Quantum State Preparation via Asymptotic Completeness

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We demonstrate that any quantum state $|\chi\rangle$ of a single mode radiation field can be prepared with arbitrarily high fidelity by interaction with a sequence of two-level atoms, prepared in a suitable initial state. No final state measurement of the atoms is needed.

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In order to encode or process quantum information, the ability to prepare quantum systems in various states is a necessary prerequisite. A recipe to create an arbitrary state of the quantized electromagnetic field has been given in [1], where the authors studied the interaction of a sequence of two-level atoms with a single mode field sustained by a high quality resonator. In this scheme, field and atoms are *entangled* with each other, after the interaction. Subsequently, in order to project the field onto the desired pure state, a measurement has to be performed on the atoms. This procedure succeeds only with the *finite* probability of the desired measurement result (e.g., detection of all atoms in the lower state), and requires the ability to detect the final atomic state with very high efficiency. Therefore, schemes that avoid the atom-field entanglement have been proposed: The method described in [2] is based on adiabatic transfer of atomic ground state Zeeman coherence (and is limited by the number of available Zeeman levels), and in [3] an appropriate, time-dependent cavity QED interaction is designed to create the desired field state. Both methods are essentially guided by general principles of coherent control: The atom-field system is transferred from a well-defined initial state to a well-defined target state through the controlled modulation and application of atom-cavity coupling strengths and classical driving fields.

In our present contribution, we shall demonstrate that an arbitrary target state of the quantized single mode radiation field can actually be prepared under a simple, *time-independent* atom-field interaction, *without* any final state projection, and *independently* of the initial state of the field. A necessary condition is the experimentalist's ability to entangle a small number N of atoms prior to the atom-field interaction [4,5]. The larger N, the better the fidelity of the target state preparation, which approaches 100% exponentially fast in N.

Our method is an application of a recent mathematical result [6] on quantum Markov chains, which are described by a quantum system in subsequent interaction with a sequence of infinitely many identical other quantum systems. As an example of such a quantum Markov chain, we will examine the photon field in a single mode resonator, i.e., a quantum harmonic oscillator, that resonantly interacts (via the Jaynes-Cummings Hamiltonian) with a sequence of N two-level atoms. Furthermore, we assume full control over the initial N-atom state $|\psi_0\rangle$, which may also include entanglement between different atoms (i.e., $|\psi_0\rangle$ need not be a product of single-atom states). Mathematics predicts [6], in the limit $N \to \infty$ of infinitely many atoms, that any field state $|\chi\rangle$ can be created by choosing an appropriate initial atomic state $|\psi_0\rangle$, irrespectively of the initial state of the field. We will show that, given a finite number N of atoms, the state preparation is still possible with very high accuracy, and will also devise a strategy to find the optimal initial atomic state in order to prepare a given field state $|\chi\rangle$ with maximum fidelity.

The above physical example of a quantum Markov chain is experimentally realized in the micromaser [7]. Here the atoms cross the cavity one after the other, so that at most one atom is present in the cavity at any time. For simplicity, we assume the interaction time t_{int} (i.e., the time of flight through the cavity) to be the same for each single atom, a situation realized to high accuracy in state-of-the-art experiments [8]. Furthermore, we neglect dissipation due to incoherent coupling of the cavity field to the environment, for the sake of transparency of our subsequent arguments. With the high quality cavities presently at use in the laboratory [8], this is a good approximation, since average photon lifetimes of about 4 orders of magnitude larger than t_{int} can be achieved.

Under these premises, the total atoms-field interaction is described by the following unitary operator:

$$U_N = e^{-i\phi(a^\dagger\sigma_N + a\sigma_N^\dagger)} \cdots e^{-i\phi(a^\dagger\sigma_1 + a\sigma_1^\dagger)}. \qquad (1)$$

Here a and $a^\dagger$ are the photon annihilation and creation operators, and $\sigma_i = |d\rangle_{ii}\langle u|$ and $\sigma_i^\dagger = |u\rangle_{ii}\langle d|$ are the ladder operators for the ith two-level atom, with upper and lower level $|u\rangle_i$ and $|d\rangle_i$, respectively. $\phi = gt_{\text{int}}$ is the vacuum Rabi angle, with g the strength of the atom-field coupling. If ϕ fulfills a $|n_t\rangle$-trapping state condition [8], i.e., $\phi = k\pi/\sqrt{n_t + 1}$, with $k \in \mathbf{Z}$, it follows from Eq. (1) that $\langle n, \psi|U_N|n_0, \psi_0\rangle = 0$ for all initial and final N-atom

states $|\psi_0\rangle$ and $|\psi\rangle$, and all photon numbers $n > n_t$ and $n_0 \leq n_t$, that is, there is no way of increasing the photon number population above n_t. Hence, if we want to prepare field states including photon numbers higher than n, we must avoid $|n_t\rangle$-trapping states with $n_t \leq n$ through a proper choice of the vacuum Rabi angle ϕ.

In the limit of infinitely many atoms, $N \rightarrow \infty$, the atoms-field interaction described by Eq. (1) has (if ϕ does not fulfill a trapping state condition) the property of *asymptotic completeness* [6]: Every observable A of the photon field develops (in the Heisenberg picture) into an observable M_A of the atoms:

$$\lim_{N\rightarrow\infty} U_N^\dagger (A \otimes \mathbf{1}) U_N = \mathbf{1} \otimes M_A \,. \tag{2}$$

In other words, the field will lose the memory about its initial state after the interaction with infinitely many atoms: The final field state ρ, i.e., the expectation value $\langle A\rangle = \mathrm{tr}\{A\rho\}$ of any photon field observable A after the atoms-field interaction, is completely determined by the initial atomic state, irrespectively of the initial field state. In particular, asymptotic completeness, Eq. (2), ensures that any field state can be prepared by choosing an appropriate initial atomic state. This can easily be seen when considering $A = |\chi\rangle\langle\chi|$, the projector on the desired field state $|\chi\rangle$. Then, after unitary evolution, also $M_{|\chi\rangle\langle\chi|}$ is a projector (in the N-atom Hilbert space). If we now choose any state $|\psi_0\rangle$ from the range of $M_{|\chi\rangle\langle\chi|}$ as the initial atomic state, the expectation value of $|\chi\rangle\langle\chi|$ after the atoms-field interaction, i.e., the probability of finding the field finally in the desired state $|\chi\rangle$, is 1. In the Schrödinger picture, this reads

$$\lim_{N\rightarrow\infty} U_N |\chi_0\rangle \otimes |\psi_0\rangle = |\chi\rangle \otimes |\psi_{\chi_0}\rangle, \qquad \forall\, |\chi_0\rangle \,. \tag{3}$$

Obviously, the final state shows *no entanglement* of the field with the atoms. Therefore, a measurement of the atomic state will not influence the photon field. Furthermore, the desired field state $|\chi\rangle$ is created *irrespectively of the initial field state* $|\chi_0\rangle$. Since the information about the initial field state cannot be lost during a unitary evolution, it is completely transferred to the final atomic state $|\psi_{\chi_0}\rangle$.

Since in reality we cannot handle an infinite number of atoms, the question arises how fast the limit of asymptotic completeness will be reached: How many atoms do we need to prepare the desired field state $|\chi\rangle$ within a given level of accuracy? In order to quantify the accuracy of the field state preparation, we will use the *fidelity* F of the final state with respect to the desired field state $|\chi\rangle$. F is defined as the expectation value of the projector $P = |\chi\rangle\langle\chi| \otimes \mathbf{1}$, that is, the probability of finding the state $|\chi\rangle$ when performing a measurement. To calculate the fidelity F, we evaluate P in the Heisenberg picture:

$$U_N^\dagger (|\chi\rangle\langle\chi| \otimes \mathbf{1}) U_N = \sum_{m,n=0}^{\infty} |m\rangle\langle n| \otimes M^{(mn)}, \tag{4}$$

where $|m\rangle, |n\rangle$ are number states of the photon field, with photon numbers m and n, respectively. This expression, together with Eq. (1), defines the operators $M^{(mn)}$, which act in the 2^N-dimensional Hilbert space of the N atoms. The fidelity F of the final state, given the initial atomic state $|\psi_0\rangle$, then reads

$$F = \langle\psi_0|M^{(\rho_0)}|\psi_0\rangle, \tag{5}$$

where the atomic operator $M^{(\rho_0)}$ depends on the initial field state ρ_0 (which may also be a mixed state),

$$M^{(\rho_0)} = \sum_{m,n=0}^{\infty} \langle n|\rho_0|m\rangle M^{(mn)}. \tag{6}$$

Hence, the maximum fidelity $F_{\max}$ is obtained as the largest eigenvalue of $M^{(\rho_0)}$, and the associated eigenvector $|\psi_0\rangle$ is the optimal initial atomic state. From the asymptotic completeness, Eq. (2), we know that in the limit $N \rightarrow \infty$, $M^{(\rho_0)}$ does not depend on the initial field state ρ_0, and its eigenvalues are 0 and 1.

In the following, we will consider the vacuum $|0\rangle$ as initial field state, so that $M^{(\rho_0)} = M^{(00)}$. In this case, it is useful to look also at the time-reversed process: Given an arbitrary field state $|\chi\rangle$, how can we create the vacuum state $|0\rangle$? It seems to be clear that this is achieved most efficiently by injecting all atoms in the lower state $|d\rangle$, so that each atom can absorb the maximum amount of energy from the photon field, which will then eventually end up in the vacuum $|0\rangle$. From this intuitive argument, the generalized time-reversal symmetry [9] of the atoms-field interaction, Eq. (1), immediately leads us to the following conjecture: *The optimal strategy to prepare a given field state* $|\chi\rangle$*, starting from the vacuum* $|0\rangle$ *as initial state, is such that all atoms leave the cavity in the lower state* $|d \ldots d\rangle$.

In this case, the maximum fidelity $F_{\max}$ would be equal to the maximum overlap F' of the total final state with $|\chi\rangle \otimes |d \ldots d\rangle$:

$$\begin{aligned} F' &= \max_{|\psi_0''\rangle} |\langle\chi, d \ldots d|U_N|0, \psi_0''\rangle|^2 \\ &= \langle\psi_0'|\psi_0'\rangle \,, \end{aligned} \tag{7}$$

where $|\psi_0'\rangle$ is the (unnormalized) optimal initial atomic state

$$|\psi_0'\rangle = \langle 0|U_N^\dagger|\chi, d \ldots d\rangle \,. \tag{8}$$

Since F' refers to a definite atomic final state $|\psi\rangle = |d \ldots d\rangle$, whereas $F_{\max}$ includes an additional maximization over all possible atomic final states, it is clear that F' gives a lower bound for the maximum fidelity $F_{\max}$:

$$F_{\max} \geq F'. \tag{9}$$

Equality holds if and only if the above conjecture: *the optimal final atomic state is* $|d \ldots d\rangle$ is true.

In the case of a number state, $|\chi\rangle = |n\rangle$, Eqs. (7), (8), and (1) give the following expression for F':

$$F'(n) = \prod_{j=1}^{n} \sin^2(\phi\sqrt{j}) \sum_{k_0+\ldots+k_n=N-n} \prod_{i=0}^{n} \cos^{2k_i}(\phi\sqrt{i}) \tag{10a}$$

$$= 1 - \sum_{k=1}^{n} \cos^{2N}(\phi\sqrt{k}) \times \prod_{\substack{i=1 \\ i\neq k}}^{n} \frac{\sin^2(\phi\sqrt{i})}{\cos^2(\phi\sqrt{k}) - \cos^2(\phi\sqrt{i})}. \tag{10b}$$

Equation (10b) is equivalent to Eq. (10a) [if $\cos^2(\phi\sqrt{k}) \neq \cos^2(\phi\sqrt{i})$ for $k \neq i$], but involves a summation over only one variable k. In the general case $|\chi\rangle = \sum_{i=0}^{n} c_i|i\rangle$, one can show that

$$F' = \sum_{i=0}^{n} |c_i|^2 F'(i). \tag{11}$$

Since $F'(i) \leq F'(j)$ for $i > j$, any field state including photon numbers not higher than n can be prepared with fidelity $F \geq F'(n)$.

From Eq. (10b), we see that $F' \to 1$ when $N \to \infty$ [if $\cos^2(\phi\sqrt{k}) < 1$ for $k \leq n$, i.e., no trapping states], so the above conjecture $F_{\max} = F'$ is at least valid asymptotically. In order to test its validity for finite N, Fig. 1 shows a comparison between $F_{\max}$ and F', for the preparation of various number states $|\chi\rangle = |n\rangle$, $n = 1, \ldots, 5$, out of the vacuum $|0\rangle$, with $N = 10$ atoms, as a function of the vacuum Rabi angle ϕ. For most values of ϕ, no difference between $F_{\max}$ and F' can be detected, which confirms the validity of our conjecture in most cases. However, we observe also some deviations, mostly near values of ϕ corresponding to $|n_t = n\rangle$-trapping states. (An exception, which is not yet fully understood, is the case $n = 1$, where the deviations are observed near the $|2\rangle$-trapping states.) This is not surprising: Since in the case of a $|n_t\rangle$-trapping state the photon number cannot exceed n_t, we could argue that the optimal way of creating the state $|n_t\rangle$ would be to inject all atoms in the upper state $|u\rangle$. This would lead to a final atomic state different from $|d \ldots d\rangle$, which shows that the above intuitive argument cannot be right in the case of trapping states.

If the desired field state $|\chi\rangle$ is not a number state (Fig. 2), the agreement of $F_{\max}$ and F' is still good, but not exact—also not for values of ϕ far away from the relevant trapping state conditions. Hence, for finite N, the conjecture $F_{\max} = F'$ is exactly valid only for number states, if ϕ is not close to a trapping state.

Finally, Figs. 3 and 4 demonstrate how effective our state preparation works: Figure 3 shows the maximum fidelity $F_{\max}$ for the preparation of $|\chi\rangle = |5\rangle$ and $|\chi\rangle = \sum_{i=0}^{5} |i\rangle/\sqrt{6}$, as a function of the number N of atoms. Two distinct initial states of the cavity field are treated, a pure (Fig. 3a) and a mixed (Fig. 3b) state. In both cases, the straight lines in the semilogarithmic plot demonstrate that the optimum fidelity 1 is approached *exponentially* fast when increasing N, which is already suggested by the appearance of N as an exponent in Eq. (10b). Also note that the rate of the exponential approach to $F_{\max} = 1$ depends on the initial state of the field, but is of the same order of magnitude in both cases.

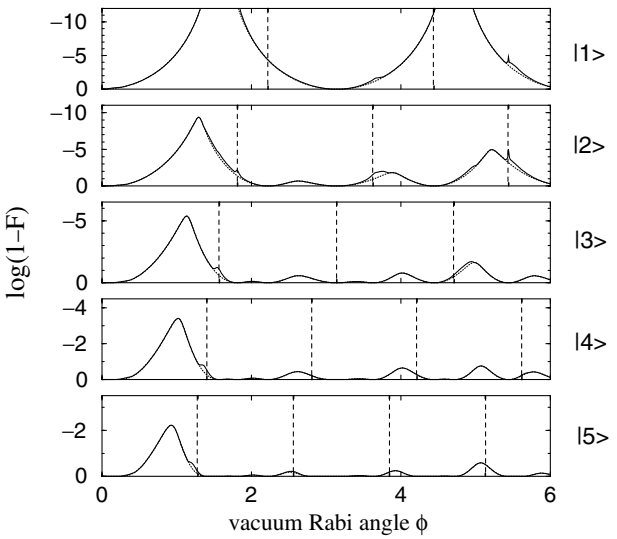

FIG. 1. Maximum fidelity $F_{\max}$ (solid line) and F' (dotted line) for the preparation of the cavity field state $|\chi\rangle = |1\rangle, \ldots, |5\rangle$ with a sequence of $N = 10$ atoms injected into the resonator, as a function of the vacuum Rabi angle ϕ. The initial field state is the vacuum $|0\rangle$. The fairly good agreement of $F_{\max}$ with F' (note the logarithmic scale) shows that our conjecture "the optimal final atomic state is $|d \ldots d\rangle$" is valid for most values of ϕ—in particular, for the optimum regime below the first trapping state of the field. Deviations of $F_{\max}$ from F' can (in most cases) be traced back to $|n\rangle$-trapping states, which occur at integer multiples of $\phi = \pi/\sqrt{n+1}$ (vertical dashed lines).

To get an idea of how many atoms have to be entangled prior to the interaction for a given desired fidelity of a

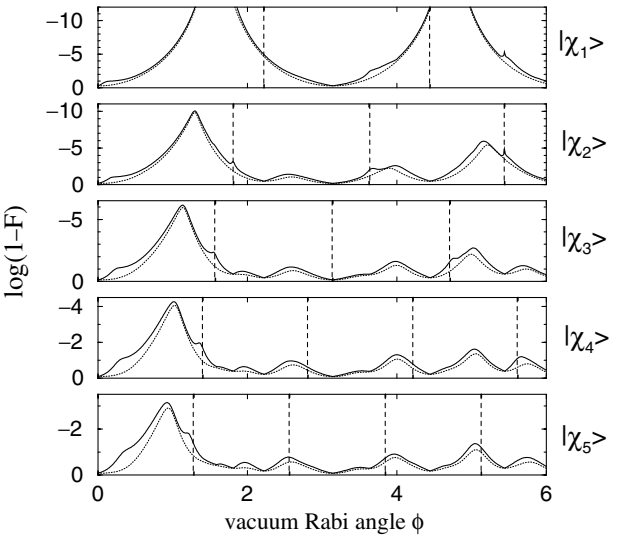

FIG. 2. Maximum fidelity $F_{\max}$ (solid line) and F' (dotted line) for the preparation of the truncated phase states $|\chi_n\rangle = \sum_{i=0}^{n} |i\rangle/\sqrt{n+1}$, $n = 1, \ldots, 5$, of the cavity field, with $N = 10$ atoms injected into the resonator. The initial field state is the vacuum $|0\rangle$. The agreement of $F_{\max}$ with F' is still good on the logarithmic scale. The $|n\rangle$-trapping states are denoted by the vertical dashed lines.

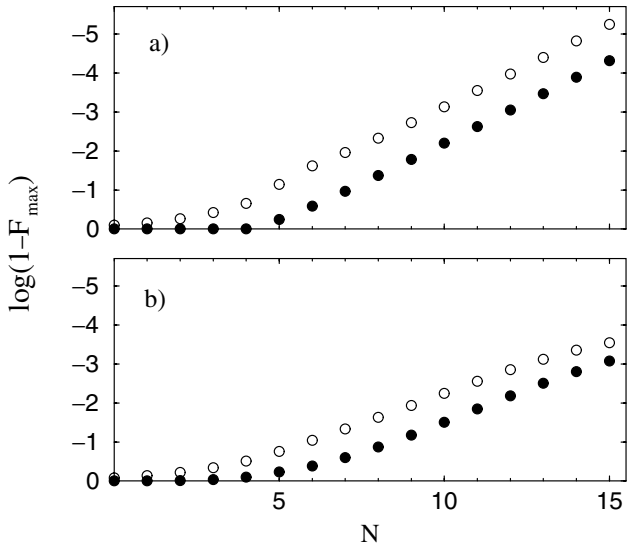

FIG. 3. Maximum fidelity $F_{\max}$ for the preparation of the 5-photon state $|\chi\rangle = |5\rangle$ (filled circles), and of the truncated phase state $|\chi\rangle = \sum_{i=0}^{5} |i\rangle/\sqrt{6}$ (open circles) of the cavity field, respectively, as a function of the number N of atoms injected into the resonator. Vacuum Rabi angle $\phi = 0.91$. Initial field state: (a) vacuum $|0\rangle$, (b) thermal equilibrium with average photon number $\langle n\rangle = 0.55$ (corresponding to a temperature of $T \simeq 1$ K in the microwave regime). In both cases, $F_{\max}$ approaches the ideal value 1 *exponentially* fast.

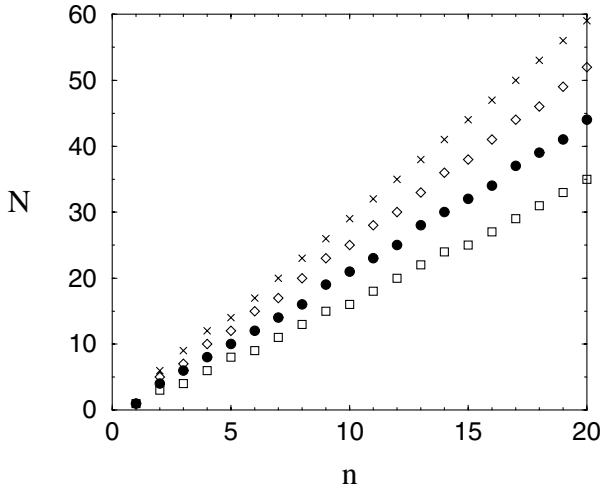

FIG. 4. Minimum number of atoms N needed to prepare the number state $|n\rangle$ out of the cavity field vacuum $|0\rangle$ with fidelity $F'(n) \geq 1 - \epsilon$, $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ (from bottom to top). In each case, the optimum value of the vacuum Rabi angle ϕ (compare Fig. 1) was chosen. To prepare $|n\rangle$ with uncertainty $\epsilon < 10^{-2}$ ($\epsilon < 10^{-4}$), $N \simeq 2n$ ($N \simeq 3n$) atoms suffice.

target number state, Fig. 4 shows the minimum number N of atoms needed such that $F'(n) > 1 - \epsilon$, for $\epsilon = 10^{-1}$, 10^{-2}, 10^{-3}, and 10^{-4}. For each value of n, the optimal value of the vacuum Rabi angle (below the first trapping state, see Fig. 1) was chosen. Remember that, by virtue of Eqs. (9) and (11), $F'(n)$ gives a lower bound for the maximum fidelity $F_{\max}$ of preparing an arbitrary field state including photon numbers not higher than n, with the vacuum $|0\rangle$ as the initial state of the field. As obvious from the plot, an uncertainty $1 - F_{\max} < 10^{-2}$ (10^{-4}) is achieved injecting about $2n$ ($3n$) atoms.

Since the entanglement of a larger number of atoms remains an experimentally formidable task, let us conclude with some remarks on a finite fidelity $F_0 = \langle\psi_0|\rho_a|\psi_0\rangle$ of the initial state preparation—with ρ_a describing the imperfect atomic initial state. Equation (5) is then generalized by $F = \mathrm{tr}\{M^{(\rho_0)}\rho_a\}$, and, consequently, $F \geq F_{\max}F_0$, with the above estimations on $F_{\max}$ unaffected. Given the recent experimental result on the entanglement of four particles [4], and an entangling procedure that should operate on even larger particle numbers [5], we are therefore confident that our novel approach to quantum state preparation opens an experimentally practicable perspective.

As a matter of fact, the experimental implementation of our scheme can be considerably facilitated for the preparation of number states, provided we drop the above requirement of the optimal atomic initial state that maximizes F *for given* N. Asymptotic completeness, Eq. (2), then implies that a sequence of *pairs* of subsequent atoms with the appropriate two-particle entanglement [10] is sufficient to prepare these specific target states (at the expense of a larger number N of atoms to be injected for a desired fidelity F). As a by-product, a steady flux of such entangled atom pairs through the cavity could also be used to stabilize number states against dissipation, if the flux is high enough compared to the cavity decay rate.

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[9] The inverse of the atoms-field interaction Eq. (1) is obtained by (i) reversing the order of the atoms and (ii) applying the transformation $|u\rangle \rightarrow -|u\rangle$ on each atom.
[10] According to Eq. (1), the initial atomic sequence $|\psi_0\rangle = |\psi\rangle \otimes \cdots \otimes |\psi\rangle$ of $N/2$ identically prepared atom pairs $|\psi\rangle = \cos(\phi\sqrt{n+1})|u\rangle \otimes |d\rangle - \cos(\phi\sqrt{n})|d\rangle \otimes |u\rangle$ (unnormalized), leaves the field state $|n\rangle$ invariant. From the independence of the final and initial field state in the limit of asymptotic completeness, Eq. (2), it follows that for any initial field state, the final field state approaches $|n\rangle$ in the limit $N \rightarrow \infty$.