# Entropic Uncertainty Relations in Quantum Physics

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# Contents

# 1 Preliminaries

In this paper by a Hilbert space we will always mean a complex Hilbert space of finite dimension with inner product $(\cdot|\cdot)$ which is linear in the second entry, unless explicitely mentioned otherwise.
For the sake of completeness we will first recall what a Hilbert space is, and to do so we must introduce two concepts that will serve only to avoid any possible unclearity.

**Definition 1.1** Let $\mathcal{V}$ be a linear space with norm $\|\cdot\|$. A sequence $(x_n)_n$ in $\mathcal{V}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer $N$ such that $\|x_n - x_m\| < \varepsilon$ for every $n, m > N$. If every Cauchy sequence $(x_n)_n$ in $\mathcal{V}$ converges to an element in $\mathcal{V}$, then we say that $\mathcal{V}$ is *complete.*

If $\mathcal{H}$ is a linear space with inner product $(\cdot|\cdot)$, then $\|\xi\| := (\xi|\xi)^{\frac{1}{2}}$ defines a norm on $\mathcal{H}$. This observation should make the following definition unambiguous.

**Definition 1.2** A *Hilbert space* is a complete linear space with inner product.

The next lemma certifies that we will need the notion of completeness for preserving clearity only, and that it will not be needed in any proof, since the only linear spaces we regard are ones that are finite dimensional. We omit the proof because it is a basic result in linear algebra.

**Lemma 1.3** *Any normed linear space of finite dimension is complete.*

## 1.1 Operators and algebras

**Definition 1.4** Let $\mathcal{A}$ and $\mathcal{B}$ be normed linear spaces. An *operator* $\mathcal{A} \to \mathcal{B}$ is a linear map from $\mathcal{A}$ into $\mathcal{B}$. An operator on $\mathcal{A}$ is a linear map $\mathcal{A} \to \mathcal{A}$.

**Definition 1.5** An *algebra* $\mathcal{A}$ is a linear space endowed with a vector multiplication such that $\mathcal{A}$ is closed under the multiplication, and the following conditions hold:

1. $\lambda(xy) = (\lambda x)y = x(\lambda y), \ \ \lambda \in \mathbb{C}, \, x, y \in \mathcal{A};$
2. $x(yz) = (xy)z, \ \ x, y, z \in \mathcal{A};$
3. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz, \ \ x, y, z \in \mathcal{A}.$

**Definition 1.6** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $V$ a linear map from $\mathcal{H}$ into $\mathcal{K}$. The *adjoint* or *dual* of $V$ is the unique linear map $V^*$ from $\mathcal{K}$ into $\mathcal{H}$ that satisfies $(V^*\xi|\psi) = (\xi|V\psi)$ for every $\xi \in \mathcal{K}$ and every $\psi \in \mathcal{H}$.

**Definition 1.7** Let $\mathcal{H}$ be a Hilbert space. By $\mathcal{L}(\mathcal{H})$ we denote the set of all operators on $\mathcal{H}$.
The operator $\xi \mapsto \xi$ on $\mathcal{H}$ we call the *identity* on $\mathcal{H}$ and we denote it by $\mathbf{1}$.
The map $a \mapsto a^*$, where $a \in \mathcal{L}(\mathcal{H})$, is called the *$^*$-operation* of $\mathcal{L}(\mathcal{H})$.
The *operator norm* of an element $a$ in $\mathcal{L}(\mathcal{H})$ is defined as

$$\|a\| := \sup\{\, \|a\xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}.$$

With the operator norm $\mathcal{L}(\mathcal{H})$ is a normed linear space.

Whenever we speak of an algebra we will think of an algebra of operators on a Hilbert space $\mathcal{H}$. Furthermore we will always assume that the algebra contains the identity on $\mathcal{H}$.

**Definition 1.8** Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a subset of $\mathcal{L}(\mathcal{H})$. The *commutant* $\mathcal{M}'$ of $\mathcal{M}$ is defined as the set of all operators on $\mathcal{H}$ that commute with every operator in $\mathcal{M}$. The *center* of $\mathcal{M}$ is $\mathcal{M} \cap \mathcal{M}'$.

**Definition 1.9** Let $\mathcal{A}$ be an algebra.

1. $\mathcal{A}$ is called *Abelian* or *commutative* if $ab = ba$ for all $a, b \in \mathcal{A}$;

2. $\mathcal{A}$ is called a *$^*$-algebra* if it is closed under the $^*$-operation;

3. $\mathcal{A}$ is called a *factor* if it is a $^*$-algebra with trivial center $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}\mathbf{1}$.

**Definition 1.10** Let $\mathcal{H}$ be a Hilbert space and let $a$ be an operator on $\mathcal{H}$.

1. $a$ is called *Hermitian* if $a^* = a$;

2. $a$ is called *normal* if $a^*a = aa^*$;

3. $a$ is called a *projection* if $a$ is Hermitian and $a^2 = a$;

4. $a$ is called *positive* if $(\xi|a\xi) \geq 0$ for every $\xi \in \mathcal{H}$; we write $a \geq 0$.

If $\mathcal{A}$ is a $^*$-algebra of operators on $\mathcal{H}$ then the collection of Hermitian elements in $\mathcal{A}$ is denoted by $\mathcal{A}_h$ and the collection of positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}_+$.

**Definition 1.11** Let $\mathcal{H}$ be a Hilbert space and let $a$ be an operator on $\mathcal{H}$. $\mathrm{Ker}(a - \alpha\mathbf{1}) = \{\xi \in \mathcal{H} : (a - \alpha\mathbf{1})\xi = 0\}$ is a linear subspace of $\mathcal{H}$ for every $\alpha \in \mathbb{C}$. If $\mathrm{Ker}(a - \alpha\mathbf{1}) \neq \{0\}$ then $\alpha$ and the projection on $\mathrm{Ker}(a - \alpha\mathbf{1})$ are called a *spectral value* respectively a *spectral projection* of $a$. The collection of spectral values of an operator $a$ is called the *spectrum* of $a$.

The following theorem, the spectral theorem for normal operators on a finite dimensional Hilbert space, is an important result in the basics of linear algebra. A proof can be found in about any introductory textbook on linear algebra and for that reason is omitted.

**Theorem 1.12 (Finite dimensional spectral theorem)** *Let $\mathcal{H}$ be a Hilbert space of finite dimension and let $a$ be a normal operator on $\mathcal{H}$. Then there are projections $p_i$ and complex numbers $\alpha_i$ such that*

$$p_i p_j = \delta_{ij} p_i, \qquad \sum_i p_i = \mathbf{1}, \qquad \alpha_i \neq \alpha_j \text{ if } i \neq j,$$

*and*

$$a = \sum_i \alpha_i p_i.$$

**Lemma 1.13** *Let $\mathcal{A}$ be a $^*$-algebra of operators on a Hilbert space $\mathcal{H}$. If $a \in \mathcal{A}$ is normal, then any spectral projection of $a$ is in $\mathcal{A}$.*

*Proof:* Let $\mathcal{A}$ be a $^*$-algebra and let $a \in \mathcal{A}$. By Theorem 1.12 there are spectral projections $p_i$ and complex numbers $\alpha_i$ such that $a = \sum_i \alpha_i p_i$, $p_i p_j = \delta_{ij} p_i$, $\sum_i p_i = \mathbf{1}$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. For every index $k$ we have

$$\prod_{j \neq k} (a - \alpha_j) = \prod_{j \neq k} (\alpha_k - \alpha_j) p_k,$$

and since $\prod_{j \neq k} (\alpha_k - \alpha_j) \neq 0$ we find that every spectral projection $p_k$ is in $\mathcal{A}$.□

**Definition 1.14** Let $\mathcal{H}$ be a Hilbert space and let $a$ be a normal operator on $\mathcal{H}$. For any map $f : \mathbb{C} \to \mathbb{C}$ we define

$$f(a) := \sum_i f(\alpha_i) p_i,$$

where $\alpha_i$ and $p_i$ are the spectral values respectively spectral projections of $a$.

**Proposition 1.15** *Let $\mathcal{A}$ be a $^*$-algebra and $a \in \mathcal{A}$. The following conditions are equivalent:*

1. *$a$ is positive;*
2. *$a$ is Hermitian and all the spectral values of $a$ are positive;*
3. *there is an operator $w \in \mathcal{A}$ such that $a = w^* w$.*

*Proof:* Let $\mathcal{A}$ be a $^*$-algebra and let $a \in \mathcal{A}$.
Suppose $a$ is positive, then $a$ is Hermitian; we do not prove this here, but refer to [10], page 195. The definition of positivity and Theorem 1.12 imply that all spectral values of $a$ are positive.
Suppose $a$ is Hermitian and all the spectral values of $a$ are positive. According to Lemma 1.13 all spectral projections of $a$ are in $\mathcal{A}$, and therefore, if $a = \sum_i \alpha_i p_i$ is the spectral decomposition of $a$, the operator $w$ defined by $w := \sum_i \alpha^{1/2} p_i$ is in $\mathcal{A}$, and $a = w^* w$.
Suppose there is an operator $w \in \mathcal{A}$ such that $a = w^* w$. Then $(\xi | a\xi) = (w\xi | w\xi) \geq 0$ for every $\xi \in \mathcal{H}$, i.e. $a$ is positive. □

## 1.2 States and density operators

**Theorem 1.16** *Let $\mathcal{H}$ be a Hilbert space.*

1. *If $a, b \in \mathcal{L}(\mathcal{H})$, then for any orthonormal basis $\{\varphi_n\}$ in $\mathcal{H}$, the sum*

$$\sum_n (\varphi_n | a^* b \varphi_n),$$

   *denoted by $(a|b)_2$, is independent of the orthonormal basis chosen.*

2. *$\mathcal{L}(\mathcal{H})$ with inner product $(\cdot|\cdot)_2$ is a Hilbert space.*

*Proof:* [10], page 210, Theorem VI.22. □

**Definition 1.17** Let $\mathcal{H}$ be a Hilbert space. The *trace* of an element $a \in \mathcal{L}(\mathcal{H})$ is defined as

$$\operatorname{tr} a := (\mathbf{1}|a)_2.$$

In Definition 1.7 we mentioned the operator norm on the algebra $\mathcal{L}(\mathcal{H})$ of operators on a certain Hilbert space $\mathcal{H}$. Here we introduce two more norms on $\mathcal{L}(\mathcal{H})$ that will come in hand later on.

**Definition 1.18** Let $\mathcal{H}$ be a Hilbert space. For every $x$ in $\mathcal{L}(\mathcal{H})$ we define

$$\|x\|_1 := \operatorname{tr} \sqrt{x^* x},$$

and

$$\|x\|_2 := (x|x)_2^{1/2}.$$

From Theorem 1.16 it is clear that $\| \cdot \|_2$ is a norm. For a discussion on $\| \cdot \|_1$ we refer to [10], page 209, Theorem VI.20.

**Definition 1.19** Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and let $S$ be a map $\mathcal{A} \to \mathcal{B}$.

1. $S$ is called *positive* if $S(a) \geq 0$ for each positive $a \in \mathcal{A}$;
2. $S$ is called *unital* if $S(\mathbf{1}) = \mathbf{1}$.

**Definition 1.20** Let $\mathcal{V}$ be a linear space. By $\mathcal{V}^*$ we denote the linear space of all linear maps from $\mathcal{V}$ into $\mathbb{C}$, and this linear space we call the *dual space* of $\mathcal{V}$.

The following theorem accomplishes a 1-1 correspondence between elements in a Hilbert space $\mathcal{H}$ and elements in the dual space $\mathcal{H}^*$.

**Theorem 1.21** *Let $\mathcal{H}$ be a Hilbert space of finite dimension. For any element $\varphi \in \mathcal{H}^*$ there is a unique element $s \in \mathcal{H}$ such that $\varphi(x) = (s|x)$ for every $x \in \mathcal{H}$.*

*Proof:* Let $\mathcal{H}$ be a Hilbert space of dimension $n$. We choose an orthonormal base $\{e_i\}_{i=1}^n$ in $\mathcal{H}$. Suppose $\varphi \in \mathcal{H}^*$. $\sum_{i=1}^n \overline{\varphi(e_i)} e_i$ is an element in $\mathcal{H}$, let us call it $s$.

$$(s|x) = \sum_{i=1}^n \varphi(e_i)(e_i|x) = \varphi\Big(\sum_{i=1}^n e_i(e_i|x)\Big) = \varphi(x)$$

for any $x \in \mathcal{H}$.
If $s, t \in \mathcal{H}$ such that $\varphi(x) = (s|x) = (t|x)$ for every $x \in \mathcal{H}$, then in particular by choosing $x = s - t$ we find $0 = (s|x) - (t|x) = \|s - t\|^2$, $s = t$. □

**Definition 1.22** Let $\mathcal{A}$ be a $^*$-algebra. A *state* on $\mathcal{A}$ is a positive, unital, linear map $\mathcal{A} \to \mathbb{C}$. The collection of states on $\mathcal{A}$ we denote by $\mathcal{A}^*_{+,1}$.

**Definition 1.23** Let $\mathcal{H}$ be a Hilbert space. An element $\rho \in \mathcal{L}(\mathcal{H})$ is called a *density operator* if $\rho$ is positive and of unit trace.

Let $\mathcal{A}$ be a $^*$-algebra. Let $\varphi \in \mathcal{A}^*$ and let $\sigma \in \mathcal{A}$ such that $\varphi(x) = (\sigma|x)_2$ for every $x \in \mathcal{A}$. We can write $\sigma = \sigma_1 + i\sigma_2$ and $\varphi = \varphi_1 - i\varphi_2$, where

$$\sigma_1 := \frac{1}{2}(\sigma + \sigma^*), \qquad \sigma_2 := \frac{1}{2i}(\sigma - \sigma^*), \tag{1}$$

$$\varphi_1(x) := (\sigma_1|x)_2, \qquad \varphi_2(x) := (\sigma_2|x)_2, \qquad (x \in \mathcal{A}). \tag{2}$$

We note that $\sigma_1, \sigma_2 \in \mathcal{A}_h$. $\mathcal{A}_h$ with the inner product $(\cdot|\cdot)_2$ is a real Hilbert space, and every element in $\mathcal{A}_h^*$ is a real valued functional.
Let $j_+$ and $j_-$ be maps $\mathbb{R} \to \mathbb{R}$ defined by

$$j_+ : x \mapsto \begin{cases} x & ,x \geq 0 \\ 0 & ,x \leq 0 \end{cases} \qquad j_- : x \mapsto \begin{cases} 0 & ,x \geq 0 \\ -x & ,x \leq 0. \end{cases}$$

Then for any $a \in \mathcal{A}_h$ we have $a_+, a_- \in \mathcal{A}_+$ , $a = a_+ - a_-$ and $a_+a_- = 0$, where we adopted the shorthand notation

$$a_+ := j_+(a) \qquad a_- := j_-(a). \tag{3}$$

The observation $a_+a_- = 0$ is implied by the fact that for any two spectral values $\alpha$ and $\beta$ of $a$, $\mathrm{Ker}(a - \alpha\mathbf{1}) \cap \mathrm{Ker}(a - \beta\mathbf{1}) = \{0\}$ whenever $\alpha \neq \beta$.

**Lemma 1.24** *Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and let $S : \mathcal{A} \to \mathcal{B}$ be a positive linear map. Then*

1. *$S(a)$ is Hermitian for every Hermitian $a \in \mathcal{A}$;*
2. *$S(a^*) = S(a)^*$ for every $a \in \mathcal{A}$.*

*Proof:* 1: Let $a \in \mathcal{A}$ be Hermitian. We adopt the denotation used in equation (3) and we recall that $a_+, a_- \in \mathcal{A}_+$ and $a = a_+ - a_-$. Using the linearity and positivity of $S$ together with the fact that positive elements in $\mathcal{B}$ are Hermitian (Proposition 1.15) we see that $S(a)$ is Hermitian.
2: Let $a \in \mathcal{A}$. Define $a_1 := \frac{1}{2}(a + a^*)$ and $a_2 := \frac{1}{2i}(a - a^*)$, then $a_1, a_2 \in \mathcal{A}_h$ and $a = a_1 + ia_2$. Using 1, we see that $S(a^*) = S(a_1 - ia_2) = S(a_1) - iS(a_2) = (S(a_1) + iS(a_2))^* = S(a)^*$. □

**Proposition 1.25** *Let $\mathcal{A}$ be a $^*$-algebra. Let $\varphi \in \mathcal{A}^*$ and $\sigma \in \mathcal{A}$ such that $\varphi(x) = (\sigma|x)_2$ for every $x \in \mathcal{A}$, then:*

1. *$\varphi(x) \in \mathbb{R}$ for every $x \in \mathcal{A}_h$ if and only if $\sigma$ is Hermitian;*
2. *$\varphi$ is positive if and only if $\sigma$ is positive;*
3. *$\varphi$ is unital if and only if $\sigma$ is of unit trace.*

*Proof:* 1: Suppose $\varphi(x) \in \mathbb{R}$ for every $x \in \mathcal{A}_h$. Adopting denotations according to equations (1) and (2), $(\sigma_2|x)_2 = \varphi_2(x) = 0$ for every $x \in \mathcal{A}_h$, in particular $\|\sigma_2\|_2^2 = (\sigma_2|\sigma_2)_2 = 0$, meaning that $\sigma = \sigma_1 \in \mathcal{A}_h$.
If on the other hand $\sigma \in \mathcal{A}_h$, then, since $\mathcal{A}_h$ with inner product $(\cdot|\cdot)_2$ is a real Hilbert space, $\varphi$ is real-valued on $\mathcal{A}_h$.
2: Suppose $\varphi$ is positive. According to Lemma 1.24 $\varphi(x) \in \mathbb{R}$ for every $x \in \mathcal{A}_h$, and a subsequent use of 1 yields that $\sigma \in \mathcal{A}_h$. We adopt the denotation of equation (3). $\sigma_+$ and $\sigma_-$ are positive elements in $\mathcal{A}$, and so

$$0 \leq \varphi(\sigma_-) = \mathrm{tr}\,(\sigma_+\sigma_-) - \mathrm{tr}\,(\sigma_-\sigma_-) = -\|\sigma_-\|_2^2,$$

which means that $\sigma = \sigma_+$ is positive.
Suppose $\sigma$ is positive, then by Proposition 1.15 there is an element $w$ in $\mathcal{A}$ such that $\sigma = w^*w$. If $x \in \mathcal{A}_+$, then likewise there is an element $y$ in $\mathcal{A}$ such that $x = y^*y$, and so

$$\varphi(x) = \mathrm{tr}\,(w^*wy^*y) = \mathrm{tr}\,((wy^*)^*(wy^*)) = \|wy^*\|_2^2 \geq 0.$$

3: $\varphi(\mathbf{1})^* = (\mathbf{1}|\sigma)_2 = \mathrm{tr}\ \sigma$, thus $\varphi$ is unital if and only if $\sigma$ is of unit trace. □

Apparently there is a 1-1 correspondence between states on a $^*$-algebra $\mathcal{A}$ and density operators in $\mathcal{A}$.

## 1.3 Direct sums and tensor products

**Definition 1.26** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with inner products $(\cdot|\cdot)_{\mathcal{H}_1}$ respectively $(\cdot|\cdot)_{\mathcal{H}_2}$. The *direct sum* of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the Cartesian product of $\mathcal{H}_1$ and $\mathcal{H}_2$ endowed with the usual vector addition and scalar multiplication and the inner product

$$((x_1, x_2)|(y_1, y_2)) = (x_1|y_1)_{\mathcal{H}_1} + (x_2|y_2)_{\mathcal{H}_2}.$$

The direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $\mathcal{H}_1 \oplus \mathcal{H}_2$, and instead of writing $(x_1, x_2)$ for an element in the direct sum we write $x_1 \oplus x_2$.

Let $\mathcal{V}$ and $\mathcal{W}$ be two linear spaces. We introduce symbols $\otimes$ and $\cdot + \cdot$, and with these for $x_1, \ldots, x_n \in \mathcal{V}$ and $y_1, \ldots, y_n \in \mathcal{W}$ we make formal expressions

$$x_1 \otimes y_1 \cdot + \cdot x_2 \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot x_n \otimes y_n,$$

which we abbreviate by writing $\sum_{i=1}^n x_i \otimes y_i$. Among these expressions we introduce a relation $\sim$ subject to the rules

1. $x_1 \otimes y_1 \cdot + \cdot x_2 \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot x_n \otimes y_n$
   $$\sim x_{1'} \otimes y_{1'} \cdot + \cdot x_{2'} \otimes y_{2'} \cdot + \cdot \ldots \cdot + \cdot x_{n'} \otimes y_{n'},$$
   where $1', 2', \ldots, n'$ denotes any permutation of the integers $1, 2, \ldots, n$;

2. $(x_1 + x_1') \otimes y_1 \cdot + \cdot x_2 \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot x_n \otimes y_n$
   $$\sim x_1 \otimes y_1 \cdot + \cdot x_1' \otimes y_1 \cdot + \cdot x_2 \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot x_n \otimes y_n;$$

3. $x_1 \otimes (y_1 + y_1') \cdot + \cdot x_2 \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot x_n \otimes y_n$
   $$\sim x_1 \otimes y_1 \cdot + \cdot x_1 \otimes y_1' \cdot + \cdot x_2 \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot x_n \otimes y_n;$$

4. $(c_1 x_1) \otimes y_1 \cdot + \cdot (c_2 x_2) \otimes y_2 \cdot + \cdot \ldots \cdot + \cdot (c_n x_n) \otimes y_n$
   $$\sim x_1 \otimes (c_1 y_1) \cdot + \cdot x_2 \otimes (c_2 y_2) \cdot + \cdot \ldots \cdot + \cdot x_n \otimes (c_n y_n),$$
   where $c_1, c_2, \ldots, c_n$ complex numbers.

Two expressions $\sum_{i=1}^n x_i \otimes y_i$ and $\sum_{j=1}^m v_j \otimes w_j$ we call equivalent if one can be transformed into the other by a finite number of successive applications of the rules 1, 2, 3 and 4, and we write

$$\sum_{i=1}^n x_i \otimes y_i \simeq \sum_{j=1}^m v_j \otimes w_j.$$

**Lemma 1.27** *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces. For any $x_1, \ldots, x_n \in \mathcal{V}$ and any $y_1, \ldots, y_n \in \mathcal{W}$ the expression $\sum_{i=1}^n x_i \otimes y_i$ is equivalent to either $0 \otimes 0$ or to an expression $\sum_{i=1}^m v_i \otimes w_i$ in which both the $v_1, \ldots, v_m \in \mathcal{V}$ and the $w_1, \ldots, w_m \in \mathcal{W}$ are linearly independent.*

*Proof:* Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and let $x_1, \ldots, x_n \in \mathcal{V}$ and $y_1, \ldots, y_n \in \mathcal{W}$. If for instance we have $x_1 = \sum_{i=2}^n c_i x_i$, where $c_2, \ldots, c_n$ complex numbers, then

$$\begin{aligned} x_1 \otimes y_1 \cdot + \cdot \sum_{i=2}^n x_i \otimes y_i &\simeq \Big(\sum_{i=2}^n c_i x_i\Big) \otimes y_1 \cdot + \cdot \sum_{i=2}^n x_i \otimes y_i \\ &\simeq \sum_{i=2}^n (c_i x_i) \otimes y_1 \cdot + \cdot \sum_{i=2}^n x_i \otimes y_i \\ &\simeq \sum_{i=2}^n x_i \otimes (c_i y_1) \cdot + \cdot \sum_{i=2}^n x_i \otimes y_i \\ &\simeq \sum_{i=2}^n x_i \otimes (c_i y_1 + y_i), \end{aligned}$$

the expression $\sum_{i=1}^n x_i \otimes y_i$ is equivalent to an expression consisting of $n-1$ terms. This shows that if the sets $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are linearly dependent, then by repeatedly applying the above procedure we are able to obtain an expression $\sum_{i=1}^m v_i \otimes w_i$ in which both the $v_1, \ldots, v_m \in \mathcal{V}$ and the $w_1, \ldots, w_m \in \mathcal{W}$ are linearly independent or $v \otimes 0$ for some $v \in \mathcal{V}$ or $0 \otimes w$ for some $w \in \mathcal{W}$. But $v \otimes 0 \simeq (0v) \otimes 0 \simeq 0 \otimes 0$, and similarly $0 \otimes w \simeq 0 \otimes 0$. □

**Definition 1.28** Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces. Let $\mathcal{VW}$ denote the collection of expressions $\sum_{i=1}^{n} x_i \otimes y_i$ where $x_1, \ldots, x_n \in \mathcal{V}$ and $y_1, \ldots, y_n \in \mathcal{W}$, and let $N$ denote the set of all such expressions that are equivalent to $0 \otimes 0$.
We define the *algebraic tensor product* of $\mathcal{V}$ and $\mathcal{W}$ as the collection of sets

$$z + N := \{z + \nu : \nu \in N\}, \qquad (z \in \mathcal{VW}),$$

endowed with the vector addition

$$(x + N) + (y + N) := x + y + N, \qquad (x, y \in \mathcal{VW}),$$

and the scalar multiplication

$$\lambda(x + N) := \lambda x + N, \qquad (x \in \mathcal{VW}, \lambda \in \mathbb{C}).$$

Often for an expression $z$ in $\mathcal{VW}$ we will identify $z$ with the set $z + N$. The algebraic tensor product of $\mathcal{V}$ and $\mathcal{W}$ we denote by $\mathcal{V} \odot \mathcal{W}$.

**Lemma 1.29** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with inner products $(\cdot|\cdot)_{\mathcal{H}_1}$ respectively $(\cdot|\cdot)_{\mathcal{H}_2}$. The sesquilinear form*

$$\Big( \sum_{i=1}^{n} \xi_{1i} \otimes \xi_{2i} \Big| \sum_{j=1}^{k} \psi_{1j} \otimes \psi_{2j} \Big) = \sum_{i=1}^{n} \sum_{j=1}^{k} (\xi_{1i}|\psi_{1j})_{\mathcal{H}_1} (\xi_{2i}|\psi_{2j})_{\mathcal{H}_2}, \tag{4}$$

*where $\xi_{1i}, \psi_{1j} \in \mathcal{H}_1$ and $\xi_{2i}, \psi_{2j} \in \mathcal{H}_2$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, k$, is an inner product on the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.*

*Proof:* Let $\sum_i \xi_{1i} \otimes \xi_{2i}$ and $\sum_j \psi_{1j} \otimes \psi_{2j}$ in $\mathcal{H}_1 \odot \mathcal{H}_2$. We may assume there are enough vectors $\xi_{1i}$ such that every $\psi_{1j}$ can be written as a linear combination of vectors $\xi_{1i}$, and then there are vectors $\phi_{2i}$ in $\mathcal{H}_2$ such that $\sum_j \psi_{1j} \otimes \psi_{2j} = \sum_i \xi_{1i} \otimes \phi_{2i}$. It is easy to see now that the sesquilinear form (4) is conjugate linear in the first entry and linear in the second. By Lemma 1.27 we may assume that the vectors $\xi_{1i}$ are linearly independent, and the vectors $\xi_{2i}$ are so too. With the Gram-Schmidt orthogonalisation procedure we can find orthonormal sets of vectors $\{\varphi_{1j}\}_j$ in $\mathcal{H}_1$ and $\{\varphi_{2k}\}_k$ in $\mathcal{H}_2$ and complex coefficients $c_{jk}$ such that $\sum_i \xi_{1i} \otimes \xi_{2i} = \sum_i c_{jk} \varphi_{1j} \otimes \varphi_{2k}$. We have

$$\begin{aligned} (\sum_i \xi_{1i} \otimes \xi_{2i} | \sum_i \xi_{1i} \otimes \xi_{2i}) &= \sum_{jk} \sum_{st} \bar{c}_{jk} c_{st} (\varphi_{1j}|\varphi_{1s})_{\mathcal{H}_1} (\varphi_{2k}|\varphi_{2t})_{\mathcal{H}_2} \\ &= \sum_{jk} |c_{jk}|^2 \\ &\geq 0 \end{aligned}$$

with equality if and only if $\sum_i \xi_{1i} \otimes \xi_{2i} = 0 \otimes 0$. □

**Definition 1.30** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. By the *tensor product* of $\mathcal{H}_1$ and $\mathcal{H}_2$ we mean the algebraic tensor product $\mathcal{H}_1 \odot \mathcal{H}_2$ with inner product defined by (4). We denote the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{H}_1 \otimes \mathcal{H}_2$.

**Proposition 1.31** *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then $\mathcal{L}(\mathcal{H}_1) \odot \mathcal{L}(\mathcal{H}_2) = \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Furthermore, for any $a_1 \in \mathcal{L}(\mathcal{H}_1)$ and any $a_2 \in \mathcal{L}(\mathcal{H}_2)$ there exists a unique operator $a_1 \otimes a_2 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfying*

$$(a_1 \otimes a_2)(\xi_1 \otimes \xi_2) = a_1\xi_1 \otimes a_2\xi_2 \quad \text{for all } \xi_i \in \mathcal{H}_i, \; i = 1, 2,$$

*and for that operator*

$$\|a_1 \otimes a_2\| = \|a_1\|\|a_2\|.$$

*Proof:* [9], page 98, Proposition 16.1, and [11], page 185. □

**Definition 1.32** Let $\mathcal{A}$ be a $^*$-algebra. For every positive integer $n$ we define $M_n(\mathcal{A})$ as the algebra of $n \times n$-matrices $[a_{ij}]$ with entries $a_{ij}$ in $\mathcal{A}$, or equivalently, as the algebraic tensor product of the algebra of complex $n \times n$-matrices and the $^*$-algebra $\mathcal{A}$.

## 1.4 Completely positive maps

**Definition 1.33** Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras. A linear map $\pi$ from $\mathcal{A}$ into $\mathcal{B}$ is called a *$^*$-homomorphism* if it is $^*$-preserving and $\pi(xy) = \pi(x)\pi(y)$ for every $x, y \in \mathcal{A}$.
A *representation* of a $^*$-algebra $\mathcal{A}$ is a pair $\{\pi, \mathcal{R}\}$ where $\mathcal{R}$ is a Hilbert space and $\pi$ is a $^*$-homomorphism from $\mathcal{A}$ into $\mathcal{L}(\mathcal{R})$.

A representation $\{\pi, \mathcal{R}\}$ of a $^*$-algebra $\mathcal{A}$ is called non-degenerate if for every $\xi \in \mathcal{R}$ there is a $a \in \mathcal{A}$ such that $\pi(a)\xi \neq 0$. Whenever we speak of a representation we will assume that it is non-degenerate unless explicitely mentioned otherwise.

**Definition 1.34** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A linear map $V$ from $\mathcal{H}$ into $\mathcal{K}$ is called an *isometry* if $V^*V = \mathbf{1}$, where $\mathbf{1}$ the identity on $\mathcal{H}$.

**Definition 1.35** Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and $S$ a linear map from $\mathcal{A}$ into $\mathcal{B}$. For every positive integer $n$ let $S^{(n)}$ be the map from $M_n(\mathcal{A})$ into $M_n(\mathcal{B})$ defined by $S^{(n)} : [a_{ij}] \mapsto [S(a_{ij})]$. If $S^{(n)}$ is positive then we say $S$ is *$n$-positive.* If $S$ is $n$-positive for every positive integer $n$, then we say $S$ is *completely positive.*

**Theorem 1.36 (Stinespring)** *Let $\mathcal{A}$ be a $^*$-algebra and $\mathcal{H}$ a Hilbert space, both of finite dimension.*

1. *If $\{\pi, \mathcal{R}\}$ is a representation of $\mathcal{A}$ and $V$ is a linear map from $\mathcal{H}$ into $\mathcal{R}$, then the map $S : \mathcal{A} \ni a \mapsto V^*\pi(a)V \in \mathcal{L}(\mathcal{H})$ is completely positive.*
2. *If $S$ is a completely positive map from $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$, then there exist a representation $\{\pi, \mathcal{R}\}$ of $\mathcal{A}$ and a linear map $V$ from $\mathcal{H}$ into $\mathcal{R}$ such that*

$$S(a) = V^*\pi(a)V, \qquad (a \in \mathcal{A}),$$

$$\mathcal{R} = \pi(\mathcal{A})V\mathcal{H}.$$

*If in addition $S$ is unital, then $V$ is an isometry.*

*Proof:* [11], pages 194-199, Theorem 3.6 and Remark 3.7 □

**Remark 1.37** Since both the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ in Theorem 1.36 are finite dimensional, the Hilbert space $\mathcal{R}$ we can choose to be of finite dimension as well.

**Definition 1.38** Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras. The *embedding* of $\mathcal{A}$ into $\mathcal{A} \odot \mathcal{B}$ is the map $e_{\mathcal{B}}$ from $\mathcal{A}$ into $\mathcal{A} \odot \mathcal{B}$, defined by

$$e_{\mathcal{B}} : a \mapsto a \otimes \mathbf{1}.$$

The *embedding* of $\mathcal{B}$ into $\mathcal{A} \odot \mathcal{B}$ is the map $e_{\mathcal{A}}$ from $\mathcal{B}$ into $\mathcal{A} \odot \mathcal{B}$, defined by

$$e_{\mathcal{A}} : b \mapsto \mathbf{1} \otimes b.$$

We state two more properties of completely positive maps in the following lemma. The first one is known as the Schwarz inequality and we use it to show the validity of the second one, a property known as the multiplication theorem, which we will use later to illustrate our motivations.

**Lemma 1.39** *Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and $S$ a completely positive, unital map from $\mathcal{A}$ into $\mathcal{B}$, then:*

1. *$S(a^*)S(a) \leq S(a^*a)$ for every $a \in \mathcal{A}$;*
2. *if $S(a^*a) = S(a^*)S(a)$ for some $a \in \mathcal{A}$ then $S(ba) = S(b)S(a)$ and $S(a^*b) = S(a^*)S(b)$ for all $b \in \mathcal{A}$.*

*Proof:* 1: [11], page 199, Corollary 3.8.
2: Suppose $a \in \mathcal{A}$ and $S(a^*a) = S(a^*)S(a)$. For any $b \in \mathcal{A}$ and any $t \in \mathbb{R}$ we have

$$S((a^* + tb)(a + tb^*)) = S(a^*)S(a) + tS(ba + a^*b^*) + t^2S(bb^*),$$

and by the Schwarz inequality

$$S((a^* + tb)(a + tb^*)) \geq S(a^*)S(a) + t(S(b)S(a) + S(a^*)S(b^*)) + t^2S(b)S(b^*).$$

This equality and inequality hold for all $t \in \mathbb{R}$, which implies that

$$S(ba + a^*b^*) \geq S(b)S(a) + S(a^*)S(b^*).$$

Replacing $a$ by $ia$ and $b$ by $ib$ yields that the opposite is also true, so that we have equality, and replacing only $b$ by $ib$ then yields that $S(ba) = S(b)S(a)$ and $S(a^*b^*) = S(a^*)S(b^*)$. □

# 2 Measurements

## 2.1 Physical systems and measuring devices

Any physical system, in particular any quantum mechanical system, we represent by a non-commutative generalisation of a classical probability space. Such a space, a quantum probability space [1], is a pair $\{\mathcal{A}, \varphi\}$, where $\mathcal{A}$ is a $^*$-algebra and $\varphi$ is a state on $\mathcal{A}$. If such a pair $\{\mathcal{A}, \varphi\}$ happens to describe a classical physical system, then the algebra $\mathcal{A}$ is commutative.

**Remark 2.1** A physical system we will identify with a pair $\{\mathcal{A}, \varphi\}$, where $\mathcal{A}$ is a $^*$-algebra and $\varphi$ is a state on $\mathcal{A}$. Sometimes we will drop the state $\varphi$ and only mention the algebra $\mathcal{A}$.

**Definition 2.2** Let $\{\mathcal{A}, \varphi\}$ be a physical system. The observable quantities or *observables* of $\{\mathcal{A}, \varphi\}$ are the Hermitian operators in $\mathcal{A}$. For any Hermitian operator $a$ in $\mathcal{A}$ we can write down a spectral decomposition $a = \sum_i \alpha_i p_i$; the *event* "observable $a$ takes on value $\alpha_j$" is given by the spectral projection $p_j$ and has probability $\varphi(p_j)$. The *sure event* is represented by $\mathbf{1}$.

One can obtain information on a physical system by performing a measurement on it. This action involves a measuring device which generates classical output. The measuring device, being a physical system itself, we characterize by an Abelian $^*$-algebra since it is only the classical output the measuring device generates that is relevant to us.
We restrict ourselves to regarding measuring devices with a finite number of possible outcomes, since any of those can be characterised by a $^*$-algebra of operators on finite dimensional Hilbert space.

If $\mathcal{A}$ is a physical system on which we perform a measurement by using a measuring device $\mathcal{C}$, then we regard the physical system and the measuring device together, that is we regard the physical system consisting of the $^*$-algebra $\mathcal{A} \otimes \mathcal{C}$. Since $\mathcal{C}$ is Abelian, any element in $\mathcal{C}$ is normal and according to Lemma 1.13 therefore all the spectral projections of an element in $\mathcal{C}$ are in $\mathcal{C}$ as well. Since all the spectral projections in $\mathcal{C}$ commute with one another we can choose a base of $\mathcal{C}$ consisting of mutually perpendicular projections, i.e. for any two elements $p$ and $q$ in the base, $pq = 0$ if $p \neq q$. If $\{e_i\}_{i=1}^n$ is such a base, then any element $x$ in $\mathcal{A} \otimes \mathcal{C}$ we can write in the form $x = \sum_{i=1}^n x_i \otimes e_i$, where $x_i$ in $\mathcal{A}$ for $i = 1, \ldots, n$, or $x = (x_i)_{i=1}^n$ for short.

The concept of measurement will be the topic of this section, and we will discuss it in two different frameworks, the Heisenberg picture and the Schrödinger picture.

[1] This generalisation of a classical probability space to a quantum probability space is obtained without any strain; one can find a very enlightening elaboration in [5].

## 2.2 The Heisenberg picture

In the Heisenberg picture the observables change with time while the state of the system remains the same. If $\{\mathcal{A}, \varphi\}$ is a physical system on which we perform a measurement $S$ with a measuring device $\{\mathcal{C}, \gamma\}$, then before the measurement is performed we have the physical system $\{\mathcal{A}, \varphi\}$, and afterwards we have the composite system $\mathcal{A} \otimes \mathcal{C}$ in some state $\psi$. An observable $x$ in the composite system $\{\mathcal{A} \otimes \mathcal{C}, \psi\}$ is mapped to an observable $S(x)$ in $\{\mathcal{A}, \varphi\}$.
In order to make $S$ reflect the behaviour of a measurement, we have the following definition.

**Definition 2.3** A *measurement* on a physical system $\mathcal{A}$ with measuring device $\mathcal{C}$ is an operator $S$ from $\mathcal{A} \otimes \mathcal{C}$ into $\mathcal{A}$, such that:

1. $S$ is completely positive;
2. $S$ is unital.

An interpretation of this definition we postpone until we discuss the Schrödinger picture.

## 2.3 The Schrödinger picture

In the Schrödinger picture it are not the observables but the states that change with time. If we perform a measurement on a physical system $\mathcal{A}$ that is in a certain state $\varphi$, and we perform this measurement by using a measuring device $\{\mathcal{C}, \gamma\}$, then in the Schrödinger picture we go from $\{\mathcal{A}, \varphi\}$ to the composite system $\mathcal{A} \otimes \mathcal{C}$ in some state or other.
We take a closer look at this picture. Suppose we expect a physical system $\mathcal{A}$ to be in either of two states, for example we expect it to be in a state $\varphi$ with probability $p$ and with probability $1-p$ in some other state $\psi$. Now we perform a measurement on the system, a map $S'$ from the states on $\mathcal{A}$ to the states on $\mathcal{A} \otimes \mathcal{C}$, where $\mathcal{C}$ our measuring device. After the measurement we expect the composite system $\mathcal{A} \otimes \mathcal{C}$ to be in a state $S'(\varphi)$ with probability $p$ and in a state $S'(\psi)$ with probability $1-p$. In other words, we demand $S'$ to be an affine map on the collection of states on $\mathcal{A}$.

**Definition 2.4** A subset $C$ of a linear space is *convex* if $\lambda x + (1-\lambda)y$ is in $C$ for every $x, y$ in $C$ and $0 \leq \lambda \leq 1$.
A map $f$ defined on a convex subset $C$ of a linear space is *affine* if

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$$

for every $x, y$ in $C$ and $0 \leq \lambda \leq 1$.

Since there is a 1-1 correspondence between states on a $^*$-algebra and density operators in that $^*$-algebra we can formulate a measurement in the Schrödinger picture as an affine transformation of density operators. If $S'$ represents a measurement in the Schrödinger picture mapping states to states, then by $S^*$ we will denote the same measurement mapping density operators to density operators. If the measurement is one on a physical system $\mathcal{A}$ and $\varphi$ is a state on $\mathcal{A}$ corresponding to a density operator $\sigma$ in the sense that

$$\varphi(x) = (\sigma|x)_2, \qquad (x \in \mathcal{A}),$$

then we have the relation

$$S'(\varphi)(x) = (S^*\sigma|x)_2, \qquad (x \in \mathcal{A}).$$

The collection of density operators in a $^*$-algebra is convex. We will now demonstrate that any affine map defined on the collection of density operators in a $^*$-algebra can be uniquely extended to a linear map defined on the entire $^*$-algebra.

**Theorem 2.5** *Let $\mathcal{A}$ be a $^*$-algebra and let $f$ be an affine map defined on the collection of density operators in $\mathcal{A}$, then there is a unique linear map $\tilde{f}$ defined on $\mathcal{A}$ such that $\tilde{f}(\sigma) = f(\sigma)$ for every density operator $\sigma$ in $\mathcal{A}$.*

*Proof:* We start by extending $f$ to a map $f_+$ defined on $\mathcal{A}_+$, the positive elements in $\mathcal{A}$, by

$$f_+ : a \mapsto \left\{ \begin{array}{cl} 0 & , a = 0, \\ \mathrm{tr}\,(a) f(a/\mathrm{tr}\,(a)) & , a > 0. \end{array} \right.$$

It is not hard to verify that $f_+$ coincides with $f$ on the collection of density operators in $\mathcal{A}$, and that $f_+(a+b) = f_+(a) + f_+(b)$ and $f_+(\lambda a) = \lambda f_+(a)$ for every $\lambda \geq 0$ and $a, b \in \mathcal{A}_+$.

We proceed by extending $f_+$ to a map $f_h$ defined on $\mathcal{A}_h$, the Hermitian elements in $\mathcal{A}$, by

$$f_h : a \mapsto f_+(a_+) - f_+(a_-),$$

where $a_+$ and $a_-$ denote the positive parts of $a$ respectively $-a$, like in equation (3). $f_h$ is linear. Indeed, if $a \in \mathcal{A}_h$ and $\lambda$ is a positive, real number then $\lambda a = \lambda a_+ - \lambda a_-$ and

$$f_h(\lambda a) = f_+(\lambda a_+) - f_+(\lambda a_-) = \lambda f_h(a).$$

If $a \in \mathcal{A}_h$ and $\lambda$ is a negative, real number then $\lambda a = (-\lambda)a_- - (-\lambda)a_+$ and

$$f_h(\lambda a) = f_+(-\lambda a_-) - f_+(-\lambda a_+) = \lambda f_h(a).$$

To see that $f_h$ is additive take $a, b \in \mathcal{A}_h$. Then $(a+b)_+ - (a+b)_- = a + b = a_+ - a_- + b_+ - b_-$, so when rearranging terms

$$(a+b)_+ + a_- + b_- = (a+b)_- + a_+ + b_+,$$

and applying $f_+$

$$f_+((a+b)_+) + f_+(a_-) + f_+(b_-) = f_+((a+b)_-) + f_+(a_+) + f_+(b_+),$$

we obtain

$$\begin{aligned} f_h(a+b) &= f_+((a+b)_+) - f_+((a+b)_-) \\ &= f_+(a_+) - f_+(a_-) + f_+(b_+) - f_+(b_-) \\ &= f_h(a) + f_h(b). \end{aligned}$$

Finally, we extend $f_h$ to a map $\tilde{f}$ defined on the entire algebra $\mathcal{A}$ by

$$\tilde{f} : a \mapsto f_h(a_1) + i f_h(a_2),$$

where

$$a_1 = \frac{1}{2}(a + a^*), \qquad a_2 = \frac{1}{2i}(a - a^*).$$

$\tilde{f}$ is linear and coincides with $f$ on the collection of density operators in $\mathcal{A}$.
It is easy to see that this extension $\tilde{f}$ of $f$ is unique. Suppose that $g$ is a linear map defined on $\mathcal{A}$ such that $g(\sigma) = f(\sigma)$ for every density operator $\sigma$ in $\mathcal{A}$. Then for any non-zero element $a$ in $\mathcal{A}_+$ we have

$$g(a) = \mathrm{tr}\,(a) g(a/\mathrm{tr}\,(a)) = \mathrm{tr}\,(a) f(a/\mathrm{tr}\,(a)) = f_+(a),$$

which means that $g$ coincides with $f_+$ on $\mathcal{A}_+$. Just as straightforward, $g$ coincides with $f_h$ on $\mathcal{A}_h$ and with $\tilde{f}$ on $\mathcal{A}$. □

A measurement $S^*$, an affine map defined on the density operators in a physical system $\mathcal{A}$, we can extend to a linear map $S^*$ defined on the entire $^*$-algebra $\mathcal{A}$. It is immediate that the extended $S^*$ is positive and trace-preserving.
As a last consideration on measurements in the Schrödinger picture we take the outside world into account. If we incorporate any outside environment into the picture of a measurement $S^*$ on a physical system $\mathcal{A}$, say one represented by the algebra $M_n(\mathbb{C})$ for some positive integer $n$, then by $S^*$ a density operator in the system $M_n(\mathcal{A})$ must be mapped to a density operator. This means that $S^*$ extended to the entire $^*$-algebra $\mathcal{A}$ must be $n$-positive, and it must be so for every positive integer $n$, that is, $S^*$ must be completely positive.
If we dualize the measurement as given in the Schrödinger picture then we obtain the measurement given in the Heisenberg picture. The measurement $S^*$ extended to the entire $^*$-algebra $\mathcal{A}$ we can dualize to obtain an operator $S$ from $\mathcal{A} \otimes \mathcal{C}$ into $\mathcal{A}$ according to the relation

$$(S^* a | x)_2 = (a | S(x))_2, \qquad (a \in \mathcal{A},\, x \in \mathcal{A} \otimes \mathcal{C}).$$

$S$ is completely positive and unital:

**Proposition 2.6** *Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and let $S$ be an operator from $\mathcal{B}$ into $\mathcal{A}$ with dual $S^* : \mathcal{A} \to \mathcal{B}$.*

1. *if $S^*$ is positive, then $S$ is positive;*
2. *if $S^*$ is completely positive, then $S$ is completely positive;*
3. *$S$ is unital if and only if $S^*$ is trace-preserving.*

*Proof:* 1: Suppose $S^*$ is positive. We can use point 2 of Proposition 1.25 to observe that for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$, both positive, we have $0 \leq (S^*a|b)_2 = (a|S(b))_2$. This means that $S$ is positive.
2: If $S^*$ is completely positive, then $S^*$ is $n$-positive for every positive integer $n$ and according to point 1 so is $S$. Therefore $S$ is completely positive.
3: For every $a$ in $\mathcal{A}$ we have

$$(S^*a|\mathbf{1})_2 = (a|S(\mathbf{1}))_2,$$

so $S(\mathbf{1}) = \mathbf{1}$ if and only if $(S^*x|\mathbf{1})_2 = (x|\mathbf{1})_2$ for every $x \in \mathcal{A}$, i.e. $S$ is unital if and only if $S^*$ is trace-preserving. □

We summarize some of the above into a definition of a measurement in the Schrödinger picture.

**Definition 2.7** A *measurement* on a physical system $\mathcal{A}$ with a measuring device $\mathcal{C}$ is an affine transformation $S'$ from the states on $\mathcal{A}$ into the states on $\mathcal{A} \otimes \mathcal{C}$, or equivalently an affine transformation $S^*$ from the density operators in $\mathcal{A}$ into the density operators in $\mathcal{A} \otimes \mathcal{C}$. These $S'$ and $S^*$ are related to a completely positive, unital map $S : \mathcal{A} \otimes \mathcal{C} \to \mathcal{A}$, the measurement in the Heisenberg picture, according to the relations

$$S' : \varphi \mapsto \varphi \circ S$$

and

$$(S^*\sigma|x)_2 = (\sigma|S(x))_2.$$

**Definition 2.8** Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras. The linear maps $\mathrm{tr}_{\mathcal{A}}$ and $\mathrm{tr}_{\mathcal{B}}$ defined on $\mathcal{A} \odot \mathcal{B}$ by

$$\mathrm{tr}_{\mathcal{A}} : a \otimes b \mapsto \mathrm{tr}\,(a)b$$

and

$$\mathrm{tr}_{\mathcal{B}} : a \otimes b \mapsto a\,\mathrm{tr}\,(b)$$

are called a *partial trace over* $\mathcal{A}$ respectively a *partial trace over* $\mathcal{B}$.

**Lemma 2.9** *The dual of an embedding is a partial trace.*

*Proof:* Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras. Let $e_{\mathcal{A}}$ be the embedding of $\mathcal{B}$ into $\mathcal{A} \odot \mathcal{B}$ and $\mathrm{tr}_{\mathcal{A}}$ the partial trace over $\mathcal{A}$ defined on $\mathcal{A} \odot \mathcal{B}$. For any $x \in \mathcal{A}$ and any $b, y \in \mathcal{B}$ we find

$$(x \otimes y|e_{\mathcal{A}}(b))_2 = (x \otimes y|\mathbf{1} \otimes b)_2 = (x|\mathbf{1})_2(y|b)_2 = ((\mathbf{1}|x)_2 y|b)_2,$$

so $e^*_{\mathcal{A}}(x \otimes y) = (\mathbf{1}|x)_2 y = \mathrm{tr}_{\mathcal{A}}\,(x \otimes y)$. This means that $e^*_{\mathcal{A}} = \mathrm{tr}_{\mathcal{A}}$.
In the same way we find that $e^*_{\mathcal{B}} = \mathrm{tr}_{\mathcal{B}}$. □

# 3 Entropy

Before we arrive at this treatise's highlight, which is an entropic uncertainty relation for a single measurement, we need to discuss the subject of entropy as a measure of uncertainty. Of coarse, when talking about an uncertainty relation one needs a measure to express uncertainty, and although there are many such thinkable measures we will restrict ourselves to the one that is known as entropy.[2] We consider the Shannon entropy, the von Neumann entropy and the quantum relative entropy. We mention the properties that make them suitable as a measure of uncertainty, and we derive some results we will need to come to an entropic uncertainty relation.

## 3.1 The Shannon entropy and the von Neumann entropy

**Definition 3.1** The *Shannon entropy* of a probability distribution $p = (p_1, \dots, p_m)$ is defined as

$$H(p) := -\sum_{i=1}^{m} p_i \log p_i.$$

In this definition we adopted the convention that $0 \log 0 = 0$. We will uphold this convention from here on.

A probability distribution is understood to be a state on a commutative space, and since we regard spaces that are commutative as well as spaces that are not, we need a counterpart of the Shannon entropy that applies to both:

**Definition 3.2** The *von Neumann entropy* of a density operator $\sigma$ is defined as

$$H(\sigma) := -\mathrm{tr}\, \sigma \log \sigma.$$

The von Neumann entropy is a generalisation of the Shannon entropy in the sense that if we interpret a density operator $\rho$ in an Abelian algebra as a probability distribution $p$, then the von Neumann entropy of $\rho$ equals the Shannon entropy of $p$.

A first feature of the von Neumann entropy we might notice as a suitable property for a measure of uncertainty is that it is invariant under permutations of the outcomes. If we exchange the spectral values of a density operator with respect to its spectral projections, the outcomes, then it does not change the entropy: the von Neumann entropy of the density operator before the permutation equals that of the density operator after the permutation.

The von Neumann entropy takes on its minimum value in case of least uncertainty, and its maximum value in case of most uncertainty. From its definition

[2] In [12] a comprehensive discussion is given on uncertainty measures in the context of the uncertainty principle.

it is clear that the von Neumann entropy is non-negative. If $\sigma$ is a density operator with spectral values $\sigma_i$, then we observe that

$$\begin{aligned}\sum_i \sigma_i \log \sigma_i &\leq \sum_i \sigma \log \max_j \sigma_j \\ &= \log \max_j \sigma_j,\end{aligned}$$

so

$$H(\sigma) \geq -\log \max_j \sigma_j.$$

This inequality in particular implies that the von Neumann entropy of a density operator is zero if and only if one spectral value equals 1 and the others are zero. Then the density operator describes a situation of least uncertainty. The other extreme is a situation of total uncertainty. In the classical case this corresponds to a uniform distribution, a probability distribution $p = (p_1, \dots, p_n)$ where every $p_i = \frac{1}{n}$. In quantum physics a density operator corresponds to a situation of total uncertainty if the spectral values are uniformly distributed, that is, if it equals $\mathbf{1}/n$ where $n = \operatorname{tr} \mathbf{1}$. For any density operator $\sigma$ on a $n$-dimensional Hilbert space we have

$$-H(\sigma) + \log n = \operatorname{tr} \sigma \log \sigma - \operatorname{tr} \sigma \log(\mathbf{1}/n) = \operatorname{tr} \sigma(\log \sigma - \log(\mathbf{1}/n)).$$

The expression on the right is the relative entropy $H(\sigma \| \mathbf{1}/n)$ of $\sigma$ relative to $\mathbf{1}/n$, a notion we will discuss in the following section, and where we will see that $H(\sigma \| \mathbf{1}/n)$ is always strictly positive, unless $\sigma$ equals $\mathbf{1}/n$ when it will be zero. Therefore, the von Neumann entropy of $\sigma$ is smaller than or equal to $\log n$ with equality if and only if $\sigma = \mathbf{1}/n$.

For future use we will introduce some notation considering states in systems that are compositions of smaller systems. Let $\mathcal{A}$ and $\mathcal{B}$ be physical systems. We use a density operator to describe the state the composite system $\mathcal{A} \otimes \mathcal{B}$ is in, and say this density operator is $\sigma$ then more explicitely we will denote it by $\sigma_{\mathcal{AB}}$. We can ignore either of the systems $\mathcal{A}$ and $\mathcal{B}$, i.e. take a partial trace, to be left with a density operator in $\mathcal{B}$ respectively $\mathcal{A}$, and this density operator we will denote by $\sigma_{\mathcal{B}}$ respectively $\sigma_{\mathcal{A}}$.

The following lemma tells us that if we regard two systems that are independent, the entropy makes no difference between regarding the two systems separately or regarding them as one larger system. If the one system is in a state $\sigma_1$ and the other is in a state $\sigma_2$, since the two systems are independent they together are in a state $\sigma_1 \otimes \sigma_2$, and the entropy of $\sigma_1$ plus the entropy of $\sigma_2$ equals the entropy of $\sigma_1 \otimes \sigma_2$. However, if the two systems are not independent, then regarding the two systems separately means ignoring any correlation between the two systems. Then the state $\sigma$ of the composite system does not equal $\sigma_1 \otimes \sigma_2$, and the entropy of $\sigma$ will be smaller than the sum of the entropy of $\sigma_1$ and the entropy of $\sigma_2$.

**Lemma 3.3 (Subadditivity)** *Let $\mathcal{A}$ and $\mathcal{B}$ be physical systems and let $\sigma_{\mathcal{AB}}$ be a density operator in the composite system $\mathcal{A} \otimes \mathcal{B}$. Then*

$$H(\sigma_{\mathcal{AB}}) \leq H(\sigma_{\mathcal{A}}) + H(\sigma_{\mathcal{B}}),$$

*with equality if and only if* $\sigma_{\mathcal{AB}} = \sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}$.

The proof of Lemma 3.3 relies on two results we will come across in the following section, point 1 of Proposition 3.8 and Theorem 3.7. It is a straightforward consequence of those two results.

Regard a density operator $\sigma$. We exchange its spectral values with repect to the outcomes to obtain a different density operator $\sigma^{(1)}$. For any number $\lambda$ between 0 and 1 we expect the entropy of the mixture $\lambda\sigma + (1-\lambda)\sigma^{(1)}$ to be greater than the entropy of $\sigma$. If we create more density operators $\sigma^{(k)}$ like we created $\sigma^{(1)}$, and we take positive numbers $\lambda_k$ such that $\sum_k \lambda_k = 1$, then we expect to have

$$H\Big(\sum_k \lambda_k \sigma^{(k)}\Big) \geq H(\sigma).$$

We noted earlier on that the entropy is invariant under permutations, so $H(\sigma) = H(\sigma^{(k)})$ for every $\sigma^{(k)}$. Therefore we can write

$$H\Big(\sum_k \lambda_k \sigma^{(k)}\Big) \geq \sum_k \lambda_k H(\sigma^{(k)}).$$

In fact, this happens to be so, not only when the $\sigma^{(k)}$ are obtained from $\sigma$ by performing permutations on the spectral values, but also for just any density operators $\sigma^{(k)}$. This property is known as the concavity of the von Neumann entropy.

**Theorem 3.4 (Concavity)** *Let* $\sigma_1, \ldots, \sigma_m$ *be density operators in a* $^*$*-algebra* $\mathcal{A}$ *and let* $\lambda_1, \ldots, \lambda_m$ *be positive real numbers that add up to 1, then*

$$\sum_{i=1}^{m} \lambda_i H(\sigma_i) \leq H\Big(\sum_{i=1}^{m} \lambda_i \sigma_i\Big),$$

*with equality if and only if for all the indices* $i$ *the density operators* $\sigma_i$ *for which* $\lambda_i \neq 0$ *are equal.*

*Proof:* Choose an orthonormal base $\{e_i\}_{i=1}^m$ in the Hilbert space $\mathbb{C}^m$, and for every $i = 1, \ldots, m$ let $E_i$ be the projection on $e_i$. Let $\mathcal{B}$ denote the algebra $M_m(\mathbb{C})$. Define a state $\sigma_{\mathcal{AB}}$ in $\mathcal{A} \otimes \mathcal{B}$ by

$$\sigma_{\mathcal{AB}} := \sum_{i=1}^{m} \lambda_i \sigma_i \otimes E_i.$$

Note that we have

$$\sigma_{\mathcal{A}} = \sum_{i=1}^{m} \lambda_i \sigma_i,$$

$$\sigma_{\mathcal{B}} = \sum_{i=1}^{m} \lambda_i E_i$$

and

$$H(\sigma_{\mathcal{AB}}) = H\Big(\sum_{i=1}^{m} \lambda_i E_i\Big) + \sum_{i=1}^{m} \lambda_i H(\sigma_i).$$

Applying Lemma 3.3 then yields

$$\sum_{i=1}^{m} \lambda_i H(\sigma_i) \leq H\Big(\sum_{i=1}^{m} \lambda_i \sigma_i\Big),$$

with equality if and only if $\sigma_{\mathcal{AB}} = \sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}$, i.e. all the $\sigma_i$ for which $\lambda_i \neq 0$ are equal. □

The following theorem shows that if we vary a density operator $\rho$ by a small amount, then the von Neumann entropy of $\rho$ changes, but this change is bounded from above, and the bound depends on how much $\rho$ is varied. We make variations in density operators quantitative by introducing a distance on the collection of density operators in a certain $^*$-algebra. Such a distance is the map $(\rho, \sigma) \mapsto \|\rho - \sigma\|_1$.

**Theorem 3.5 (Fannes' inequality)** *Let $\mathcal{A}$ be a $^*$-algebra of operators on a Hilbert space of dimension $n$. For any two density operators $\rho$ and $\sigma$ in $\mathcal{A}$*

$$|H(\rho) - H(\sigma)| \leq \|\rho - \sigma\|_1 \log n + \frac{1}{e}.$$

*Proof:* [7], page 512, Theorem 11.6. □

## 3.2 The quantum relative entropy

**Definition 3.6** Let $\mathcal{A}$ be a $^*$-algebra and let $\rho$ and $\sigma$ be two density operators in $\mathcal{A}$. The *quantum relative entropy* of $\rho$ relative to $\sigma$ is defined as

$$H(\rho\|\sigma) := \operatorname{tr} \rho(\log \rho - \log \sigma).$$

**Theorem 3.7 (Klein's inequality)** *Let $\mathcal{A}$ be a $^*$-algebra. For any two density operators $\rho$ and $\sigma$ in $\mathcal{A}$*

$$H(\rho\|\sigma) \geq 0,$$

*with equality if and only if $\rho = \sigma$.*

*Proof:* [7], page 511, Theorem 11.7. □

Theorem 3.7 might give the impression that the relative entropy serves as a metric. However, it should be noted that the relative entropy is not symmetric, that is, in general $H(\sigma\|\rho) \neq H(\rho\|\sigma)$.
Unlike the von Neumann entropy, the quantum relative entropy is not bounded from above. As a matter of fact, $H(\rho\|\sigma)$ is infinite if the support of $\rho$ is not contained in the support of $\sigma$, i.e. if $\{\xi : \rho\xi \neq 0\} \not\subseteq \{\xi : \sigma\xi \neq 0\}$.

**Proposition 3.8** *Let $\mathcal{A}$ and $\mathcal{B}$ be physical systems. Let $\sigma_{\mathcal{AB}}$ be a density operator in the composite system $\mathcal{A} \otimes \mathcal{B}$, let $\rho_{\mathcal{A}}$ be a density operator in $\mathcal{A}$ and let $\rho_{\mathcal{B}}$ be a density operator in $\mathcal{B}$. Then:*

*1.* $H(\sigma_{\mathcal{AB}} \| \sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}) = H(\sigma_{\mathcal{A}}) + H(\sigma_{\mathcal{B}}) - H(\sigma_{\mathcal{AB}})$*;*

*2.* $H(\sigma_{\mathcal{AB}} \| \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = H(\sigma_{\mathcal{AB}} \| \sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}) + H(\sigma_{\mathcal{A}} \| \rho_{\mathcal{A}}) + H(\sigma_{\mathcal{B}} \| \rho_{\mathcal{B}})$*.*

*Proof:* By writing down the spectral decomposition of $\sigma_{\mathcal{AB}}$ explicitely and using Definition 1.14 we find

$$\log(\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = (\log \rho_{\mathcal{A}}) \otimes \mathbf{1} + \mathbf{1} \otimes (\log \rho_{\mathcal{B}}),$$

and so

$$\begin{aligned} H(\sigma_{\mathcal{AB}} \| \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) &= -H(\sigma_{\mathcal{AB}}) - \operatorname{tr}(\sigma_{\mathcal{AB}}(\log \rho_{\mathcal{A}} \otimes \mathbf{1})) - \operatorname{tr}(\sigma_{\mathcal{AB}}(\mathbf{1} \otimes \log \rho_{\mathcal{B}})) \\ &= -H(\sigma_{\mathcal{AB}}) - \operatorname{tr}(\sigma_{\mathcal{A}} \log \rho_{\mathcal{A}}) - \operatorname{tr}(\sigma_{\mathcal{B}} \log \rho_{\mathcal{B}}) \\ &= -H(\sigma_{\mathcal{AB}}) + H(\sigma_{\mathcal{A}}) + H(\sigma_{\mathcal{A}} \| \rho_{\mathcal{A}}) + H(\sigma_{\mathcal{B}}) + H(\sigma_{\mathcal{B}} \| \rho_{\mathcal{B}}), \end{aligned}$$

where in the second line we have used Lemma 2.9.
We obtain 1. by taking $\rho_{\mathcal{A}} = \sigma_{\mathcal{A}}$ and $\rho_{\mathcal{B}} = \sigma_{\mathcal{B}}$, and then we obtain 2. by substituting 1. in the above equation. □

We can not take the above results further, as we will see in the following example. The limitation that we demonstrate here will turn out to be a limitation on the uncertainty relation we will eventually derive.

**Example 3.9** Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras, $\mathcal{A}$ equal to the algebra $M_2(\mathbb{C})$ of $2 \times 2$ matrices and $\mathcal{B}$ equal to the Abelian algebra $\mathcal{C}_2$ of complex funtions on a set of two elements. Let $p_1$ and $p_2$ be projections in $\mathcal{A}$ such that $p_1 p_2 = 0$ and $\operatorname{tr} p_1 = \operatorname{tr} p_2 = 1$, and let $\rho_{\mathcal{AB}}$ be the density operator $(\frac{1}{2}p_1, \frac{1}{2}p_2)$ in $\mathcal{A} \otimes \mathcal{B}$ and $\sigma_{\mathcal{AB}}$ the density operator $(p_1, 0)$.
We have

$$H(\sigma_{\mathcal{AB}} \| \rho_{\mathcal{AB}}) = \operatorname{tr} p_1(\log p_1 - \log(p_1/2)) = \log 2,$$

and when taking partial traces

$$H(\sigma_{\mathcal{A}} \| \rho_{\mathcal{A}}) = H(p_1 \| \mathbf{1}/2) = -\log(1/2) = \log 2$$

and

$$H(\sigma_{\mathcal{B}} \| \rho_{\mathcal{B}}) = H((1,0) \| (1/2, 1/2)) = -\log(1/2) = \log 2,$$

so

$$2H(\sigma_{\mathcal{AB}} \| \rho_{\mathcal{AB}}) = H(\sigma_{\mathcal{A}} \| \rho_{\mathcal{A}}) + H(\sigma_{\mathcal{B}} \| \rho_{\mathcal{B}}).$$

Regard a system that is a composition of two physical systems $\mathcal{A}$ and $\mathcal{B}$. If two density operators $\sigma_{\mathcal{AB}}$ and $\rho_{\mathcal{AB}}$ are used to describe the system $\mathcal{A} \otimes \mathcal{B}$, say by two different people, then it is intuitively clear that these two density operators are easier to distinguish than the two density operators $\sigma_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ that are obtained after casting out the subsystem $\mathcal{B}$. This thought is the content of the following example.

**Example 3.10 (Monotonicity under partial traces)** Let $\mathcal{A}$ and $\mathcal{B}$ be physical systems. For any two density operators $\sigma_{\mathcal{AB}}$ and $\rho_{\mathcal{AB}}$ in the composite system $\mathcal{A} \otimes \mathcal{B}$ we have

$$H(\sigma_{\mathcal{AB}} \| \rho_{\mathcal{AB}}) \geq H(\sigma_{\mathcal{A}} \| \rho_{\mathcal{A}}).$$

This example is a special case of the general result that quantum relative entropy is decreasing under completely positive, unital maps. Indeed, a partial trace is the dual of an embedding, which is a completely positive, unital map.

**Theorem 3.11** *Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and let $S$ be a completely positive, unital map from $\mathcal{B}$ into $\mathcal{A}$. Then for any two density operators $\sigma_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ in $\mathcal{A}$*

$$H(S^*\sigma_{\mathcal{A}} \| S^*\rho_{\mathcal{A}}) \leq H(\sigma_{\mathcal{A}} \| \rho_{\mathcal{A}}).$$

The proof of this theorem is rather lengthy. It will be the subject of section 3.3.

A direct consequence of the monotonicity of the quantum relative entropy under partial traces, Example 3.10, is the joint convexity of the quantum relative entropy.

**Theorem 3.12 (Joint convexity)** *Let $\rho_1, \ldots, \rho_n, \sigma_1, \ldots, \sigma_n$ be density operators in a $^*$-algebra $\mathcal{A}$ and let $\lambda_1, \ldots, \lambda_n$ be positive numbers that add up to 1. Then*

$$H\Big(\sum_{i=1}^{n} \lambda_i \rho_i \Big\| \sum_{i=1}^{n} \lambda_i \sigma_i\Big) \leq \sum_{i=1}^{n} \lambda_i H(\rho_i \| \sigma_i).$$

*Proof:* Define density operators $\rho$ and $\sigma$ in $M_n(\mathcal{A})$ by

$$\rho = \begin{pmatrix} \lambda_1\rho_1 & & \\ & \ddots & \\ & & \lambda_n\rho_n \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \lambda_1\sigma_1 & & \\ & \ddots & \\ & & \lambda_n\sigma_n \end{pmatrix}.$$

Regarding $H(\rho\|\sigma)$ and applying Example 3.10 directly yields the joint convexity property. □

## 3.3 Proof of the monotonicity of quantum relative entropy

Here we prove the monotonicity of the quantum relative entropy under completely positive, unital maps, Theorem 3.11. In this pursuit we closely follow [1] by Ahlswede and Löber who in turn say to have closely followed [2] (concerning Lemma 3.13) and [8]. The proof requires some preparatory work, which we will start with.

**Lemma 3.13** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $V$ a linear map from a $\mathcal{H}_1$ into $\mathcal{H}_2$ with the property $\|V\| \leq 1$, and $x$ a positive operator on $\mathcal{H}_2$. For any $\mu \in [0, 1]$,*

$$(V^*xV)^\mu \geq V^*x^\mu V.$$

*Proof:* Let $a := (\mathbf{1} - V^*V)^{\frac{1}{2}}$ and $b := (\mathbf{1} - VV^*)^{\frac{1}{2}}$, and define an operator $U$ from $\mathcal{H}_1 \oplus \mathcal{H}_2$ into $\mathcal{H}_2 \oplus \mathcal{H}_1$ by

$$U = \begin{pmatrix} V & b \\ a & -V^* \end{pmatrix}$$

and an operator $X$ on $\mathcal{H}_2 \oplus \mathcal{H}_1$ by

$$X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose a positive number $\lambda$ large enough such that

$$Y := \begin{pmatrix} V^*xV & 0 \\ 0 & \lambda\mathbf{1} \end{pmatrix} \geq \begin{pmatrix} V^*xV & V^*xb \\ bxV & bxb \end{pmatrix} = U^*XU.$$

We note that $V^*\xi \oplus b\xi$ is an eigenvector of $U^*XU$ with eigenvalue $\alpha$ if $\xi$ is an eigenvector of $x$ with eigenvalue $\alpha$. We have

$$Y^\mu \geq (U^*XU)^\mu = U^*X^\mu U = U^* \begin{pmatrix} x^\mu & 0 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} V^*x^\mu V & V^*x^\mu b \\ bx^\mu V & bx^\mu b \end{pmatrix}$$

for any $\mu \in [0, 1]$. Consequently, $(V^*xV)^\mu \geq V^*x^\mu V$. □

**Lemma 3.14** *Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and let $S$ be a completely positive, unital map from $\mathcal{B}$ into $\mathcal{A}$. If $\sigma_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ are density operators in $\mathcal{A}$, $\rho_{\mathcal{A}}$ invertible, then for any $\mu \in [0, 1]$ and any $x \in \mathcal{B}$*

$$\operatorname{tr} S(x)^* \sigma_{\mathcal{A}}^\mu S(x) \rho_{\mathcal{A}}^{1-\mu} \leq \operatorname{tr} x^* \sigma_{\mathcal{B}}^\mu x \rho_{\mathcal{B}}^{1-\mu},$$

*where $\sigma_{\mathcal{B}} = S^*\sigma_{\mathcal{A}}$ and $\rho_{\mathcal{B}} = S^*\rho_{\mathcal{A}}$.*

*Proof:* Define a map $V$ from $\mathcal{B}$ into $\mathcal{A}$ by

$$V : b \mapsto S(b\rho_{\mathcal{B}}^{-\frac{1}{2}})\rho_{\mathcal{A}}^{\frac{1}{2}}.$$

$V$ is a contraction, that is, $\|V\| \leq 1$:

$$\begin{aligned} \|V(b)\|_2^2 &= \operatorname{tr} \rho_{\mathcal{A}} S(b\rho_{\mathcal{B}}^{-\frac{1}{2}})^* S(b\rho_{\mathcal{B}}^{-\frac{1}{2}}) \\ &\leq \operatorname{tr} \rho_{\mathcal{A}} S((b\rho_{\mathcal{B}}^{-\frac{1}{2}})^* b\rho_{\mathcal{B}}^{-\frac{1}{2}}) \\ &= \operatorname{tr} S^*(\rho_{\mathcal{A}}) \rho_{\mathcal{B}}^{-\frac{1}{2}} b^* b \rho_{\mathcal{B}}^{-\frac{1}{2}} \\ &= \|b\|_2^2 \end{aligned}$$

for any $b \in \mathcal{B}$, where the inequality results from Schwarz, i.e. point 1 of Lemma 1.39.
We define two more maps,

$$W_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} : a \mapsto \sigma_{\mathcal{A}} a \rho_{\mathcal{A}}^{-1},$$

$$W_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} : b \mapsto \sigma_{\mathcal{B}} b \rho_{\mathcal{B}}^{-1}.$$

$W_{\mathcal{A}}$ and $W_{\mathcal{B}}$ are positive. For example, for any $a \in \mathcal{A}$

$$(a|W_{\mathcal{A}}a)_2 = \operatorname{tr} a^* \sigma_{\mathcal{A}} a \rho_{\mathcal{A}}^{-1} = \operatorname{tr} \rho_{\mathcal{A}}^{-\frac{1}{2}} a^* \sigma_{\mathcal{A}}^{\frac{1}{2}} \sigma_{\mathcal{A}}^{\frac{1}{2}} a \rho_{\mathcal{A}}^{-\frac{1}{2}} = (\sigma_{\mathcal{A}}^{\frac{1}{2}} a \rho_{\mathcal{A}}^{-\frac{1}{2}} | \sigma_{\mathcal{A}}^{\frac{1}{2}} a \rho_{\mathcal{A}}^{-\frac{1}{2}})_2 \geq 0.$$

Observe that if $p$ and $q$ are spectral projections of $\sigma_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ respectively, then $pq$ is an eigenvector of $W_{\mathcal{A}}$. So for any $\mu \geq 0$ we have $W_{\mathcal{A}}^{\mu} a = \sigma_{\mathcal{A}}^{\mu} a \rho_{\mathcal{A}}^{-\mu}$ and $W_{\mathcal{B}}^{\mu} b = \sigma_{\mathcal{B}}^{\mu} b \rho_{\mathcal{B}}^{-\mu}$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Furthermore, for any $x \in \mathcal{B}$,

$$\begin{aligned}
(x|V^* W_{\mathcal{A}} V x)_2 &= \operatorname{tr} \rho_{\mathcal{A}}^{\frac{1}{2}} S(x\rho_{\mathcal{B}}^{-\frac{1}{2}})^* \sigma_{\mathcal{A}} S(x\rho_{\mathcal{B}}^{-\frac{1}{2}}) \rho_{\mathcal{A}}^{-\frac{1}{2}} \\
&\leq \operatorname{tr} \sigma_{\mathcal{A}} S(x\rho_{\mathcal{B}}^{-1} x^*) \\
&= \operatorname{tr} \sigma_{\mathcal{B}} x \rho_{\mathcal{B}}^{-1} x^* \\
&= (x|W_{\mathcal{B}} x)_2,
\end{aligned}$$

where we used Schwarz again, so $V^* W_{\mathcal{A}} V \leq W_{\mathcal{B}}$.
$W_{\mathcal{A}}$ and $W_{\mathcal{B}}$ are positive and $\|V\| \leq 1$, so according to Lemma 3.13 for any $\mu \in [0,1]$ we have $V^* W_{\mathcal{A}}^{\mu} V \leq (V^* W_{\mathcal{A}} V)^{\mu} \leq W_{\mathcal{B}}^{\mu}$, and so for any $x \in \mathcal{B}$

$$\begin{aligned}
\operatorname{tr} \rho_{\mathcal{A}}^{\frac{1}{2}} S(x)^* \sigma_{\mathcal{A}}^{\mu} S(x) \rho_{\mathcal{A}}^{\frac{1}{2}-\mu} &= (V x \rho_{\mathcal{B}}^{\frac{1}{2}} | W_{\mathcal{A}}^{\mu} V x \rho_{\mathcal{B}}^{\frac{1}{2}})_2 \\
&\leq (x \rho_{\mathcal{B}}^{\frac{1}{2}} | W_{\mathcal{B}}^{\mu} x \rho_{\mathcal{B}}^{\frac{1}{2}})_2 \\
&= \operatorname{tr} x^* \sigma_{\mathcal{B}}^{\mu} x \rho_{\mathcal{B}}^{1-\mu}. \qquad \square
\end{aligned}$$

**Example 3.15** Adopting notations from Lemma 3.14,

$$\operatorname{tr} \sigma_{\mathcal{A}}^{\mu} \rho_{\mathcal{A}}^{1-\mu} \leq \operatorname{tr} \sigma_{\mathcal{B}}^{\mu} \rho_{\mathcal{B}}^{1-\mu}.$$

Now we are ready to prove that the quantum relative entropy is decreasing under completely positive, unital maps.

*Proof:* Suppose $\sigma$ and $\rho$ are density operators on a finite dimensional Hilbert space and $0 \leq \mu \leq 1$, then we observe that

$$\frac{d}{d\mu} \operatorname{tr} \sigma^{\mu} \rho^{1-\mu} = \operatorname{tr} \sigma^{\mu} (\ln \sigma - \ln \rho) \rho^{1-\mu},$$

and in the limit $\mu$ going to 1 from below this expression becomes the quantum relative entropy $H(\sigma \| \rho) = \operatorname{tr} \sigma (\ln \sigma - \ln \rho)$.
Let $\mathcal{A}$ and $\mathcal{B}$ be $^*$-algebras and let $S$ be a completely positive, unital map from $\mathcal{B}$ into $\mathcal{A}$. Let $\sigma_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ be density operators in $\mathcal{A}$, and denote $S^* \sigma_{\mathcal{A}}$ and $S^* \rho_{\mathcal{A}}$ by $\sigma_{\mathcal{B}}$ respectively $\rho_{\mathcal{B}}$. Without loss of generality we may assume that $\rho_{\mathcal{A}}$ is invertible. Using Example 3.15 we obtain

$$\begin{aligned}
\lim_{\mu \uparrow 1} \frac{d}{d\mu} \operatorname{tr} \sigma_{\mathcal{A}}^{\mu} \rho_{\mathcal{A}}^{1-\mu} &= \lim_{\mu \uparrow 1} \frac{1}{1-\mu} (1 - \operatorname{tr} \sigma_{\mathcal{A}}^{\mu} \rho_{\mathcal{A}}^{1-\mu}) \\
&\geq \lim_{\mu \uparrow 1} \frac{1}{1-\mu} (1 - \operatorname{tr} \sigma_{\mathcal{B}}^{\mu} \rho_{\mathcal{B}}^{1-\mu}) \\
&= \lim_{\mu \uparrow 1} \frac{d}{d\mu} \operatorname{tr} \sigma_{\mathcal{B}}^{\mu} \rho_{\mathcal{B}}^{1-\mu}.
\end{aligned}$$

This finishes the proof. □

# 4 Entropic uncertainty relations

## 4.1 The Heisenberg uncertainty principle

From daily experience we know that it is possible to obtain information about a classical system without disturbing it. To be more precise, we can perform such a measurement that would we ignore the outcome of the measurement, the state of the classical system would be unaltered. A similar argument holds for systems that are partially classical and partially quantum physical. As we will now see, it is only a purely quantum physical system that does not allow a measurement that yields information about the system but leaves the state of the system unaltered in case we would ignore this information.

**Definition 4.1** A measurement $S$ on a physical system $\mathcal{A}$ with a measuring device $\mathcal{C}$ we call *non-interfering* if $S(a \otimes \mathbf{1}) = a$ for every $a \in \mathcal{A}$.

**Theorem 4.2** *Let $S$ be a non-interfering measurement on a physical system $\mathcal{A}$ with a measuring device $\mathcal{C}$, then $S(\mathbf{1} \otimes c)$ is in the center of $\mathcal{A}$ for every $c$ in $\mathcal{C}$. Furthermore, if $\mathcal{A}$ is a factor, then $c \mapsto S(\mathbf{1} \otimes c)$ is a state on $\mathcal{C}$ times the identity of $\mathcal{A}$.*

*Proof:* By Lemma 1.39, for all $a \in \mathcal{A}$ and all $c \in \mathcal{C}$,

$$S(\mathbf{1} \otimes c)a = S(\mathbf{1} \otimes c)S(a \otimes \mathbf{1}) = S(a \otimes c).$$

Replacing $a$ by $a^*$ and $c$ by $c^*$, and taking the $^*$ of each term in these equalities, we obtain

$$a\, S(\mathbf{1} \otimes c) = S(a \otimes c).$$

So for every $c \in \mathcal{C}$, $S(\mathbf{1} \otimes c)$ is in the center of $\mathcal{A}$.
If $\mathcal{A}$ is a factor, then its center equals $\mathbb{C}\mathbf{1}$, and the map $c \mapsto S(\mathbf{1} \otimes c)$ evidently is a state on $\mathcal{C}$ times the identity of $\mathcal{A}$. □

A non-interfering measurement by definition is one that does not change the state of the system if the outcome is ignored. According to the previous theorem, if the system is purely quantum physical, i.e. if it is a factor, then the outcome of the measurement does not depend on the initial state of the system and so the measurement supplies us no information about the system.

In the remaining sections we pursue to make the above observation quantitative by means of an uncertainty relation for a single measurement. Our first endeavour will involve an uncertainty relation regarding two measurements, and therefore that relation will be introduced first.

## 4.2 An entropic uncertainty relation for a pair of measurements

There is an entropic uncertainty relation for a pair of measurements. It relates the Shannon entropy of our expectations on the outcome of the measurements.

The measurements in question are of a particular kind, and we start by regarding those measurements.

**Definition 4.3** A collection $\{X_i\}_{i=1}^n$ of positive operators in an algebra $\mathcal{A}$ with the property $\sum_{i=1}^n X_i = \mathbf{1}$, where $\mathbf{1}$ is the identity in $\mathcal{A}$, is called a *positive operator valued measure*, or *POVM* for short.

A POVM induces a particular kind of measurement:

**Lemma 4.4** *If $\{X_i\}_{i=1}^n$ is a POVM in a ${}^*$-algebra $\mathcal{A}$ and $\mathcal{C}$ is an Abelian ${}^*$-algebra of dimension $n$, then the map $S$ defined by*

$$S : (x_i)_{i=1}^n \mapsto \sum_{i=1}^n X_i^{\frac{1}{2}} x_i X_i^{\frac{1}{2}} \tag{5}$$

*is a completely positive, unital map from $\mathcal{A} \otimes \mathcal{C}$ into $\mathcal{A}$.*

*Proof:* Let $\mathcal{H}$ be the Hilbert space on which $\mathcal{A}$ acts. We choose a base $\{e_i\}_{i=1}^n$ of projections in $\mathcal{C}$ with the property $e_i e_j = \delta_{ij} e_i$. Next we define a map $V$ from $\mathcal{H}$ into the Hilbert space $\mathcal{H} \otimes \mathcal{C}$ by

$$V : \xi \mapsto \sum_{i=1}^n X_i^{\frac{1}{2}} \xi \otimes e_i.$$

Without difficulty it is verified that $V$ is an isometry and that

$$V^*(x_i)_{i=1}^n V = \sum_{i=1}^n X_i^{\frac{1}{2}} x_i X_i^{\frac{1}{2}},$$

where $(x_i)_{i=1}^n = \sum_{i=1}^n x_i \otimes e_i$. According to Theorem 1.36 this finishes the proof. □

In the Schrödinger picture the measurement given by (5) reads

$$S^* : \sigma \mapsto (X_i^{\frac{1}{2}} \sigma X_i^{\frac{1}{2}})_{i=1}^n. \tag{6}$$

The entropic uncertainty relation that is regarded here is concerned with this kind of measurement. Actually, it concerns our expectations on the outcome of the measurements. The map $S^*$ yields a state on the composite system that consists of the physical system and the measuring device, and when ignoring the physical system we are left with a state on the measuring device which reflects our expectations on the outcome. This state is the classical probability distribution

$$p_i = \operatorname{tr} X_i^{\frac{1}{2}} \sigma X_i^{\frac{1}{2}}, \qquad i = 1, \dots, n,$$

where $\sigma$ is the initial state of the physical system. The Shannon entropy of this probability distribution we will denote by $H(\mathbf{X}, \sigma)$, where $\mathbf{X} = \{X_i\}_{i=1}^n$. If $\mathbf{Y} = \{Y_i\}_{i=1}^m$ is another POVM, then there is a sharp lower bound for the sum $H(\mathbf{X}, \sigma) + H(\mathbf{Y}, \sigma)$ of the two entropies, and this lower bound does not depend on the initial state $\sigma$.

**Theorem 4.5** *Let $\mathcal{A}$ be a physical system and let $\mathbf{X} = \{X_i\}_{i=1}^m$ and $\mathbf{Y} = \{Y_i\}_{i=1}^n$ be POVM's in $\mathcal{A}$. For any density operator $\rho$ in $\mathcal{A}$*

$$H(\mathbf{X}, \rho) + H(\mathbf{Y}, \rho) \geq -2 \log \max_{i,j} \|X_i^{\frac{1}{2}} Y_j^{\frac{1}{2}}\|.$$

*Proof:* [4], page 9, Corollary 2.6. □

In the case that every $X_i$ in $\mathbf{X}$ is a projection on a vector $\xi_i$ and the $\xi_i$ form an orthonormal set, and likewise for $\mathbf{Y}$, we have an uncertainty relation for two non-degenerate observables. The lower bound then is non-trivial, i.e. strictly positive, if and only if the two observables have no eigenvectors in common. The uncertainty relation in this form is first derived by Maassen and Uffink [6] pursuing a conjecture of Kraus [3].

## 4.3 An entropic uncertainty relation for a single measurement, I

In a first attempt to derive an uncertainty relation for a single measurement, we take the uncertainty relation for two measurements as given by Theorem 4.5 and we choose the measurements to be the same. The uncertainty relation then reads

$$H(\mathbf{X}, \rho) \geq -\log \max_{i,j} \|X_i^{\frac{1}{2}} X_j^{\frac{1}{2}}\| \tag{7}$$

for any density operator $\rho$.

However, for any $i, j$

$$\begin{aligned} \|X_i^{\frac{1}{2}} X_j^{\frac{1}{2}}\| &\leq \|X_i^{\frac{1}{2}}\|\|X_j^{\frac{1}{2}}\| \\ &\leq \max_i \|X_i^{\frac{1}{2}}\|^2 \\ &= \max_i \|X_i\|, \end{aligned}$$

so

$$\max_{i,j} \|X_i^{\frac{1}{2}} X_j^{\frac{1}{2}}\| = \max_i \|X_i\|,$$

and the uncertainty relation for a single measurement as given by (7) becomes

$$H(\mathbf{X}, \rho) \geq -\log \max_i \|X_i\|. \tag{8}$$

It is clear that this cannot serve as a uncertainty relation that is typical for quantum physics. The lower bound only depends on the norms of the elements $X_i$ in the POVM $\mathbf{X}$, and not on the multiplicative structure of the algebra $\mathcal{A}$ that represents the physical system, that is whether the system is classical or quantum mechanical. In fact, we can derive inequality (8) straight from the definition of the Shannon entropy:

For any probability distribution $(q_1, \ldots, q_n)$ we have

$$\begin{aligned} \sum_{i=1}^{n} q_i \log q_i &\leq \sum_{i=1}^{n} q_i \log \max_j q_j \\ &= \log \max_j q_j \,. \end{aligned}$$

Observing that for any density operator $\rho$ and any element $X_i$ in the POVM $\mathbf{X}$ the norm of $X_i$ is greater than the trace of $\rho X_i$,

$$\|X_i\| \geq \operatorname{tr} \rho X_i,$$

we obtain

$$H(\mathbf{X}, \rho) \geq -\log \max_i \|X_i\|.$$

## 4.4 An entropic uncertainty relation for a single measurement, II

Our second attempt has an entirely different approach. We focus on relative entropy instead of von Neumann or Shannon entropy, and perhaps not surprising we use the monotonicity property of relative entropy under completely positive, unital maps, Theorem 3.11. The relation we derive unfortunately does not apply to every measurement, but only to those that know a specific kind of state. This state has to be an equilibrium state, a state that is returned if the outcome of the measurement is ignored. Furthermore, it has to be such a state that after the measurement there is a final state in which the physical system and the measuring device are independent. This limitation relates to Example 3.9.

**Definition 4.6** Let $S$ be a measurement on a physical system $\mathcal{A}$ with a measuring device $\mathcal{C}$. We define the maps

$$T : \mathcal{A} \to \mathcal{A} : a \mapsto S(a \otimes \mathbf{1}),$$

$$Q : \mathcal{C} \to \mathcal{A} : c \mapsto S(\mathbf{1} \otimes c).$$

The map $T$ is interpreted as performing a measurement $S$ but ignoring the outcome given by the measuring device; the map $Q$ is interpreted as performing measurement $S$ and then ignoring the physical system.

**Theorem 4.7** *Let $S$ be a measurement on a physical system $\mathcal{A}$ with measuring device $\mathcal{C}$. If there is a density operator $\lambda_{\mathcal{A}}$ in $\mathcal{A}$ and a density operator $\lambda_{\mathcal{C}}$ in $\mathcal{C}$ such that $S^*\lambda_{\mathcal{A}} = \lambda_{\mathcal{A}} \otimes \lambda_{\mathcal{C}}$, then*

$$H(\rho_{\mathcal{A}} \| \lambda_{\mathcal{A}}) - H(T^*\rho_{\mathcal{A}} \| T^*\lambda_{\mathcal{A}}) \geq H(Q^*\rho_{\mathcal{A}} \| Q^*\lambda_{\mathcal{A}})$$

*for every density operator $\rho_{\mathcal{A}}$ in $\mathcal{A}$.*

*Proof:* Let $\rho_{\mathcal{A}}$ be a density operator in $\mathcal{A}$, then there is a density operator $\sigma_{\mathcal{AC}}$ such that $S^*\rho_{\mathcal{A}} = \sigma_{\mathcal{AC}}$, and then $T^*\rho_{\mathcal{A}} = \sigma_{\mathcal{A}}$ and $Q^*\rho_{\mathcal{A}} = \sigma_{\mathcal{C}}$. Suppose there are density operators $\lambda_{\mathcal{A}}$ in $\mathcal{A}$ and $\lambda_{\mathcal{C}}$ in $\mathcal{C}$ such that $S^*\lambda_{\mathcal{A}} = \lambda_{\mathcal{A}} \otimes \lambda_{\mathcal{C}}$. Then we have

$$\begin{aligned} H(\rho_{\mathcal{A}} \| \lambda_{\mathcal{A}}) &\geq H(S^*\rho_{\mathcal{A}} \| S^*\lambda_{\mathcal{A}}) \\ &= H(\sigma_{\mathcal{AC}} \| \lambda_{\mathcal{A}} \otimes \lambda_{\mathcal{C}}) \\ &\geq H(\sigma_{\mathcal{A}} \| \lambda_{\mathcal{A}}) + H(\sigma_{\mathcal{C}} \| \lambda_{\mathcal{C}}) \\ &= H(T^*\rho_{\mathcal{A}} \| T^*\lambda_{\mathcal{A}}) + H(Q^*\rho_{\mathcal{A}} \| Q^*\lambda_{\mathcal{A}}). \end{aligned}$$

The first inequality is a consequence of the monotonicity of relative entropy under measurements, Theorem 3.11, the second inequality comes from applying Proposition 3.8. □

This uncertainty relation relates the information generated by a measurement to the alteration of the state of the physical system due to the measurement. The amount of information contained in the classical output of the measuring device is bounded from above by the amount of change of the state of the physical system subject to the measurement. In case the measurement is non-interfering, the left hand side of the inequality in Theorem 4.7 is zero, but it might be so that the measurement is one to which Theorem 4.7 does not apply. If the measurement is one to which Theorem 4.7 does apply and it is non-interfering, then evidently it supplies us no information whether the physical system is purely quantum physical or not. So as far as such measurements are concerned, Theorem 4.7 succeeds in making quantitative the thought contained in the Heisenberg uncertainty principle as stated in Theorem 4.2.

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# 5 Summary

In this paper we pursue to find an entropic uncertainty relation for a single measurement. We consider a measurement on a physical system using a measuring device that yields classical output. In the Schrödinger picture a state on the physical system then is transformed to a state on the composite system of physical system and measuring device. We use entropy as a measure of uncertainty. In our pursuit we take two different approaches. The first starts from the entropic uncertainty relation for two measurements by Krishna and Parthasarathy, a generalisation of the Maassen-Uffink inequality. By taking the two measurements in this inequality to be the same, we hope to derive an uncertainty relation for a single measurement. This approach, however, appears to be fruitless. The obtained inequality for a single measurement fails to say anything quantitative in the context of the Heisenberg uncertainty principle.

The second approach appears to be of some succes. We use the monotonicity of the quantum relative entropy under completely positive, unital maps to obtain an uncertainty relation in terms of the quantum relative entropy. This relation entirely succeeds in reflecting the idea of the Heisenberg uncertainty principle in a quantitative way, though with the shortcoming that it does not apply to all measurements, but only to those that know a certain kind of state. This state has to be an equilibrium state, a state that is returned if the outcome of the measurement is ignored. Furthermore, it has to be such a state that after the measurement there is a final state in which the physical system and the measuring device are independent.