Quasi-classical limit

The ***quasi-classical limit*** of quantum mechanics refers, roughly speaking, to the limit $\hbar \to 0$. Of course, $\hbar$ is a dimensionful *constant*, but in practice one studies the semiclassical regime of a given quantum theory by forming a dimensionless combination of $\hbar$ and other parameters; this combination then re-enters the theory as if it were a dimensionless version of $\hbar$ that can indeed be varied.

The oldest example of this procedure is Planck's radiation formula. Indeed, the observation of Einstein [5] and Planck [8] that in the limit $\hbar\nu/kT \to 0$ this formula converges to the classical equipartition law may well be the first use of the $\hbar \to 0$ limit of quantum theory; note that Einstein put $\hbar\nu/kT \to 0$ by letting $\nu \to 0$ at fixed T and $\hbar$, whereas Planck took $T \to \infty$ at fixed ν and $\hbar$.

Another example is the one-particle Schrödinger equation, where one may pass to dimensionless parameters by introducing a typical energy scale ϵ and a typical length scale λ. In terms of the dimensionless variable $\tilde{x} = x/\lambda$, the rescaled Hamiltonian H/ϵ is then dimensionless and contains $\hbar$ through the dimensionless variable $\tilde{\hbar} = \hbar/\lambda\sqrt{2m\epsilon}$. In particular, large mass means effectively small $\hbar$.

Finally, as perhaps first remarked by Bogoliubov [1], averages of N single-particle operators satisfy commutation relations in which $\hbar$ has been replaced by $\hbar/N$, so that the limit $\hbar \to 0$ is effectively equivalent to the limit $N \to \infty$. This remark lies at the basis of the quantum theory of macroscopic observables (see [19] and references therein).

The quasi-classical limit has has two separate aims, which should be sharply distinguished conceptually (although there is considerable overlap in the mathematical techniques that are used):

1. The approximation of solutions to the quantum-mechanical equations of motion (e.g., the Schrödinger equation) by solutions of the corresponding classical equations.
2. The derivation of of classical mechanics, and more generally the explanation of the appearance of the classical world, from quantum theory.

The first application is mathematically sophisticated but is conceptually quite straightforward. The best-known technique is the *WKB approximation*, which goes back to Wentzel [11], Kramers [7] and Brillouin [3] in 1926. In the case of the time-independent Schrödinger equation, one postulates that the wave function has the form

$$\Psi(x) = a_\hbar(x) e^{\frac{i}{\hbar} S(x)}, \tag{1}$$

where S is independent of $\hbar$, substitutes this Ansatz into the Schrödinger equation, and expands in powers of $\hbar$. At lowest order this yields the (time-independent) Hamilton-Jacobi equation $H(\partial S/\partial x, x) = E$, where H is the classical Hamiltonian. This equation is supplemented by the so-called (homogeneous) transport equation

$$\left(\frac{1}{2}\Delta S + \sum_k \frac{\partial S}{\partial x^k}\frac{\partial}{\partial x^k} \right) a_0 = 0. \tag{2}$$

Higher-order terms in $\hbar$ yield further, inhomogeneous transport equations for the expansion coefficients $a_j(x)$ in $a_\hbar = \sum_j a_j \hbar^j$. These can be solved in a recursive way, starting with (2). There are various problems with this method, the main ones being convergence and the fact that in most cases of interest the Ansatz (1) is only valid locally (in x), leading to problems with caustics. These problems have been addressed in a sophisticated field of mathematics called *microlocal analysis* [15, 18, 21]. The WKB method is of little use for chaotic systems and has to be replaced by techniques surrounding the so-called *Gutzwiller trace formula*; see [16, 14].

Another insight dating back to the early days of (mature) quantum theory is *Ehrenfest's Theorem* from 1927 [4], which states that for any wave function Ψ (in the domain of the position operator and of $\partial V(x)/\partial x^j$, where V is the potential) one has

$$m\frac{d^2}{dt^2}\langle x^j\rangle(t) = -\left\langle \frac{\partial V(x)}{\partial x^j} \right\rangle(t), \tag{3}$$

where the brackets $\langle \cdots \rangle(t)$ denote expectation values in the time-dependent state $\Psi(t)$. This looks like Newton's second law, with the tiny but crucial difference that this law should have $(\partial V/\partial x^j)(\langle x\rangle(t))$

on the right-hand side. For further developments in this direction see [17], as well as the literature on microlocal analysis just cited. In particular, Egorov's Theorem in microlocal analysis is closely related to Ehrenfest's: it states that for a large class of Hamiltonians and classical observables f one has $Q(f)(t) = Q(f_t) + O(\hbar)$. Here $Q(f)$ is the Weyl quantization of f (see → Quantization) and the left-hand side evolves according to the quantum equation of motion, whereas the right-hand side follows the classical one.

The last early idea we mention is the *Wigner function*, introduced in 1932 [12]. Namely, each wave function Ψ (or, more generally, each density matrix) defines a function W_Ψ on classical phase space, defined by

$$W_\Psi(p,q) = \int_{\mathbb{R}^n} d^n v\, e^{ipv} \overline{\Psi(q + \tfrac{1}{2}\hbar v)} \Psi(q - \tfrac{1}{2}\hbar v). \tag{4}$$

This function has the property

$$(\Psi, Q(f)\Psi) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi)^n}\, W_\Psi(p,q) f(p,q), \tag{5}$$

where $(\, , \,)$ is the inner product in the Hilbert space $L^2(\mathbb{R}^n)$ and $Q(f)$ is the Weyl quantization of f as before. Thus the Wigner function transforms quantum-mechanical expectation values into classical ones, with the proviso that W_Ψ may fail to be positive and therefore cannot strictly be interpreted as a classical phase space distribution. Nonetheless, it is an extremely effective tool for studying the $\hbar \to 0$ limit [13].

The second application of the quasi-classical limit, i.e. to the explanation of the classical world, is a very deep and largely unsolved problem (cf. [19]) for a survey). To their credit, also here many of the key ideas date back to the founders of quantum mechanics.

Bohr's → *correspondence principle* [2, 10] was, in its original form, not concerned with the classical limit of electronic orbits (but rather with the emitted radiation, which for wide orbits behaves approximately classically). However, at a later stage it was transformed into the general idea that large quantum numbers should give rise to classical behaviour. Applied to atoms, this idea works if it is combined with Schrödinger's suggestion that particle behaviour emerges from wave mechanics by looking at wave packets [9] (see [20] for a modern account). In particular, semiclassical motion emerges if a localized wave packet is formed as a superposition of tens of thousands of energy eigenfunctions with similarly large quantum numbers. Such a wave packet initially follows a time-evolution with almost classical periodicity, but subsequently spreads out after a number of orbits. During this second stage the (Born) probability distribution approximately fills the classical orbit. On a much longer time scale one sees *wave packet revival*, in that the wave packet recovers its initial localization. Then the whole cycle starts once again. See [22] for a popular account and [23] for a technical review. Another successful application of the correspondence principle is to the classical limit of quantum partition functions [24].

Heisenberg's famous 1927 paper [6] not only contained his uncertainty relations, but also suggested that the classical world emerged from quantum mechanics through observation: *'Die Bahn entsteht erst dadurch, daß wir sie beobachten.'* (*'The trajectory only comes into existence because we observe it.'* This idea has to be combined with the quasi-classical limit in order to have the beginning of an explanation of classical physics from quantum theory. Here modern methods of → decoherence and → consistent histories play an important role.

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