Addition of Freely Independent Random Variables

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Abstract

A direct proof is given of Voiculescu's addition theorem for freely independent real-valued random variables, using resolvents of self-adjoint operators. In contrast to the original proof, no assumption is made on the existence of moments above the second.

§0. Introduction

The concept of independent random variables lies at the heart of classical probability. Via independent sequences it leads to the Gauss and Poisson distributions, and via independent increments of a process to stochastic calculus. Classical, commutative independence of random variables amounts to a factorisation property of probability spaces. Algebraically this corresponds to a *tensor* product decomposition of function algebras.

At the opposite, non-commutative extreme Voiculescu discovered in 1983 the notion of 'free independence' of random variables, which corresponds to a *free* product of von Neumann algebras [Voi 1]. He showed that this notion leads naturally to analogues of the Gauss and Poisson distributions, very different in form from the classical ones [Voi 1], [Voi 5]. For instance the free analogue of the Gauss curve is a semi-ellipse. He also showed that $n \times n$-matrices whose entries are independent Gaussian random variables become freely independent in the limit of large n [Voi 4]. This explains the older result ([Wig 1,2], [Arn 1,2], [Wac], [Jon]) that the eigenvalue distributions of such random matrices with probability 1 tend to the semi-ellipse form as $n \to \infty$.

The relevance of free independence for non-commutative probability theory was realised by Kümmerer, who, with Speicher, developed a stochastic calculus based on free independence [KSp], [Spe 1,2].

In this paper we consider the addition problem: Which is the probability distribution μ of the sum $X_1 + X_2$ of two freely independent random variables, given the distributions μ_1 and μ_2 of the summands ? This problem was solved by Voiculescu himself in 1986 for the case of bounded, not necessarily self-adjoint random variables, relying on the existence of all the moments of the probability distributions μ_1 and μ_2 [Voi 2]. The result is an explicit calculation procedure for the 'free convolution product' of two probability distributions. In this procedure a central role is played by the Cauchy transform $G(z)$ of a distribution μ, which equals the expectation of the resolvent of the associated operator X. If we take X self-adjoint, μ is a probability measure on $\mathbf{R}$ and we may write:

$$G(z) := \int_{-\infty}^{\infty} \frac{\mu(dx)}{z-x} = \mathbf{E}\left((z-X)^{-1}\right).$$

This formula points at a direct way to find the free convolution product of μ_1 and μ_2: calculate the expectation of the resolvent of $X_1 + X_2$. In the present paper we follow this approach, which indeed leads to quite a direct proof. As a bonus— since we may lean on the classical results on resolvents of unbounded self-adjoint operators— our result extends beyond the measures of finite support. For technical reasons we had to assume finite variance.

A drawback of the method is that it does not work for the multiplication problem— also solved by Voiculescu [Voi 3]— since the product $X_1 X_2$ of two self-adjoint, freely independent random variables can be self-adjoint only if one of them is a constant.

The assumption of finite variance can possibly be weakened. It would be of some interest to do so, since all self-similar distributions apart from the semi-ellipse and the point measures have infinite variance.

As to the complex analysis of the addition problem, it turns out to be advantageous to consider instead of $G(z)$ the reciprocal $F(z) = 1/G(z)$. One may thus exploit some good properties of F; e.g., F increases the imaginary part, and collapses to the identity function for $X = 0$.

Consideration of unbounded supports leads to the following difficulty. Voiculescu's free convolution involves the inverse of G (or F) with respect to composition of functions. However, invertibility of G cannot be assumed. (In fact, it is equivalent to infinite divisibility of the distribution μ ([Voi 2]; see also Theorem 6.1).) Hence Voiculescu found himself obliged to consider G^{-1} only on some neighbourhood of 0 (corresponding to F^{-1} on some neighbourhood of ∞). Now, for bounded random variables this presents no problem. But for unbounded ones F need not be invertible on any neighbourhood of ∞. For this reason we formulate the addition theorem entirely in terms of F itself, avoiding to mention its inverse.

The addition theorem leads to a central limit theorem for freely independent, identically distributed random variables of finite variance. (For bounded support this theorem was proved in [Voi 1]; cf. also [Voi 5].)

Finally, as in [Voi 2] and [Voi 5] infinitely divisible distributions are considered and a free Lévy-Khinchin formula proved. The conclusion is that every infinitely divisible random variable (of finite variance) is composed of a 'semi-ellipse' part and a combination of 'free Poisson' parts. These free Poisson distributions can be readily calculated, as was done in [Voi 5]. They were first found in 1967 by Marchenko and Pastur [MaP] in the context of random matrices, and rediscovered in the West ten years later [GrS], [Wac].

This paper consists of six sections. The first contains some preliminaries on free independence, confined to the self-adjoint case, and gives the 'standard' cyclic representation of a pair of freely independent random variables. In the second we gather some facts about Cauchy transforms. In Section 3 we construct the free convolution product $F_1 \boxplus F_2$ of the reciprocal Cauchy transforms F_1 and F_2 of probability measures μ_1 and μ_2 of finite variance. In Section 4 it is shown that $F_1 \boxplus F_2(z) = \mathbf{E}((z - (\overline{X_1 + X_2}))^{-1})^{-1}$, where X_1 and X_2 are freely independent random variables with distributions μ_1 and μ_2 respectively, and the bar denotes operator closure. Section 5 contains the central limit theorem and Section 6 the free Lévy-Khinchin formula.

§1. Free independence of random variables and free products of von Neumann algebras

To fix terminology, we recall in this section some definitions and results from [Voi 1], in an adapted form.

By a (real-valued) *random variable* we shall mean a self-adjoint operator X on a Hilbert space $\mathcal{H}$ in which a particular unit vector ξ has been singled out. Via the functional calculus of spectral theory such an operator determines an embedding ι_X of the commutative C*-algebra $C(\overline{\mathbf{R}})$ of continuous functions on the one-point compactification $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ of $\mathbf{R}$ to the bounded operators on $\mathcal{H}$:

$$\iota_X(f) = f(X).$$

We shall consider the spectral measure μ of X, which is determined by

$$\langle \xi, \iota_X(f)\xi \rangle = \int_{-\infty}^{\infty} f(x)\mu(dx), \qquad (f \in C(\overline{\mathbf{R}})),$$

as the *probability distribution* of X, and we shall think of $\langle \xi, \iota_X(f)\xi \rangle$ as the *expectation value* of the (complex-valued) 'random variable' $f(X)$, which is a bounded normal operator on $\mathcal{H}$. (Note that we do not suppose X itself to be bounded.) The embedding ι_X naturally extends from $C(\overline{\mathbf{R}})$ to the commutative von Neumann algebra $\mathcal{A}_X = L^\infty(\mathbf{R}, \mu)$, on which μ determines a faithful normal trace $f \mapsto \int f d\mu$.

Two real-valued random variables X_1 and X_2 are said to be *independent* if for all $f_1, f_2 \in C(\overline{\mathbf{R}})$

$$\langle \xi, f_j(X_j)\xi \rangle = 0,\ (j = 1, 2) \quad \Longrightarrow \quad \langle \xi, f_1(X_1)f_2(X_2)\xi \rangle = 0.$$

Now let us consider the von Neumann algebra $\mathcal{A}$ generated by $\iota_1(\mathcal{A}_1) := \iota_{X_1}(\mathcal{A}_{X_1})$ and $\iota_2(\mathcal{A}_2) := \iota_{X_2}(\mathcal{A}_{X_2})$. We may assume that ξ is cyclic for $\mathcal{A}$; we discard an irrelevant subspace if necessary:

$$\mathcal{H} = \overline{\mathcal{A}\xi}.$$

If X_1 and X_2 commute, then $\mathcal{A}$ is isomorphic to $L^\infty(\mathbf{R}^2, \mu_1 \otimes \mu_2)$,i.e.:

$$\mathcal{A} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2.$$

In this sense commutative independence is related to tensor products of von Neumann algebras.

Voiculescu's notion of free independence [Voi 1] is much stronger than classical independence, and incompatible with commutativity. The random variables X_1 and X_2 on $(\mathcal{H}, \xi)$ are said to be *freely independent* if for all $n \in \mathbf{N}$ and all alternating sequences $i_1, i_2, \cdots, i_n$ of 1's and 2's, i.e.,

$$i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_n,$$

and for all $f_k \in C(\overline{\mathbf{R}})$, $(k = 1, \cdots, n)$, one has

$$\langle \xi, f_k(X_{i_k})\xi \rangle = 0,\ \ (k = 1, \cdots, n) \quad \Longrightarrow \quad \langle \xi, f_1(X_{i_1})f_2(X_{i_2}) \cdots f_n(X_{i_n})\xi \rangle = 0.$$

The main point of this section is that for freely independent X_1 and X_2 the algebra $\mathcal{A}$ is isomorphic to Ching's free product of $\mathcal{A}_1$ and $\mathcal{A}_2$ [Chi], both endowed with the cyclic trace vector $1 \in C(\overline{\mathbf{R}})$:

$$\mathcal{A} \simeq \mathcal{A}_1 \star \mathcal{A}_2.$$

We shall introduce this free product in an explicit standard form [Voi 1]. Let

$$\mathcal{K}_j = L^2(\mathbf{R}, \mu_j); \qquad \mathcal{K}_j^0 = 1^\perp = \{\, f \in \mathcal{K}_j \mid \int f d\mu_j = 0 \,\}.$$

$$\mathcal{H} = \mathbf{C} \oplus \big(\mathcal{K}_1^0 \oplus \mathcal{K}_2^0\big) \oplus \big((\mathcal{K}_1^0 \otimes \mathcal{K}_2^0) \oplus (\mathcal{K}_2^0 \otimes \mathcal{K}_1^0)\big) \oplus \big((\mathcal{K}_1^0 \otimes \mathcal{K}_2^0 \otimes \mathcal{K}_1^0) \oplus (\mathcal{K}_2^0 \otimes \mathcal{K}_1^0 \otimes \mathcal{K}_2^0)\big) \oplus \cdots.$$

$$\xi = 1 \oplus 0 \oplus 0 \oplus \cdots.$$

$$\iota_j(f)\xi = (\textstyle\int f d\mu_j)\xi \ \oplus\ (f - (\int f d\mu_j) \cdot 1);$$

$$\iota_j(f) f_{i_1} \otimes \cdots \otimes f_{i_n} =$$

$$\begin{cases} (\int f d\mu_j) \cdot f_{i_1} \otimes \cdots \otimes f_{i_n} \ \oplus\ (f - \int f d\mu_j) \otimes f_{i_1} \otimes \cdots \otimes f_{i_n}, & \text{if } j \neq i_1; \\ (\int f f_j d\mu_j) \cdot f_{i_2} \otimes \cdots \otimes f_{i_n} \ \oplus\ (f f_j - \int f f_j d\mu_j) \otimes f_{i_2} \otimes \cdots \otimes f_{i_n}, & \text{if } j = i_1. \end{cases}$$

Let us also define embeddings ι_1' and ι_2' in the same way as above, except that the operations act from the right instead of the left. Let $\mathcal{A}$ be the von Neumann algebra generated by $\iota_1(\mathcal{A}_1)$ and $\iota_2(\mathcal{A}_2)$, and let $\mathcal{A}'$ be generated by $\iota_1'(\mathcal{A}_1)$ and $\iota_2'(\mathcal{A}_2)$. Then $\mathcal{A}$ and $\mathcal{A}'$ are each other's commutant, and ξ is a cyclic trace vector for both of them. For the sake of clarity we shall now give a proof of Voiculescu's observation that every pair of freely independent random variables gives rise to the above structure [Voi 1].

Proposition 1.1. *Let freely independent random variables X_1 and X_2 on some Hilbert space $\tilde{\mathcal{H}}$ with cyclic vector $\tilde{\xi}$ have probability distributions μ_1 and μ_2 respectively. Then there exists a unitary map $U : \mathcal{H} \to \tilde{\mathcal{H}}$ with the properties*

$$U\iota_j(f) = f(X_j)U, \qquad (f \in C(\overline{\mathbf{R}})),$$

and

$$U\xi = \tilde{\xi}.$$

Proof. For $n \in \mathbf{N}$ and any alternating sequence $i_1, \cdots, i_n$ of 1's and 2's, and for $f_k \in C(\overline{\mathbf{R}})$, $(k = 1, \cdots, n)$, with $\int f_k d\mu_{i_k} = 0$, define $U_0\xi = \tilde{\xi}$ and

$$U_0\big(f_1 \otimes f_2 \otimes \cdots \otimes f_n\big) = f_1(X_{i_1})f_2(X_{i_2})\cdots f_n(X_{i_n})\tilde{\xi}.$$

Then U_0 extends in a unique way to a unitary map $U : \mathcal{H} \to \tilde{\mathcal{H}}$. Indeed, for alternating sequences $i_1, \cdots, i_n$ and $j_1, \cdots, j_m$ and functions $f_k \in C(\overline{\mathbf{R}})$, $g_l \in C(\overline{\mathbf{R}})$ with $\int f_k d\mu_{i_k} = 0$, $\int g_l d\mu_{j_l} = 0$, $(k = 1, \cdots, n;\ l = 1, \cdots, m)$, we have

$$\langle U_0 f_1 \otimes \cdots \otimes f_n, U_0 g_1 \otimes \cdots \otimes g_m\rangle = \langle f_1(X_{i_1})\cdots f_n(X_{i_n})\tilde{\xi}, g_1(X_{j_1})\cdots g_m(X_{j_m})\tilde{\xi}\rangle$$

$$= \langle \tilde{\xi}, f_n(X_{i_n})^* \cdots f_1(X_{i_1})^* g_1(X_{j_1})\cdots g_m(X_{j_m})\tilde{\xi}\rangle.$$

If $i_1 \neq j_1$, then the sequence $i_n, \cdots, i_1, j_1, \cdots, j_m$ is alternating, so that the inner product is zero. If $i_1 = j_1$, we write

$$\overline{f_1}g_1 = h_1 + \langle f_1, g_1\rangle \cdot 1,$$

so that $\int h_1 d\mu_{i_1} = 0$. Upon substitution, the term with $h_1(X_{i_1})$ yields zero, since the sequence $i_n, \cdots, i_1, j_2, \cdots, j_m$ is alternating. With the other term we proceed inductively, to find that the inner product is zero unless $n = m$ and $i_1 = j_1$, in which case

$$\langle U_0 f_1 \otimes \cdots, \otimes f_n, U_0 g_1 \otimes \cdots \otimes g_m\rangle = \langle f_1, g_1\rangle \cdots \langle f_n, g_n\rangle = \langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m\rangle.$$

The stated properties of U are now easy to verify.

□

§2. The reciprocal Cauchy transform

In the context of free independence it turns out to be natural to consider the expectation values of functions $f_z \in C(\overline{\mathbf{R}})$ of the form

$$f_z(x) = \frac{1}{z - x}, \qquad (\operatorname{Im} z > 0).$$

In particular they play a key role in the addition of freely independent random variables, in much the same way as do the expectation values of $x \mapsto e^{itx}$, $(t \in \mathbf{R})$, in the addition of independent commuting random variables.

Let $\mathbf{C}^+$ ($\mathbf{C}^-$) denote the open upper (lower) half of the complex plane. If μ is a finite positive measure on $\mathbf{R}$, then its Cauchy transform

$$G(z) = \int_{-\infty}^{\infty} \frac{\mu(dx)}{z - x}, \qquad (\operatorname{Im} z > 0),$$

is a holomorphic function $\mathbf{C}^+ \to \mathbf{C}^-$ with the property

$$\limsup_{y\to\infty} y|G(iy)| < \infty. \tag{2.1}$$

Conversely every holomorphic function $\mathbf{C}^+ \to \mathbf{C}^-$ with this property is the Cauchy transform of some finite positive measure on $\mathbf{R}$, and the lim sup in (2.1) equals $\mu(\mathbf{R})$. (Theorem 3 in Chapter VI of [AkG]). The inverse correspondence is given by Stieltjes' inversion formula:

$$\mu(B) = -\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_B \operatorname{Im} G(x+i\varepsilon)dx,$$

valid for all Borel sets $B \in \mathbf{R}$ for which $\mu(\partial B) = 0$ ([AkG]).
We shall be mainly interested in the *reciprocal Cauchy transform*

$$F(z) = \frac{1}{G(z)}.$$

If μ is the point measure δ_0 at 0, then F is the identity map $z \mapsto z$. If $\mu = \delta_a$, ($a \in \mathbf{R}$), then $F(z) = z - a$, whereas $F(z) = z + ib$ corresponds to the probability measure with density $x \mapsto b/\pi(x^2 + b^2)$, a measure with infinite variance.

Basic properties

The remainder of this section contains some facts about the Cauchy transform and its reciprocal which we shall need, and which are not easily available in the literature.
For $a \in \mathbf{R}$, let $\mathbf{C}_a^+$ and $\mathbf{C}_a^-$ denote the open half-planes

$$\mathbf{C}_a^+ = \left\{\, z \in \mathbf{C} \;\middle|\; \operatorname{Im} z > a \,\right\} \quad \text{and} \quad \mathbf{C}_a^- = \left\{\, z \in \mathbf{C} \;\middle|\; \operatorname{Im} z < a \,\right\}.$$

We shall denote their closures by $\overline{\mathbf{C}_a^+}$ and $\overline{\mathbf{C}_a^-}$.
By $\mathcal{P}$ we shall denote the class of all probability measures on $\mathbf{R}$; by $\mathcal{P}^2$ the class of probability measures with finite variance and by $\mathcal{P}_0^2$ those of finite variance and zero mean. The corresponding classes of reciprocal Cauchy transforms will be denoted by $\mathcal{F}$, $\mathcal{F}^2$ and $\mathcal{F}_0^2$ respectively.
Our first proposition characterises the class $\mathcal{F}$.

Proposition 2.1. *Let F be holomorphic $\mathbf{C}^+ \to \mathbf{C}^+$. A necessary and sufficient condition for F to be the reciprocal Cauchy transform of some probability measure μ on $\mathbf{R}$ is that*

$$\inf_{z\in\mathbf{C}^+} \frac{\operatorname{Im} F(z)}{\operatorname{Im} z} = 1. \tag{2.2}$$

In particular every $F \in \mathcal{F}$ increases the imaginary part:

$$\operatorname{Im} F(z) \geq \operatorname{Im} z.$$

As will be clear from the proof, equality is reached in some point $z \in \mathbf{C}^+$ if and only if μ is a point measure.

Proof of necessity. Let F be the reciprocal Cauchy transform of some probability measure μ on $\mathbf{R}$. Fix $z \in \mathbf{C}^+$, and consider $f_z : z \mapsto 1/(z-x)$ as a vector in the Hilbert space $L^2(\mathbf{R}, \mu)$. Then we may write $G(z) = \langle 1, f_z\rangle$, and the equality

$$\operatorname{Im}\frac{1}{z-x} = -\frac{\operatorname{Im} z}{|z-x|^2}, \quad (x \in \mathbf{R}),$$

implies that $\operatorname{Im} G(z) = -\operatorname{Im} z \cdot \|f_z\|^2$. Therefore

$$\frac{\operatorname{Im} F(z)}{\operatorname{Im} z} = \frac{1}{\operatorname{Im} z} \cdot \frac{-\operatorname{Im} G(z)}{|G(z)|^2} = \frac{\|f_z\|^2}{|\langle 1, f_z\rangle|^2}. \tag{2.3}$$

By the Cauchy-Schwartz inequality it follows that $\operatorname{Im} F(z)/\operatorname{Im} z$ only takes values ≥ 1. To show that values arbitrarily close to 1 are actually assumed, observe that for $y > 0$,

$$iy\langle 1, f_{iy}\rangle = iyG(iy) = \int_{-\infty}^{\infty} \frac{iy}{iy - x}\mu(dx),$$

and

$$y^2\|f_{iy}\|^2 = \int_{-\infty}^{\infty} \frac{y^2}{x^2 + y^2}\mu(dx).$$

If we take $y \to \infty$, both integrals tend to 1, and so does the quotient

$$\frac{y^2\|f_{iy}\|^2}{|iy\langle 1, f_{iy}\rangle|^2} = \frac{\|f_{iy}\|^2}{|\langle 1, f_{iy}\rangle|^2} = \frac{\operatorname{Im} F(iy)}{y}.$$

This proves (2.2).

□

From (2.3) one sees that $\operatorname{Im} F(z) = \operatorname{Im} z$ for some $z \in \mathbf{C}^+$if and only if f_z is constant almost everywhere with respect to μ, that is if μ is a point measure.

Proof of sufficiency. Let $F : \mathbf{C}^+ \to \mathbf{C}^+$ be holomorphic and suppose that (2.2) holds. Then $\operatorname{Im} F(z) \geq \operatorname{Im} z$, and also $|F(z)| \geq \operatorname{Im} z$. So $G := 1/F$ takes values in $\mathbf{C}^-$ and satisfies: $|G(z)| \leq 1/\operatorname{Im} z$. Hence

$$\limsup_{y\to\infty}\; y|G(iy)| \leq 1,$$

and it follows (cf. (2.1)) that G is the Cauchy transform of some positive measure μ on $\mathbf{R}$ with $\mu(\mathbf{R}) \leq 1$. Since G is nonzero, we have $\mu(\mathbf{R}) > 0$. But then $\mu/\mu(\mathbf{R})$ is a probability measure , and the 'necessity' proof above shows that

$$\inf_{z\in\mathbf{C}^+} \frac{\operatorname{Im} F(z)}{\operatorname{Im} z} = \frac{1}{\mu(\mathbf{R})}.$$

By assumption, this infimum is 1, and μ is a probability measure .

□

The next proposition characterises the class $\mathcal{F}_0^2$.

Proposition 2.2. *Let F be a holomorphic function $\mathbf{C}^+ \to \mathbf{C}^+$. Then the following statements are equivalent.*

(a) F is the reciprocal Cauchy transform of a probability measure on $\mathbf{R}$ with finite variance and zero mean:

$$\int_{-\infty}^{\infty} x^2\mu(dx) < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} x\mu(dx) = 0\;;$$

(b) There exists a finite positive measure ρ on $\mathbf{R}$ such that for all $z \in \mathbf{C}^+$:

$$F(z) = z + \int_{-\infty}^{\infty} \frac{\rho(dx)}{x - z}\;;$$

(c) *There exists a positive number C such that for all $z \in \mathbf{C}^+$:*

$$|F(z) - z| \leq \frac{C}{\operatorname{Im} z}.$$

Moreover, the variance σ^2 of μ in (a), the total weight $\rho(\mathbf{R})$ of ρ in (b) and the (smallest possible) constant C in (c) are all equal.

For the proof it is useful to introduce the function

$$C_F : (0, \infty) \to \mathbf{C} : y \mapsto y^2 \left(\frac{1}{F(iy)} - \frac{1}{iy} \right) = \frac{iy}{F(iy)} (F(iy) - iy).$$

In case F is the reciprocal Cauchy transform of some probability measure μ on $\mathbf{R}$, the limiting behaviour of $C_F(y)$ as $y \to \infty$ gives information on the integrals $\int x^2 \mu(dx)$ and $x\mu(dx)$. Indeed one has

$$C_F(y) = y^2 \int_{-\infty}^{\infty} \left(\frac{1}{iy - x} - \frac{1}{iy} \right) \mu(dx) = \int_{-\infty}^{\infty} \frac{-xy^2 + ix^2 y}{x^2 + y^2} \mu(dx).$$

In particular the function $y \mapsto y \operatorname{Im} C_F(y)$ is nondecreasing and

$$\sup_{y>0} y \operatorname{Im} C_F(y) = \lim_{y \to \infty} y \operatorname{Im} C_F(y) = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{y^2}{x^2 + y^2} x^2 \mu(dx) = \int_{-\infty}^{\infty} x^2 \mu(dx). \tag{2.4}$$

Here we allow the value ∞ on both sides. Furthermore, if $\int_{-\infty}^{\infty} x^2 \mu(dx) < \infty$, then by the dominated convergence theorem,

$$\int_{-\infty}^{\infty} x\mu(dx) = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{y^2}{x^2 + y^2} x\mu(dx) = - \lim_{y \to \infty} \operatorname{Re} C_F(y). \tag{2.5}$$

Proof of Proposition 2.2. (a)$\Longrightarrow$(b). If $F \in \mathcal{F}_0^2$, then by (2.4) and (2.5) both the real and the imaginary part of $C_F(y)$ tend to zero as $y \to \infty$. Since $|C_F(y)| = y \left| \frac{iy}{F(iy)} - 1 \right|$, it follows that

$$\lim_{y \to \infty} \frac{F(iy)}{iy} = 1.$$

Therefore

$$\sigma^2 = \lim_{y \to \infty} y |C_F(y)| = \lim_{y \to \infty} y \left| \frac{iy}{F(iy)} \right| \cdot |F(iy) - iy| = \lim_{y \to \infty} y |F(iy) - iy|. \tag{2.6}$$

This equation says that the function $z \mapsto F(z) - z$ satisfies (2.1), and is therefore the Cauchy transform of some finite positive measure ρ on $\mathbf{R}$ with $\rho(\mathbf{R}) = \sigma^2$. This proves (b).

(b)$\Longrightarrow$(c). If F is of the form (b), then

$$|F(z) - z| \leq \int_{-\infty}^{\infty} \frac{\rho(dx)}{|z - x|} \leq \frac{\rho(\mathbf{R})}{\operatorname{Im} z}.$$

(c)$\Longrightarrow$(a). Since F is holomorphic $\mathbf{C}^+ \to \mathbf{C}^+$, it can be written in Nevanlinna's integral form [AkG]:

$$F(z) = a + bz + \int_{-\infty}^{\infty} \frac{1 + xz}{x - z} \tau(dx), \tag{2.7}$$

where $a, b \in \mathbf{R}$ with $b \geq 0$ and τ is a finite positive measure. Putting $z = iy$, $(y > 0)$ we find that

$$y \operatorname{Im} \big(F(iy) - iy\big) = y^2 \left((b-1) + \int_{-\infty}^{\infty} \frac{x^2+1}{x^2+y^2} \tau(dx)\right).$$

As $y \to \infty$, the integral tends to zero. By the assumption (c), the whole expression must remain bounded, which can only be the case if $b = 1$. But then by (2.7) F must increase the imaginary part:

$$\operatorname{Im} F(z) \geq \operatorname{Im} z.$$

Moreover, (c) implies that $F(z)$ and z can be brought arbitrarily close together, so by Proposition 2.1 F is the reciprocal Cauchy transform of some probability measure μ on $\mathbf{R}$.
Again by (c) this measure μ must have the properties

$$\int_{-\infty}^{\infty} x^2 \mu(dx) \leq \limsup_{y \to \infty} \, y|C_F(y)| = \limsup_{y \to \infty} \, y|F(iy) - iy| \leq C,$$

and

$$\int_{-\infty}^{\infty} x\mu(dx) = - \lim_{y \to \infty} \operatorname{Re} C_F(y) = 0.$$

Finally it is clear from the above that

$$\sigma^2 \geq \rho(\mathbf{R}) \geq C \geq \sigma^2,$$

so that these three numbers must be equal.

□

We now prove two lemmas about invertibility of reciprocal Cauchy transforms of measures and certain related functions, to be called φ-functions (cf. Section 3). The lemmas act in opposite directions: from reciprocal Cauchy transforms of probability measures to φ-functions, and vice versa.

Lemma 2.3. *Let $C > 0$ and let $\varphi : \mathbf{C}^+ \to \mathbf{C}^-$ be analytic with*

$$|\varphi(z)| \leq \frac{C}{\operatorname{Im} z}.$$

Then the function $K : \mathbf{C}^+ \to \mathbf{C} : u \mapsto u + \varphi(u)$ takes every value in $\mathbf{C}^+$ precisely once. The inverse $K^{-1} : \mathbf{C}^+ \to \mathbf{C}^+$ thus defined is of class $\mathcal{F}_0^2$ with variance $\sigma^2 \leq C$.

Proof. Fix $z \in \mathbf{C}^+$and put $r = C/\operatorname{Im} z$. Let Γ_1 denote the semicircle in the half-plane $\mathbf{C}^+_{\operatorname{Im} z}$ with centre z and radius r. Let Γ_2 be some smooth curve connecting $z - r$ to $z + r$ inside the strip $\left\{ u \in \mathbf{C} \,\middle|\, 0 < \operatorname{Im} u < \operatorname{Im} z \right\}$. Then the closed curve Γ composed of Γ_1 and Γ_2 encircles the point z once. If we let u run through Γ, then on Γ_1 its image $K(u)$ stays close to u itself:

$$|K(u) - u| = |\varphi(u)| \leq \frac{C}{\operatorname{Im} u} < \frac{C}{\operatorname{Im} z} = r,$$

and on Γ_2 it lies below z. Hence $K(u)$ winds around z once, and it follows that inside the curve Γ there is a unique point u_0 with $K(u_0) = z$. Outside Γ there is no such point by the same inequality. This proves the first statement of the lemma.

Putting $F(z) = u_0$ defines an analytic function $F : \mathbf{C}^+ \to \mathbf{C}^+$ satisfying

$$|F(z) - z| = |u_0 - z| < r = \frac{C}{\operatorname{Im} z}.$$

So F is of class $\mathcal{F}_0^2$, ($\sigma^2 \leq C$) by Prop. 2.2.

$\square$

Lemma 2.4. *Let μ be a probability measure on $\mathbf{R}$ with mean 0, variance σ^2 and reciprocal Cauchy transform F. Then the restriction of F to $\mathbf{C}_\sigma^+$ takes every value in $\mathbf{C}_{2\sigma}^+$ precisely once. The inverse function $F^{-1} : \mathbf{C}_{2\sigma}^+ \to \mathbf{C}_\sigma^+$ thus defined satisfies*

$$|F^{-1}(u) - u| \big(=: |\varphi(u)|\big) \leq \frac{2\sigma^2}{\operatorname{Im} u}.$$

Proof. Fix $u \in \mathbf{C}_{2\sigma}^+$, and consider the circle Γ with centre u and radius r satisfying

$$r + \frac{\sigma^2}{r} < \operatorname{Im} u. \tag{2.7}$$

Let H be the half-plane $\mathbf{C}_{\sigma^2/r} = \left\{ z \,\middle|\, \operatorname{Im} z > \sigma^2/r \right\}$. By (2.7) Γ lies entirely inside H, and by Prop. 2.2 one has for all $z \in H$,

$$|F(z) - z| \leq \frac{\sigma^2}{\operatorname{Im} z} < r.$$

So precisely as in the foregoing proof, when z runs through Γ, its image $F(z)$ drags along, and winds once around u as well. The function F, being analytic, must assume the value u in precisely one point z_0 inside Γ. Outside Γ, but in H, the value u is not taken. We may draw two conclusions.
By putting $r = \sigma$, so that $H = \mathbf{C}_\sigma^+$, we obtain the first statement in the lemma.
By putting $r = 2\sigma^2/\operatorname{Im} u$, we obtain the second:

$$|F^{-1}(u) - u| = |z_0 - u| < r = \frac{2\sigma^2}{\operatorname{Im} u}.$$

(Note that, in the latter case, $r < \sigma < \frac{1}{2}\operatorname{Im} u$, so that indeed (2.7) holds.)

$\square$

Finally we prove a continuity theorem for the Cauchy transform.

Theorem 2.5. *Let μ and $\mu_1, \mu_2, \mu_3 \cdots$ be probability measures on $\mathbf{R}$ with Cauchy transforms G and $G_1, G_2, G_3, \cdots$. Then $\mu_n \to \mu$ weakly as $n \to \infty$ if and only if there exists $y > 0$ such that*

$$\forall_{x \in \mathbf{R}}: \qquad \lim_{n \to \infty} \operatorname{Im} G_n(x + iy) = \operatorname{Im} G(x + iy). \tag{2.8}$$

Proof. Adapting Theorem 7.6 in [Bil] to the Cauchy transform, we must prove the following two statements.

(i) For each positive value of y the function $x \mapsto \operatorname{Im} G(x+iy)$ determines the measure μ uniquely;

(ii) if for some $y > 0$ (2.8) holds, then the sequence (μ_n) is tight.

(i): For each $y > 0$ the function $g_y : x \mapsto -(1/\pi)\operatorname{Im} G(x + iy)$ is the convolution of μ with the function $h_y : x \mapsto (1/\pi)y/(x^2 + y^2)$. Since the Fourier transform $\hat{h}_y : t \mapsto \exp(-2y|t|)$ of h_y is nowhere zero, convolution with h_y is an injective mapping.

(ii): Fix $y > 0$, and let γ and γ_n be the probability measures on $\mathbf{R}$ with densities g_y and $g_{n,y} : x \mapsto -(1/\pi)\mathrm{Im}\, G_n(x+iy)$ respectively. We can express γ and γ_n in terms of the primitive function $H_y(x) = (1/\pi)\mathrm{arctg}\,(x/y)$ of h_y. By Fubini's theorem we have for $a \le b$:

$$\begin{aligned}\gamma[a,b] &= -\frac{1}{\pi}\int_a^b \mathrm{Im}\, G(u+iy)du = -\frac{1}{\pi}\int_a^b\int_{-\infty}^{\infty} \mathrm{Im}\,\frac{\mu(dx)}{u+iy-x}du \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_a^b \frac{y\,du}{(u-x)^2+y^2}\right)\mu(dx) = \int_{-\infty}^{\infty}\left(H_y(x-a)-H_y(x-b)\right)\mu(dx),\end{aligned}$$

and the same for γ_n and μ_n.

Now for $\varepsilon > 0$ let us choose L so large that

$$|x| > L \quad\Longrightarrow\quad |H_y(x) - \tfrac{1}{2}| < \tfrac{1}{2}\varepsilon.$$

By choosing K large we can make $\mu[-K,K]$ larger than $1-\varepsilon$. Then

$$\begin{aligned}\gamma[-K-L,K+L] &= \tfrac{1}{2}\int_{-\infty}^{\infty}\left(H_y(x-K-L)-H_y(x+K+L)\right)\mu(dx) \\ &\ge (1-\varepsilon)\mu[-K,K] > (1-\varepsilon)^2.\end{aligned}$$

By bounded convergence we can choose N large enough that $\gamma_n[-K-L,K+L] > (1-\varepsilon)^3$ for all $n \ge N$. Finally, putting $M = K+2L$,

$$\begin{aligned}1-\gamma_n[-K-L,K+L] &= \int_{-\infty}^{\infty}\big(1-\tfrac{1}{2}(H_y(x-K-L)-H_y(x+K+L))\big)\mu_n(dx) \\ &\ge \int_{\mathbf{R}\setminus[-M,M]}(1-\varepsilon)\mu_n(dx) = (1-\varepsilon)(1-\mu_n[-M,M]).\end{aligned}$$

Therefore we have (with M independent of N),

$$1-\mu_n[-M,M] \le \frac{1-(1-\varepsilon)^3}{1-\varepsilon},$$

which can be made arbitrarily small. Hence (μ_n) is tight and the continuity theorem is proved.

$\square$

§3. The free convolution product

In this section we shall establish a binary operation ' $\boxplus$ ' on the space $\mathcal{P}^2$ of all probability measures on $\mathbf{R}$ with finite variance and on the space $\mathcal{F}^2$ of their reciprocal Cauchy transforms . This operation will later turn out to play the role of a convolution product with respect to the addition of free random variables. It generalises Voiculescu's free convolution product from the measures of bounded support to those of finite variance.

Definition 3.1. Let $F_1, F_2 \in \mathcal{F}^2$ and $z \in \mathbf{C}^+$. A quadruple $(z,u;z_1,z_2)$ of points in $\mathbf{C}^+$ will be called a *paralellogram for the pair* (F_1,F_2) if

$$z+u = z_1+z_2, \quad \text{and} \quad F_1(z_1) = F_2(z_2) = u.$$

Note that $\mathrm{Im}\, u \ge \mathrm{Im}\, z_1, \mathrm{Im}\, z_2 \ge \mathrm{Im}\, z$, since F_1 and F_2 increase imaginary parts (Lemma 2.1). For this reason we shall say that the above paralellogram is *based on z*.

Theorem 3.2. *Let μ_1 and μ_2 be probability measures on the real line with finite variances σ_1^2 and σ_2^2 and reciprocal Cauchy transforms F_1 and F_2. Then for all $z \in \mathbf{C}^+$there exists a unique paralellogram for the pair (F_1, F_2) based on z. Moreover the map $z \mapsto u$ defines a third function $F \in \mathcal{F}^2$, reciprocal Cauchy transform of a probability measure μ on $\mathbf{R}$ with variance*

$$\sigma^2 = \sigma_1^2 + \sigma_2^2.$$

We shall call F the *free convolution product* of F_1 and F_2, and apply the same terminology to the corresponding probability measures . Notation:

$$F = F_1 \boxplus F_2; \qquad \mu = \mu_1 \boxplus \mu_2.$$

By Lemma 2.4 it makes sense to define for $\mathrm{Im}\, u > 2\sigma$:

$$\varphi(u) = F^{-1}(u) - u \quad \text{and} \quad \varphi_j(u) = F_j^{-1}(u) - u, \quad (j = 1, 2).$$

Corollary 3.3. *In the situation of Theorem 3.2 we have for all $u \in \mathbf{C}^+_{2\sigma}$:*

$$\varphi(u) = \varphi_1(u) + \varphi_2(u). \tag{3.1}$$

Proof. Take $u \in \mathbf{C}^+_{2\sigma}$. Then by Lemma 2.4 there are unique points $z \in \mathbf{C}^+_\sigma$, $z_1 \in \mathbf{C}^+_{\sigma_1}$ and $z_2 \in \mathbf{C}^+_{\sigma_2}$ such that $F(z) = F_1(z_1) = F_2(z_2) = u$. Now, some paralellogram based on z must exist. This can only be $(z, u; z_1, z_2)$, therefore

$$z + u = z_1 + z_2,$$

or equivalently

$$\varphi(u) = F^{-1}(u) - u = z - u = (z_1 - u) + (z_2 - u) = \varphi_1(u) + \varphi_2(u).$$

$\square$

One may say that for the free convolution product F plays the role of a characteristic function and φ that of its logarithm. In the case that μ has compact support, the expansion of $\varphi(u)$ in terms of u^{-1} is Voiculescu's R-series, [Voi 2].

If μ is the point measure in 0, then $F = \mathrm{id}$ and $\varphi = 0$. If μ is the point measure at $a \in \mathbf{R}$, then free convolution by μ (or F) is translation over a in the real direction, and $\varphi = a$.

Proof of the theorem. First suppose that $F_1, F_2 \in \mathcal{F}_0^2$ and for $j = 1, 2$ let $\tilde{F}_j$ be obtained from F_j by translation over $a_j \in \mathbf{R}$: $\tilde{F}_j(z) = F_j(z - a_j)$. Then $(z, u; z_1, z_2)$ is a paralellogram for (F_1, F_2) if and only if the quadruple

$$(z + a_1 + a_2, u; z_1 + a_1, z_2 + a_2)$$

is a paralellogram for the pair $(\tilde{F}_1, \tilde{F}_2)$. Since all functions in $\mathcal{F}^2$ can be obtained by translation from a function in $\mathcal{F}_0^2$, and since the variance $\int x^2 d\mu - (\int x d\mu)^2$ of a measure μ does not change under translation, it suffices therefore to prove the theorem for $F \in \mathcal{F}_0^2$.

So let $F_1, F_2 \in \mathcal{F}_0^2$. For $\varepsilon \geq 0$ let $\mathcal{R}_\varepsilon$ denote the Riemann surface

$$\mathcal{R}_\varepsilon = \left\{ (z_1, z_2) \in (\mathbf{C}_\varepsilon^+)^2 \;\middle|\; F_1(z_1) = F_2(z_2) \right\}.$$

Define

$$\vartheta \colon \mathcal{R}_0 \to \mathbf{C} \colon (z_1, z_2) \mapsto z_1 + z_2 - F_1(z_1).$$

Let ϑ_ε denote the restriction of ϑ to $\overline{\mathcal{R}_\varepsilon}$. For $\varepsilon > 0$, the map ϑ_ε is analytic on a neighbourhood of $\overline{\mathcal{R}_\varepsilon}$ and maps its boundary $\partial\mathcal{R}_\varepsilon$ entirely into $\overline{\mathbf{C}_\varepsilon^-}$; indeed

$$\begin{aligned} \operatorname{Im} z_1 = \varepsilon &\quad\Longrightarrow\quad \operatorname{Im}\vartheta(z_1,z_2) = \varepsilon + (\operatorname{Im} z_2 - \operatorname{Im} F_2(z_2)) \leq \varepsilon; \\ \operatorname{Im} z_2 = \varepsilon &\quad\Longrightarrow\quad \operatorname{Im}\vartheta(z_1,z_2) = \varepsilon + (\operatorname{Im} z_1 - \operatorname{Im} F_1(z_1)) \leq \varepsilon. \end{aligned}$$

Now let $n_\varepsilon(z)$ denote the number of times that ϑ_ε takes the value z (counted with multiplicity). It is a well-known property of holomorphic mappings between Riemann surfaces that such a number can only change its value on the image of the domain's boundary, i.e. on $\vartheta_\varepsilon(\partial\mathcal{R}_\varepsilon) \subset \overline{\mathbf{C}_\varepsilon^-}$. Hence n_ε is constant on $\mathbf{C}_\varepsilon^+$. This result is independent of ε, so ϑ takes every value in $\mathbf{C}^+$ a constant number of times. We shall show that this constant number is 1.

Choose z well above the real line, say $\operatorname{Im} z > M := 4\max(\sigma_1,\sigma_2)$. Then any solution $(z_1,z_2) \in \mathcal{R}_0$ of the equation

$$\vartheta(z_1,z_2) = z \tag{3.2}$$

must lie in $\mathbf{C}_M^+ \times \mathbf{C}_M^+$ since it must satisfy

$$\operatorname{Im} z_1 = \operatorname{Im}\big(\vartheta(z_1,z_2) - (z_2 - F_2(z_2))\big) = \operatorname{Im} z + \operatorname{Im} F_2(z_2) - \operatorname{Im} z_2 \geq \operatorname{Im} z,$$

and the same for $\operatorname{Im} z_2$. By Lemma 2.4 the solution (z_1,z_2) is therefore uniquely determined by the single complex number $u := F_1(z_1) = F_2(z_2)$. It suffices now to show that the equation (3.2), which can be written as

$$F_1^{-1}(u) + F_2^{-1}(u) = z + u \tag{3.3}$$

has a unique solution for $u \in \mathbf{C}_M^+$.

We write (3.3) as

$$u - z + (F_1^{-1}(u) - u) + (F_2^{-1}(u) - u) = 0. \tag{3.4}$$

Now let Γ denote the circle with centre z and radius $M/2$. By Lemma 2.4 we have for $\operatorname{Im} u > M/2$,

$$|F_j^{-1}(u) - u| \leq \frac{2\sigma_j^2}{\operatorname{Im} u} < \frac{4\sigma_j^2}{M} \leq \frac{1}{4}M. \qquad (j = 1,2). \tag{3.5}$$

Then, as u runs through Γ, the left hand side of (3.4) winds around 0 once; so it must take the value 0 once inside Γ. Outside Γ, but in $\mathbf{C}_{M/2}^+$ it cannot become 0 by (3.5). Hence (3.3) has a single solution u in $\mathbf{C}_M^+$.

We have now proved existence and uniqueness of the paralellogram $(z,u;z_1,z_2)$ for all $z \in \mathbf{C}^+$.

Putting $F(z) = u$ defines an analytic function $F : \mathbf{C}^+ \to \mathbf{C}^+$. We shall show that $F \in \mathcal{F}_0^2$. By Prop. 2.2

$$\begin{aligned} |F(z) - z| = |u - z| \leq |u - z_1| + |u - z_2| &= |F(z_1) - z_1| + |F(z_2) - z_2| \\ \leq \frac{\sigma_1^2}{\operatorname{Im} z_1} + \frac{\sigma_2^2}{\operatorname{Im} z_2} &\leq \frac{\sigma_1^2 + \sigma_2^2}{\operatorname{Im} z}. \end{aligned}$$

Hence $F \in \mathcal{F}_0^2$ and μ has variance $\sigma^2 \leq \sigma_1^2 + \sigma_2^2$.

To see that actually σ^2 is equal to $\sigma_1^2 + \sigma_2^2$, we consider the integral representations of F_1, F_2 and F in terms of finite positive measures ρ_1, ρ_2 and ρ which exist by Prop.

2.2. Multiplication by $-z$ of both sides of the paralellogram relation $u - z = (u - z_1) + (u - z_2)$ leads to

$$\int_{-\infty}^{\infty} \frac{z}{z-x}\rho(dx) = \int_{-\infty}^{\infty} \frac{z}{z_1 - x}\rho_1(dx) + \int_{-\infty}^{\infty} \frac{z}{z_2 - x}\rho_2(dx).$$

Now put $z = i\eta$ and let η go to ∞. Then by Prop. 2.2(c) the paralellogram shrinks to zero in size, so that in the above equation all three integrands tend to 1 pointwise in x. By bounded convergence it follows that

$$\rho(\mathbf{R}) = \rho_1(\mathbf{R}) + \rho_2(\mathbf{R}) \qquad \text{and hence} \qquad \sigma^2 = \sigma_1^2 + \sigma_2^2.$$

□

§4. The addition theorem

We now formulate the main theorem of this paper.

Theorem 4.1. *Let X_1 and X_2 be freely independent random variables on some Hilbert space $\mathcal{H}$ with distinguished vector ξ, cyclic for X_1 and X_2. Suppose that X_1 and X_2 have distributions μ_1 and μ_2 with variances σ_1^2 and σ_2^2. Then the closure of the operator*

$$X = X_1 + X_2$$

defined on $\mathrm{Dom}\,(X_1) \cap \mathrm{Dom}\,(X_2)$ *is self-adjoint and its probability distribution μ on $(\mathcal{H}, \xi)$ is given by*

$$\mu = \mu_1 \boxplus \mu_2.$$

In particular in the region $\left\{ z \in \mathbf{C} \,\middle|\, \mathrm{Im}\, z > 2\sqrt{\sigma_1^2 + \sigma_2^2} \right\}$ the φ-functions related to μ, μ_1 and μ_2 satisfy

$$\varphi = \varphi_1 + \varphi_2.$$

In view of theorem (1.1) we may assume that $(\mathcal{H}, \xi)$ is the standard representation space described in Section 1.

Since the variances of X_1 and X_2 are finite, for $f \in C(\overline{\mathbf{R}})$ the function xf is square integrable with respect to μ_1 and μ_2, so that we meet no difficulties in applying the defining expressions for ι_j in Section 1 to the function $x \mapsto x$. We thus have explicit expressions at our disposal for the operators X_1 and X_2 on the dense subspace $\mathcal{D}$ of $\mathcal{H}$ consisting of all finite linear combinations of alternating tensor products of functions in $C(\overline{\mathbf{R}})$.

By adding appropriate constants to X_1 and X_2 we may reduce to the situation that

$$\langle \xi, X_1 \xi \rangle = \langle \xi, X_2 \xi \rangle = 0.$$

We shall prove the theorem by showing that $\mathcal{D}$ is a core for X and that the closure $\overline{X}$ of X satisfies

$$\langle \xi, (z - \overline{X})^{-1} \xi \rangle^{-1} = (F_1 \boxplus F_2)(z). \tag{4.1}$$

We fix $z \in \mathbf{C}^+$, and we set out to find the vector $(z - \overline{X})^{-1}\xi$ explicitly.

Consider the paralellogram $(z, u; z_1, z_2)$ based on z for the pair (F_1, F_2) of reciprocal Cauchy transforms of μ_1 and μ_2. Define functions f_1 and f_2 in $C(\overline{\mathbf{R}})$ by

$$f_j(x) = \frac{u}{z_j - x} - 1, \qquad (j = 1, 2).$$

In what follows, we consider f_j as a vector in $\mathcal{K}_j := L^2(\mathbf{R}, \mu_j)$.

Lemma 4.2. *The functions f_1 and f_2 have the properties*

$$f_1, f_2 \perp 1, \quad \textit{and} \quad \|f_1\| \cdot \|f_2\| < 1.$$

Proof. We have

$$\langle 1, f_j \rangle = \int_{-\infty}^{\infty} f_j(x)\mu_j(dx) = u \cdot \int_{-\infty}^{\infty} \frac{\mu_j(dx)}{z_j - x} - 1 = \frac{u}{F_j(z_j)} - 1 = 0, \quad (j = 1, 2).$$

So $f_j \perp 1$. By relation (2.3) it follows that

$$\|f_j + 1\|^2 = |u|^2 \cdot \int_{-\infty}^{\infty} \frac{\mu_j(dx)}{|z_j - x|^2} = |u|^2 \cdot \left(\frac{1}{|F_j(z_j)|^2} \cdot \frac{\operatorname{Im} F_j(z_j)}{\operatorname{Im} z_j} \right) = \frac{\operatorname{Im} u}{\operatorname{Im} z_j}.$$

By Pythagoras' law we thus have

$$\|f_j\|^2 = \|f_j + 1\|^2 - 1 = \frac{\operatorname{Im} u - \operatorname{Im} z_j}{\operatorname{Im} z_j}.$$

Therefore $\|f_1\| \cdot \|f_2\| < 1$ holds if and only if

$$(\operatorname{Im} u - \operatorname{Im} z_1)(\operatorname{Im} u - \operatorname{Im} z_2) < \operatorname{Im} z_1 \cdot \operatorname{Im} z_2.$$

The latter relation can be written in the form

$$\operatorname{Im} u \cdot \operatorname{Im} (z_1 + z_2 - u) > 0,$$

in which it is ostensibly valid, since both u and $z_1 + z_2 - u = z$ lie in $\mathbf{C}^+$.

$\square$

Let $(a_n)_{n=1}^{\infty}$ be the alternating sequence of 1's and 2's starting with $a_1 = 1$. Let us denote by a bar the tranposition $1 \mapsto 2$, $2 \mapsto 1$ on the set $\{1, 2\}$: $\overline{1} = 2$ and $\overline{2} = 1$. We define, still for our fixed $z \in \mathbf{C}^+$and the functions f_1 and f_2 related to it:

$$\pi_1(0) = \pi_2(0) = \xi,$$

and for $n \geq 1$:

$$\pi_1(n) = f_1 \otimes f_2 \otimes f_1 \otimes \cdots \otimes f_{a_n},$$

$$\pi_2(n) = f_2 \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_{\overline{a_n}}.$$

Let P_1 be the orthogonal projection in $\mathcal{H}$ onto the subspace of those vectors which 'start in $\mathcal{K}_1^0$':

$$\mathbf{C} \oplus \mathcal{K}_1^0 \oplus \left(\mathcal{K}_1^0 \otimes \mathcal{K}_2^0\right) \oplus \left(\mathcal{K}_1^0 \otimes \mathcal{K}_2^0 \otimes \mathcal{K}_1^0\right) \oplus \left(\mathcal{K}_1^0 \otimes \mathcal{K}_2^0 \otimes \mathcal{K}_1^0 \otimes \mathcal{K}_2^0\right) \oplus \cdots\cdots,$$

and let P_2 be the same, with 1's and 2's interchanged.

Let us furthermore define vectors $\psi_N(z) \in \mathcal{D}$ by

$$\psi_N(z) = \frac{1}{u} \left(\xi \oplus \bigoplus_{n=1}^{N} (\pi_1(n) \oplus \pi_2(n)) \right).$$

By Lemma 4.2 we have $\|f_1 \otimes f_2\| < 1$, so that $\psi_N(z)$ tends to a finite limit $\psi(z)$ as N tends to infinity.

Lemma 4.3. *For $j = 1, 2$ we have*

$$\lim_{N\to\infty} (z_j - X_j - uP_{\overline{j}})\psi_N(z) = 0.$$

Proof. Since for $x \in \mathbf{R}$,

$$(z_j - x)f_j(x) = (z_j - x)\left(\frac{u}{z_j - x} - 1\right) = u - z_j + x,$$

we have for $n = 0, 1, 2, \cdots$ (cf. Section 1):

$$\begin{aligned}(z_j - X_j)\pi_j(n+1) &= (z_j - X_j)f_j \otimes \pi_{\overline{j}}(n) \\ &= (u - z_j)\pi_{\overline{j}}(n) + (x)_j \otimes \pi_{\overline{j}}(n)\end{aligned}.$$

On the other hand,

$$(z_j - X_j)\pi_{\overline{j}}(n) = z_j\pi_{\overline{j}}(n) + (-x)_j \otimes \pi_{\overline{j}}(n),$$

so that

$$(z_j - X_j)(\pi_j(n+1) \oplus \pi_{\overline{j}}(n)) = u\pi_{\overline{j}}(n) = uP_{\overline{j}}(\pi_j(n+1) \oplus \pi_{\overline{j}}(n)).$$

Summation over n from 0 to N yields

$$(z_j - X_j - uP_{\overline{j}})\big(u\psi_N(z) + \pi_j(N+1)\big) = 0.$$

Therefore, again by the same calculation,

$$(z_j - X_j - uP_{\overline{j}})\psi_N(z) = -\frac{z_j - X_j}{u}\pi_j(N+1) = -\frac{u - z_j}{u}\pi_{\overline{j}}(N) - (x)_j \otimes \pi_{\overline{j}}(N).$$

Since $\lim_{N\to\infty} \pi_{\overline{j}}(N) = 0$, the result follows.

□

Proposition 4.4. *The operator X is essentially self-adjoint on $\mathcal{D}$. The vector $\psi(z)$ lies in the domain of its closure $\overline{X}$ and satisfies*

$$(z - \overline{X})\psi(z) = \xi. \tag{4.2}$$

Proof. First we note that X, being a symmetric operator defined on a dense domain, is closable. Further, since $(P_1 + P_2) = \mathbf{1} + |\xi\rangle\langle\xi|$, Lemma 4.3 yields

$$\begin{aligned}(z - X)\psi_N(z) &= ((z_1 + z_2 - u) - (X_1 + X_2))\psi_N(z) \\ &= \big((z_1 - X_1 - uP_1) + (z_2 - X_2 - uP_2)\big)\psi_N(z) + \xi \longrightarrow \xi, \quad (N \to \infty).\end{aligned}$$

So $\psi(z)$ lies in the domain of $\overline{X}$, and (4.2) holds.

To prove that $\overline{X}$ is self-adjoint, is suffices to show that the range of $z - \overline{X}$ is dense for all $z \in \mathbf{C} \setminus \mathbf{R}$. As X commutes with complex conjugation, we may restrict our attention to $z \in \mathbf{C}^+$. Now, any vector in $\mathcal{H}$ can be approximated by elements of the form $A'\xi$, where A' is a finite sum of products of operators $\iota'_j(g)$ with $j = 1, 2$ and $g \in C(\overline{\mathbf{R}})$ (cf. Section 1). Such a vector $A'\xi$ lies in $\mathcal{D}$ and satisfies

$$A'\xi = A'\big((z - \overline{X})\psi(z)\big) = (z - \overline{X})A'\psi(z).$$

Hence $A'\xi$ lies in the range of $z - \overline{X}$. We conclude that the latter is dense in $\mathcal{H}$, and that $\overline{X}$ is self-adjoint.

□

We now finish the proof of theorem 4.1. Let F be the reciprocal Cauchy transform of the probability distribution of $\overline{X}$. Then for $z \in \mathbf{C}^+$

$$\frac{1}{F(z)} = \langle \xi, (z - \overline{X})^{-1} \xi \rangle = \langle \xi, \psi(z) \rangle = \frac{1}{u}.$$

By its definition, the number u is $F_1 \boxplus F_2(z)$. Hence

$$F(z) = F_1 \boxplus F_2(z),$$

and the same holds (by the definition of $\boxplus$) for the associated measures. The statement about the φ-functions follows by Corollary 3.3.

$\square$

§5. A free central limit theorem

Sums of large numbers of freely independent random variables of finite variance tend to take a semiellipse distribution. The semiellipse distribution was first encountered by Wigner [Wig] when he was studying spectra of large random matrices. The *Wigner distribution* ω_σ with standard deviation σ is defined by

$$\omega_\sigma(dx) = w_\sigma(x) dx,$$

where

$$w_\sigma(x) = \begin{cases} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2} & \text{if } |x| \le 2\sigma; \\ 0 & \text{if } |x| > 2\sigma. \end{cases}$$

Lemma 5.1. *The Wigner distribution ω_σ has the following φ-function:*

$$\varphi(u) = \frac{\sigma^2}{u}. \tag{5.1}$$

Proof. The inverse of the function $K_\sigma : \mathbf{C}^+ \to \mathbf{C}^+ : u \mapsto u + \dfrac{\sigma^2}{u}$ is easily seen to be

$$F_\sigma = K_\sigma^{-1} : \mathbf{C}^+ \to \mathbf{C}^+ : z \mapsto \frac{2\sigma^2}{z - \sqrt{z^2 - 4\sigma^2}},$$

where we take the square root to be $\mathbf{C}^+$-valued. But this is the reciprocal Cauchy transform of ω_σ by Stieltjes' inversion formula. Indeed $F_\sigma \in \mathcal{F}$ and

$$\lim_{\varepsilon \downarrow 0} -\frac{1}{\pi} \mathrm{Im} \frac{1}{F_\sigma(x + i\varepsilon)} = w_\sigma(x).$$

$\square$

We now formulate the free central limit theorem. For a probability measure μ on $\mathbf{R}$ we denote by $D_\lambda \mu$ its dilation by a factor λ :

$$D_\lambda \mu(A) = \mu(\lambda^{-1} A), \quad (A \subset \mathbf{R} \text{ measurable}).$$

Theorem 5.2. *Let μ be a probability measure on $\mathbf{R}$ with mean 0 and variance σ^2, and for $n \in \mathbf{N}$ let*

$$\mu_n = D_{1/\sqrt{n}}\, \mu \boxplus \cdots \boxplus D_{1/\sqrt{n}}\, \mu, \qquad (n \text{ times}).$$

Then

$$\textit{weak-}\lim_{n\to\infty} \mu_n = \omega_\sigma.$$

Remarks. 1. In this paper we do not consider more than two freely independent random variables at a time. But actually this theorem can be read to refer to a sequence $X_1, X_2, X_3, \cdots$ of freely independent random variables, all having distribution μ. Then μ_n is the distribution of

$$S_n := \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n).$$

2. Although the proof following below is a bit technical, the reason why the above central limit theorem holds is simple. First note that every φ-function goes like σ^2/u high above the real line. Indeed we have $z := F^{-1}(u) \sim u$ and by Prop. 2.2(b),

$$\varphi(u) = F^{-1}(u) - u = z - F(z) \sim \frac{\sigma^2}{z} \sim \frac{\sigma^2}{u}.$$

Now, due to the scaling law $\varphi_{D_\lambda \mu}(u) = \lambda \varphi(\lambda^{-1}u)$, φ_{μ_n} picks out precisely this asymptotic part:

$$\varphi_{\mu_n}(u) = n\varphi_{D_{1/\sqrt{n}}}(u) = \sqrt{n}\varphi(\sqrt{n}u) \longrightarrow \frac{\sigma^2}{u}, \quad (n \to \infty).$$

Proof of Theorem 5.2.. Let F, $\tilde{F}_n$ and F_n denote the reciprocal Cauchy transforms of μ, $D_{1/\sqrt{n}}\,\mu$ and μ_n respectively. Denote the associated φ-functions by φ, $\tilde{\varphi}_n$ and φ_n. Let, as in the proof of Lemma 5.1, F_σ denote the reciprocal Cauchy transform of ω_σ. By the continuity theorem 2.5 it suffices to show that for some $M > 0$ and all $z \in \mathbf{C}_M^+$:

$$\lim_{n\to\infty} F_n(z) = F_\sigma(z).$$

Since for M large enough the inverse $K_\sigma : u \mapsto u + \sigma^2/u$ has a derivative close to 1 on $\mathbf{C}_M^+$, the above is equivalent with

$$\lim_{n\to\infty} K_\sigma \circ F_n(z) = z. \tag{5.2}$$

Now, fix $z \in \mathbf{C}_M^+$ and (for $n > 4\sigma^2/M^2$) put $u_n = F_n(z)$ and $z_n = \tilde{F}_n^{-1}(u_n)$. Then $z - u_n = \varphi_n(u_n)$ and $z_n - u_n = \tilde{\varphi}_n(u_n)$. Hence by an n-fold application of the addition theorem 4.1,

$$z - u_n = n(z_n - u_n).$$

Note that also

$$|z - u_n| \le \frac{\sigma^2}{M}, \quad \text{and} \quad \operatorname{Im} u_n > M.$$

Therefore, by the scaling property $F_{D_\lambda \mu}(z) = \lambda F(\lambda^{-1}z)$ and the integral representation Prop. 2.2(b) of F,

$$\begin{aligned} z - u_n &= n(z_n - \tilde{F}_n(z_n)) = n(z_n - \frac{1}{\sqrt{n}}F(\sqrt{n}z_n)) \\ &= \sqrt{n}(\sqrt{n}z_n - F(\sqrt{n}z_n)) = \sqrt{n} \cdot \int_{-\infty}^{\infty} \frac{\rho(dx)}{\sqrt{n}z_n - x} = \\ &= \int_{-\infty}^{\infty} \frac{\rho(dx)}{z_n - x/\sqrt{n}}. \end{aligned}$$

Hence

$$|z - K_\sigma \circ F_n(z)| = |z - K_\sigma(u_n)| = |z - u_n - \frac{\sigma^2}{u_n}|$$
$$\leq \int_{-\infty}^{\infty} \left| \frac{1}{z_n - x/\sqrt{n}} - \frac{1}{u_n} \right| \rho(dx) = \frac{1}{|u_n|} \int_{-\infty}^{\infty} \frac{|u_n - z_n + x/\sqrt{n}|}{|z_n - x/\sqrt{n}|} \rho(dx).$$

The integrand on the right hand side is uniformly bounded and tends to zero pointwise as n tends to infinity. So (5.2) follows by the bounded convergence theorem, and the theorem is proved.

□

The following result puts the Wigner distribution in perspective. It has been known for some time already [Voi 1], but we are now in a position to give an easy proof.

Consider the left shift S on the Hilbert space $l^2(\mathbf{N})$ with distinguished unit vector

$$\delta\colon n \mapsto \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

and consider the bounded self-adjoint operator $W_\sigma := \sigma(S + S^*)$.

Lemma 5.3. *For $\sigma > 0$ the probability distribution of W_σ on $(l^2(\mathbf{N}), \delta)$ is the Wigner distribution ω_σ.*

Proof. We shall prove this for $W := S + S^*$, which is clearly sufficient. Fix $z \in \mathbf{C}^+$and put $u = K_1^{-1}(z)$. Then u is the solution of the equation $u + u^{-1} = z$ with $\operatorname{Im} u > 0$, hence $|u| > 1$. Consider the vector $\psi \in l^2$ given by

$$\psi_n = u^{-n}.$$

One has

$$\big((z - W)\psi\big)_1 = \big((z - (S + S^*))\psi\big)_1 = z\psi_1 - \psi_2$$

$$= zu^{-1} - u^{-2} = u^{-1}(z - u^{-1}) = u^{-1}u = 1,$$

and for $n > 1$,

$$\big((z - W)\psi\big)_n = z\psi_n - (\psi_{n+1} + \psi_{n-1}) = zu^{-n}(z - u - u^{-1}) = 0.$$

So $(z - W)\psi = \delta$. Therefore

$$F(z) := \langle \delta, (z - W)^{-1}\delta \rangle^{-1} = \langle \delta, \psi \rangle^{-1} = \frac{1}{\psi_1} = u = K_1^{-1}(z).$$

So $F^{-1}(u) - u = K_1(u) - u = \dfrac{1}{u}$. By Lemma 5.1, F is the reciprocal Cauchy transform of ω_1.

□

The sum $S+S^*$ is a free analogue of the position operator $a+a^*$ of the harmonic oscillator, a Gaussian random variable on (l^2,δ).
An infinite free product of copies of (l^2,δ) with shifts on them forms a natural framework for a free counterpart to the theory of Brownian motion [Voi 1]. This point of view has been elaborated by Kümmerer and Speicher [KSp], [Spe 1,2].

§6. Infinite divisibility

A probability measure μ on $\mathbf{R}$ will be called *(freely) infinitely divisible* if for all $n \in \mathbf{N}$ there exists $\mu_{1/n}$ such that

$$\mu = \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}, \qquad (n \text{ times}). \tag{6.1}$$

The following theorem gives a complete characterisation. (Cf. also [Voi 2]).

Theorem 6.1. *Every holomorphic function $\varphi\colon \mathbf{C}^+ \to \mathbf{C}^-$ with the property that for some $a \in \mathbf{R}$ and $C > 0$,*

$$|\varphi(z) - a| \le \frac{C}{\operatorname{Im} z}, \qquad (z \in \mathbf{C}^+), \tag{6.2}$$

is the φ-function of an infinitely divisible distribution with variance not greater than C. Conversely, if a probability measure μ of variance σ^2 is infinitely divisible, then the associated φ-function $\mathbf{C}^+_{2\sigma} \to \mathbf{C}^-$ extends to a holomorphic function $\mathbf{C}^+ \to \mathbf{C}^-$ satisfying (6.2) with $C = \sigma^2$.

Proof. If $\varphi\colon \mathbf{C}^+ \to \mathbf{C}^-$ satisfies (6.2), then the same holds for $\varphi_{1/n} := (1/n)\varphi$ (with a constant C/n). Lemma 2.3 associates functions F and $F_{1/n}$ to them, reciprocal Cauchy transforms of probability measures μ and $\mu_{1/n}$ with variances $\sigma^2 \le C$ and $\sigma^2/n \le C/n$ respectively, which satisfy (6.1) because of the additivity of φ-functions.

Conversely, let μ with variance σ^2 be infinitely divisible. Let its n-th free convolution root $\mu_{1/n}$ have the φ-function $\varphi_{1/n}$. Since these φ-functions may be added, we have for $\operatorname{Im} u > 2\sigma$:

$$\varphi(u) = n\varphi_{1/n}(u).$$

However, since $\mu_{1/n}$ has variance σ^2/n, $\varphi_{1/n}$ is analytic on $\mathbf{C}^+_{2\sigma/\sqrt{n}}$. Therefore the above equation defines an analytic extension of φ to

$$\bigcup_{n=1}^{\infty} \mathbf{C}^+_{2\sigma/\sqrt{n}} = \mathbf{C}^+.$$

The inequality (6.2) with $C = \sigma^2$ follows from Lemma 2.2.

$\square$

Via Nevalinna's integral representation the simple characterisation given above, leads to an explicit formula, a free analogue of the classical Lévy-Khinchin formula.

Theorem 6.2. *Let μ be an infinitely divisible probability measure in the free sense, with variance σ^2. Then the associated φ-function $\mathbf{C}^+ \to \mathbf{C}^-$ is of the form*

$$\varphi(u) = a + \int_{-\infty}^{\infty} \frac{\nu(dx)}{u - x}, \tag{6.3}$$

where ν is a positive ('Lévy'-) measure on $\mathbf{R}$ with total weight $\nu(\mathbf{R}) = \sigma^2$.

Moreover, there exists a weakly continuous free convolution semigroup μ_t, $(t \geq 0)$ with $\mu_0 = \delta$ and $\mu_1 = \mu$. One has

$$\nu(dx) = \text{weak-}\lim_{t \downarrow 0} \frac{1}{t} x^2 \mu_t(dx). \tag{6.4}$$

Proof. The first statement (except the specification of $\nu(\mathbf{R})$) follows from Theorem 6.1 and the integral representation formula for analytic functions $\mathbf{C}^+ \to \mathbf{C}^-$ (cf. Section 2, in particular around relation (2.1)).

The free convolution semigroup (μ_t) is obtained from the additive semigroup $(t\varphi)_{t\geq 0}$ by the procedure in Lemma 2.3. Weak continuity of $t \mapsto \mu_t$ follows from the pointwise continuity of $t \mapsto F_t$ (Lemma 2.5). Since the total weight $\nu(\mathbf{R}) = \sigma^2$ of ν can be found from (6.4), it remains to prove the latter.

By Lemma 2.5, (6.4) is equivalent with

$$\mathrm{Im} \int_{-\infty}^{\infty} \frac{\nu(dx)}{z - x} = \lim_{t \downarrow 0} \mathrm{Im}\, \frac{1}{t} \int_{-\infty}^{\infty} \frac{x^2 \mu_t(dx)}{z - x}. \tag{6.5}$$

Now consider the defining identity for F_t:

$$F_t(z) + t\varphi(F_t(z)) = z. \tag{6.6}$$

From the continuity of $t \mapsto F_t(z)$ it follows that $t \mapsto F_t(z) = z - t\varphi(F_t(z))$ is differentiable at $t = 0$. Then, differentiating (6.6) we find that

$$\varphi(z) = -\frac{\partial}{\partial t} F_t(z)\Big|_{t=0}.$$

Denoting the Cauchy transform of μ_t by G_t (so that $F_t = 1/G_t$), we proceed as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\nu(dx)}{z - x} &= \varphi(z) = -\frac{\partial}{\partial t}\left(\frac{1}{G_t(z)}\right)\Big|_{t=0} = \frac{1}{G_0(z)^2} \frac{\partial}{\partial t} G_t(z)\Big|_{t=0} \\ &= z^2 \lim_{t \downarrow 0} \frac{1}{t}\big(G_t(z) - G_0(z)\big). \end{aligned} \tag{6.7}$$

Now observe that

$$\mathrm{Im}\, z^2\big(G_t(z) - G_0(z)\big) = \mathrm{Im} \int_{-\infty}^{\infty} \frac{z^2}{z - x}\big(\mu_t(dx) - \mu_0(dx)\big)$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} \frac{x^2}{z - x} \mu_t(dx) - \mathrm{Im} \int_{-\infty}^{\infty} \frac{x^2}{z - x} \mu_0(dx) + \mathrm{Im} \int_{-\infty}^{\infty} \frac{z^2 - x^2}{z - x}\big(\mu_t(dx) - \mu_0(dx)\big).$$

In the right hand side the second term is zero since $\mu_0 = \delta$. The third vanishes as well, since $(z^2 - x^2)/(z - x) = z + x$, $\mathrm{Im}\, x = 0$ and $\mu_t(\mathbf{R}) = \mu_0(\mathbf{R}) = 1$. Substitution of the first term into (6.7) yields (6.5).

$\square$

Interpretation of the Lévy-measure

The additive semigroup $(t\varphi)_{t\geq 0}$ determines a free convolution semigroup $(\mu_t)_{t\geq 0}$, which may then be used to construct a *free stochastic process.* Examples of such processes have now been given by several workers [Voi 1], [KSp], [Spe 2]. We shall now describe the role of a and ν in (6.3) in terms of this process.

The real constant a describes the *drift* of the process. Indeed,

$$\int_{-\infty}^{\infty} x\mu_t(dx) = ta,$$

since addition of a constant to φ acts as a shift on F and on μ, whereas without the constant μ has mean zero.

The weight $\nu(\{0\})$ put in the origin describes the Wigner component of the process. Indeed, $a = 0$ and $\nu = \sigma^2\delta_0$ leads to $\varphi(u) = \sigma^2/u$, and hence to the Wigner distribution ω_σ. In view of the central limit theorem in Section 5, we may consider $(\omega_{\sqrt{t}})_{t\geq 0}$ as the distribution of *free Brownian motion* [KSp], [GSS].

In analogy with the classical Lévy-Khinchin formula, we now interpret the contribution of ν on $\mathbf{R}\backslash\{0\}$ as a *free Poisson process*: if $\nu = \delta_b$, $(b \neq 0)$, then μ_t describes a (compensated) Poisson process of *jump size* b and *intensity* $1/b^2$.

We finish this section with a calculation of the distribution semigroup of the (uncompensated) free Poisson process.

Put $\nu = \delta_b$ and $a = b^{-1}$, and substitute this into (6.3):

$$\varphi(u) = \frac{1}{b} + \frac{1}{u-b} = \frac{u}{b(u-b)}.$$

The reciprocal Cauchy transform F_t of μ_t is obtained by inverting the function

$$u \mapsto u + t\varphi(u) = \frac{bu^2 - b^2u + tu}{b(u-b)};$$

i.e. by solving the quadratic equation

$$zu - bz = bu^2 - b^2u + tu$$

for u under the condition $\operatorname{Im} u > 0$. If we solve for u^{-1} we obtain $G_t(z) = 1/F_t(z)$:

$$G_t(z) = \frac{1}{2z}\left(1 - \frac{t}{b^2} + zb - \frac{1}{b}\sqrt{\left(z - \frac{(b-\sqrt{t})^2}{b}\right)\left(z - \frac{(b-\sqrt{t})^2}{b}\right)}\right).$$

Applying Stieltjes' inversion formula we obtain

$$\mu_t = f(t)\delta_0 + \mu_t^{\text{abs.cont.}},$$

where

$$f(t) = \begin{cases} 1 - \frac{t}{b^2} & \text{if } t < b^2; \\ 0 & \text{if } t \geq b^2; \end{cases}$$

and

$$\mu_t^{\text{abs.cont.}}(dx) = h_t(x)dx,$$

with

$$h_t(x) = \begin{cases} \frac{1}{2\pi bx}\sqrt{-\left(x - \frac{(b-\sqrt{t})^2}{b}\right)\left(x - \frac{(b-\sqrt{t})^2}{b}\right)} & \text{if } (b-\sqrt{t})^2 \leq bx \leq (b+\sqrt{t})^2; \\ 0 & \text{otherwise.} \end{cases}$$

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