C^1_\lambda(A)$ and if we define

$$\phi(x,y) := b\sigma(\ell(x), \ell(y)),$$

then $\pi_*\phi = b\sigma$ and

$$\begin{aligned}
\phi(x,y) &= b\sigma(\ell(x), \ell(y)) \\
&= \sigma[\ell(x), \ell(y)] \\
&= \tau([\ell(x), \ell(y)] - \ell\pi[\ell(x), \ell(y)]) \\
&= \tau([\ell(x), \ell(y)] - \ell[x,y]) \\
&= \phi_\tau(x,y). \quad \square
\end{aligned}$$

Consider the map $A/I^2 \to A/I$, with nilpotent kernel I/I^2. On periodic cyclic cohomology, this gives an isomorphism $HP^*(A/I) \to HP^*(A/I^2)$. Using the five lemma and the exact sequence in periodic cyclic cohomology, it follows that $HP^*(A,I) \cong HP^*(A,I^2)$. Then, considering the natural maps

$$HC^{2n}(A) \to \lim_{\to} HC^{2j}(A) \cong HP^0(A)$$

and

$$HC^{2n+1}(A) \to \lim_{\to} HC^{2j+1}(A) \cong HP^1(A)$$

and the excision theorem this amounts to

Lemma 3.5 (Nistor). *The diagram*

$$\begin{array}{ccccccc}
HC^0(I) & \longrightarrow & HC^0(A,I^2) & \xrightarrow{\partial} & HC^1(A/I^2) & \longleftarrow & HC^1(A/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
HP^0(I) & \xrightarrow{\sim} & HP^0(A,I^2) & \xrightarrow{\partial} & HP^1(A/I^2) & \xleftarrow{\sim} & HP^1(A/I)
\end{array}$$

commutes. Therefore, a class $[\tau] \in HP^0(I)$ coming from a trace gets mapped by ∂ to the image of $[\phi_\tau]$ under the Goodwillie isomorphism.

3.3 The Chern Character

Let A be a unital k-algebra and $e \in \text{Idem } A$. For each n, we define a $2n$-cycle $c(e)$ in $\text{Tot}CC_{**}(M_\infty(A))$ by first setting

$$z_i := (-1)^i \frac{(2i)!}{i!} e^{\otimes 2i+1}$$
$$y_i := (-1)^{i-1} \frac{(2i)!}{2(i!)} e^{\otimes 2i},$$

and then defining

$$c_{2n}(e) := z_0 + \sum_{i=1}^{n} (y_i + z_i) \in \bigoplus_{i=0}^{2n} M_\infty(A)^{\otimes i+1} = \text{Tot}CC_{**}(M_\infty(A))_{2n}.$$

Proposition 3.6 (Connes). *For any $e \in \text{Idem } A$, $c_{2n}(e)$ is cycle, so it defines an element of $HC_{2n}(M_\infty(A))$. The association*

$$e \mapsto \text{Tr}_*[c_{2n}(e)],$$

is well defined and induces a natural transformation $\text{ch}_{0,n} : K_0(A) \to HC_{2n}(A)$.

Proof. Clearly, we have $\lambda_{2i}(e^{\otimes 2i+1}) = e^{\otimes 2i+1}$ and in particular $1 - \lambda_0(y_0) = 0$. Thus to check that $c_{2n}(e)$ is a cycle, it suffices to show that $b(y_i) = -(1-\lambda)z_i$ and $b'(z_i) = \mathcal{N} y_{i-1}$. This is immediate since in odd degrees $\lambda_{2i-1}(e^{\otimes 2i}) = -e^{\otimes 2i}$, so $(1-\lambda)z_i = 2z_i$. Since e is an idempotent, $b(e^{\otimes 2i+1}) = e^{\otimes 2i}$ and $b(y_i) = -2z_i$. For b' we have $b'(e^{\otimes 2i}) = e^{\otimes 2i-1}$, so $b' z_i = 2iy_{i-1} = \mathcal{N} y_{i-1}$.
Applying Tr_* yields a class in $HC_{2n}(A)$. To check that we have a homomorphism, we must show that for $g \in GL(A)$ $\text{ch}_{0,n}(geg^{-1}) = \text{ch}_{0,n}(e)$ and $\text{ch}_{0,n}(e \oplus f) = \text{ch}_{0,n}(e) + \text{ch}_{0,n}(f)$. The first equality follows from corollary 2.29, which shows that

$$[c_{2n}(geg^{-1})] = g_*[c_{2n}(e)] = [c_{2n}(e)] \in HC_{2n}(M(A)).$$

The second equality follows from the definition of Tr. In $M(A)$ we have $e \oplus f = e \oplus 0 + 0 \oplus f$. Therefore

$$(e \oplus f)^{\otimes n} = (e \oplus 0)^{\otimes n} + (0 \oplus f)^{\otimes n} + \text{mixed tensors},$$

where "mixed tensors" indicates that both $e \oplus 0$ and $0 \oplus f$ occur. Choose n such that $e, f \in M_n(A)$ and let E_{ij} be the matrices consisting of 0's, except for a 1 in the (i,j)-slot. Writing

$$e \oplus 0 = \sum_{i,j=0}^{n} E_{ij} e_{ij}, \qquad 0 \oplus f = \sum_{i,j=n+1}^{2n} E_{ij} f_{ij},$$

we see that

$$\mathrm{Tr}((e \oplus 0) \otimes (0 \oplus f)) = \sum_{i,j=0}^{n} \sum_{k,l=n+1}^{2n} \mathrm{Tr}(E_{ij}E_{lk}) e_{ij} \otimes f_{lk} = 0,$$

because $E_{ij}E_{lk} = \delta_{jl}E_{ik}$. Since for any mixed tensor $a_0 \otimes ... \otimes a_n$ there is a j such that $a_j = e \oplus 0$ and $a_{j+1} = 0 \oplus f$, it follows that Tr vanishes on mixed tensors. Therefore, $\mathrm{Tr}_*[c_{2n}(e \oplus f)] = \mathrm{Tr}_*[c_{2n}(e)] + \mathrm{Tr}_*[c_{2n}(f)]$. □

From the definition of $\mathrm{ch}_{0,n}$, it is immediate that $S \circ \mathrm{ch}_{0,n} = \mathrm{ch}_{0,n-1}$. From the proof of the proposition it follows that

$$c(e) := y_0 + \sum_{i=1}^{\infty} (y_i + z_i)$$

defines a cycle in $CC_{**}^{per}(M_\infty(A))$. Composition with Tr_* then gives a natural homomorphism

$$\mathrm{Ch}_0 : K_0(A) \to HP_0(A),$$

called the *Chern character* on K_0.

On K_1 we can carry out a similar construction. An element $u \in GL(A)$ comes from an element in $GL(r, A)$ for some r. Let

$$x_i := (i+1)!(u^{-1} - 1) \otimes (u - 1) \otimes ... \otimes (u^{-1} - 1) \otimes (u - 1) \in M_r(A)^{\otimes 2i+2}$$
$$w_i := (i+1)!(u - 1) \otimes (u^{-1} - 1) \otimes ... \otimes (u^{-1} - 1) \otimes (u - 1) \in M_r(A)^{\otimes 2i+1}.$$

For each n, we define a $2n+1$-cycle in $\mathrm{Tot}CC_{**}(M_r(A))$ by

$$c_{2n+1}(u) := \sum_{i=0}^{n} (x_i + w_i) \in \bigoplus_{i=0}^{2n+1} M_r(A)^{\otimes i+1} = \mathrm{Tot}CC_{**}(M_r(A))_{2n+1}.$$

Proposition 3.7 (Karoubi). *For any $u \in GL(r, A)$, $c_{2n+1}(u)$ is a cycle and thus defines an element of $HC_{2n+1}(M_r(A))$. The association*

$$u \mapsto \mathrm{Tr}_*[c_{2n+1}(u)]$$

is well defined and induces a natural transformation $\mathrm{ch}_{1,n} : K_1(A) \to HC_{2n+1}(A)$.

Proof. Since we have

$$(u^{-1} - 1)(u - 1) = (u - 1)(u^{-1} - 1) = -(u - 1) - (u^{-1} - 1),$$

it follows that $b(x_i) = -(1 - \lambda)w_i$ by a simple computation. From the same identity it follows that $\mathcal{N}(x_i) = ix_i - i\lambda x_i = b' w_{i+1}$.
To prove that we obtain a homomorphism, recall that $K_1(A) \cong H_1(GL(A), \mathbb{Z})$ and that

$$HC_n(\mathbb{Z}[\mathbb{B}_* GL(A)]) \cong \bigoplus_{i=0}^{\infty} H_{n-2i}(GL(A), \mathbb{Z}).$$

This last direct sum is actually finite, since there are no homology groups in negative degrees. For any group Γ, denote by

$$i : H_1(\Gamma, \mathbb{Z}) \to HC_{2n+1}(\mathbb{Z}[\mathbb{B}_*\Gamma]),$$

the inclusion. Also consider, for fixed r, the *fusion map* $\mathcal{F} : \mathbb{Z}[GL(r,A)] \to M_r(A)$, defined by viewing a formal sum as an actual sum. There is a sequence of cyclic modules

$$\mathbb{Z}[\mathbb{B}_*GL(r,A)] \xrightarrow{\iota} \mathbb{Z}[GL(r,A)]^\sharp \xrightarrow{\mathcal{F}} M_r(A)^\sharp \xrightarrow{\mathrm{Tr}} A^\sharp \quad (*).$$

It suffices to show that for the element x in the Laurent polynomial ring $R := k[x,x^{-1}] = k[\mathbb{Z}]$, the image $[c_{2n+1}(x)]$ of $[x] \in H_1(R^\times, \mathbb{Z})$ (with $R^\times := GL(1,R)$), coincides with the element $\mathcal{F}_*\iota_* i([x]) \in HC_{2n+1}(R)$. Since then, for any $u \in GL(r,A)$, the unique homomorphism

$$k[x,x^{-1}] \to M_r(A),$$

satisfying $x \mapsto u$ determines a commutative diagram

$$\begin{array}{ccccccc}
H_1(GL(r,A),\mathbb{Z}) & \xrightarrow{i} & HC_{2n+1}(\mathbb{Z}[\mathbb{B}_*GL(r,A)]) & \xrightarrow{\iota_*} & HC_{2n+1}(\mathbb{Z}[GL(r,A)]^\sharp) & \xrightarrow{\mathcal{F}_*} & HC_{2n+1}(M_r(A)^\sharp) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_1(R^\times,\mathbb{Z}) & \xrightarrow{i} & HC_{2n+1}(\mathbb{Z}[\mathbb{B}_*R^\times]) & \xrightarrow{\iota_*} & HC_{2n+1}(\mathbb{Z}[R^\times]^\sharp) & \xrightarrow{\mathcal{F}_*} & HC_{2n+1}(k[R^\times]^\sharp)
\end{array}$$

and it follows that for any $u \in GL(r,A)$, we have $[c_{2n+1}(u)] = \mathcal{F}_*\iota_* i([u])$. Since we saw that

$$\mathrm{Tr}_*(a_0 \oplus 1 \otimes ... \otimes a_n \oplus 1) = \mathrm{Tr}_*(a_0 \otimes ... \otimes a_n) + \mathrm{Tr}_*(1 \otimes ... \otimes 1) = \mathrm{Tr}_*(a_0 \otimes ... \otimes a_n),$$

the sequence (*) is compatible with the inclusions $GL(r,A) \to GL(r+1,A)$, and we get a well defined homomorphism

$$\lim_{r\to\infty} H_1(GL(r,A),\mathbb{Z}) = K_1(A) \to HC_{2n+1}(A)$$
$$[u] \mapsto \mathrm{Tr}_*([c_{2n+1}(u)]).$$

We proceed by identifying a representative of $c_{2n+1}(x)$ in the normalized mixed complex of R. Since R is unital, one has $R_+ \cong R \times k$ by

$$\phi(a,n) = (a + n.1, n)$$

and thus, applying corollary 2.44,

$$\mathrm{Tot}CC_{**}(R^\sharp) \cong \mathrm{Tot}MC_{**}(N(R_+,R)) \cong \mathrm{Tot}MC_{**}(N(R \times k, R)).$$

In $MC_{**}(N(R_+,R))$, $z_i + w_i$ corresponds to $(x^{-1}-1,1) \otimes w_i$, which by ϕ_* gets mapped to

$$(x^{-1},1) \otimes x \otimes (x^{-1} \otimes x)^{\otimes i}.$$

Then, using the projection $B \times k \to R$ we see that in $MC_{**}(N(R))$, $z_i + w_i$ corresponds to $(x^{-1} \otimes x)^{\otimes i+1}$. Thus the homology class of $c_{2n+1}(x)$ is the class of the cycle

$$\sum_{i=0}^{n} (i+1)!(x^{-1} \otimes x)^{i+1} \in MC_{**}(N(R)),$$

which is the image under ι of

$$\overline{x} := \sum_{i=0}^{n} (i+1)!(x, x^{-1}, ..., x).$$

Consider the unique map $\chi : \mathbb{Z} \to R^{\times}$ sending 1 to x, and the diagram

$$\begin{array}{ccc} H_1(R^{\times}, \mathbb{Z}) & \xrightarrow{i} & HC_{2n+1}(\mathbb{Z}[\mathbb{B}_* R^{\times}]) \\ \chi_* \Big\uparrow & & \Big\uparrow \chi_* \\ H_1(\mathbb{Z}, \mathbb{Z}) & \xrightarrow{i} & HC_{2n+1}(\mathbb{Z}[\mathbb{B}_* \mathbb{Z}]) \end{array}$$

it induces. The groups $H_j(\mathbb{Z}, \mathbb{Z})$ vanish for $j \geq 2$, as the relevant classifying space $\mathbb{BZ}$ is homotopy equivalent to the circle S^1. Therefore $i : H_1(\mathbb{Z}, \mathbb{Z}) \to HC_{2n+1}(\mathbb{Z}[\mathbb{B}_* \mathbb{Z}])$ is an isomorphism for all n and its inverse is the canonical projection $\pi : HC_{2n+1}(\mathbb{Z}[\mathbb{B}_* \mathbb{Z}]) \to H_1(\mathbb{Z}, \mathbb{Z})$. The element

$$\overline{1} := \sum_{i=0}^{n} (i+1)!(1, -1, ..., 1),$$

satisfies $\chi(\overline{1}) = \overline{x}$ and $\pi(\overline{1}) = 1$, whose class generates $H_1(\mathbb{Z}, \mathbb{Z})$. Therefore, since $\chi_*([1]) = [x]$, we have

$$i([x]) = i \circ \chi_*([1]) = \chi_* \circ i([1]) = \chi_*([\overline{1}]) = [\overline{x}],$$

which implies

$$[c_{2n+1}(x)] = \mathcal{F}_* \circ \iota_* \circ i([x]).$$

This completes the proof. □

The proof of the proposition shows that

$$c_1(u) := \sum_{i=0}^{\infty} (x_i + w_i)$$

is a cycle in $\mathrm{ToT}CC_{**}^{per}(M_r(A))$ and that the association $[u] \mapsto \mathrm{Tr}_*[c_1(u)]$ defines a natural homomorphism

$$\mathrm{Ch}_1 : K_1(A) \to HP_1(A)$$

called the *Chern character* on K_1. For a complex algebra A, the algebraic homotopy invariance of HP_1 implies that $\mathrm{Ch}_1(uv) = \mathrm{Ch}_1(u)$ for $v \in E(A)$. Since we can transform uw into $u \oplus w$ by multiplication with elements in $E(A)$, the proof that we obtain a homomorphism is much easier in this case. Connes proved this theorem for complex algebras in [4]. The proof we gave here is due to Karoubi, who constructed a Chern character on all the Quillen K-groups $K_i(R)$ in [16]. Although we haven't defined the higher K-groups here, this proof goes through basically unchanged for these groups.
For the study of the exact sequences associated with a two sided ideal $I \subset A$, it is necessary to construct relative Chern characters $\mathrm{Ch}_j : K_j(A, I) \to HP_j(A, I)$, and, in view of the excision results for K_0 and HP, a Chern character $\mathrm{Ch}_0 : K_0(I) \to HP_0(I)$ for non-unital rings. These relative characters can be obtained by looking at the ring $D(A, I)$. The diagram

$$\begin{array}{ccc} D(A,I) & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\pi} & A/I \end{array}$$

commutes, and therefore, at the level of cyclic modules, $p_{1*} : D(A, I)^\sharp \to A^\sharp$ maps $\ker p_{2*}$ to $\ker \pi_* = (A, I)^\sharp$. Naturality of the Chern character implies that $\mathrm{Ch}_j : K_j(D(A, I)) \to HP_j(D(A, I))$ maps $\ker p_{2*} = K_j(A, I)$ to $\ker p_{2*}$. Therefore, it makes sense to define $\mathrm{Ch}_j : K_j(A, I) \to HP_j(A, I)$ to be the composite

$$K_j(A, I) \longrightarrow K_j(D(A, I)) \xrightarrow{\mathrm{Ch}_j} HP_j(D(A, I)) \xrightarrow{p_{1*}} HP_j(A, I).$$

Proposition 3.8. *For $j = 0, 1$, the diagram*

$$\begin{array}{ccccc} K_j(A,I) & \longrightarrow & K_j(A) & \longrightarrow & K_j(A/I) \\ {\scriptstyle \mathrm{Ch}_j}\downarrow & & {\scriptstyle \mathrm{Ch}_j}\downarrow & & {\scriptstyle \mathrm{Ch}_j}\downarrow \\ HP_j(A,I) & \longrightarrow & HP_j(A) & \longrightarrow & HP_j(A/I) \end{array}$$

commutes.

Proof. The square on the right commutes by naturality of Ch_j. The square on the left commutes since the map $K_j(A, I) \to K_j(A)$ is essentially the projection $p_{1*} : K_j(D(A, I)) \to K_j(A)$ restricted to $K_j(A, I)$. Commutativity thus follows from the definition of the relative Chern character. $\square$

For non-unital rings, naturality gives a Chern character, since the diagram

$$\begin{array}{ccc} K_j(I_+) & \xrightarrow{\rho_*} & K_j(k) \\ \Big\downarrow \mathrm{Ch}_j & & \Big\downarrow \mathrm{Ch}_j \\ HP_j(I_+) & \xrightarrow{\rho_*} & HP_j(k) \end{array}$$

commutes, and therefore Ch_j maps $K_j(I) = \ker \rho_*$ to $HP_j(I) = \ker \rho_*$. The excision isomorphisms for K_0 and HP_0 imply that the diagram

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \Big\downarrow \mathrm{Ch}_0 & & \Big\downarrow \mathrm{Ch}_0 & & \Big\downarrow \mathrm{Ch}_0 \\ HP_0(I) & \longrightarrow & HP_0(A) & \longrightarrow & HP_0(A/I) \end{array}$$

commutes.

Using the Chern character, we can generalize the pairings from section 3.1 by composing the Chern character with the canonical pairing

$$HC_n(A) \otimes_k HC^n(A) \to k,$$

defined by evaluating a cyclic cocycle on a cyclic cycle.

Theorem 3.9. *The pairings*

$$K_0(A) \otimes_{\mathbb{Z}} HC^{2n}(A) \to k, \qquad K_1(A) \otimes_{\mathbb{Z}} HC^{2n+1}(A) \to k,$$

generalize the pairings

$$HH^0(A) \otimes_{\mathbb{Z}} K_0(A) \to k, \qquad HH^1(A) \otimes_{\mathbb{Z}} K_1(A) \to k.$$

Proof. Since $HH^0(A) \cong HC^0(A)$, we clearly have a generalization of the pairing with traces. In the odd case, the Chern character of $[u] \in K_1(A)$ in $HC_1(A)$ equals

$$\mathrm{Tr}_*((u^{-1} - 1) \otimes (u - 1)),$$

because b_1' is surjective (since A is unital), so we can forget about the degree 0 part. We show that this is cohomologous to $\mathrm{Tr}_*(u^{-1} \otimes u)$. Since Tr_* is a chain map, it suffices to show that $(u^{-1} - 1) \otimes (u - 1)$ is cohomologous to $u^{-1} \otimes u$. We can do this by adding elements in the images of b and $1 - \lambda$.

$$\begin{aligned} &[(u^{-1} - 1) \otimes (u - 1)] \\ &\quad = [(u^{-1} - 1) \otimes (u - 1) + (1 - \lambda)(1 \otimes u) + b(1 \otimes (u + u^{-1} - 1) \otimes 1)] \\ &\quad = [(u^{-1} - 1) \otimes (u - 1) + 1 \otimes u - u \otimes 1 + u \otimes 1 + u^{-1} \otimes 1 - 1 \otimes 1] \\ &\quad = [u^{-1} \otimes u]. \quad \square \end{aligned}$$

3.4 The universal extension

We want to prove that the Chern character yields a natural transformation from the exact sequence in lower algebraic K-theory to the six term exact sequence in periodic cyclic homology. We do this by constructing an exact sequence of algebras for which the result holds, and then use a universal property of this sequence.

Definition 3.10. Let V be a vector space over $\mathbb{C}$. The *tensor algebra* $T(V)$ over V is the algebra

$$T(V) := \bigoplus_{i=0}^{\infty} V^{\otimes i}, \quad V^{\otimes 0} = \mathbb{C},$$

with the obvious addition operation and the product defined by concatenation of tensors.
The *free non-commutative algebra on n symbols* $\mathbb{C}\langle a_1, .., a_n\rangle$ is the tensor algebra of the vector space $\mathbb{C}a_1 \oplus ... \oplus \mathbb{C}a_n$.

The following result is due to Loday and Quillen, who proved it in [18].

Theorem 3.11 (Loday-Quillen). *Let V be a complex vector space and $T(V)$ its tensor algebra. The inclusion $\mathbb{C} \to T(V)$ given by $z \mapsto z.1$ induces an isomorphism in periodic cyclic (co)homology.*

Proof. Because the map $z \mapsto z.1$ is a split injection, it suffices to show that $HP_0(T(V)) \cong \mathbb{C}$ and $HP_1(T(V)) = 0$. To do this, we use the Tor interpretation of Hochschild homology, cf. proposition 2.19. For cohomology the argument is exactly dual. We first argue that the standard b'-resolution of $T(V)$ as a $T(V)^e$-module can be replaced by a finite resolution. The map $b'_1 : T(V) \otimes T(V) \to T(V)$ is just the multiplication map. Its kernel is the the $T(V)^e$-module generated by the expressions $x \otimes 1 - 1 \otimes x$ for $x \in T(V)$. This is the image of the module $T(V) \otimes V \otimes T(V)$ under the map b'_2, where V is considered part of $T(V)$. For notational convenience, we will denote the multiplication in $T(V)$ by $(a_0, a_1) \mapsto a_0a_1$. It suffices to show that elements of the form $(v_0...v_n) \otimes 1 - 1 \otimes (v_0...v_n)$, with $v_i \in V$, are in the image of b'_2. This is immediate, since

$$b'_2(v_0...v_{n-1}) \otimes v_n \otimes 1 + 1 \otimes v_0 \otimes (v_1...v_n) = (v_0...v_n) \otimes 1 - 1 \otimes (v_0...v_n).$$

So we have a resolution

$$0 \longrightarrow T(V) \otimes V \otimes T(V) \longrightarrow T(V) \otimes T(V) \longrightarrow T(V) \longrightarrow 0.$$

Since V is a vector space, $T(V) \otimes V \otimes T(V)$ is a free $T(V)^e$-module (for a basis of V gives a basis of this module), hence projective. Tensoring with $T(V)$ over $T(V)^e$ yields the complex

$$0 \longrightarrow T(V) \otimes V \longrightarrow T(V) \longrightarrow 0,$$

with the nontrivial map given by $a \otimes v \mapsto av - va$. We have shown that the embedding of this complex into the normalized b-complex of $T(V)$ induces an isomorphism in homology. In the normalized mixed complex, the action of $B : T(V) \to T(V) \otimes \overline{T(V)}$ is given by $a \mapsto 1 \otimes a$. Modulo the image of b_2 we have

$$a_0 \otimes a_1 a_2 = a_0 a_1 \otimes a_2 + a_2 a_0 \otimes a_1,$$

so in particular the map $HH_0(T(V)) \to HH_1(T(V))$ induced by B is given by

$$\tilde{B}(v_0 ... v_n) = \sum_{i=0}^{n} (v_{i+1} ... v_n v_0 ... v_{i-1}) \otimes v_i.$$

If we consider $\tilde{B}$ as a map $T(V) \to T(V) \otimes V$, then $HP_*(T(V))$ is the homology of the complex

$$... \xrightarrow{b_1} T(V) \xrightarrow{\tilde{B}} T(V) \otimes V \xrightarrow{b_1} ...$$

To prove the statement of the theorem, we examine this complex a bit more carefully. It inherits a grading from the grading of $T(V)$, and if we want the maps to preserve the grading, then the degree m part of $T(V) \otimes V$ is $V^{\otimes m+1}$. The complex consisting of the positively graded part of $T(V)$ and all of $T(V) \otimes V$ is contractible, for one checks that

$$H_0 : T(V)_{>0} \to T(V) \otimes V \quad \text{and} \quad H_1 : T(V) \otimes V \to T(V)_{>0}$$

defined by

$$H_0(v_1 ... v_m) := -\frac{1}{m} \sum_{j=1}^{m} j v_{j+1} ... v_m v_0 ... v_{j-1} \otimes v_j,$$

and

$$H_1(v_1 ... v_{m+1}) := \frac{1}{m+1} v_1 ... v_m,$$

satisfy

$$H_0 b_1 + \tilde{B} H_1 = \mathrm{id}_{V^{\otimes m+1}}, \quad b_1 H_0 + H_1 \tilde{B} = \mathrm{id}_{V^{\otimes m}}.$$

It follows that $HP_0(T(V)) \cong \mathbb{C}$, while $HP_1(T(V)) = 0$. $\square$

To be able to compute the cyclic homology of the universal extension we are about to construct, we need a specific algebra of operators, defined as follows. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with basis $\{e_0, e_1, ...\}$ and $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. The *shift operator* on $\mathcal{H}$ is the linear operator defined by $Se_i := e_{i+1}$. Clearly S is injective and its adjoint (with respect to the standard inner product) is determined by $S^* e_i = e_{i-1}$. This map is not injective, but we have the relations

$$S^* S = 1, \quad 1 - SS^* = p_0,$$

where $p_0 : \mathcal{H} \to \mathcal{H}$ is projection on the subspace $\mathbb{C}e_0$.

Definition 3.12. The (algebraic) *Toeplitz algebra* is the subalgebra of $B(\mathcal{H})$ generated by S and S^*.

Viewing elements of $B(\mathcal{H})$ as matrices, it is clear that $M_\infty(\mathbb{C}) \subset \mathcal{T}$. It is the two-sided ideal generated by $1 - SS^*$. There is a homomorphism $q : \mathcal{T} \to \mathbb{C}[z, z^{-1}]$ defined by sending S to z and S^* to z^{-1}. Let I denote the kernel of this map. Clearly $M_\infty(\mathbb{C}) \subset I$. Therefore there is a surjection $Q : \mathcal{T}/M_\infty(\mathbb{C}) \to \mathbb{C}[z, z^{-1}]$ sending the class of $x \in \mathcal{T}$ to $q(x)$. On the other hand, there is a linear splitting $s : \mathbb{C}[z, z^{-1}] \to \mathcal{T}$ sending z to S and z^{-1} to S^*. It is clear that $M_\infty(\mathbb{C}) \cap \operatorname{im} s = 0$, since a polynomial in S and S^* has infinite dimensional range. Therefore s defines an injection $\mathbb{C}[z, z^{-1}] \to \mathcal{T}/M_\infty(\mathbb{C})$. Since $sQ = 1$, Q is injective and hence an isomorphism. Therefore $I = M_\infty(\mathbb{C})$. Let $\mathbb{C}\langle a, b\rangle$ be the free noncommutative algebra on 2 symbols. There is a homomorphism $\mathbb{C}\langle a, b\rangle \to \mathbb{C}[z, z^{-1}]$ defined by sending a to z and b to z^{-1}. Let J be the kernel of this map and $\phi_0 : \mathbb{C}\langle a, b\rangle \to \mathcal{T}$ the homomorphism defined by $a \mapsto S$ and $b \mapsto S^*$. By restriction we get a homomorphism $\phi : J \to M_\infty(\mathbb{C})$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \mathbb{C}\langle a, b\rangle & \longrightarrow & \mathbb{C}[z, z^{-1}] & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi_0} & & \Vert & & \\
0 & \longrightarrow & M_\infty(\mathbb{C}) & \longrightarrow & \mathcal{T} & \longrightarrow & \mathbb{C}[z, z^{-1}] & \longrightarrow & 0.
\end{array}$$

Proposition 3.13. *$HP^*(J)$ is singly generated by the trace* $\operatorname{Tr} \circ \phi$.

Proof. We want to show that the map $\mathbb{C} \to \mathcal{T}$ defined by $z \mapsto z.1$ induces an isomorphism in in periodic cyclic (co)homology. This suffices because there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{z \mapsto z.1} & \mathbb{C}\langle a, b\rangle \\
{\scriptstyle z \mapsto z.1}\Big\downarrow & \swarrow{\scriptstyle \phi_0} & \\
\mathcal{T} & &
\end{array}$$

and hence ϕ_0 induces an isomorphism in periodic cyclic (co)homology. It follows (with the five lemma) that ϕ induces an isomorphism as well, which immediately implies the statement of the proposition. Again, to prove our desired result, it suffices to show that $HP_0(\mathcal{T}) \cong \mathbb{C}$ and $HP_1(\mathcal{T}) = 0$, because there is a splitting of $z \mapsto z.1$, defined by sending both S and S^* to 1.

We examine the six term exact sequence

$$\begin{array}{ccccc} HP^0(M_\infty(\mathbb{C})) & \longleftarrow & HP^0(\mathcal{T}) & \longleftarrow & HP^0(\mathbb{C}[z,z^{-1}]) \\ \partial \big\downarrow & & & & \big\uparrow \\ HP^1(\mathbb{C}[z,z^{-1}]) & \longrightarrow & HP^1(\mathcal{T}) & \longrightarrow & HP^1(M_\infty(\mathbb{C})) \end{array}$$

associated to the Toeplitz extension (the bottom row of the diagram above). We know that

$$\mathrm{Tr}_* : HP^*(\mathbb{C}) \to HP^*(M_\infty(\mathbb{C}))$$

is an isomorphism and since id : $\mathbb{C} \to \mathbb{C}$ generates $HP^*(\mathbb{C})$ we know that Tr : $M_\infty(\mathbb{C}) \to \mathbb{C}$ generates $HP^0(M_\infty(\mathbb{C}))$. Also, $M_\infty(\mathbb{C})^2 = M_\infty(\mathbb{C})$, because this algebra is locally unital, and the diagram of lemma 3.5 becomes

$$\begin{array}{ccc} HC^0(\mathcal{T}, M_\infty(\mathbb{C})) & \xrightarrow{\partial} & HC^1(\mathbb{C}[z,z^{-1}]) \\ \big\downarrow & & \big\downarrow \\ HP^0(M_\infty(\mathbb{C})) & \xrightarrow{\partial} & HP^1(\mathbb{C}[z,z^{-1}]). \end{array}$$

Thus ∂ maps Tr to ϕ_{Tr}. So to show that ∂ is an isomorphism, it suffices to show that ϕ_{Tr} is non-trivial, since we also know that $HP^1(\mathbb{C}[z,z^{-1}]) \cong \mathbb{C}$. To do this, we compute the pairing of ϕ_{Tr} with $[z] \in K_1(\mathbb{C}[z,z^{-1}])$, using the linear map $s : \mathbb{C}[z,z^{-1}] \to \mathcal{T}$:

$$\phi_{\mathrm{Tr}}([z]) = \phi_{\mathrm{Tr}}(z, z^{-1}) = \mathrm{Tr}([S, S^*]) = \mathrm{Tr}(SS^* - 1) = -1.$$

Thus the six term exact sequence becomes

$$\begin{array}{ccccc} \mathbb{C} & \xleftarrow{0} & HP^0(\mathcal{T}) & \xleftarrow{\sim} & \mathbb{C} \\ \sim \big\downarrow & & & & \big\uparrow \\ \mathbb{C} & \xrightarrow{0} & HP^1(\mathcal{T}) & \longrightarrow & 0 \end{array}$$

and we read off our desired result. □

The following theorem can be regarded as an abstract cohomological index formula. We will use it in the next chapter, in the proof of the Connes-Moscovici index theorem.

Theorem 3.14 (Nistor). *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

be an exact sequence of complex algebras. Then for any $\psi \in HP^0(I)$ *and any* $[u] \in K_1(A/I)$ *we have*

$$\psi_*(\mathrm{Ind}[u]) = (\partial\psi)_*([u]).$$

Proof. First observe that the theorem is true for the universal extension

$$0 \longrightarrow J \longrightarrow \mathbb{C}\langle a, b\rangle \longrightarrow \mathbb{C}[z, z^{-1}] \longrightarrow 0,$$

for $HP^0(J)$ is generated by the trace $\mathrm{Tr}\circ\phi$, which satisfies $\mathrm{Tr}\circ\phi([\mathbb{C}\langle a, b\rangle, J]) = 0$ because

$$\mathrm{Tr} \circ \phi([\mathbb{C}\langle a, b\rangle, J]) \subset \mathrm{Tr}([\mathcal{T}, M_\infty(\mathbb{C})]) = 0.$$

So the theorem follows from lemmas 3.2 and 3.5. We will use this particular case to prove the general case. Let $[u] \in K_1(A/I)$ be given by an invertible matrix $u \in M_n(A/I)$. By Morita invariance we may assume $u \in A/I$ and u determines a unique homomorphism $\eta : \mathbb{C}[z, z^{-1}] \to A/I$. By choosing liftings $a_0, b_0 \in A$ of u and u^{-1}, we get a homomorphism $\phi_0 : \mathbb{C}\langle a, b\rangle \to A$, and its restriction J gives a map $\phi : J \to I$. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \mathbb{C}\langle a, b\rangle & \longrightarrow & \mathbb{C}[z, z^{-1}] & \longrightarrow & 0 \\
 & & \downarrow \phi & & \downarrow \phi_0 & & \downarrow \eta & & \\
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0
\end{array}$$

commutes and naturality of Ind and ∂ give

$$\phi_* \circ \mathrm{Ind} = \mathrm{Ind} \circ \eta_* : K_1(\mathbb{C}[z, z^{-1}]) \to K_0(I),$$

and

$$\partial \circ \phi^* = \eta^* \circ \partial : HP^0(I) \to HP^1(\mathbb{C}[z, z^{-1}]).$$

Since the theorem holds for the cocycle $\phi^*(\psi)$, we can compute

$$\begin{aligned}
\psi_*(\mathrm{Ind}[u]) &= \psi_*(\mathrm{Ind} \circ \eta_*[z]) \\
&= \psi_*(\phi_* \circ \mathrm{Ind}[z]) \\
&= (\phi^*(\psi))_*(\mathrm{Ind}[z]) \\
&= (\partial \circ \phi^*(\psi))_*[z] \\
&= (\eta^* \circ \partial(\psi))_*[z] \\
&= (\partial\psi)_*(\eta_*[z]) \\
&= (\partial\psi)_*([u]). \quad \square
\end{aligned}$$

Corollary 3.15. *The following diagram commutes:*

$$\begin{array}{ccccccccccc}
K_1(A, I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(A/I) & \xrightarrow{\mathrm{Ind}} & K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
HP_1(A, I) & \longrightarrow & HP_1(A) & \longrightarrow & HP_1(A/I) & \xrightarrow{\partial} & HP_0(I) & \longrightarrow & HP_0(A) & \longrightarrow & HP_0(A/I).
\end{array}$$

Proof. In view of the results of the previous section, we only have to show that $\mathrm{Ch}_0 \circ \mathrm{Ind} = \partial \circ \mathrm{Ch}_1$. And by the same reasoning as above, it suffices to prove the relation for the universal extension. For all $\psi \in HP^0(J)$ and $[u] \in K_1(\mathbb{C}[z, z^{-1}])$ we have

$$\psi(\mathrm{Ch}_0\mathrm{Ind}[u]) = \partial\psi(\mathrm{Ch}_1[u]),$$

which is just a reformulation of the previous theorem. By definition of the boundary map in homology and cohomology, we have

$$\partial\psi(\mathrm{Ch}_1([u])) = \psi(\partial\mathrm{Ch}_1([u])),$$

and thus

$$\psi(\mathrm{Ch}_0(\mathrm{Ind}[u])) = \psi(\partial\mathrm{Ch}_1([u])).$$

The groups $HP^0(J)$ and $HP_0(J)$ are both isomorphic to $\mathbb{C}$ and the canonical pairing is nontrivial. Hence it is perfect, so that $\mathrm{Ch}_0(\mathrm{Ind}[u]) = \partial\mathrm{Ch}_1([u])$. □

3.5 Some remarks on the topological case

For locally convex algebras, there is a natural transformation $HP_*(A) \to HP_*^c(A)$, defined by the inclusion of double complexes $CC_{**}^{per}(A^\sharp) \hookrightarrow CC_{**}^{per}(A_c^\sharp)$. Using the Chern character, this gives a natural transformation

$$\mathrm{Ch}_i^c : K_i(A) \to HP_i^c(A)$$

and a pairing

$$K_i(A) \otimes_{\mathbb{Z}} HP_c^i(A) \to \mathbb{C},$$

which is intensively studied by Connes in [4]. To prove the results of the previous paragraph we need to restrict ourselves to m-algebras, since the excision theorem in periodic cyclic homology only extends to this class of locally convex algebras. Thus, we need m-algebra versions of the Toeplitz and universal extensions.

The universal extension is in a way the easiest to handle. For a tensor algebra $T(V)$, it is convenient to topologize it by the seminorms induced by seminorms $p : V \to \mathbb{C}$ on V. These extend to $T(V)$ by $p(a_0 \otimes .... \otimes a_n) = p(a_0)...p(a_n)$. The completion is denoted $\hat{T}(V)$. This m-algebra has the following universal property (analogous to the universal property of its algebraic version): Given an m-algebra A and a continuous map $\sigma : V \to A$, there is a unique homomorphism of m-algebras $\hat{T}(V) \to A$ extending σ. Let $\mathcal{O}(\mathbb{C}^*)$ be the algebra of holomorphic functions on the punctured plane. This is an m-algebra in the usual seminorms, given by

$$p_n^K(f) := \sum_{i=0}^{n} \frac{1}{i!} \sup_K \| \frac{\partial^i f}{\partial z^i} \|,$$

for each compact set $K \subset \mathbb{C}^*$. It can be identified with the algebra of Laurent series

$$\{\sum_{n\in\mathbb{Z}} a_n z^n : \sum_{n\in\mathbb{Z}} |a_n| x^n < \infty, \quad \forall x \in \mathbb{R}_{>0}\}.$$

This algebra also has a universal property, analogous to the universal property of $\mathbb{C}[z, z^{-1}]$. Given an invertible element u in an m-algebra A, there is a unique homomorphism of m-algebras $\eta : \mathcal{O}(\mathbb{C}^*) \to A$ such that $\eta(z) = u$. By abuse of notation, we denote by $\mathbb{C}\langle a, b\rangle$ the m-algebra $\hat{T}(V)$ where $V = \mathbb{C}a \oplus \mathbb{C}b$. The unique map $\eta_0 : \mathbb{C}\langle a, b\rangle \to \mathcal{O}(\mathbb{C}^*)$ given by $\eta_0(a) = z, \eta_0(b) = z^{-1}$, is surjective. This is because there is an obvious splitting. One has to check convergence, but this works by the very definitions of these algebras. Denote the kernel of this map by $\hat{J}$. The m-algebra version of the universal extension is then

$$0 \longrightarrow \hat{J} \longrightarrow \mathbb{C}\langle a, b\rangle \longrightarrow \mathcal{O}(\mathbb{C}^*) \longrightarrow 0.$$

For the Toeplitz extension, we take $C^\infty(S^1)$, the completion of $\mathbb{C}[z, z^{-1}]$ in the seminorms $p_n(f) := \sum_{i=0}^n \frac{1}{i!} \sup_{z\in S^1} \|f^{(i)}(z)\|$. $M_\infty(\mathbb{C})$ is replaced by the algebra $\hat{\mathcal{K}}$ of *smooth compact operators*. This is the completion of $M_\infty(\mathbb{C})$ in the seminorms

$$p_n(a_{ij}) = \sum_{i,j} |1 + i + j|^n |a_{ij}|.$$

The *smooth Toeplitz algebra* $\hat{\mathcal{T}}$ is then defined by equipping the direct sum $\mathcal{K} \oplus C^\infty(S^1)$ with the product determined by letting z act as the shift and z^{-1} as its adjoint. Then the *smooth Toeplitz extension* reads

$$0 \longrightarrow \hat{\mathcal{K}} \longrightarrow \hat{\mathcal{T}} \longrightarrow C^\infty(S^1) \longrightarrow 0.$$

The inclusion $\mathcal{O}(\mathbb{C}^*) \hookrightarrow C^\infty(S^1)$ induces an isomorphism in continuous periodic cyclic (co)homology, which one checks using differential forms. Moreover $HP_c^*(\hat{\mathcal{K}}) \cong HP_c^*(\mathbb{C})$. By diffotopy invariance, $HP_*(\hat{T}(V)) \cong HP_*(\mathbb{C})$, and the proof of proposition 3.13 carries over to this setting, using continuous periodic cyclic cohomology. There are no obstructions in proving the analogue of theorem 3.14 and its corollary, then, because of the universal property of $\mathcal{O}(\mathbb{C}^*)$.
Passing to topological K-theory is a more delicate matter, for several reasons. In the first place, the topological K-theory we discussed in section 1.7 is not suitable for the type of theorem we want to prove here, because periodic cyclic (co)homology is ill-behaved on a large class of C^*-algebras: For a commutative C^*-algebra $C(X)$, one has

$$HP^0(C(X)) = \{\text{Radon measures on } X\}, \quad HP^1(C(X)) = 0.$$

For a proof, see [4]. Thus, proving something in this category would not be a very applicable result. Defining diffotopy-invariant and excisive topological K-theory for locally convex algebras is done by Cuntz in [8]. If the extension in question satisfies Bott periodicity, then it can be shown that the six term exact sequence of topological K-theory is mapped naturally to the six-term exact sequence in periodic cyclic homology. However, for the boundary map $\mathrm{Ind}_0 : K_0(A/I) \to K_1(I)$, a factor of $2\pi i$ must be taken into account. Since we will not use this result, we will not go into it here.

3.6 The Gohberg-Krein theorem

We can now prove theorem 1.40, the Gohberg-Krein index theorem for Toeplitz operators. If we take the C^*-algebra completion of the Toeplitz extension, we recover the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

of C^*-algebras, from section 1.8. The theorem states that the boundary map in topological K-theory maps the symbol of a Fredholm Toeplitz operator to its index. We will prove this statement for the smooth Toeplitz extension and algebraic K-theory. In section 3.1, we saw that the index map associated to the Calkin extension maps the symbol of a Fredholm operator to its index. Since the Toeplitz extension is a subextension, the index map for this extension sends a Fredholm Toeplitz operator to its index. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{\mathcal{K}} & \longrightarrow & \hat{\mathcal{T}} & \longrightarrow & C^\infty(S^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T} & \longrightarrow & C(S^1) & \longrightarrow & 0
\end{array}$$

and the fact that the map $K_0(\hat{\mathcal{K}}) \to K_0(\mathcal{K})$ is the identity then imply that the class $[f] \in K_1(C^\infty(S^1))$ of a smooth symbol is mapped to its index. But by homotopy invariance of K_0, the image of a symbol f under the map $\mathrm{Ind} : K_1(C(S^1)) \to K_0(\mathcal{K})$ depends only on the class $[f] \in K_1^{top}(C(S^1))$. By the density theorem, the smooth symbols exhaust this topological K-group, and every continuous non-vanishing function on S^1 is homotopic to a smooth one. Since the winding number is a homotopy invariant, theorem 1.40 is implied by

Theorem 3.16 (Smooth Gohberg-Krein index theorem). *Let*

$$0 \longrightarrow \hat{\mathcal{K}} \longrightarrow \hat{\mathcal{T}} \longrightarrow C^\infty(S^1) \longrightarrow 0$$

be the smooth Toeplitz extension and $\mathrm{Tr} : \hat{\mathcal{K}} \to \mathbb{C}$ *the canonical trace. The index of a smooth Fredholm Toeplitz operator with symbol* f *is given by*

$$\mathrm{Tr}_*\mathrm{Ind}([f]) = -\frac{1}{2\pi i}\int_{S^1}\frac{df}{f}.$$

Proof. We have $\mathrm{Tr}_*\mathrm{Ind}([f]) = (\partial\mathrm{Tr})_*([f])$. As in the proof of proposition 3.13, we have $\phi_{\mathrm{Tr}}([z]) = -1$. By Connes' theorem, $HP_c^1(C^\infty(S^1)) \cong \mathbb{C}$, and this group is the dual of $H^1_{DR}(S^1)$, which is generated by the functional

$$[\omega] \mapsto \int_{S^1}\omega.$$

Because

$$-1 = \phi_{\mathrm{Tr}*}([z]) = \phi_{\mathrm{Tr}}(\mathrm{Ch}([z])) = \phi_{\mathrm{Tr}}(zdz^{-1}) = -\frac{1}{2\pi i}\int_{S^1}\frac{dz}{z}$$

under the Connes isomorphism, we must have

$$\phi_{\mathrm{Tr}*}([f]) = -\frac{1}{2\pi i}\int_{S^1}\frac{df}{f}$$

for all invertible symbols f. □

This result illustrates, that theorem 3.14 is very useful in proving index theorems. The two cases we considered are very simple compared to to the Atiyah-Singer and Connes-Moscovici index theorems. With the machinery developed so far, we can now turn to the proof of the index theorem for coverings.

Chapter 4

The Connes-Moscovici theorem

We now have at our disposal all the tools to give a detailed account of Nistor's proof of the higher index theorem for covering spaces, given in [20]. We will closely follow his line of proof.
The theorem concerns the K-theoretic index of certain elliptic operators on a noncompact manifold. To obtain a numerical invariant, we pair the index with periodic cyclic cohomology. Theorem 3.14 will be used to compute this pairing, reducing the computation of Ind to the computation of ∂. We will use the properties of ∂ obtained in chapter 2, with respect to the external product in discrete periodic cyclic cohomology. The interplay between the discrete and continuous theories will be of crucial importance.The index theorem of Atiyah and Singer will also play a prominent role.
The proof given is entirely different from the proof given by Connes and Moscovici in [7]. First we give a brief overview of the Atiyah-Singer theorem and the necessary concepts and results from algebraic topology. Then we introduce etale groupoids, because these provide a convenient framework to work with manifolds and discrete groups at the same time.

4.1 The Atiyah-Singer index theorem

In this section, we give a brief overview of one of the landmark theorems of twentieth century mathematics. We will only need the K-theoretic formulation of this theorem, which can be given by means of a quite simple formula. The discussion here is mainly based on Van Erp's discussion of the C^*-algebraic proof of the index theorem in [12], and what we present here are the parts of his exposition that are relevant to the understanding of the proof of the higher index theorem we are concerned with here.
First we will say a few words about classical pseudo-differential operators. Let M be a manifold and U an open subset of M, homeomorphic to an open subset in

$\mathbb{R}^n$. Consider functions $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ which are positively homogeneous of order k in ξ and allow a formal expansion

$$p(x,\xi) = \sum_{i=m}^{-\infty} p_i(x,\xi).$$

The p_i are required to satisfy the bound

$$|D_\xi^\alpha D_x^\beta p_i(x,\xi)| \leq C_{\alpha\beta}^K (1+|\xi|)^{i-|\alpha|},$$

for all multi indices α and β and each compact subset $K \subset U$ on which the constants $C_{\alpha\beta}^K$ depend. The integer i is called the *order* of p_i. Functions p satisfying these technical requirements are called *classical symbols*. They act on compactly supported distributions u on U by

$$p(x,D)u(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{u}(\xi)d\xi.$$

Such operators are called *pseudodifferential operators on U*. The leading coefficient of $p(x,\xi)$ restricted to $C^\infty(U \times S^{n-1})$ is called the *principal symbol* of $P = p(x,D)$ and is denoted $\sigma(P)$. It turns out that this part of the symbol behaves well under coordinate transformations. The following definition generalizes the notion of pseudodifferential operator to smooth manifolds.

Definition 4.1. Let M be a compact Riemannian manifold, $\mathcal{E}(M)$ the space of compactly supported distributions on M and $\mathcal{D}(M)$ the space of all distributions on M. A linear map

$$P : \mathcal{E}(M) \to \mathcal{D}(M)$$

is called a *pseudo-differential operator of order m* if for any local chart $U \subset M$, the restriction of P to U is a pseudo-differential operator of order m in U.

Note that every operator of order k is also of order m for all $m \geq k$. We denote by $\Psi^k(M)$ the space of order at most k pseudo-differential operators on M. Fixing a finite measure on M, $\Psi^0(M)$ acts on the infinite dimensional separable Hilbert space $L^2(M)$ by bounded operators. Therefore, we can complete $\Psi^0(M)$ to a C^*-algebra, denoted $\overline{\Psi^0(M)}$. The principal symbol map $\sigma : \Psi^0(M) \to C^\infty(S^*M)$ is well defined and turns out to be an algebra homomorphism that is continuous with respect to the Hilbert space norm on $B(L^2(M))$ and the supremum norm on $C^\infty(S^*M)$. Therefore it extends to the completions of both algebras in these norms. There kernel of σ turns out to be $\mathcal{K}(L^2(M))$. In other words, there is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\Psi^0(M)} \longrightarrow C(S^*M) \longrightarrow 0$$

of C^*-algebras.

A pseudodifferential operator P on M is called *elliptic* if its principal symbol $\sigma(P)$ is invertible in $C^\infty(S^*M)$ (and thus also in $C(S^*M)$). One can show show that such operators are Fredholm, when considered as operators in $L^2(M)$. In the C^*-algebraic proof, it is shown that

$$\mathrm{Ind}_0 : K_1^{top}(C(S^*M)) \to K_0(\mathcal{K}) \cong \mathbb{Z}$$

maps the K_1^{top}-class of the principal symbol $\sigma(P)$ of an elliptic operator to its index. The theorem consists in identifying this index as an integral of certain differential forms on M, which we will now discuss.

The differential forms involved are called characteristic classes. The study of these classes is, among other things, the subject of algebraic topology. We will recall some of the classical theory of characteristic classes, of which the Chern character discussed in chapter 3 is a modern algebraic analogue. Let $I_r(\mathbb{C})$ be the algebra of symmetric polynomials in r variables. It contains the polynomials

$$\sigma_p(z_1, ..., z_r) = \sum_{i=1}^{r} z_i^p,$$

and these can be chosen as the generators of this algebra.
Let E be an r-dimensional complex vector bundle over M, equipped with a connection ∇, with curvature matrix k_∇. Viewing k_∇ as an $\Omega(M)$-linear map on the space of E-valued forms on M, we define the *characteristic class associated to* $f \in I_r(\mathbb{C})$ as the cohomology class in $\bigoplus_{k=0}^{\infty} H_{DR}^{2k}(M)$ of the form

$$f(\lambda_1(k_\nabla), ..., \lambda_r(k_\nabla)).$$

Here $\lambda_i(k_\nabla)$ denotes the i-the eigenvalue of k_∇.
These cohomology classes turn out to be independent of the connection ∇. If E is given by the idempotent $e \in M_k(C^\infty(M))$, we can always endow it with the connection ede, where d is the DeRham differential. This connection has curvature $edede$ and the k-th classical Chern class $\mathrm{ch}_{2k}(E)$ of E is the characteristic class associated to the polynomial $(-1)^k k!(2\pi i)^{-k}\sigma_k$. The classical Chern character is then

$$\sum_{k=0}^{\infty} \mathrm{ch}_{2k}(E) \in \bigoplus_{k=0}^{\infty} H_{DR}^{2k}(M),$$

which is actually a finite sum. Under the Connes isomorphism $\chi : HP_0^c(C^\infty(M)) \to \bigoplus_{k=0}^{\infty} H_{DR}^{2k}(M)$ we have the relation

$$\chi(\mathrm{Ch}(e)) = \sum_{k=0}^{\infty} (2\pi i)^k \mathrm{ch}_{2k}(e),$$

between the classical Chern character ch and the Chern character in periodic cyclic homology. Recall that in classical topological K-theory, $K^1(M) \cong K^0(\Sigma M)$, where ΣM is the smooth suspension of M. The cohomology groups of ΣM are just those of M with a degree shift: $H_{DR}^*(\Sigma M) = H_{DR}^{*-1}(M)$. Using the Chern character on K^0, we obtain a Chern character

$$\mathrm{ch} : K^1(M) \to \bigoplus_{k=0}^{\infty} H_{DR}^{2k+1}(M).$$

Using the natural map

$$K_1(C^\infty(M)) \to K_1(C(M)) \to K^1(M),$$

which by the density theorem is surjective, we have

$$\chi(\mathrm{Ch}([u])) := \sum_{k=0}^{\infty} (2\pi i)^k \mathrm{ch}_{2k-1}([u]).$$

The most important theorem about the classical Chern character states that it detects all of the torsion free part of topological K-theory.

Theorem 4.2 (Chern isomorphism). *Let M be a smooth manifold. The classical Chern character* $\mathrm{ch} : K^*(M) \to \bigoplus_{i=0}^{\infty} H^{*+2i}(M, \mathbb{C})$ *induces an isomorphism*

$$K^*(M) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}_* \otimes \mathrm{id}} \bigoplus_{i=0}^{\infty} H_{DR}^{*+2i}(M, \mathbb{C}).$$

Corollary 4.3. *For a smooth manifold M, the map*

$$\mathrm{Ch}_* \otimes \mathrm{id} : K_*(C^\infty(M)) \otimes \mathbb{C} \to HP_*(C^\infty(M))$$

on algebraic K-theory is surjective.

Proof. By the previous theorem, the image classical Chern character

$$\mathrm{ch} : K^*(M) \to \bigoplus_{i=0}^{\infty} H^{*+2i}(M, \mathbb{C}) \cong HP_*(C^\infty(M))$$

generates $HP_*(C^\infty(M))$. By the density theorem the composite

$$K_*(C^\infty(M)) \to K_*(C(M)) \to K^*(M)$$

is surjective. Therefore the statement follows by the relation between the classical Chern character ch and the Chern character on algebraic K-theory, described before. □

There is another characteristic class of importance to us. The *Todd class* is the characteristic class associated to the polynomial

$$\prod_{i=1}^{r} \frac{z_i}{1 - e^{-z_i}},$$

where we read the exponential as being expanded "as far as necessary" (since we will plug in differential forms on a finite dimensional manifold, their powers will vanish beyond the dimension of M). The Todd class of M is defined as the the Todd class of the complexification of T^*M, lifted to S^*M, and is denoted $\mathrm{Td}(M)$. Its component in $H^{2k}(S^*M)$ is denoted $\mathrm{Td}(M)_{2k}$.
Given that the boundary map in K-theory maps the K_1^{top}-class of the symbol of an elliptic operator to its index, we can formulate the Atiyah-Singer index theorem as

$$\mathrm{Ind}_0([\sigma(P)]) = (-1)^n \int_{T^*M} \mathrm{ch}(\sigma(P)) \wedge \mathrm{Td}(M).$$

In this formulation, the forms $\mathrm{ch}(\sigma(P))$ and $\mathrm{Td}(M)$ are actually the pull backs under the projection $T^*M \to S^*M$ of the forms on S^*M.
Since we want to work with cyclic homology, we need to get rid of the C^*-algebras and pass to dense subalgebras. To do so, we need the following facts. A classical pseudodifferential operator with vanishing symbol is of order -1. Let n be the dimension of M. It is known that $\Psi^{-1}(M) \subset \mathcal{L}_p$, for any $p > n$. Since the symbol of order -1 operators vanishes, the following definition makes sense.

Definition 4.4. We define the *Atiyah-Singer algebra* E_{AS} as the algebra $\Psi^0(M) + \mathcal{L}_{n+1}$.

The algebras $\mathcal{L}_p$ are complete in the norms $\|T\|_p := \mathrm{Tr}(|T|^p)^{\frac{1}{p}}$. Using this norm and the seminorms E_{AS} inherits from $C^\infty(S^*M)$ via σ, it becomes a complete locally convex algebra. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{L}_{n+1} \longrightarrow E_{AS} \longrightarrow C^\infty(S^*M) \longrightarrow 0$$

of complete locally convex algebras, which we will refer to as the *Atiyah-Singer exact sequence.*
On $\mathcal{L}_p$ we can define cyclic cocycles $\mathrm{Tr}_n \in HC^{2n}(\mathcal{L}_p)$ by the equality

$$\mathrm{Tr}_n(a_0, ..., a_{2n}) := \frac{n!}{(2n)!}\mathrm{Tr}(a_0 ... a_{2n}),$$

because $\mathcal{L}_p^p \subset \mathcal{L}_1$. By chasing the cyclic double complex, one shows that $S\mathrm{Tr}_n = \mathrm{Tr}_{n+1}$, so we get a class in $HP^0(\mathcal{L}_{n+1})$. By definition of the Chern character, we have for $[e] \in K_0(\mathcal{L}_p)$ that $\mathrm{Tr}_{n*}([e]) = \mathrm{Tr}(e)$. The constants are chosen so that they cancel out. Thus, as with the Gohberg-Krein index theorem, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_{n+1} & \longrightarrow & E_{AS} & \longrightarrow & C^\infty(S^*M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \overline{\Psi^0(M)} & \longrightarrow & C(S^*M) & \longrightarrow & 0.
\end{array}$$

The map $\text{Ind} : K_1(C(S^*M) \to K_0(\mathcal{K})$ maps the symbol of an elliptic pseudodifferential operator to its index, because, for an invertible element $u \in C(S^*M)$, $\text{Ind}([u]) \in K_0(\mathcal{K})$ depends only on the class of u in $K_1^{top}(C(S^*M))$. Since the diagram

$$\begin{array}{ccc} K_0(\mathcal{L}_{n+1}) & \xrightarrow{\text{Tr}_{n*}} & \mathbb{Z} \\ \text{id}\downarrow & & \| \\ K_0(\mathcal{K}) & \xrightarrow{\sim} & \mathbb{Z} \end{array}$$

commutes, we can rephrase the Atiyah-Singer index theorem as follows.

Theorem 4.5 (Atiyah-Singer index theorem in algebraic K-theory). *Let P be a matrix of order zero elliptic pseudodifferential operators, with symbol $u = \sigma(P)$. We have*

$$\text{Tr}_{n*}\text{Ind}[u] = (-1)^n \int_{S^*M} \text{ch}([u]) \wedge \text{Td}(M),$$

for $[u] \in K_1(C^\infty(S^*M))$.

The wedge product in this formula is to be understood as wedging the appropriate components of $\text{ch}([u])$ and $\text{Td}(M)$ to get a $2n-1$-form.
As mentioned in the introduction, elliptic operators are not necessarily Fredholm as operators in $L^2(M)$, and the symbol does not define a class in $K^1(M)$. Therefore we need to put restrictions on the type of elliptic operator we are interested in. In the sequel, we will be interest in covering spaces $\tilde{M} \to M$, with M compact, and elliptic operators that are invariant under the action of the group of covering transformations Γ. If the group of covering transformations is infinite, then $\tilde{M}$ is noncompact. In the next section, we discuss the basic properties of coverings. Out of the Atiyah-Singer algebra of M, we will construct another algebra, the Connes-Moscovici algebra E_{CM}, encoding information about the covering and invariant operators on $\tilde{M}$. This algebra fits into an extension

$$0 \longrightarrow \mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma] \longrightarrow E_{CM} \longrightarrow C^\infty(S^*M) \longrightarrow 0$$

and a Γ-invariant elliptic pseudodifferential operator has a symbol in $C^\infty(S^*M)$ and an index in $K_0(\mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma])$, as the image of the K_1-class of the symbol. To construct this extension, and compute its periodic cyclic homology, we need the formalism of étale groupoids.

4.2 Covering spaces

The Connes-Moscovici theorem is a statement about covering spaces, and the integral that computes the index, involves forms given by the Chern character

and the Todd class, and also the pull back of a group cocycle under the *classifying map* of the covering. This section is devoted to the understanding of this map. At first sight, it seems a quite uncomputable object, so I felt the need of clarifying some things.
First we recall some well known facts about coverings.

Definition 4.6. A *covering space* is given by a continuous map $p : \tilde{Y} \to Y$, such that each $y \in Y$ has an open neighbourhood U, with the property that $p^{-1}(U) = \bigcup_{j\in J} V_j$ is a disjoint union for which $p|_{V_j} \to U$ is a homeomorphism. A *covering transformation* is a homeomorphism $\phi : \tilde{Y} \to \tilde{Y}$ such that $p \circ \phi = p$. The covering transformations form a group $\mathrm{Cov}(\tilde{Y}/Y)$, and the covering is called *normal* if $\mathrm{Cov}(\tilde{Y}/Y)$ acts transitively on each fibre.
A covering $\tilde{Y} \to Y$ is called *universal* if it has the following property. For any covering $q : \tilde{X} \to Y$, $y \in Y$, $\tilde{y} \in p^{-1}(y)$ and $\tilde{x} \in q^{-1}(y)$, there is a unique map $F : \tilde{Y} \to \tilde{X}$ such that $q \circ F = p$ and $F(\tilde{y}) = \tilde{x}$.

Clearly, a universal covering is unique up to homeomorphism and we have the following existence result.

Theorem 4.7. *Let M be connected topological manifold. Then there exists a universal covering $p : \tilde{M} \to M$, and $\mathrm{Cov}(\tilde{M}/M) \cong \pi_1(M)$. Moreover, if $\tilde{X} \to M$ is any covering with $\tilde{X}$ simply connected, then it is isomorphic to the universal covering of M.*

Given a discrete group Γ, denote by $\mathrm{Cov}_\Gamma(M)$ the set of isomorphism classes of normal coverings of M with group Γ. A remark on the choice of basepoints in a space is in order here. Many constructions in algebraic topology require singling out a specific point in the space, to guarantee certain uniqueness properties. An example is the fundamental group, which is computed relative to a basepoint $m \in M$. If M is connected, the isomorphism class of $\pi_1(M)$ does not depend on this choice, although it is essential in the construction. This is why we omitted it in the previous theorem. The pairs (M, m) form a category, in which the morphisms are continuous maps mapping the basepoint to the basepoint. Note that, in this category, the universal property of the universal covering guarantees the existence of a *unique* map $F : \tilde{M} \to \tilde{X}$ to any covering space $\tilde{X}$ of M.

Proposition 4.8. *Let M be a connected topological manifold and Γ a discrete group. There is a bijection* $\mathrm{Hom}(\pi_1(M, m_0), \Gamma) \leftrightarrow \mathrm{Cov}_\Gamma(M)$.

Proof. Let $q : \tilde{X} \to M$ be a normal covering and $p : \tilde{M} \to M$ the universal covering of M. Let m_0 be the basepoint of M and choose $\tilde{x}_0 \in q^{-1}(m_0)$, $\tilde{m}_0 \in p^{-1}(m_0)$ to be the basepoints of $\tilde{X}$ and $\tilde{M}$. If $F : \tilde{M} \to \tilde{X}$ is the map obtained by the universal property of p, then define $f : \pi_1(M) \to \Gamma$ by

$$F(g(\tilde{m})) := f(g)F(\tilde{m}) = f(g)\tilde{x}.$$

The map f exists, since the covering q is normal, and it is well defined since there can be at most one covering transformation mapping $\tilde{x}$ to $F(g(\tilde{m}))$. It is a homomorphism because F is a covering transformation.

Next we show that given $f : \pi_1(M) \to \Gamma$, we can construct a normal covering $\tilde{X} \to M$. Using f, we define an equivalence relation on $\Gamma \times \tilde{M}$ by

$$(\gamma_0, \tilde{m}_0) \sim (\gamma_1, \tilde{m}_1) \Leftrightarrow \exists g \in \pi_1(M) \quad (\gamma_0, \tilde{m}_0) = (\gamma_1 f(g^{-1}), g\tilde{m}_1),$$

and set

$$\tilde{X} := (\Gamma \times \tilde{M}) / \sim .$$

The covering map $q : \tilde{X} \to M$ is given by $q(\gamma, \tilde{m}) = p(\tilde{m})$. Γ acts by left translation, which is compatible with the equivalence relation, and this action defines covering transformations. Since

$$(\gamma_0, g\tilde{m}) = (\gamma_0 f(g), \tilde{m}) = ((\gamma_0 f(g) \gamma_0^{-1})\gamma_0, \tilde{m}),$$

all covering transformations are obtained in this way, the fibre is isomorphic to Γ, and the action is transitive. So the covering is normal.
It remains to show that these constructions are inverse to each other. Having constructed $\tilde{X}$ from $f : \pi_1(M) \to \Gamma$, we choose the basepoint to be $(e, \tilde{m}_0)$. Then it is immediate that the map F is given by $\tilde{m} \mapsto (e, \tilde{m})$. Thus

$$F(g\tilde{m}) = (e, g\tilde{m}) = (f(g), \tilde{m}) = f(g)(e, \tilde{m}),$$

and we recover $f : \pi_1(M, m_0) \to \Gamma$. Conversely, if f is the map associated to $q : \tilde{X} \to M$, then the map $F : \tilde{M} \to \tilde{X}$ defines an isomorphism

$$\begin{aligned} \Gamma \times \tilde{M} / \sim &\to \tilde{X} \\ (\gamma, \tilde{m}) &\mapsto \gamma F(\tilde{m}). \end{aligned}$$

This map is surjective, because Γ acts transitively, and it is injective because

$$\gamma F(\tilde{m}) = \gamma' F(\tilde{m}')$$

implies that $\tilde{m} = g\tilde{m}'$ for some $g \in \pi_1(M, m_0)$, since $\pi_1(M, m_0)$ acts transitively and F is fibre preserving. Thus

$$f(g)F(\tilde{m}') = F(g\tilde{m}') = F(\tilde{m}) = \gamma^{-1}\gamma' F(\tilde{m}'),$$

so $\gamma^{-1}\gamma' = f(g)$. Then $(\gamma, \tilde{m}) = (\gamma f(g), g^{-1}\tilde{m}) = (\gamma', \tilde{m}')$. $\square$

We can now obtain the characterization of normal coverings of M with group Γ, in terms of homotopy classes of maps $M \to \mathbb{B}\Gamma$. In section 2.10 we encountered a contraction of the simplicial module $k[\mathbb{E}_*\Gamma]$ and this contraction gives rise to a contraction of the space $\mathbb{E}\Gamma$. Since Γ acts freely on $\mathbb{E}\Gamma$ and $\mathbb{E}\Gamma/\Gamma \cong \mathbb{B}\Gamma$ (this corresponds to the isomorphism $k[\mathbb{E}_*\Gamma] \otimes k \cong k[\mathbb{B}_*\Gamma]$), we have a normal covering $\mathbb{E}\Gamma \to \mathbb{B}\Gamma$. Since $\mathbb{E}\Gamma$ is simply connected, theorem 4.7 implies that $\pi_1(\mathbb{B}\Gamma) \cong \Gamma$. Therefore a map $M \to \mathbb{B}\Gamma$ determines an element of $\mathrm{Hom}(\pi_1(M), \Gamma)$, which depends only its homotopy class.
For the proof of the following proposition, it is important to mention that the

notion of classifying space we introduced in section 2.10 is a bit too restrictive. In algebraic topology, the classifying space of a discrete group is defined as follows. Let X be a contractible space such that Γ acts freely and properly discontinuously on X, and such that X/Γ is paracompact. Then X/Γ is called a classifying space for Γ. This notion is well defined up to homotopy equivalence, and the space $\mathbb{B}\Gamma$ is indeed a classifying space for Γ. In the proof of the lemma, we will encounter another model for this space, in the case $\Gamma = \pi_1(M)$, for some manifold M.

Proposition 4.9 (Milnor). *There is a bijection* $[M, \mathbb{B}\Gamma] \leftrightarrow \mathrm{Cov}_\Gamma(X)$.

Proof. (Sketch)It suffices to show that the map defined above is a bijection

$$[M, \mathbb{B}\Gamma] \leftrightarrow \mathrm{Hom}(\pi_1(M), \Gamma).$$

To provide an inverse, note that a homomorphism $\pi_1(M) \to \Gamma$ induces a continuous map of classifying spaces $\mathbb{B}\pi_1(M) \to \mathbb{B}\Gamma$. It suffices to provide a continuous map $\phi : M \to \mathbb{B}\pi_1(M)$ which induces the identity on the fundamental groups. Suppose we have constructed this map. Then it is immediate that every homomorphism $\pi_1(M) \to \Gamma$ comes from a continuous map $M \to \mathbb{B}\Gamma$. Injectivity follows from the fact hat if two continuous maps $M \to \mathbb{B}\Gamma$ induce the same map on π_1, then they induce the same map on all homotopy groups, since $\pi_n(\mathbb{B}\Gamma) = 0$ for $n \geq 2$, and therefore are homotopic.
Now let us sketch how to construct ϕ. Consider the space obtained from M be the following inductive process: For each non trivial element in $\pi_2(M)$, attach a 3-cell using a representive as the gluing map. The space so obtained has vainshing π_2. Apply the same process to this space with π_3, an so on for all n. In this way one obtains a space X for which $M \subset X$ and $\pi_n(X) = \pi_n(\mathbb{B}\Gamma)$ for all n. The inclusion $M \to X$ by construction induces the identity on π_1, and therefore X is homotopy equivalent to $\mathbb{B}\Gamma$. □

The classifying map of a covering is the unique element in $[M, \mathbb{B}\Gamma]$ determining the covering. Now that we know what a classifying map is, we can turn to étale groupoids. These provide a convenient framework for covering spaces and the cyclic homology of the algebras we associate with them.

4.3 Étale groupoids

There is a formalism that allows one to treat smooth manifolds and discrete groups on equal terms. By associating appropriate categories to them, they become part of the world of étale groupoids. We will describe what these are, give the relevant examples, and state a couple of theorems that relate cyclic homology to ordinary topology and group homology.

Definition 4.10. A groupoid is a small category in which every morphism is invertible. A groupoid $\mathcal{G}$ is called *smooth étale* if its set of objects obj $\mathcal{G}$ and its set of morphisms Mor$\mathcal{G}$ are smooth manifolds and the domain and range maps

$$d : \text{Mor}\mathcal{G} \to \text{obj } \mathcal{G}, \quad r : \text{Mor}\mathcal{G} \to \text{obj } \mathcal{G},$$

associating to a morphism its domain and range, respectively, are local diffeomorphisms.
An *étale morphism of groupoids* is a functor $F : \mathcal{G}_1 \to \mathcal{G}_2$ that is a local diffeomorphism on both obj $\mathcal{G}_1$ and Mor$\mathcal{G}_1$.
To an étale groupoid $\mathcal{G}$ we associate the algebra $C_c^\infty(\text{Mor}\mathcal{G})$ of smooth compactly supported functions on Mor$\mathcal{G}$, with the convolution product

$$f_0 * f_1(g) := \sum_{r(\gamma)=r(g)} f_0(\gamma) f_1(\gamma^{-1} g).$$

This is usually denoted $C_c^\infty(\mathcal{G})$.

Note that the convolution product is well defined: supp $f_0 \cup$supp f_1 is compact, and we can cover it with a finite number of open sets on which r is a local diffeomorphism. In each of these open sets, there is at most one γ with $r(\gamma) = r(g)$. Thus the sum defining the convolution product is finite.
An étale morphism of groupoids $F : \mathcal{G}_1 \to \mathcal{G}_2$ does *not* in general induce an algebra homomorphism $C_c^\infty(\mathcal{G}_2) \to C_c^\infty(\mathcal{G}_1)$ or $C_c^\infty(\mathcal{G}_1) \to C_c^\infty(\mathcal{G}_2)$. If F is injective on obj $\mathcal{G}_1$, then there is a map $C_c^\infty(\mathcal{G}_1) \to C_c^\infty(\mathcal{G}_2)$. It is given by

$$F_*(f)(g) := \sum_{\gamma \in F^{-1}(g)} f(\gamma),$$

which is well defined because f has compact support and F is a local diffeomorphism. It is straightforward to check that F_* is an algebra homomorphism (here injectivity of F is needed).
Since groupoids are categories, we can associate classifying spaces with them. These will be useful, because the cyclic homology of convolution algebras turns out to be related to the singular homology of $\mathbb{B}\mathcal{G}$. If one has a nice model of $\mathbb{B}\mathcal{G}$, one can thus get information about the cyclic homology of the groupoid in question. We now turn to the main examples.

To a discrete group Γ we associate the small category of definition 2.71. Clearly, the classifying space of this category is just $\mathbb{B}\Gamma$.
The convolution algebra $C_c^\infty(\Gamma)$ is isomorphic to the group algebra $\mathbb{C}[\Gamma]$. This is because Γ has the discrete topology, so a compactly supported function is non zero on only a finite number of elements. Therefore we can define a map $C_c^\infty(\Gamma) \to \mathbb{C}[\Gamma]$ by $f \mapsto \sum_{\gamma\in\Gamma} f(\gamma)\gamma$, which is clearly invertible. It is an algebra

map, because the product $f_0 * f_1$ gets mapped to

$$\begin{aligned}\sum_{\gamma\in\Gamma} f_0 * f_1(\gamma)\gamma &= \sum_{\gamma\in\Gamma}\sum_{r(\gamma)=r(g)} f_0(g)f_1(g^{-1}\gamma)\gamma\\ &= \sum_{\gamma\in\Gamma}\sum_{g\in\Gamma} f_0(g)f_1(g^{-1}\gamma)\gamma\\ &= \sum_{\gamma\in\Gamma}\sum_{g\in\Gamma} f_0(g)f_1(\gamma)g\gamma\\ &= (\sum_{g\in\Gamma} f_0(g)g)(\sum_{\gamma\in\Gamma} f_1(\gamma)\gamma),\end{aligned}$$

and this calculation is reversible.

The most straightforward example of an étale groupoid $\mathcal{M}$, is the one given by a smooth manifold M. If one takes obj $\mathcal{M} = M$, $\mathrm{Mor}(m,m) = \{\mathrm{id}_m\}$ and $\mathrm{Mor}(m_0,m_1) = \emptyset$ if $m_0 \neq m_1$, all conditions are met. The classifying space of this category is just M itself. The convolution algebra $C_c^\infty(\mathcal{M})$ is also familiar to us, as it is $C_c^\infty(M)$. Because any morphism is only composable with itself, and the convolution product reduces to the ordinary pointwise product.
A related object is the *gluing groupoid* $\mathcal{G}_\mathcal{U}$ associated to an open covering $\mathcal{U} = \{U_i\}_{\imath\in\mathcal{I}}$ of M. It is defined as

$$\text{obj } \mathcal{G}_\mathcal{U} := \bigcup_{i\in\mathcal{I}} U_i \times \{i\}, \quad \mathrm{Mor}\mathcal{G}_\mathcal{U} := \bigcup_{i,j\in\mathcal{I}\times\mathcal{I}} U_i \cap U_j \times (i,j).$$

The classifying space of $\mathcal{G}_\mathcal{U}$ turns out to be homotopy equivalent to M.
Gluing groupoids allow us to construct maps

$$F_{\mathrm{Tr}} : HP_c^*(C_c^\infty(\mathcal{G}_1)) \to HP_c^*(C_c^\infty(\mathcal{G}_2)),$$

for any étale morphism of groupoids $F : \mathcal{G}_1 \to \mathcal{G}_2$. We need antoher example to do this.

Let $\mathcal{I}$ be a discrete set and R an equivalence relation on $\mathcal{I}$. The groupoid $R_\mathcal{I}$ has $\mathcal{I}$ as its set of objects, and there is a morphism between two objects if and only if they are related. If $\mathcal{I}$ is finite of order k and all objects of $\mathcal{I}$ are equivalent, then $C_c^\infty(R_\mathcal{I})$ is isomorphic to $M_k(\mathbb{C})$. Since $\mathrm{Mor}R_\mathcal{I} = \mathcal{I}\times\mathcal{I}$, we can define a map $C_c^\infty(R_\mathcal{I}) \to M_k(\mathbb{C})$ by $f \mapsto (f(j,i))_{ij}$, after identifying $\mathcal{I}$ with $\{1,...,k\}$. Again it is clear that this is invertible. The convolution product is

$$\begin{aligned}f_0 * f_1(j,i) &= \sum_{r(\gamma)=r(j,i)} f_0(\gamma)f_1(\gamma^{-1}(j,i))\\ &= \sum_{k=1}^n f_0(k,i)f_1((i,k)\circ(j,i))\\ &= \sum_{k=1}^n f_0(k,i)f_1(j,k),\end{aligned}$$

which coincides under the given bijection with matrix multiplication. If $\mathcal{I}$ is countable, we obtain in the same way an isomorphism $C_c^\infty(R_\mathcal{I}) \cong M_\infty(\mathbb{C})$.
If $\mathcal{G}_\mathcal{U}$ is the gluing groupoid defined above, and $R_\mathcal{I}$ is the groupoid associated to the total equivalence relation on the index set, then the obvious injection is an etale morphism of groupoids $\mathcal{G}_\mathcal{U} \to \mathcal{M} \times R_\mathcal{I}$.
This is a special case of the following construction on arbitrary groupoids $\mathcal{G}$. Take a countable trivializing open cover $\mathcal{U}$ of obj $\mathcal{G}$. Then there is an injective etale morphism $G_\mathcal{U} \to \mathcal{G} \times R_\mathcal{I}$, given by $(x, i) \mapsto x$ and $(x, i, j) \mapsto \mathrm{id}_X$. Therefore any etale morphism $f : \mathcal{G}_1 \to \mathcal{G}_2$ induces a map $\mathcal{G}_1 \times R_\mathcal{I} \to \mathcal{G}_2 \times R_\mathcal{I}$ and taking convolution algebras gives an algebra map

$$C_c^\infty(\mathcal{G}_1) \otimes M_\infty(\mathbb{C}) \to C_c^\infty(\mathcal{G}_2) \otimes M_\infty(\mathbb{C}).$$

We denote by $f_{\mathrm{Tr}} : HP_c^*(C_c^\infty(\mathcal{G}_1)) \to HP_c^*(C_c^\infty(\mathcal{G}_2))$ the induced map on continuous periodic cyclic cohomology, composed with the Morita invariance isomorphisms Tr.
Recall that the *orientation sheaf* of a topological manifold is the sheaf of sections of its orientation bundle. We have the following theorem on etale groupoids, that we will use only in the special cases of a group and a manifold.

Theorem 4.11 (Nistor). *If $\mathcal{G}$ is a Hausdorff étale groupoid of dimension n and $\mathcal{O}$ the complexified orientation sheaf of $\mathbb{B}\mathcal{G}$ then there is an embedding*

$$\Phi : H^{*+n}(\mathbb{B}\mathcal{G}, \mathcal{O}) \to HP_c^*(C_c^\infty(\mathcal{G})).$$

If $f : \mathcal{G}_1 \to \mathcal{G}_2$ is an etale morphism of groupoids, then the diagram

$$\begin{array}{ccc} H^{*+n}(\mathbb{B}\mathcal{G}_2, \mathcal{O}_2) & \xrightarrow{\Phi} & HP_c^*(C_c^\infty(\mathcal{G}_2)) \\ \mathbb{B}f^* \downarrow & & \downarrow f_{\mathrm{Tr}} \\ H^{*+n}(\mathbb{B}\mathcal{G}_1, \mathcal{O}_1) & \xrightarrow{\Phi} & HP_c^*(C_c^\infty(\mathcal{G}_1)) \end{array}$$

commutes. Also, Φ turns the external product of Čech cohomology into the external product in periodic cyclic cohomology.

In the cases of interest to us, we can describe this map explicitly. For a group it is the embedding ι of section 2.10. In the case of a manifold, we need to recall the Poincare duality theorem.

Theorem 4.12. *Let M be a smooth manifold of dimension n. The pairing*

$$\begin{array}{c} H_{DR}^i(M) \times H_{DR}^{n-i}(M) \to \mathbb{C} \\ ([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta \end{array}.$$

is nondegenerate. This in turn gives an isomorphism $H_{DR}^i(M) \cong H_{n-i}(M)$ of de Rham cohomology with singular homology.

Note that in the case that M is even dimensional, Poincare duality gives pairings of $\bigoplus_{j=0}^{\infty} H_{DR}^{2j}$ and $\bigoplus_{j=0}^{\infty} H_{DR}^{2j+1}$ with themselves, while if M is odd dimensional, we can pair these graded vector spaces with each other. The map Φ from theorem 4.11 is determined by the formula

$$\langle \Phi(\omega), \eta \rangle = \int_M \omega \wedge \chi(\eta),$$

where χ is the Hochschild-Konstant-Rosenberg-Connes map.

4.4 The Mischenko idempotent

Suppose we are giving a normal covering $q : \tilde{M} \to M$, with group Γ. This is the same thing as a principal Γ-bundle, and is thus given by an element of $H^1(M, \Gamma)$, see appendix D. We lift this covering to S^*M, the sphere bundle of M, by pulling it back through the canonical map $p : S^*M \to M$. The lifting is also denoted $q : S^*\tilde{M} \to S^*M$. If $f : M \to \mathbb{B}\Gamma$ is the classifying map, then $f \circ p : S^*M \to \mathbb{B}\Gamma$ classifies the lifted covering. The cocycle $\gamma_{ij} \in H^1(M, \Gamma)$, associated to the trivializing cover $\mathcal{U} := \{U_i\}$ of M, lifts to the cocycle $\gamma_{ij} \circ p \in H^1(S^*M, \Gamma)$, associated to the cover $\mathcal{V} := p^{-1}(\mathcal{U})$ of S^*M. The covers $\mathcal{U}$ and $\mathcal{V}$ are finite and consist of k open sets. Let $\phi_i : V_i \to \mathbb{R}$ be functions such that their squares form a partition of unity subordinate to $\mathcal{V}$. Then we can define an element $P = (P)_{ij}$ of $M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma]$ by

$$(P)_{ij} := \phi_i \phi_j \otimes \gamma_{ij}.$$

Multiplying P with itself gives

$$\begin{aligned}(P^2)_{ij} &= \sum_{l=1}^{k} \phi_i \phi_l \phi_l \phi_j \otimes \gamma_{il} \gamma_{lj} \\ &= \phi_i \phi_j \otimes \gamma_{ij} = (P)_{ij},\end{aligned}$$

so P is an idempotent, called the *Mischenko idempotent*. Let $\mathcal{W}$ be another trivializing cover (consisting of m open sets) with coycle δ_{rs} and partition of unity ψ_r^2. Passing to the common refinement $\{U_i \cap W_r\}$, there are functions $\eta_{ir} : U_i \cap W_r \to \Gamma$ such that

$$\gamma_{ij} = \eta_{ir} \delta_{rs} \eta_{js}^{-1} \text{ on } U_i \cap W_r \cap U_j \cap W_s,$$

because $[\delta_{rs}] = [\gamma_{ij}] \in H^1(S^*M, \Gamma)$. The functions $\phi_i^2 \psi_r^2 : U_i \cap W_r \to \mathbb{R}$ form a partition of unity subordinate to the common refinement. Let Q be the Mischenko idempotent associated to $\mathcal{W}$. By enlarging P or Q by adding zeroes, we assume $P, Q \in M_n(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma]$ for $n = \max\{k, m\}$. The matrix $(g)_{ir} = \phi_i \psi_r \otimes \eta_{ir}$ is invertible with inverse $(g^{-1})_{ri} = \psi_r \phi_i \otimes \eta_{ir}^{-1}$. One checks by calculation that $gQg^{-1} = P$.
The above discussion motivates the definition of the map

$$\lambda : C^\infty(S^*M) \to M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma],$$

by $f \mapsto fP$, for it shows that the induced map

$$\lambda^* : HP^*(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma]) \to HP^*(C^\infty(S^*M)),$$

(which is in fact $\lambda^* \circ \mathrm{Tr}^*$) is independent of the choices made. This morphism will play a crucial role in the proof of the theorem, and we proceed by identifying its action on HP. Consider the gluing groupoid $\mathcal{G}_\mathcal{V}$ and the projection $t : \mathcal{G}_\mathcal{V} \to S^*M$. This induces a homotopy equivalence of classifying spaces and thus by theorem 4.11 an isomorphism

$$t_{\mathrm{Tr}} : HP_c^*(C^\infty(M)) \to HP_c^*(C_c^\infty(\mathcal{G}_\mathcal{V})).$$

If $l : \mathcal{G}_\mathcal{V} \to S^*M \times R_\mathcal{I}$ is the canonical map, then $t_{\mathrm{Tr}} = \mathrm{Tr}^{*-1} \circ l^* \circ \mathrm{Tr}^*$, so l^* is also an isomorphism. Now define an injective etale morphism

$$\begin{aligned} g : \mathcal{G}_\mathcal{V} &\to \mathcal{G}_\mathcal{V} \times \Gamma \\ (x, i, j) &\mapsto (x, i, j, \gamma_{ij}). \end{aligned}$$

Using the homotopy equivalence of S^*M and $\mathbb{B}\mathcal{G}_\mathcal{V}$, we obtain a continuous map $h_0 : S^*M \to S^*M \times \mathbb{B}\Gamma$. h_0 is uniquely determined by the commutative diagram

$$\begin{array}{ccc} \mathbb{B}\mathcal{G}_\mathcal{V} & \xrightarrow{\mathbb{B}t} & S^*M \\ \downarrow{\scriptstyle \mathbb{B}g} & & \downarrow{\scriptstyle h_0} \\ \mathbb{B}\mathcal{G}_\mathcal{V} \times \mathbb{B}\Gamma & \xrightarrow{\mathbb{B}t \times \mathrm{id}} & S^*M \times \mathbb{B}\Gamma. \end{array}$$

For its action on homology, only the homotopy class of h_0 matters. The following lemma gives us just that.

Lemma 4.13. *Let $f : S^*M \to \mathbb{B}\Gamma$ be the classifying map. The map h_0 coincides up to homotopy with the product* $\mathrm{id}_{S^*M} \times f$*.*

Proof. The assertion on the first coordinate is obvious. Denote the second coordinate function by $h_1 : S^*M \to \mathbb{B}\Gamma$. It is induced by the morphism $(x, i, j) \mapsto \gamma_{ij}$, which is a morphism of topological groupoids. Since it factors as $\mathcal{G}_V \to \mathcal{G}_U \to \Gamma$, we may replace S^*M by M. To prove that h_1 is homotopic to f, it suffices to check that they induce the same map $\pi_1(M) \to \Gamma$. Since the covering $\tilde{M} \to M$ is determined by its restriction to loops, it suffices to prove the claim for the circle S^1. Covering S^1 with two connected intervals intersecting in two neighbourhoods of 1, we may asssume that the cocycle is the identity on one of these, and γ on the other. But then it is clear from the definition of h_1 and the classifying homomorphism that these coincide. $\square$

To identify λ^*, we consider the morphism of algebras

$$\nu : C^\infty(S^*M) \to C_c^\infty(\mathcal{G}_\mathcal{V})$$

given by $\nu(g)(x,i,j) := g(x)\phi_i(x)\phi_j(x)$. If we compose this map with the map l defined above, the result

$$C^\infty(S^*M) \xrightarrow{\nu} C_c^\infty(\mathcal{G}_\mathcal{V}) \xrightarrow{l_*} C_c^\infty(S^*M \times R_\mathcal{I}) \cong M_k(C^\infty(S^*M))$$

is given by

$$(l \circ \nu(f))_{ij} = (\phi_i\phi_j f),$$

and $\mathrm{Tr}(l_* \circ \nu(f)) = f$. ν^* is in fact unitarily equivalent to the upper left corner embedding, and therefore, ν^* is the inverse of t_{Tr}. The étale morphism $l \times \mathrm{id} : \mathcal{G}_\mathcal{V} \times \Gamma \to \mathcal{G}_\mathcal{V} \times R_\mathcal{I} \times \Gamma$ is injective and thus induces an algebra morphism

$$l_* \otimes \mathrm{id} = (l \times \mathrm{id})_* : C_c^\infty(\mathcal{G}_\mathcal{V} \times \Gamma) \to M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma].$$

Moreover, we have that

$$(l_* \times \mathrm{id}) \circ g_* \circ \nu f = (l_* \times \mathrm{id})(\nu f \otimes \gamma_{ij}) = \phi_i\phi_j f \otimes \gamma_{ij} = \lambda f.$$

Now we can give an explicit description of the morphism λ^*.

Proposition 4.14. *The composition*

$$H^{*-1}(S^*M \times \mathbb{B}\Gamma) \longrightarrow HP_c^*(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma]) \xrightarrow{\lambda^*} HP_c^*(C^\infty(S^*M)) \xrightarrow{\sim} H^{*-1}(S^*M)$$

equals $\Phi^{-1} \circ \lambda^* \circ \Phi = (\mathrm{id} \times f)^*$.

Proof. Because $\nu^* = (t_{\mathrm{Tr}})^{-1}$, thereom 4.11 implies that $\Phi^{-1} \circ \nu^* \circ \Phi = (\mathbb{B}t)^{*-1}$. By the same theorem $g^* \circ \Phi = \Phi \circ (\mathbb{B}g)^*$ and $(l \times \mathrm{id})^* \circ \Phi = \Phi \circ \mathbb{B}(l \times \mathrm{id})^*$. Since λ equals the composite $(l \times \mathrm{id})_* \circ g_* \circ \nu$, we compute

$$\Phi^{-1} \circ \lambda^* \circ \Phi = \Phi^{-1} \circ \nu^* \circ \Phi(\mathbb{B}g)^* \circ \mathbb{B}(l \times \mathrm{id})^* = (\mathbb{B}t)^{*-1} \circ (\mathbb{B}g)^* \circ \mathbb{B}(l \times \mathrm{id})^* = h_0^*.$$

The last equality holds because $l_{\mathrm{Tr}} = \mathrm{Tr}^{*-1} \circ l^*$, and thus $\Phi \circ \mathbb{B}l = \mathrm{Tr}^{*-1} \circ l^* \circ \Phi$. It follows that

$$\Phi \circ \mathbb{B}t = \mathrm{Tr}^{*-1} \circ l^* \circ \mathrm{Tr}^* \circ \Phi = \Phi \circ \mathbb{B}l \circ \Phi^{-1} \circ \mathrm{Tr}^* \circ \Phi,$$

and since Φ is injective, we have $\mathbb{B}t = \mathbb{B}l \circ \Phi^{-1} \circ \mathrm{Tr}^* \circ \Phi$. Recalling that λ^* is actually $\lambda^* \circ \mathrm{Tr}^*$, the diagram determining h_0 and lemma 4.13 then complete the proof. □

4.5 Four preparatory lemmas

In this section, we will provide all the computational tools to prove the theorem. We will use both discrete and continuous periodic cyclic cohomology. Therefore, we need to make sure that we can safely switch between the two (in our situation, of course). Recall from section 3.5 that for a locally convex algebra A, the inclusion of the discrete into the the continuous cyclic double complex induces natural transformations

$$HP_*(A) \to HP_*^c(A), \quad \text{and} \quad HP_c^*(A) \to HP^*(A).$$

In general these maps need not be injective nor surjective.

Lemma 4.15. *Let A be a locally convex algebra, for which the Chern character gives a surjective map*

$$\mathrm{Ch}_* \otimes \mathrm{id} : K_*(A) \otimes \mathbb{C} \to HP_*^c(A)$$

after tensoring with $\mathbb{C}$. Then the natural map $HP_*(A) \to HP_*^c(A)$ is surjective. If, moreover, the pairing

$$HP_c^*(A) \otimes HP_*^c(A) \to \mathbb{C}$$

is nondegenerate, then the natural map $HP_c^*(A) \to HP^*(A)$ is injective.

Proof. By definition of the continuous Chern character, the diagram

$$\begin{array}{ccc} K_*(A) & \xrightarrow{\mathrm{Ch}_*} & HP_*(A) \\ {\scriptstyle \mathrm{Ch}_*}\downarrow & \swarrow & \\ HP_*^c(A) & & \end{array}$$

commutes. Since the downward arrow on the left is surjective, the bottom arrow is surjective as well.
In cohomology we reason as follows: Let $[f]_c \in HP_c^*(A)$ be nonzero. By surjectivity of $\mathrm{Ch} \otimes \mathrm{id}$, the image of Ch generates $HP_*(A)$ as a$\mathbb{C}$-vector space. By nondegeneracy of the pairing

$$HP_c^*(A) \otimes HP_*^c(A) \to \mathbb{C},$$

there is an element $x \in K_*(A)$ with $\langle [f]_c, \mathrm{Ch}_*(x)_c \rangle_c \neq 0$. By definition of the continuous pairing, there is an equality

$$\langle [f]_c, \mathrm{Ch}_*(x)_c \rangle_c = \langle [f], \mathrm{Ch}_*(x) \rangle,$$

where $[f] \in HP^*(A)$ denotes the image of $[f]_c$ under the natural map. Since $[f]$ pairs nontrivially with $\mathrm{Ch}_*(x)$, it must be nontrivial, so the natural map is injective. □

Corollary 4.16. *For a smooth manifold M, there is an inclusion*

$$HP_c^*(C^\infty(M)) \subset HP^*(C^\infty(M)).$$

Proof. By corollary 4.3 and Poincare duality the hypotheses of the previous lemma are satisfied. □

We will use the Atiyah-Singer index formula to determine what the boundary morphism ∂_{AS} of the Atiyah-Singer exact sequence does to the continuous cohomology class $\mathrm{Tr}_n \in HP^0(\mathcal{L}_{n+1})$. For this we need the following lemma of Nistor, which is a corollary to a rather technical theorem in [20].

Lemma 4.17. *Suppose we are given an exact sequence of locally convex algebras*

$$0 \longrightarrow \mathcal{L}_{n+1}\hat{\otimes}A \longrightarrow E \longrightarrow B \longrightarrow 0.$$

Let ξ *be a continuous cyclic cocycle on* B*. Then the boundary map*

$$\partial : HP^0(\mathcal{L}_{n+1}\hat{\otimes}A) \to HP^1(B),$$

from the discrete *periodic cyclic cohomology exact sequence maps the class* $\mathrm{Tr}_n \otimes \xi$ *to a continuous class.*

Lemma 4.18. $\partial_{AS}(\mathrm{Tr}_n)$ *defines a class in* $HP^1_c(C^\infty(S^*M))$ *and this class is represented by*

$$(-1)^n \sum_k (2\pi i)^{k-n}\Phi(\mathrm{Td}(M)_{2k}).$$

Proof. The first statement follows immediately from the previous lemma. To proof the second statement we use the fact that the image of the Chern character

$$\mathrm{Ch}_1 : K_1(C^\infty(S^*M) \to HP_1^c(C^\infty(S^*(M))$$

generates $HP_*(C^\infty(M))$. To prove that two classes in HP^1 are equal, it suffices to show that their pairings with $\mathrm{Ch}_1([u])$ coincide, for any u. We have

$$\begin{aligned}
&\langle (-1)^n \sum_k (2\pi i)^{k-n}\Phi(\mathrm{Td}(M)_{2k}), \mathrm{Ch}_1([u])\rangle \\
&\qquad = \langle (-1)^n \sum_k (2\pi i)^{k-n}\Phi(\mathrm{Td}(M)_{2k}), \sum_k (2\pi i)^k \chi^{-1}\mathrm{ch}_{2k-1}([u])\rangle \\
&\qquad = (-1)^n \int_{S^*M} \sum_k (2\pi i)^{k-n}\mathrm{Td}(M)_{2k} \wedge \sum_k (2\pi i)^k \mathrm{ch}_{2k-1}([u]) \\
&\qquad = (-1)^n \int_{S^*M} \mathrm{ch}([u]) \wedge \mathrm{Td}(M) \\
&\qquad = \mathrm{Tr}_{n*}\mathrm{Ind}([u]) \\
&\qquad = (\partial_{AS}\mathrm{Tr}_n)_*([u]) \\
&\qquad = \langle \partial_{AS}\mathrm{Tr}_{n*}, \mathrm{Ch}_1[u]\rangle. \quad \square
\end{aligned}$$

From the Atiyah-Singer exact sequence, we construct another exact sequence, which will be used to prove the theorem. Recall that we consider a Galois covering $\tilde{M} \to M$ with covering group Γ. We lift this covering to S^*M, and cover S^*M with k open sets. First we take matrix algebras and tensor the Atiyah-Singer exact sequence with the group algebra $\mathbb{C}[\Gamma]$, to obtain

$$0 \longrightarrow M_k(\mathcal{L}_{n+1}) \otimes \mathbb{C}[\Gamma] \longrightarrow M_k(E_{AS}) \otimes \mathbb{C}[\Gamma] \xrightarrow{\sigma} M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma] \longrightarrow 0.$$

The morphism

$$\lambda : C^\infty(M) \to M_k(C^\infty(M)) \otimes \mathbb{C}[\Gamma]$$

can be viewed as a morphism

$$C^\infty(S^*M) \to P(M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma])P,$$

where P is the Mischenko idempotent used to define λ. Restricting σ to this subalgebra allows us to define the *Connes-Moscovici algebra* as

$$E_{CM} := \{(T, f) \in M_k(E_{AS}) \otimes \mathbb{C}[\Gamma] \oplus C^\infty(S^*M) : \sigma(T) = \lambda(f)\}.$$

By construction, there is an exact sequence

$$0 \longrightarrow P(M_k(\mathcal{L}_{n+1}) \otimes \mathbb{C}[\Gamma])P \longrightarrow E_{CM} \longrightarrow C^\infty(S^*M) \longrightarrow 0.$$

This sequence can be written as

$$0 \longrightarrow \mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma] \longrightarrow E_{CM} \longrightarrow C^\infty(S^*M) \longrightarrow 0,$$

because $P(M_k(\mathcal{L}_{n+1}) \otimes \mathbb{C}[\Gamma])P$ is isomorphic to $\mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma]$.

The following proposition is the last step in providing all the computational tools for the final calculation. As with the Atiyah-Singer exact sequence, it determines a class in $HP^1_c(\mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma])$ given by the boundary map ∂_{CM} associated to the *discrete* cyclic homology exact sequence.

Lemma 4.19. *For any cyclic cocycle $\eta \in HP^*(\mathbb{C}[\Gamma]) = HP^*_c(\mathbb{C}[\Gamma])$, $\partial_{CM}(\mathrm{Tr}_n \otimes \eta)$ defines a class in $HP^1_c(C^\infty(S^*M))$ and this class is represented by*

$$\partial_{CM}(\mathrm{Tr}_n \otimes \eta) = \lambda^*((-1)^n \sum_k (2\pi i)^{k-n} \Phi(\mathrm{Td}(M)_{2k}) \otimes \eta).$$

Proof. Let

$$\partial : HP^0(\mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma]) \to HP^1(C^\infty(S^*M) \otimes \mathbb{C}[\Gamma])$$

be the boundary map associated to the top row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_k(\mathcal{L}_{n+1}) \otimes \mathbb{C}[\Gamma] & \longrightarrow & M_k(E_{AS}) \otimes \mathbb{C}[\Gamma] & \xrightarrow{\sigma} & M_k(C^\infty(S^*M)) \otimes \mathbb{C}[\Gamma] & \longrightarrow & 0 \\
 & & \uparrow \lambda & & \uparrow \lambda & & \uparrow \lambda & & \\
0 & \longrightarrow & \mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma] & \longrightarrow & E_{CM} & \longrightarrow & C^\infty(S^*M) & \longrightarrow & 0,
\end{array}$$

where the action of λ on E_{CM} is defined in the obvious way. By theorem 2.70 we find $\partial(\mathrm{Tr}_n \otimes \eta) = \partial_{AS}(\mathrm{Tr}_n) \otimes \eta$. Since the bottom row of the above diagram is the Connes-Moscovici exact sequence, proposition 4.18 and naturality of the boundary map amount to the representation given in the statement. The fact that in defines a class in $HP^1_c(C^\infty(S^*M))$ is again immediate from lemma 4.17. $\square$

4.6 The index of Γ-invariant elliptic operators

The proof of the Connes-Moscovici theorem is now reduced to a straightforward calculation. Before we give this calculation, we first give a translation to the original setting.
The algebra E_{CM} acts on the Hilbert space $P(L^2(M)\otimes\ell^2(\Gamma))^k$. Choose sections $\beta_i : U_i \to \tilde{M}$ for the projection $p : \tilde{M} \to M$. For each $\tilde{x} \in \tilde{M}$, define $\alpha_i(\tilde{x})$ as the unique element of Γ satisfying $\alpha_i(\tilde{x})\beta_i(p(\tilde{x})) = \tilde{x}$. The space $P(L^2(M)\otimes\ell^2(\Gamma))^k$ is isomorphic to $L^2(\tilde{M})$, by the isomorphism

$$\sum_{l=1}^{k}(\phi_l\phi_i f_i \otimes \gamma_{li}\delta_i)(\tilde{x}) = \sum_{i,l=1}^{k} \phi_l\phi_i(x)\delta_i(\gamma_{li}(x)\alpha_i(\tilde{x})),$$

where we wrote x for $p(\tilde{x})$. This map commutes with the action of Γ, as one easily checks. Thus, since E_{CM} acts Γ-invariantly on $P(L^2(M) \otimes \ell^2(\Gamma))$, it can be viewed as an algebra of operators on $L^2(\tilde{M})$, that commute with the right-action of Γ. For more on the analytic properties of E_{CM}, see [7], paragraph 5.
To prove the theorem, we will compute the pairing of the cyclic cocycle

$$\mathrm{Tr}_{n*} \otimes \xi \in HP^0_c(C^\infty(S^*M)) \subset HP^0(C^\infty(S^*M)),$$

for even group cocycles $\xi \in H^{2q}(\Gamma, \mathbb{C})$, with the K-theory index $\mathrm{Ind}([u])$ for $[u] \in K_1(C^\infty(S^*M))$. We compute the pairing of discrete periodic cyclic homology with K-theory, but the previous lemmas assure that we keep working with continuous classes. This is important, because only the continuous pairing is expressible by means of integrals.

Theorem 4.20. *Let M be an n-dimensional smooth manifold and $\tilde{M} \to M$ a normal covering with covering group Γ and classifying map $f : M \to \mathbb{B}\Gamma$. Let P be a matrix of Γ-invariant elliptic differential operators on $\tilde{M}$, and $\xi \in H^{2q}(\Gamma, \mathbb{C}) \cong H^{2q}(\mathbb{B}\Gamma)$ be a group cocycle. Then*

$$(\mathrm{Tr}_n \otimes \xi)_*\mathrm{Ind}([u]) = \frac{(-1)^n}{(2\pi i)^q}\int_{S^*M} \mathrm{ch}([u]) \wedge \mathrm{Td}(M) \wedge f^*(\xi).$$

Proof. We can just compute:

$$
\begin{aligned}
&(\mathrm{Tr}_n \otimes \xi)_* \mathrm{Ind}([u]) \\
&\quad = \partial_{CM}(\mathrm{Tr}_n \otimes \xi)_*[u] \\
&\quad = \lambda^*(\partial_{AS}(\mathrm{Tr}_n) \otimes \Phi(\xi))_*[u] \\
&\quad = \lambda^*(\Phi(\Phi^{-1}(\partial_{AS}\mathrm{Tr}_n) \times \xi)_*[u] \\
&\quad = (\Phi(\mathrm{id} \times f)^* \Phi^{-1}(\partial_{AS}\mathrm{Tr}_n) \times \xi)_*[u] \\
&\quad = \Phi(\Phi^{-1}(\partial_{AS}\mathrm{Tr}_n) \wedge f^*(\xi))_*[u] \\
&\quad = \int_{S^*M} \Phi^{-1}(\partial_{AS}\mathrm{Tr}_n) \wedge f^*(\xi) \wedge \chi(\mathrm{Ch}([u])) \\
&\quad = \int_{S^*M} \Phi^{-1}((-1)^n \sum_k (2\pi i)^{k-n} \Phi(\mathrm{Td}(M)_{2k})) \wedge f^*(\xi) \wedge \chi(\mathrm{Ch}([u])) \\
&\quad = (-1)^n \sum_{k+j=n-q} (2\pi i)^{k-n} \int_{S^*M} \mathrm{Td}(M)_{2k} \wedge f^*(\xi) \wedge \chi(\mathrm{Ch}([u]))_{2j-1} \\
&\quad = (-1)^n \sum_{k+j=n-q} (2\pi i)^{-q} \int_{S^*M} \mathrm{Td}(M)_{2k} \wedge f^*(\xi) \wedge \mathrm{ch}([u])_{2j-1} \\
&\quad = \frac{(-1)^n}{(2\pi i)^q} \int_{S^*M} \mathrm{ch}([u]) \wedge \mathrm{Td}(M) \wedge f^*(\xi). \quad \square
\end{aligned}
$$

This formula differs from the original one given in [7], due to the different normalization we chose for the Chern character.

Our work is now done, although we must be aware of some unclear points in the exposition. The isomorphism $P(M_k(\mathcal{L}_{n+1}) \otimes \mathbb{C}[\Gamma]P \cong \mathcal{L}_{n+1} \otimes \mathbb{C}[\Gamma]$ is nontrivial, and we have not succeeded in finding an explicit expression for it. This is important, because the K_0-group of the ideal in Connes-Moscovici extension is the receptacle for the K-theory index of a Γ-invariant elliptic operator. To relate the theorem to the original formulation of Connes and Moscovici, this is crucial, for their index lives in $K_0(\hat{\mathcal{K}} \otimes \mathbb{C}[\Gamma])$. The algebras $\hat{\mathcal{K}}$ and $\mathcal{L}_{n+1}$ are K_0-equivalent, so this relates our index to theirs. Also the original formulation expresses the index as an integral over T^*M, and this can be achieved by pulling back all forms through the inclusion $S^*M \to T^*M$.

Appendix A

Locally convex algebras

The basic properties of locally convex algebras are discussed. Throughout this section, $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. A more complete discussion of locally convex spaces and tensor products can be found in [25].

Definition A.1. Let A be an $\mathbb{F}$-vector space. A *seminorm* on A is a function $p : A \to \mathbb{R}_{\geq 0}$ satisfying

$$p(x+y) \leq p(x) + p(y) \qquad p(\lambda x) = |\lambda| p(x) \qquad x, y \in A, \lambda \in \mathbb{F}.$$

A collection $\mathcal{P}$ of seminorms on A is said to be *directed* if for every $p, q \in \mathcal{P}$ there exists $r \in \mathcal{P}$ and $M \in \mathbb{R}_{\leq 0}$ such that $\max\{p(x), q(x)\} \leq Mr(x)$ for all $x \in A$.

For a seminorm p and $\varepsilon > 0$, we can consider the set

$$V_{p,\varepsilon} := \{a \in A : p(a) < \varepsilon\},$$

and we note that this set is convex, since, for $t \in [0,1]$

$$p(ta + (1-t)b) \leq tp(a) + (1-t)p(b) < t\varepsilon + (1-t)\varepsilon = \varepsilon.$$

Given a directed collection $\mathcal{P}$ of seminorms on A, we can define a topology on A by letting

$$\mathcal{B} := \{V_{p,\varepsilon} : p \in \mathcal{P}, \varepsilon \in \mathbb{R}_+\}$$

be a neighbourhood basis of 0. A neighbourhood basis of an arbitrary point $a \in A$ will be given by the sets

$$\{a + V_{p,\varepsilon} : V_{p,\varepsilon} \in \mathcal{B}\}.$$

The fact that $\mathcal{P}$ is directed implies that these sets indeed form the basis for a topology. For seminorms $p_k, k = 0, ..., n$ there is a seminorm r and a number M

such that $p_k(x) \leq Mr(x)$ for all $x \in A$ (by inductively applying the property of directedness). Then it follows that

$$\bigcap_{k=0}^{n} V_{p_k,\varepsilon_k} \supset V_{r,\frac{\varepsilon}{M}}.$$

Addition is continuous for this topology, but to show this we need some observations. The product topology on $A \times A$ is the the topology induced by the collection of seminorms

$$\mathcal{P}^2 := \{p \times q : p, q \in \mathcal{P}, p \times q(x) := p(x) + q(x)\}.$$

This is seen as follows. The sets $(a + V_{p,\varepsilon}) \times (b + V_{p,\delta})$ form a basis for the product topology. It is clear that $V_{p,\varepsilon} \times V_{q,\delta} \subset V_{p\times q,\varepsilon+\delta}$. On the other hand we have $V_{p\times q,\varepsilon} \subset V_{p,\varepsilon} \times V_{q,\varepsilon}$, so the two collections generate the same topology. Now let $(a, b) \in A \times A$ with $p(a + b) < \varepsilon$ (that is, $a + b \in V_{p,\varepsilon}$). Then the set

$$W := (a, b) + V_{p\times p,\delta},$$

with $\delta := \varepsilon - p(a + b)$ is open in $A \times A$ and $\{x + y : x, y \in W\} \subset V_{p,\varepsilon}$. Multiplication $\mathbb{F} \times A \to A$ is also continuous for this topology. To show this it is more convenient to work with the product topology. Let $p(\lambda x) < \varepsilon$, i.e. $\lambda x \in V_{p,\varepsilon}$ and

$$\delta_1 := \frac{\varepsilon - |\lambda|p(x)}{p(x) + \sqrt{\varepsilon - |\lambda|p(x)}} \qquad \delta_2 := \frac{\varepsilon - |\lambda|p(x)}{|\lambda| + \sqrt{\varepsilon - |\lambda|p(x)}}.$$

Then the set

$$W := (\{\lambda\} + V_{|.|,\delta_1}) \times (\{x\} + V_{p,\delta_2})$$

has the property that

$$\{\mu y : (\mu, y) \in W\} \subset V_{p,\varepsilon}.$$

This is because

$$\begin{aligned}
p((\lambda + \mu)(x + y)) &= p(\lambda x + \mu x + \lambda y + \mu y) \\
&\leq |\lambda|p(x) + |\mu|p(x) + |\lambda|p(y) + |\mu|p(y) \\
&\leq |\lambda|p(x) + \delta_1 p(x) + \delta_2|\lambda| + \delta_1\delta_2 \\
&\leq 2\varepsilon - |\lambda|p(x) + \frac{(\varepsilon - |\lambda|p(x))^2}{\varepsilon} \\
&= 3\varepsilon - |\lambda|p(x) + \frac{(|\lambda|p(x))^2 - 2\varepsilon|\lambda|p(x)}{\varepsilon} \\
&< 3\varepsilon - |\lambda|p(x) + \frac{(|\lambda|p(x))^2 - 2\varepsilon|\lambda|p(x)}{|\lambda|p(x)} \\
&= \varepsilon.
\end{aligned}$$

Note that the topology induced by $\mathcal{P}$ is Hausdorff if and only if $p(x) = 0$ for all $p \in \mathcal{P}$ implies $x = 0$.

Proposition A.2. *A linear map $l : (V, \mathcal{P}) \to (W, \mathcal{Q})$ between locally convex spaces is continuous if and only if for every seminorm $q \in \mathcal{Q}$ there exist a seminorm $p \in \mathcal{P}$ and a constant $C_{pq} \in \mathbb{R}_{>0}$ such that for all $v \in V$*

$$q(l(v)) \leq C_{pq} p(v).$$

Definition A.3. Let A be a topological space that is also an $\mathbb{F}$-vector space, such that the functions

$$(a, b) \mapsto a + b \qquad (\lambda, a) \mapsto \lambda a,$$

are continuous. A is called *locally convex* if there exist a collection $\mathcal{P}$ of seminorms on A, such that the topology on A is induced by $\mathcal{P}$.
An $\mathbb{F}$-algebra A is called a *locally convex algebra* if it is locally convex as an $\mathbb{F}$-vector space and if the multiplication

$$\begin{aligned} A \times A &\to A \\ (a, b) &\mapsto ab \end{aligned}$$

is continuous.

The most important example of a locally convex algebra is the algebra $C^\infty(M)$ of complex valued differentiable functions on a manifold M. The topology is defined by the seminorms

$$p_{K,n}(f) := \sup_K \|D^n f\|,$$

where $K \subset M$ is a compact set. This topology is Hausdorff. If M itself is compact, then this topology coincides with the norm topology. This example has some more nice properties. Without proof we mention the following result.

Proposition A.4. *A locally convex space A is metrizable if and only if it is Hausdorff and there exists a countable set $\mathcal{P}$ of seminorms of A that induce its topology.*

Given a bijection $\mathbb{N} \to \mathcal{P}$, the metric can be given by

$$d(x, y) := \sum_{n=0}^{\infty} \frac{1}{2^n} p_n(x - y).$$

Definition A.5. A locally convex space resp. algebra that is complete and metrizable is called a *Frechet space*, resp. a *Frechet algebra*.

When dealing with locally convex algebras, it is convenient if the multiplication $A \times A \to A$ is well behaved with respect to the seminorms defining the topology.

Definition A.6. A locally convex algebra A is called an *m-algebra* if its topology can be given by a collection $\mathcal{P}$ of *submultiplicative* seminorms. That is, for any $x, y \in A$ and any seminorm $p \in \mathcal{P}$, we have

$$p(xy) \leq p(x)p(y).$$

The class of m-algebras contains a lot of interesting examples. The Frechet algebras $C^\infty(M)$, for M a smooth manifold, are m-algebras. It also contains the more familiar algebras described in the following definition. Recall that an involution on an algebra is a linear map $x \mapsto x^*$ that is of order 2 and antimultiplicative, that is, $(xy)^* = y^*x^*$.

Definition A.7. A locally convex vector space is called *normable* if its topology can be given by a norm, that is, a seminorm $\|.\|$ satisfying $\|x\| = 0$ if and only if $x = 0$. A normable algebra that is an m-algebra in this norm is called a *Banach algebra*. A C^**-algebra* is a Banach algebra with an involution $x \mapsto x^*$ satisfying $\|xx^*\| = \|x\|^2$.

Now we turn to the issue of tensor products of locally convex spaces. For finite dimensional vector spaces V and W, the algebraic tensor product $V \otimes W$ is again a finite dimensional vector space. Therefore it is complete. For infinite dimensional vector spaces, this need not be the case. In many cases the completion is not even unique!
We will discuss only one completion of the algebraic tensor product $V \otimes W$ of locally convex spaces.

Definition A.8. Let $(V, \mathcal{P})$ and $(W, \mathcal{Q})$ be locally convex vector spaces. For seminorms $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, define

$$p \otimes q(x) := \inf\{\sum_i p(x_i)q(x'_i) : x = \sum_i x_i \otimes x'_i\},$$

for $x \in V \otimes W$. $p \otimes q$ is a seminorm on $V \otimes W$. The *projective tensor product* $V \hat{\otimes} W$ is the completion of $V \otimes W$ in the seminorms $\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\}$.

Grothendieck showed in [14] that the projective tensor product has the following universal property:

Theorem A.9. *Let X be a locally convex space. A bilinear map $B : V \times W \to X$ is continuous if and only if the corresponding linear map $V \otimes W \to X$ is continuous for the projective tensor product seminorms.*

In particular, the canonical map $V \times W \to V \otimes W$ is continuous. It follows that in a locally convex algebra A, the multiplication map extends to a continuous linear map $A \hat{\otimes} A \to A$.

Appendix B

Homological algebra

B.1 Double complexes

We use a lot of homological algebra in this paper, and a good reference for homological algebra on ordinary complexes is [15].To study cyclic homology, double complexes are indispensable. In this section we state a few generalities about them, the most important of which is the so called "double complex lemma". . First of all we formally define the notion.

Definition B.1. Let k be a commutative and unital ring. A *double complex of k-modules* is a $\mathbb{Z}^2$-graded k-module

$$X = \bigoplus_{(m,n)\in\mathbb{Z}^2} X_{mn},$$

equipped with two endomorphisms (called *differentials*)

$$d^{(v)}_{mn} : X_{mn} \to X_{m(n-1)}, \quad d^{(h)}_{mn} : X_{mn} \to X_{(m-1)n},$$

satisfying

$$d^{(v)2} = d^{(h)2} = d^{(v)}d^{(h)} + d^{(h)}d^{(v)} = 0.$$

$d^{(v)}$ is called the *vertical* differential and $d^{(h)}$ the *horizontal* differential. This is because a double complex can be depicted as a commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow d^{(v)} & & \downarrow d^{(v)} & & \downarrow d^{(v)} & & \downarrow d^{(v)} \\
X_{02} & \xleftarrow{d^{(h)}} & X_{12} & \xleftarrow{d^{(h)}} & X_{22} & \xleftarrow{d^{(h)}} & X_{32} & \xleftarrow{d^{(h)}} & \dots \\
\downarrow d^{(v)} & & \downarrow d^{(v)} & & \downarrow d^{(v)} & & \downarrow d^{(v)} & & \\
X_{01} & \xleftarrow{d^{(h)}} & X_{11} & \xleftarrow{d^{(h)}} & X_{21} & \xleftarrow{d^{(h)}} & X_{31} & \xleftarrow{d^{(h)}} & \dots \\
\downarrow d^{(v)} & & \downarrow d^{(v)} & & \downarrow d^{(v)} & & \downarrow d^{(v)} & & \\
X_{00} & \xleftarrow{d^{(h)}} & X_{10} & \xleftarrow{d^{(h)}} & X_{20} & \xleftarrow{d^{(h)}} & X_{30} & \xleftarrow{d^{(h)}} & \dots
\end{array}$$

which in this picture is chosen to live only in the first quadrant.
Given a double complex X, there are two ways of assigning an ordinary complex to it:

$$\mathrm{Tot}(X)_n := \bigoplus_{p+q=n} X_{pq}, \quad \mathrm{ToT}(X)_n := \prod_{p+q=n} X_{pq},$$

where in both cases the differential is given by $d = d^{(v)} + d^{(h)}$. d satisfies $d^2 = 0$, i.e. is an ordinary differential, because of the conditions on $d^{(v)}$ and $d^{(h)}$. These complexes are called the *total complexes* of X. Thus one can associate homology groups to double complexes, using either of the total complexes. If the complex lives in the first quadrant, both total complexes coincide. In that case we write $H_*(X)$ for the homology groups obtained in this way. Since the maps $d^{(v)}$ and $d^{(h)}$ are ordinary differentials, one can also study the $d^{(v)}$ or $d^{(h)}$ homology of X, denoted $H_*(X, d^{(v)})$ and $H_*(X, d^{(h)})$, respectively.

Definition B.2. Let X and Y be double complexes. A *morphism of double complexes* is a map $\phi : X \to Y$ of graded k-modules, commuting with the differentials.

We can now state and prove the "double complex lemma". The proof relies on a so-called "staircase argument".

Proposition B.3. *Let X and Y be double complexes with $X_{nm} = Y_{nm} = 0$ if $n < 0$ or $m < 0$ (that is, living in the first quadrant). Let $\phi : X \to Y$ be a morphism of double complexes. Suppose that the induced map ϕ_* on either the $d^{(v)}$ or the $d^{(h)}$-homology is an isomorphism. Then the induced map on the homology of* $\mathrm{Tot}(X)$ *is an isomorphism.*

Proof. We prove the assertion assuming that ϕ_* is an isomorphism on the $d^{(v)}$-homology. We need the following observation: Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\phi & \longrightarrow & X & \longrightarrow & \text{im }\phi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \text{im }\phi & \longrightarrow & Y & \longrightarrow & \text{coker }\phi & \longrightarrow & 0.
\end{array}$$

Applying $H_*(-, d^{(v)})$, the vertical homology functor, to this diagram and using the long exact homology sequence, we see that $\phi_* : H_*(X, d^{(v)}) \to H_*(Y, d^{(v)})$ being an isomorphism implies

$$H_*(\ker\phi, d^{(v)}) = H_*(\text{coker }\phi, d^{(v)}) = 0$$

and

$$H_*(X, d^{(v)}) \cong H_*(\text{im }\phi, d^{(v)}) \cong H_*(Y, d^{(v)}).$$

It also shows that to prove that $\phi_* : H_*(X) \to H_*(Y)$ is an isomorphism, it suffices to show that $H_*(\ker\phi) = H_*(\text{coker }\phi) = 0$. Both $\ker\phi$ and coker ϕ have acyclic columns, and we show that for a double complex D with acyclic columns $H_n(D) = 0$ for all n .

Acyclicity of the columns means that $d^{(v)} : D_{i1} \to D_{i0}$ is surjective. Therefore $H_0(D) = 0$, trivially. Now suppose $x = \sum_{i=0}^{n} x_i$ is a cycle in $\text{Tot}(D)_n$, with $x_i \in D_{i(n-i)}$. Since $d^{(v)} : D_{1n} \to D_{0n}$ is surjective, there is $y_0 \in D_{1n}$ with $d^{(v)} y_0 = x_0$. We have

$$d^{(v)} d^{(h)} y_0 = -d^{(h)} d^{(v)} y_0 = -d^{(h)} x_0 = d^{(v)} x_1,$$

because x is a cycle. Therefore $d^{(v)}(x_1 - d^{(h)} y_0) = 0$ and there exists $y_1 \in D_{2(n-2)}$ with $d^{(v)} y_1 = x_1 - d^{(h)} y_0$. In other words $d^{(v)} y_1 + d^{(h)} y_0 = x_1$. $d^{(h)} y_0 + d^{(v)} y_1 = x_1$. We can continue this process for y_1 and find y_2 with the right properties. Since the complex lives in the first quadrant, this process terminates at y_n, and x is a boundary, so $H_n(D) = 0$ for all n. $\square$

This proposition has some very useful consequences. If two morphisms $\phi, \psi : X \to Y$ induce the same map on the vertical homologies, then $\phi - \psi$ induces the zero map, so the inclusion of double complexes $\ker(\phi - \psi) \to X$ induces an isomorphism on the vertical homology and thus an isomorphism on the total homology. This implies that $\phi - \psi$ induces the zero map on the total homology, so ϕ and ψ induce the same map there. The same holds of course for the horizontal homology. This property is in fact equivalent to the statement of the proposition.

Suppose we are given a double complex X with acyclic rows, living in the first quadrant. Consider the double complex Y consisting of zeroes except for the first column, in which it is the first column of X modulo im $d^{(h)}$. Then the projection $X \to Y$ is a morphism of double complexes that induces an isomorphism on the horizontal homology. Therefore it induces an isomorphism on the homology of the total complexes, which for Y is just Y itself. Thus in this case, the homology can be computed from an ordinary complex.

B.2 Yoneda's Ext

Connes' theorem on the interpretation of cyclic homology as a derived functor requires some familiarity with the functors Tor and Ext from abstract homological algebra. Let R be any ring, and M an R-module. Recall that a *projective resolution* of M is an exact sequence

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of R-modules, with all P_i projective. Such resolutions exist, because every free module is projective, and every module is the image of a free one. Similarly, an *injective resolution* of M is an exact sequence

$$\ldots \longleftarrow I_1 \longleftarrow I_0 \longleftarrow M \longleftarrow 0$$

with all the I_i injective. Such resolutions always exist because every module is a submodule of an injective one. These resolutions have the following nice property.

Proposition B.4. *Let M and M' be R-modules and $\phi_{-1} : M \to M'$ an R-module homomorphism. Choose projective resolutions (resp. injective resolutions) $P =< P_n, \partial_n >$ and $P' =< P'_n, d_n >$ for M and M' respectively. Then ϕ_{-1} extends to a chain map $\phi : P \to P'$. If $\psi : P \to P'$ is another such extension, then ϕ and ψ are homotopic.*

This property assures us that the following definition makes sense.

Definition B.5. Let M and N be R-modules. Let

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

be an injective resolution of N. Apply to this resolution the functor $\mathrm{Hom}_R(M, -)$ and drop the $\mathrm{Hom}_R(M, N)$ term. Then $\mathrm{Ext}^n_R(M, N)$ is the n-th homology module of the resulting complex. Equivalently, one takes a projective resolution of M, applies the functor $\mathrm{Hom}_R(-, N)$ to it, drops the $\mathrm{Hom}_R(M, N)$ term and takes homology. With a projective resolution of M, to which we apply $- \otimes_R N$ and drop the $M \otimes_R N$ term, we define $\mathrm{Tor}^R_n(M, N)$ as the n-th homology module of the resulting complex.

Additivity of the functors $\mathrm{Hom}_R(-, N)$, $\mathrm{Hom}_R(M, -)$ and $- \otimes_R N$ and the above proposition show that this definition is independent of the resolution chosen, and that Tor and Ext are bifunctors. A short exact sequence of R-modules gives rise to long exact sequences in both variables of these functors. Using the categorical descriptions of projectivity and injectivity, Tor and Ext can be defined in arbitrary abelian categories.
It was shown by Yoneda, that in an abelian category $\mathcal{C}$, the groups $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$ can be recoverd as groups of equivalence classes of n-extensions $(X_i)_{i=0}^{n-1}$

$$0 \longrightarrow Y \longrightarrow X_0 \longrightarrow \ldots \longrightarrow X_{n-1} \longrightarrow X \longrightarrow 0$$

of X by Y. We will treat this theory here for cyclic vector spaces (i.e. cyclic modules over a field).

Definition B.6. Let X and Y be cyclic vector spaces. An *n-extension* $(E_i)_{i=0}^{n-1}$ *of X by Y* is an exact sequence

$$0 \longrightarrow Y \longrightarrow E_0 \longrightarrow \ldots \longrightarrow E_{n-1} \longrightarrow X \longrightarrow 0$$

of cyclic vector spaces. Let (E_i, ϕ_i) and (F_i, ψ_i) be n-extensions of X by Y. A *morphism of n-extensions* is a sequence of morphisms of cyclic vector spaces $\chi_j : E_j \to F_j$, such that the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_0 & \xrightarrow{\phi_0} & \ldots & \xrightarrow{\phi_{n-2}} & E_{n-1} & \xrightarrow{\phi_{n-1}} & X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \chi_n} & & & & \downarrow{\scriptstyle \chi_1} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & F_0 & \xrightarrow{\psi_0} & \ldots & \xrightarrow{\psi_{n-2}} & F_{n-1} & \xrightarrow{\psi_{n-1}} & X & \longrightarrow & 0
\end{array}$$

commutes. An n-extension (E_i) is said to be *equivalent* to (F_i) if there exists a morphism between them.

Unfortunately, equivalence of n-extensions is not an equivalence relation, since it fails to be symmetric for $n \geq 2$. Therefore we define an equivalence relation on the set of all n-extensions as the equivalence relation generated by equivalence. Suggestively, we denote the set of equivalence classes by $\mathrm{Ext}^n_\Lambda(X, Y)$. This set carries an abelian group structure. $(E_i, \phi_i) + (F_i, \psi_i)$ is defined to be the n-extension (G_i), where $G_i = E_i \oplus F_i$ for $1 \leq i \leq n-2$, with the obvious maps. To define G_0, let i_e and i_f be the embeddings of Y in E_0 and F_0 respectively. Let

$$G_0 := E_0 \oplus F_0 / \{(i_e(y), -i_f(y)) : y \in Y\},$$

such that the maps $Y \to G_0$ induced by $y \mapsto (i_e(y), 0)$ and $y \mapsto (0, i_f(y))$ coincide and are injective. Denote the map so defined by i_g. The map $(e_0, f_0) \mapsto (\phi_0(e_0), \psi_0(f_0))$ has im i_g as its kernel and its image is $\ker \phi_1 \oplus \psi_1$. G_{n-1} is defined as

$$G_{n-1} := \{(e_{n-1}, f_{n-1}) \in E_{n-1} \oplus F_{n-1} : \phi_{n-1}(e_{n-1}) = \psi_{n-1}(f_{n-1})\},$$

the mapping cone of ϕ_{n-1} and ψ_{n-1}. It is straightforward to check that this is well defined on equivalence classes. It also carries a k-module structure. Multiplication by an element of k is a linear map $Y \to Y$, and can thus be used to define another

We will now describe the isomorphism of Ext using injectives with this one. Let

$$0 \longrightarrow Y \longrightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \ldots$$

be an injective resolution of Y by cyclic vector spaces and apply $\mathrm{Mor}_\Lambda(X, -)$ to it. Since $\mathrm{Ext}^n_\Lambda(X, Y)$ is the n-th homology group of the complex

$$\mathrm{Mor}_\Lambda(X, I_0) \xrightarrow{d_0} \mathrm{Mor}_\Lambda(X, I_1) \xrightarrow{d_1} \mathrm{Mor}_\Lambda(X, I_2) \xrightarrow{d_2} \ldots$$

an element in $\mathrm{Ext}^n_\Lambda(X,Y)$ is represented by a morphism $f : X \to I_n$ such that $d_n f = 0$. Let $E_{n-1} := \{(i,x) \in I_{n-1} \oplus X : d_{n-1}(i) = f(x)\}$. The projection $p_2 : E_{n-1} \to X$ on the second coordinate is surjective, because $d_n f(x) = 0$, and $\ker d_n = \operatorname{im} d_{n-1}$, so there is an $i \in I_{n-1}$ with $f(x) = d_{n-1}(i)$. Moreover, its kernel is $\{(i,0) \in E_{n-1} : d_{n-1}(i) = 0\} \cong \ker d_{n-1}$, which is the image of the map $\phi_{n-2} : I_{n-2} \to E_{n-1}$ given by $i \mapsto (d_{n-2}(i), 0)$. Thus

$$0 \longrightarrow Y \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \dots$$

$$\longrightarrow I_{n-2} \xrightarrow{\phi_{n-2}} E_{n-1} \xrightarrow{p_2} X \longrightarrow 0$$

is an n-extension of X by Y. If we replace f by $f + d_{n-1}g$ for some $g : X \to I_{n-1}$, and F_{n-1} is the last element in the extension associated to $f + d_{n-1}g$, then the map $(i,x) \mapsto (i + g(x), x)$ is an isomorphism $E_{n-1} \to F_{n-1}$. This map clearly respects p_2 and commutes with the ϕ_{n-2}'s, so the two extensions are equivalent. To go in the other direction, let (E_i) be an n-extension of X by Y. Recall that there exists a chain map $\chi_i : E_i \to I_i$ such that the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & Y & \longrightarrow & E_0 & \longrightarrow & \dots \\
 & & \| & & \downarrow{\scriptstyle \chi_0} & & \\
0 & \longrightarrow & Y & \longrightarrow & I_0 & \xrightarrow{d_0} & \dots
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow & E_{n-1} & \longrightarrow & X & \longrightarrow & 0 & & \\
 & \downarrow{\scriptstyle \chi_{n-1}} & & \downarrow{\scriptstyle \chi_n} & & \downarrow{\scriptstyle \chi_{n+1}} & & \\
\longrightarrow & I_{n-1} & \xrightarrow{d_{n-1}} & I_n & \xrightarrow{d_n} & I_{n+1} & \longrightarrow & \dots
\end{array}$$

commutes. From this it is immediate that $d_n \chi_n = 0$, so it defines an element in $\mathrm{Ext}^n_\Lambda(X,Y)$. It is straightforward to check that this inverts the construction above, as well as it is to show that the both of them define natural homomorphisms. This isomorphism allows us to carry over the k-vector space structure of Ext using injectives to the one of Ext using extensions.
The boundary map in the exact Ext-sequences takes a particularly elegant form in our current description.

Proposition B.7. *Let V be a cyclic module and*

$$0 \longrightarrow W \longrightarrow X \longrightarrow Y \longrightarrow 0$$

an exact sequence of cyclic vector spaces. Viewing this sequence as defining an element of $[Y,W] \in \mathrm{Ext}^1_\Lambda(Y,W)$, the boundary maps

$$\partial : \mathrm{Ext}^n_\Lambda(V,Y) \to \mathrm{Ext}^{n+1}_\Lambda(V,W), \quad \partial : \mathrm{Ext}^n_\Lambda(W,V) \to \mathrm{Ext}^{n+1}_\Lambda(Y,V)$$

are given by the Yoneda product with $[Y, W]$ *on the left and* $(-1)^{n+1}[Y, W]$ *on the right, respectively.*

This and other properties are described in [15], in much more detail then we did here. For our purposes the account given here suffices.

Appendix C

Failure of excision for K_1

Excision does not hold for K_1, and we give a counterexample, given in exercise 2.5.20 in Rosenberg's book [23]. It uses the fact that the relative K-theory $K_1(R, I)$ is isomorphic to the quotient group $GL(R, I)/E(R, I)$. Here $GL(R, I) = \ker(GL(R) \to GL(R/I))$ and $E(R, I)$ is the smallest normal subgroup of $GL(R, I)$ containing the matrices $e_{ij}(x)$ with $x \in I$. There is a relative form of the Whitehead lemma, stating that

$$E(R, I) = [E(R), E(R, I)] = [GL(R), E(R, I)].$$

We also use, that for a commutative local ring R, the determinant induces an isomorphism $K_1(R) \to R^\times$, the unit group of R. For more details see [23], chapter 2.

Proposition C.1. *Let k be a field and define*

$$R := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(k) \right\}, R' := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in M_2(k) \right\}$$

$$I := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in M_2(k) \right\}$$

Then we have $K_1(R, I) = 1$, while $K_1(R', I) \cong k$. Therefore there is no excision theorem for K_1, and $K_1(R, I)$ depends on both I and R.

Proof. There are split extensions

$$0 \longrightarrow I \longrightarrow R \longrightarrow k \times k \longrightarrow 0 \quad (1)$$

with splitting map $(x_1, x_2) \mapsto \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ and

$$0 \longrightarrow I \longrightarrow R' \longrightarrow k \longrightarrow 0 \quad (2)$$

with splitting map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Moreover we have $R' \cong k[t]/(t^2)$ by mapping $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ to $a + bt$. This map is clearly a bijection and it is a homomorphism since

$$(a + bt)(c + dt) = ac + (bc + ad)t$$

while

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} ac & bc + ad \\ 0 & ac \end{pmatrix}$$

$k[t]/(t^2)$ is a commutative local ring with maximal ideal (t), which clearly corresponds to $I \subset R'$. It follows that

$$K_1(R') \cong (k[t]/(t^2))^\times \cong k^\times \oplus k$$

under the isomorphism $a + bt \mapsto (a, a^{-1}b)$. By the split exact sequence (2), we have a split exact sequence

$$1 \longrightarrow K_1(R', I) \longrightarrow k^\times \oplus k \longrightarrow k^\times \longrightarrow 1$$

such that $K_1(R', I) \cong k$, where k is viewed as an additive group.
To show that $K_1(R, I) = 1$, observe that for any $x, y \in I$ we have $xy = 0$. Let $A \in GL(R, I)$ and choose a representative $A = (a_{ij}) \in GL(n, R)$. Since $GL(R, I) = \ker(GL(R) \to GL(R/I))$, we have $a_{11} = 1 + x$ for some $x \in I$. Therefore,

$$(1 + x)(1 - x) = 1 - x^2 = 1$$

and a_{11} is invertible. Thus by subtracting $a_{i1}a_{11}^{-1}$ times the first row from the i-th row for $i = 2, ..., n$ and thereafter subtracting $a_{1j}a_{11}^{-1}$ times the first column from the i-th column, we can reduce A to a matrix of the form

$$\begin{pmatrix} a_{11} & 0 \\ 0 & A' \end{pmatrix}$$

Since I is an ideal, all these operations can be carried out by multiplication with matrices in $E(R, I)$. By induction it follows that A can be reduced to a diagonal matrix in $GL(R, I)$. By lemma 1.13, matrices of the form

$$\text{diag}\ (1, ..., 1, a, a^{-1}, 1, ..., 1)$$

are elements of $E(R)$, and these can be used to change all the diagonal entries $a_{ii} = 1 + x_i$ for $i \geq 2$ into 1's. The above argument shows that the image of $K_1(R, I)$ in $K_1(R)$ is generated by the image of

$$\{x \in R^\times : q(x) = 1\} \subset GL(R, I) \subset GL(R)$$

By the split exact sequence (1), we have a split exact sequence

$$1 \longrightarrow K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \longrightarrow 1$$

It follows that $K_1(R,I)$ is generated by the image of $\{x \in R^\times : q(x) = 1\}$ in $K_1(R,I)$. Therefore it suffices to show that this image is trivial. If k has more than 2 elements, then we choose $a \neq d \in k^\times$ and write

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{\frac{a}{d}-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \frac{-b}{ad} \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-b}{\frac{a}{d}-1} \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{\frac{a}{d}-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & \frac{b}{\frac{a}{d}-1} \\ 0 & 1 \end{pmatrix}^{-1}$$

such that all our generators are elements of $[GL(R), E(R,I)]$ (since $\frac{b}{\frac{a}{d}-1} \in I$), and thus, by the relative Whitehead lemma, in $E(R,I)$, showing $K_1(R,I) = 1$. If k has two elements, then necessarily $b = 0$, such that we can draw the same conclusion. □

Appendix D

Fibre bundles

We will introduce a class of maps between topological spaces, with which a lot of examples can be described in a very nice way. Examples of fibre bundles include vector bundles, covering spaces and and principal bundles. The definition below describes fibre bundles with topological structure groups. Since discrete groups are just topological groups with the discrete topology, this is not a restriction. Recall that a topological group G acts *effectively* on space F, if no element other than the identity acts as the identity. Good texts on algebraic topology are [24] and [11].

Definition D.1. Let X, E and F be topological spaces, G a topological group acting effectively on F, and $p : E \to Y$ a continuous map. p is called a *fibre bundle with structure group G and fibre F* if the following conditions are met:

- There is an open covering $\mathcal{U} := \{U_i\}_{i \in \mathcal{I}}$ of Y, such that for each $y \in Y$ there is an $i \in \mathcal{I}$ with $y \in U_i$ and a homeomorphism $\phi_i : p^{-1}(U_i) \to U_i \times F$ such that the diagram

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times F \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle p_{U_i}} & \\ U_i & & \end{array}$$

  commutes.

- For each $i, j \in \mathcal{I}$, there is a continuous map $\gamma_{ij} : U_i \cap U_j \to G$ such that

$$\phi_j \circ \phi_i^{-1} : U_i \cap U_j \times F \to U_i \cap U_j \times F$$

  is given by

$$\phi_i \circ \phi_j^{-1}(u, f) = (u, \gamma_{ij}(u) f).$$

A *morphism of fibre bundles* is a fibre preserving continuous map that commutes with the action of G.

The second condition in the definition implies that $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$. Such functions are called *G-valued Čech 1-cocycles on $\mathcal{U}$*. The set of all these cocycles is denoted $Z^1(\mathcal{U}, G)$. If γ_{ij} is of the form $\gamma_i\gamma_j^{-1}$, for functions $\gamma_j : U_j \to G$, then the bundle is isomorphic to $Y \times F$, for $e \mapsto (\phi_i(e), p(e))$ is well defined (i.e. independent α) and defines a bundle isomorphism. As expected, these γ_{ij}'s are called coboundaries, and the set of all coboundaries is denoted $B^1(\mathcal{U}, G)$. In a similar way, one sees that if the cocycles γ and δ satisfy $\delta_{ij} = \eta_i\gamma_{ij}\eta_j^{-1}$, then the bundles are isomorphic. This leads to looking at the set

$$H^1(\mathcal{U}, G) := Z^1(\mathcal{U}, G)/B^1(\mathcal{U}, G),$$

the *first Čech cohomology set of $\mathcal{U}$ with values in G*. This is *not* a group unless G is abelian. Given an action of G on F, $H^1(\mathcal{U}, G)$ classifies the fibre bundles over Y with fiber F, group G and covering $\mathcal{U}$. It would be nice to get rid of the dependence on the covering $\mathcal{U}$, to be able to classify all bundles over Y. This is done by a limiting process. A covering $\mathcal{V} := (V_j)_{j\in\mathcal{J}}$ is called a *refinement* of $\mathcal{U}$ if for each $j \in \mathcal{J}$ there is an $i \in \mathcal{I}$ such that $V_j \subset U_i$. A choice of i for each $j \in \mathcal{J}$ defines a *refinement map* $\sigma : \mathcal{J} \to \mathcal{I}$ and induces a map of cohomology sets

$$\sigma^* : H^1(\mathcal{U}, G) \to H^1(\mathcal{V}, G),$$

which turns out to be injective and independent of σ. Since any two coverings have common refinement,

$$\{U_i \cap V_j : i \in \mathcal{I}, j \in \mathcal{J}\},$$

for example, the cohomology groups form a directed system and we define

$$H^1(Y, G) := \lim_{\to} H^1(\mathcal{U}, G),$$

the direct limit being with respect to all coverings and refinement maps. Since the refinement maps are injective, an element of $H^1(Y, G)$ can be represented by choosing a covering of Y and a cocycle on that covering. Given an action of G on F, $H^1(Y, G)$ classifies the fibre bundles over Y with group G and fibre F.

Examples of fibre bundles are numerous, and we will discuss the ones of interest to us. If M is an n-dimensional manifold, then an n-dimensional $\mathbb{F}$-vector bundle is a fibre bundle with fibre $\mathbb{F}^n$ and group $GL(n, \mathbb{F})$. Fixing a Riemannian metric on M, we can consider the *cosphere bundle*

$$S^*M := \{(x, \omega) \in T^*M : |\omega| = 1\},$$

of M, which is a submanifold of T^*M, and a fibre bundle with fibre S^{n-1} and group $O(n)$.
A *principal bundle* is a fibre bundle with fibre G, where, locally, the action is

given by left translation. If $G = \Gamma$ is a discrete group, a principal Γ bundle over a topological space Y is just a normal covering space.

We will now give an interpretation of topological K-theory for commutative C^*-algebras, using the Serre-Swan and Gel'fand Naimark theorems. This explains the origins of topological K-theory. Our discussion will not be too detailed, because the subject is vast and is not of much relevance to us in this paper.

Theorem D.2 (Gel'fand-Naimark). *The category of locally compact Hausdorff spaces and proper continuous maps is dual to the category of commutative C^*-algebras and $*$-homomorphisms.*

"Dual" means that there are contravariant functors relating these categories. One direction is easy for $C_0(X)$ the ring of compactly supported continuous complex valued functions on X is easily shown to be a C^*-algebra in the sup-norm. The other direction is harder. For a unital C^*-algebra A, one considers the space of characters

$$\Delta(A) := \{\omega : A \to \mathbb{C} : \omega \text{ a } *\text{-homomorphism}\}$$

and topologizes it with the relative topology on the dual A^* given by pointwise convergence. This is then a compact Hausdorff space and $C(\Delta(A)) \cong A$ as C^*-algebras. For the nonunital case one needs to modify the argument and finds a locally compact Hausdorff space X such that $A \cong C_0(X)$. Of course this direction is the more powerful one, since it allows one to treat abstract commutative C^*-algebras as function spaces.

Definition D.3. Let X be a topological space, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An $\mathbb{F}$-*vector bundle over* X consists of a space E and a continuous open surjection $p : E \to X$, such that

1.) Each fiber $p^{-1}(x)$ is a finite dimensional vector space over $\mathbb{F}$;

2.) There are continuous maps $E \times E \to E$ and $\mathbb{F} \times E \to E$, which restrict to addition and scalar multiplication on each fiber.

A *morphism* of vector bundles is a map $f : E \to E'$ with $p'f = p$ and which is linear on each fiber. Given two bundles E and E', we can construct a bundle $E \oplus E'$ called their *Whitney sum*, defined by

$$E \oplus E' := \{(x, x') \in E \times E' : p(x) = p'(x')\}$$

for which there then is a well defined projection to X. A bundle E is said to be *locally trivial* if each $x \in X$ has a neighbourhood U and an isomorphism $\phi : p^{-1}(U) \to U \times \mathbb{F}^r$.

Thus, if X is connected and E a locally trivial bundle over X, the *rank* of E is well defined, being the dimension of some fiber. Considering the set of vector bundles over some space X, we see that their isomorphism classes form an

abelian monoid (addition induced by the Whitney sum) with zero-element the trivial bundle of rank 0. Denote this monoid by $\mathrm{Vect}_{\mathbb{F}}(X)$, and its Grothendieck group $G(\mathrm{Vect}_{\mathbb{F}}(X))$ by $K^0(X)$. Apart from this section, we will only consider complex K-theory of spaces, so this notation suffices.
If X is connected then $\tilde{K}^0(X) := \ker(\mathrm{rank}\ K^0(X) \to \mathbb{Z})$. If $f : X \to Y$ is a continuous function, then it induces a monoid homomorphism $\mathrm{Vect}_{\mathbb{F}}(Y) \to \mathrm{Vect}_{\mathbb{F}}(X)$ by defining

$$f^*(E) := \{(x, e) \in X \times E : f(x) = p(x)\}$$

so we get a map $K^0(Y) \to K^0(X)$.
If X is taken to be compact Hausdorff, there exists an intimate relation between the $\mathbb{F}$-vector bundles over X and projective modules over $C^{\mathbb{F}}(X)$. In fact, the above defined topological K-theory is obtained by specializing the algebraic theory to the case $R = C^{\mathbb{F}}(X)$.

Theorem D.4 (Serre-Swan). *Let X be compact Hausdorff, E an $\mathbb{F}$- bundle over X. Define*

$$\Gamma(X, E) := \{s : X \to E \text{ continuous } : p \circ s = id_X\}$$

This is a $C^{\mathbb{F}}(X)$-module and it is finitely generated and projective. Moreover, if S is a finitely generated projective $C^{\mathbb{F}}(X)$-module, then there exists an $\mathbb{F}$-bundle E_S over X such that $S \cong \Gamma(X, E_S)$ as $C^{\mathbb{F}}(X)$-modules. The functor $E \rightsquigarrow \Gamma(X, E)$ is an equivalence of categories $\mathrm{Vect}_{\mathbb{F}}(X) \leftrightsquigarrow \mathrm{Proj}\ C^{\mathbb{F}}(X)$. It induces an isomorphism $K^0(X) \to K_0(C^F(X))$.

The proof uses that every vector bundle is a direct summand in a trivial bundle. This corresponds to projective modules being direct summands free modules which correspond to trivial bundles. One extends topological K-theory to locally compact Hausdorff spaces by considering functions vanishing at infinity, that is, by considering the C^*-algebra $C_0(X)$.

Bibliography

[1] P.Baum, A.Connes, N.Higson, *Classifying spaces for proper group actions and the K-theory of group C^*-algebras*, C^*-algebras, 1943-1993 A fifty year celebration, Contemporary Mathematics 167 (1994), AMS, Providence, RI 241-291.

[2] M.Benameur, J.Brodzki, V.Nistor, *Cyclic homology and pseudodifferential operators*, Quantization of singular symplectic quotients, 21-46, Progr. Math. 198, Birkhuser, Basel, 2001.

[3] D. Burghelea, *The cyclic homology of group rings*, Comment. Mathem. Helv. 60 (1985), 354-365.

[4] A.Connes, *Noncommutative differential geometry*, IHES Publ. Math. 62 (1985), 257-360.

[5] A.Connes, *Noncommutative geometry*, Academic Press, Harcourt Brace and company, 1994.

[6] A.Connes, *Cohomologie cyclique et foncteurs* Ext^n, C.R. Acad.Sci.Paris Ser. I.Math. 296 (1983), 953-958.

[7] A.Connes and H.Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology 29 (1990), 345-388.

[8] J.Cuntz, G. Skandalis, B.Tsygan, *Cyclic homology in noncommutative geometry*, Encyclopaedia of Mathematical Sciences 121, Springer Berlin Heidelberg New York, 2004.

[9] J.Cuntz, *Excision in periodic cyclic theory for topological algebras*, Cyclic homology and noncommutative geomtery, Fields Institute communications 17 (1997), 43-53.

[10] J.Cuntz and D. Quillen, *Excision in bivariant periodic cyclic cohomology*, Inventiones Mathematicae 127 (1997), 67-98.

[11] J.F. Davis, P.Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics 35, American Mathematical Society, 2001.

[12] E.van Erp, *The Atiyah-singer Index Theorem, C^*-algebraic K-theory, and quantization*, master thesis, UvA, 2000.

[13] T.G. Goodwillie, *Cyclic homology and the free loop space*, Topology 24 (1985), 187-215.

[14] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the AMS 16 (1955), Providence R.I., MR 17, 763.

[15] P.J. Hilton, U.Stammbach, *A course in homological algebra*, GTM 4, Springer-Verlag New York, 1971.

[16] M.Karoubi, *Homologie cyclique et K-théorie*, Astérisque 149 (1987).

[17] J.L.Loday, *Cyclic homology*, Grundlehren der Mathematische Wissenschaften 310, Springer-Verlag Berlin Heidelberg New York, 1992.

[18] J.L.Loday and D.Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helvetici 59 (1984), 565-591.

[19] N.P. Landsman, *Lecture notes on C^*-algebras and K-theory.*

[20] V.Nistor, *Higher index theorems and the boundary map in cyclic cohomology*, Documenta Mathematica 2 (1997), 263-295.

[21] N.C.Phillips, *K-theory for Frechet algebras*, International Journal of Mathematics, vol. 2(1)(1991), 77-129.

[22] M.Rordam, F.Larsen, N.J.Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts 49, 2000.

[23] J.Rosenberg, *Algebraic K-theory and its applications*, GTM 147, Springer-Verlag New York, 1994.

[24] J.J.Rotman, *An introduction to algebraic topology*, GTM 119, Springer-Verlag New York, 1988.

[25] E. Thomas, *Nuclear spaces and topological tensor products*, lecture notes.

[26] M.Wodzicki, *Excision in cyclic homology and in rational algebraic K-theory*, Ann. of Math. 129 (1989), 591-640.