Quantization (systematic)

The term ***quantization*** (in the sense described here) refers to attempts to construct a mathematical description of a quantum system from its formulation as a classical system (which is supposed to be known). Such attempts go back to the pioneers of the old quantum theory (Planck, Einstein, Bohr, Sommerfeld); see [16] and → Quantization: historical. (The opposite procedure is the subject of the → quasi-classical limit.)

The modern era of quantization theory started with Heisenberg's famous paper [5] from 1925, in which he proposed the idea of a 'quantum-theoretical reinterpretation (*Umdeutung*) of classical observables.' All later work on quantization may be said to consist of various different implementations of this idea.

The first successful such implementation consisted of the position and momentum operators introduced by Schrödinger [9], i.e. $\hat{q}^j = x^j$ and $\hat{p}_j = -i\hbar\partial/\partial x^j$, seen (in modern parlance) as unbounded operators on the Hilbert space $L^2(\mathbb{R}^3)$. Substituting these expressions into the classical Hamiltonian yields the left-hand side of the → Schrödinger equation. These operators satisfy the so-called *canonical commutation relations*

$$[\hat{p}_j, \hat{q}^k] = -i\hbar\delta^k_j, \tag{1}$$

along with $[\hat{p}_j, \hat{p}_k] = 0$ and $[\hat{q}^j, \hat{q}^k] = 0$. This fact formed the basis of the various equivalence proofs of matrix mechanics and wave mechanics that were given at the time by Schrödinger, Dirac, and Pauli; the first genuine mathematical proof of this equivalence is due to von Neumann [8].

Approaches to quantization that are based on the canonical commutation relations are usually called *canonical quantization.* Dirac [3, 4] made the important observation that the canonical commutation relations resemble the Poisson brackets in classical mechanics. He suggested that a quantization map $f \mapsto Q(f)$ (in which a function f on phase space, seen as a classical observable, is replaced by some operator on a Hilbert space interpreted as the corresponding quantum observable) should satisfy the condition

$$\frac{i}{\hbar}[Q(f), Q(g)] = Q(\{f, g\}). \tag{2}$$

This is indeed the case for $f(p,q) = p_j$ or q^k and $g(p,q)$ likewise, provided we follow Schrödinger in putting $Q(p_j) = \hat{p}_j$ and $Q(q^j) = \hat{q}^j$. For more complicated observables, however, Dirac's condition turns out to hold only asymptotically as $\hbar \to 0$. For example, in the first systematic account of the quantization of a particle moving in flat space, Weyl [11] proposed that a function f on classical phase space $\mathbb{R}^{2n}$ corresponds to the operator

$$Q(f)\Psi(x) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi\hbar)^n}\, e^{ip(x-q)/\hbar} f\left(p, \tfrac{1}{2}(x+q)\right) \Psi(q). \tag{3}$$

on $L^2(\mathbb{R}^n)$. This reproduces Schrödinger's position and momentum operators, but satisfies (2) only if f and g are at most quadratic in p and q (and according to the so-called Groenewold–van Hove theorem prescriptions different from Weyl's will not fare better). This violation of Dirac's condition is well understood now, since is has been recognized that the essence of the process of quantization is that it yields a *deformation* of the classical algebra of observables [1, 2]. The idea of deformation quantization is particularly relevant to physics in the framework of → algebraic quantum theory [14, 17] (see also [13] for other aspects of Weyl quantization).

The quantization problem on phase spaces other than $\mathbb{R}^{2n}$ (or, more generally, cotangent bundles of Riemannian manifolds, to which Weyl's quantization method is easily generalized [14]) has to be treated by different means. In fact, even on flat space one can sympathize with Mackey's lamentation that 'Simple and elegant as this model [i.e. canonical quantization] is, it appears at first sight to be quite arbitrary and ad hoc. It is difficult to understand how anyone could have guessed it and by no means obvious how to modify it to fit a model for space different from $\mathbb{R}^n$.' ([15], p. 283). Mackey himself explained and generalized canonical quantization on the basis of symmetry arguments that apply whenever a symmetry group G acts on configuration space Q (with associated phase space T^*Q); for flat space $Q = \mathbb{R}^3$ ones takes $G = E(3) = SO(3) \ltimes \mathbb{R}^3$, the Euclidean symmetry group of rigid translations and rotations. Mackey's generalization of the canonical commutation relations (1) consists of his notion of a *system of imprimitivity.* Given an action of a group G on a space Q, such a system consists of a

Hilbert space H, a unitary representation U of G on H, and a projection-valued measure $E \mapsto P(E)$ on Q with values in H, such that

$$U(x)P(E)U(x)^{-1} = P(xE), \tag{4}$$

for all $x \in G$ and all (Borel) sets $E \subset Q$. One notices that position and momentum are assigned a quite different role in this procedure: the former are replaced by the projection-valued measure $E \mapsto P(E)$, whereas the latter are treated as the (infinitesimal) generators of symmetries. Each irreducible system of imprimitivity provides a valid quantization of a particle moving on Q. Mackey's imprimitivity theorem classifies all possibilities; for example, for $Q = \mathbb{R}^3$and $G = E(3)$ one finds that each irreducible representation of $SO(3)$ yields a possible quantization. This is Mackey's explanation of spin. More generally, if $Q = G/K$ is a homogeneous G-space with stability group K, then each irreducible representation of K induces a system of imprimitivity and hence a quantization of the system (and *vice versa*). Let us note that the modern way of understanding this method involves groupoids and their C^*-algebras, which not only lead to a vast generalization of Mackey's approach but in addition put it under the umbrella of deformation quantization [14].

Geometric quantization is a method that starts from the symplectic (or, in old-fashioned language, 'canonical') structure of phase space. This method was independently introduced by Kostant [6] and Souriau [10] and is still being developed; cf. [12, 18]. Although its formalism is quite general, geometric quantization is most effective in the presence of a Lie group acting canonically and transitively on phase space. If successful, the method then yields a representation of the Lie algebra of this group, whose elements play the role of quantum observables.

The procedure starts with a phase space M (i.e. a symplectic manifold), and as a first step towards a quantum theory one constructs a map $f \mapsto Q^{\mathrm{pre}}(f)$ from functions on M to operators on the Hilbert space $L^2(M)$. This map turns out to satisfy Dirac's condition (2) exactly. In the special case $M = \mathbb{R}^{2n}$, it is given by

$$Q^{\mathrm{pre}}(f)\Phi = -i\hbar\{f, \Phi\} + \left(f - \sum_j p_j \frac{\partial f}{\partial p_j} \right) \Phi, \tag{5}$$

where $\{f, \Phi\}$ is the Poisson bracket (which makes sense if $\Phi \in L^2(\mathbb{R}^{2n})$ is assumed differentiable). Unfortunately, the Hilbert space is wrong and the ensuing representation of the canonical commutation relations $Q^{\mathrm{pre}}(q^k) = q^k + i\hbar\partial/\partial p_k$ and $Q^{\mathrm{pre}}(p_j) = -i\hbar\partial/\partial q^j$ is highly reducible: it contains an infinite number of copies of the Schrödinger representation on $L^2(\mathbb{R}^n)$. The second step of the method therefore involves a procedure to cut down the size of the Hilbert space $L^2(M)$ by a certain geometric technique, but through this step only some of the operators (5) remain well defined. Those that are still satisfy (2), however, which fact lies at the basis of the construction of Lie algebra representations from geometric quantization. Despite some successes in that direction, with considerable impact on mathematics, the method of geometric quantization remains unfinished and somewhat unsatisfactory for physics.

Like geometric quantization, *phase space quantization* starts with the Hilbert space $L^2(M)$, but instead of (5) one constructs a quantization map $f \mapsto Q^{\mathrm{p}}(f)$ by

$$Q^{\mathrm{p}}(f) = pfp, \tag{6}$$

where p is a suitable projection operator on $L^2(M)$ (so that the operator $Q^{\mathrm{p}}(f)$ effectively acts on $pL^2(M)$). This projection is constructed from a so-called reproducing kernel K on $L^2(M)$, and has the form $p\Phi(z) = \int_M dw\, K(z, w)\Phi(w)$. This kernel, in turn, comes from a family of $\rightarrow$ coherent states - here construed as maps $z \mapsto \Psi_z$ from M to the set of unit vectors in an auxiliary Hilbert space H - by means of $K(z, w) = (\Psi_z, \Psi_w)$ (the inner product in H). See [12, 14]. The best-know example is $M = \mathbb{R}^{2n}$ with coherent states $\Psi^{\hbar}_{(p,q)}(x) = (\pi\hbar)^{-n/4} \exp((-(x - q)^2 + ip(2x - q))/2\hbar)$ in $H = L^2(\mathbb{R}^n)$, yielding what is often called *Berezin quantization* Q^{B} on $\mathbb{R}^{2n}$. It has the advantage over Weyl quantization and geometric quantization of being positive (in the sense that $(\Phi, Q^{\mathrm{B}}(f)\Phi) \geq 0$ for all Φ whenever $f \geq 0$) and bounded (i.e. $Q^{\mathrm{B}}(f)$ is a bounded operator if f is a bounded function on M).

Quantization theory remains a very active area of research in physics and mathematics [12].

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