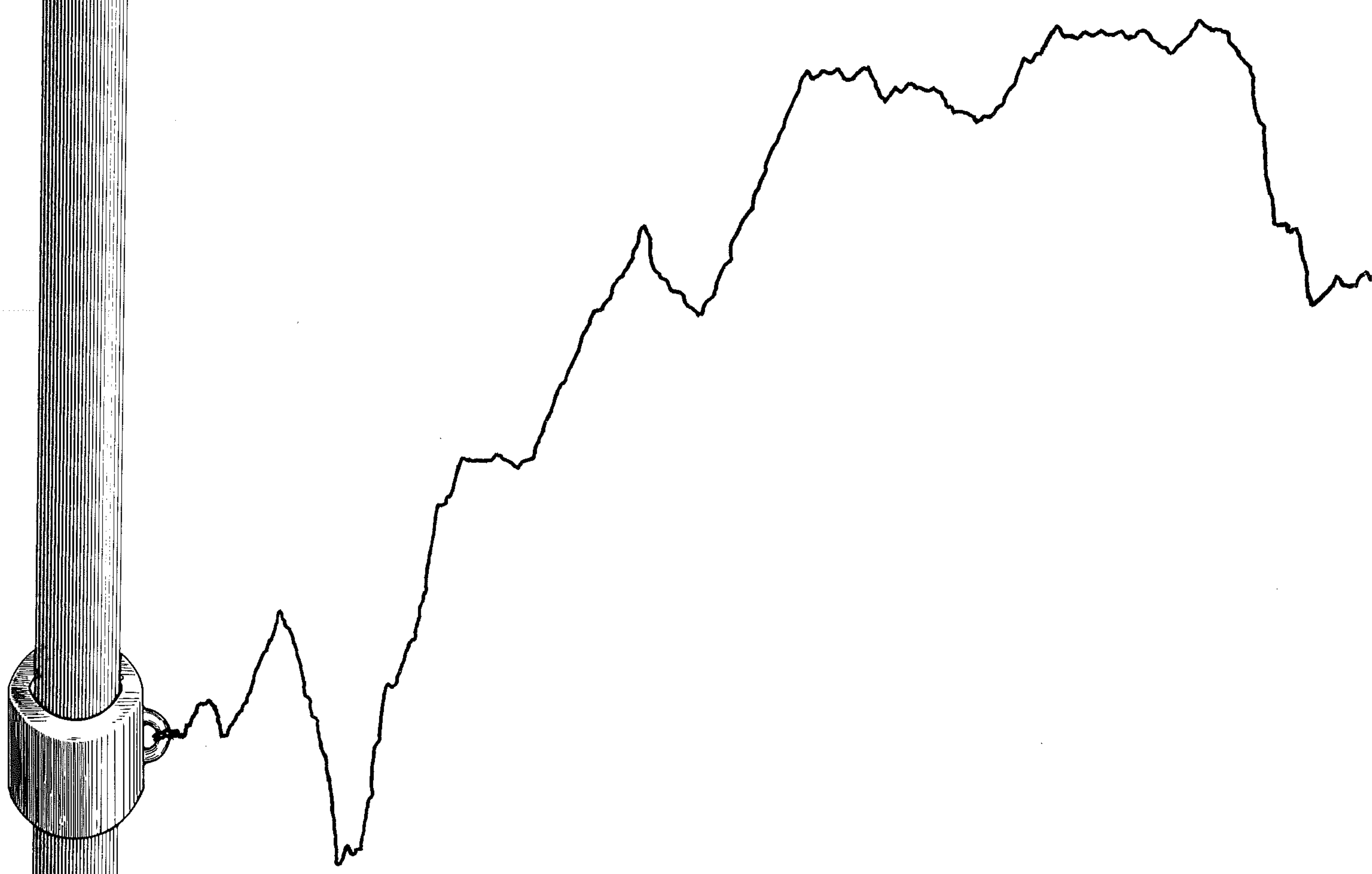
ON A CLASS OF QUANTUM LANGEVIN EQUATIONS AND THE QUESTION OF APPROACH TO EQUILIBRIUM



Hans Maassen

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RIJKSUNIVERSITEIT TE GRONINGEN

ON A CLASS OF QUANTUM LANGEVIN EQUATIONS AND THE QUESTION OF APPROACH TO EQUILIBRIUM

PROEFSCHRIFT

ter verkrijging van het doctoraat
in de wiskunde en natuurwetenschappen
aan de Rijksuniversiteit te Groningen
op gezag van de Rector Magnificus
Dr. L. J. Engels
in het openbaar te verdedigen op
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door

JOHANNES DOMINICUS MARIA MAASSEN

geboren te Arnhem

PROMOTOREN: Prof. N.M. Hugenholtz
Prof. J.T. Lewis

STELLINGEN

1. Onder de realistische kwantummechanische processen benaderen de oplossingen van Langevin-vergelijkingen het meest het idee van een dissipatief proces.

2. In tegenstelling tot een bewering van van Hemmen in diens proefschrift, heeft het door hem bestudeerde harmonische kristal niet één, maar oneindig veel toestanden, die aan de klassieke KMS-konditie voldoen.

   J.L. van Hemmen: Dynamics and ergodicity of the infinite harmonic crystal. Proefschrift, Groningen, 1976.

3. Het verschijnsel, dat door Frigerio en Lewis "kwantum-thermisch geheugen" is genoemd, namelijk het niet-Markoviaans zijn van kwantummechanische processen in thermisch evenwicht, kan beter eenvoudig "kwantumgeheugen" worden genoemd, omdat het zich ook bij temperatuur nul voordoet.

4. Een goede maat voor het geheugen (het niet-Markoviaans karakter) van het Hilbertruimte-proces $\{U^n\psi\}_{n=-\infty}^{\infty}$, waarbij ψ een eenheidsvektor in een Hilbertruimte is, en U een unitaire transformatie daarvan, is de hoek θ tussen de "verleden" deelruimte $D_- = H_- \cap \{\psi\}^\perp$ en de "toekomstige" deelruimte $D_+ = H_+ \cap \{\psi\}^\perp$, waarbij H_- en H_+ de gesloten deelruimten zijn, gegenereerd voor respektievelijk $\{U^n\psi\}_{n\leqslant 0}$ en $\{U^n\psi\}_{n\geqslant 0}$.
Als de maat μ op $[0,2\pi]$, gedefinieerd door $\langle\psi,U^n\psi\rangle = \int_0^{2\pi}\exp(in\varphi)\,\mu(d\varphi)$ kan worden geschreven als $\mu(d\varphi) = w(\varphi)d\varphi/2\pi$, met $\log w \in L^2$, dan geldt

$$\cos^2\theta \leqslant \sum_{n=1}^{\infty} n|c_n|^2 ,$$

waarbij $\{c_n\}$ de rij van Fourier-koëfficiënten van $\log w$ is.

5. Hoewel de grammatika en de semantiek van de moedertaal het denken op de langere termijn beïnvloeden, is het niet zo, dat we in het alledaagse spraak-gebruik de gedachte laten bepalen door de vorm van de zin die haar tot uitdrukking zal brengen.

6. Het verdient aanbeveling, objekten, waarvan het bestaan alleen met behulp van het lemma van Zorn kan worden aangetoond, niet in konstrukties in de fysika op te nemen.

7. Een oplossing van de tweelingparadox in de relativiteitstheorie is de konstatering, dat de reiziger, na lange tijd eenparig weggereisd te zijn, bij het aanzetten van de motor die zijn ruimteschip moet gaan omkeren, zijn thuisgebleven tweelingbroer zo hoog in een gravitatieveld plaatst, dat deze hem kwa leeftijd snel voorbijstreeft.

8. In de wiskunde is het onderscheid tussen globale theorie en gezwets in de ruimte gelegen in het probleemoplossend vermogen ervan.

9. De verwarrende ekwivalentie van het sneller reizen door de tijd en de vertraging van de levensprocessen, leidt tot het goed verdedigbaar zijn van diametraal tegengestelde uitdrukkingswijzen voor zo een gebeuren.

   R. Kousbroek: Anathema's I, Meulenhoff, zesde vermeerderde druk, 1979.

10. Er bestaat een tendens onder pas afgestudeerde artsen, andere specialismen te kiezen dan dat van de huisartsgeneeskunde, omdat zich in de huisartsenpraktijk het probleem het meest nijpend voordoet, dat de huidige geneeskunde op het merendeel van de haar gestelde vragen geen antwoord heeft.

11. Om de wereld eens van een andere kant te bekijken, is het niet noodzakelijk, naar Australië te reizen.

12. De bewegingen, die de fysikus bestudeert, zijn vaak gracieuzer, dan die hij maakt.

   G. Zukav: The dancing Wu-Li masters, Bantam Books, 1979.

aan mijn vader

CONTENTS

INTRODUCTION AND SUMMARY

A few years ago, M. Kac put forward the question, whether the solutions of certain quantummechanical Langevin equations approach thermal equilibrium. In this thesis a partial, positive answer to this question will be given.

Let us sketch the background, and explain the meaning of the question. In 1908, Langevin proposed his now famous equation for the description of the irregular motion of dust particles, suspended in a liquid. For the case that the particle is also subject to a conservative force, derived from a potential v, this equation is the following:

$$m \frac{d^2}{dt^2} Q_t + f \frac{d}{dt} Q_t + v'(Q_t) = E_t. \tag{1}$$

Here, m is the particle's mass, f a friction coefficient of the liquid, and E denotes what is now known as "white noise", the Gaussian generalised stochastic process, with covariance given by

$$\langle E_t E_s \rangle = 2\, f k T\, \delta(t-s). \tag{2}$$

By δ we mean Dirac's delta function. The constants k and T are Boltzmann's constant and the temperature of the liquid respectively.

In 1930, Uhlenbeck and Ornstein [UhO 30] constructed the solution of the Langevin equation (1). It is a Markov process with values in the phase space of the particle. Because of the Markov property, the associated probability density on this phase space satisfies a diffusion equation. If we suppose that the potential v is of a cup-like form, the latter equation has a single stationary solution, towards which all other solutions converge as time goes on. This stationary probability density turns out to be the thermal equilibrium, or Gibbs, probability distribution, associated with the potential v.

Thus it may be said that the solution of the Langevin equation (1) approaches thermal equilibrium.

The probabilistic theory of Brownian motion thus being established, two further fundamental problems came into view. On the one hand, it was not clear, whether and how the theory could be derived from the first principles of classical mechanics and, on the other hand, some authors wondered, what could be a suitable corresponding theory in quantum mechanics. Somewhat surprisingly, answers to both questions were provided by harmonic oscillator models.

In 1965, Ford, Kac and Mazur showed that, in a chain of coupled harmonic oscillators, one of the oscillators can be made to satisfy (1) to arbitrary accuracy by an appropriate choice of the coupling strengths, [FKM 65]. A similar result was obtained by Ullersma, [Ull 66]. A quantum theory of friction and noise was now easily obtained by quantisation of the oscillators in the chain. The equation of motion, satisfied by an element of this FKM chain, was called the "quantum Langevin equation". It is formally identical with (1), but now Q_t is a self-adjoint operator on some Hilbert space, and E is an operator-valued distribution. The defining relation (2) is replaced by the commutation relation

$$[E_t , E_s] = 2\,i\,f\hbar\,\delta'(t-s)\,\mathbb{1}\,. \tag{3}$$

If the entire chain is in thermal equilibrium, E has covariance

$$\langle E_t E_s \rangle = 2f \int_{-\infty}^{\infty} \frac{\hbar\omega}{1-e^{-\hbar\omega/kT}}\, e^{i\omega(t-s)}\,\frac{d\omega}{2\pi}\,. \tag{4}$$

As (4) indicates already, this quantum version of the Ornstein-Uhlenbeck process is not a Markov process. This fact deprives one of the tool by which to prove that it approaches thermal equilibrium. The question, whether or not this is nevertheless the case, concerns us here.

As a matter of fact, approach to the quantummechanical Gibbs state is not to be expected for nonzero values of f, on physical grounds. Indeed, the noise and friction terms in (1) will continually induce transitions between the energy levels of the oscillator considered, so that these will be broadened and shifted. Only in the limit $f \downarrow 0$ one may hope to find the quantum Gibbs state. For this reason, the question of approach to equilibrium was posed by R. Benguria and M. Kac, [BeK 81] in the following form.

For which v and T does the solution $\{Q_t\}$ of the quantum Langevin equation have the property that, irrespective of the initial state $\langle\cdot\rangle$,

$$\lim_{f\downarrow 0}\ \lim_{t\to\infty} \langle \exp(-i\lambda Q_t)\rangle = \langle \exp(-i\lambda x)\rangle_T\ ? \tag{5}$$

Here, $\langle\cdot\rangle_T$ denotes the ordinary quantummechanical Gibbs state on $\mathcal{L}(L^2(\mathbb{R}))$, associated with the potential v.

For a harmonic potential v, the equation (1) can be explicitly solved, and the solution indeed satisfies (5).

In the present thesis, as in [BeK 81], the problem is attacked using perturbation theory around the harmonic potential. In [BeK 81], the unperturbed equilibrium state was chosen as an initial state, and v was chosen to be given by

$$v(x) = \tfrac{1}{2}\,a x^2 + \varepsilon \exp(\lambda x)\,. \tag{6}$$

A power series expansion in ε was considered of both sides of (5), a few coefficients were explicitly computed, and seen to be equal. This strongly suggested that (5) is indeed valid, at least for the above choices of the initial state and the potential.

Here, it will be proved that for a general class of - convex - perturbations of the harmonic potential, the limit as $t\to\infty$ indeed exists for all initial states of the oscillator, (cf. § II.8), and that the limit is close to the Gibbs state for f small, (cf. § II.9). However, the limit $f\downarrow 0$ cannot be taken, because for every potential and temperature the proof of the statement concerning the limit $t\to\infty$ ceases to be valid below some positive value of f. Our proof is based on the existence of a "Møller" isomorphism between the perturbed and the unperturbed model. Such isomorphisms were also considered in an earlier paper, [Maa 82a].

As to the physical application of the quantum Langevin equation, we note that it does not lie in the description of dust particles, suspended in a liquid. Indeed, the parameter $\beta = \hbar\sqrt{a/m}/kT$, (cf. Appendix A), measuring the importance of quantum effects, is extremely small in this case. Instead, one should think of electrons in an atom or molecule. The noise then originates from the vacuum and thermal fluctuations in the electromagnetic field, whereas the friction is caused by radiation reaction. We shall prove a result in this direction: an oscillator, coupled to a massless scalar field, satisfies the quantum Langevin equation in the limit where it becomes a point charge.

This thesis consists of three chapters.

Chapter I treats a classical mechanical model, which goes back to Lamb, [Lam 00], and was revived by Lewis and Thomas, [LeT 75]. It consists of a one-dimensional oscillator, attached to a semi-infinite string, which is heated to a certain positive temperature (cf. front page). The oscillator in this model satisfies the Langevin equation (1) exactly; the model is isomorphic to, but more transparent than, the limiting FKM chain model.

In Chapter II, the harmonic string model is quantised, and perturbations are added to the potential. For each value of the temperature and the friction coefficient, a class of perturbations is delineated, for which the perturbed and the unperturbed evolutions are isomorphic. An estimate is derived for the difference between the Gibbs distribution and the actual equilibrium distribution.

In Chapter III it is shown that the abovementioned model of a point charge oscillator in a massless scalar field is isomorphic to the string model, constructed in the Chapters I and II, and thus satisfies the quantum Langevin equation.

Chapter I

LANGEVIN EQUATIONS IN CLASSICAL MECHANICS

In this chapter, a classical mechanical model, obeying a Langevin equation, is treated. In preparation of a quantummechanical treatment, a Poisson bracket structure on its phase space is introduced. The question of approach to equilibrium is studied, both with and without the use of the Markov property.

§ 1. NEWTONIAN FRICTION

Consider the physical system of a massive particle on a line, the motion $t \mapsto Q_t$ of which is subject to a linear friction force $-\eta \dot{Q}_t$, with $\eta > 0$.

We shall choose the mass of the particle equal to 1, and suppose that there also acts upon it a conservative force, derived from a potential v on the line. For the moment, let these two forces be all that the particle is subject to. Then, according to Newton's law, Q_t satisfies the differential equation

$$\ddot{Q}_t + \eta \dot{Q}_t + v'(Q_t) = 0. \tag{1.1}$$

We assume that v is continuously differentiable and that v' satisfies the Lipschitz condition

$$\exists_{k>0} \forall_{\lambda_1,\lambda_2 \in \mathbb{R}}: \left|v'(\lambda_1) - v'(\lambda_2)\right| \leq k \cdot \left|\lambda_1 - \lambda_2\right| , \tag{1.2}$$

then there is a unique solution to (1.1) for any choice of the initial position and velocity, Q_0 and $\dot{Q}_0$. Let $S_t : \mathbb{R}^2 \to \mathbb{R}^2$ denote the map $\{Q_0, \dot{Q}_0\} \mapsto \{Q_t, \dot{Q}_t\}$, $(t \in \mathbb{R})$. By the nature of this definition, we have $S_t \circ S_s = S_{t+s}$, $(t,s \in \mathbb{R})$, i.e. $\{S_t\}_{t\in\mathbb{R}}$ is a flow on $\mathbb{R}^2$. Let us call $\mathbb{R}^2$ the *phase space*, and $\{S_t\}_{t\in\mathbb{R}}$ the *phase flow* of our system.

Approach to equilibrium

We are interested in the question, what happens if t becomes large. Now, a friction force is a dissipative force. Under its influence the particle can only loose energy, or stand still. Indeed, from (1.1) one derives that

$$\frac{d}{dt}\left(v(Q_t) + \tfrac{1}{2}\dot{Q}_t^2\right) = \dot{Q}_t\left(v'(Q_t) + \ddot{Q}_t\right) = -\eta\dot{Q}_t^2 \leq 0. \qquad (1.3)$$

From this one would expect that, if v is of a form, as drawn in Fig. 1, the particle's motion will slow down, and its position will approach one of the equilibrium points of v, i.e. one of the zeroes of v'. To prove this, we need the following lemma. Let $h: \mathbb{R}^2 \to \mathbb{R}$ be the energy function

$$h(q,p) = v(q) + \tfrac{1}{2}p^2.$$

Let $E = \{\lambda \in \mathbb{R} \mid v'(\lambda) = 0\}$.

Lemma 1.1. For all $x \in \mathbb{R}^2$, all $t \geq 0$:

$$h(S_t x) \leq h(x), \qquad (1.4)$$

and

$$h(S_t x) = h(x) \Rightarrow S_t x = x \text{ and } x \in E\times\{0\}. \qquad (1.5)$$

Proof. (1.4) follows directly from (1.3). To prove (1.5), let $S_s x = \{Q_s, \dot{Q}_s\}$, and suppose that $h(S_t x) = h(x)$, for some $t \geqslant 0$. Then, by (1.3):

$$0 = \int_0^t \frac{d}{ds}\, h(S_s x)\, ds = -\eta \int_0^t \dot{Q}_s^2\, ds.$$

It follows that $\dot{Q}_s = 0$ for $0 \leq s \leq t$, therefore $\dot{Q}_0 = \dot{Q}_t = 0$ and $Q_0 = Q_t$, so $S_t x = x$. Moreover, because also $\ddot{Q}_s = 0$ for $0 \leq s \leq t$, we have $v'(Q_s) = 0$ by (1.1). So $x \in E\times\{0\}$. □

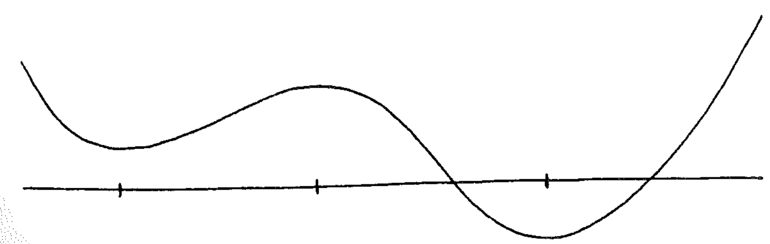

Fig. 1.
Potential with three equilibrium points.

Theorem 1.2. (variant of Liapunov's theorem). Suppose that

$$\lim_{\lambda\to\pm\infty} v(\lambda) = \infty,$$

and that the set E of zeroes of v' is finite. Then for all $x \in \mathbb{R}^2$ there is $e \in E$, such that

$$\lim_{t\to\infty} S_t x = \{e,0\}.$$

Proof. By lemma 1.1, the orbit $\{S_t x\}_{t\geq 0}$ remains inside the compact set K, defined by

$$K = \{y \in \mathbb{R}^2 \mid h(y) \leq h(x)\}.$$

Consider

$$L = \bigcap_{T\geq 0} \overline{\{S_t x \mid t \geq T\}}.$$

L is the set of those points in $\mathbb{R}^2$ that are the limit of a sequence $S_{t_n} x$ with $t_n \to \infty$, (called the ω-limit points of x). We claim that $L \subset E\times\{0\}$. Let $y \in L$, say $y = \lim S_{t_n} x$. Then, because $h(S_t x)$ is decreasing in t, for all $s \in \mathbb{R}$:

$$h(S_s y) = \lim_{n\to\infty} h(S_s \circ S_{t_n} x) = \lim_{n\to\infty} h(S_{s+t_n} x) = \inf_n h(S_{s+t_n} x) = \inf_t h(S_t x) =$$

$$= \inf_n h(S_{t_n} x) = \lim_{n\to\infty} h(S_{t_n} x) = h(y).$$

By Lemma 1.1, $y \in E\times\{0\}$. To show that $S_t x$ tends to one point of $E\times\{0\}$, define, for $\varepsilon > 0$:

$$U_\varepsilon = \left\{\{q,p\} \in \mathbb{R}^2 \,\middle|\, |p| < \varepsilon \text{ and } |q-e| < \varepsilon\ \exists_e \in E\right\}.$$

Suppose now that for some $\varepsilon > 0$ and for arbitrarily large t, there are points $S_t x \notin U_\varepsilon$. Then there is a sequence $\{S_{t_n} x\}_{n\in\mathbb{N}}$ outside U_ε with $t_n \to \infty$. But because $S_{t_n} x \in K$, and $K\setminus U_\varepsilon$ is compact, $K\setminus U_\varepsilon$ must contain a limit point y of some subsequence of $\{S_{t_n}\}$. But then $y \in L$, and L lies inside U_ε, so we have a contradiction. It follows that for all $\varepsilon > 0$ there is $T \geq 0$, such that

$$\{S_t x \mid t \geq T\} \subset U_\varepsilon.$$

Now, for ε small enough, U_ε consists of as many disconnected squares as E has points, and $\{S_t x \mid t \geq T\}$, being connected, must lie inside one of these. The statement follows. □

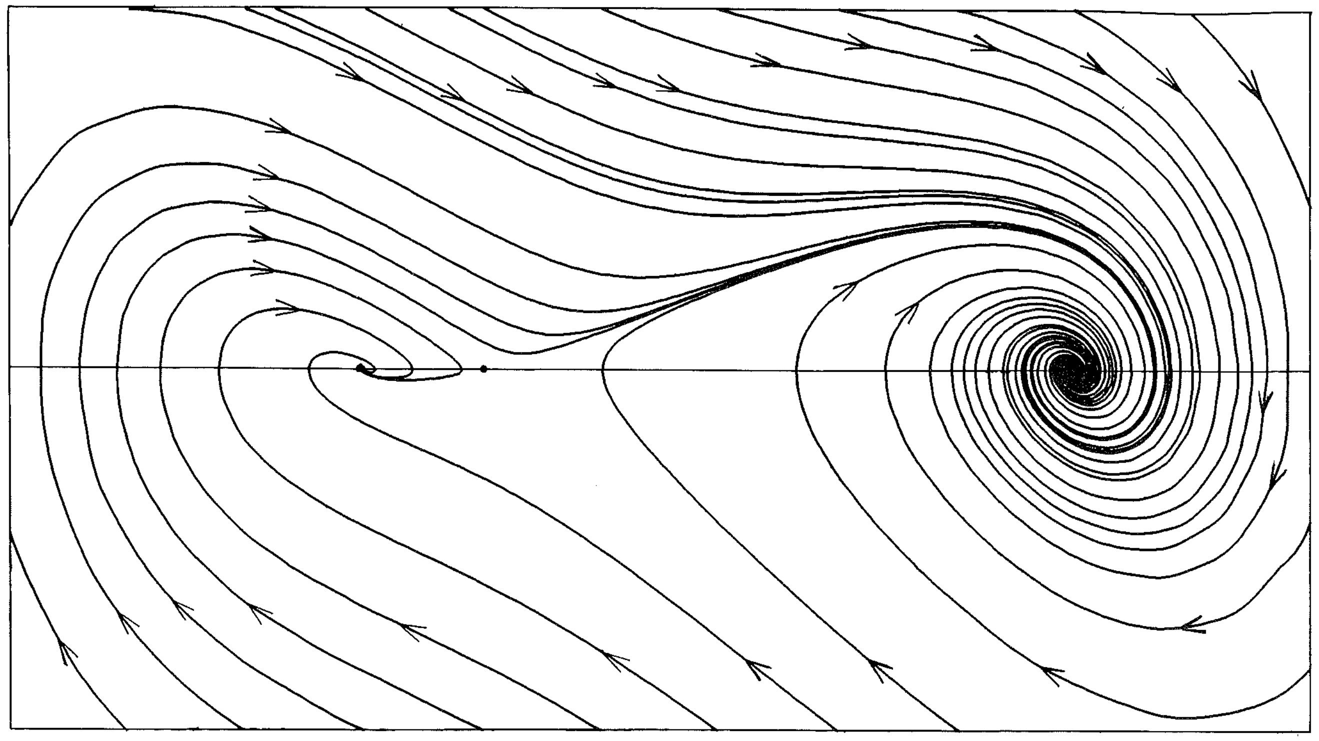

Fig. 2. Stream lines of a phase flow $\{S_t\}$.

§ 2. EMBEDDING INTO A HAMILTONIAN SYSTEM

Can the system $S := \left\{\mathbb{R}^2, \{S_t\}\right\}$, described in the previous section, be cast into the mould of Hamiltonian mechanics?

Hamiltonian mechanics is the classical theory of isolated mechanical systems. Such systems are described by a phase space, on which a real-valued function (the Hamiltonian), a two-form and a flow are defined. The three are related by the canonical equations of motion, and the Hamiltonian and the two-form are conserved by the flow.

S is not such a Hamiltonian system. Indeed, h cannot serve as a Hamiltonian, because h is not conserved by $\{S_t\}$. Neither is it possible to find some other function $\tilde{h}$ on $\mathbb{R}^2$, such that the canonical equations

$$\dot{Q}_t = \frac{\partial \tilde{h}}{\partial p}(Q_t,P_t) \text{ and } \dot{P}_t = -\frac{\partial \tilde{h}}{\partial q}(Q_t,P_t) \tag{2.1}$$

have the orbits of $\{S_t\}$ as their solutions. Indeed, the flow defined by (2.1) would preserve the two-form

$$\sigma_{\mathbb{R}^2} = dq \wedge dp, \tag{2.2}$$

whereas $\{S_t\}$ does not. Hamiltonian mechanics has a distinct frictionless flavour. But, as we shall see, S admits an embedding into a Hamiltonian system. If this were not the case, S could never describe a physical system. Indeed, for every physical system one can find a smallest isolated system containing it, and this is described by a Hamiltonian system R, say. This yields an embedding of S into R.

The Lamb model

The Hamiltonian system we shall consider, was proposed by Lamb in 1900 [Lam 00] as an early model for spontaneous emission of radiation. Recently, Lewis and Thomas [LeT 75] showed that this system could be used as a model for a small system in a heat bath, satisfying the Langevin equation. In this and the next chapter, we shall extend these results to anharmonic potentials, and a quantum version of the system.

Let us first give a sketch. Imagine a massive ring of mass 1 that can slide without friction (!) along a bar. To this ring there is attached a string, extending infinitely far, and pulled tight to a tension η. The mass of the string per unit of length is also η, so that waves can travel along it at unit speed. The direction of the bar is perpendicular to that of the string in its rest position. An irregular spring connects the ring with some point, held fixed, and exerts on it a force $-v'(Q)$, if $Q \in \mathbb{R}$ is the position of the ring, measured along the bar. We suppose that v satisfies the conditions of theorem 1.2. Henceforth, we shall refer to the ring as

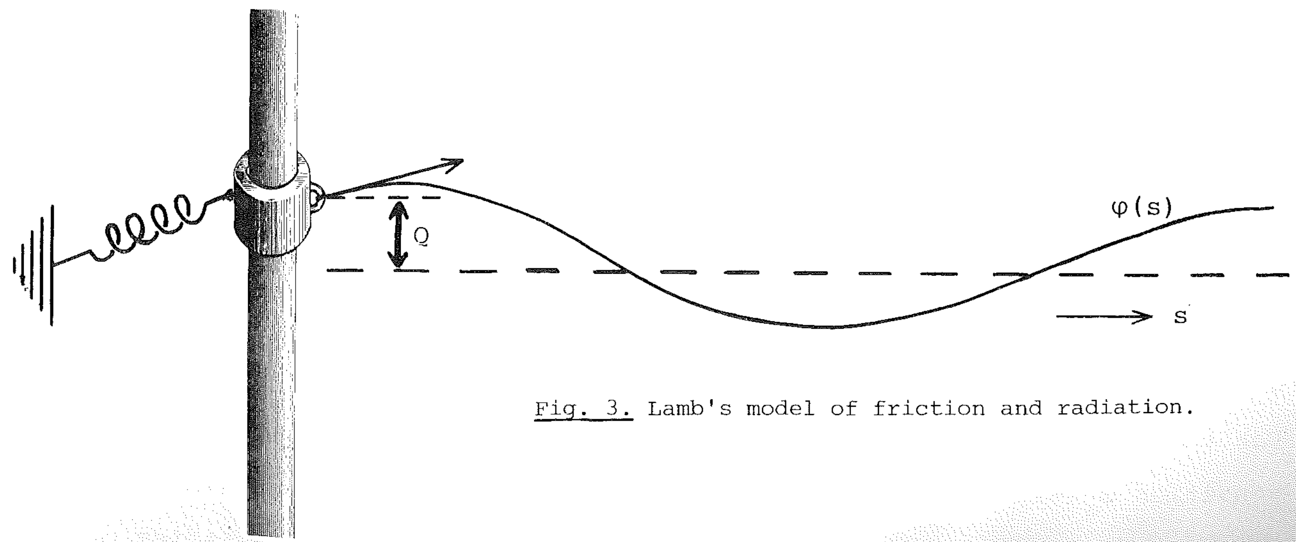

Fig. 3. Lamb's model of friction and radiation.

"the oscillator". Let $\varphi_t(s)$, $(s \geq 0)$ be the shape of the string at time t, $(t \in \mathbb{R})$. $\{\varphi_t\}$ will satisfy the one-dimensional wave equation

$$\ddot{\varphi}_t = \varphi_t'' , \quad (t \in \mathbb{R}), \tag{2.3}$$

where the dot denotes time differentiation, and the prime differentiation w.r.t. the spatial distance s. As the position of the ring at time t is given by $\varphi_t(0)$, we have

$$\ddot{\varphi}_t(0) = \eta\varphi_t'(0) - v'(\varphi_t(0)), \tag{2.4}$$

where $\eta\varphi_t'(0)$ is the force, exerted by the string (cf. Fig. 3). We shall see in Theorem 2.2 that the behaviour (2.3) and (2.4) prescribe, is indeed possible. But first, let us assume this is so, and show that the oscillator does indeed behave like the particle in § 1.

A simulation of friction

All solutions of the wave equation (2.3) are of the form

$$\varphi_t(s) = a(t-s) + b(t+s). \tag{2.5}$$

If we add $\eta\dot{\varphi}_t(0) + v'(\varphi_t(0))$ to either side of (2.4), we obtain the equation

$$\ddot{\varphi}_t(0) + \eta\dot{\varphi}_t(0) + v'(\varphi_t(0)) = \eta(\dot{\varphi}_t(0) + \varphi_t'(0)). \tag{2.6}$$

Now, the right-hand side of (2.6) equals, by (2.5)

$$\eta\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)\left(b(t+s) + a(t-s)\right)\Big|_{s=0} = 2\eta b'(t),$$

and therefore the position $Q_t = \varphi_t(0)$ of the oscillator satisfies the differential equation

$$\ddot{Q}_t + \eta\dot{Q}_t + v'(Q_t) = 2\eta b'(t), \quad (t \in \mathbb{R}). \tag{2.7}$$

This is the basic equation of the Lamb model.

Now, choose any $\{q,p\} \in \mathbb{R}^2$ and prepare the system as follows. Let, before time 0, the string be horizontal at a height q. Exactly at time 0, give the oscillator a jolt, transferring to it a momentum p, and let go. For $t \geq 0$, the equations of motion take over. From (2.5), and the initial condition $\varphi_0'(s) = \dot{\varphi}_0(s) = 0, (s > 0)$, it follows that b is constant on $(0,\infty)$, and a on $(-\infty,0)$. Then (2.7) implies that $\{Q_t\}_{t>0}$ satisfies our

equation (1.1) for a particle on a line with friction, and therefore

$$\{Q_t, \dot{Q}_t\} = S_t(q,p), \quad (t \geq 0), \tag{2.8}$$

as announced.

It should be noted that there is nothing inherently "irreversible" about this process. Indeed, if we halt the oscillator and the string at a time $t \geq 0$, and, in the fashion of Losschmidt, reverse the velocities of both, and let go again, the oscillator will retrace its path. Moreover, by an argument parallel to the one that validated (2.7), we can show that the following relation holds as well:

$$\ddot{Q}_t - \eta\dot{Q}_t + v'(Q_t) = -2\eta a'(t), \quad (t \in \mathbb{R}). \tag{2.9}$$

If we would be able to impose the above described situation at time zero as a final, instead of an initial condition, then, because a is constant on $(-\infty,0)$, $\{Q_t\}$ satisfies the negative-friction differential equation

$$\ddot{Q}_t - \eta\dot{Q}_t + v'(Q_t) = 0, \quad (t \leq 0).$$

Its solution is given by

$$\{Q_t, \dot{Q}_t\} = S_{-t}(q,p), \quad (t \leq 0). \tag{2.10}$$

So the Lamb model can simulate friction behaviour backwards as well as forwards in time.

Solution of the equations of motion

The remainder of this section will be devoted to a construction of the Lamb model as a Hamiltonian system, showing that the equations of motion (2.3) and (2.4) indeed determine a flow on a well-defined phase space.

Definition 2.1. Let Ω be one of the intervals $(-\infty,\infty)$ or $[0,\infty)$. By $C_K^m(\Omega)$, $(m \in \mathbb{N})$, we shall mean the space of all C^m-functions (i.e. m times continuously differentiable functions) $\Omega \to \mathbb{R}$ of compact support. By $C_0^m(\Omega)$ we denote the Banach space of C^m-functions on Ω, the first m derivatives of which vanish at infinity.

Theorem 2.2. Let v and E be as in Theorem 1.2. Let $\varphi : [0,\infty) \to \mathbb{R}$ be such that $\varphi' \in C_0^1([0,\infty))$, and let $\pi \in C_0^1([0,\infty))$. Suppose that

$$\varphi''(0) - \eta\varphi'(0) + v'(\varphi(0)) = 0. \tag{2.11}$$

Then there are C^2-functions a and b, uniquely determined up to a constant difference, with $a', b' \in C_0^1(\mathbb{R})$, such that the family $\{\varphi_t\}_{t\in\mathbb{R}}$, given by

$$\varphi_t(s) = a(t-s) + b(t+s), \quad (s \geq 0), \tag{2.12}$$

solves the equations of motion (2.3) and (2.4), with initial conditions

$$\varphi_0 = \varphi \text{ and } \dot{\varphi}_0 = \pi. \tag{2.13}$$

Moreover, the limits

$$\lim_{t\to\pm\infty} (a(t) + b(t)) =: e_\pm \tag{2.14}$$

exist and are points of E.

Definition 2.3. Let Φ_0 be the space of pairs $\varphi \oplus \pi$, satisfying the conditions of the above theorem. We shall call a and b the *output function* and the *input function* of $\varphi\oplus\pi$. The points e_+ and e_- will be called the *equilibrium points at* $t = \pm\infty$.

Proof. Let $\psi : [0,\infty) \to \mathbb{R}$ be such that $\psi' = \pi$. (ψ is determined up to an additive constant.) Define

$$b(t) = \tfrac{1}{2}(\varphi(t) + \psi(t)), \quad (t \geq 0), \tag{2.15}$$

and $$a(t) = \tfrac{1}{2}(\varphi(-t) - \psi(-t)), \quad (t \leq 0). \tag{2.16}$$

Let, for $t \leq 0$, $b(t)$ be the solution of the (non-autonomous) differential equation

$$b''(t) - \eta b'(t) = -\Big(a''(t) + \eta a'(t) + v'(a(t) + b(t))\Big), \tag{2.17}$$

with boundary values for b and b' in 0 given by (2.15). Let, for $t \geq 0$, $a(t)$ be the solution of

$$a''(t) + \eta a'(t) = -\Big(b''(t) - \eta b'(t) + v'(a(t) + b(t))\Big), \tag{2.18}$$

with boundary values for a and a' in 0 given by (2.16). We shall show that a and b are C^2.

By definition, they are C^2 on $\mathbb{R}\setminus\{0\}$. Also by definition, a, a', b and b' are continuous in 0. Because of (2.11), a'' and b'' are continuous there too: indeed

$$\lim_{t\uparrow 0} b''(t) = \lim_{t\uparrow 0}\Big(\eta(b'(t) - a'(t)) - v'(b(t) + a(t)) - a''(t)\Big) =$$

$$= \eta(b'(0) - a'(0)) - v'(a(0) + b(0)) - \lim_{t\downarrow 0} a''(t) =$$

$$= \eta\varphi'(0) - v'(\varphi(0)) - \lim_{t\downarrow 0} \tfrac{1}{2}(\varphi''(t) - \psi''(t)) =$$

$$= \varphi''(0) - \tfrac{1}{2}(\varphi''(0) - \psi''(0)) = \tfrac{1}{2}(\varphi''(0) + \psi''(0)) = \lim_{t\downarrow 0} b''(t).$$

An analogous argument holds for a.

So a and b are C^2, and $\{\varphi_t\}$, given by (2.12), satisfies the wave equation $\ddot{\varphi}_t = \varphi_t''$. The equation of motion (2.4) is valid by construction ((2.17) and (2.18)). Finally, note that $b' \upharpoonright [0,\infty)$ is of compact support, because φ' and ψ' are. Let $T = \sup\{t \mid b'(t) \neq 0\}$. Then, for $t \geq T$, $b(t)$ is constant, and (2.18) becomes the un-driven friction equation (1.1) for $a(t)$ in the potential $v(\cdot + c)$, ($c = b(t)$). It follows from Theorem 1.2 that $\{a(t), a'(t)\} \to \{e_+ - c, 0\}$ $(t\to\infty)$ for some $e_+ \in E$. But then also $a''(t) \to 0$, $(t\to\infty)$ by (2.18). So $a' \in C_0^1(\mathbb{R})$ and $a(t) + b(t) \to e_+$, $(t\to\infty)$. Analogously, we see that $b' \in C_0^1(\mathbb{R})$ and $a(t) + b(t) \to e_-$, $(t \to -\infty)$. □

Define $F_t(\varphi\oplus\pi) = \varphi_t \oplus \dot{\varphi}_t$. Let the Hamiltonian H_Φ of the Lamb model be given by

$$H_\Phi(\varphi\oplus\pi) = v(\varphi(0)) + \tfrac{1}{2}\pi(0)^2 + \tfrac{1}{2}\eta\int_0^\infty \Big(\varphi'(s)^2 + \pi(s)^2\Big)ds,$$

$$\text{for } \varphi', \pi \in C_K^0([0,\infty)).$$

Let σ_Φ be the antisymmetric bilinear form on $\Phi_1 := \{\varphi\oplus\pi \mid \varphi', \pi \in C_K^0([0,\infty))\}$, defined by

$$\sigma_\Phi(\varphi_1\oplus\pi_1, \varphi_2\oplus\pi_2) = \varphi_1(0)\pi_2(0) - \varphi_2(0)\pi_1(0) + \eta\int_0^\infty \Big(\varphi_1(s)\pi_2(s) - \varphi_2(s)\pi_1(s)\Big)ds.$$

Lemma 2.4. For all $\varphi_0\oplus\pi_0 \in \Phi_0$ and $\varphi_1\oplus\pi_1 \in \Phi_1$ we have

$$\sigma_\Phi\Big(\frac{d}{dt}F_t(\varphi_0\oplus\pi_0)\Big|_{t=0}, \varphi_1\oplus\pi_1\Big) = \frac{d}{d\lambda}H_\Phi\Big(\varphi_0\oplus\pi_0 + \lambda\,\varphi_1\oplus\pi_1\Big)\Big|_{\lambda=0} \tag{2.19}$$

Remark. (2.19) is a version of the canonical equations of motion for a Hamiltonian system (cf. § 4).

Proof of Lemma 2.4. For $\varphi_0 \oplus \pi_0 \in \Phi_0$, we have

$$\frac{d}{dt} F_t(\varphi_0 \oplus \pi_0) = \pi_0 \oplus \varphi_0'' \in \Phi_1 ,$$

and, with $\varphi_1 \oplus \pi_1 \in \Phi_1$:

$$\sigma_\Phi(\pi_0 \oplus \varphi_0'', \varphi_1 \oplus \pi_1) = \pi_0(0)\pi_1(0) - \varphi_0''(0)\varphi_1(0) + \eta \int_0^\infty (\pi_0\pi_1 - \varphi_0''\varphi_1)\, ds$$

$$= \pi_0(0)\pi_1(0) + \varphi_1(0)\Big(-\varphi_0''(0) + \eta\varphi_0'(0)\Big) + \eta \int_0^\infty (\pi_0\pi_1 + \varphi_0'\varphi_1')\, ds$$

$$= \pi_0(0)\pi_1(0) + v(\varphi_0(0))\varphi_1(0) + \eta \int_0^\infty \Big(\varphi_0'\varphi_1' + \pi_0\pi_1\Big)\, ds .$$

On the other hand,

$$\frac{d}{d\lambda} H_\Phi\Big(\varphi_0 \oplus \pi_0 + \lambda\, \varphi_1 \oplus \pi_1\Big)\Big|_{\lambda=0} = \frac{d}{d\lambda}\Big(v(\varphi_0(0) + \lambda\varphi_1(0)) + \tfrac{1}{2}(\pi_0(0) + \lambda\pi_1(0))^2$$

$$+ \tfrac{1}{2}\eta \int_0^\infty \Big((\varphi_0' + \lambda\varphi_1')^2 + (\pi_0 + \lambda\pi_1)^2\Big)\, ds\Big)\Big|_{\lambda=0}$$

$$= \varphi_1(0)v'(\varphi_0(0)) + \pi_0(0)\pi_1(0) + \eta \int_0^\infty (\varphi_0'\varphi_1' + \pi_0\pi_1)\, ds.$$

Comparison yields (2.19). □

Corollary 2.5. $\{F_t\}$ conserves the energy H_Φ.

Remark. It also follows that $\{F_t\}_{t\in\mathbb{R}}$ is a symplectic flow, i.e. F_t conserves σ_Φ in a local sense. We shall not need this result, however.

Proof. Let $L(\varphi\oplus\pi) = \frac{d}{dt} F_t(\varphi\oplus\pi)\Big|_{t=0}$. Then by (2.19), $\frac{d}{dt} H_\Phi\Big(F_t(\varphi\oplus\pi)\Big)\Big|_{t=0} = \frac{d}{d\lambda} H_\Phi\Big(\varphi\oplus\pi + \lambda L(\varphi\oplus\pi)\Big)\Big|_{\lambda=0} = \sigma_\Phi\Big(L(\varphi\oplus\pi), L(\varphi\oplus\pi)\Big) = 0$, because σ_Φ is anti-symmetric. □

Corollary 2.6. Let $\varphi\oplus\pi$ be a point of Φ_0, and let b and a be its input and output function, e_- and e_+ its equilibrium points at time $\pm\infty$. Then

$$\eta \int_{-\infty}^\infty b'(s)^2\, ds + v(e_-) = H_\Phi(\varphi\oplus\pi) = \eta \int_{-\infty}^\infty a'(s)^2\, ds + v(e_+). \qquad (2.20)$$

Proof. Because of energy conservation we have for all $t \in \mathbb{R}$;

$$H_\Phi(\varphi\oplus\pi) = H_\Phi(\varphi_t \oplus \dot\varphi_t) = v(\varphi_t(0)) + \tfrac{1}{2}\dot\varphi_t(0)^2 + \tfrac{1}{2}\eta \int_0^\infty \Big(\varphi_t'(s)^2 + \dot\varphi_t(s)^2\Big)\, ds$$

$$= v(a(t) + b(t)) + \tfrac{1}{2}(a'(t) + b'(t))^2$$

$$+ \tfrac{1}{2}\eta \int_0^\infty \Big(\Big(b'(t+s) - a'(t-s)\Big)^2 + \Big(b'(t+s) + a'(t-s)\Big)^2\Big)\, ds$$

$$= v(a(t) + b(t)) + \tfrac{1}{2}(a'(t) + b'(t))^2 + \eta \int_t^\infty b'(u)^2\, du + \eta \int_{-\infty}^t a'(u)^2\, du.$$

We obtain (2.20) by taking the limits $t \to \infty$ and $t \to -\infty$. □

§ 3. A TRANSLATION REPRESENTATION

In the previous section we have seen that a "dead" string can absorb the oscillator's energy in such a way that it looks as though the oscillator is subjected to friction.

So far, so good. There exists a classical mechanical model that can imitate friction. Why should this be of any interest? As we shall discuss in this section, this model is not just an imitation, but the prototype of a friction process, in the sense that, if ever an oscillator undergoes friction, its environment must behave very much like the string. This general idea is exemplified by a result due to [LeT 74] on the harmonic oscillator case. In the case that $v(\lambda) = \frac{1}{2}\lambda^2$, it can be shown that there is precisely one (minimal, linear) Hamiltonian system, inducing the behaviour $\{S_t\}$, as determined by this choice of v, on a part of itself. This system is the unitary dilation of the semigroup $\{S_t\}$ according to Sz.- Nagy. We shall give a definition shortly. On the other hand, there may be a host of possible environments for a harmonic oscillator that induce this frictional behaviour only approximately.

The input function as a representer

We shall now pursue our investigations of the string model, taking into account also those initial states that are not "dead", i.e. that do not yield zero on the r.h.s. of the basic equation (2.7). We shall find a convenient representation of the system's phase space as $L^2(\mathbb{R})$.

Consider the correspondence $\varphi \oplus \pi \mapsto b$, adjoining to a state of the string model its input function (Def. 2.3). If $\varphi \oplus \pi \in \Phi_0$, the image of $F_t(\varphi \oplus \pi)$ under this map is obtained by translation of the image b of $\varphi \oplus \pi$ to the left:

$$F_t(\varphi \oplus \pi) \mapsto \tilde{T}_t b\,;\quad (\tilde{T}_t b)(s) = b(s+t). \tag{3.1}$$

Proof. The function b is the input function of $\varphi \oplus \pi$ if for some function a and all $u \in \mathbb{R}$: $(F_u(\varphi \oplus \pi))(s) = a(u-s) + b(u+s)$. Choose $t \in \mathbb{R}$. Then $(F_u(F_t(\varphi \oplus \pi)))(s) = (F_{u+t}(\varphi \oplus \pi))(s) = a(u+t-s) + b(u+t+s) = (\tilde{T}_t a)(u-s) + (\tilde{T}_t b)(u+s)$. Therefore $\tilde{T}_t b$ is the input function of $F_t(\varphi \oplus \pi)$. □

By corollary 2.6, the energy $H_\Phi(\varphi \oplus \pi)$ is equal to $\eta\|b'\|^2 + v(e_-)$, and if v has only one equilibrium point e, say with $v(e) = 0$, we even have

$$H_\Phi(\varphi \oplus \pi) = \eta\|b'\|^2. \tag{3.2}$$

It seems very attractive, therefore, to label the phase space point $\varphi \oplus \pi$ by b; this would yield a convenient translation representation of the string model because of (3.1), with a simple Hamiltonian (3.2).

However, the correspondence $\varphi \oplus \pi \mapsto b$ may be a very awkward one. It is not necessarily continuous; thinking backward in time, one may imagine how a tiny change in $\varphi \oplus \pi$ may make the oscillator "decide" to go to another equilibrium point. Nor is it necessarily invertible. For instance, if $b = 0$, it may be that $\varphi \oplus \pi = (e \cdot 1) \oplus 0$ for any $e \in E$. More generally, given b, $\varphi \oplus \pi$ may be found by solving the basic equation (2.7) for $\{Q_t\}$, with different "initial conditions at time $-\infty$". It is this ambiguity that has to be dealt with to prove approach to equilibrium (cf. § 9).

None of the above inconveniences plagues the harmonic string model. This we obtain if we choose v to be a quadratic. Let us put

$$v(\lambda) = \tfrac{1}{2}\lambda^2. \tag{3.3}$$

Then H_Φ, defined in § 2, becomes a quadratic form on Φ_0 and, being strictly positive, it defines a norm on Φ_0, which we shall call the "energy norm":

$$\|\varphi \oplus \pi\|_{H_\Phi} := (2H_\Phi(\varphi \oplus \pi))^{\frac{1}{2}}. \tag{3.4}$$

Let Φ be the completion of Φ_0 in this norm. (An example of an element of $\Phi \setminus \Phi_0$ is the initial state $(q\cdot 1) \oplus (p \cdot \delta)$, $(q,p \in \mathbb{R})$, described in § 2, where the string is at rest at a height q and the oscillator has a momentum p). Φ becomes a real Hilbert space when endowed with the inner product

$$\langle \varphi_1 \oplus \pi_1, \varphi_2 \oplus \pi_2 \rangle_{H_\Phi} = \tfrac{1}{2}\Big(\varphi_1(0)\varphi_2(0) + \pi_1(0)\pi_2(0)\Big) + \tfrac{1}{2}\eta\int_0^\infty (\varphi_1'\varphi_2' + \pi_1\pi_2)\,ds. \tag{3.5}$$

Now, if we adjoin to $\varphi \oplus \pi \in \Phi$ the function $\sqrt{2\eta}\,b'$, where b is the input function, we have the convenient representation of $\{\Phi, F\}$ as translations on $L^2(\mathbb{R})$, as announced.

Definition 3.1. Let the functions p, q and r be defined as follows. If $t > 0$, $p(t) = q(t) = r(t) = 0$, and on $(-\infty, 0]$, p, q and r are solutions of the differential equation

$$f'' - \eta f' + f = 0, \tag{3.6}$$

determined by the initial conditions

$r(0) = \sqrt{2\eta}$ and $r'(0) = 0$,

$q(0) = 0$ and $q'(0) = -\sqrt{2\eta}$,

$p(0) = \sqrt{2\eta}$ and $p'(0) = -\eta\sqrt{2\eta}$.

We note that, on $(-\infty, 0]$, $p = -q'$ and $q = r'$.

Theorem 3.2. Let $v(\lambda) = \frac{1}{2}\lambda^2$. For $\varphi \oplus \pi \in \Phi_0$, the input function b, determined up to a constant, is given by $b(t) = \frac{1}{2}(\varphi(t) + \psi(t))$, $(t \geq 0)$, where $\psi' = \pi$ (ψ is determined up to a constant), and:

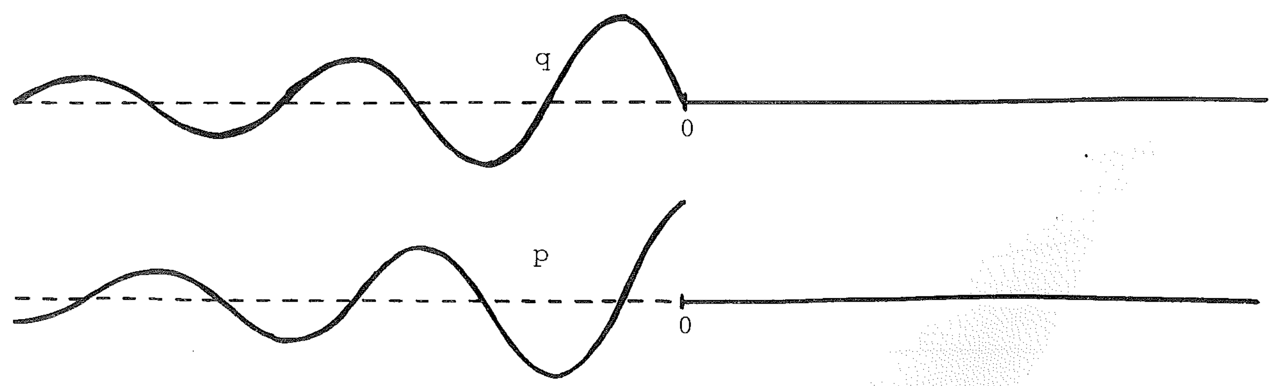

Fig. 4. The functions q and p.

$$b(t) = -a(t) - \sqrt{2\eta}\int_t^0 q(t-s)\,a'(s)\,ds + \frac{1}{\sqrt{2\eta}}\Big(\varphi(0)\,r(t) - \pi(0)\,q(t)\Big) \tag{3.7}$$

for $t < 0$, where $a(t) = \frac{1}{2}(\varphi(-t) - \psi(-t))$, $(t \leq 0)$. Moreover, the map i_Φ: $\varphi \oplus \pi \mapsto \sqrt{2\eta}\,b'$ is linear and extends by continuity to an isomorphism of the real Hilbert spaces Φ and $L^2(\mathbb{R})$, satisfying

$$i_\Phi \circ F_t = \tilde{T}_t \circ i_\Phi\,, \quad (t \in \mathbb{R}). \tag{3.8}$$

Remark. Define δ by: $\delta(s) = 0$ for $s > 0$ and $\delta(0) = 1$. Then $1 \oplus 0$ and $0 \oplus \delta$ are points of Φ. Indeed, they can be written as energy norm limits of points in Φ_0 as follows:

$$1 \oplus 0 = \lim_{n\to\infty} \varphi_n \oplus 0; \quad 0 \oplus \delta = \lim_{n\to\infty} 0 \oplus \pi_n,$$

where $\{\varphi_n\}$, $\{\pi_n\}$ are as drawn in Fig. 5. By (3.7) we have (note that $r' = q$):

$$i_\Phi(1 \oplus 0) = q \quad \text{and} \quad i_\Phi(0 \oplus \delta) = p. \tag{3.9}$$

This motivates the names of the functions q and p.

For the proof of the theorem we need two lemmas.

Lemma 3.3. Let $f:(-\infty,0] \to \mathbb{R}$ be continuous. The solution b of the differential equation

$$b'' - \eta b' + b = f \quad \text{on } (-\infty, 0]$$

with boundary conditions $b(0) = b'(0) = 0$, is given by

$$b(t) = \frac{1}{\sqrt{2\eta}}\int_t^0 q(t-s)\,f(s)\,ds. \tag{3.10}$$

Proof. If b is defined by (3.10), we have $b'(t) = (2\eta)^{-\frac{1}{2}}\int_t^0 (q'(t-s)\,f(s)\,ds$, because $q(0) = 0$, and $b''(t) = f(t) + (2\eta)^{-\frac{1}{2}}\int_t^0 q''(t-s)\,f(s)\,ds$, because

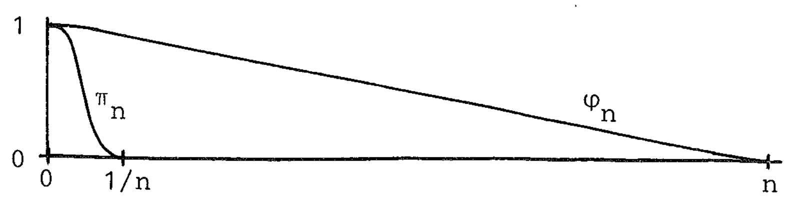

Fig. 5. $\{\varphi_n\}$ and $\{\pi_n\}$, approaching 1 and δ.

$q'(0) = -\sqrt{2\eta}$. It follows that

$$(b'' - \eta b' + b)(t) = f(t) + \frac{1}{\sqrt{2\eta}}\int_t^0 (q'' - \eta q' + q)(t-s)\,f(s)\,ds = f(t).$$

□

Lemma 3.4. The linear span of $\{\tilde{T}_t q \mid t \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$.

Proof. This is a consequence of the L^2-version of Wiener's theorem, saying that, if the Fourier transform $\hat{q}$ of q is nonzero almost everywhere (a.e.), $\{\tilde{T}_t q \mid t \in \mathbb{R}\}$ spans a dense subspace of $L^2(\mathbb{R})$. Indeed, because $q'' - \eta q' + q = \sqrt{2\eta}\,\delta$ in the distribution sense, we have for all $\omega \in \mathbb{R}$,

$$\hat{q}(\omega) = (-\omega^2 + i\eta\omega + 1)^{-1} \cdot \sqrt{2\eta} \neq 0. \tag{3.11}$$

Wiener's theorem is proved as follows. Suppose $f \in L^2$ is orthogonal to $\tilde{T}_t q$ for all $t \in \mathbb{R}$. Then for all t:

$$0 = \langle f, \tilde{T}_t q\rangle = \int_{-\infty}^{\infty} \hat{f}(\omega)^*\,\hat{q}(\omega)\,e^{i\omega t}\,\frac{d\omega}{2\pi} = 0,$$

i.e., the Fourier transform of $\hat{f}^*\hat{q}$ is zero. But then this function is zero itself, and because $\hat{q}$ is nonzero a.e., we have $\hat{f} = 0$ a.e., so $\hat{f} = 0$ as an L^2-function, hence $f = 0$ in L^2. The statement follows. □

Proof of theorem 3.2. From the proof of theorem 2.2 we see that, in order to show that we have the right function b, we must check

(i) b and b' are continuous in 0.

(ii) $(b'' - \eta b' + b)(t) = -(a'' + \eta a' + a)(t)$, $(t < 0)$.

Now, from (3.7) it follows that $\lim_{t\uparrow 0} b(t) = -a(0) + \varphi(0) = b(0)$ and $\lim_{t\uparrow 0} b'(t) = -a'(0) + \pi(0) = b'(0)$. To prove (ii), we let $(\partial^2 - \eta\partial + 1)$ act on both sides of (3.7), putting $t < 0$. The r- and q-terms are annihilated and, by lemma 3.3,

$$(b'' - \eta b' + b)(t) = -(a'' - \eta a' + a)(t) - 2\eta a'(t) = -(a'' + \eta a' + a)(t).$$

To prove the second part of the theorem, consider i_Φ: $\varphi \oplus \pi \mapsto \sqrt{2\eta}\,b'$. Clearly, it is linear. By (3.2) we have

$$\| i_\Phi(\varphi \oplus \pi)\|^2 = 2\eta\| b'\|^2 = 2H_\Phi(\varphi \oplus \pi) = \|\varphi \oplus \pi\|^2_{H_\Phi}.$$

So i_Φ is isometric. From (3.1) if follows that $i_\Phi \circ F_t = \tilde{T}_t \cdot i_\Phi$, on Φ_0. By continuity, i_Φ extends to an isometry from Φ to $L^2(\mathbb{R})$, $i_\Phi\Phi$ being a closed and translation invariant subspace of $L^2(\mathbb{R})$. Now, because $1 \oplus 0 \in \Phi$, we have

$q \in i_\Phi \Phi$, and by Lemma 3.4 it follows that $i_\Phi \Phi = L^2(\mathbb{R})$. □

The very convenient form $L^2(\mathbb{R})$, that our system has taken via i_Φ makes it attractive, to think of i_Φ as an identification, and to talk of functions $x \in L^2(\mathbb{R})$ as phase space points, evolving like $t \mapsto \tilde{T}_t x$. But then we must have a way of knowing what the position and momentum of the oscillator are, if the system is in a state $x \in L^2(\mathbb{R})$.

Definition 3.5. Let $Q_t: L^2(\mathbb{R}) \to \mathbb{R}$ and $P_t: L^2(\mathbb{R}) \to \mathbb{R}$ be given by

$$Q_t(i_\Phi(\varphi \oplus \pi)) = \varphi_t(0) \text{ and } P_t(i_\Phi(\varphi \oplus \pi)) = \pi_t(0),$$

where $\varphi_t \oplus \pi_t = F_t(\varphi \oplus \pi)$. Let $T_t = \tilde{T}_{-t}$; T_t denotes translation to the right.

Lemma 3.6. For all $x \in L^2(\mathbb{R})$ we have

$$Q_t(x) = \langle x, T_t q\rangle \text{ and } P_t(x) = \langle x, T_t p\rangle. \tag{3.12}$$

Proof. From (3.5) if follows that for $\varphi \oplus \pi \in \Phi_0$:

$$\varphi(0) = \langle 1 \oplus 0, \varphi \oplus \pi\rangle_{H_\Phi} \text{ and } \pi(0) = \langle 0 \oplus \delta, \varphi \oplus \pi\rangle_{H_\Phi}.$$

Therefore, if $\varphi_t \oplus \pi_t = F_t(\varphi \oplus \pi)$, and $x = i_\Phi(\varphi \oplus \pi)$,

$$Q_t(x) = \varphi_t(0) = \langle 1 \oplus 0, \varphi_t \oplus \pi_t\rangle_{H_\Phi} = \langle 1 \oplus 0, F_t(\varphi \oplus \pi)\rangle_{H_\Phi} =$$
$$= \langle q, \tilde{T}_t x\rangle = \langle x, T_t q\rangle.$$

Analogously, $P_t(x) = \langle x, T_t p\rangle$. □

Dilations according to Sz. - Nagy

We shall now proceed to show that the linear string model, as we have constructed it, is the dilation of the semigroup $\{S_t\}_{t \geq 0}$ of linear transformations on $\mathbb{R}^2$ defined in § 1, if we choose the quadratic potential (3.3). $\{S_t\}$ is given by

$$S_t = \exp\left(t\begin{pmatrix} 0 & 1 \\ -1 & -\eta \end{pmatrix}\right), \quad (t \geq 0). \tag{3.13}$$

Definition 3.7. Let $\{S_t\}_{t \geq 0}$ be a semigroup of contractions on a real (complex) Hilbert space M. A triple $\{H, j, \{U_t\}_{t \in \mathbb{R}}\}$ is called a *dilation of* $\{S_t\}$ *(in the sense of Sz. - Nagy)*, if H is a real (complex) Hilbert space, $\{U_t\}_{t \in \mathbb{R}}$ a group of orthogonal (unitary) transformations $H \to H$, and j an isometry $M \to H$, such that

$$j^* U_t j = S_t, \quad (t \geq 0). \tag{3.14}$$

$\{H, j, \{U_t\}\}$ is called *minimal* if the span of $\{U_t m \mid t \in \mathbb{R}, m \in M\}$ is dense in H.

Lemma 3.8. Let $j: \mathbb{R}^2 \to L^2(\mathbb{R})$ be given by

$$j(x_1, x_2) = x_1 q + x_2 p. \tag{3.15}$$

Then $\{L^2(\mathbb{R}), j, \{\tilde{T}_t\}\}$ is a minimal Sz. - Nagy dilation of $\{S_t\}$.

Proof. Let $j': \mathbb{R}^2 \to \Phi: \{x_1, x_2\} \mapsto (x_1 1) \oplus (x_2 \delta)$. Then $j = i_\Phi \circ j'$, j' is isometric, and

$$(j')^*(\varphi \oplus \pi) = \{\varphi(0), \pi(0)\}.$$

By construction, we have, if $\varphi_t \oplus \dot{\varphi}_t = F_t(j' x)$:

$$(j')^*(F_t(j' x)) = \{\varphi_t(0), \dot{\varphi}_t(0)\} = S_t x.$$

So $(j')^* \circ F_t \circ j' = S_t$, and it follows that

$$j^* \circ \tilde{T}_t \circ j = S_t$$

by theorem 3.2. The dilation is minimal by Lemma (3.4). □

Theorem 3.9 (Sz. - Nagy). Let $\{S_t\}_{t \geq 0}$ be a strongly continuous semigroup of contractions on a real (complex) Hilbert space M. Up to isomorphism, $\{S_t\}$ has a unique minimal dilation.

Proof. For the general proof of the existence of such a dilation, we refer to the literature ([SzN 53], [EvL 77]). In the special case (3.13) we have already given one (Lemma 3.8). We shall now prove uniqueness.

Suppose both $\{\tilde{H}, \tilde{j}, \{\tilde{U}_t\}\}$ and $\{H, j, \{U_t\}\}$ are minimal dilations of the semigroup $\{S_t\}$ of contractions on M. For $n \geq 0$; $t_i, a_i \in \mathbb{R}$ $(\mathbb{C})$, and $m_i \in M$, $(i = 1,\dots,n)$, define

$$V_0\left(\sum_{i=1}^n a_i U_{t_i} j m_i\right) = \sum_{i=1}^n a_i \tilde{U}_{t_i} \tilde{j} m_i. \tag{3.16}$$

This definition makes sense because, if $\sum_{i=1}^n a_i U_{t_i} j m_i = 0$, then also

$$\left\| \sum_{i=1}^n a_i \tilde{U}_{t_i} \tilde{j} m_i \right\|^2 = \sum_{i,k=1}^n a_i^* a_k \langle \tilde{U}_{t_i} \tilde{j} m_i, \tilde{U}_{t_k} \tilde{j} m_k\rangle =$$

$$= \sum_{i,k=1}^{n} a_i^* a_k \langle m_i, \tilde{j}^* \tilde{U}_{t_k - t_i} \tilde{j} m_k \rangle = \sum_{i,k=1}^{n} a_i^* a_k \langle m_i, j^* U_{t_k - t_i} j m_k \rangle =$$

$$= \| \sum_{i=1}^{n} a_i U_{t_i} j m \|^2 = 0.$$

The same computation shows that V_0 is isometric. As both dilations are minimal, V_0 has dense domain and range, and hence extends to an isomorphism $V: H \to H'$. By (3.16), and continuity, we have

$$V \circ U_t = \tilde{U}_t \circ V \text{ and } V \circ j = \tilde{j}. \qquad \square$$

§ 4. THE POISSON BRACKET

In Chapter II we shall give a quantum-mechanical treatment of the Lamb model, in terms of the abstract description that we have introduced in the previous section. This means that we shall have to "quantise" the Hamiltonian system whose phase space is $L^2(\mathbb{R})$, whose time evolution is translation to the left, and whose Hamiltonian is the function $H(x) = \frac{1}{2}\|x\|^2$.

Now, quantisation is not a well-defined and straightforward procedure yielding a quantum system for every Hamiltonian system. Neither is there a clear-cut definition of a Hamiltonian system, of a sufficient generality to encompass the present infinite-dimensional case. (Not even in [ChM 74]). Therefore some care has to be taken. It is for this reason that we include this section, a rather extensive discussion of the Hamiltonian formalism for this infinite dimensional special case. We use the notation of books like [Thi 77] and [Arn 78], adapting it to infinite dimension. The latter case was studied in [ChM 74].

We start by defining a symplectic form on a large part Ψ of $L^2(\mathbb{R})$. A justification of this choice will be given after Lemma 4.4.

Definition 4.1. Let $W_1(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ of those functions y, for which also $y' \in L^2(\mathbb{R})$. Let Ψ be the image of $W_1(\mathbb{R})$ under the map $y \mapsto y'$. Note that this map is injective. (Indeed, if $y_1' = y_2'$, then $y_1 - y_2$ is a constant, and if y_1 and y_2 are both in $L^2(\mathbb{R})$, this constant must be zero). So we may define $I: \Psi \to W_1(\mathbb{R})$ as the inverse map $y' \mapsto y$. Now let $\tilde{\sigma}: \Psi \times \Psi \to \mathbb{R}$ be given by

$$\tilde{\sigma}(x,y) = \langle I x, y \rangle. \qquad (4.1)$$

$\tilde{\sigma}$ corresponds to σ_Φ via the identification i_Φ (see § 3). Indeed, the linear flow preserves σ_Φ, and if $\varphi_1 \oplus \pi_1$ and $\varphi_2 \oplus \pi_2$ are points of Φ_0, and b_1 and b_2 are their input functions, one checks in a way, resembling the proof of Corollary 2.6 that

$$\sigma_\Phi(\varphi_1 \oplus \pi_1, \varphi_2 \oplus \pi_2) = 2\eta \int_{-\infty}^{\infty} b_1 b_2' \, dt. \qquad (4.2)$$

Now, $i_\Phi(\varphi_j \oplus \pi_j) = \sqrt{2\eta}\, b_j'$, and therefore

$$\sigma_\Phi(\varphi_1 \oplus \pi_1, \varphi_2 \oplus \pi_2) = \tilde{\sigma}(i_\Phi(\varphi_1 \oplus \pi_1), i_\Phi(\varphi_2 \oplus \pi_2)).$$

We note, however, that this fact does not justify our choice of $\tilde{\sigma}$, because σ_Φ has come falling out of the blue as well.

Our phase space is now Ψ. By an *observable* we shall mean a function $\Psi \to \mathbb{C}$.

Definition 4.2. Let $F: \Psi \to \mathbb{C}$. By the *gradient* $(DF)_x$ *of* F *at* $x \in \Psi$ we mean the continuous linear functional on Ψ, if it exists, that satisfies

$$\forall_{u \in \Psi}: (DF)_x(u) = \frac{d}{d\lambda} F(x + \lambda u)\Big|_{\lambda=0}. \qquad (4.3)$$

Let $\Psi_{\mathbb{C}}$ be the complexification of Ψ, and extend $\tilde{\sigma}$ to a bilinear form on $\Psi_{\mathbb{C}}$. We call an observable F *$\tilde{\sigma}$-smooth in* x, if $(DF)_x$ exists and there is $y \in \Psi_{\mathbb{C}}$ such that for all $u \in \Psi$

$$\tilde{\sigma}(y,u) = (DF)_x(u). \qquad (4.4)$$

The set of points x in which F is $\tilde{\sigma}$-smooth we call the *domain of* X_F ($\mathrm{Dom}(X_F)$). For all $x \in \mathrm{Dom}(X_F)$, y is uniquely determined by (4.4). Define

$$X_F(x) = y. \qquad (4.5)$$

We call X_F *the vector field associated to* F. If F and G are observables, we define the *Poisson bracket* $\{F,G\}$ of F and G by

$$\{F,G\}: \mathrm{Dom}(X_F) \cap \mathrm{Dom}(X_G) \to \mathbb{C}: \{F,G\}(x) = \tilde{\sigma}(X_F(x), X_G(x)). \qquad (4.6)$$

The following lemma provides us with a large, translation invariant, set of everywhere $\tilde{\sigma}$-smooth observables, on which the Poisson bracket can act freely, yielding observables in the same set. We introduce the following notation.

Let $\mathcal{S}$ be Schwartz's class of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ of rapid decrease. Let $\mathbb{A}_0$ be the space of functions $F\colon \Psi \to \mathbb{C}$ of the form

$$F(x) = \sum_{j=1}^{n} c_j e^{i\langle f_j, x\rangle}, \ (n \in \mathbb{N};\ c_j \in \mathbb{C},\ f_j \in \mathcal{S},\ (j = 1,\dots,n)). \tag{4.7}$$

Let τ_t, $(t \in \mathbb{R})$ act on an observable F as follows

$$\tau_t(F)(x) = F(\tilde{T}_t x) . \tag{4.8}$$

Let $\sigma\colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be given by

$$\sigma(f,g) = \int_{-\infty}^{\infty} f g' \, dt . \tag{4.9}$$

Let $\{T_t\colon \mathcal{S} \to \mathcal{S}\}_{t\in\mathbb{R}}$ denote translation to the right:

$$(T_t f)(s) = f(s-t). \tag{4.10}$$

Lemma 4.3.

(a) Let $f,g \in \mathcal{S}$ and consider the observables $\langle f, \cdot\rangle$ and $\langle g, \cdot\rangle$. We have

$$\{\langle f,\cdot\rangle, \langle g,\cdot\rangle\} = \sigma(f,g)\cdot 1, \tag{4.11}$$

where 1 is the constant function 1 on Ψ.

(b) If $f \in \mathcal{S}$, then

$$\tau_t(\langle f,\cdot\rangle) = \langle T_t f, \cdot\rangle . \tag{4.12}$$

(c) $\mathbb{A}_0$ is a Lie algebra under the Poisson bracket operation, and $\tau_t(\mathbb{A}_0) \subset \mathbb{A}_0$, $\forall_t$.

(d) $\mathrm{Dom}(X_H) = \{x \in \Psi \mid x' \in L^2(\mathbb{R})\}$, and for all $x \in \mathrm{Dom}(X_H)$:

$$\frac{d}{dt} \tilde{T}_t x \Big|_{t=0} = x' = X_H(x). \tag{4.13}$$

Moreover, for all $F \in \mathbb{A}_0$:

$$\frac{d}{dt} \tau_t(F) \Big|_{t=0} = \{F,H\} \text{ on } \mathrm{Dom}(X_H). \tag{4.14}$$

Proof. (a) Let $f \in \mathcal{S}$. Then for all $x \in \Psi$:

$$X_{\langle f,\cdot\rangle}(x) = f'. \tag{4.15}$$

Indeed, because $f \in \mathcal{S} \subset W_1(\mathbb{R})$, we have $f' \in \Psi$, and also $I(f') = f$. Therefore for all $u \in \Psi$, all $x \in \Psi$:

$$\tilde{\sigma}(f',u) = \langle I(f'),u\rangle = \langle f,u\rangle = \frac{d}{d\lambda} \langle f, x+\lambda u\rangle \Big|_{\lambda=0} = (D(\langle f,\cdot\rangle))_x(u) . \tag{4.16}$$

This implies (4.15). It follows that, if $f,g \in \mathcal{S}$ we have for all $x \in \Psi$

$$\{\langle f,\cdot\rangle, \langle g,\cdot\rangle\}(x) = \tilde{\sigma}(X_{\langle f,\cdot\rangle}(x), X_{\langle g,\cdot\rangle}(x)) = \tilde{\sigma}(f',g') = \langle I(f'),g'\rangle =$$

$$= \int f g' \, dt = \sigma(f,g) .$$

(b) $\tau_t(\langle f,\cdot\rangle)(x) = \langle f, \tilde{T}_t x\rangle = \langle T_t f, x\rangle$.

(c) Let $f \in \mathcal{S}$ and $F(x) = \exp(i\langle f, x\rangle)$. Then

$$(DF)_x(u) = \frac{d}{d\lambda} e^{i\langle f, x+\lambda u\rangle} \Big|_{\lambda=0} = i\langle f,u\rangle e^{i\langle f,x\rangle} = \tilde{\sigma}(i e^{i\langle f,x\rangle} f', u) \tag{4.17}$$

by (4.16). We conclude that F is everywhere $\tilde{\sigma}$-smooth and

$$X_F(x) = i e^{i\langle f,x\rangle} f' . \tag{4.18}$$

Now, if also $g \in \mathcal{S}$ and $G(x) = e^{i\langle g,x\rangle}$, we have

$$\{F,G\}(x) = \tilde{\sigma}(X_F(x), X_G(x)) = -e^{i\langle f+g,x\rangle} \tilde{\sigma}(f',g') = -e^{i\langle f+g,x\rangle} \sigma(f,g). \tag{4.19}$$

From the bilinearity of the Poisson bracket we conclude that $\{F,G\}$ is defined and in $\mathbb{A}_0$ for all $F,G \in \mathbb{A}_0$. We have to check the validity of the Jacobi identity

$$\{\{F,G\}\ K\} + \{\{G,K\}\ F\} + \{\{K,F\}\ G\} = 0, \ (F,\ G,\ K \in \mathbb{A}_0). \tag{4.20}$$

It suffices to prove (4.20) for F,G as before and $K(x) = e^{i\langle k,x\rangle}$ with $k \in \mathcal{S}$. The left-hand side of (4.20) can be worked out using (4.19); it equals

$$-e^{i\langle f+g+k,\cdot\rangle}\Big(\sigma(f,g)\,\sigma(f+g,k) + \sigma(g,k)\,\sigma(g+k,f) + \sigma(k,f)\,\sigma(k+f,g)\Big).$$

One checks that, by the antisymmetry of σ, this is indeed equal to zero. Clearly, $\mathbb{A}_0$ is τ-invariant.

(d) First note that, for all $x,u \in \Psi$:

$$(DH)_x(u) = \frac{d}{d\lambda} H(x+\lambda u) \Big|_{\lambda=0} = \tfrac{1}{2} \frac{d}{d\lambda} \| x+\lambda u\|^2 \Big|_{\lambda=0} = \langle x,u\rangle .$$

Now, $u \mapsto \langle x,u\rangle$ is of the form $u \mapsto \tilde{\sigma}(y,u) = \langle I y, u\rangle$ if and only if $x = I y$ for some

$y \in \Psi$, which is the case if and only if $x' \in L^2(\mathbb{R})$. If this is so, then $X_H(x) = y = x'$. Of course, $x' = \frac{d}{dt}\tilde{T}_t x\Big|_{t=0}$. This proves the first statement.

To prove the second, put $F(x) = \exp(i\langle f,x\rangle)$, (this is enough), and suppose that $x \in \mathrm{Dom}(X_H)$. Then

$$\frac{d}{dt}\tau_t(F)(x)\Big|_{t=0} = \frac{d}{dt}F(\tilde{T}_t x)\Big|_{t=0} = \frac{d}{dt}\exp(i\langle f,\tilde{T}_t x\rangle)\Big|_{t=0} =$$

$$= i e^{i\langle f,x\rangle}\left(\frac{d}{dt}\langle f,\tilde{T}_t x\rangle\Big|_{t=0}\right) = i e^{i\langle f,x\rangle}\langle f,x'\rangle =$$

$$= i e^{i\langle f,x\rangle}\tilde{\sigma}(f',x') = \tilde{\sigma}(i e^{i\langle f,x\rangle}f',x') = \tilde{\sigma}(X_F(x), X_H(x)) =$$

$$= \{F,H\}(x). \qquad \square$$

We are now in a position to formulate a justification for our choice of the symplectic form $\tilde{\sigma}$ on $\Psi \subset L^2(\mathbb{R})$. Our system has to be such that the vector field associated to the Hamiltonian generates the time evolution:

$$\frac{d}{dt}\tilde{T}_t x\Big|_{t=0} = X_H(x).$$

The vector field X_H is related to $\tilde{\sigma}$ and H as follows:

$$\tilde{\sigma}(X_H(x), \cdot) = (DH)_x.$$

Combining, we get:

$$\forall_u:\ \tilde{\sigma}\left(\frac{d}{dt}\tilde{T}_t x\Big|_{t=0}, u\right) = (DH)_x(u). \tag{4.21}$$

This is the geometric form of the canonical equations of motion of a Hamiltonian system (compare Lemma 2.4). Now, $\frac{d}{dt}\tilde{T}_t x\Big|_{t=0} = x'$ and $(DH)_x(u) = \langle x,u\rangle$; therefore

$$\forall_x\forall_u:\ \tilde{\sigma}(x',u) = \langle x,u\rangle,$$

and it follows that (put $y = x'$):

$$\forall_y\forall_u:\ \tilde{\sigma}(y,u) = \langle I y,u\rangle.$$

We shall conclude this section by showing the connection between (4.21) and the traditional form of the canonical equations of motion,

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i}, \qquad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad (i = 1,\dots,n). \tag{4.22}$$

Let σ_n be the symplectic form on $\mathbb{R}^{2n}$, given by

$$\sigma_n\left(\begin{pmatrix}q\\p\end{pmatrix}, \begin{pmatrix}q'\\p'\end{pmatrix}\right) = \left\langle\begin{pmatrix}0 & -\mathbb{1}_n\\ \mathbb{1}_n & 0\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}, \begin{pmatrix}q'\\p'\end{pmatrix}\right\rangle,$$

where $q, q', p, p' \in \mathbb{R}^n$, and $\mathbb{1}_n$ is the $n\times n$ identity matrix. Then the differential equation for a curve $(q(t), p(t))_{t\in\mathbb{R}}$ in $\mathbb{R}^{2n}$, given by

$$\forall_{u\in\mathbb{R}^{2n}}:\ \sigma_n\left(\frac{d}{dt}\begin{pmatrix}q(t)\\p(t)\end{pmatrix}, u\right) = (DH)_{(q(t),\,p(t))}(u),$$

is equivalent with

$$\frac{d}{dt}\begin{pmatrix}0 & -\mathbb{1}_n\\ \mathbb{1}_n & 0\end{pmatrix}\cdot\begin{pmatrix}q(t)\\p(t)\end{pmatrix} = \begin{pmatrix}\partial H/\partial q\\ \partial H/\partial p\end{pmatrix}(q(t), p(t)).$$

This is a vector form of the canonical equations (4.22).

§ 5. A GIBBS MEASURE ON PHASE SPACE

In this section we shall consider thermal equilibrium states of the Lamb model. We shall show that, in such a state, the input functions are graphs of Brownian motion, and, that the motion of the oscillator is governed by a Langevin equation.

Thermal equilibrium

We choose some positive temperature T, and we absorb the constant T into the unit of string deflection. This unit will be $(kT/a)^{\frac{1}{2}}$, where k is Boltzmann's constant and a the spring constant of the oscillator. (See the appendix for the use of units, constants and parameters.)

A mathematical description of Gibbs' canonical ensemble at temperature T, is provided by a probability measure on the space of all possible configurations of the model, its phase space. We shall call it the *Gibbs measure*, and to find it we shall use a heuristic argument. After finishing the construction, we shall verify that the state of the system, thus obtained, satisfies the classical KMS condition, introduced by Gallavotti and Verboven [GaV 75]. The analogous construction, for the case of a lattice of harmonic oscillators, has been treated in detail by Van Hemmen [Hem 76].

First we shall argue that $L^2(\mathbb{R})$ is too small a phase space to carry the Gibbs measure. Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal set in $L^2(\mathbb{R})$. We introduce coordinates on $L^2(\mathbb{R})$ by associating to the point x the sequence $\{\xi_j\}_{j=1}^{\infty}$, defined by

$$x = \sum_{j=1}^{\infty} \xi_j e_j .$$

In these coordinates, the Hamiltonian takes the form

$$H(x) = \sum_{j=1}^{\infty} \tfrac{1}{2} \xi_j^2 .$$

Neglecting the fact that these coordinates are not canonical w.r.t. $\tilde{\sigma}$, we try to write down a Gibbs measure in the fashion customary for finite Hamiltonian systems:

$$\mu(dx) := \mu'(d\{\xi_j\}) := Z^{-1} \cdot \exp -\Big(\tfrac{1}{2} \sum_{j=1}^{\infty} \xi_j^2\Big) d\xi_1\, d\xi_2\, d\xi_3 \dots .$$

This does not define a measure on the set of all sequences $\{\xi_j\}_{j=1}^{\infty}$, because we cannot make sense of the expression "$d\xi_1\ d\xi_2\, d\xi_3 \dots$". However, the following definition does:

$$\mu'(d\{\xi_j\}) := \prod_{j=1}^{\infty} \Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi_j^2} d\xi_j\Big) .$$

Indeed, a product of countably many copies of a probability measure on $\mathbb{R}$ defines a probability measure on $\mathbb{R}^{\mathbb{N}}$. We have thus obtained a probability space $\{\mathbb{R}^{\mathbb{N}}, \mu'\}$, and we shall transfer μ' back onto phase space to get the Gibbs measure μ.

Let $\mathbb{E}_{\mu}$ $(\mathbb{E}_{\mu'})$ denote expectation w.r.t. μ (μ'). We would like the following to hold:

$$\mathbb{E}_{\mu}\Big(e^{i\langle \lambda e_j, \cdot\rangle}\Big) = \mathbb{E}_{\mu'}\Big(e^{i\lambda\xi_j}\Big) = \int_{-\infty}^{\infty} e^{i\lambda\xi_j} \Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi_j^2} d\xi_j\Big) =$$

$$= e^{-\frac{1}{2}\lambda^2}.$$

This should hold, not only for the vectors λe_j, $(j \in \mathbb{N}, \lambda \in \mathbb{R})$, - they were arbitrary in any case -, but for all $f \in L^2(\mathbb{R})$. I.e., for $f = \|f\| \cdot f_1$,

$$\mathbb{E}\,(e^{i\langle f, \cdot\rangle}) = e^{-\frac{1}{2}\|f\|^2} . \tag{5.1a}$$

Relation (5.1a) is a requirement that our Gibbs probability measure should satisfy. In words:

<u>Rule</u>: *The linear observable* $\langle f, \cdot\rangle$ *should become a Gaussian random variable with mean zero and variance* $\|f\|^2$.

<u>Infinite energy configurations</u>

On $\{\mathbb{R}^{\mathbb{N}}, \mu'\}$, the ξ_j, $(j = 1,2,3\dots)$, are independent Gaussian random variables of variance 1 and mean 0. The law of large numbers says that, with probability 1,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \xi_j^2 = 1.$$

The probability that the sum $\sum_{j=1}^{\infty} \xi_j^2$ converges, is therefore zero:

$$\mu'(\ell^2) = \mu'\Big(\Big\{\{\xi_j\}_{j=1}^{\infty} \,\Big|\, \sum_{j=1}^{\infty} \xi_j^2 < \infty\Big\}\Big) = 0.$$

This means for our Gibbs measure μ that

$$\mu(L^2(\mathbb{R})) = 0. \tag{5.2}$$

So our entire phase space has Gibbs measure zero.

This result, surprising as it may be at first sight, can be readily understood in physical terms as follows.

All configurations of the Lamb model that are given by L^2-functions b', have finite energy. Now, if one heats up the system to a positive temperature, thermal fluctuations will occur in a homogeneous way over the full, infinite length of the string, and certainly their total energy will *not* be finite. Therefore the vast majority of ensemble elements (almost all configurations w.r.t. μ), will lie outside of $L^2(\mathbb{R})$.

As is well known in the classical theory of fields, already a finite stretch of a string (which is a field in one dimension), in thermal equilibrium carries an infinite amount of energy. Indeed, a finite piece of string has an infinite number of degrees of freedom, and, according to the principle of equipartition, each of them gets an amount $\frac{1}{2}$ kT of energy.

Notice that the above argument is a re-phrasing of the probabilistic argument adduced before, now applied to a finite interval. To drive this point somewhat further, let $\{f_j\}_{j=1}^{\infty}$ be an (incomplete) orthonormal system in $L^2(\mathbb{R})$ of functions, vanishing outside the interval $[s_1,s_2]$. The $\langle f_j,\cdot\rangle$ are Gaussian with variance 1 (and energy $\frac{1}{2}$ kT). As j increases, the Fourier transforms $\hat{f}_j$ will become broader and broader. (Indeed, for all $a \geq 0$, $\int_{-a}^{a}|\hat{f}_j(\omega)|^2 d\omega \to 0$, $(j\to\infty)$, because the operator $A_a := \chi_{[-a,a]}\cdot \mathcal{F}\cdot\chi_{[s_1,s_2]}$ is Hilbert-Schmidt and $\|A_a f_j\|^2 = \int_{-a}^{a}|\hat{f}_j(\omega)|^2 d\omega/2\pi$.) This means that the random variables $\langle f_j,\cdot\rangle$ get their variance (energy) from higher and higher frequencies, as j increases. The fact that there is no cut-off mechanism at high frequencies to prevent the energy $\Sigma\,\frac{1}{2}\,kT\,\mathbb{E}(\langle f_j,\cdot\rangle^2)$ of the stretch of string between s_1 and s_2 from diverging, is called an "ultraviolet catastrophe". The one we have here is of an ancient type, the same as that of the infinite energy density of the electromagnetic field in its classical description, that has prompted Planck's hypothesis of a "quantum of radiation". The effect of this hypothesis was that the high frequency degrees of freedom got "frozen". And, as a matter of fact, in the quantummechanical Lamb model which we shall consider later, a finite stretch of string has a finite energy.

A measure on $\mathcal{S}'$

Having seen that $L^2(\mathbb{R})$ is too small a space to carry the Gibbs measure, we proceed to construct the latter on a larger space, the space $\mathcal{S}'$ of all tempered distributions. It will not bother us if - as is in fact the case - this space is much larger than necessary: the measure will be concentrated (i.e., will put weight 1) on the relevant part. So let $\mathcal{S}'$ be our phase space.

Functions in $\mathcal{A}_0$, i.e. functions of the form (cf. § 4),

$$F(x) = \sum_{j=1}^{n} c_j\, e^{i\langle x,f_j\rangle}, \quad (c_j \in \mathbb{C},\ f_j \in \mathcal{S}),$$

can be extended in an obvious manner to functions on $\mathcal{S}'$.

Definition 5.1. By a *cylinder set* $A \subset \mathcal{S}'$ we mean a set of the form

$$A = \{x \in \mathcal{S}' \mid (\langle x,f_1\rangle, \ldots, \langle x,f_n\rangle) \in B\},$$

where $n \geq 0$, B is some Borel subset of $\mathbb{R}^n$, and $f_1,\ldots,f_n \in \mathcal{S}$. Let Σ be the

σ-field, generated by the cylinder sets. A *cylinder measure* on $\mathcal{S}'$ is a measure on the measure space $\{\mathcal{S}',\Sigma\}$.

Because $\mathcal{S}'$ is a co-nuclear space, any cylinder measure extends to a unique Borel measure. But, when speaking of measures on $\mathcal{S}'$, we shall always mean cylinder measures. A *probability measure* is a positive measure of total weight 1.

Definition 5.2. If μ is a probability measure on $\mathcal{S}'$, *the Fourier transform* $\hat{\mu}$ of μ is the function $\mathcal{S} \to \mathbb{C}$, defined by

$$\hat{\mu}(f) := \int_{x\in\mathcal{S}'} e^{i\langle x,f\rangle}\,\mu(dx).$$

Remark. Let μ be a probability measure on $\mathcal{S}'$. Its Fourier transform $\hat{\mu}$ has the properties

(i) $\hat{\mu}$ is continuous $\mathcal{S} \to \mathbb{C}$,

(ii) $\hat{\mu}(0) = 1$,

(iii) $\hat{\mu}$ is positive definite, i.e. for all $n \geq 0$; $c_j \in \mathbb{C}$, $f_j \in \mathcal{S}$ $(j = 1,\ldots,n)$:

$$\sum_{j,k=1}^{n} c_j^* c_k\, \hat{\mu}(f_j - f_k) \geq 0.$$

Theorem 5.3. (Minlos). Every function $\mathcal{S} \to \mathbb{C}$ having the properties (i), (ii), and (iii) is the Fourier transform of a unique probability measure on $\mathcal{S}'$. For the proof we refer to the literature (e.g. [Min 59], [Hid 80], [Hem 76]).

Lemma 5.4. The function $C\colon \mathcal{S} \to \mathbb{C}\colon f \mapsto \exp(-\frac{1}{2}\|f\|^2)$ has the properties (i), (ii) and (iii).

Proof. Let $H_F = \bigoplus_{n=0}^{\infty} L^2_{symm}(\mathbb{R}^n)$, the symmetric Fock space over the complex Hilbert space $L^2(\mathbb{R})$, and define the map $\mathrm{Exp}_F\colon \mathcal{S} \to H_F$ by

$$\mathrm{Exp}_F(f) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

Then for all f and g in $\mathcal{S}$ we have

$$\langle \mathrm{Exp}_F f, \mathrm{Exp}_F g\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\langle f,g\rangle^n = \exp(\langle f,g\rangle).$$

It follows that for all $m \in \mathbb{N}$; $c_j \in \mathbb{C}$, $f_j \in \mathcal{S}$, $(j = 1,\ldots,n)$:

$$\sum_{j,k=1}^{m} c_j^* c_k C(f_j - f_k) = \sum_{j,k=1}^{m} \left(c_j^* e^{-\frac{1}{2}\|f_k\|^2}\right)\cdot\left(c_k e^{-\frac{1}{2}\|f_k\|^2}\right)\cdot e^{\langle f_j, f_k\rangle} =$$

$$= \left\| \sum_{k=1}^{m} c_k e^{-\frac{1}{2}\|f_k\|^2} \operatorname{Exp}_F(f_k)\right\|^2 \geqslant 0.$$

So C is positive definite. Clearly, C also has the properties (i) and (ii). □

Combining Theorem 5.3 with Lemma 5.4 we obtain our Gibbs measure. Let μ_B be the measure on $\mathcal{S}'$, given by

$$\int_{x\in\mathcal{S}'} e^{i\langle x,f\rangle} \mu_B(dx) = e^{-\frac{1}{2}\|f\|^2}. \tag{5.1b}$$

Let H_B be the Hilbert space $L^2(\mathcal{S}', \mu_B)$. (The letter B refers to Brownian motion (cf. § 6)).

Lemma 5.5. For all $f,g \in \mathcal{S}$, $\langle f,\cdot\rangle$ and $\langle g,\cdot\rangle$ are in H_B, and

$$\int_{x\in\mathcal{S}'} \langle f,x\rangle\langle g,x\rangle \mu_B(dx) = \langle f,g\rangle. \tag{5.3}$$

Proof. We have for all $f \in \mathcal{S}$

$$\int_{x\in\mathcal{S}'} \langle f,x\rangle^2 \mu(dx) = -\frac{d^2}{d\lambda^2}\int_{x\in\mathcal{S}'} e^{i\lambda\langle x,f\rangle}\mu(dx)\Big|_{\lambda=0} = -\frac{d^2}{d\lambda^2} e^{-\frac{1}{2}\lambda^2\|f\|^2}\Big|_{\lambda=0} = \|f\|^2.$$

(5.3) follows by polarisation. □

The map $E_0 : f \mapsto \langle f,\cdot\rangle$ is a Gaussian-random-variable-valued distribution with covariance

$$\mathbb{E}(E_0(f)\,E_0(g)) = \langle f,g\rangle. \tag{5.4}$$

Such a distribution is called a generalised Gaussian stochastic process, and if it has the covariance (5.4) it is called *white noise*.

From (5.4) it follows that E_0 extends to an isometry $L^2(\mathbb{R}) \to H_B$, which we shall also call E_0.

Often (5.4) is formally written as

$$\mathbb{E}(E_t E_s) = \delta(t-s).$$

One regains (5.4) from this if one puts

$$E_0(f) = \int f(t)\, E_t\, dt.$$

Sometimes we shall consider $E_0(f)$ as an (unbounded) self-adjoint operator on H. The $\exp(i\,E_0(f))$ span A_0. We shall consider A_0 as a (commutative) *-algebra of multiplication operators on H_B. Let $\mathfrak{M}_0 = A_0''$, the strong closure of A_0 in $\mathcal{L}(H_B)$.

Lemma 5.6. $\mathfrak{M}_0 = L^\infty(\mathcal{S}',\mu_B)$ and $A_0 1$ is dense in H_B.

Proof. Let $\Sigma^{(0)}_{\mu_B} = \{A \in \Sigma \mid \mu_B(A) = 0\}$, and let $\Sigma_{\mu_B} = \Sigma / \Sigma^{(0)}_{\mu_B}$, be the measure algebra, i.e. the algebra of μ_B-equivalence classes of measurable subsets of $\mathcal{S}'$. Then the characteristic functions χ_A with $A \in \Sigma_{\mu_B}$ are the projectors in $L^\infty(\mathcal{S}',\mu_B)$. Because $\mathfrak{M}_0$ is a strongly closed (i.e. von Neumann-) subalgebra of $L^\infty(\mathcal{S}',\mu_B)$, the projectors in $\mathfrak{M}_0$ are the characteristic functions of a sub-measure-algebra Σ'_{μ_B} of Σ_{μ_B}. We claim that $\Sigma_{\mu_B} = \Sigma'_{\mu_B}$. Let $S \subset \mathbb{R}$ be an interval. Then there is a sequence $\{F_n\}_{n\in\mathbb{N}}$ of linear combinations of $\{e^{ia\cdot} \mid a \in \mathbb{R}\}$, such that $F_n(\lambda) \to \chi_S(\lambda)$, $(n\to\infty)$ for almost all $\lambda \in \mathbb{R}$. It follows that for almost all $x \in \mathcal{S}'$ and all $f \in \mathcal{S}$:

$$F_n(\langle f,x\rangle) \xrightarrow[n\to\infty]{} \chi_S(\langle f,x\rangle).$$

But this implies that $\chi_S(\langle f,\cdot\rangle)$ is in the strong closure $\mathfrak{M}_0$ of A_0, so $\{x \mid \langle f,x\rangle \in I\} \in \Sigma'_{\mu_B}$. And because Σ_{μ_B} is the smallest σ-field containing all such sets, Σ'_{μ_B} must be equal to Σ_{μ_B}. So $\mathfrak{M}_0 = L^\infty(\mathcal{S}',\mu_B)$. Finally, $A_0 1$ is dense in $\mathfrak{M}_0 1 = L^\infty(\mathcal{S}',\mu_B)$, which again is dense in $L^2(\mathcal{S}',\mu_B) = H_B$. □

Decomposition of H_B

Lemma 5.7. There is a unitary map $i_F : H_F \to H_B$, such that

$$\forall_{f\in\mathcal{S}}: \; i_F \operatorname{Exp}_F(f) = \exp(-i\langle f,\cdot\rangle + \tfrac{1}{2}\|f\|^2). \tag{5.5}$$

Proof. Let us call the r.h.s. of (5.5): $\operatorname{Exp}_B(f)$. Then

$$\forall_{f,g\in\mathcal{S}}: \langle \operatorname{Exp}_F(f), \operatorname{Exp}_F(g)\rangle = \langle \operatorname{Exp}_B(f), \operatorname{Exp}_B(g)\rangle = \exp(\langle f,g\rangle).$$

Now, define, for $c_j \in \mathbb{C}$, $f_j \in \mathcal{S}$, $(j=1,\ldots,m)$:

$$i_F \sum_{j=1}^{m} c_j \operatorname{Exp}_F(f_j) = \sum_{j=1}^{m} c_j \operatorname{Exp}_B(f_j).$$

This definition makes sense, for suppose that $\sum c_j \operatorname{Exp}_F(f_j) = 0$; then also

$$\Big\| \sum_{j=1}^{m} c_j \, \mathrm{Exp}_B(f_j) \Big\|^2 = \sum_{j,k=1}^{m} c_j^* \, c_k \langle \mathrm{Exp}_B(f_j), \mathrm{Exp}_B(f_k) \rangle =$$

$$= \sum_{j,k=1}^{m} c_j^* \, c_k \langle \mathrm{Exp}_F(f_j), \mathrm{Exp}_F(f_k) \rangle = \Big\| \sum_{j=1}^{m} c_j \, \mathrm{Exp}_F(f_j) \Big\|^2 = 0.$$

The same computation shows that i_F preserves the norm, so i_F extends to a unitary map from the closed linear span of the $\mathrm{Exp}_F(f)$, to that of the $\mathrm{Exp}_B(f)$. By Lemma 5.6, the latter space is H_B. It is not hard to see that the former is H_F.

□

Remarks. The maps Exp_F and Exp_B are called Kolmogorov decompositions of the positive definite kernel $\mathcal{S} \times \mathcal{S} \to \mathbb{R} : \{f,g\} \mapsto \exp(\langle f,g \rangle)$, [EvL 77], (cf. App. B).

Under the isomorphism i_F, the decomposition of H_F into its levels $L^2_{\mathrm{symm}}(\mathbb{R}^n)$ corresponds to the Wiener-Kakutani decomposition of $H_B = L^2(\mathcal{S}', \mu_B)$.

Already in the classical mechanical context, Fock space comes onto the scene. We shall see that, due to this fact, quantisation can be a gradual affair.

The classical KMS condition

If ω is a state (i.e. a positive functional of norm 1) on A_0, it is said to satisfy the *classical KMS condition* with respect to the evolution τ, at a temperature T, if

$$\forall_{F,G \in A_0} : \frac{1}{kT} \cdot \frac{d}{dt} \omega(\tau_t(F)\, G) \Big|_{t=0} = \omega(\{F,G\}). \tag{5.6}$$

This condition was introduced by Gallavotti and Verboven [GaV 75] as an analogue of the Kubo-Martin-Schwinger (KMS) condition in quantum mechanics, introduced by Haag, Hugenholtz, and Winnink [HHW 67], (cf. Chapter II). Whereas the quantum KMS condition characterises Gibbs states without mention of a density operator, the classical KMS condition characterises Gibbs states without mention of a Liouville density function. If the system under consideration is infinite, density operators and density functions no longer exist, but the KMS conditions retain their meaning.

A probability measure μ on $\mathcal{S}'$ determines a state ω_μ on A_0 as follows:

$$\omega_\mu(F) = \int_{x \in \mathcal{S}'} F(x)\, \mu(dx).$$

Let $\omega_B = \omega_{\mu_B}$.

Proposition 5.8. The state ω_B satisfies the classical KMS condition w.r.t. the evolution τ at temperature T.

Proof. Notice that we have put kT = 1 at the beginning of this section. It suffices to prove (5.6) for $F = \exp(i\langle f, \cdot \rangle)$ and $G = \exp(i\langle g, \cdot \rangle)$ with $f,g \in \mathcal{S}$. With these substitutions, and using (4.19) in the proof of Lemma 4.3, we transform (5.6) into

$$\forall_{f,g \in \mathcal{S}} : \frac{d}{dt} \hat{\mu}(T_t f + g) \Big|_{t=0} = -\sigma(f,g)\, \hat{\mu}(f+g). \tag{5.7}$$

Now, (5.7) indeed holds for $\hat{\mu}(\cdot) = \exp(-\frac{1}{2}\|\cdot\|^2)$, because

$$\frac{d}{dt} \langle T_t f, g \rangle \Big|_{t=0} = \sigma(f,g). \tag{5.8}$$

□

Remark. The measure μ_B is not the only measure to satisfy the classical KMS condition. If we translate μ_B on $\mathcal{S}'$ over multiples of the distribution $1 : f \mapsto \int_{-\infty}^{\infty} f(t)\, dt$, we obtain other "KMS measures". We may also take convex combinations of such translates. Indeed, if $\mu_c(A) = \mu_B(A - c \cdot 1)$, where $A - c \cdot 1 = \{x - c \cdot 1 \mid x \in A\}$, then

$$\hat{\mu}_c(f) = e^{i c \langle 1,f \rangle} \hat{\mu}_B(f). \tag{5.9}$$

And because $\langle 1, T_t f + g \rangle = \langle 1, f + g \rangle$, μ_c also satisfies (5.7). Generally, a probability measure satisfies (5.7) if and only if it is of the form

$$\int_{c \in \mathbb{R}} \mu_c \, \nu(dc), \tag{5.10}$$

where ν is a probability measure on $\mathbb{R}$.

In the state ω_{μ_c}, the string in our model is tilted over its whole length, with a slope c, apart from the random fluctuations due to the thermal excitation. However, in § 2 we stipulated that "the direction of the bar is perpendicular to that of the string in its rest position", and we shall not allow the model to deviate from this. Therefore we shall choose $\nu = \delta$ in (5.10), and consider only the state ω_B.

§ 6. THE LANGEVIN EQUATION

We have seen that E_0 is white noise. For the time being, the index "0" has no meaning; later it will distinguish E_0 from its quantummechanical analogues $E_\beta (\beta > 0)$.

Let us define the family $\{B_t\}_{t \in \mathbb{R}}$ by

$$B_t = \begin{cases} E_0(\chi_{[0,t]}) & \text{for } t \geq 0, \text{ and} \\ -E_0(\chi_{[t,0]}) & \text{for } t \leq 0. \end{cases}$$

Then $\{B_t\}$ is standard Brownian motion. Now, formally we have

$$B_t' = E_t = E_0(\delta_t) = \langle \delta_t, x \rangle = x(t) = \sqrt{2\eta}\, b'(t). \tag{6.1}$$

It may be said, therefore, that the input function b has become a Brownian motion as a result of our heating up the Lamb model.

As we shall now prove, the basic equation (2.7) of the Lamb model, with $v(\lambda) = \frac{1}{2}\lambda^2$, turns into the Langevin equation

$$\frac{d^2}{dt^2} Q_t + \eta \frac{d}{dt} Q_t + Q_t = \sqrt{2\eta}\, E_t. \tag{6.2}$$

Theorem 6.1 (Langevin equation). Let E_0 be as defined in § 5, (cf. (5.4)), and $q \in L^2(\mathbb{R})$ as defined in def. 3.1. Define

$$Q_t = E_0(T_t q). \tag{6.3}$$

Then for all $f \in \mathcal{S}$ we have

$$\int_{-\infty}^{\infty} (f'' - \eta f' + f)(t)\; Q_t\, dt = \sqrt{2\eta}\, E_0(f). \tag{6.4}$$

We shall prove this using a lemma.

Lemma 6.2. The maps

$$A: f \mapsto \frac{1}{\sqrt{2\eta}}\, q * f \quad \text{and} \quad B: f \mapsto f'' - \eta f' + f$$

both leave $\mathcal{S}$ invariant, and, considered as maps $\mathcal{S} \to \mathcal{S}$, they are each other's inverse.

Proof. We note that, because $q(0) = 0$,

$$i\omega \hat{q}(\omega) = \int_{-\infty}^{0} i\omega e^{i\omega t} q(t)\, dt = -\int_{-\infty}^{0} e^{i\omega t} q'(t)\, dt.$$

Therefore, now using the fact that $q'(0) = -\sqrt{2\eta}$:

$$(i\omega)^2 \hat{q}(\omega) = -\int_{-\infty}^{0} i\omega e^{i\omega t} q'(t)\, dt = \int_{-\infty}^{0} q''(t)\, e^{i\omega t} dt + \sqrt{2\eta}.$$

It follows that

$$((i\omega)^2 + i\eta\omega + 1)\, \hat{q}(\omega) = \int_{-\infty}^{\infty} (q'' - \eta q' + q)(t) e^{i\omega t} + \sqrt{2\eta} = \sqrt{2\eta}.$$

So $\hat{q}(\omega) = \sqrt{2\eta}((i\omega)^2 + i\eta\omega + 1)^{-1}$.

Now $(Af)\hat{\ }(\omega) = \left(\frac{\hat{q}}{\sqrt{2\eta}} \cdot \hat{f}\right)(\omega) = \hat{f}(\omega)\left((i\omega)^2 + i\eta\omega + 1\right)^{-1}$. Because $\hat{\mathcal{S}} = \mathcal{S}$, and division by a polynomial without real zeroes keeps $\hat{f}$ in $\mathcal{S}$, we have $A\mathcal{S} \subset \mathcal{S}$. Clearly, $B\mathcal{S} \subset \mathcal{S}$. Moreover, $(Bf)\hat{\ }(\omega) = \left((i\omega)^2 + i\eta\omega + 1\right)\hat{f}(\omega)$. It follows that $(A \circ B)f = f$, $\forall_{f \in \mathcal{S}}$ and $A^{-1} = B$.

□

Proof of the theorem. Choose $f \in \mathcal{S}$ and put $g = Bf$. Then

$$\int_{-\infty}^{\infty} (f'' - \eta f' + f)(t)\, T_t q\, dt = \int_{-\infty}^{\infty} g(t)\, q(\cdot - t)\, dt = g * q = \sqrt{2\eta}\, Ag = \sqrt{2\eta}\, AB f = \sqrt{2\eta}\, f.$$

Now, because E_0 is an isomorphism between $L^2(\mathbb{R})$ and its image inside H_B, we may write

$$\int_{-\infty}^{\infty} (f'' - \eta f' + f)(t)\, E_0(T_t q)\, dt = \sqrt{2\eta}\, E_0(f).$$

□

In the theory of stochastic differential equations (6.2) would appear in the following, different form.

Theorem 6.1'. (Langevin equation, stochastic differential form). Let E_0 and q be as before, and let $p = -q'$. Define

$$Q_t = E_0(t_t q) \text{ and } P_t = E_0(T_t p). \tag{6.5}$$

Then for all $s,t \in \mathbb{R}$ with $s \leq t$

$$Q_t - Q_s = \int_s^t P_u\, du, \text{ and} \tag{6.6a}$$

$$P_t - P_s = \int_s^t (-\eta P_u - Q_u)\, du + \sqrt{2\eta}\, (B_t - B_s). \qquad (6.6b)$$

Usually, (6.6) us abbreviated as

$$dQ_t = P_t\, dt\ ;\ dP_t = (-\eta P_t - Q_t)\, dt + \sqrt{2\eta}\, dB_t\ ; \qquad (6.7)$$

A formal manipulation leads from (6.7) to (6.2). (Divide by dt and put $\dot{B}_t = E_t$). We note, however, that this manipulation does not make a strict sense, because $t \mapsto P_t$ is not L^2-differentiable; whereas both (6.2) and (6.7) have a well-defined meaning, namely (6.4) and (6.6) respectively.

Proof of theorem 6.1'. Because E_0 is an isometry, it suffices to prove that, for almost all $y \in \mathbb{R}$

$$q(y-t) - q(y-s) = \int_s^t p(y-u)\, du, \quad \text{and} \qquad (6.8a)$$

$$p(y-t) - p(y-s) = \int_s^t (-\eta p(y-u) - q(y-u))\, du + \sqrt{2\eta}\, \chi_{[s,t]}(y). \qquad (6.8b)$$

Now, (6.8a) is a direct consequence of the fact that $p = -q' \in L^2(\mathbb{R})$. To check (6.8b), first put $y > t$, and note that both sides equal zero. If $y < s$, we have, because $q'' - \eta q' + q = 0$ on $(-\infty, 0)$:

$$\text{l.h.s.} = \int_s^t q''(y-u)\, du = \int_s^t (\eta q'(Y-u) - q(y-u))\, du = \text{r.h.s.}$$

Finally, for $s \leqslant y \leqslant t$ we obtain

$$\text{l.h.s.} = p(y-t) = -q'(y-t) = -q'(0) + \int_y^t q''(y-u)\, du =$$

$$= \sqrt{2\eta}\, \chi_{[s,t]}(y) + \int_s^t (-\eta p(y-u) - q(y-u))\, du. \qquad \square$$

§ 7. THE STRING MODEL WITH AN ANHARMONIC OSCILLATOR

Our study of the string model with a harmonic potential has led to an identification of this model and the standard representation of Brownian motion as a flow on $\{\mathcal{S}', \mu_B\}$. This flow is called "the flow of Brownian motion" following N. Wiener, and is the group $\{T_t^*\}_{t \in \mathbb{R}}$, given by

$$\langle (T_t^* x), f \rangle = \langle x, T_t f \rangle. \qquad (7.1)$$

It determines a unitary group $\{\mathcal{T}_t\}_{t \in \mathbb{R}}$ on H_B by

$$(\mathcal{T}_t \varphi)(x) = \varphi(T_t^* x). \qquad (7.2)$$

The generator of this group is an anti-self-adjoint operator $\mathcal{L}$, whose action on A_0 can be expressed as

$$\mathcal{L}G = \{G, H\}, (G \in A_0). \qquad (7.3)$$

In this section we shall perturb the flow of the harmonic string model by adding a term to its potential:

$$v(\lambda) = \tfrac{1}{2}\lambda^2 + w(\lambda). \qquad (7.4)$$

Perturbed dynamics and the Møller operator

We shall explain why, under mild conditions for w, the oscillator again satisfies a Langevin equation under the perturbed flow. The analogous quantummechanical situation will be studied in detail in Chapter II. Here we only indicate the main lines.

Suppose that w is once continuously differentiable, and bounded from below. Let μ_B^w be given by

$$\mu_B^w = N \cdot \exp(-w(Q_0)) \cdot \mu_B\ , \qquad (7.5)$$

with $N > 0$ chosen such that $\mu_B^w(\mathcal{S}') = 1$. Define

$$H_B^w = L^2(\mathcal{S}', \mu_B^w).$$

Then $\{\mathcal{T}_t^w = \exp(t\mathcal{L}_w)\}_{t \in \mathbb{R}}$, with $\mathcal{L}_w$ given by

$$\mathcal{L}_w = \mathcal{L} + \{\cdot, w(Q_0)\}, \qquad (7.6)$$

is a unitary group on H_B^w. Under the isomorphism i_Φ of § 3, $\mathcal{T}_t^w$ corresponds to the transpose onto the observables, of the flow $\{F_t\}$ from § 2, associated to the potential (7.4); i.e., we have

$$\forall_{G \in A_0}: G\Big(i_\Phi(F_t(\varphi \oplus \pi))\Big) = (\mathcal{T}_t^w G)(i_\Phi(\varphi \oplus \pi)). \qquad (7.7)$$

Now, consider the limit

$$\Omega_w^{(0)} G = \lim_{t \to \infty} \mathcal{T}_{-t}^w \circ \mathcal{T}_t G, \ (G \in A_0). \qquad (7.8)$$

It is not hard to show that, under the condition

$$\int_{-\infty}^{\infty} w'(\lambda)^2 e^{-\frac{1}{2}\lambda^2} d\lambda < \infty , \tag{7.9}$$

the limit (7.8) exists for each $G \in A_0$. The map $\Omega_W^{(0)}$, thus defined, extends to an isometry

$$\Omega_W : H_B \to H_B^W , \tag{7.10}$$

satisfying

$$\Omega_W \mathfrak{T}_t = \mathfrak{T}_t^W \Omega_W , \tag{7.11}$$

$$\Omega_W \mathfrak{M}_0 = \mathfrak{M}_0 \Omega_W , \quad \text{and} \tag{7.12}$$

$$\Omega_W Q_0 = Q_0 + \int_{-\infty}^{0} \frac{1}{\sqrt{2\eta}} q(s) \mathfrak{T}_s^W(w'(Q_0))\, ds . \tag{7.13}$$

Now, put

$$\tilde{Q}_t^W = \mathfrak{T}_t^W Q_0 . \tag{7.14}$$

Then, from (7.11) and (7.13) it follows that

$$\Omega_W Q_t = \tilde{Q}_t^W + \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}} q(s-t) w'(\tilde{Q}_s^W) ds . \tag{7.15}$$

Integrating both sides of (7.15) with $f \in \mathcal{S}$, and using Lemma 6.2, we obtain

$$\forall_{f \in \mathcal{S}} : \int_{-\infty}^{\infty} (f'' - \eta f' + f)(t) \tilde{Q}_t^W dt + \int_{-\infty}^{\infty} f(t) w'(\tilde{Q}_t^W)\, dt = \sqrt{2\eta}\, \Omega_W E_0(f) . \tag{7.16}$$

Now, with respect to μ_B^W, $\Omega_W \circ E_0$ is white noise, because of (7.10):

$$\int_{\mathcal{S}'} (\Omega_W E_0(f))^2 d\mu_B^W = \int_{\mathcal{S}'} E_0(f)^2 d\mu_B = \|f\|^2 . \tag{7.17}$$

Therefore, (7.16) is the distribution form of a Langevin equation, namely,

$$\frac{d^2}{dt^2} \tilde{Q}_t^W + \eta \frac{d}{dt} \tilde{Q}_t^W + v'(\tilde{Q}_t^W) = \sqrt{2\eta}\, \Omega_W E_{0,t} . \tag{7.18}$$

An alternative form is

$$d\begin{pmatrix} \tilde{Q}_t^W \\ \tilde{P}_t^W \end{pmatrix} = \begin{pmatrix} \tilde{P}_t^W \\ -\tilde{P}_t^W - v'(\tilde{Q}_t^W) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2\eta} \end{pmatrix} d\tilde{B}_t^W . \tag{7.19}$$

Here, $\tilde{B}_t^W := \Omega_W B_t$ is a Brownian motion.

§ 8. A RESULT ON APPROACH TO EQUILIBRIUM, BASED ON THE MARKOV PROPERTY

Consider the particle on the line, introduced in § 1. In this section, we suppose that it is subject to a stochastic force $\sqrt{2\eta}\, E_t = \sqrt{2\eta}\, \dot{B}_t$, where E is white noise, besides the frictional force and the conservative force $-v'(Q_t)$.

We shall no longer bother about the origin of the friction and noise terms; we have seen that these terms can be brought about by a string, attached to the particle. But perhaps some other heat bath may do the job just as well. The motion of the particle will be governed by the stochastic differential equation, (cf. 7.19),

$$d\begin{pmatrix} Q_t \\ P_t \end{pmatrix} = \begin{pmatrix} P_t \\ -\eta P_t - v'(Q_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2\eta} \end{pmatrix} dB_t . \tag{8.1}$$

We shall refer to (8.1) as the *Langevin equation with potential* v, and to the solution $\{Q_t, P_t\}_{t \geq 0}$ as the *Ornstein-Uhlenbeck process with potential* v.

A basic theorem in the theory of stochastic differential equations (cf., for instance, [GiS 72]), implies that, if v' satisfies the Lipschitz condition (1.2), then, given any probability measure ν on $\mathbb{R}^2$ with finite second moments, (8.1) has a solution $\{Q_t, P_t\}_{t \geq 0}$ with continuous sample paths and with initial probability distribution ν, i.e. with the property that for all Borel sets $A \subset \mathbb{R}^2$

$$\mathbb{P}(\{Q_0, P_0\} \in A) = \nu(A) . \tag{8.2}$$

(Here, $\mathbb{P}(\cdot)$ stands for "the probability that $\cdot$"). This solution is unique in the sense that every solution of (8.1) has an almost surely continuous version that equals $\{Q_t, P_t\}_{t \geq 0}$ with probability 1.

As in § 1, we are interested in the question, what happens if t becomes large.

Now, it is generally believed that, in the limit of weak coupling between a system and a large reservoir at constant temperature, the system is driven to its *Gibbs state*. This state is described by a measure on the phase space of the system, called the *Gibbs measure*. In the present case, this phase space is $\mathbb{R}^2$, and the Gibbs measure, ν_V say, is given by

$$\nu_V(dq\, dp) = N \cdot \exp(-(v(q) + \tfrac{1}{2} p^2))\, dq\, dp . \tag{8.3}$$

It turns out that, by virtue of the special, singular coupling between the string and the oscillator in the Lamb model, the string drives the oscillator to its Gibbs state for all positive values of the coupling constant η. In fact, this is a property of the Langevin equation (8.1) and the measure (8.3). Let us give a definition of this property of approach to equilibrium.

Definition 8.1. A measure ν_0 on $\mathbb{R}^2$ will be called an *attracting equilibrium measure* for the Langevin equation (8.1) if the following holds:

(i) The solution $\{Q_t, P_t\}_{t \geq 0}$ with initial probability ν_0 has probability distribution ν_0 at all times.

(ii) For all probability measures ν on $\mathbb{R}^2$, absolutely continuous w.r.t. ν_0, the solution $\{Q_t, P_t\}_{t \geq 0}$ with initial probability distribution ν, satisfies

$$\forall_{A \in \mathcal{B}(\mathbb{R}^2)}: \lim_{t\to\infty} \mathbb{P}(\{Q_t, P_t\} \in A) = \nu_0(A). \qquad (8.4)$$

For the fact that the Gibbs measure is an attracting equilibrium measure in this sense, proofs have been given in various degrees of rigour, all making essential use of the independence of the pieces of noise signal on disjoint time intervals. This independence leads to a predictable evolution of the probability distribution on $\mathbb{R}^2$, described by a partial differential equation, the Fokker-Planck equation, which can be studied by semigroup techniques.

Below we shall give a sketch of a proof by Tropper [Tro 77]. First, we given an outline of the necessary background.

Markov processes and transition probabilities.

The Ornstein-Uhlenbeck process, like any solution of a stochastic differential equation with a white noise source, is a Markov process (cf. [GiS 72]). This means that the probability $\pi_{t,s}(x,A)$ for the particle to be inside the Borel set $A \subset \mathbb{R}^2$ at a time $s \geq t$, given that it is in $x \subset \mathbb{R}^2$ at time t, is not dependent on the previous history of the particle. If, moreover, $\pi_{t,s}(x,A)$ only depends on t and s via their difference $u = s - t$, the process is called a *time-homogeneous Markov process*, and one may speak of $\pi_u(x,A)$ in an obvious sense. As the Langevin equation contains no explicit

time dependence, the Ornstein-Uhlenbeck process is indeed time-homogeneous. The function π: $\{t,x,A\} \mapsto \pi_t(x,A)$ has the following properties.

(a) $x \mapsto \pi_t(x,A)$ is measurable on $\mathbb{R}^2$ for all $t \geq 0$ and all $A \in \mathcal{B}(\mathbb{R}^2)$,

(b) $A \mapsto \pi_t(x,A)$ is a probability measure on $\mathbb{R}^2$ for all $t \geq 0$ and all $x \in \mathbb{R}^2$,

(c) For all $s,t \geq 0$, all $x \in \mathbb{R}^2$ and all $A \in \mathcal{B}(\mathbb{R}^2)$:

$$\pi_{s+t}(x,A) = \int_{y \in \mathbb{R}^2} \pi_s(x,dy)\,\pi_t(y,A). \qquad (8.5)$$

Property (c) is called the *Chapman-Kolmogorov equation*. A function π: $[0,\infty) \times \mathbb{R}^2 \times \mathcal{B}(\mathbb{R}^2) \to [0,1]$, satisfying (a), (b), and (c) is called a *transition probability* on $\mathbb{R}^2$.

The Fokker-Planck semigroup

Let π be the transition probability of the Ornstein-Uhlenbeck process, and define, for $t \geq 0$ and $f \in L^\infty(\mathbb{R}^2)$:

$$(Z_t f)(x) = \int_{y \in \mathbb{R}^2} \pi_t(x,dy)\, f(y). \qquad (8.6)$$

Then Z_t is linear, and maps positive functions on positive functions. By (b), $Z_t 1 = 1$ and

$$\| Z_t f \|_\infty \leq \| f \|_\infty ,$$

and by (a) and (c),

$$\forall_{s,t \geq 0}: Z_{t+s} = Z_t \circ Z_s ;$$

i.e., $\{Z_t\}_{t \geq 0}$ is a semigroup of positivity preserving contractions on $L^\infty(\mathbb{R}^2)$. Let G be its generator. One computes that G is the differential operator

$$G = \eta\left(\frac{\partial^2}{\partial p^2} - p\frac{\partial}{\partial p}\right) + \left(p\frac{\partial}{\partial q} - v'(q)\frac{\partial}{\partial p}\right), \qquad (8.7)$$

with the formal adjoint

$$G^* = \eta\frac{\partial}{\partial p}\left(\frac{\partial}{\partial p} + p\right) + \left(-p\frac{\partial}{\partial q} + v'(q)\frac{\partial}{\partial p}\right). \qquad (8.8)$$

The semigroup $\{Z_t\}$ is the main tool in the proof of the following theorem by Tropper [Tro 77].

Theorem 8.2. Suppose that

(i) v is three times continuously differentiable,

(ii) v" and v"' are bounded,

(iii) $\exp(-v) \in L^1(\mathbb{R})$, and

(iv) $(v')^2 \cdot \exp(-v) \in L^1(\mathbb{R})$.

Then the Gibbs measure (8.3) is an attracting equilibrium measure for the Langevin equation with potential v .

Proof. For the proof we refer to [Tro 77]. We shall indicate the main steps below. From (8.6) it follows that $\pi_t(x,A) = (Z_t \chi_A)(x)$. Let $\{Q_t, P_t\}$ be the solution of (8.1) with initial probability distribution ν. Then

$$\mathbb{P}(\{Q_t, P_t\} \in A) = \int_{x \in \mathbb{R}^2} \nu(dx)\, \pi_t(x,A) = \int_{x \in \mathbb{R}} (Z_t \chi_A)\, d\nu . \tag{8.9}$$

Let $h(q,p) = v(q) + \frac{1}{2}p^2$. One checks that $G^*(e^{-h}) = 0$. Suppose that ν is the Gibbs measure ν_v itself. Then

$$\frac{d}{dt}\mathbb{P}(\{Q_t, P_t\} \in A) = \frac{d}{dt}\left(N \cdot \int_A Z_t^*(e^{-h})\, dq\, dp\right) = 0.$$

This proves (i) of def. 8.1. It also shows that ν_v is an invariant measure for Z_t^*, $(t \geq 0)$. Now, consider the Hilbert space $L^2(\mathbb{R}^2, \nu_v)$. $\{Z_t\}$ extends to a contraction semigroup on this space, and the claim is that for all $f \in L^2(\mathbb{R}^2, \nu_v)$

$$\underset{t \to \infty}{\text{weak-lim}}\; Z_t f = \langle f,1\rangle \cdot 1. \tag{8.10}$$

To prove this, one notes that, for $f \in C^2(\mathbb{R}^2)$,

$$G(f^2) - 2f(Gf) = \eta \frac{\partial^2}{\partial p^2}(f^2) - 2\eta f \frac{\partial^2 f}{\partial p^2} = 2\eta\left(\frac{\partial f}{\partial p}\right)^2 .$$

Therefore

$$\frac{d}{dt}\|Z_t f\|^2\Big|_{t=0} = 2\langle f, Gf\rangle = N \cdot \int_{\mathbb{R}^2} 2f(Gf)\, e^{-h}\, dq\, dp =$$

$$= N \cdot \int_{\mathbb{R}^2}\left(G(f^2) - 2\eta\left(\frac{\partial f}{\partial p}\right)^2\right)e^{-h}\, dq\, dp = -2\eta \left\|\frac{\partial f}{\partial p}\right\|^2 .$$

So for all $t \geq 0$, all $f \in C^2(\mathbb{R}^2)$

$$\frac{d}{dt}\|Z_t f\|^2 = -2\eta \left\|\frac{\partial}{\partial p} Z_t f\right\|^2 . \tag{8.11}$$

This equation suggests that $Z_t f$ will loose norm until it becomes p-independent.

Indeed, it can be shown that $Z_{t_n} \to \gamma$ weakly for some sequence $\{t_n\}_{n \in \mathbb{N}}$ with $t_n \to \infty$, where γ is a function of q alone. But then, it turns out, for $t \geq 0$, $Z_t \gamma$ also has to be p-independent. It follows that, as a function in $L^2(\mathbb{R}^2, \nu_v)$, the vector $(\partial/\partial p)G\gamma$ is zero, and by (8.7) this equals the vector $\partial\gamma/\partial q$. Now, if $\exp(-h)$ is sufficiently nonzero, γ must be a constant. It can be shown that necessarily $Z_t f \to \gamma$ weakly. As $\langle 1, Z_t f\rangle = \langle 1,f\rangle$ for all t , we have $\gamma = \langle 1,f\rangle \cdot 1$.

Finally, suppose $\nu \ll \nu_v$. Then $\nu = g\,\nu_v$, $g \in L^1(\nu_v)$. Approximating g by L^2-functions one concludes

$$\mathbb{P}(\{Q_t, P_t\} \in A) = \int_{\mathbb{R}^2} (Z_t \chi_A)\, g\, d\nu_v \to \nu_v(A), \quad (t \to \infty).$$

□

Remark. It is illustrative to compare potentials like v_1 and v_2, drawn in Fig. 6. For v_1, there are two attracting equilibrium measures $\nu_\pm$, given by

$$\nu_\pm(dq\, dp) = N_\pm \cdot \theta(\pm q) \cdot e^{-h(q,p)}\, dq\, dp. \tag{8.12}$$

(N_+ and N_- are normalisation constants). The Gibbs measure is a convex combination of ν_+ and ν_-, but it is not an attracting equilibrium measure itself, because the barrier cannot be crossed, and the probabilities to find the particle on either side, remain constant. The above proof goes wrong for v_1, because $\partial\gamma/\partial q = 0$ in $L^2(\exp(-h)\, dq\, dp)$ does not imply that γ is a constant. (And indeed, potentials like v_1 are excluded by the conditions of the theorem).

However, the theorem implies that potentials like v_2 do not prevent the approach to the Gibbs measure, however high the barrier may be.

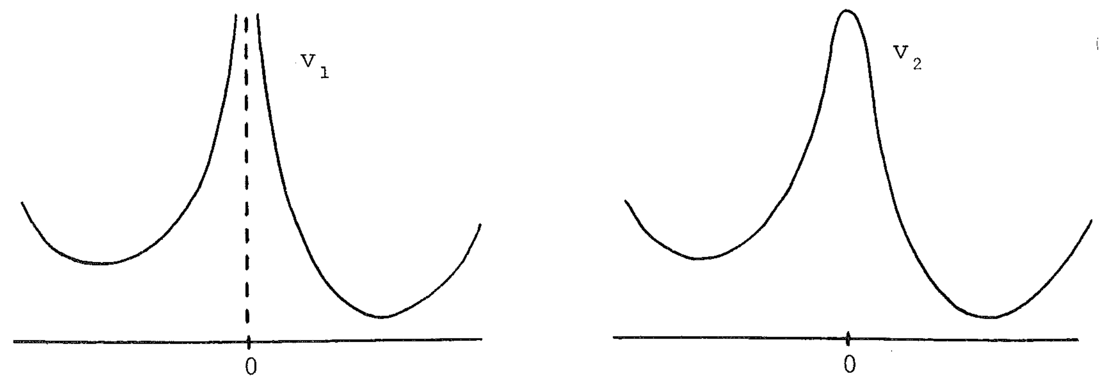

Fig. 6. Potential v_1 with infinitely high barrier, and potential v_2 with very high barrier.

§ 9. A RESULT ON APPROACH TO EQUILIBRIUM, BASED ON THE MIXING PROPERTY.

It could be maintained that the approach to the ergodic problem, used in the previous section, is an unnatural one. It starts by "projecting out" the heat bath variables to yield a reduced stochastic evolution of the oscillator, and then the asymptotic properties of this evolution are studied. Why not study directly the ergodic properties of the total system, including the heat bath?

The latter point of view will be exploited in this section. As a matter of fact, the result obtained here will be much weaker than Theorem 8.2, (and this explains why the Ornstein-Uhlenbeck process is not usually treated along these lines!). However, in quantum mechanics the oscillator's evolution in the string model is not a Markov process. It has no reduced description, as is provided by the Fokker-Planck semigroup in the classical case, and the "total system approach" seems to be the only one available, (cf. § II.5). When following this approach, it will be illustrative to have a comparable classical treatment at hand.

We shall preferably formulate matters in terms whose meaning remains unaltered in quantum mechanics. For instance, we shall speak of a "W^*-dynamical system" instead of a dynamical system in the sense of probability theory. Upon quantisation, the W^*-dynamical system of Brownian motion barely changes.

Mixing W^*-dynamical systems.

The flow of Brownian motion has very nice ergodic properties. It is mixing, and hence ergodic. It has become, moreover, the prototype of a "Kolmogorov flow" and a "Bernoulli flow".

The evolution of the Lamb model at a positive temperature, being isomorphic to Brownian motion, also deserves these epitheta, as well as the flows of some other Gaussian models. For a lattice of harmonic oscillators these results were proved by van Hemmen [Hem 76], and Lanford and Lebowitz [LaL 75].

For our purposes, the mixing property is of greatest interest. It implies approach to equilibrium for all functionals of Brownian motion, notably including the nonlinear ones.

A function F in $\mathcal{A}_0$ or in $\mathfrak{M}_0 = L^\infty(\mathcal{S}',\mu_B)$ will be viewed as a multiplication operator on H_B, not as a vector in H_B. The corresponding vector will be denoted by F 1. We extend the evolution $\{\tau_t\}$ to all operators on H_B by defining

$$\tau_t(A) = \mathcal{U}_t\, A\, \mathcal{U}_t^* \ . \qquad (9.1)$$

Definition 9.1. By a W^*-dynamical system we mean a quadruple $\{H,\mathfrak{M},\{\alpha_t\}_{t\in\mathbb{R}},\xi\}$, where H is a Hilbert space, $\mathfrak{M}$ a von Neumann algebra of operators on H, $\{\alpha_t\}_{t\in\mathbb{R}}$ a group of *-automorphisms of $\mathfrak{M}$, and ξ a vector in H, satisfying

(i) $\overline{\mathfrak{M}\xi} = H$, (i.e., ξ is cyclic for $\mathfrak{M}$),

(ii) $\forall_{A\in\mathfrak{M}}\forall_{t\in\mathbb{R}}$: $\langle\xi,\alpha_t(A)\xi\rangle = \langle\xi,A\,\xi\rangle$.

(A von Neumann algebra is a strongly closed, self-adjoint algebra of operators. As an abstract algebra, it is a W^*-algebra. For a general background in operator algebras, see, for instance, [BrR 79]).

For example, if $\{\Phi,\mu\}$ is a probability space, and $\{\varphi_t\}_{t\in\mathbb{R}}$ a flow on Φ, then

$$\{L^2(\Phi,\mu),\ L^\infty(\Phi,\mu),\ \{F\mapsto F\circ\varphi_t\},\ 1\} \qquad (9.2)$$

is a W^*-dynamical system.

Definition 9.2. A W^*-dynamical system will be called *mixing* if for all $A\in\mathfrak{M}$, and all unit vectors $\psi\in H$,

$$\lim_{t\to\pm\infty} \langle\psi,\alpha_t(A)\psi\rangle = \langle\xi,A\,\xi\rangle . \qquad (9.3)$$

Recall that a dynamical system $\{\Phi,\mu,\{\varphi_t\}\}$ is called mixing if for all measurable subsets $S_1,S_2\subset\Phi$

$$\lim_{t\to\pm\infty} \mu(S_1\cap\varphi_t(S_2)) = \mu(S_1)\,\mu(S_2) . \qquad (9.4)$$

This is equivalent with the mixing property of the W^*-dynamical system (9.2).

Proposition 9.3. The W^*-dynamical system of Brownian motion, $\{H_B,\mathfrak{M}_0,\tau,1\}$, is mixing.

We shall first prove a lemma.

Lemma 9.4. For all $\varphi,\psi \in H_B$: $\lim\limits_{t\to\pm\infty} \langle\varphi,\mathcal{T}_t\,\psi\rangle = \langle\varphi,1\rangle\langle 1,\psi\rangle$. (9.5)

Proof. For all $f,g \in \mathcal{S}$, $\langle f, T_t\, g\rangle \to 0$, $(t\to\pm\infty)$, and therefore

$$\langle \mathrm{Exp}_B(f),\ \mathcal{T}_t\ \mathrm{Exp}_B(g)\rangle = \langle \mathrm{Exp}_B(f),\ \mathrm{Exp}_B(T_t\, g)\rangle = \exp(\langle f\,,T_t\, g\rangle) \to 1,\ (t\to\pm\infty).$$

Now, $1 = \langle \mathrm{Exp}_B(f), 1\rangle\langle 1\,, \mathrm{Exp}_B(g)\rangle$ and the statement follows by linear continuation.

□

Proof of Proposition 9.3. Let $F \in \mathfrak{M}_0$ and $\psi \in H_B$, such that $\|\psi\| = 1$. Let $\varepsilon > 0$. As 1 is cyclic for $\mathfrak{M}_0$, there is $G \in \mathfrak{M}_0$, such that $\|\psi - G\,1\| < \varepsilon$ and $\|G\,1\| = 1$. By Lemma 9.4 we have, for t sufficiently large,

$$|\langle G^*G\,1\,,\mathcal{T}_t\ F\,1\rangle - \langle G^*G\,1,1\rangle\langle 1,F\,1\rangle| < \varepsilon\,.$$

Now, $\langle G^*G\,1\,,\ \mathcal{T}_t\,F\,1\rangle = \langle G\,1\,,\ \tau_t(F)\,G\,1\rangle$ and $\langle G^*G\,1\,,1\rangle = 1$. Therefore, for t sufficiently large,

$$|\langle\psi,\tau_t(F)\psi\rangle - \langle 1,F\,1\rangle| < \varepsilon(1+2\|F\|_\infty)\,.$$

□

(The proof is taylored to prepare for the proof of Prop. II.5.2).

A consequence of Prop. 9.3., is that for all probability measures $\mu \ll \mu_B$ (absolutely continuous w.r.t. μ_B), and all measurable subsets A of $\mathcal{S}'$,

$$\lim_{t\to\pm\infty} \mu(T_t^*(A)) = \mu_B(A)\,. \qquad (9.6)$$

Approach to equilibrium (harmonic case)

We exploit the mixing property to prove approach to equilibrium. The theorems are special cases of Theorem 8.2, and only the proofs are of interest.

Theorem 9.5. The Gaussian measure on $\mathbb{R}^2$, given by

$$\nu_G(dq\,dp) = \frac{1}{2\pi}\exp(-\tfrac{1}{2}(q^2+p^2))\,dq\,dp \qquad (9.7)$$

is an attracting equilibrium measure for the Langevin equation with potential $v(\lambda) = \frac{1}{2}\lambda^2$.

Proof. Let $\{Q_t,P_t\}_{t\geq 0}$ be as given in Theorem 6.1', and define

$$X_t\ :\ \mathcal{S}' \to \mathbb{R}^2\ :\ x \mapsto \{Q_t(x),\ P_t(x)\}.$$

Then for all $S \in \mathcal{B}(\mathbb{R}^2)$ and all $t \in \mathbb{R}$,

$$\mathbb{P}_{\mu_B}(\{Q_t,P_t\} \in S) = \mu_B(X_t^{-1}(S)) = \nu_G(S)\,.$$

This proves the first statement, (cf. Def. 8.1). Now, let $\nu \ll \nu_G$. By the uniqueness theorem of solutions of stochastic differential equations, it suffices to show that there exists a solution of the Langevin equation with initial probability distribution ν, and tending to ν_G in the sense of (8.4).

By the Radon-Nikodym theorem, there is $g \in L^1(\mathbb{R},\nu_G)$, such that $\nu = g\,\nu_G$. Define the measure μ on $\mathcal{S}'$ by

$$\mu = g(Q_0,P_0)\mu_B\,. \qquad (9.8)$$

Then the process $\{Q_t,P_t\}_{t\geq 0}$, taken w.r.t. the probability measure μ meets our requirements. Indeed, for all $S \in \mathcal{B}(\mathbb{R}^2)$ we have,

$$\mathbb{P}_\mu(\{Q_0,P_0\} \in S) = \mu(X_0^{-1}(S)) = \int_{X_0^{-1}(S)} (g\circ X_0)\,d\mu_B = \int_S g\,d\nu_G = \nu(S)\,. \qquad (9.9)$$

Moreover, $\{X_t = \{Q_t,P_t\}\}_{t\in\mathbb{R}}$ is a *stationary* process, i.e.

$$X_t(x) = X_0(T_t^*\,x)\,. \qquad (9.10)$$

It follows that

$$\mathbb{P}_\mu(\{Q_t,P_t\} \in S) = \mu(X_t^{-1}(S)) = \mu(T_{-t}^*(X_0^{-1}(S)) \xrightarrow[t\to\infty]{} \mu_B(X_0^{-1}(S)) = \nu_G(S)\,,$$

by the mixing property, in the form (9.6).

Finally, we must check that $\{Q_t,P_t\}_{t\geq 0}$ satisfies (8.1) also when taken with respect to $\mu = g(Q_0,P_0)\mu_B$. Now, $\{B_t\}_{t\geq 0}$ is statistically independent of Q_0 and P_0, so $\{B_t\}_{t\geq 0}$ is a Brownian motion w.r.t. μ as well as w.r.t. μ_B.

□

Approach to equilibrium (anharmonic potentials)

The essential feature, going into the above proof, is the stationarity, the "sweeping along" with the flow, (9.10) of the solution. Indeed, by (9.6), any measurable subset of $\mathcal{S}'$ is "smeared out" to a constant in a weak sense, by the flow $\{T_t^*\}_{t\in\mathbb{R}}$.

We shall exploit this fact further, in generalising Theorem 9.5 to a class of perturbations of the harmonic potential. Put

$$v(\lambda) = \tfrac{1}{2}\lambda^2 + w(\lambda)\,. \qquad (9.11)$$

Theorem 9.6. Let $\{Q_t, P_t\}_{t \in \mathbb{R}}$ be as before, and let w be differentiable, with bounded derivative, satisfying the Lipschitz condition

$$|w'(\lambda_1) - w'(\lambda_2)| \leqslant k \cdot |\lambda_1 - \lambda_2|, \tag{9.12}$$

where

$$k < \frac{|\int q\, dt|}{\int |q|\, dt} . \tag{9.13}$$

Then there is a unique function $Q_0^W \in H_B$, such that $\{Q_t^W = \mathcal{T}_t\, Q_0^W\}$ solves the equation

$$Q_t^W = Q_t - \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}}\, q(s-t) w'(Q_s^W)\, ds . \tag{9.14}$$

Moreover, the process $\{Q_t^W, P_t^W\}_{t \in \mathbb{R}}$, with

$$P_t^W = P_t - \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}}\, p(s-t) w'(Q_s^W)\, ds , \tag{9.15}$$

is the unique stationary solution of the Langevin equation with potential (9.11). The probability measure ν_W on $\mathbb{R}^2$, given by

$$\nu_W(S) = \mathbb{P}_{\mu_B}(\{Q_t^W, P_t^W\} \in S), \tag{9.16}$$

is an attracting equilibrium measure for this equation.

Proof. For almost all $x \in \mathcal{S}'$ w.r.t. μ_B, $t \mapsto Q_t(x)$ is continuous, so we need only prove existence and uniqueness of $t \mapsto Q_t^W(x)$, satisfying (9.14), for a fixed, continuous $t \mapsto Q_t(x)$. We use a method, known as "Picard's method", [Nel 67]. Let $C_b^0(\mathbb{R})$ be the Banach space of bounded and continuous functions $\mathbb{R} \to \mathbb{R}$ with the sup norm. Choose $x \in \mathcal{S}'$ such that $t \mapsto Q_t(x)$ is continuous, and define $\theta_x : C_b^0(\mathbb{R}) \to C_b^0(\mathbb{R})$ by

$$(\theta_x f)(t) = - \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}}\, q(s-t) w'(Q_s(x) + f(s))\, ds .$$

Then, for all $f, g \in C_b^0(\mathbb{R})$,

$$\|\theta_x f - \theta_x g\| = \sup_t \left| \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}}\, q(s-t) \Big(w'(Q_s(x) + f(s)) - w'(Q_s(x) + g(s)) \Big)\, ds \right|$$

$$\leqslant \sup_t \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}}\, |q(s-t)| \cdot k \cdot |f(s) - g(s)|\, ds \leqslant k \cdot \|f-g\| \cdot \frac{\int |q|\, ds}{|\int q\, ds|},$$

because $\int q\, ds = \sqrt{2\eta}$. By (9.13), we have $\|\theta_x f - \theta_x g\| \leqslant c\|f-g\|$ with $c < 1$, so θ_x has a unique fixed point $f_x \in C_b^0(\mathbb{R})$. Define

$$Q_t^W(x) = Q_t(x) + f_x(t). \tag{9.17}$$

Then $\{Q_t^W\}$ solves (9.14). It is stationary, because it is uniquely determined. By substitution into (8.1), (or rather its integrated version, like (6.6)), one checks that $\{Q_t^W, P_t^W\}$ is a stationary solution of the Langevin equation, (see also Def. II.4.1). A repetition of the arguments, used in the proof of Theorem 9.5, shows that ν_W is an attracting equilibrium measure. Especially, we note that $Q_t^W(x)$ depends only on $\{Q_s(x)\}_{s \leqslant t}$. □

Remarks 1. If w satisfies (9.12) with (9.13), then v is strictly convex. Indeed, note that $k < 1$, hence, for $\lambda_1 > \lambda_2$,

$$v'(\lambda_1) - v'(\lambda_2) = (\lambda_1 - \lambda_2) + (w'(\lambda_1) - w'(\lambda_2)) \geqslant (1-k)(\lambda_1 - \lambda_2) > 0 .$$

So v' is strictly increasing, and v is strictly convex.

It follows that v has only one equilibrium point e, (i.e. a zero of v').

2. The above "pathwise" approach can only work if v has only a single equilibrium point. For example, put $x = 0$. Then $Q_t(x) = 0\ \forall_t$, and for any $e \in \mathbb{R}$, such that $v'(e) = e + w'(e) = 0$, the constant function $Q_t^W(x) = e$ satisfies (9.14) in the phase space point $x = 0$. Uniqueness of $\{Q_t^W(x)\}$ therefore implies uniqueness of e.

This shows the weakness of the pathwise approach to the existence and uniqueness question for solutions of the Langevin equation. Indeed, we do not need uniqueness of the solution in each point of phase space, but only for some class, of measure one, of rather rough input functions.

3. From Theorem 8.2 we know that the probability distribution ν_W of $\{Q_0^W, P_0^W\}$ is the Gibbs measure (8.3), (which is called ν_V there). This can be concluded also without use of the Markov property and semigroup analysis, from (7.5), (7.14) and (7.18).

However, to try and verify directly from (9.14) and (9.15) that ν_W is the Gibbs measure, is a tremendous undertaking, not unlike the efforts made by Benguria and Kac in [BeK 81].

Chapter II

QUANTUM LANGEVIN EQUATIONS

In this chapter, a quantummechanical version is treated of the string model, considered in Chapter I. The operator differential equation that is satisfied by the oscillator in this quantum model, has been given the name of "quantum Langevin equation". The question of approach to equilibrium for its solutions, is studied using perturbation theory. It is shown that for a general class of - convex - perturbations of the harmonic potential, the oscillator is driven to a certain limit state. This state is close to the Gibbs state, both for small friction coefficients and for high temperatures.

§ 1. REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS

Let us formulate how a damped oscillator in a heat bath is to be described in quantum mechanics. We take the point of view, put forward in [FKM 65], that one should not try to artificially impose friction on the ordinary, frictionless quantum oscillator, but that one should put the oscillator in a quantummechanical, friction-producing, environment. The advantage is that, in this way, one only needs a procedure for the quantisation of Hamiltonian systems.

Therefore, we shall quantise the Lamb model. We recall that the Lamb model is isomorphic to a limit of models, considered in [FKM 65]. This section, and the next, will reproduce the results obtained there, notably the commutation relation and the correlation of the "quantum noise".

In quantising, we shall apply the rule of Dirac that "Poisson brackets are to be replaced by commutators":

$$\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot] . \tag{1.1}$$

Here, $\hbar$ is Planck's constant, divided by 2π, $\{\cdot,\cdot\}$ is the Poisson bracket, acting on classical observables, and $[\cdot,\cdot]$, defined by $[A,B] = AB - BA$,

is the commutator, acting on operators on a Hilbert space, which is as yet to be constructed. The rule (1.1) we shall apply not to all classical observables - it was not meant for them all - but to a large class of linear ones, namely the linear and continuous functionals $x \mapsto \langle f,x\rangle$ on $\mathcal{S}'$, where $f \in \mathcal{S}'' = \mathcal{S}$.

By Lemma 4.3 of the previous chapter, we have

$$\{\langle f,\cdot\rangle, \langle g,\cdot\rangle\} = \sigma(f,g)\,1 . \tag{1.2}$$

Let us postulate then, that to every $f \in \mathcal{S}$ there shall correspond a self-adjoint operator $E(f)$ on some Hilbert space H, such that for all $f,g \in \mathcal{S}$,

$$[E(f), E(g)] = i\,\sigma(f,g)\,\mathbb{1} . \tag{1.3}$$

Here, $\mathbb{1}$ is the identity operator on H. The constant $\hbar$ in (1.1) has been omitted to make $E(f)$ dimensionless.

Given such a correspondence E, let $W(f)$ be defined by

$$W(f) = \exp(-i\,E(f)) . \tag{1.4}$$

Then W satisfies

$$W(f+g) = \exp(\tfrac{i}{2}\,\sigma(f,g))\,W(f)\,W(g) . \tag{1.5}$$

The relation (1.3) is called the *canonical commutation relation* (CCR) over the symplectic space $\{\mathcal{S},\sigma\}$.

Definition 1.1. By a *cyclic representation of the CCR over* $\{\mathcal{S},\sigma\}$ one means a triple $\{H,W,\xi\}$, where H is a Hilbert space, W a strongly continuous map from $\mathcal{S}$ to the unitary operators on H, and ξ a unit vector in H, such that (1.5) holds, and the linear span of the vectors $W(f)\xi$, $(f \in \mathcal{S})$, is dense in H.

The Stone-von Neumann uniqueness theorem says that, for a finite dimensional symplectic space, all cyclic representations of the CCR are unitarily equivalent (differing only in the choice of the cyclic vector ξ).

However, $\mathcal{S}$ is infinite dimensional. This reflects the fact that the Lamb model has an infinite number of degrees of freedom. There is an abundance of different (i,e, unitarily inequivalent) cyclic representations of the CCR over $\{\mathcal{S},\sigma\}$, and we still have to specify which of these is going to yield our quantummechanical Lamb model.

The following concept will help us in classifying the cyclic representations of the CCR.

Let $\{H,W,\xi\}$ be such a representation, and define its *generating functional* C by

$$C : \mathcal{S} \to \mathbb{C} : f \mapsto \langle \xi, W(f)\, \xi \rangle . \tag{1.6}$$

The C has the properties

(i) C is continuous

(ii) $C(0) = 1$

(iii) For all $n \geq 0$; $c_j \in \mathbb{C}$, $f_j \in \mathcal{S}$, $(j = 1,\dots,m)$:

$$\sum_{j=1}^{m} c_j c_k^* e^{\frac{i}{2}\sigma(f_j,f_k)} C(f_j - f_k) \geq 0. \tag{1.7}$$

(Indeed, the r.h.s. of (1.7) is equal to $\|\Sigma c_j W(f_j)\xi\|^2$.)

Proposition 1.2. Every $C : \mathcal{S} \to \mathbb{C}$, satisfying (i), (ii), and (iii) is the generating functional of a cyclic representation of the CCR over $\{\mathcal{S},\sigma\}$. The latter is determined up to unitary equivalence.

Proof. Let $V : \mathcal{S} \to H$ be a minimal Kolmogorov decomposition of the positive definite kernel (cf. App. B),

$$\{f,g\} \mapsto \exp(\tfrac{i}{2}\sigma(f,g))\, C(f-g). \tag{1.8}$$

Then the map $W(f)$, defined by

$$W(f)\, V(g) = \exp(\tfrac{-i}{2}\sigma(f,g))\, V(f+g), \tag{1.9}$$

extends to a unitary map on H. The W's satisfy (1.5). By (1.9) and the minimality of V, V(0) is a cyclic vector for the $W(f)$, $(f \in \mathcal{S})$. Conversely, suppose that $\{\tilde{H},\tilde{W},\tilde{\xi}\}$ is a cyclic representation of the CCR. Then $f \mapsto \tilde{W}(f)\tilde{\xi}$ is a minimal Kolmogorov decomposition of (1.9), and it follows that $\{\tilde{H},\tilde{W},\tilde{\xi}\}$ is unitarily equivalent to $\{H,W,V(0)\}$. □

We conclude that it is sufficient to specify a generating functional C, in order that our representation should be determined. Now, certain of these functionals are of a special physical interest: those determining a thermal equilibrium state of the Lamb model. We shall speak of ground states as thermal equilibrium states at temperature zero.

Let us introduce the following notation. For $\beta \in (0,\infty)$, let $\Lambda(\beta)$ denote the strip

$$\Lambda(\beta) = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z \leq \beta\}. \tag{1.10}$$

Let $\Lambda(\infty)$ be the closed upper half plane. By $\mathfrak{C}(\Lambda(\beta))$ we shall denote the space of all bounded and continuous functions $\Lambda(\beta) \to \mathbb{C}$, which are analytic on the interior of $\Lambda(\beta)$.

Definition 1.3. A generating functional C on $\{\mathcal{S},\sigma\}$ is said to satisfy the $\{T,\beta\}$- *KMS condition*, with $\beta \in (0,\infty]$, if the function

$$t \mapsto \exp\left(-\tfrac{i}{2}\sigma(f,T_t g)\right) \cdot C(f + T_t g)$$

extends to a function $G_{f,g} \in \mathfrak{C}(\Lambda(\beta))$, satisfying

$$G_{f,g}(t + i\beta) = \exp\left(\tfrac{i}{2}\sigma(f,T_t g)\right) \cdot C(f + T_t g),$$

if $\beta < \infty$.

Remarks

1. A theorem by Slawny [Sla 72] says that the C^*-algebras, generated by the $W(f)$, $(f \in \mathcal{S})$ in different representations of the CCR, are isomorphic. This implies that there exists one abstract C^*-algebra, of which a representation is given by each of the maps $W : \mathcal{S} \to \mathcal{L}(H)$. This algebra is called "the CCR-algebra over $\{\mathcal{S},\sigma\}$", and is the norm closure of the linear span of certain elements $\mathbb{W}(f)$, $(f \in \mathcal{S})$, where $\mathbb{W}$ satisfies (1.5). There is a one-to-one correspondence between the generating functionals C and states ω on this algebra via

   $$C(f) = \omega(\mathbb{W}(f)).$$

   Definition 1.3 for $\beta \in (0,\infty)$, is equivalent with the KMS-condition for the state ω. For $\beta = \infty$ it says that ω is a ground state.
2. In the above, $\mathcal{S}$ can be replaced by any real-linear space, σ by any anti-symmetric form on $\mathcal{S}$, and T by any σ-preserving flow on $\mathcal{S}$.
3. Especially if $\mathcal{S}$ is Schwartz's class, and $\sigma = 0$, C satisfies the conditions of Minlos' theorem (Theorem 5.3 of Ch. I), and W is a unitary representation of $\mathcal{S}$. A natural choice is

   $$H = L^2(\mathcal{S}',\mu),\ (\hat{\mu} = C),\ \text{and}\ W(f)(x) = e^{i\langle x,f\rangle}.$$

§ 2. QUANTUM WHITE NOISE

In this section we shall find all generating functionals, satisfying the $\{T,\beta\}$-KMS condition for some $\beta \in (0,\infty]$, and explicitly construct the associated representations of the canonical commutation relation. The construction is similar to that of Araki and Woods for the free Bose gas [ArW 63], and it generalises the construction of a Bose field over the "Boson single particle space" $\{\mathcal{S},\sigma,T\}$ in the tradition of Segal and Weinless, [Seg 59] , [Wei 69], to the case of a positive temperature.

Generating functionals at $T \geq 0$.

Let the functions ρ_β, $\beta \in [-\infty,\infty]\backslash\{0\}$ be defined as follows

$$\rho_\beta(k) = \frac{k}{1-e^{-\beta k}}, \quad \beta \in (-\infty,\infty)\backslash\{0\} \tag{2.1}$$

$$\rho_{\pm\infty}(k) = \theta(\pm k)k, \tag{2.2}$$

and let $R_\beta : \mathcal{S}\times\mathcal{S} \to \mathbb{C}$ be given by

$$R_\beta(f,g) = \int_{-\infty}^{\infty} \rho_\beta(k)\,\hat{f}(k)^*\hat{g}(k)\,\frac{dk}{2\pi}. \tag{2.3}$$

Then, because $e^{-\beta k}\rho_\beta(k) = \rho_\beta(-k)$, $t \mapsto R_\beta(f,T_t g)$ extends to a function $F_{f,g} \in \mathbb{C}(\Lambda(\beta))$, satisfying $F_{f,g}(t+i\beta) = R_\beta(T_t g,f)$. Moreover, because $\rho_\beta(k) - \rho_\beta(-k) = k$, we have $\mathrm{Im}\, R_\beta(f,g) = \frac{1}{2}\sigma(f,g)$. One checks that the generating functional

$$C_\beta(f) = \exp(-\tfrac{1}{2}R_\beta(f,f)) \tag{2.4}$$

satisfies the $\{T,\beta\}$-KMS condition.

However, C_β is not the only $\{T,\beta\}$-KMS functional. If $\varphi : \mathbb{R}\to\mathbb{C}$ is continuous and positive definite, with $\varphi(0) = 1$, then the functional $C_{\beta,\varphi}$ given by

$$C_{\beta,\varphi}(f) = C_\beta(f)\cdot\varphi(\hat{f}(0)), \tag{2.5}$$

also satisfies the $\{T,\beta\}$-KMS condition, (compare (I.5.9)).

Proposition 2.1. Every $\{T,\beta\}$-KMS functional on $\{\mathcal{S},\sigma\}$ is of the form (2.5). For the proof we need a lemma:

Lemma 2.2. If $f \in \mathcal{S}$ is such that $\hat{f}(0) = 0$, then for some $g \in \mathcal{S}$:

$$f(t) = 2g(t) - g(t-1) - g(t-\sqrt{2}). \tag{2.6}$$

Proof. Put

$$\hat{g}(\lambda) = \hat{f}(\lambda)/(2-e^{i\lambda}-e^{i\lambda\sqrt{2}}). \tag{2.7}$$

By the irrationality of $\sqrt{2}$, the denominator has no zeroes, except $\lambda = 0$. What is more, outside $(-\varepsilon,\varepsilon)$, $(\varepsilon>0)$, $(2-e^{i\lambda}-e^{i\lambda\sqrt{2}})^{-1}$ is bounded by a quadratic in λ. Therefore $\hat{g} \in \mathcal{S}$. The statement follows by Fourier transformation. □

Proof of Proposition 2.1. (After [RST 70]) let C be a $\{T,\beta\}$-KMS functional on $\{\mathcal{S},\sigma\}$. We claim that for all $f,g \in \mathcal{S}$, the function

$$t \mapsto C(f+T_t g)/C_\beta(f+T_t g) \tag{2.8}$$

is a constant.

Indeed, because both $t \mapsto e^{-\frac{1}{2}i\sigma(f,T_t g)}C(f+T_t g)$ and $t \mapsto e^{-\frac{1}{2}i\sigma(f,T_t g)}C_\beta(f+T_t g)$ extend to functions in $\mathbb{C}(\Lambda(\beta))$, and because the second function is bounded from below in norm on $\Lambda(\beta)$, also (2.8) extends to a function $\gamma \in \mathbb{C}(\Lambda(\beta))$. Now, if $\beta \in (0,\infty)$, we find that $\gamma(t+i\beta) = \gamma(t)$ for all $t \in \mathbb{R}$, hence γ is a constant.

On the other hand, if $\beta = \infty$, then

$$\gamma(t) = C(g+T_{-t}f)\,/\,C_\beta(g+T_{-t}f),$$

extends to a function in $\mathbb{C}(\Lambda(-\infty))$ as well. Again, it follows that (2.8) is a constant.

Using Lemma (2.2) one derives from this that $C(f)/C_\beta(f)$ depends only on $\hat{f}(0)$; i.e., there is a continuous function $\varphi\colon \mathbb{R}\to\mathbb{C}$ with $\varphi(0) = 1$, such that $C(f) = C_\beta(f)\cdot\varphi(\hat{f}(0))$. It remains to show that φ is positive definite.

Choose a sequence $\{f_n \in \mathcal{S}\}$ with $\hat{f}_n(0) = 1$ and such that $R_\beta(f_n,f_n) \to 0$, $(n\to\infty)$. Then, for all $n \in \mathbb{N}$, and all $c_1,\ldots,c_m \in \mathbb{C}$, $f_1,\ldots,f_m \in \mathcal{S}$:

$$0 \leq \sum_{j,k=1}^{m} c_j c_k^* C((\lambda_j-\lambda_k)f_n) = \sum_{j,k=1}^{m} c_j c_k^*\varphi(\lambda_j-\lambda_k)\cdot\exp(-\tfrac{1}{2}(\lambda_j-\lambda_k)^2 R_\beta(f_n,f_n)).$$

Taking $n\to\infty$, we find that

$$\sum_{j,k=1}^{m} c_j c_k^*\varphi(\lambda_j-\lambda_k) \geq 0.$$

□

The spaces H_F *and* H_B

Consider the symplectic space $\{L^2(\mathbb{R})\ ,\ 2\ \mathrm{Im} < \cdot\ ,\ \cdot >\}$. Let the generating functional C_F on this space be given by

$$C_F(h) = \exp(-\tfrac{1}{2}\|h\|^2)\ . \tag{2.9}$$

It is well known that $\{H_F\ ,W_F\ ,1_F\}$ is the representation of the CCR over $\{L^2(\mathbb{R})\ ,\ 2\ \mathrm{Im} < \cdot\ ,\ \cdot >\}$, associated to C_F. Here, H_F is the symmetric Fock space over $L^2(\mathbb{R})$, $1_F = 1 \oplus 0 \oplus 0 \oplus \ldots$, and W_F is given by

$$W_F(h)\ \mathrm{coh}_F(\ell) = e^{-i\ \mathrm{Im}<h,\ell>}\ \mathrm{coh}_F(h+\ell)\ , \tag{2.10}$$

where $\mathrm{coh}_F : L^2(\mathbb{R}) \to H_F$ is the Kolmogorov decomposition of the kernel

$$\{h,\ell\} \mapsto e^{i\ \mathrm{Im}<h,\ell>}\ C_F(h-\ell)\ , \tag{2.11}$$

given by

$$\mathrm{coh}_F(h) = C_F(h)\ \mathrm{Exp}_F(h) = e^{-\frac{1}{2}\|h\|^2} \cdot \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\ h^{\otimes n}\ . \tag{2.12}$$

Another Kolmogorov decomposition of (2.11) is $\mathrm{coh}_B : L^2(\mathbb{R}) \to H_B$, given by

$$\mathrm{coh}_B(h) = e^{-\frac{1}{2}\|h\|^2} \cdot \mathrm{Exp}_B(h)\ , \tag{2.13}$$

where $\mathrm{Exp}_B(h) = \exp(-i< \cdot\ ,h> + \tfrac{1}{2}\|\mathrm{Re}\,h\|^2 - \tfrac{1}{2}\|\mathrm{Im}\,h\|^2 + i<\mathrm{Re}\,h\ ,\ \mathrm{Im}\,h>)$. (2.14)

($\mathrm{Exp}_B(h)$ corresponds to $:\exp(-i<\cdot\ ,h>)$: in the notation of many field theory books, e.g. [Sim 74], [GlJ 81]). We define W_B as in (2.10), with coh_F replaced by coh_B.

The structures on H_F and H_B correspond under the isomorphism i_F (cf. § 5, Ch. I).

Representations of the CCR at $T \geqslant 0$

Let us construct representations of the CCR over $\{\mathfrak{S},\sigma\}$, associated to C_β, $(0 < \beta \leqslant \infty)$ in the way, indicated in the proof of Proposition 1.2. The kernel (1.8), with C replaced by C_β, is

$$\{f,g\} \mapsto C_\beta(f)C_\beta(g)\exp(R_\beta(f,g))\ . \tag{2.15}$$

Now, let $\mathfrak{v}'_\beta : \mathfrak{S} \to L^2(\mathbb{R})$ be given by

$$(\mathfrak{v}'_\beta f)(t) = \int_{-\infty}^{\infty} e^{-ikt}\ \rho_\beta(k)^{\frac{1}{2}}\ \hat{f}(k)\ \frac{dk}{2\pi}\ . \tag{2.16}$$

Then $<\mathfrak{v}'_\beta f, \mathfrak{v}'_\beta g> = R_\beta(f,g)$, and therefore

$$f \mapsto \mathrm{coh}_B(\mathfrak{v}'_\beta f) \tag{2.17}$$

is a decomposition of (2.15).

Our representation is the following:

$$\{H_B\ ,W_B \circ \mathfrak{v}'_\beta\ ,1\}\ . \tag{2.18}$$

Proposition 2.3. Any representation of the CCR over $\{\mathfrak{S},\sigma\}$ at temperature $\geqslant 0$ is equivalent with

$$\{H_B \otimes L^2(\mathbb{R},\nu)\ ,\ f \mapsto W_B(\mathfrak{v}'_\beta(f)) \otimes e^{-i\hat{f}(0)\cdot}\ ,\ 1 \otimes 1\}\ , \tag{2.19}$$

where ν is some probability measure on $\mathbb{R}$, and $\exp(-i\hat{f}(0)\cdot)$ is a multiplication operator on $L^2(\mathbb{R},\nu)$.

Proof. This is a direct consequence of Proposition 2.1 and the above construction. □

Remark. We see that the quantisation of the Lamb model is not yet uniquely determined by Definition 1.3, as one might have expected it to be. The existence of more than one thermal equilibrium state at one temperature, a feature which the classical and the quantum Lamb model share (cf. § 5, Ch. I), is usually interpreted as the coexistence of different phases of the same material. Here, the material consists of free phonons in the string, and the phenomenon is much like Bose-Einstein condensation: part of the phonons condense into the "ground state" with "wave function" b' = 1.

In terms of the string, the measure ν, occurring in Proposition 2.3, is the probability distribution of a non-local observable: the systematic slope that is superimposed on the fluctuations in the string.

As we did in the classical case, we signal this phenomenon, but neglect it afterwards. Our interest is in a quantum white noise, comparable to the ordinary white noise, i.e., without systematic deviations.

Quantum white noise

Define $E'_\beta(f)$ by

$$\exp(-i\,E'_\beta(f)) = W_B(\mathfrak{v}'_\beta f)\ . \tag{2.20}$$

For $\beta \in (0,\infty)$, we define $\mathfrak{v}_\beta = \sqrt{\beta}\,\mathfrak{v}'_\beta$ and

$$E_\beta(f) = \sqrt{\beta}\, E'_\beta(f). \tag{2.21}$$

By construction, E_β satisfies the CCR over $\{\mathcal{S}, \beta\sigma\}$:

$$[E_\beta(f), E_\beta(g)] = i\beta\sigma(f,g)\,\mathbb{1}. \tag{2.22}$$

Its covariance is given by

$$\langle 1, E_\beta(f)\, E_\beta(g)\, 1\rangle = \int_{-\infty}^{\infty} \frac{\beta k}{1 - e^{-\beta k}}\, \hat{f}(k)^* \hat{g}(k)\, \frac{dk}{2\pi}. \tag{2.23}$$

Apart from factors, (2.22) and (2.23) are equivalent with the defining formulae for the noise, used as a source in the quantum Langevin equation, in the paper by Benguria and Kac, where this equation was introduced [BeK 81]. Moreover, our operator-valued distribution E_β is clearly a quantummechanical version of the random-variable-valued distribution E_0.

For these reasons we shall call E_β *quantum white noise at inverse temperature* β.

The factor $\sqrt{\beta}$ in (2.21) is introduced to agree with the temperature-dependent vertical length scale which we have used for the classical Lamb model from § I.5 onwards, and shall use from now on for the quantum-mechanical Lamb model, (cf. Appendix A).

Note that, as a result of this choice of scale, the "classical" white noise E_0 shows up in the limit $\beta \downarrow 0$ of (2.21).

Structure of the representation

We conclude this section by an examination of the structure of the representation $\{H_B, W_B \circ \vartheta_\beta, 1\}$. We use the terminology and results of the Tomita-Takesaki theory on faithful normal states on a von Neumann algebra, [Tak 70], [BrR 79].

For all $\beta \in (-\infty,\infty)$, let $\mathfrak{M}_\beta$ be the von Neumann algebra, generated by the operators $W_B(\vartheta'_\beta f)$, $(f \in \mathcal{S})$. By the KMS-property of C_β, the state $A \mapsto \langle 1, A1\rangle$ is a (β,τ)-KMS state on $\mathfrak{M}_\beta$. It follows that 1 is not only cyclic, but also separating for $\mathfrak{M}_\beta$, i.e.

$$\forall_{A \in \mathfrak{M}_\beta}: A1 = 0 \Rightarrow A = 0. \tag{2.24}$$

Because $\tau_t(A) = \mathfrak{T}_t A \mathfrak{T}_t^*$, $\mathfrak{T}_{i\beta}$ is the *modular operator*, associated to $\{\mathfrak{M}_\beta, 1\}$. This is to say, that there exists an anti-unitary involution J_B on H_B, such that for all $A \in \mathfrak{M}_\beta$:

$$J_B\, \mathfrak{T}_{i\beta/2}\, A1 = A^* 1. \tag{2.25}$$

J_B is called the *modular conjugation*, associated to $\{\mathfrak{M}_\beta, 1\}$. An important property of J_B is the following:

$$J_B\, \mathfrak{M}_\beta\, J_B = \mathfrak{M}'_\beta. \tag{2.26}$$

Proposition 2.4. J_B is the operation of complex conjugation on H_B. Moreover,

$$\mathfrak{M}'_\beta = \mathfrak{M}_{-\beta}. \tag{2.27}$$

Proof. Let $h \in \vartheta_\beta \mathcal{S}$. By (2.16), $h \in \mathrm{Dom}(T_{i\beta/2})$ and $T_{i\beta/2} h = h^*$. It follows that $z \mapsto \langle k, T_z h\rangle$ is in $\mathfrak{C}(\Lambda(\beta/2))$ for all $k \in L^2(\mathbb{R})$, as well as

$$z \mapsto \exp\langle k, T_z h\rangle = \langle \mathrm{Exp}_B(k), \mathfrak{T}_z\, \mathrm{Exp}_B(h)\rangle \tag{2.28}$$

Putting $z = i\beta/2$, we conclude

$$\mathfrak{T}_{i\beta/2}\, \mathrm{Exp}_B(h) = \mathrm{Exp}_B(h^*). \tag{2.29}$$

On the other hand, we read off from (2.14) that

$$\mathrm{Exp}_B(h)^* = \mathrm{Exp}_B(-h^*). \tag{2.30}$$

Combining (2.29) and (2.30), we find that

$$(\mathfrak{T}_{i\beta/2}\, W_B(h)\, 1)^* = e^{-\frac{1}{2}\|h\|^2} (\mathfrak{T}_{i\beta/2}\, \mathrm{Exp}_B(h))^* = e^{-\frac{1}{2}\|h\|^2} \mathrm{Exp}_B(-h) =$$

$$= W_B(-h)\, 1 = W_B(h)^* 1. \tag{2.31}$$

By (2.25) it follows that $J_B \varphi = \varphi^*$, $(\varphi \in H_B)$. Finally, from (2.30) one shows that $J_B W_B(h) J_B = W_B(-h^*)$, and from (2.16) that $-(\vartheta_\beta f)^* = \vartheta_{-\beta} f$, $(f \in \mathcal{S})$. Therefore, $J_B\, \mathfrak{M}_\beta\, J_B = \mathfrak{M}_{-\beta}$, and by (2.26), (2.27) follows. □

Finally, let us consider the zero temperature case. We have

$$\overline{\vartheta'_\infty \mathcal{S}} = \mathcal{F}^{-1}\big(L^2([0,\infty))\big), \tag{2.32}$$

where $\mathcal{F}$ is the Fourier transform. Let $H_F^+ \subset H_F$ be the symmetric Fock space over this "positive frequency one-particle space" (2.32), and let $H_B^+ = i_F(H_F^+)$. Then

$$\overline{\mathfrak{M}_\infty 1} = H_B^+, \tag{2.33}$$

and, because ground states are irreducible, [ArW 63],

$$\mathfrak{M}_\infty = \mathcal{L}(H_B^+), \tag{2.34}$$

i.e., the set of *all* bounded operators on H_B^+.

§ 3. THE QUANTUM LANGEVIN EQUATION WITH HARMONIC POTENTIAL

Consider the classical Lamb model with harmonic potential $v(\lambda) = \frac{1}{2}\lambda^2$. The position and the momentum of the oscillator at some time t, are represented by the linear observables $\langle T_t q, \cdot\rangle$ and $\langle T_t p, \cdot\rangle$, (cf. § I.3).

To find the position and momentum operators in the quantum model, let us simply apply the quantisation map E_β, and write

$$Q_t^{(\beta,\eta)} := E_\beta\left(T_t q^{(\eta)}\right), \quad \text{and} \tag{3.1}$$

$$P_t^{(\beta,\eta)} := E_\beta\left(T_t p^{(\eta)}\right). \tag{3.2}$$

(We explicitly indicate the dependence of q and p on η). The expressions on the r.h.s. of (3.1) and (3.2) are not a priori defined, because $q^{(\eta)}$ and $p^{(\eta)}$ are not in $\mathfrak{S}$. However, E_β has sufficient continuity for its domain to be extended to contain $T_t q^{(\eta)}$. The family of operators $\{Q_t^{(\beta,\eta)}\}_{t\in\mathbb{R}}$, thus defined, satisfies the distribution version of the Langevin equation with the quantum white noise E_β as a source:

$$\frac{d^2}{dt^2} Q_t^{(\beta,\eta)} + \eta \frac{d}{dt} Q_t^{(\beta,\eta)} + Q_t^{(\beta,\eta)} = \sqrt{2\eta}\, E_{\beta,t}. \tag{3.3}$$

To (3.2), on the other hand, no meaning can reasonably be attributed if $\beta \neq 0$. In the quantum Lamb model, the oscillator moves about too wildly to have a momentum operator.

This section will be devoted to the proof of the above statements. We shall slightly generalise them, to make them applicable to all quantum stochastic differential equations like (3.3) with a polynomial in d/dt on the l.h.s.

But first, let us note that on a formal level, (3.3) is entirely trivial as the following computation shows:

$$\left(\frac{d^2}{dt^2} + \eta\frac{d}{dt} + 1\right) E_\beta\left(T_t q^{(\eta)}\right) = E_\beta\left(T_t\left((\partial^2 - \eta\partial + 1)q^{(\eta)}\right)\right) =$$

$$= \sqrt{2\eta}\, E_\beta(T_t \delta) = \sqrt{2\eta}\, E_{\beta,t}.$$

Linear quantum stochastic differential equations

We extend the domain of E_β as follows. Define the norm $\|\cdot\|_\beta$ on $\mathfrak{S}$ by

$$\|f\|_\beta = \|\mathfrak{D}_\beta f\|. \tag{3.4}$$

(Let $\mathfrak{D}_0 f = f$). Let $\overline{\mathfrak{S}}^\beta$ be the completion of $\mathfrak{S}$ in this norm. Clearly, $\mathfrak{D}_\beta$ extends to an isometry

$$\overline{\mathfrak{D}}_\beta : \overline{\mathfrak{S}}^\beta \to L^2(\mathbb{R}).$$

Now, define E_B by $W_B(h) = \exp(-i E_B(h))$, and $\overline{E}_\beta$ by

$$\overline{E}_\beta = \begin{cases} E_B \circ \overline{\mathfrak{D}'_\beta}, & (\beta = \pm\infty), \\ E_B \circ \overline{\mathfrak{D}_\beta}, & (\beta \in \mathbb{R}). \end{cases} \tag{3.5}$$

A linear observable f can be *quantised* if and only if it is in $\overline{\mathfrak{S}}^\beta$. Its *quantisation* is the operator $\overline{E}_\beta(f)$.

Now, let $\mathfrak{P}$ be a polynomial of degree $n \geq 1$ with real coefficients, and without zeroes on the imaginary axis. Consider the formal differential equation

$$\mathfrak{P}\left(\frac{d}{dt}\right) X_t = E_{\beta,t}. \tag{3.6}$$

Here, $E_{\beta,t}$ is to be read as $E_\beta(\delta_t)$, where $\delta_t(s) = \delta(s-t)$, and δ is Dirac's delta function. Clearly, $\delta_t \notin \overline{\mathfrak{S}}^\beta$, so $E_{\beta,t}$ does not exist as an operator. Equation (3.6) will be interpreted in a distribution sense (compare § I.6).

Definition 3.1. Let $\beta \in [0,\infty]$. A family $\{X_t\}_{t\in\mathbb{R}}$ of self-adjoint operators on H_B is said to *solve the (quantum) stochastic differential equation* (3.6), if there exists a dense linear subspace D of H_B containing 1, of analytic vectors for each of the X_t, $(t \in \mathbb{R})$, and $E_\beta(f)$, $(f \in \mathfrak{S})$, such that for all $\varphi \in D$ the function $t \mapsto \|X_t\varphi\|$ is tempered, and for all $f \in \mathfrak{S}$:

$$\int_{-\infty}^{\infty} (\mathfrak{P}(-\partial) f)(t) X_t \varphi\, dt = E_\beta(f)\varphi. \tag{3.7}$$

Remark. With this definition, $\{X_t\}_{t\in\mathbb{R}}$ is automatically *stationary*, i.e.,

$$\forall_{t\in\mathbb{R}} : X_t = \tau_t(X_0). \tag{3.8}$$

Indeed, the requirement of temperedness for $t \mapsto \|X_t\varphi\|$ rules out homogeneous solutions, because these diverge exponentially as $t \to \infty$ or as $t \to -\infty$. Now, if $\{X_s\}_{s \in \mathbb{R}}$ is a solution, $\{\tau_{-t}(X_{s+t})\}_{s \in \mathbb{R}}$ also is one, and hence (3.8) holds.

Because of this fact, we may as well say that *the operator* X_0 *satisfies* (3.6).

Now, given $\mathfrak{P}$, let the function $q_{\mathfrak{P}}$ be the unique tempered solution of

$$\mathfrak{P}(-\partial)\, q_{\mathfrak{P}} = \delta. \tag{3.9}$$

Proposition 3.2. The (quantum) stochastic differential equation (3.6) has a solution if and only if

$$q_{\mathfrak{P}} \in \overline{\mathcal{S}}^{\beta}. \tag{3.10}$$

This solution is unique and given by

$$X_t = \overline{E}_\beta(T_t\, q_{\mathfrak{P}}). \tag{3.11}$$

To prove this proposition, a lemma is needed.

Lemma 3.3. Let $\ell \in L^2(\mathbb{R})$. Then $\mathrm{coh}_B(\ell)$ is an analytic vector for each of the operators $E_B(h)$, $(h \in L^2(\mathbb{R}))$. The map

$$h \mapsto E_B(h)\, \mathrm{coh}_B(\ell) \tag{3.12}$$

is continuous.

Corollary 3.4. If $t \mapsto h_t$ is a bounded and continuous curve in $\overline{\mathcal{S}}^{\beta}$, and $g \in L^1(\mathbb{R})$, then for all $\ell \in L^2(\mathbb{R})$:

$$\overline{E}_\beta\Big(\Big(\int_{-\infty}^{\infty} g(t) h_t\, dt\Big)\, \mathrm{coh}(\ell)\Big) = \int_{-\infty}^{\infty} g(t)\Big(\overline{E}_\beta(h_t)\, \mathrm{coh}(\ell)\Big)\, dt.$$

Proof. Approximate g by stepfunctions, and use the continuity of (3.12), and of $\overline{\mathfrak{D}}_\beta$. □

Proof of Lemma 3.3. For all $h, \ell \in L^2(\mathbb{R})$, the function $f_{h,\ell} : \mathbb{R} \to \mathbb{C}$, defined by

$$f_{h,\ell}(\lambda) = \langle \mathrm{coh}_B(\ell),\, W_B(\lambda h)\, \mathrm{coh}_B(\ell)\rangle = \exp(-\tfrac{1}{2}\lambda^2\|h\|^2 - 2i\lambda\, \mathrm{Im}\langle h, \ell\rangle)$$

is the restriction to $\mathbb{R}$ of an entire analytic function. It follows that $\mathrm{coh}_B(\ell)$ is an analytic vector for $E_B(h)$. Now,

$$\|E_B(h)\mathrm{coh}_B(\ell)\|^2 = -f''_{h,\ell}(0) = \|h\|^2 + 4(\mathrm{Im}\langle h,\ell\rangle)^2 \leqslant \|h\|^2(1 + 4\|\ell\|^2).$$

So $h \mapsto E_B(h)\mathrm{coh}(\ell)$ is continuous. □

Proof of Proposition 3.2. Suppose $q_{\mathfrak{P}} \in \overline{\mathcal{S}}^{\beta}$. The maps $f \mapsto q_{\mathfrak{P}} * f$ and $f \mapsto \mathfrak{P}(-\partial) f$ are each other's inverse as maps $\mathcal{S} \to \mathcal{S}$ (this is similar to Lemma I.6.2), and therefore, for all $f \in \mathcal{S}$, and all $\ell \in L^2(\mathbb{R})$:

$$E_\beta(f)\mathrm{coh}_B(\ell) = E_\beta(q_{\mathfrak{P}} * \mathfrak{P}(-\partial) f)\mathrm{coh}_B(\ell) = E_\beta\Big(\Big(\int_{-\infty}^{\infty} (\mathfrak{P}(-\partial)f)(t)\, T_t\, q_{\mathfrak{P}}\, dt\Big)\, \mathrm{coh}_B(\ell)\Big) =$$

$$= \int_{-\infty}^{\infty} (\mathfrak{P}(-\partial)f)(t)\Big(\overline{E}_\beta(T_t\, q_{\mathfrak{P}})\, \mathrm{coh}_B(\ell)\Big)\, dt\,. \tag{3.13}$$

Here, we used Corollary 3.4 and the $\overline{\mathcal{S}}^{\beta}$-continuity of $t \mapsto T_t\, q_{\mathfrak{P}}$. The conditions of Definition 3.1 are satisfied with X_t given by (3.11), and D by

$$D = \mathrm{span}\Big(\mathrm{coh}_B(L^2(\mathbb{R}))\Big). \tag{3.14}$$

The uniqueness of the solution follows from the fact that $\mathfrak{P}$ has no zeroes in $i\mathbb{R}$, (see remark, preceding Prop. 3.2).

Conversely, suppose that $\{X_t\}$ is a solution of (3.6). Then $X_t = \mathfrak{T}_t X_0 \mathfrak{T}_t^*$ and $1 \in \mathrm{Dom}(X_0)$. Let ν be the measure on $\mathbb{R}$, satisfying

$$\langle X_0 1, X_t 1\rangle = \int_{-\infty}^{\infty} e^{ikt}\, \nu(dk). \tag{3.15}$$

Then (3.7) implies that for all $f \in \mathcal{S}$,

$$\|E_\beta(f)\,1\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathfrak{P}(-\partial)f)(t)\,(\mathfrak{P}(-\partial)f)(s)\Big(\int_{-\infty}^{\infty} e^{ik(s-t)}\, \nu(dk)\Big)\, ds\, dt =$$

$$= \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \cdot |\mathfrak{P}(ik)|^2\, \nu(dk).$$

On the other hand,

$$\|E_\beta(f)\,1\|^2 = (\beta \text{ or } 1) \cdot \int_{-\infty}^{\infty} \rho_\beta(k)\, |\hat{f}(k)|^2\, \frac{dk}{2\pi}\,.$$

Therefore,

$$\nu(dk) = (\beta \text{ or } 1) \cdot \frac{\rho_\beta(k)}{|\mathfrak{P}(ik)|^2} \cdot \frac{dk}{2\pi}\,,$$

and $\|q_{\mathfrak{P}}\|_\beta^2 = \nu(\mathbb{R}) = \|X_0 1\|^2 < \infty$. □

Remarks. If $\beta = 0$, then $\|q_{\mathfrak{P}}\|_\beta^2 = \|q_{\mathfrak{P}}\|^2$ is finite for all $\mathfrak{P}$ of degree $n \geqslant 1$. But, if $\beta \neq 0$, n has to be strictly greater than 1. As a consequence, the

Ornstein-Uhlenbeck velocity process, which is the solution of (3.6) with $\mathcal{P}(d/dt) = \eta + d/dt$, has no quantum version. (We recall that the Ornstein-Uhlenbeck velocity process is the stochastic process $\{P_t\}_{t \in \mathbb{R}}$, where P_t is the velocity at time t of a particle on a line, subject to noise and friction only).

In the model we are considering, the polynomial $\mathcal{P}$ is

$$\mathcal{P} = \frac{1}{\sqrt{2\eta}} \mathcal{P}_\eta, \text{ with } \mathcal{P}_\eta(x) = x^2 + \eta x + 1. \tag{3.16}$$

We have

$$\|q^{(\eta)}\|^2_\beta = \int_{-\infty}^{\infty} \frac{\beta k}{1 - e^{-\beta k}} \cdot \frac{2\eta}{(k^2-1)^2 + \eta^2 k^2} \cdot \frac{dk}{2\pi} < \infty. \tag{3.17}$$

However,

$$\|p^{(\eta)}\|^2_\beta = \int_{-\infty}^{\infty} \frac{\beta k}{1 - e^{-\beta k}} \cdot \frac{2\eta k^2}{(k^2-1)^2 + \eta^2 k^2} \cdot \frac{dk}{2\pi} \tag{3.18}$$

diverges if $\beta \neq 0$. Hence $p^{(\eta)} \notin \mathcal{S}^\beta$ for $\beta \neq 0$, and there is no momentum operator for the oscillator in the quantum string model.

In the same vein, there does not exist quantum Brownian motion of the type $B_t^{(\beta)} = \bar{E}_\beta(\chi_{[0,t]})$, because $\chi_{[0,t]} \notin \mathcal{S}^\beta$ for $\beta, t \neq 0$.

§ 4. THE QUANTUM LANGEVIN EQUATION WITH ANHARMONIC POTENTIAL

Consider the Quantum Langevin Equation (QLE)

$$\frac{d^2}{dt^2} X_t + \eta \frac{d}{dt} X_t + v'(X_t) = \sqrt{2\eta}\, E_{\beta,t}, \tag{4.1}$$

with $0 \leq \beta \leq \infty$ and $\eta > 0$. As in § I.9, we take v of the form

$$v(\lambda) = \tfrac{1}{2}\lambda^2 + w(\lambda), \tag{4.2}$$

where w is differentiable with bounded and continuous derivative. For short, we call equation (4.1) the *QLE given by* $\{\beta, \eta, w\}$.

Definition 4.1. A family $\{X_t\}_{t \in \mathbb{R}}$ of self-adjoint operators on H_B will be called *a stationary solution of the QLE given by* $\{\beta,\eta,w\}$, if for all $t \in \mathbb{R}$:

$$X_t = \tau_t(X_0), \quad \mathrm{Dom}(X_t) = \mathrm{Dom}(Q_t^{(\beta,\eta)}),$$

and for all $\varphi \in \mathrm{Dom}(Q_t^{(\beta,\eta)})$

$$X_t\varphi + \int_{-\infty}^{t} \frac{1}{\sqrt{2\eta}} q(s-t) w'(X_s)\varphi\, ds = Q_t^{(\beta,\eta)}\varphi. \tag{4.3}$$

Motivation. Suppose that (4.3) holds. Again let $D \subset H_B$ be the span of $\mathrm{coh}_B(L^2(\mathbb{R}))$. As the integral in (4.3) is uniformly bounded in φ, the function $t \mapsto \|X_t\varphi\|$ is tempered for all $\varphi \in D$. Therefore, for all $g \in \mathcal{S}$,

$$\int_{-\infty}^{\infty} g(t) X_t \varphi\, dt + \int_{-\infty}^{\infty} g(t) \left(\int_{-\infty}^{t} \frac{q(s-t)}{\sqrt{2\eta}} w'(X_s)\varphi\, ds \right) dt = \int_{-\infty}^{\infty} g(t) Q_t^{(\beta,\eta)} \varphi\, dt. \tag{4.4}$$

Now, put $f = \frac{1}{\sqrt{2\eta}} q * g$. By Lemma I.6.2, f takes all values in $\mathcal{S}$, and $g = f'' - \eta f' + f$. Therefore, for all $f \in \mathcal{S}$, all $\varphi \in D$:

$$\int_{-\infty}^{\infty} (f'' - \eta f' + f)(t) X_t \varphi\, dt + \int_{-\infty}^{\infty} f(t) w'(X_t)\varphi\, dt = \sqrt{2\eta}\, E_\beta(f)\varphi, \tag{4.5}$$

because $\{Q_t^{(\beta,\eta)}\}$ solves the unperturbed QLE.

Evidently, (4.5) is a distribution version of (4.1) in exactly the same manner as (3.7) is a distribution version of (3.6). For this reason, we shall deal with (4.3), which is a well-defined, and tractible version of the QLE. Note that, in fact, all domain difficulties have disappeared, because the integral in (4.3) is a bounded operator. The form (4.3) of the QLE was also used in [BeK 81].

The following existence and uniqueness theorem for stationary solutions of the QLE is analogous to the "pathwise perturbation" result, Theorem I.9.6. Because of noncommutativity a little more care has to be taken. We introduce some notation.

Let M be the Banach space of all complex measures μ on $\mathbb{R}$ of finite total variation, and satisfying

$$\forall_{S \in \mathcal{B}(\mathbb{R})} : \mu(-S) = \mu(S)^*. \tag{4.6}$$

As the norm $\|\mu\|_M$ of μ we take its total variation. The total variation measure will be denoted by μ^+. Especially, $\mu^+(\mathbb{R}) = \|\mu\|_M$. Let $\hat{M}$ be the linear space of real functions of the form

$$f(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} \mu(dx), \quad (\mu \in M). \tag{4.7}$$

$\hat{M}$ is a Banach space in the norm $\|f\|_{\hat{M}} := \|\mu\|_M$.

Theorem 4.2. Let $w \in \hat{M}$ be such that also $w'' \in \hat{M}$, and let $v(\lambda) = \frac{1}{2}\lambda^2 + w(\lambda)$. Let $\eta > 0$ and suppose that

$$\|w''\|_{\hat{M}} < \frac{|\int q^{(\eta)}\,dt|}{\int |q^{(\eta)}|\,dt}\,. \tag{4.8}$$

Then, for all $\beta \in [0,\infty]$ the QLE given by $\{\beta,\eta,w\}$ has a unique stationary solution

$$\{Q_t^{(\beta,\eta,w)} = \tau_t(Q^{(\beta,\eta,w)})\}_{t\,\in\,\mathbb{R}}\,.$$

Lemma 4.3. Let Q and A be self-adjoint operators on some Hilbert space H, one of which, A, is bounded. Suppose that f and f' are in $\hat{M}$. Then

$$\|f(Q+A) - f(Q)\| \leqslant \|f'\|_{\hat{M}} \cdot \|A\|\,.$$

Proof. First note that for all $\lambda \geqslant 0$,

$$\left\|e^{i\lambda(Q+A)} - e^{i\lambda Q}\right\| = \left\|e^{i\lambda(Q+A)}e^{-i\lambda Q} - \mathbb{1}\right\| = \left\|\int_0^\lambda e^{i\lambda'(Q+A)}(iA)\,e^{-i\lambda' Q}\,d\lambda'\right\| \leqslant \lambda \cdot \|A\|.$$

Now, let $f = \hat{\mu}$, with $\mu \in M$ such that $\int |\lambda|\,\mu^+(d\lambda) < \infty$. Then

$$\|f(Q+A) - f(Q)\| = \left\|\int_{-\infty}^{\infty} (e^{i\lambda(Q+A)} - e^{i\lambda Q})\,\mu(d\lambda)\right\| \leqslant \int_{-\infty}^{\infty} |\lambda| \cdot \|A\| \cdot \mu^+(d\lambda) = \|f'\|_{\hat{M}} \cdot \|A\|\,.$$

□

Proof of the theorem. We drop indices β and η. Let $\mathfrak{M}^{(s.a.)}_{(-\infty,0]}$ be the Banach space of all self-adjoint elements of the algebra $\mathfrak{M}_{(-\infty,0]} = \{Q_t \mid t \in (-\infty,0]\}''$, and define the (nonlinear) map $\theta : \mathfrak{M}^{(s.a.)}_{(-\infty,0]} \to \mathfrak{M}^{(s.a.)}_{(-\infty,0]}$ by

$$\theta(A) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\eta}}\, q(s) w'(\tau_s(Q+A))\,ds\,. \tag{4.9}$$

Then, for all $A,B \in \mathfrak{M}^{(s.a.)}_{(-\infty,0]}$,

$$\|\theta(A) - \theta(B)\| = \left\|\int_{-\infty}^{0} \frac{1}{\sqrt{2\eta}}\, q(s-t)\tau_s(w'(Q+A) - w'(Q+B))\,ds\right\|$$

$$\leqslant \frac{1}{\sqrt{2\eta}} \int_{-\infty}^{0} |q(s)| \cdot \|w'(Q+A) - w'(Q+B)\|\,ds \leqslant \|w''\|_{\hat{M}} \cdot \frac{\int|q|\,ds}{|\int q\,ds|} \cdot \|A-B\|,$$

because $\int q\,ds = \sqrt{2\eta}$. From (4.8) it follows that $\|\theta(A) - \theta(B)\| < c\|A-B\|$ with $c < 1$, and therefore θ has a unique fixed point $A_w \in \mathfrak{M}^{(s.a.)}_{(-\infty,0]}$. Put

$$Q^w = Q + A_w\,. \tag{4.10}$$

Then $Q_t^w := \tau_t(Q^w)$ is the unique solution of (4.3).

□

Remarks. 1. The condition (4.8) is somewhat more restrictive than the Lipschitz condition (I.9.12) with (I.9.13), which suffices in the classical case. Indeed, we have, if $w, w'' \in \hat{M}$,

$$\sup_{\lambda_1 \neq \lambda_2} \frac{|w'(\lambda_1) - w'(\lambda_2)|}{|\lambda_1 - \lambda_2|} \leqslant \sup |w''(\lambda)| \leqslant \|w''\|_{\hat{M}}\,. \tag{4.11}$$

We do not know, whether the weakening of the result in comparison with the classical case is inevitable, or just a matter of insufficient methods.

2. Note that Q^w is affiliated with $\mathfrak{M}_{(-\infty,0]}$, to be called the "past" in the next section.

§ 5. APPROACH TO EQUILIBRIUM

Breakdown of the Markov property

The solutions of the quantum Langevin equations, treated in the previous sections, are not Markov processes.

We shall briefly explain this statement.

In the theory of quantum stochastic processes that is emerging in the present time, (cf., for instance, [Dav 76], [Lin 76], [Acc 75], [Lin 79], [AFL 82]), several quantummechanical generalisations of the concept of a Markov process have been given. As they are all practically equivalent, let us cite the one in [Acc 75], that has a convenient algebraic form.

A W^*-*stochastic process* for a physical system that has a von Neumann algebra $\mathfrak{N}$ as its observable algebra, is a family $\{j_t\}_{t\,\in\,\mathbb{R}}$ of *-morphisms from $\mathfrak{N}$ into a ("big") von Neumann algebra $\mathfrak{M}$ on some Hilbert space H, together with a cyclic vector $\xi \in H$. The process is called *stationary* if there is a group $\{\alpha_t\}_{t\,\in\,\mathbb{R}}$ of *-automorphisms of $\mathfrak{M}$, such that $j_{t+s} = \alpha_t \circ j_s$ and $\langle\xi, \alpha_t(A)\xi\rangle = \langle\xi, A\xi\rangle$, $(t,s \in \mathbb{R};\ A \in \mathfrak{M})$.

Given such a stationary W^*-stochastic process, one defines, for $I \subset \mathbb{R}$,

$$\mathfrak{M}_I = \{j_t(\mathfrak{N}) \mid t \in I\}''\,. \tag{5.1}$$

We shall suppose that $\mathfrak{M}_{\mathbb{R}} = \mathfrak{M}$. The algebras $\mathfrak{M}_{(-\infty,0]}$, $\mathfrak{M}_{\{0\}}$ and $\mathfrak{M}_{[0,\infty)}$ are called the *past*, the *present* and the *future* respectively.

The process is called a *Markov process* if

$$\mathbb{E}_{(-\infty,0]}\mathfrak{M}_{[0,\infty)} = \mathfrak{M}_{\{0\}}, \tag{5.2}$$

where $\mathbb{E}_{(-\infty,0]}$ is a *conditional expectation* w.r.t. the past, *compatible with* ξ, i.e. a projection $\mathfrak{M} \to \mathfrak{M}_{(-\infty,0]}$, such that for all $A \in \mathfrak{M}$ and all $B \in \mathfrak{M}_{(-\infty,0]}$,

$$\mathbb{E}_{(-\infty,0]}(A)B = \mathbb{E}_{(-\infty,0]}(AB), \quad \text{and} \tag{5.3}$$

$$\langle \xi, \mathbb{E}_{(-\infty,0]}(A)\xi\rangle = \langle \xi, A\xi\rangle. \tag{5.4}$$

The relation (5.2) says that the expectation value of any future variable, based on a knowledge of the entire past, only depends on the present. Stated in these terms, it coincides with the probabilistic definition of a Markov process (cf. § I.8, [Doo 53]). The generalisation lies in the applicability of (5.2) to non-commutative algebras of "stochastic variables".

A necessary precondition for (5.2) is the existence of a conditional expectation onto the past. Now, a theorem of Takesaki says that, if ξ is separating for $\mathfrak{M}$, and α is the associated modular automorphism group, there only exist conditional expectations onto subalgebras of $\mathfrak{M}$, which are α-invariant, [Tak 72]. The past $\mathfrak{M}_{(-\infty,0]}$ is not such a subalgebra, unless $\mathfrak{M} = \mathfrak{M}_{\{0\}}$. But then the entire process collapses to the process of an isolated system. Therefore, in thermal equilibrium, there are no proper quantum Markov processes.

In the case of the quantum string model, the situation is even worse than this. Not only in a representation of the CCR over $\{\mathfrak{S},\sigma\}$, associated to thermal equilibrium, but in any representation, there does not exist a momentum operator. The algebra $\mathbb{N}$ merely consists of functions of the position of the oscillator, and is clearly too small for (5.2) to hold.

The relevance of the Markov property (5.2) lies in the fact that, if it holds, a family $\{Z_t : \mathbb{N} \to \mathbb{N}\}_{t \geq 0}$ can be defined by

$$Z_t = j_0^{-1} \circ E_{(-\infty,0]} \circ j_t, \tag{5.5}$$

and constitutes a semigroup of completely positive maps on the small

system's algebra $\mathbb{N}$. It provides a "reduced description" of the small system's dynamics, and thus it generalises the Fokker-Planck semigroup for an ordinary Markov process, (cf. § I.8). It has been shown, [AFL 82] that the Markov property is not only a sufficient, but also a necessary condition for the existence of such a reduced description.

The quantum string model does not admit a reduced description of the dynamics of the oscillator. It seems likely that this is the reason, why so many attempts at a quantummechanical treatment of the damped (harmonic) oscillator *in isolation*, have met with severe difficulties, [Mes 79],[Dek 81].

Use of the mixing property

Because initial probability distributions of the oscillator's position do not determine the later distributions, Definition I.8.1 of an attracting equilibrium measure does not apply in quantum mechanics. We replace it by the following.

Definition 5.1. A measure ν_0 on $\mathbb{R}$ will be called an *attracting equilibrium measure for the QLE (4.1)* if there exists a stationary solution $\{X_t\}_{t \in \mathbb{R}}$ of this equation, such that for all times t

$$\langle 1, e^{-i\lambda X_t} 1\rangle = \hat{\nu}_0(\lambda), \quad (\lambda \in \mathbb{R}), \tag{5.6}$$

and for all $\psi \in H_B$ with $\|\psi\| = 1$,

$$\lim_{t\to\infty} \langle \psi, e^{-i\lambda X_t}\psi\rangle = \hat{\nu}_0(\lambda), \quad (\lambda \in \mathbb{R}). \tag{5.7}$$

Proposition 5.2. Let $\beta \in (-\infty,\infty)$. Then for all $A \in \mathfrak{M}_\beta$ and all unit vectors $\psi \in H_B$,

$$\lim_{t\to\pm\infty} \langle \psi, \tau_t(A)\psi\rangle = \langle 1, A1\rangle. \tag{5.8}$$

Proof. In the proof of Proposition I.9.3, put $F = A \in \mathfrak{M}_\beta$ and let $G \in \mathfrak{M}_{-\beta}$. (For $\beta = 0$ this changes nothing.) Use the cyclicity of 1 for $\mathfrak{M}_{-\beta}$ and the fact that $\mathfrak{M}_{-\beta} = \mathfrak{M}'_\beta$. □

Remarks. *1.* Proposition 5.2. says that $\{H_B, \mathfrak{M}_\beta, \tau, 1\}$ is mixing, (cf. Def. I.9.2).
2. For $\beta = \infty$, 1 is not cyclic for $\mathfrak{M}'_\beta$, and $\{H_B^+, \mathfrak{M}_\infty, \tau, 1\}$ is not mixing.

Indeed, because $\mathfrak{M}_\infty = \mathcal{L}(H_B^+)$, (cf. (2.34)), the projection operator P_1 on $1 \in H_B$ is in $\mathfrak{M}_\infty$. But certainly (5.8) is not valid for $A = P_1$.

Corollary 5.3. Let $\{X_t\}_{t \in \mathbb{R}}$ be a stationary solution of the QLE (4.1), and let ν_0 be its probability distribution, defined by (5.6). Suppose that $\exp(-i\lambda X_t) \in \mathfrak{M}_\beta$, $(\lambda, t \in \mathbb{R})$. Then ν_0 is an attracting equilibrium measure for the QLE.

Conclusions and questions

If η and w satisfy (4.8), and $\beta \in (0,\infty)$, then the solution of the QLE approaches the equilibrium distribution $\nu_{\beta,\eta,w}$, given by

$$\hat{\nu}_{\beta,\eta,w}(\lambda) = \langle 1, \exp(-i\lambda Q^{(\beta,\eta,w)})\, 1\rangle . \tag{5.9}$$

The following questions are of interest.

1. Is $\nu_{\beta,\eta,w}$ close to the thermal equilibrium distribution of an isolated quantum oscillator?
2. Is $\nu_{\beta,\eta,w}$ absolutely continuous w.r.t. Lebesgue measure, and, conversely, are there no sets of $\nu_{\beta,\eta,w}$-measure zero and positive Lebesgue measure, ("*forbidden areas*")?

In terms of the W^*-stochastic process, the second question can be rephrased as follows. It is natural to define

$$\mathfrak{N}_{\beta,\eta,w} = L^\infty(\mathbb{R}, \nu_{\beta,\eta,w}), \text{ and}$$

$$j_t = \mathfrak{N}_{\beta,\eta,w} \to \mathfrak{M}_\beta : j_t(f) = f(Q_t^{(\beta,\eta,w)}) .$$

Does the following hold:

$$\mathfrak{N}_{\beta,\eta,w} = L^\infty(\mathbb{R})\ ? \tag{5.10}$$

To answer the questions 1 and 2 above, a more constructive approach is needed than the mere application of a fixed point theorem. This approach is the subject of the following sections.

§ 6. EXISTENCE AND INVERTIBILITY OF MØLLER MORPHISMS

A natural way to solve the quantum Langevin equation with an anharmonic potential, is to study the anharmonic quantummechanical Lamb model. It differs from the harmonic Lamb model by a term $\beta^{-1} w(Q^{(\beta,\eta)})$ in its

Hamiltonian, and it has therefore a different time evolution, $\tilde{\tau}$ say. Our question is, whether $\tilde{\tau}$ inherits the mixing property from τ. This is a question in the general class of stability problems for infinite quantum systems, (cf. [BrR 81]). In this section we shall prove two stability theorems for W^*-dynamical systems, (Theorems 6.1 and 6.4), applying the concept of a Møller morphism. It turns out that the modular theory of [Tak 70] gives a little extra information.

Let $\{H,\mathfrak{M},\alpha,\xi\}$ be any W^*-dynamical system. The group $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ is implemented by a strongly continuous group of unitaries $\{U_t\}_{t \in \mathbb{R}}$, defined on $\mathfrak{M}\xi$ by

$$U_t A\xi = \alpha_t(A)\,\xi , \tag{6.1}$$

and extended continuously to $\overline{\mathfrak{M}\xi} = H$. Let $\mathfrak{H}$ be the generator of $\{U_t\}_{t \in \mathbb{R}}$. Then for all $A \in \mathfrak{M}$,

$$\alpha_t(A) = \exp(it\mathfrak{H})\, A \exp(-it\mathfrak{H}) .$$

Now, let $V = V^* \in \mathfrak{M}$, and define for all $A \in \mathfrak{M}$,

$$\tilde{\alpha}_t(A) = \exp(it(\mathfrak{H}+V))\, A \exp(-it(\mathfrak{H}+V)) . \tag{6.2}$$

In its turn, $\{\tilde{\alpha}_t\}_{t \in \mathbb{R}}$ is a *-automorphism group of $\mathfrak{M}$, and we call it the *perturbation of* α *by* V. It may be the case that again there is a vector $\tilde{\xi} \in H$, cyclic for $\mathfrak{M}$, and such that for all $A \in \mathfrak{M}$,

$$\langle\tilde{\xi}, \tilde{\alpha}_t(A)\tilde{\xi}\rangle = \langle\tilde{\xi}, A\,\tilde{\xi}\rangle . \tag{6.3}$$

Then $\{H,\mathfrak{M},\tilde{\alpha},\tilde{\xi}\}$ is another W^*-dynamical system, and another strongly continuous group of unitaries $\{\tilde{U}_t\}_{t \in \mathbb{R}}$ is defined by

$$\tilde{U}_t A\tilde{\xi} = \tilde{\alpha}_t(A)\,\tilde{\xi} . \tag{6.4}$$

We note that, in general, $\tilde{U}_t \neq \exp(it(\mathfrak{H}+V))$.

We want to know, whether the mixing property is stable for perturbations of the above type.

It was proposed by Robinson [Rob 73], to consider the limit

$$\gamma_0(A) := \lim_{t\to\infty} \tilde{\alpha}_{-t} \circ \alpha_t(A) , \tag{6.5}$$

in the norm topology of $\mathfrak{M}$. He showed that, if for a *norm-dense* subset A of $\mathfrak{M}$, it is true that

$$\forall_{A \in \mathfrak{A}}: \int_0^\infty \| [V,\alpha_t(A)] \| \, dt < \infty, \qquad (6.6)$$

then the limit (6.5) exists for all $A \in \mathfrak{A}$, and γ_0 extends to an isometric *-morphism $\gamma : \mathfrak{M} \to \mathfrak{M}$, satisfying

$$\gamma \circ \alpha_t = \tilde{\alpha}_t \circ \gamma \; . \qquad (6.7)$$

This morphism he called a Møller morphism, by analogy with the Møller operators in scattering theory.

Now, if γ turns out to be invertible, the perturbed evolution $\tilde{\alpha}$ is similar to α, because of (6.7), and hence inherits its ergodic properties.

Actually, Robinson considered the more general case of a C^*-algebra, but he supposed that $t \mapsto \alpha_t(A)$ is continuous in the operator norm. In the case of the CCR, the "free" evolution τ does not have this continuity property. Indeed, $\|W(f) - W(g)\| = 2$ if $f \neq g$, hence $t \mapsto W(T_t f)$ is not norm-continuous. As a consequence, the C^*-algebra, generated by the W's, will not be invariant for the perturbed evolution.

Moreover, in our case of the W^*-dynamical system of quantum white noise, the convergence property (6.6) can be proved at best for a *strongly dense* subset $\mathfrak{A}$ of $\mathfrak{M}$, and then γ_0 does not automatically extend to a map $\mathfrak{M} \to \mathfrak{M}$. If we know a $\tilde{\xi}$ to exist, however, that satisfies (6.3), something can nevertheless be proved.

From (6.2) it follows that (cf. [Rob 73]),

$$\tilde{\alpha}_{-t} \circ \alpha_t(A) = A + i \int_0^t \tilde{\alpha}_{-s}([V,\alpha_s(A)]) \, ds \; . \qquad (6.8)$$

Theorem 6.1. (Weak stability theorem). Let $\{H,\mathfrak{M},\alpha,\xi\}$ be a mixing W^*-dynamical system, and let $\tilde{\alpha}$ be the perturbation of α by V, where $V = V^* \in \mathfrak{M}$. Suppose that $\tilde{\xi} \in H$ exists, satisfying (6.3), and assume that (6.6) holds for a strongly dense sub-*-algebra $\mathfrak{A}$ of $\mathfrak{M}$, containing $\mathbb{1}$.

Then there is an isometry $\Omega : H \to H$ with the properties

(i) $\Omega \xi = \tilde{\xi}$,

(ii) $\Omega U_t = \tilde{U}_t \Omega$,

(iii) for all $A \in \mathfrak{A}$, $\Omega A = A\Omega + i \int_0^\infty \tilde{\alpha}_{-t}([V,\alpha_t(A)]) \, \Omega \, dt$.

Proof. Let $0 \leq s \leq t$. Then, for all $A \in \mathfrak{A}$,

$$\left\| \tilde{\alpha}_{-t} \circ \alpha_t(A) - \tilde{\alpha}_{-s} \circ \alpha_s(A) \right\| = \left\| \int_s^t \tilde{\alpha}_{-u}([V,\alpha_u(A)]) \, du \right\| \leq \int_s^t \left\| [V,\alpha_u(A)] \right\| du \, . \qquad (6.9)$$

By (6.6) this can be made arbitrarily small by choosing s large. Therefore, $t \mapsto \tilde{\alpha}_{-t} \circ \alpha_t(A)$ is Cauchy in the operator norm, and the limit (6.5) exists for all $A \in \mathfrak{A}$. Being the norm-limit of a family of isometric *-morphisms $\mathfrak{A} \to \mathfrak{M}$, γ_0 is an isometric *-morphism $\mathfrak{A} \to \mathfrak{M}$ itself. It satisfies the intertwining relation

$$\gamma_0 \circ \alpha_t = \tilde{\alpha}_t \circ \gamma_0 \; . \qquad (6.10)$$

Now, for all $A \in \mathfrak{A}$ we have

$$\langle \tilde{\xi}, \gamma_0(A) \tilde{\xi} \rangle = \lim_{t\to\infty} \langle \tilde{\xi}, \tilde{\alpha}_{-t} \circ \alpha_t(A) \tilde{\xi} \rangle = \lim_{t\to\infty} \langle \tilde{\xi}, \alpha_t(A) \tilde{\xi} \rangle = \langle \xi, A \xi \rangle \, , \qquad (6.11)$$

by (6.3) and the mixing property of $\{H,\mathfrak{M},\alpha,\xi\}$. Define $\Omega_0 : A\xi \to \gamma_0(A)\tilde{\xi}$ by

$$\Omega_0 A \xi = \gamma_0(A) \tilde{\xi} \; . \qquad (6.12)$$

Then, for all $A \in \mathfrak{A}$, by (6.11), and because γ_0 is a *-morphism,

$$\|\Omega_0 A \xi\|^2 = \|\gamma_0(A)\tilde{\xi}\|^2 = \langle \tilde{\xi}, \gamma_0(A)^* \gamma_0(A) \tilde{\xi} \rangle = \langle \tilde{\xi}, \gamma_0(A^*A) \tilde{\xi} \rangle = \langle \xi, A^*A \xi \rangle = \|A\xi\|^2 .$$

And, because $\mathfrak{A}$ is strongly dense in $\mathfrak{M}$, and $\overline{\mathfrak{M}\xi} = H$, ξ is cyclic for $\mathfrak{A}$, and Ω_0 extends to an isometry $\Omega : H \to H$, with range

$$\Omega H = \overline{\gamma_0(\mathfrak{A})\tilde{\xi}} \; . \qquad (6.13)$$

Finally, let us check the properties, claimed by the theorem.

(i). $\Omega\xi = \gamma_0(\mathbb{1})\tilde{\xi} = \tilde{\xi}$, because $\gamma_0(\mathbb{1}) = \mathbb{1}$.

(ii). For all $A \in \mathfrak{A}$, we have, by (6.10), $\Omega U_t A\xi = \Omega\alpha_t(A)\xi = \gamma_0(\alpha_t(A))\tilde{\xi} = \tilde{\alpha}_t(\gamma_0(A))\tilde{\xi} = \tilde{U}_t \gamma_0(A)\tilde{\xi} = \tilde{U}_t \Omega A\xi$. Because $\overline{\mathfrak{A}\xi} = H$, the statement follows.

(iii). For all $A,B \in \mathfrak{A}$ we have $\Omega AB\xi = \gamma_0(AB)\tilde{\xi} = \gamma_0(A)\gamma_0(B)\tilde{\xi} = \gamma_0(A)\Omega B\xi$. Therefore, for all $A \in \mathfrak{A}$,

$$\Omega A = \gamma_0(A) \, \Omega \; . \qquad (6.14)$$

By (6.8), we have

$$\gamma_0(A) = A + i \int_0^\infty \tilde{\alpha}_{-t}([V,\alpha_t(A)]) dt \; . \qquad (6.15)$$

The statement (iii) follows. □

Thermal equilibrium

In the case of thermal equilibrium, some more structure can be added to the above. It turns out that this structure helps resolving the subtle question, whether γ_0 extends to a *-automorphism γ of $\mathfrak{M}$ as soon as Ω is unitary. This question is of interest, because we need the inverse of γ to construct a solution to the QLE.

Suppose that ξ is not only cyclic, but also separating for $\mathfrak{M}$ (cf. (2.24)), and that α is the associated modular automorphism group of $\mathfrak{M}$. If this is the case, we say that $\{H,\mathfrak{M},\alpha,\xi\}$ is a *W^*-dynamical system in thermal equilibrium*. By the perturbation theory of cyclic and separating vectors for a von Neumann algebra, [Ara 76], there is a vector $\tilde{\xi}$, cyclic and separating for $\mathfrak{M}$, such that $\{H,\mathfrak{M},\tilde{\alpha},\tilde{\xi}\}$ is again a W^*-dynamical system in thermal equilibrium. Let J, $\tilde{J}$, Δ and $\tilde{\Delta}$ be the modular conjugations and modular operators respectively. Then (cf. [Ara 76]),

$$\tilde{J} = J. \tag{6.16}$$

Lemma 6.2. Suppose that, in the situation of Theorem 6.1, $\{H,\mathfrak{M},\alpha,\xi\}$ is a W^*-dynamical system in thermal equilibrium. Then

$$J\Omega = \Omega J. \tag{6.17}$$

Proof. We have $\Delta^{\frac{1}{2}} = U_{i\beta/2}$ and $\tilde{\Delta}^{\frac{1}{2}} = \tilde{U}_{i\beta/2}$. For all $A \in \mathfrak{M}$, $A\xi \in \mathrm{Dom}(\Delta^{\frac{1}{2}})$. Now, because $\Omega U_t = \tilde{U}_t\Omega$,

$$\Omega\Delta^{\frac{1}{2}}A\,\xi = \tilde{\Delta}^{\frac{1}{2}}\Omega\,A\,\xi\,.$$

It follows that for all $A \in \mathbb{A}$,

$$\tilde{J}\Omega\Delta^{\frac{1}{2}}A\xi = \tilde{J}\tilde{\Delta}^{\frac{1}{2}}\Omega A\xi = \tilde{J}\tilde{\Delta}^{\frac{1}{2}}\gamma_0(A)\tilde{\xi} = \gamma_0(A)^*\tilde{\xi} = \gamma_0(A^*)\tilde{\xi} = \Omega A^*\xi = \Omega J\Delta^{\frac{1}{2}}A\xi\,. \tag{6.18}$$

Now, $\mathbb{A}\xi$ is dense in H, and therefore $\Delta^{\frac{1}{2}}\mathbb{A}\xi = J(J\Delta^{\frac{1}{2}})\mathbb{A}\xi = J\mathbb{A}^*\xi = J\mathbb{A}\xi$ is dense in $JH = H$. By (6.18) it follows that $\tilde{J}\Omega = \Omega J$, and by (6.16), that $J\Omega = \Omega J$. □

Theorem 6.3. In the situation of Theorem 6.1, let $\{H,\mathfrak{M},\alpha,\xi\}$ be in thermal equilibrium. Then the following statements are equivalent:

(i) The *-morphism $\gamma_0 : \mathbb{A} \to \mathfrak{M}$ extends to a *-automorphism γ of $\mathfrak{M}$,

(ii) $\gamma_0(\mathbb{A})' \subset \mathfrak{M}'$,

(iii) $\Omega H = H$.

Proof. (i) ⇒ (ii): Suppose (i) holds. Let $B \in \gamma_0(\mathbb{A})'$. Then $[B,C] = 0$ for all C in the strongly dense subset $\gamma(\mathbb{A})$ of $\mathfrak{M}$. Because $C \mapsto [B,C]$ is strongly continuous, $[B,C] = 0$ for all $C \in \mathfrak{M}$, so $B \in \mathfrak{M}'$.

(ii) ⇒ (iii): Suppose (ii) holds and let $P = \Omega\Omega^*$. P is the orthogonal projection on $\overline{\gamma_0(\mathbb{A})\tilde{\xi}}$, so $P \in \gamma_0(\mathbb{A})'$, and therefore $P \in \mathfrak{M}'$ by (ii). Now, $P\tilde{\xi} = \tilde{\xi}$, so $(P - \mathbb{1})\tilde{\xi} = 0$, and because $\tilde{\xi}$ is separating for $\mathfrak{M}'$, $P = \mathbb{1}$. It follows that $\Omega H = H$.

(iii) ⇒ (i): Suppose Ω is unitary. Define $\gamma(A) = \Omega A\Omega^*$ for all $A \in \mathfrak{M}$. Then for all $A \in \mathbb{A}$ we have $\gamma(A) = \Omega A\Omega^* = \gamma_0(A)\Omega\Omega^* = \gamma_0(A)$ by (6.14). So γ extends γ_0. Clearly, γ is a *-morphism $\mathfrak{M}\to\mathfrak{M}$. We shall show that it is surjective. Choose $A \in \mathfrak{M}$ and let $B = \Omega^*A\Omega$. We claim that $B \in \mathfrak{M}$; then $\gamma(B) = A$. Indeed, for all $C \in \mathbb{A}$, we have by Lemma 6.2,

$$[JCJ\,,\,B] = JCJ\Omega^*A\Omega - \Omega^*A\Omega JCJ = \Omega^*J(\Omega C\Omega^*JAJ - JAJ\Omega C\Omega^*)J\Omega =$$

$$= \Omega^*J([\gamma_0(C)\,,\,JAJ])J\Omega = 0,$$

because $\gamma_0(C) \in \mathfrak{M}$ and $JAJ \in \mathfrak{M}'$. It follows that $B \in (J\mathbb{A}J)' = (J\mathfrak{M}J)' = \mathfrak{M}'' = \mathfrak{M}$. □

The Dyson series

If we assume a much stronger decay of the commutator $[\alpha_t(V),\cdot\,]$ than we did in (6.6), we can prove more about Ω.

For $A \in \mathfrak{M}$, consider the norm limit

$$\tilde{\gamma}_0(A) := \lim_{t\to\infty} \alpha_{-t}\circ\tilde{\alpha}_t(A)\,. \tag{6.19}$$

The existence of $\tilde{\gamma}_0$ is generally much harder to prove on the basis of assumptions concerning the system $\{H,\mathfrak{M},\alpha,\xi\}$, than that of γ_0. This is seen in (6.9): the troublesome evolution $\tilde{\alpha}$ luckily drops out. Such a thing does not happen when one deals with the limit (6.19). In this case one needs a convergence assumption on the entire Dyson series for $\alpha_{-t}\circ\tilde{\alpha}_t(A)$, (cf. [Rob 73]),

$$\alpha_{-t}\circ\tilde{\alpha}_t(A) = \sum_{n=0}^{\infty}(-i)^n \int_{0\geq t_1\geq\ldots\geq t_n\geq(-t)} dt_1\ldots dt_n \quad [\alpha_{t_n}(V),[\ldots[\alpha_{t_1}(V),A]..]]. \tag{6.20}$$

The convergence assumption is that for all $A \in \mathbb{A}$,

$$\sum_{n=0}^{\infty} \int_{0\geq t_1\geq\ldots\geq t_n} dt_1\ldots dt_n \left\|[\alpha_{t_n}(V),[\ldots[\alpha_{t_1}(V),A]..]]\right\| < \infty\,. \tag{6.21}$$

Theorem 6.4 (Strong stability theorem). In the situation of Theorem 6.1, assume that (6.21) holds for all $A \in \mathfrak{A}$. Then Ω is unitary and the map $\gamma : B \mapsto \Omega B \Omega^*$ is a *-automorphism of $\mathfrak{M}$.

The following nice lemma helps us to prove this.

Lemma 6.5. In the situation of Theorem 6.4, we have, for all $A,B \in \mathfrak{A}$,

$$\lim_{t\to\infty} \langle \tilde{\xi}, \alpha_t(B)\tilde{\alpha}_t(A)\tilde{\xi}\rangle = \lim_{t\to\infty} \langle \xi, \alpha_t(B)\tilde{\alpha}_t(A)\xi\rangle \ . \qquad (6.22)$$

Proof. Using the expansion (6.20), we may write

$$\langle \tilde{\xi}, \alpha_t(B)\tilde{\alpha}_t(A)\tilde{\xi}\rangle - \langle \xi, \alpha_t(B)\tilde{\alpha}_t(A)\xi\rangle = \langle \tilde{\xi}, \alpha_t\Big(B\alpha_{-t}\circ\tilde{\alpha}_t(A)\Big)\tilde{\xi}\rangle - \langle \xi, B\alpha_{-t}\circ\tilde{\alpha}_t(A)\xi\rangle =$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{0\geq t_1\geq\ldots\geq t_n>-\infty} dt_1\ldots dt_n \ \theta(t+t_n)\Big(\langle \tilde{\xi}, \alpha_t(BA(t_1,\ldots,t_n))\tilde{\xi}\rangle - \langle \xi, BA(t_1,\ldots,t_n)\xi\rangle\Big) . \qquad (6.23)$$

Here, $A(t_1,\ldots,t_n) = [\alpha_{t_n}(V),[\ldots \alpha_{t_1}(V),A]..]]$, and θ is Heaviside's function.

Now, by the mixing property, the integrand in (6.23) tends to zero for all $n \in \mathbb{N}$, $t_1,\ldots,t_n \in \mathbb{R}$. Its absolute value is bounded by the function

$$2\|B\|\cdot\|A(t_1,\ldots,t_n)\| \ ,$$

which is summable by assumption (6.21). The statement (6.22) now follows by the dominated convergence theorem.

□

Proof of Theorem 6.4. By looking only at the term with $n=1$ in (6.21), we see that (6.21) implies (6.6). Therefore, γ_0 and Ω exist and have the properties derived in the proof of Theorem 6.1. Now, let again $0 \leq s \leq t$. Then for all $A \in \mathfrak{A}$,

$$\|\alpha_{-t}\circ\tilde{\alpha}_t(A) - \alpha_{-s}\circ\tilde{\alpha}_s(A)\| =$$

$$= \Big\| \sum_{n=0}^{\infty} (-i)^n \int_{0\geq t_1\geq\ldots\geq t_n} dt_1\ldots dt_n \ (\theta(t+t_n) - \theta(s+t_n))A(t_1,\ldots,t_n) \Big\|$$

$$\leq \sum_{n=0}^{\infty} \int_{0\geq t_1\geq\ldots\geq t_n} dt_1\ldots dt_n \ \chi_{[s,t]}(-t_n)\, \|A(t_1,\ldots,t_n)\| \ .$$

By (6.21) this can be made arbitrarily small by choosing s large. So $t \mapsto \alpha_{-t}\circ\tilde{\alpha}_t(A)$ is Cauchy, and $\tilde{\gamma}_0$ exists as an isometric *-morphism $\mathfrak{A} \to \mathfrak{M}$.

Now, if we put $B = \mathbb{1}$ in (6.22) we see that, for all $A \in \mathfrak{A}$,

$$\langle \xi, \tilde{\gamma}_0(A)\xi\rangle = \lim_{t\to\infty} \langle \xi, \alpha_{-t}\circ\tilde{\alpha}_t(A)\xi\rangle = \lim_{t\to\infty} \langle \xi, \tilde{\alpha}_t(A)\xi\rangle = \langle \tilde{\xi}, A\tilde{\xi}\rangle . \qquad (6.24)$$

Define

$$\tilde{\Omega}_0 : A\tilde{\xi} \to \tilde{\gamma}_0(A)\xi : A\tilde{\xi} \mapsto \tilde{\gamma}_0(A)\,\xi \ . \qquad (6.25)$$

By (6.24), $\tilde{\Omega}_0$ also extends to an isometry $\tilde{\Omega} : H \to H$.

Now, by (6.22) we have for all $A,B \in \mathfrak{A}$,

$$\langle B^*\xi, \tilde{\Omega}A\tilde{\xi}\rangle = \langle \xi, B\tilde{\gamma}_0(A)\xi\rangle = \lim_{t\to\infty} \langle \xi, B\alpha_{-t}\circ\tilde{\alpha}_t(A)\xi\rangle = \lim_{t\to\infty} \langle \xi, \alpha_t(B)\tilde{\alpha}_t(A)\xi\rangle =$$

$$\lim_{t\to\infty} \langle \tilde{\xi}, \alpha_t(B)\tilde{\alpha}_t(A)\tilde{\xi}\rangle = \lim_{t\to\infty} \langle \tilde{\xi}, \tilde{\alpha}_{-t}\circ\alpha_t(B)A\tilde{\xi}\rangle = \langle \tilde{\xi}, \gamma_0(B)A\tilde{\xi}\rangle = \langle \gamma_0(B^*)\tilde{\xi}, A\tilde{\xi}\rangle =$$

$$= \langle \Omega B^*\xi, A\tilde{\xi}\rangle \ .$$

From the fact that ξ and $\tilde{\xi}$ are both cyclic for $\mathfrak{A}$, it now follows that

$$\tilde{\Omega} = \Omega^* . \qquad (6.26)$$

Being both isometric, Ω and $\tilde{\Omega}$ must be unitary, and each other's inverse.

Like in (6.14), we have $\tilde{\Omega}A = \tilde{\gamma}_0(A)\tilde{\Omega}$. For all $A \in \mathfrak{A}$, $\Omega A \Omega^* = \gamma_0(A) \in \mathfrak{M}$, and $\Omega^* A \Omega = \tilde{\gamma}_0(A) \in \mathfrak{M}$. Let $\gamma : B \mapsto \Omega B \Omega^*$, $(B \in \mathfrak{M})$. Then γ extends γ_0 and γ^{-1} extends $\tilde{\gamma}_0$. To see that γ is a *-automorphism of $\mathfrak{M}$, note that γ and γ^{-1} are strongly continuous, and therefore

$$\gamma(\mathfrak{M}) = \gamma(\mathfrak{A}'') \subset \gamma(\mathfrak{A})'' = \gamma_0(\mathfrak{A})'' \subset \mathfrak{M}'' = \mathfrak{M}.$$

By the same argument, $\gamma^{-1}(\mathfrak{M}) \subset \mathfrak{M}$.

□

§ 7. CONVERGENCE OF THE DYSON SERIES

We shall now show that the stability theorems, proved in the previous section, apply to the Lamb model. To this end, we check the validity of the "weak" and the "strong" integrability conditions (6.6) and (6.21), for the following choice of V:

$$V = \beta^{-1} w(Q_0^{(\beta,\eta)}) \ . \qquad (7.1)$$

Here w is a bounded function $\mathbb{R} \to \mathbb{R}$, properties of which will be specified. As the strongly dense sub-*-algebra of $\mathfrak{M}_\beta$ figuring in (6.6) and (6.21), we take $\mathfrak{A}_{\beta,\eta}$, the *-algebra, finitely generated by the operators

$$W_B \circ \vartheta_\beta(\lambda T_t q^{(\eta)}) = \exp(-i\lambda Q_t^{(\beta,\eta)}) , \quad (t,\lambda \in \mathbb{R}) . \qquad (7.2)$$

Let us introduce the abbreviation

$$W_\beta = W_B \circ \vartheta_\beta . \tag{7.3}$$

Generally speaking, the weak condition (6.6) is not very restrictive indeed. It is satisfied for many local perturbations of the dynamics of many infinite systems. However, there is generally no hope of satisfying (6.21). The curious fact that for our string model, and then for a restricted class of w's, (6.21) does hold, is based on the exponential fall-off of the commutator of the position variable $Q^{(\beta,\eta)}$ of the harmonic oscillator, attached to the string, taken at different times:

$$\| [Q_t^{(\beta,\eta)}, Q_s^{(\beta,\eta)}] \| \leqslant a\, e^{-b\cdot|t-s|} , \tag{7.4}$$

with a,b > 0 properly chosen.

Lemma 7.1. For all $\eta > 0$ there are positive constants a and b, such that for all $t \in \mathbb{R}$,

$$|\sigma(q^{(\eta)}, T_t q^{(\eta)})| \leqslant a\, e^{-b|t|}. \tag{7.5}$$

These constants necessarily satisfy

$$a > b . \tag{7.6}$$

Example. In the underdamped case, (i.e., $\eta < 2$), one may choose $a = (1-\eta^2/4)^{-\frac{1}{2}}$ and $b = \eta/2$.

Proof. We omit superscripts η. Let $\mathfrak{P}_\eta(x) = x^2 + \eta x + 1$, as in (3.16). We have, for all $t > 0$,

$$\mathfrak{P}_\eta\left(\frac{d}{dt}\right) \sigma(q, T_t q) = \mathfrak{P}_\eta\left(\frac{d}{dt}\right) \int_{-\infty}^{0} p(s)\, q(s-t)\, ds = 0, \tag{7.7}$$

because $\mathfrak{P}_\eta(-\partial) q = 0$ on $(-\infty, 0)$. So $t \mapsto \sigma(q, T_t q)$ satisfies the damped harmonic oscillator equation on $(0,\infty)$, and because $\sigma(q, T_{-t} q) = -\sigma(q, T_t q)$, (7.5) follows. Now, note that $\sigma(q,q) = 0$ and $\frac{d}{dt}\sigma(q, T_t q)\Big|_{t=0} = \sigma(q,p) = 1$. Therefore

$$\sigma(q, T_t q) = \frac{1}{\sqrt{2\eta}}\, q(-t) , \quad (t \geqslant 0). \tag{7.8}$$

It follows that $\int_0^\infty \sigma(q, T_t q)\, dt = (2\eta)^{-\frac{1}{2}} \hat{q}(0) = 1$. On the other hand, $\int_0^\infty |\sigma(q, T_t q)|\, dt < \int_0^\infty a\, e^{-bt}\, dt = a/b$ by (7.5). Therefore, $1 < a/b$, i.e., $a > b$. □

The following lemmas, 7.2 and 7.3, indicate for which choices of $\{\beta, \eta, w\}$ the weak and the strong integrability conditions are satisfied.

Lemma 7.2. Suppose that $w \in \hat{M}$ is such that also $w' \in \hat{M}$. Then for all $A \in \mathfrak{A}_{\beta,\eta}$,

$$\int_{-\infty}^{\infty} \| [w(Q_0^{(\beta,\eta)}), \tau_t(A)] \|\, dt < \infty . \tag{7.9}$$

Proof. Note that for all $f,g \in \mathfrak{F}^\beta$,

$$\| [W_\beta(f), W_\beta(g)] \| = 2\, |\sin(\tfrac{1}{2}\beta\,\sigma(f,g))| \leqslant \beta |\sigma(f,g)| .$$

Now, it suffices to prove (7.9) for $A = W_\beta(\lambda q^{(\eta)})$, $(\lambda \in \mathbb{R})$. Let $w = \hat{\mu}$, with $\mu \in M$. Then,

$$\| [w(Q_0^{(\beta,\eta)}), \tau_t(W_\beta(\lambda q^{(\eta)}))] \| = \left\| \int_{-\infty}^{\infty} [W_\beta(\lambda' q^{(\eta)}), W_\beta(\lambda T_t q^{(\eta)})]\, \mu(d\lambda') \right\|$$

$$\leqslant \int_{-\infty}^{\infty} \beta |\lambda\lambda'\, \sigma(q^{(\eta)}, T_t q^{(\eta)})| \cdot \mu^+(d\lambda') = \beta |\lambda\, \sigma(q^{(\eta)}, T_t q^{(\eta)})| \cdot \|w'\|_{\hat{M}} . \tag{7.10}$$

By Lemma 7.1, this is an integrable function of t. □

Lemma 7.3. Let $w \in \hat{M}$ be such that also w' and w" are in $\hat{M}$. Suppose that

$$2\beta^{-1} \|w\|_{\hat{M}} + a \|w''\|_{\hat{M}} < b, \tag{7.11}$$

where a and b are positive constants, satisfying (7.5). Then, for all $A \in \mathfrak{A}_{\beta,\eta}$,

$$\sum_{n=0}^{\infty} \int_{0 \geqslant t_1 \geqslant \ldots \geqslant t_n} dt_1 \ldots dt_n \left\| [\tfrac{1}{\beta} w(Q_{t_n}^{(\beta,\eta)}), [\ldots [\tfrac{1}{\beta} w(Q_{t_1}^{(\beta,\eta)}), A]..]] \right\| < \infty. \tag{7.12}$$

We shall prove this lemma in several steps.

Let $\rho(n)$, $(n \in \mathbb{N})$, denote the set of ordered sequences $r = \{r_1, r_2, \ldots, r_p\}$ of integer numbers, satisfying

$$0 < r_1 < r_2 < \ldots < r_p < n.$$

Here, p is simply the length of r. Let $\rho(n)$ also contain the empty sequence.

Lemma 7.4. For all $f_0, \ldots, f_n \in \mathfrak{F}^\beta$,

$$\| [W_\beta(f_n),[\,..[W_\beta(f_1),W_\beta(f_0)]\,..]] \| \leqslant$$

$$\leqslant 2^n \cdot \sum_{r \in \rho(n)} |\tfrac{1}{2}\beta\,\sigma(f_0,f_{r_1})| \times |\tfrac{1}{2}\beta\,\sigma(f_{r_1},f_{r_2})| \times \cdots \times |\tfrac{1}{2}\beta\,\sigma(f_{r_p},f_n)| \;.$$

Proof. Repeated use of the equality

$$[W_\beta(f),W_\beta(g)] = -2\,i\,\sin(\tfrac{1}{2}\beta\,\sigma(f,g)) \cdot W_\beta(f+g),$$

yields: $\| [W_\beta(f_n),[\,..[W_\beta(f_1),W_\beta(f_0)]\,..]] \| = 2^n s_1 \ldots s_n$, where $s_k = |\sin(\tfrac{1}{2}\beta\sigma(f_k,f_{k-1}+\,..+f_0))|$, $(k = 1,\ldots,n)$. Now, let $\sigma_{jk} = \tfrac{1}{2}\beta|\sigma(f_j,f_k)|$. Then s_k satisfies the following two bounds:

$$s_k \leqslant 1, \tag{7.13}$$

and

$$s_k \leqslant \sum_{j=0}^{k-1} \sigma_{jk} \,. \tag{7.14}$$

We claim that these bounds imply that

$$s_1 \ldots s_n \leqslant \sum_{r \in \rho(n)} \sigma_{0r_1}\,\sigma_{r_1 r_2} \ldots \sigma_{r_p n} \,. \tag{7.15}$$

We proceed by induction. In the first place, (7.15) is valid for $n = 1$ by (7.14): $s_1 \leqslant \sigma_{01}$. Now, suppose that (7.15) holds for all n up to some integer m. Then, by (7.14) and (7.13) respectively,

$$s_1 \ldots s_m\, s_{m+1} \leqslant s_1 \ldots s_m\, (\sigma_{0,m+1} + \ldots + \sigma_{m,m+1})$$

$$\leqslant \sigma_{0,m+1} + s_1\,\sigma_{1,m+1} + s_1 s_2\,\sigma_{2,m+1} + \ldots + s_1 .. s_m\,\sigma_{m,m+1} \,.$$

Now we apply the induction hypothesis, (7.15) for $n = 1,\ldots,m$, and conclude that

$$s_1 \ldots s_m\, s_{m+1} \leqslant \sigma_{0,m+1} + \sum_{n=1}^{m} \Big(\sum_{r \in \rho(n)} \sigma_{0r_1} \ldots \sigma_{r_p n} \Big) \sigma_{n,m+1} =$$

$$= \sum_{r \in \rho(m+1)} \sigma_{0r_1}\,\sigma_{r_1 r_2} \ldots \sigma_{r_p,m+1} \,.$$

We conclude that (7.15) also holds for $n = m+1$, and the statement follows by induction on m . □

Lemma 7.5. Let $\beta,\eta > 0$ and let $a,b > 0$ be such that $|\sigma(q^{(\eta)},T_t\,q^{(\eta)}| \leqslant \leqslant a\,\exp(-b|t|)$. Choose $n,m \in \mathbb{N}$, and let $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$, $t_1,\ldots,t_n \in \mathbb{R}$, $\kappa_1,\ldots,\kappa_m \in \mathbb{R}$, and $s_1,\ldots,s_m \in \mathbb{R}$. Then

$$\Big\| \Big[W_\beta(\lambda_n\,T_{t_n}\,q^{(\eta)}), \Big[\cdots \Big[W_\beta(\lambda_1\,T_{t_1}\,q^{(\eta)}), W_\beta\Big(\sum_{k=1}^{m} \kappa_k\,T_{s_k}\,q^{(\eta)} \Big) \Big] \cdot\cdot \Big] \Big] \Big\|$$

$$\leqslant \Big(\sum_{k=1}^{m} |\kappa_k| \cdot e^{-b\,s_k} \Big) \cdot \beta\,a\,|\lambda_n| \cdot e^{b\,t_n} \prod_{\ell=1}^{n-1} (2 + \beta\,a\,\lambda_\ell^2) \,. \tag{7.16}$$

Proof. Let us omit indices β and η, and denote $T_t\,q$ by q_t. An application of Lemma 7.4 with $f_0 = \sum_{k=1}^m \kappa_k\,q_{s_k}$ and $f_\ell = \lambda_\ell\,q_{t_\ell}$, $(\ell = 1,\ldots,n)$, gives the following upper bound for the left hand side (l.h.s.) of (7.16):

$$\text{(l.h.s.)} \leqslant 2^n \sum_{r \in \rho(n)} \tfrac{1}{2}\beta \Big| \sigma\Big(\sum_{k=1}^{m} \kappa_k\,q_{s_k}, \lambda_{r_1}\,q_{t_{r_1}} \Big) \Big| \cdot$$

$$\cdot\, \tfrac{1}{2}\beta \Big| \sigma(\lambda_{r_1}\,q_{t_{r_1}}, \lambda_{r_2}\,q_{t_{r_2}}) \Big| \cdot \ldots \cdot \tfrac{1}{2}\beta \Big| \sigma(\lambda_{r_p}\,q_{t_{r_p}}, \lambda_n\,q_{t_n}) \Big| \;.$$

This can again be estimated by the use of the bounds on $\sigma(q_s,q_t)$, given by

$$|\sigma(q_s,q_t)| \leqslant a\,e^{-b|s-t|} \leqslant a\,e^{\pm b(s-t)},$$

as follows:

$$\text{(l.h.s.)} \leqslant 2^n \sum_{r \in \rho(n)} \Big(\tfrac{1}{2}\beta\,a \sum_{k=1}^{m} |\kappa_k\,\lambda_{r_1}|\, e^{b(t_{r_1} - s_k)} \Big) \times$$

$$\times\, \tfrac{1}{2}\beta\,a\,|\lambda_{r_1}\,\lambda_{r_2}|\, e^{b(t_{r_2} - t_{r_1})} \times \cdots \times \tfrac{1}{2}\beta\,a\,|\lambda_{r_p}\,\lambda_n|\, e^{b(t_n - t_{r_p})} =$$

$$= 2^n \Big(\sum_{k=1}^{m} |\kappa_k|\, e^{-b\,s_k} \Big) \cdot \tfrac{1}{2}\beta\,a\,|\lambda_n| \cdot e^{b\,t_n} \cdot \sum_{r \in \rho(n)} \Big(\tfrac{1}{2}\beta\,a\,\lambda_{r_1}^2 \Big) \times \cdots \times \Big(\tfrac{1}{2}\beta\,a\,\lambda_{r_p}^2 \Big).$$

Now, the sum over $r \in \rho(n)$ is in fact a sum over all subsets of $\{1,\ldots,n-1\}$, and therefore we have

$$\sum_{r \in \rho(n)} (\tfrac{1}{2}\beta\,a\,\lambda_{r_1}^2) \times \cdots \times (\tfrac{1}{2}\beta\,a\,\lambda_{r_p}^2) = \prod_{\ell=1}^{n-1} (1 + \tfrac{1}{2}\beta\,a\,\lambda_\ell^2) \,.$$

By distributing the 2^n over the factors, we obtain (7.16). □

Proof of Lemma 7.3. Let $\mu \in M$ be such that $\int \exp(-i\,\lambda\,x)\,\mu(dx) = w(\lambda)$. It suffices to prove (7.12) for all A of the form

$$A = W\Big(\sum_{k=1}^{m} \kappa_k\,q_{s_k} \Big) \;. \tag{7.17}$$

Applying Lemma 7.5, we find that for all $\{t_\ell\}_{\ell=1}^n$ the following holds:

$$\|[w(Q_{t_n}),[\cdots[w(Q_{t_1}),A]\cdots]]\| \leq \int_{\lambda_n\in\mathbb{R}} \mu^+(d\lambda_n)\cdots\int_{\lambda_1\in\mathbb{R}}\mu^+(d\lambda_1)\times$$

$$\left\|\left[W(\lambda_n q_{t_n}),\left[\cdots\left[W(\lambda_1 q_{t_1}),W\left(\sum_{k=1}^m \kappa_k q_{s_k}\right)\right]\cdots\right]\right]\right\|$$

$$\leq \left(\sum_{k=1}^m |\kappa_k| e^{-b s_k}\right)\cdot\left(\int_{\lambda_n\in\mathbb{R}}\beta a|\lambda_n|\mu^+(d\lambda_n)\right)\cdot\prod_{\ell=1}^{n-1}\left(\int_{\lambda_\ell\in\mathbb{R}}(2+\beta a\lambda_\ell^2)\,\mu^+(d\lambda_\ell)\right)\times e^{b t_n} =$$

$$= c_A\cdot\beta a\|w'\|_{\hat{M}}\cdot(2\|w\|_{\hat{M}}+\beta a\|w''\|_{\hat{M}})^{n-1}\cdot e^{b t_n}. \qquad (7.18)$$

Here $c_A = \Sigma_{k=1}^m|\kappa_k|\exp(-b\,s_k)$ is a positive constant, determined by the choice (7.17) of A.

Now note that

$$\int_{0\geq t_1\geq\cdots\geq t_n} dt_1\cdots dt_n\, e^{b t_n} = \int_{-\infty}^{0} dt_1\, e^{b t_1}\int_{-\infty}^{t_1} dt_2\, e^{b(t_2-t_1)}\cdots\int_{-\infty}^{t_{n-1}} dt_n\, e^{b(t_n-t_{n-1})} = b^{-n}.$$

Therefore the sum in (7.12) is bounded by

$$\|A\| + c_A\, a\, b^{-1}\|w'\|_{\hat{M}}\sum_{n=1}^{\infty}(2\beta^{-1}\|w\|_{\hat{M}} + a\|w''\|_{\hat{M}})^{n-1}\, b^{-(n-1)}.$$

This is a finite number if (7.11) holds. □

8. CONCLUSIONS

Let us see, what can be concluded from the analysis in the preceding sections.

The quantum Langevin equation at positive temperature is entirely determined by the choice of a friction coefficient $\eta > 0$, an inverse temperature $\beta > 0$, and a function w. The following three cases deserve a separate discussion.

A. The functions w and w' are in $\hat{M}$. In this case the weak stability theorem applies.

B. The functions w, w', and w" are in $\hat{M}$, and

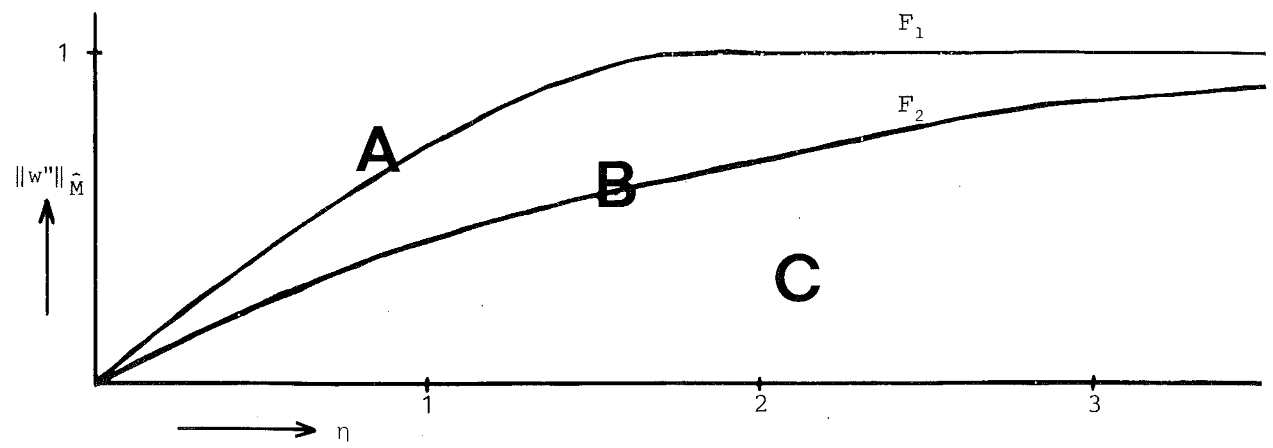

Fig. 7. Three regions of interest in the $\{\eta,\|w''\|_{\hat{M}}\}$-plane. The curves are the graphs of F_1 and F_2, given by

$$F_1(\eta) = \left(\int_0^\infty |\sigma(q^{(\eta)},T_t q^{(\eta)})|\,dt\right)^{-1} = \mathrm{tgh}\left(\frac{\pi}{4}\cdot\frac{\eta}{\sqrt{1-\eta^2/4}}\right) \text{ if } 0\leq\eta\leq 2, \text{ and}$$

$$F_1(\eta) = 1 \text{ if } \eta\geq 2.$$

$$F_2(\eta) = \sup\{b/a\,|\,a,b>0 \text{ and } \forall_{t\geq 0}: |\sigma(q^{(\eta)},T_t q^{(\eta)})|\leq a\,e^{-bt}\}.$$

— — — — — — — — —

$$\|w''\|_{\hat{M}} < \frac{\int q^{(\eta)}dt}{\int|q^{(\eta)}|dt} \quad \left(=F_1(\eta)\right). \qquad (8.1)$$

(Cf. Fig. 7). There is a unique solution to the QLE, (Theorem 4.2).

C. The functions w, w' and w" are in $\hat{M}$, and there are a,b > 0 such that

$$|\sigma(q^{(\eta)},T_t q^{(\eta)})|\leq a\,e^{-b|t|} \text{ and } 2\beta^{-1}\|w\|_{\hat{M}} + a\|w''\|_{\hat{M}} < b. \qquad (8.2)$$

In this case the strong stability theorem applies. In Fig. 7 the region indicated by the letter C is that part of the $\{\eta,\|w''\|_{\hat{M}}\}$-plane, where (8.2) is satisfied for β sufficiently large, and a,b properly chosen. Note that $C\Rightarrow B\Rightarrow A$.

A. *Application of the weak stability theorem*

Consider the W*-dynamical system at thermal equilibrium, given by $\{H_B,\mathfrak{M}_\beta,\tau,1\}$. Choose $w\in\hat{M}$ such that also $w'\in\hat{M}$, and let $\tilde{\tau}$ be the perturbation of τ by $\beta^{-1}w(Q_0^{(\beta,\eta)})$. Let the perturbed W*-dynamical system at thermal equilibrium be $\{H_B,\mathfrak{M}_\beta,\tilde{\tau},\tilde{\xi}\}$, with $\tilde{\xi}>0$ a.e.

By Lemma 7.2, Theorem 6.1, there exist a *-morphism $\gamma_0 : A_{\beta,\eta}\to\mathfrak{M}_\beta$ and an isometry $\Omega : H_B\to H_B$ with the following properties:

$\Omega\, 1 = \widetilde{\xi} \quad , \gamma_0(\mathbb{1}) = \mathbb{1},$

$\Omega\, \mathfrak{T}_t = \widetilde{\mathfrak{T}}_t\, \Omega,\ \gamma_0 \circ \tau_t = \widetilde{\tau}_t \circ \gamma_0\,,$

$$\gamma_0(W_\beta(\lambda q^{(\eta)}))\Omega = \Omega\, W_\beta(\lambda q^{(\eta)}), \text{ and} \tag{8.3}$$

$$\gamma_0(W_\beta(\lambda q^{(\eta)})) = W_\beta(\lambda q^{(\eta)}) + \beta^{-1}\int_0^\infty \widetilde{\tau}_{-t}\Big([w(Q_0^{(\beta,\eta)}),W_\beta(\lambda T_t\, q^{(\eta)})]\Big)dt. \tag{8.4}$$

Let $\mathcal{L}_\eta$ denote the real-linear span of $\{T_t q^{(\eta)}\,|\,t \in \mathbb{R}\}$. Then $f \mapsto \gamma_0 \circ W_\beta(f)$ is a representation of the CCR over the symplectic space $\{\mathcal{L}_\eta, \beta\sigma\}$. However, $\widetilde{\xi}$ is not necessarily a cyclic vector. Indeed, it is cyclic if and only if

$$\overline{\gamma_0(A_{\beta,\eta})\widetilde{\xi}} = H_B\,. \tag{8.5}$$

By (8.3) this is equivalent with the unitarity of Ω.

On the other hand, we have, for all $f \in \mathcal{L}_\eta$,

$$\langle\widetilde{\xi},\gamma_0 \circ W_\beta(f)\widetilde{\xi}\rangle = \langle\Omega 1,\gamma_0 \circ W_\beta(f)\Omega 1\rangle = \langle\Omega 1,\Omega W_\beta(f)1\rangle = \langle 1,W_\beta(f)1\rangle =$$
$$= \exp(-\tfrac{1}{2}\|f\|_\beta^2)\,. \tag{8.6}$$

Now, let $\widetilde{E}_\beta$ be given by $\gamma_0(W_\beta(\lambda f)) = \exp(-i\lambda\widetilde{E}_\beta(f))$, $(f \in \mathcal{L}_\eta)$. Then, with respect to $\widetilde{\xi}$, $\widetilde{E}_\beta$ has all the properties of white noise, except that it has some unknown behaviour on the, not necessarily vanishing, subspace $(\Omega H_B)^\perp$. For this reason we shall call $\widetilde{E}_\beta$ *improper quantum white noise*. Note that

$$\Omega^*\widetilde{E}_\beta(f)\Omega = \Omega^*\Omega E_\beta(f) = E_\beta(f)\,. \tag{8.7}$$

Therefore, if Ω is unitary, $\widetilde{E}_\beta$ is unitarily equivalent with quantum white noise E_β.

Definition 8.1. We shall say that the family $\{X_t\}_{t\in\mathbb{R}}$ of self-adjoint operators on H_B satisfies the *improper QLE given by* $\{\beta,\eta,w\}$, if for all $t \in \mathbb{R}$,

$\mathrm{Dom}(X_t) = \mathrm{Dom}(\widetilde{E}(T_t\, q^{(\eta)}))\ ,\ X_t = \widetilde{\tau}_t(X_0)\,,$

and for all $\varphi \in \mathrm{Dom}(X_t)$

$$X_t\varphi + \int_{-\infty}^{t} \frac{q(s-t)}{\sqrt{2\eta}}\, w'(X_s)\varphi\, ds = \widetilde{E}_\beta(T_t\, q^{(\eta)})\varphi. \tag{8.8}$$

Remark. In comparison with Definition 4.1 of the (proper) QLE, the noise term $Q_t^{(\beta,\eta)} = E_\beta(T_t\, q^{(\eta)})$ has been replaced by the improper quantum white noise term $\widetilde{E}_\beta(T_t\, q^{(\eta)})$, and the evolution τ by the evolution $\widetilde{\tau}$.

Proposition 8.2. If $\eta > 0$, $\beta \in (0,\infty]$ and $w,w' \in \hat{M}$, then the family $\{\widetilde{\tau}_t(Q^{(\beta,\eta)})\}_{t\in\mathbb{R}}$ satisfies the improper QLE.

Proof. We omit indices β and η. Let $w = \hat{\mu}$ with $\mu \in M$. Take $\varphi \in H_B$ and consider the difference

$$\gamma_0\left(\frac{W(\lambda q) - \mathbb{1}}{-i\lambda}\right)\varphi - \left(\frac{W(\lambda q) - \mathbb{1}}{-i\lambda}\right)\varphi\,. \tag{8.9}$$

By (8.4) this is equal to

$$\int_0^\infty (-i\beta\lambda)^{-1}\,\widetilde{\tau}_{-t}\Big([w(Q),W(\lambda\, T_t\, q)]\Big)\varphi\, dt. \tag{8.10}$$

Now,

$$[w(Q),W(\lambda\, T_t\, q)] = \int_{-\infty}^{\infty} [W(\lambda' q),W(\lambda\, T_t\, q)]\mu(d\lambda') =$$

$$= \int_{-\infty}^{\infty} (e^{-i\beta\lambda\,\lambda'\sigma(q,T_t q)} - 1)W(\lambda T_t q)\, W(\lambda' q)\,\mu(d\lambda')\,. \tag{8.11}$$

The integrand in (8.10) is bounded in norm by $|\sigma(q,T_t\, q)|\cdot\|w'\|_{\hat{M}}\cdot\|\varphi\|$, which is clearly independent of λ and integrable. By (8.11), as λ tends to zero, the integrand in (8.10) tends to

$$\widetilde{\tau}_{-t}\Big(\int_{-\infty}^{\infty} \lambda'\sigma(q,T_t\, q)\, W(\lambda' q)\mu(d\lambda')\Big)\varphi = \sigma(q,T_t\, q)\,\widetilde{\tau}_{-t}(w'(Q))\,\varphi\,.$$

It follows by the dominated convergence theorem that the limit of (8.10), as $\lambda \to 0$, is equal to

$$\int_0^\infty \sigma(q,T_t\, q)\,\widetilde{\tau}_{-t}(w'(Q))\,\varphi\, dt. \tag{8.12}$$

Therefore, the same holds for (8.9). As (8.12) is finite for all φ, it follows that the limit, as $\lambda \to 0$, of the first term of (8.9) exists precisely for those φ, for which the limit of the second does; i.e.,

$\mathrm{Dom}(\widetilde{E}(q)) = \mathrm{Dom}(E(q))\,.$

If φ is in this common domain, we have

$$\tilde{E}(q)\varphi - E(q)\varphi = \int_0^\infty \sigma(q,T_t q)\tilde{\tau}_{-t}(w'(Q))\varphi\, dt .$$

The statement is proved by noting that $\sigma(q,T_t q) = q(-t)/\sqrt{2\eta}$, $(t \geqslant 0)$, that $Q = E(q)$, and that $\tilde{\tau}_t(\tilde{E}(q)) = \tilde{E}(T_t q)$. □

Remark. Proposition 8.2 says that

$$Q + \int_{-\infty}^0 \frac{1}{\sqrt{2\eta}} q(t) w'(\tilde{\tau}_t(Q))dt = \tilde{E}_\beta(q). \tag{8.13}$$

By (8.7) it follows that

$$\Omega^* Q\Omega + \int_{-\infty}^0 \frac{1}{\sqrt{2\eta}} q(t)\tau_t(\Omega^* w'(Q)\Omega)dt = Q. \tag{8.14}$$

Now, let us suppose for a moment that we could prove, by whatever means, that Ω is unitary. Then we have

$$\Omega^* Q\Omega + \int_{-\infty}^0 \frac{1}{\sqrt{2\eta}} q(t)\tau_t(w'(\Omega^* Q\Omega))dt = Q, \tag{8.15}$$

i.e., $\{\tau_t(\Omega^* Q\Omega)\}$ satisfies the (proper) QLE, (cf. Def. 4.1).

Moreover, by Theorem 6.3, $\gamma : A \mapsto \Omega A\Omega^*$ is a *-automorphism of $\mathfrak{M}_\beta$. Therefore, $\Omega^* W(\lambda q)\Omega = \gamma^{-1}(W(\lambda q)) \in \mathfrak{M}_\beta$. By the mixing property it follows that for all unit vectors $\psi \in H_B$,

$$\langle\psi,\tau_t(\Omega^* W(\lambda q)\Omega)\psi\rangle \xrightarrow[t\to\pm\infty]{} \langle 1,\Omega^* W(\lambda q)\Omega 1\rangle = \langle\tilde{\xi},W(\lambda q)\tilde{\xi}\rangle. \tag{8.16}$$

So, provided that Ω is unitary, we have a solution of the QLE with absolutely continuous spectrum, filling out the whole real line, a solution which approaches equilibrium under the evolution τ, the equilibrium distribution $\nu_{\beta,\eta,w}$ being given by

$$\hat{\nu}_{\beta,\eta,w}(\lambda) = \langle\tilde{\xi},W(\lambda q)\tilde{\xi}\rangle . \tag{8.17}$$

B. *The existence and uniqueness theorem*

Suppose now, that $\eta > 0$ and $w,w',w'' \in \hat{M}$, where $\|w''\|_{\hat{M}}$ satisfies the inequaltiy (8.1), (see also Fig. 7). Then we have, apart from the solution $\{\tilde{\tau}_t(Q)\}_{t\in\mathbb{R}}$ of the improper QLE, a unique solution $\{\tau_t(Q^W)\}_{t\in\mathbb{R}}$ of the proper QLE. The latter can be written concisely as follows

$$Q^W + \int_{-\infty}^0 \frac{1}{\sqrt{2\eta}} q(t)\tau_t(Q^W)dt = Q. \tag{8.18}$$

Now, one might be tempted to think that, in some reasonable sense, these solutions have to be the same. However, attempts to prove this have failed thus far. It is not clear, how (8.18), together with (8.14), should lead to (8.15), (and hence, by the uniqueness of Q^W, to the conclusion that $Q^W = \Omega^* Q\Omega$).

C. *Application of the strong stability theorem*

Let us finally suppose that $w,w',w'' \in \hat{M}$ are such that (8.2) holds. By Lemma 7.3, the strong stability theorem applies. It says that Ω is unitary and that $A \mapsto \Omega A\Omega^*$ is a *-automorphism of $\mathfrak{M}_\beta$.

All the conclusions at the end of A follow, especially (8.16) and (8.17). This answers the two questions, left open at the end of § 5, at least for $\{\beta,\eta,w\}$ in the region C.

§ 9. EQUILIBRIUM DISTRIBUTION VERSUS GIBBS DISTRIBUTION

We remind the reader of the question, posed in [BeK 81], which motivated the present work. It is the following.

Can one prove that the QLE has a solution $\{Q_t^{(\beta,\eta,w)}\}_{t\in\mathbb{R}}$ with the property that, irrespective of the initial state vector ψ,

$$\lim_{\eta\downarrow 0}\lim_{t\to\infty} \langle\psi,\exp(-i\lambda Q_t^{(\beta,\eta,w)})\psi\rangle = \frac{\mathrm{tr}(e^{-i\lambda\cdot}\exp(-H_{S,\beta}^W))}{\mathrm{tr}\exp(-H_{S,\beta}^W)} \;? \tag{9.1}$$

Here, $H_{S,\beta}^W = \frac{1}{2}x^2 + w(x) - \beta^2\partial^2/\partial x^2$ is the (Schrödinger) Hamiltonian of the anharmonic quantum oscillator in isolation. The r.h.s. of (9.1) is the Fourier transform of the probability distribution, $\nu_{\beta,0,w}$ say, of the position of such an oscillator in its Gibbs state. (Our units have been chosen in such a way that β occurs in places where one expects $\hbar$, cf. Appendix A).

Strictly speaking, we are able to prove (9.1) only in the trivial case

$w'' = 0$. Indeed, a look at Fig. 7 tells us that, if $w'' \neq 0$, the limit $\eta \downarrow 0$ leads us out of the region C, where our result is valid.

On the other hand, it was shown in § 8 that for all $\{\beta,\eta,w\}$ in C the limit $t \to \infty$ exists and is given by

$$\lim_{t\to\infty} \langle\psi, \exp(-i\lambda Q_t^{(\beta,\eta,w)})\psi\rangle = \langle\tilde{\xi}, W_\beta(\lambda q^{(\eta)})\tilde{\xi}\rangle \left(= \hat{\nu}_{\beta,\eta,w}(\lambda)\right). \tag{9.2}$$

While remaining inside the region C, one can get arbitrarily close to the line $\eta = 0$, by choosing $\|w''\|_{\hat{M}}$ small. In the remainder of this section we shall give an estimate of the difference

$$\left|\langle\tilde{\xi}, W_\beta(\lambda q^{(\eta)})\tilde{\xi}\rangle - \frac{\operatorname{tr}(e^{-i\lambda\cdot}\exp(-H^W_{S,\beta}))}{\operatorname{tr}\exp(-H^W_{S,\beta})}\right| = \left|\hat{\nu}_{\beta,\eta,w}(\lambda) - \hat{\nu}_{\beta,0,w}(\lambda)\right|, \tag{9.3}$$

which shows that this difference is small for small values of η.

Perturbation of the KMS state

By the $\{T,\beta\}$-KMS property of $C_\beta : f \mapsto \exp(-\frac{1}{2}R_\beta(f,f))$, the state $\omega : A \mapsto \langle 1, A\, 1\rangle$ is a $\{\tau,\beta\}$-KMS state on $\mathfrak{M}_\beta = W_\beta(\mathfrak{S})''$. This is to say that, for all $A,B \in \mathfrak{M}_\beta$, the function $t \mapsto \omega(A\tau_t(B))$ extends to a function $G_{A,B} \in \mathfrak{C}(\Lambda(\beta))$ with $G_{A,B}(t+i\beta) = \omega(\tau_t(B)A)$. Moreover, it has been shown that, for all $A = \{A_0,\dots,A_n\} \in \mathfrak{M}_\beta^{n+1}$, there is a unique function $G_A \in \mathfrak{C}(\Lambda_n(\beta))$, such that

$$G_A(t_1,\dots,t_n) = \omega(A_1\,\tau_{t_1}(A_1)\dots\tau_{t_n}(A_n)).$$

Here, $\Lambda_n(\beta) = \{\{z_1,\dots,z_n\} \in \mathbb{C}^n \mid 0 \leq \operatorname{Im} z_1 \leq \dots \leq \operatorname{Im} z_n \leq \beta\}$.

The state $\tilde{\omega}_{\beta,\eta,w} : A \mapsto \langle\tilde{\xi}, A\,\tilde{\xi}\rangle$ on $\mathfrak{M}_\beta$, which is KMS for the perturbation of τ by $V = \beta^{-1}w(Q^{(\beta,\eta)})$, is given by

$$\tilde{\omega}_{\beta,\eta,w}(A) = \rho_{\beta,\eta,w}(A)/\rho_{\beta,\eta,w}(\mathbb{1}), \tag{9.4}$$

where

$$\rho_{\beta,\eta,w}(A) = \sum_{n=0}^{\infty}(-1)^n \int_{0\leq s_1\leq\dots\leq s_n\leq\beta} ds_1\dots ds_n\; G_{A,V,\dots,V}(i s_1,\dots,i s_n). \tag{9.5}$$

(Cf. [Ara 76], [BrR 81]).

Now, let $F_{\beta,\eta} \in \mathfrak{C}(\Lambda(\beta))$ be the two-point function of the oscillator in the Lamb model, i.e. the extension to $\Lambda(\beta)$ of $t \mapsto \langle 1, Q_0^{(\beta,\eta)}Q_t^{(\beta,\eta)}1\rangle$:

$$F_{\beta,\eta}(z) = \int_{-\infty}^{\infty} \frac{\beta k}{1-e^{-\beta k}} \cdot \frac{2\eta}{|P_\eta(i\,k)|^2} \cdot e^{i\,k\,z}\,\frac{dk}{2\pi}.$$

Using the CCR and the form of C_β, one derives that, for all $\lambda_0,\dots,\lambda_n \in \mathbb{R}$, $t_1,\dots,t_n \in \mathbb{R}$:

$$\omega\left(W_\beta(\lambda_0 q^{(\eta)})W_\beta(\lambda_1 T_{t_1} q^{(\eta)})\dots W_\beta(\lambda_n T_{t_n} q^{(\eta)})\right) =$$

$$= \exp\left(-\sum_{j,k=0}^{n} \lambda_j\lambda_k\,\theta(j-k)\,F_{\beta,\eta}(t_j - t_k)\right). \tag{9.6}$$

Here, we have put $0 = t_0$. We define: $\theta(j-k)$ is equal to 0 if $j<k$, to $\frac{1}{2}$ if $j=k$ and to 1 if $j>k$. Because $F_{\beta,\eta} \in \mathfrak{C}(\Lambda(\beta))$, (9.6) extends to a function in $\mathfrak{C}(\Lambda_n(\beta))$, whose restriction to purely imaginary arguments $\{i\,s_1,\dots,i\,s_n\}$ is given by

$$\exp\left(-\tfrac{1}{2}\sum_{j,k=0}^{n}\lambda_j\lambda_k\,F_{\beta,\eta}(i|s_j - s_k|)\right).$$

Therefore, if $w = \hat{\mu}$ with $\mu \in M$, by (9.5),

$$\rho_{\beta,\eta,w}(W_\beta(\lambda_0\,q^{(\eta)})) = \sum_{n=0}^{\infty}(-1)^n \int_{0\leq s_1\leq\dots\leq s_n\leq\beta} ds_1\dots ds_n \int_{-\infty}^{\infty}\mu(d\lambda_1)\dots\int_{-\infty}^{\infty}\mu(d\lambda_n)$$

$$\times \exp\left(-\tfrac{1}{2}\sum_{j,k=0}^{n}\lambda_j\lambda_k\,F_{\beta,\eta}(i|s_j - s_k|)\right), \tag{9.7}$$

where, again, $s_0 = 0$.

We look for a similar formula for the isolated oscillator.

Perturbation of the harmonic oscillator

The phase space of the harmonic oscillator is the symplectic space $\{\mathbb{R}^2,\sigma_1\}$, (cf. end of § I.4). The Schrödinger representation $W_{S,\beta}$ of the CCR over $\{\mathbb{R}^2,\beta\,\sigma_1\}$ in the Hilbert space $L^2(\mathbb{R})$ is given by

$$W_{S,\beta}(\alpha_1,\alpha_2) = \exp(-\tfrac{1}{2}i\,\beta\,\alpha_1\alpha_2)\,T_{\beta\alpha_2}\,\exp(-i\,\alpha_1\,\cdot).$$

Now, let $q^{(0)} = \{1,0\} \in \mathbb{R}^2$ and let $D_t : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$D_t = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}.$$

Then the position operator $Q_t^{(\beta,0)}$ of the oscillator at time t is given by

$$\exp(-i\lambda Q_t^{(\beta,0)}) = W_{S,\beta}(\lambda D_t q^{(0)}).$$

It is well known that

$$\exp(i\,t\,H^0_{S,\beta}/\beta)\; Q_0^{(\beta,0)}\; \exp(-i\,t\,H^0_{S,\beta}/\beta) = Q_t^{(\beta,0)}. \qquad (9.8)$$

Note that $W_{S,\beta}(\lambda\, q^{(0)}) = e^{-i\lambda\cdot}$.

Now define, for $A \in \mathcal{L}(L^2(\mathbb{R}))$,

$$\omega_{S,\beta}(A) = \mathrm{tr}(A\, e^{-H^0_{S,\beta}})/\mathrm{tr}(e^{-H^0_{S,\beta}}).$$

Then $\omega_{S,\beta}(W_{S,\beta}(\alpha_1,\alpha_2)) = \exp(-(\tfrac14 \beta \coth \tfrac12 \beta)(\alpha_1^2+\alpha_2^2))$, and again we have for all $\lambda_0,\dots,\lambda_n \in \mathbb{R}$, $t_1,\dots,t_n \in \mathbb{R}$:

$$\omega_{S,\beta}\Big(W_{S,\beta}(\lambda_0 q^{(0)})\, W_{S,\beta}(\lambda_1 D_{t_1} q^{(0)}) \dots W_{S,\beta}(\lambda_n D_{t_n} q^{(0)})\Big)$$

$$= \exp\Big(-\sum_{j,k=0}^{n} \lambda_j \lambda_k \theta(j-k) F_{\beta,0}(t_j - t_k)\Big),$$

where $F_{\beta,0} \in \mathfrak{C}(\Lambda(\beta))$ is the two-point function of the harmonic oscillator, given by

$$F_{\beta,0}(z) = \frac{\beta}{2(1-e^{-\beta})}\, e^{iz} - \frac{\beta}{2(1-e^{\beta})}\, e^{-iz}.$$

The analogue of $\rho_{\beta,\eta,w}$ is $\rho_{\beta,0,w}$, given by

$$\rho_{\beta,0,w}(A) = \mathrm{tr}(A\, e^{-H^w_{S,\beta}})/\mathrm{tr}\, e^{-H^0_{S,\beta}}. \qquad (9.9)$$

By the perturbation theory of one parameter semigroups of contractions on a Hilbert space, [Kat 66], we have, (when omitting indices β and $\eta = 0$),

$$\rho_w(W_S(\lambda_0 q)) = \mathrm{tr}\Big(W_S(\lambda_0 q)\, e^{-(H_S + w(Q))}\Big)/\mathrm{tr}(e^{-H_S})$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_{0 \le s_1 \le \dots \le s_n \le \beta} ds_1 \dots ds_n \;\mathrm{tr}\Big(W_S(\lambda_0 q)\, \times$$

$$\times\, e^{-s_1 H_S/\beta}\, w(Q)\, e^{-(s_2 - s_1) H_S/\beta}\, w(Q) \dots w(Q)\, e^{-(\beta - s_n) H_S/\beta}\Big)/\mathrm{tr}(e^{-H_S}).$$

Now, by (9.8), the integrand is the restriction to purely imaginary arguments, of the function on $\Lambda_n(\beta)$, given on $\mathbb{R}^n$ by

$$\omega_S\Big(W_S(\lambda_0 q)\, w(Q_{t_1})\, w(Q_{t_2}) \dots w(Q_{t_n})\Big) =$$

$$= \int_{-\infty}^{\infty} \mu(d\lambda_1) \dots \int_{-\infty}^{\infty} \mu(d\lambda_n)\, \omega_S(W_S(\lambda_0 q) W_S(\lambda_1 q_{t_1}) \dots W_S(\lambda_n q_{t_n})) =$$

$$= \int_{-\infty}^{\infty} \mu(d\lambda_1) \dots \int_{-\infty}^{\infty} \mu(d\lambda_n) \exp\Big(-\sum_{j,k=0}^{n} \lambda_j \lambda_k\, \theta(j-k) F_{\beta,0}(t_j - t_k)\Big).$$

Therefore, (9.7), with W_β replaced by $W_{S,\beta}$, is valid for $\eta = 0$.

The estimate

Lemma 9.1. Let $\beta > 0$, $\eta > 0$. The functions $\{s,u\} \mapsto F_{\beta,\eta}(i|s-u|)$ and $\{s,u\} \mapsto F_{\beta,0}(i|s-u|) - F_{\beta,\eta}(i|s-u|)$ are positive definite kernels on $[0,\beta] \times [0,\beta]$. Moreover, for all $s \in [0,\beta]$,

$$|F_{\beta,0}(i\,s) - F_{\beta,\eta}(i\,s)| \le \eta \sum_{m=-\infty}^{\infty} \frac{2\pi|m|/\beta}{\left(\left(\frac{2\pi m}{\beta}\right)^2 + 1\right)^2}. \qquad (9.10)$$

Remark. The function $\varphi(\beta)$, occuring on the r.h.s. of (9.10) with a coefficient η, has the following behaviour, (cf. Fig. 8):

$$\text{For all } \beta > 0:\ \varphi(\beta) < \beta/2\pi \text{ and } \varphi(\beta) < 2\,\zeta(3)\left(\frac{\beta}{2\pi}\right)^3. \qquad (9.11)$$

$$\varphi(\beta) = \frac{\beta}{2\pi} + O(\beta^{-1}),\ (\beta \to \infty), \text{ and } \varphi(\beta) = 2\,\zeta(3)\left(\frac{\beta}{2\pi}\right)^3 + O(\beta^4),\ (\beta \downarrow 0). \qquad (9.12)$$

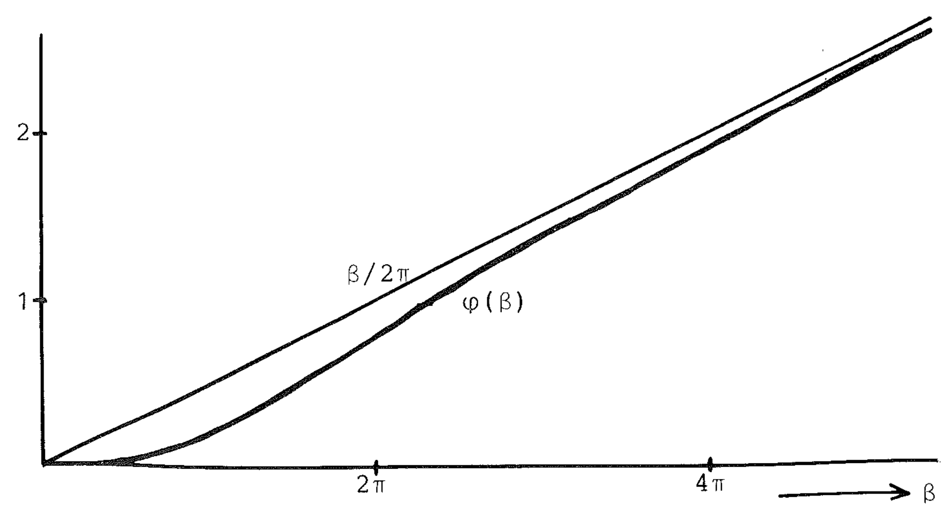

Fig. 8. Graph of $\varphi(\beta)$.

Proof. Recall that $\mathcal{P}_\eta(x) = x^2 + \eta x + 1$. A computation of the Fourier coefficients of $s \mapsto F_{\beta,\eta}(i\,s)$ on $[0,\beta]$ results in the, uniformly convergent, series expansion

$$F_{\beta,\eta}(i\,s) = \sum_{m=-\infty}^{\infty} \frac{\exp\left(\frac{2\pi i}{\beta}\, m\, s\right)}{\mathcal{P}_\eta\left(\frac{2\pi}{\beta}|m|\right)} . \tag{9.13}$$

Because the Fourier coefficients are positive, $\{s,u\} \mapsto F_{\beta,\eta}(i|s-u|)$ is positive definite. And, because the difference

$$\frac{1}{\mathcal{P}_0\left(\frac{2\pi}{\beta}|m|\right)} - \frac{1}{\mathcal{P}_\eta\left(\frac{2\pi}{\beta}|m|\right)} = \frac{2\pi|m|\eta/\beta}{\mathcal{P}_0\left(\frac{2\pi}{\beta}|m|\right)\mathcal{P}_\eta\left(\frac{2\pi}{\beta}|m|\right)} \tag{9.14}$$

is also positive, the difference of the corresponding kernels is positive definite. Finally, the right hand side of (9.14) is bounded by

$$\eta\, \frac{2\pi|m|/\beta}{\left(\left(\frac{2\pi m}{\beta}\right)^2 + 1\right)^2} ,$$

and (9.10) follows. □

Theorem 9.2. Let $\beta,\eta > 0$, and let $w,w',w'' \in \hat{M}$. Let $\nu_{\beta,\eta,w}$ be the equilibrium distribution of the oscillator in the Lamb model, and $\nu_{\beta,0,w}$ the quantum-mechanical Gibbs distribution, (cf. (9.1), (9.3)). Then, for all $\lambda \in \mathbb{R}$,

$$\left|\hat{\nu}_{\beta,\eta,w}(\lambda) - \hat{\nu}_{\beta,0,w}(\lambda)\right| \leqslant \tfrac{1}{2}\,\eta\,\varphi(\beta) \cdot \frac{\operatorname{tr}\exp(-H^0_\beta)}{\operatorname{tr}\exp(-H^w_\beta)} \cdot (\lambda^2 + 2\|w''\|_{\hat{M}})\, e^{\|w\|_{\hat{M}}} .$$

Remark. We note that $\eta\varphi(\beta) \leqslant \frac{\zeta(3)}{(2\pi)^3}\,\eta\beta^3$. Therefore, the difference (9.3) can be made small, either by choosing β small, or by choosing η small. For fixed w, decreasing β or η leads outside the region C. For $\beta = 0$ we regain the classical result

$$\nu_{0,\eta,w} = \nu_{0,0,w},$$

i.e., the Gibbs measure is independent of the friction coefficient.

Proof. Let $w = \hat{\mu}$, $\mu \in M$. We use Lemma 9.1 to estimate the difference of the un-normalised KMS states $\rho_{\beta,\eta,w}$ and $\rho_{\beta,0,w}$. By (9.7) and its version for the isolated oscillator, we have

$$|\rho_{\beta,\eta,w}(W_\beta(\lambda q^{(\eta)})) - \rho_{\beta,0,w}(W_{S,\beta}(\lambda q^{(0)}))| \leqslant$$

$$\leqslant \sum_{n=0}^{\infty} \beta^{-n} \int_{0\leqslant s_1\leqslant\ldots\leqslant s_n\leqslant\beta} ds_1\ldots ds_n \int_{-\infty}^{\infty} \mu^+(d\lambda_1)\ldots \int_{-\infty}^{\infty} \mu^+(d\lambda_n)\, \times$$

$$\times \left|\exp\left(-\tfrac{1}{2}\sum_{j,k=0}^{n} \lambda_j\lambda_k\, F_{\beta,\eta}(i|s_j - s_k|)\right) - \exp\left(-\tfrac{1}{2}\sum_{j,k=0}^{n} \lambda_j\lambda_k\, F_{\beta,0}(i|s_j - s_k|)\right)\right| \tag{9.15}$$

where $s_0 = 0$.

Now, let us call the arguments of the exp functions in (9.15) $-x$ and $-y$ respectively. Then Lemma 9.1 says that $0 \leqslant x \leqslant y$. It follows that $|e^{-x} - e^{-y}| \leqslant y - x$. Therefore, the r.h.s. of (9.15) is bounded from above by

$$\sum_{n=0}^{\infty} \beta^{-n} \int_{0\leqslant s_1\leqslant\ldots\leqslant s_n\leqslant\beta} ds_1\ldots ds_n \int_{-\infty}^{\infty} \mu^+(d\lambda_1)\ldots \int_{-\infty}^{\infty} \mu^+(d\lambda_n)\, \times$$

$$\times\ \tfrac{1}{2}\sum_{j,k=0}^{n} \lambda_j\lambda_k\left(F_{\beta,0}(i|s_j - s_k|) - F_{\beta,\eta}(i|s_j - s_k|)\right) . \tag{9.16}$$

We interchange the sum over j and k with the λ-integrals, and perform the latter. If $j \neq k$, the integral over $\lambda_j\lambda_k$ yields zero, because μ^+ is a symmetric measure. If $j = k = 0$, it yields $\lambda_0^2 \cdot \|w\|^n_{\hat{M}}$, and if $j = k \neq 0$: $\|w''\|_{\hat{M}} \cdot \|w\|^{n-1}_{\hat{M}}$. The s-integral then becomes simple to perform, because only zero remains as an argument for the functions $F_{\beta,0}$ and $F_{\beta,\eta}$. It gives a factor $\beta^n/n!$. So (9.16) is equal to

$$\tfrac{1}{2}(F_{\beta,0}(0) - F_{\beta,\eta}(0)) \cdot \sum_{n=0}^{\infty} \frac{1}{n!}(\lambda_0^2\|w\|^n_{\hat{M}} + n\|w''\|_{\hat{M}} \cdot \|w\|^{n-1}_{\hat{M}})$$

$$= \tfrac{1}{2}(F_{\beta,0}(0) - F_{\beta,\eta}(0)) \cdot (\lambda_0^2 + \|w''\|_{\hat{M}})\, e^{\|w\|_{\hat{M}}} .$$

Here is an upper bound for $|\rho_{\beta,\eta,w}(W_\beta(\lambda q^{(\eta)})) - \rho_{\beta,0,w}(W_{S,\beta}(\lambda q^{(0)}))|$. To derive from it an upper bound for the difference of the normalised states (cf. (9.4) and (9.9)), we argue as follows: If $x,y \in \mathbb{C}$ and $x_0, y_0 > 0$ are such that $|x| \leqslant x_0$ and $|y| \leqslant y_0$, then

$$\left|\frac{x}{x_0} - \frac{y}{y_0}\right| = \frac{1}{x_0 y_0}\left|xy_0 - yx_0\right| = \frac{1}{x_0 y_0}\left|x(y_0 - x_0) - x_0(y - x)\right| \leqslant \frac{1}{y_0}\left(|y_0 - x_0| + |y - x|\right).$$

Applying this inequality, we obtain

$$\left|\hat{\nu}_{\beta,\eta,w}(\lambda) - \hat{\nu}_{\beta,0,w}(\lambda)\right| \leqslant \tfrac{1}{2}\eta\varphi(\beta)\,\frac{\lambda^2 + 2\|w''\|_{\hat{M}}}{\rho_{\beta,0,w}(\mathbb{1})}\, e^{\|w\|_{\hat{M}}}.$$

The statement follows by (9.9).

□

Chapter III

A POINT CHARGE IN A QUANTUM FIELD

In this chapter it is shown that the system of a quantum harmonic oscillator, coupled to a massless scalar field, becomes equivalent to the Lamb model in the point charge limit.

This chapter is based on an earlier paper [Maa 82b].

Introduction

In 1964 Schwabl and Thirring [ScT 64] introduced an exactly solvable model for a molecule in a radiation field in order to discuss general questions concerning laser physics. The model consists of a harmonic oscillator, coupled to a massless scalar field by means of an interaction Hamiltonian

$$H_I = Q \int_{\mathbb{R}^3} \rho(x)\,\varphi(x)\,d_3x\,. \qquad (1)$$

Here, Q is the oscillator's position operator, φ the field, and ρ a certain smooth spherically symmetric function, to be interpreted as a charge density.

In fact, the authors were not so much interested in this model, as in its renormalised version, i.e. the limit of the model as ρ tends to the density $e\delta$ of a point charge. Now, if one lets ρ shrink to $e\delta$, one must meanwhile increase the spring constant a of the oscillator, in order that the relevant quantity - the two-point function - tends to a limit. Luckily, this can be done without loosing the positivity of the Hamiltonian as a function on phase space. Thus "runaway solutions", which are such a nuisance in the electromagnetic case (cf. [vKa 51]), are avoided.

Now, the formulas obtained by this procedure strongly remind one of the Lamb model. Indeed, in this chapter we shall show that the renormalised Schwabl-Thirring model and the Lamb model are isomorphic as symplectic spaces (including the symplectic flows). This makes them automatically isomorphic after quantisation.

The quantum version of the Schwabl-Thirring model in the natural Hilbert space of the uncoupled models has been constructed recently by Asao Arai [Arai 81]. We do not take over his construction here, because in the point charge limit time-evolution is no longer unitarily implementable on the abovementioned Hilbert space.

Let us proceed to the construction of the model.

The free field

As a phase space for the relevant, namely the spherically symmetric, part of the free field we choose $\Phi_0 = \mathcal{S}_{sym}(\mathbb{R}^3)^2$, the set of all pairs of spherically symmetric, infinitely differentiable and rapidly decreasing functions $\mathbb{R}^3 \to \mathbb{R}$. The natural symplectic form σ_{Φ_0} on Φ_0 is given by

$$\sigma_{\Phi_0}(\varphi_1 \oplus \pi_1, \varphi_2 \oplus \pi_2) = \int_{\mathbb{R}^3} (\varphi_1\pi_2 - \varphi_2\pi_1) d_3x.$$

For $\varphi \oplus \pi \in \Phi_0$, let $\tilde{S}_t(\varphi \oplus \pi)$ denote the phase space point $\varphi_t \oplus \dot{\varphi}_t$, where $\{\varphi_t\}_{t \in \mathbb{R}}$ is the solution of the free wave equation $\ddot{\varphi}_t = \Delta\varphi_t$ with initial conditions $\varphi_0 = \varphi$ and $\dot{\varphi}_0 = \pi$. An element x of Φ_0 can be viewed either as a point of phase space, evolving according to $t \to \tilde{S}_t x$, or as a linear functional on phase space, acting as $y \to \sigma_{\Phi_0}(x,y)$, and evolving the other way: as $t \to \tilde{S}_{-t}x =: S_t x$. Because S_t preserves σ_{Φ_0}, the triple $\{\Phi_0, \sigma_{\Phi_0}, \{S_t\}\}$ forms a *Boson single particle space* in the sense of Weinless [Wei 69] . Such spaces are the starting point for quantisation according to Segal [Seg 59].

Now, if $\psi \in \mathcal{S}_{sym}(\mathbb{R}^3)$, let ψ_r denote the even function in $\mathcal{S}(\mathbb{R})$, defined by

$$\hat{\psi}_r(\omega) = \hat{\psi}(\omega,0,0), \quad (\omega \in \mathbb{R}). \tag{2}$$

Proposition 1. Let $j : \Phi_0 \to \mathcal{S}(\mathbb{R})$ be given by

$$j(\varphi \oplus \pi) = \frac{1}{\sqrt{2\pi}} (\varphi_r' + \pi_r).$$

Then j is an isomorphism of the Boson single particle spaces $\{\Phi_0, \sigma_{\Phi_0}, \{S_t\}_{t \in \mathbb{R}}\}$ and $\{\mathcal{S}(\mathbb{R}), \sigma, \{T_t\}_{t \in \mathbb{R}}\}$.

Proof. For every $f \in \mathcal{S}(\mathbb{R})$ there are functions φ and π in $\mathcal{S}_{sym}(\mathbb{R}^3)$, such that π_r is the even part and φ_r' is the odd part of $\sqrt{2\pi}\, f$. Therefore, j is invertible.

Now, a consequence of the definition (2) is that

$$-2\pi s\psi(s,0,0) = \psi_r'(s), \quad (s \in \mathbb{R}). \tag{3}$$

Therefore, for all $x \in \mathbb{R}^3$,

$$(j^{-1}f)(x) = \varphi(x) \oplus \pi(x) = \frac{-1}{2\cdot\sqrt{2\pi}}\left(\frac{f(\|x\|) - f(-\|x\|)}{\|x\|} \oplus \frac{f'(\|x\|) - f'(-\|x\|)}{\|x\|}\right).$$

Now, one checks that $j^{-1} \circ \tilde{T}_t f = j^{-1}(f(\cdot + t))$ satisfies the defining properties of $\tilde{S}_t(\varphi \oplus \pi)$. It follows that $\tilde{S}_t \circ j^{-1} = j^{-1} \circ \tilde{T}_t$.

Finally, by (3), we have for all $\varphi_1 \oplus \pi_1, \varphi_2 \oplus \pi_2 \in \Phi_0$:

$$\sigma(j(\varphi_1 \oplus \pi_1), j(\varphi_2 \oplus \pi_2)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\varphi_{1,r}' + \pi_{1,r})(\varphi_{2,r}' + \pi_{2,r})' \, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\varphi_{1,r}' \pi_{2,r}' - \pi_{1,r}' \varphi_{2,r}') \, ds = 2\pi \int_{-\infty}^{\infty} s^2 (\varphi_1\pi_2 - \varphi_2\pi_1)(s,0,0) \, ds =$$

$$= \int_{\mathbb{R}^3} (\varphi_1\pi_2 - \varphi_2\pi_1) d_3x = \sigma_{\Phi_0}(\varphi_1 \oplus \pi_1, \varphi_2 \oplus \pi_2).$$

□

A direct consequence of the above is that $E_\beta \circ j$ is a quantisation of $\{\Phi_0, \sigma_{\Phi_0}, \{S_t\}\}$. We shall treat the interacting system of oscillator and field in the same manner.

The interacting system

Take some $\rho \in \mathcal{S}_{sym}(\mathbb{R}^3)$, such that $\rho(x) > 0$ and $\hat{\rho}(k) > 0$ for all x, $k \in \mathbb{R}^3$ and $\hat{\rho}(0) = \int \rho \, d_3x = e$. We interpret ρ as a charge density and e as a charge. Define the unrenormalised spring constant a_ρ by

$$a_\rho = 1 + \int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{\|k\|^2} \frac{d_3k}{(2\pi)^3}. \tag{4}$$

The phase space of the entire system is $\Phi = \mathbb{R}^2 \oplus \Phi_0$, and on it the natural symplectic form σ_Φ is given by

$$\sigma_\Phi(q_1 \oplus p_1 \oplus \varphi_1 \oplus \pi_1, q_2 \oplus p_2 \oplus \varphi_2 \oplus \pi_2) =$$

$$= q_1p_2 - q_2p_1 + \int_{\mathbb{R}^3} (\varphi_1\pi_2 - \varphi_2\pi_1) d_3x.$$

The evolution $\{\tilde{S}_t^{(\rho)} : \Phi \to \Phi\}_{t \in \mathbb{R}}$ is defined by $\tilde{S}_t^{(\rho)}(q \oplus p \oplus \varphi \oplus \pi) = q_t \oplus \dot{q}_t \oplus \varphi_t \oplus \dot{\varphi}_t$, where $\{q_t, \varphi_t\}$ is the solution of the differential equations

$$\ddot{q}_t + a_\rho q_t = -\int \rho\varphi_t \, d_3x \text{ , and} \tag{5a}$$

$$\ddot{\varphi}_t - \Delta\varphi_t = -q_t \rho \text{ ,} \tag{5b}$$

with initial conditions $q_0 = q$, $\dot{q}_0 = p$, $\varphi_0 = \varphi$, $\dot{\varphi}_0 = \pi$. Let $S_t^{(\rho)} = \tilde{S}_{-t}^{(\rho)}$. Again $\{\Phi, \sigma_\Phi, \{S_t^{(\rho)}\}\}$ is a Boson single particle space.

Let $u_q = 0 \oplus (-1) \oplus 0 \oplus 0$ and $u_p = 1 \oplus 0 \oplus 0 \oplus 0$. Then u_q and u_p correspond to the linear observables of position and momentum of the oscillator, in the sense that for any phase space point $x = q \oplus p \oplus \varphi \oplus \pi$, $\sigma_\Phi(u_q, x) = q$ and $\sigma_\Phi(u_p, x) = p$. Accordingly, u_q and u_p should evolve with $S_t^{(\rho)}$. Let $\zeta_\rho : \mathbb{R} \to \mathbb{R}$ be defined by

$$\zeta_\rho(t) = \Theta(-t)\sigma_\Phi(u_q, S_{-t}^{(\rho)} u_q) \text{ .}$$

Then, for $t > 0$, $\zeta_\rho(-t)$ is equal to the position of the oscillator at time t, when initially the system starts in $0 \oplus 1 \oplus 0 \oplus 0$. Now, if $a_\rho > \left\| \frac{\hat{\rho}}{\|k\|} \right\|^2$, as is assured by (4), the Hamiltonian of the system is a strictly positive quadratic form on Φ, and hence $\sigma_\Phi(x, S_t^{(\rho)} y)$ remains bounded for all $x, y \in \Phi$. It follows that $\hat{\zeta}_\rho(\omega)$ is defined for all $\omega \in \mathbb{C}$ with $\mathrm{Im}\, \omega > 0$. One then computes from (5) that

$$\zeta_\rho(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + 1 - i\omega\eta_\rho(\omega)} \frac{d\omega}{2\pi} \text{ ,}$$

where $\eta_\rho(\omega) = \frac{1}{2\pi} \int_{x \geq y} \rho_r(x)\rho_r(y)\, e^{i\omega(x-y)} dxdy$. Notice that η_ρ is infinitely differentiable, $|\eta_\rho(\omega)| \leq e^2/4\pi$, and $\mathrm{Re}\, \eta_\rho(\omega) = |\hat{\rho}_r(\omega)|^2/4\pi > 0$. It follows that, if we define

$$\|f\|_{n,m}^2 = \int_{-\infty}^{\infty} t^{2n} |f^{(m)}(t)|^2 \, dt \text{ ,}$$

then

$$\|\zeta_\rho\|_{n,0} \text{ , } \|\zeta_\rho\|_{n,1} < \infty \text{ for all } n \in \mathbb{N}. \tag{6}$$

Let $j' : \Phi \to \mathcal{S}(\mathbb{R}) : j'(q \oplus p \oplus \varphi \oplus \pi) = j(\varphi \oplus \pi)$. By $\{S_t^{(0)}\}$ we shall mean the uncoupled evolution, defined by (5) if one sets $\rho = 0$.

Theorem 2. For all $x \in \Phi$, the limit

$$\theta_\rho x := \lim_{t \to \infty} j' \circ S_{-t}^{(0)} \circ S_t^{(\rho)} x \tag{7}$$

exists in each of the norms $\|\cdot\|_{n,m}$, $(n, m \in \mathbb{N})$. The map θ_ρ thus defined is an isomorphism of the Boson single particle spaces $\{\Phi, \sigma_\Phi, S_t^{(\rho)}\}\}$ and $\{\mathcal{S}(\mathbb{R}), \sigma, \{T_t\}\}$. Furthermore,

$$\theta_\rho u_q = \rho_r * \zeta_\rho \text{ ; } \theta_\rho u_p = -\rho_r * \zeta'_\rho \text{.}$$

Sketch of the proof. The equations of motion (5) imply that

$$\frac{d}{dt} j' \circ S_{-t}^{(0)} \circ S_t^{(\rho)} x = (2\pi)^{-\frac{1}{2}} \sigma_\Phi(u_q, S_t^{(\rho)} x) \, T_{-t}\rho_r \text{ .} \tag{8}$$

We shall show that $\sigma_\Phi(u_q, S_t^{(\rho)} x)$ tends to zero as t goes to ∞, faster than any power of t. Indeed, if x is a combination of u_q and u_p, it certainly does by (6). Moreover, if $f = j(\varphi \oplus \pi) \in \mathcal{S}(\mathbb{R})$, then, by (5),

$$\sigma_\Phi(u_q, S_t^{(\rho)}(0 \oplus 0 \oplus \varphi \oplus \pi)) = \int_{-t}^{0} \zeta_\rho(s)\sigma_{\Phi_0}(0 \oplus \rho, S_{t+s}(\varphi \oplus \pi)) ds =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-t}^{0} \zeta_\rho(s)\sigma(\rho_r, T_{t+s} f) \, ds = \frac{1}{\sqrt{2\pi}} \Big(\zeta_\rho * (\theta_- \cdot (\rho'_r * f)) \Big)(-t) \text{ ,} \tag{9}$$

where $\theta_-(t) = \theta(-t)$. The r.h.s. of (9) has the claimed property, being a convolution product of rapidly decreasing functions. Now, by (8), for all $n, m \in \mathbb{N}$:

$$\int_0^\infty \left\| \frac{d}{dt} j' \circ S_{-t}^{(0)} \circ S_t^{(\rho)} x \right\|_{n,m} dt = \int_0^\infty |\sigma_\Phi(u_q, S_t^{(\rho)} x)| \cdot \|T_{-t}\rho_r\|_{n,m} \, dt$$

$$\leq C \cdot \|\rho_r\|_{n,m} \cdot \int_0^\infty t^{2n} |\sigma_\Phi(u_q, S_t^{(\rho)} x)| \, dt,$$

where C is a suitably chosen constant. The above expression is finite; therefore the limit (7) exists for all $x \in \Phi$, $n, m \in \mathbb{N}$ and $\theta_\rho x$ is in $\mathcal{S}(\mathbb{R})$ because $\mathcal{S}(\mathbb{R}) = \bigcap_{n,m} \overline{\mathcal{S}(\mathbb{R})}^{n,m}$. The invertibility of θ_ρ is shown by treating $S_{-t}^{(\rho)} \circ S_t^{(0)}$ likewise. θ_ρ is symplectic and intertwines $S_t^{(\rho)}$ and T_t. In fact, θ_ρ can be explicitly computed without difficulty [ScT 64]. Its action on u_q or u_p is obtained by putting $x = u_q$ or u_p in (8) and integrating. □

As a consequence, $E_\beta \circ \theta_\rho$ is a quantisation of the interacting system $\{\varphi, \sigma_\phi, S_t^{(\rho)}\}$. The position operator at time t of the oscillator in this quantisation is

$$Q_t^{(\beta,\rho)} := E_\beta \circ \theta_\rho \circ S_t^{(\rho)} u_q = E_\beta(T_t(\rho_r * \zeta_\rho)),$$

and its two-point function $F_{\beta,\rho}(t) := \langle 1, Q_0^{(\beta,\rho)} Q_t^{(\beta,\rho)} 1\rangle$ is

$$F_{\beta,\rho}(t) = \int_{-\infty}^{\infty} \frac{\beta\omega}{1 - e^{-\beta\omega}} \cdot \frac{|\hat{\rho}_r(\omega)|^2 e^{i\omega t}}{|-\omega^2 + 1 - i\omega\eta_\rho(\omega)|^2} \frac{d\omega}{(2\pi)^2} .$$

On the other hand, the two-point function of the Lamb model is, (cf. § II.9),

$$F_{\beta,\eta}(t) := \langle 1, Q_0^{(\beta,\eta)} Q_t^{(\beta,\eta)} 1\rangle = \int_{-\infty}^{\infty} \frac{\beta\omega}{1 - e^{-\beta\omega}} \frac{2\eta\, e^{i\omega t}}{|-\omega^2 + 1 - i\omega\eta|^2} \frac{d\omega}{2\pi} .$$

If we now fix ρ, define $\rho_\varepsilon(x) = \varepsilon^{-3} \rho(\varepsilon^{-1}x)$, and put $\eta = e^2/4\pi$ with $e = \int \rho\, dx$, then we see that

$$\lim_{\varepsilon \downarrow 0} F_{\beta,\rho_\varepsilon}(t) = F_{\beta,\eta}(t), \quad (t \in \mathbb{R}).$$

The quantisation E_β is Gaussian. Therefore all correlations of the $Q_t^{(\beta,\rho)}$ and $Q_t^{(\beta,\eta)}$, $(t \in \mathbb{R})$, are expressible as finite sums of products of two-point functions, and it follows that all correlation functions of the Schwabl-Thirring model tend to the corresponding ones of the Lamb model in the point charge limit.

Appendix A : UNITS AND PARAMETERS

A copy of the harmonic Lamb model is determined by 5 specifications: the mass m and the spring constant a of the oscillator, the mass density μ and the tension α of the string, and the temperature T.

The classical Lamb model has only one genuine parameter, however; we choose

$$\eta = \left(\frac{\mu\alpha}{ma}\right)^{\frac{1}{2}}.$$

The quantum Lamb model has two, the one above, and, say,

$$\beta = \hbar (a/m)^{\frac{1}{2}}/kT.$$

Two models with the same values of η and β differ only by some scale factors in space and in time.

In order to make the above facts manifest in our notation, we have chosen the following quantities as units.

Unit of mass	: m
Unit of time	: ω^{-1}, where $\omega = (a/m)^{\frac{1}{2}}$ is the angular frequency of the free motion of the oscillator
Unit of horizontal length	: $c\,\omega^{-1}$, where $c = (\alpha/\mu)^{\frac{1}{2}}$ is the velocity of wave propagation along the string
Unit of vertical length	: $(kT/a)^{\frac{1}{2}}$, the variance of the oscillator's position in thermal equilibrium.

An advantage of the latter choice of a vertical length unit is that the scaling equivalence $\varphi \mapsto \lambda\varphi$, $T \to \lambda^{\frac{1}{2}} T$, ($\varphi$ is the string's deflection), which is valid for large temperatures, is divided out, so that indeed only η remains as a parameter in the classical case.

A consequence of this choice is that β tends to show up everywhere, where one expects an $\hbar$, and not where one expects $1/kT$.

Appendix B : KOLMOGOROV DECOMPOSITIONS OF POSITIVE DEFINITE KERNELS

For details we refer to [EvL 77]. We consider only complex-valued kernels here.

Let V be a map from a set X to a complex Hilbert space H. Define

$$\kappa(x,y) = \langle V(x),V(y)\rangle, \quad (x,y \in X). \tag{B.1}$$

Then, for all $c_j \in \mathbb{C}$, $x_j \in X$, $(j = 1,\dots,n)$,

$$\sum_{j,k=1}^{n} c_j^* c_k \,\kappa(x_j,x_k) \geq 0. \tag{B.2}$$

Indeed, the l.h.s. is equal to

$$\left\| \sum_{k=1}^{n} c_k V(x_k) \right\|^2 . \tag{B.3}$$

Now, a map $\kappa : X \times X \to \mathbb{C}$ is called a *kernel*, and if it satisfies (B.2) it is said to be *positive definite*. A map $V : X \to H$ is called a *Kolmogorov decomposition of* κ if (B.1) holds. It is called *minimal* if the span of $V(x)$, $(x \in X)$ is dense in H.

Proposition B. Let X be a set. Every positive definite kernel $\kappa : X \times X \to \mathbb{C}$ has, up to unitary equivalence, a unique minimal Kolmogorov decomposition.

Proof. For an existence proof, cf. [EvL 77]. For the kernels considered in the present thesis, Kolmogorov decompositions are explicitly indicated. The uniqueness proof is like that of Lemma I.5.7. □

REFERENCES

[Acc 75] Accardi L.: On the non-commutative Markov property, *Funct. Anal. Appl.* 9 (1975) 1-8.

[AfL 82] Accardi L., Frigerio A., Lewis J.T.: Quantum Stochastic Processes, *Publications of the R.I.M.S., Kyoto*, 18 (1982) 97-133.

[Ara 76] Araki H.: Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita-Takesaki Theory. In *C*-Algebras and Their Applications to Statistical Mechanics and Quantum Field Theory*, D. Kastler, ed., North Holland, 1976.

[Arai 81] Arai A.: On a model of a harmonic oscillator coupled to a quantized, massless, scalar field. I, *Journ. Math. Phys.* 22 (1981) 2539.

[Arn 78] Arnol'd V.I.: Mathematical methods of classical mechanics, Springer, 1978.

[ArW 63] Araki H., Woods E.J.: Representations of the Canonical Commutation Relations Describing a Nonrelativistic Infinite Free Bose Gas, *Journ. Math. Phys.* 4 (1963) 637-662.

[BeK 81] Benguria R., Kac. M.: Quantum Langevin Equation, *Phys. Rev. Lett.* 46 (1981) 1-4.

[BrR 79] Bratteli O., Robinson D.W.: Operator Algebras and Quantum Statistical Mechanics, Volume I, Springer, 1979.

[BrR 81] Idem, Volume II, Springer, 1981.

[ChM 74] Chernoff P.R., Marsden J.E.: Properties of Infinite Dimensional Hamiltonian Systems, *Lecture Notes in Mathematics* 425, Springer, 1974.

[Dav 76] Davies E.B.: Quantum Theory of Open Systems, Academic Press, 1976.

[Dek 81] Dekker H.: Classical and quantum mechanics of the damped harmonic oscillator, Physics Reports Preprint, 1981.

[Doo 53] Doob J.L.: Stochastic Processes, Wiley & Sons, Inc., 1953.

[EvL 77] Evans D.E., Lewis J.T.: Dilations of Irreversible Evolutions in Algebraic Quantum Theory, *Communications of the Dublin Institute of Advanced Studies*, Series A, 24 (1977).

[FKM 65] Ford G.W., Kac M., Mazur P.: Statistical Mechanics of Assemblies of Coupled Oscillators, *Journ. Math. Phys.* 6 (1965) 504-515.

[GaV 75] Gallavotti G., Verboven E.: On the classical KMS boundary condition, *Nuovo Cimento* 28B (1975) 274-286.

[GiS 72] Gihman I.I., Skorohod A.V.: Stochastic Differential Equations, Springer, 1972.

[GlJ 81] Glimm J., Jaffe A.: Quantum Physics, Springer, 1981.

[Hem 76] Hemmen J.L. van: Dynamics and ergodicity of the infinite harmonic crystal, thesis, Groningen University, 1976.

[HHW 67] Haag., Hugenholtz N.M., Winnink M.: On the equilibrium states in quantum statistical mechanics, *Comm. Math. Phys.* 5 (1967) 215-236.

[Hid 80] Hida T.: Brownian Motion, Springer, 1980.

[Kat 66] Kato T.: Perturbation theory for linear operators, Springer, 1966.

[vKa 51] Kampen N.G. van: Contribution to the quantum theory of light scattering, *Mat.-Fys. Medd. Dansk Vid. Selsk.* 26 (1951) no. 15, 1-77.

[LaL 75] Lanford O.E., Lebowitz J.L.: Time evolution and ergodic properties of harmonic systems, Lecture Notes in Physics 38, 144-177, Springer 1975.

[Lam 00] Lamb H., Proc. Lond. math. Soc. 2 (1900) 88.

[LeT 74] Lewis J.T., Thomas L.C.: A characterisation of the regular solutions of a linear stochastic differential equation, *Z. f. Wahrscheinlichkeitstheorie* 30 (1974) 45-55.

[LeT 75] Lewis J.T., Thomas L.C.: How to make a heat bath, in *Functional Integration and its Applications*, ed. A.M. Arthurs, Oxford, Clarendon, 1975.

[Lin 76] Lindblad G.: On the Generators of Quantum Dynamical Semigroups, *Comm. Math. Phys.* 48 (1976) 119-130.

[Lin 79] Lindblad G.: Non-Markovian Quantum Stochastic Processes and Their Entropy, *Comm. Math. Phys.* 65 (1979) 281-294.

[Maa 82a] Maassen H.: On the invertibility of Møller morphisms; to appear in *Journ. Math. Phys.*, 1982.

[Maa 82b] Maassen H.: A quantum field acting as a heat bath; to appear in *Physics Letters A*, 1982.

[Mes 79] Messer J.: Friction in Quantum Mechanics, *Acta Physica Austriaca* 50 (1979) 75-91.

[Min 59] Minlos R.A.: Generalised random processes and their extension to a measure, *Trudy Moskov. Obsc.* 8 (1959) 497 (in russian); *Selected transl. in math. stat. and prob.* 3 (1963) 291-313.

[Nel 67] Nelson E.: Dynamical theories of Brownian motion, Princeton University Press, 1967.

[Rob 73] Robinson D.W.: Return to Equilibrium, *Comm. Math. Phys.* 31 (1973) 171-189.

[RST 70] Rocca F., Sirugue M., Testard D.: On a Class of Equilibrium States under the Kubo-Martin-Schwinger Condition, *Comm. Math. Phys.* 19 (1970) 119-141.

[ScT 64] Schwabl F., Thirring W.: Quantum Theory of Laser Radiation, *Ergebn. Exakten Naturw.* 36 (1964) 219-242.

[Seg 59] Segal I.E.: Foundations of the theory of dynamical systems of infinitely many degrees of freedom, I, *Mat.-Fys. Medd. Dansk Vid. Selsk.* 31 (1959) no. 12, 1.

[Sim 74] Simon B.: The $P(\Phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, 1974.

[Sla 72] Slawny J.: On Factor Representations and the C*-Algebra of Canonical Commutation Relations, *Comm. Math. Phys.* 24 (1972) 151-170.

[SzN 53] Sz.-Nagy B.: Sur les contractions de l'espace de Hilbert, *Acta Sci. Math. Szeged* 15 (1953) 87-92, MR 15, 326.

[Tak 70] Takesaki M.: Tomita's Theory of Modular Hilbert Algebras and its Applications, *Lecture Notes in Mathematics* 128, Springer, 1970.

[Tak 72] Takesaki M.: Conditional Expectations in von Neumann Algebras, *J. Funct. Anal.* 9 (1972) 306-321.

[Thi 77] Thirring W.: Lehrbuch der Mathematischen Physik, I, Springer, 1977.

[Tro 77] Tropper M.M.: Ergodic and Quasideterministic Properties of Finite-Dimensional Stochastic Systems, *Journ. Stat. Phys.* 17 (1977) 491-509.

[Ull 66] Ullersma P.: An exactly solvable model for Brownian motion I,II,III,IV, *Physica* 32 (1966) 27-96.

[UhO 30] Uhlenbeck G.E., Ornstein L.S.: On the theory of the Brownian motion, *Phys. Rev.* 36 (1930) 823-841.

[Wei 69] Weinless M.: Existence and Uniqueness of the Vacuum for Linear Quantised Fields, *Journ. Funct. Anal.* 4 (1969) 350-379.

Special symbols in current use, with the number of the page on which they are introduced.

SAMENVATTING

Het beschrijven van wrijvings-achtige processen is nog steeds een fundamenteel probleem in de klassieke mechanika, en al helemaal in de kwantummechanika. Beide theorieën hebben een uitstekend lopend formalisme voor het beschrijven van gesloten fysische systemen, maar in dat formalisme komen remmende werkingen, energie-absorptie, en ook overgangswaarschijnlijkheden, niet voor. Van de andere kant zijn wrijvings-achtige processen, in het domein van de klassieke mechanika, uit de dagelijkse ervaring vertrouwd, en een bekend voorbeeld van een wrijvings-achtig proces uit de kwantummechanika is het naar beneden tuimelen van een elektron door de diverse schillen van een atoom heen, onder het uitzenden van steeds anders gekleurde lichtkwanten.

Het heersende idee is, dat dit dilemma grofweg als volgt moet worden opgelost.

Als een fysisch systeem met maar weinig vrijheidsgraden (d.w.z. mogelijke onafhankelijke bewegingsrichtingen) geïsoleerd is van de buitenwereld, of alleen in kontakt staat met een ander systeem met weinig vrijheidsgraden, dan gedraagt het zich wrijvingsloos, als een perpetuum mobile. Maar als het systeem open is, d.w.z. in kontakt staat met een grote buitenwereld (of "binnenwereld" van molekulen waaruit het is opgebouwd), dan kan het, in bijzondere gevallen, wrijvingsgedrag gaan vertonen. Voor zuiver wrijvingsgedrag is een buitenwereld met oneindig veel vrijheidsgraden nodig. Het tuimelende elektron, bijvoorbeeld, staat in kontakt met het elektromagnetisch veld om hem heen, dat met zijn bewegingen meegolft.

Stel nu, dat zo'n open systeem zich in een buitenwereld bevindt waarin een zekere (positieve) temperatuur heerst. Dan zullen de thermische bewegingen van de buitenwereld een chaotische invloed op het systeem uitoefenen, "ruis" genaamd. Een beweging, onder invloed van wrijving en ruis, wordt beschreven door een Langevin-vergelijking. Dit is een uitdrukking van de wet van Newton, waarin de kracht, naast een gewoon konservatief krachtveld, bestaat uit een wrijvingskracht en een ruis-kracht.

In de kwantummechanika ruist de buitenwereld al bij temperatuur nul. Daar wordt dus wrijvingsgedrag noodzakelijk door een Langevin-vergelijking beschreven.

Het probleem dat in dit proefschrift wordt behandeld, betreft een, zeer eenvoudig, open kwantumsysteem. De vraag luidt, of de beweging van dit systeem in zoverre inderdaad een wrijvingsproces is, dat het naar een toestand van thermisch evenwicht streeft.

Het systeem staat afgebeeld in Fig. 3 op pagina 5. De schuif, in het krachtveld van de veer, is het systeem. De strak gespannen snaar is de buitenwereld. Het model is in 1900 bedacht door Lamb.

Geeft men de schuif een tik, dan gaat hij oscilleren in het krachtveld van de veer; de snaar volgt de beweging, en er loopt een golf door

weg, die niet meer terugkomt omdat de snaar oneindig lang is. De energie van de schuif raakt op, en hij komt tot stilstand. Zodoende vertoont de schuif, op zichzelf beschouwd, wrijvingsgedrag, terwijl het model als geheel wrijvingsloos beweegt. Het idee is, dat eigenlijk alle wrijving zo werkt.

Brengt men verder de snaar op een positieve temperatuur, dan komen er sidderingen in, zoals op de voorpagina getekend staan. Deze zorgen voor ruis. In hoofdstuk I wordt aangetoond, dat de schuif inderdaad aan een Langevin-vergelijking voldoet, en streeft naar een evenwichts-kansverdeling, de Gibbs-toestand. Het bewijs maakt gebruik van een wrijvings-achtige eigenschap van het kansproces dat de beweging beschrijft: de Markov-eigenschap. Deze impliceert het bestaan van overgangswaarschijnlijkheden van de ene plaats naar de andere, hetgeen leidt tot een diffusie van de kansdichtheid naar een evenwichts-kansdichtheid.

In hoofdstuk II wordt het Lamb-model kwantummechanisch behandeld. Hierbij blijkt, dat de schuif geen impuls-operator heeft, geen energie-operator, en daarom geen energieniveau's om langs naar beneden te springen. Dit is een voorbeeld van het grote probleem in de kwantummechanische beschrijving van wrijvingsprocessen: de meeste modellen, en ook de oplossingen van Langevin-vergelijkingen, hebben maar zeer weinig wrijvingsachtige eigenschappen. In het bijzonder is de beweging van de schuif geen Markov-proces. De methode voor het bewijs van streven naar evenwicht, als gebruikt in hoofdstuk I, gaat dus niet door.

Dit werd een paar jaar geleden als probleem gesteld door M. Kac. In hoofdstuk II van dit proefschrift wordt dit probleem tot op zekere hoogte opgelost, door gebruikmaking van een andere eigenschap, een van het kwantum-Lamb-model als geheel: de meng-eigenschap.

Tenslotte wordt in hoofdstuk III aangetoond dat het eerder genoemde tuimelende elektron, in een vereenvoudigd, skalair, "elektromagnetisch veld" aan dezelfde Langevin-vergelijking voldoet als het kwantum-Lamb-model. De veer speelt hierbij de rol van het atoom, en de snaar die van het veld. Dit leidt tot de verrassende konklusie dat ook het tuimelende elektron geen echt wrijvingsgedrag vertoont: de bekende overgangswaarschijnlijkheden bestaan slechts in benadering, en de energie-niveau's zijn "uitgeveegd" door de koppeling met het veld.

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Hans