Entanglement of completely symmetric quantum states

Hans Maassen

Mark Kac Seminar, October 7, 2011.

Collaboration with Burkhard Kümmerer and Bram Petri

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For a given completely symmetric state, we want to find out if it is entangled or not, and, if so, to quantify how entangled it is.

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Not only ancient, but also some more recent mathematical work in this area turns out to be surprisingly relevant to quantum information theory.

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- A measure of entanglement.

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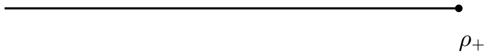

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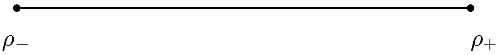

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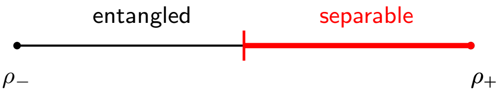

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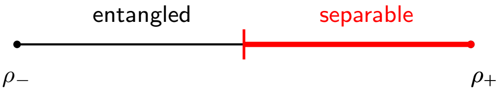

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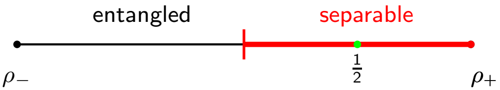

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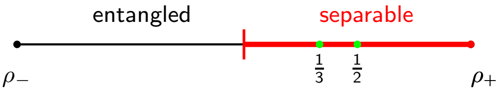

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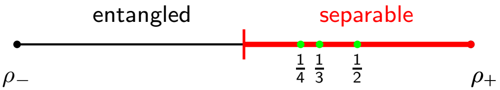

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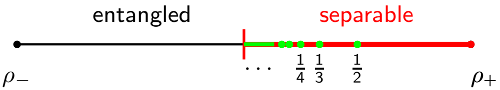

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For example

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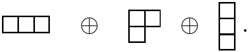
$=$ 

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The normalized version $\tau_{\mathrm{reg}} := \frac{1}{n!}\mathrm{tr}_{\mathrm{reg}}$ is the regular trace state.

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Let us now choose a minimal projection $q_i \in p_i(\mathcal{A}')$.

Then the regular representation acts irredicibly on the range of $p_i q_i$.

Interpretation of the characters χ_i

Lemma

The character $\chi_i(\sigma)$ is the trace of σ in its irreducible representation labelled by i.

Proof.

The space $p_i\mathcal{H}_{\mathrm{reg}}$ looks like $M_{d(i)} \simeq \mathbb{C}^{d(i)} \otimes \mathbb{C}^{d(i)}$.

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Now, in the regular representation we may calculate

$$\chi_i(\sigma) = \frac{n!}{d(i)} p_i(\sigma) = \frac{1}{d(i)} n!(p_i * \delta_\sigma)(e) = \frac{1}{d(i)} \mathrm{tr}_{\mathrm{reg}}(p_i * \delta_\sigma) = \mathrm{tr}_{\mathrm{reg}}(p_i * q_i * \delta_\sigma) \ ,$$

□

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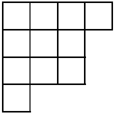
The irreducible representations of S_n (and hence also the minimal central projections and the characters) are labelled by Young frames with n boxes:

$Y =$  .

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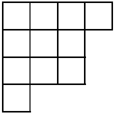
$Y =$  .

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For example:

$$d\left(\ydiagram{3,2}\right) = \frac{5!}{4 \times 3 \times 2} = 5 \quad \text{hook lengths:} \quad \begin{ytableau} 4 & 3 & 1 \\ 2 & 1 \end{ytableau} \; .$$

An exclusion principle

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Theorem

Let $n, d \in \mathbb{N}$. Let Y denote a Young frame with n boxes and height $h(Y)$. Then

$$\pi_{n,d}(p_Y) = 0 \quad \textit{iff} \quad h(Y) > d \ .$$

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Hence the above theorem generalizes this exclusion principle.

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Let ϑ be a completely symmetric state on $\mathcal{B}\big((\mathbb{C}^d)^{\otimes n}\big)$. Then ϑ is separable iff its restriction to $\mathcal{Z}$ lies in the convex hull of the restricted product states.

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Conclusion: We must calculate the shadow of the product states!

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for some positive weights μ_i with sum 1 and unit product vectors ψ_i, then since ϑ is completely symmetric, we have for all $x \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} \vartheta(x) &= \vartheta(Px) = \sum_i \mu_i \langle \psi_i, Px\psi_i \rangle \\ &= \frac{1}{n!} \sum_i \sum_{\sigma \in S_n} \int_{SU(d)} \mu_i \big\langle \pi(\sigma)(u \otimes \ldots \otimes u)\psi_i, x\, \pi(\sigma)(u \otimes \ldots \otimes u)\psi_i \big\rangle \, du \ , \end{aligned}$$

which is a convex inegral of product states, hence separable. □

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\mathrm{tr}_d^{\otimes n}(\pi(\sigma)) &= \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \pi(\sigma)\, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle \\
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since for every cycle one summation variable remains. Hence:

$$\tau_d^{\otimes n}(p_y) = \frac{d(Y)}{n!} \sum_{\sigma \in S_n} \chi_Y(\sigma) \frac{1}{d^n} \mathrm{tr}_d^{\otimes n}(\pi(\sigma)) \to \frac{d(Y)^2}{n!} \, , \qquad (d \to \infty) \, .$$

□

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The density of a product state $\psi_1 \otimes \ldots \otimes \psi_n$ with respect to the regular trace is the normalized immanant of the Gram matrix of $\psi_1, \psi_2, \ldots, \psi_n$.

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The first inequality was proved by Schur in 1918, the second was conjectured by Elliott Lieb in 1967, and is still open!

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The simplex $\mathcal{S}(\mathcal{Z}_n)$ for $n = 3$

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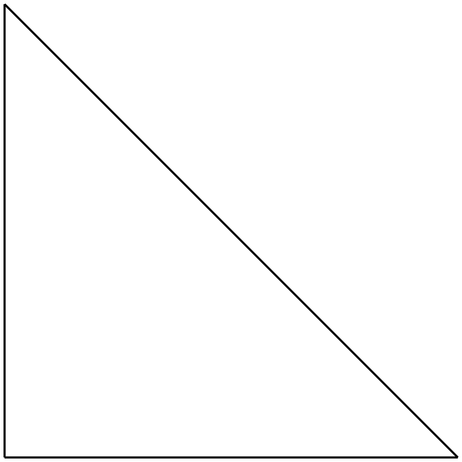

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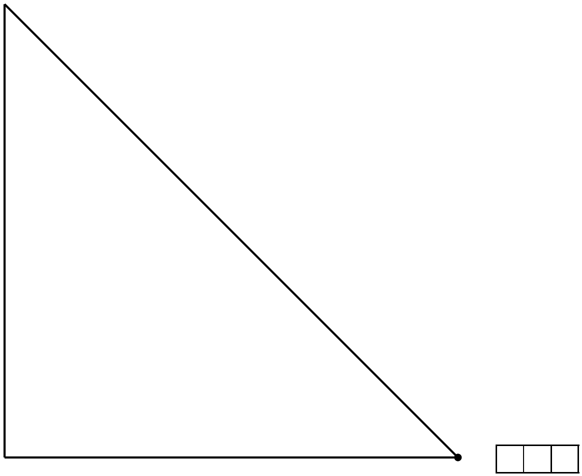

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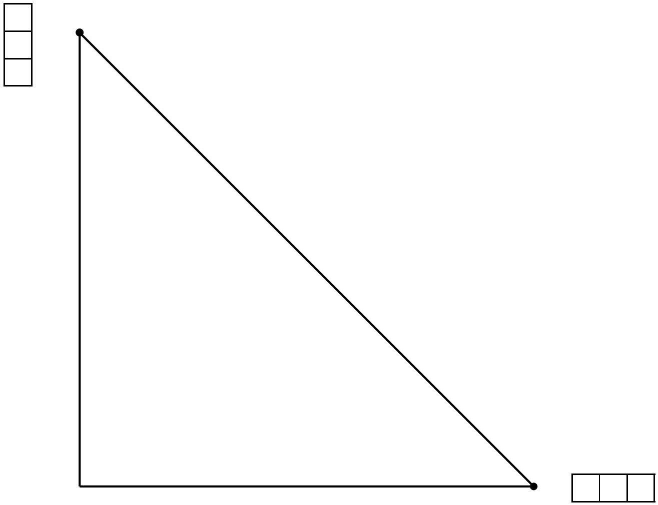

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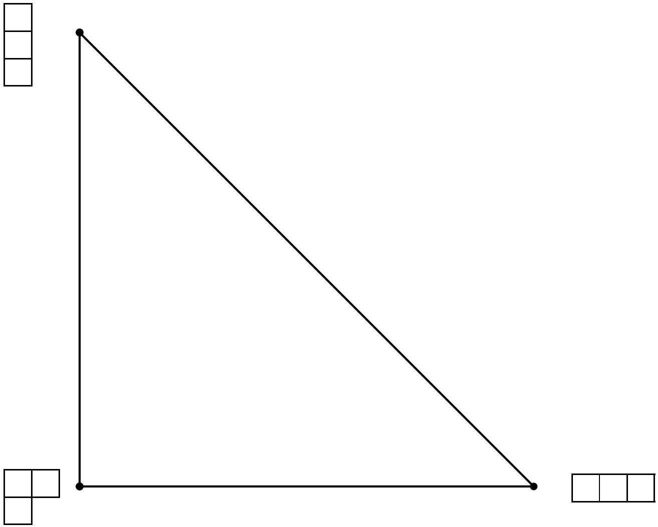

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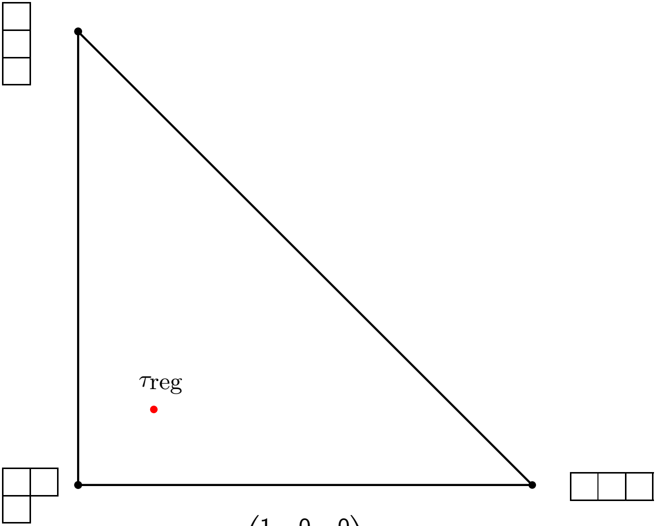

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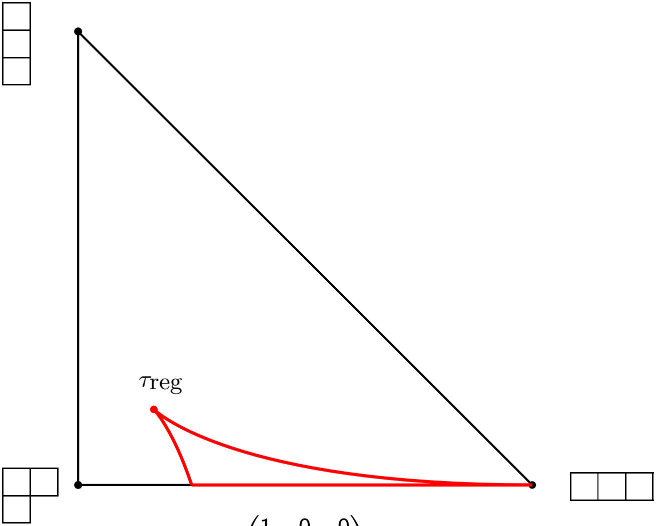

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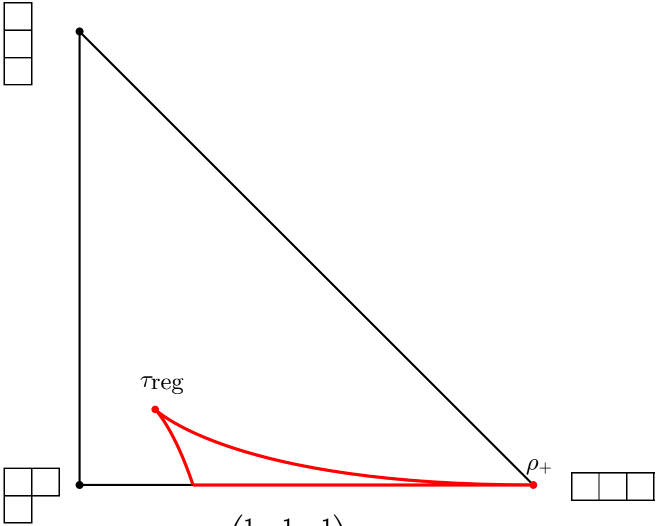

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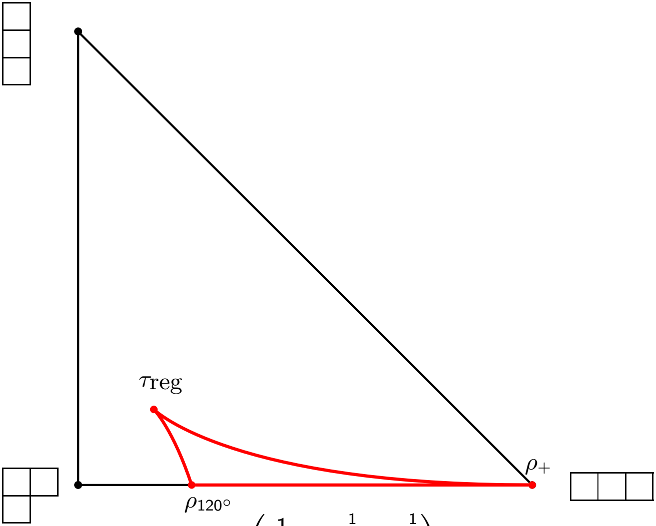

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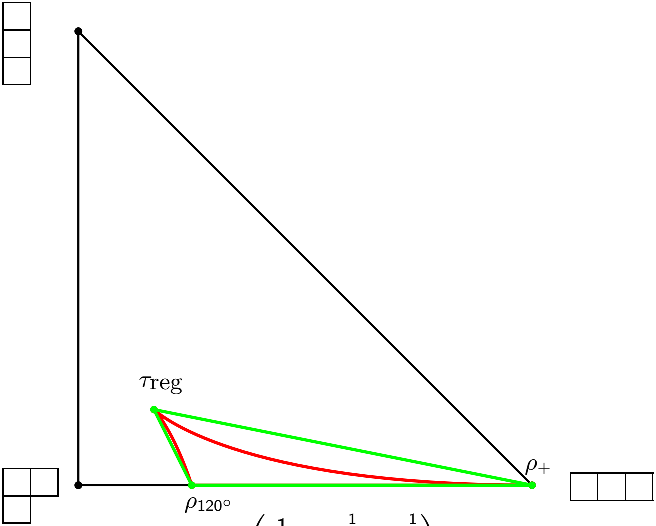

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General n: the shadow touches only one corner

□

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Theorem

Only the state $\rho_+ = \rho_{\square\square\square\square\square\square\square}$ (n boxes) is separable, all other extremal states ρ_Y on $\mathcal{Z}_{n,d}$ are entangled.

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However, product vectors with this property do not exist!
Indeed, suppose that the product vector $\psi_1 \otimes \ldots \otimes \psi_n$ is orthogonal to $p_+\mathcal{H}$.
Then it is orthogonal to all vectors of the form $\vartheta \otimes \ldots \otimes \vartheta$ with $\vartheta \in \mathbb{C}^d$:

$$0 = \langle \vartheta \otimes \ldots \otimes \vartheta, \psi_1 \otimes \ldots \otimes \psi_n \rangle = \prod_{j=1}^{n} \langle \vartheta, \psi_j \rangle \,. \quad \text{But: } \bigcup_{j=1}^{n} \{\psi_j\}^\perp \neq \mathbb{C}^d \,,$$

the left hand side having Lebesgue measure 0. $\square$

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Hence

$$\langle\psi_1\otimes\ldots\otimes\psi_n, p_-\,\psi_1\otimes\ldots\otimes\psi_n\rangle \leq \tau\mathrm{reg}(p_-) = \frac{1}{n!}\ .$$

□

The Schur and Lieb inequalities

We have $2\mathcal{P}(n) - 3$ inequalities, which divide the state space $\mathcal{S}(\mathcal{Z}_n)$ into compartments, and claim the the shadow of the product states falls into one of them.

Schur's 1918 inequality states that for all separables states ρ and all Young frames $Y \neq \{-\}$:

$$\rho(p_Y) \geq d(Y)^2 \, \rho(p_-) \; .$$

Lieb's 1967 conjecture hopes that for all separable ρ and all Young frames

$Y \neq \{+\}$:

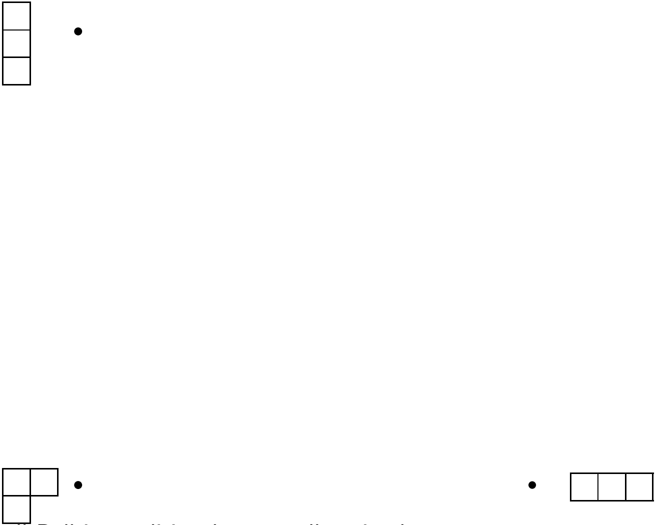
$$\rho(p_Y) \leq d(Y)^2 \, \rho(p_+) \; .$$

The last trivial inequality says that for all separable ρ:

$$\rho(p_-) \leq \rho(p_+) \; .$$

These are all Bell inequalities.

The immanant inequalities for $n = 3$ in a picture



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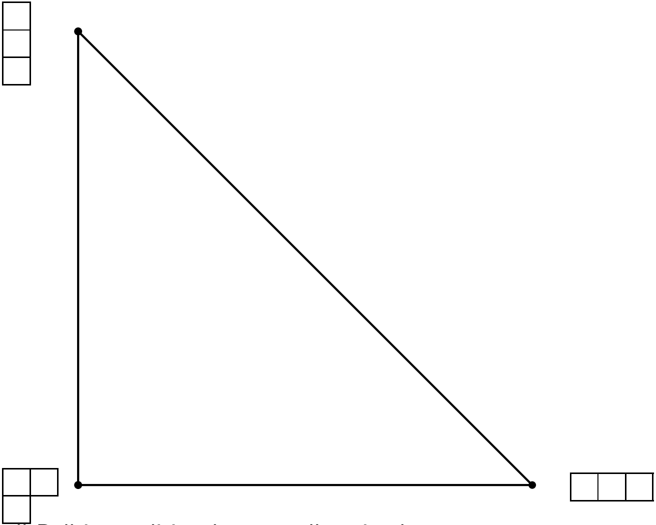

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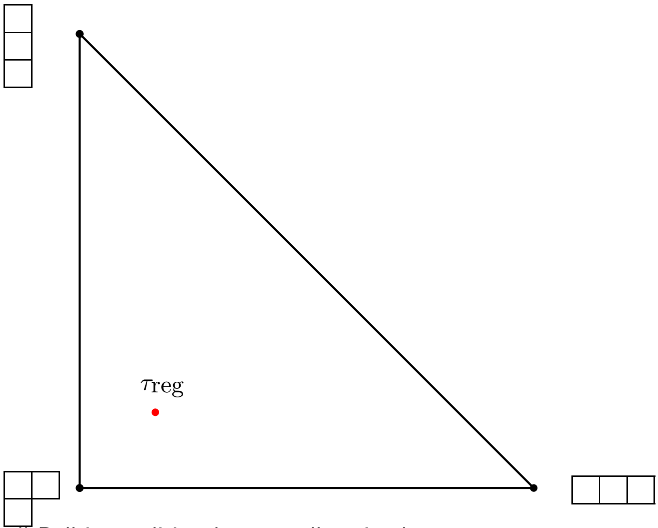

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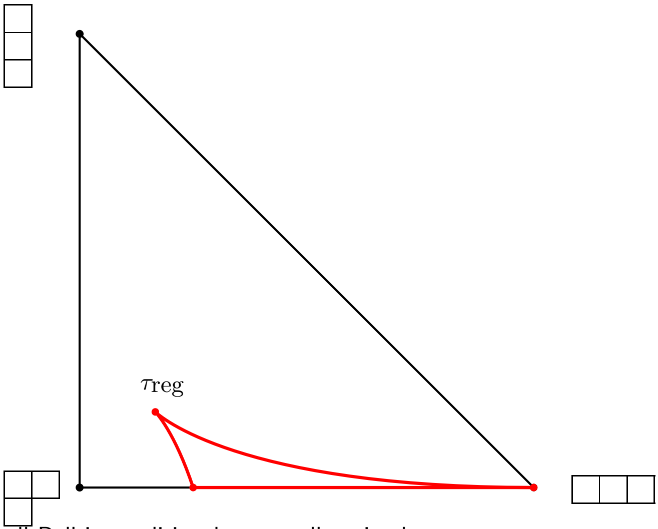

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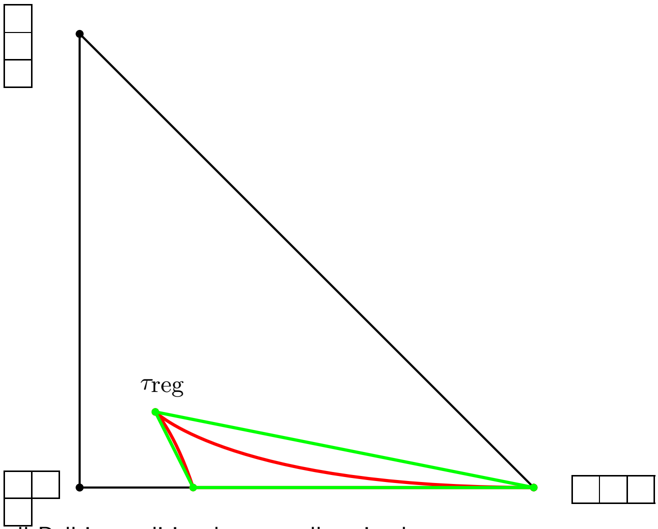

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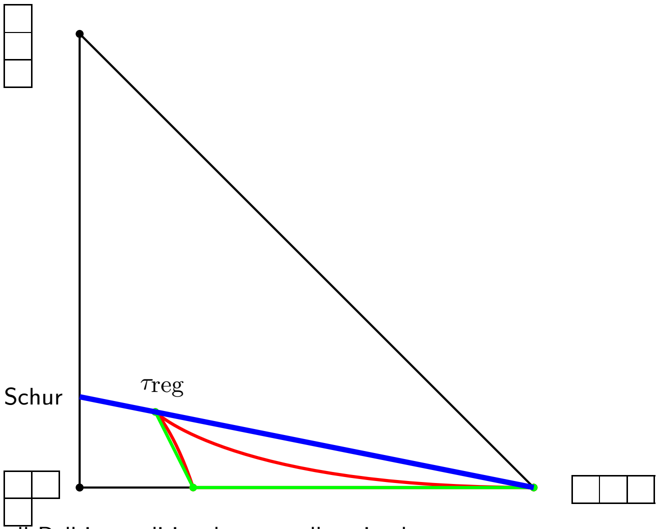

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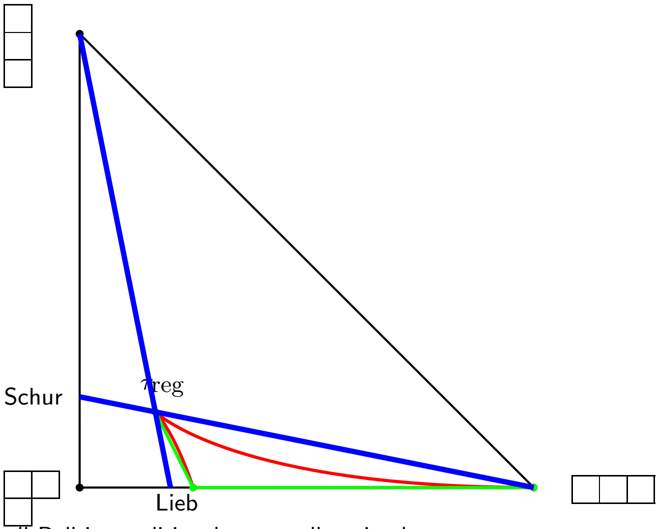

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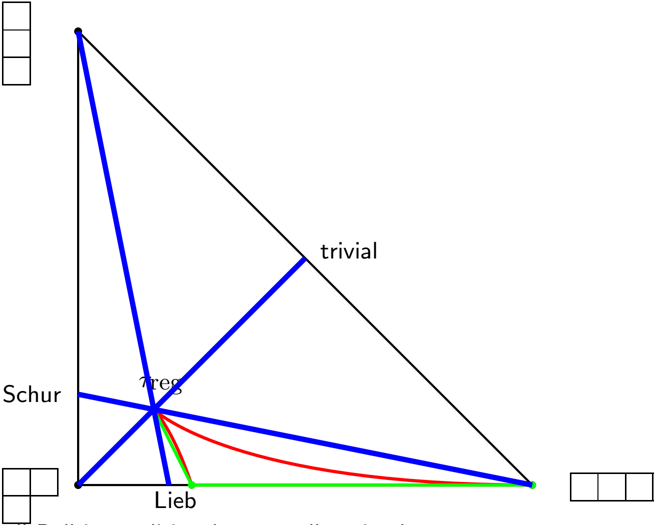

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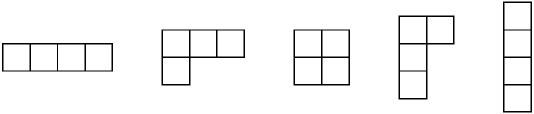
In the case $n = 4$ there are five Young diagrams:



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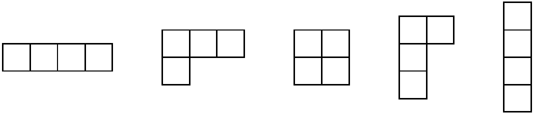

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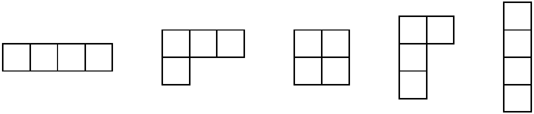

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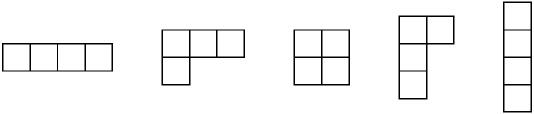

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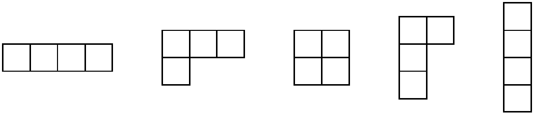

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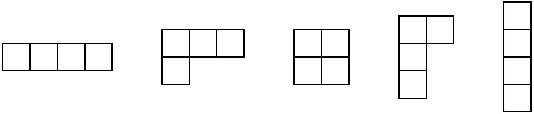

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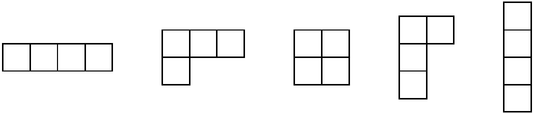

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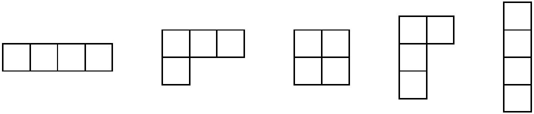

Theorem (Barrett, Hall, Loewy (1998) translated to quantum states)

The set of completely symmetric separable states on $\mathcal{B}(\mathbb{C}^d)^{\otimes 4}$ is the convex hull of 7 extreme points. These extremal states are obtained by twirling and averaging 7 configurations of unit vectors in $\mathbb{C}^d$ (with $d \geq 4$ to fit all of them). These configurations are given by the Gram matrices

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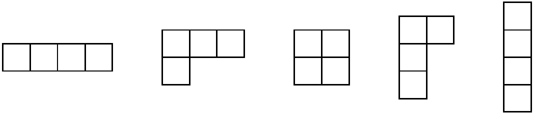

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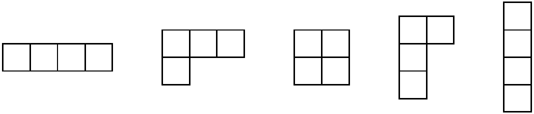

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Four qubits

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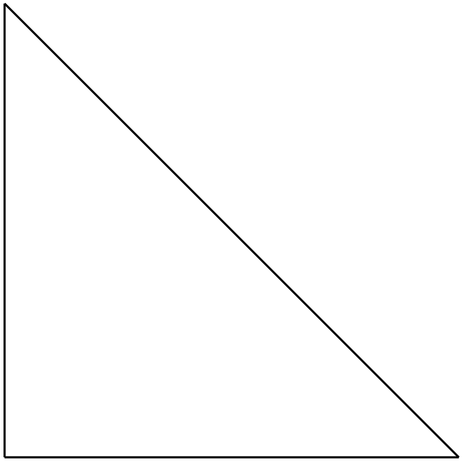

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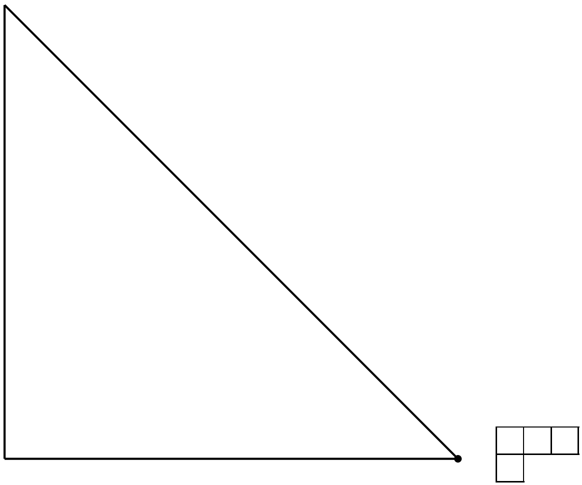

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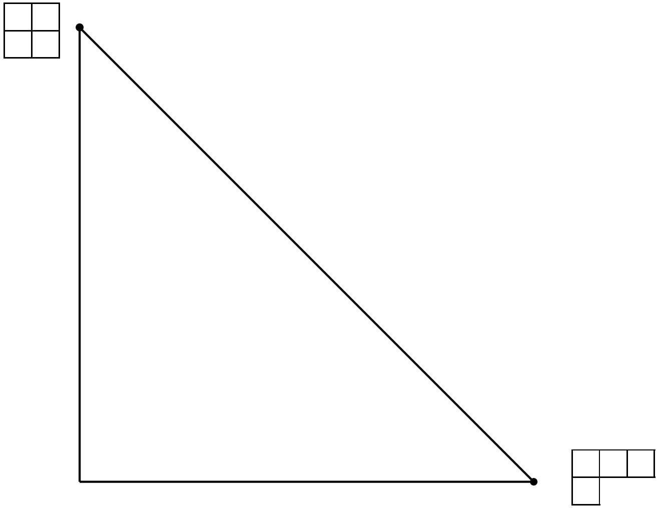

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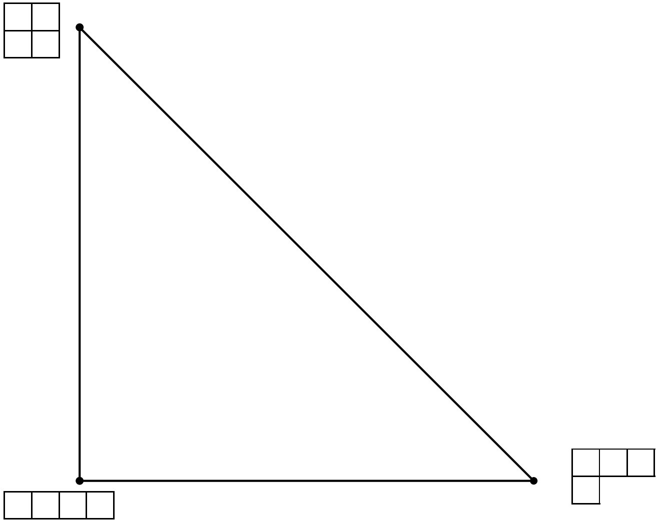

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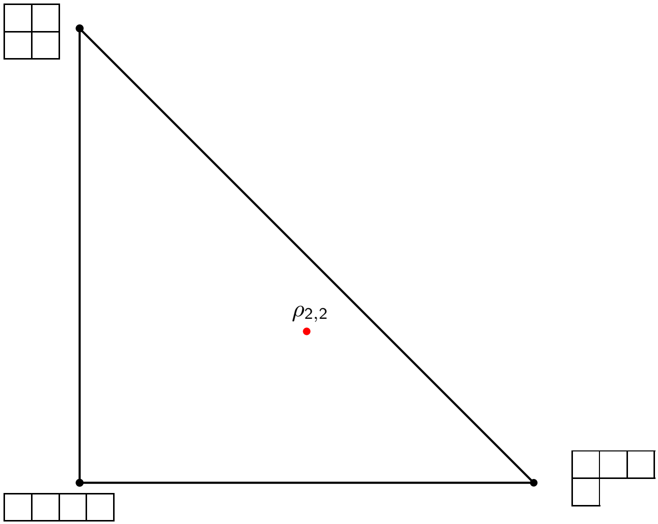

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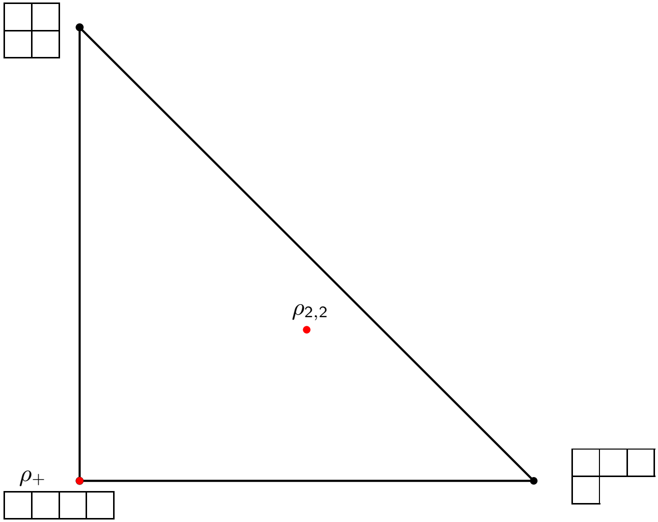

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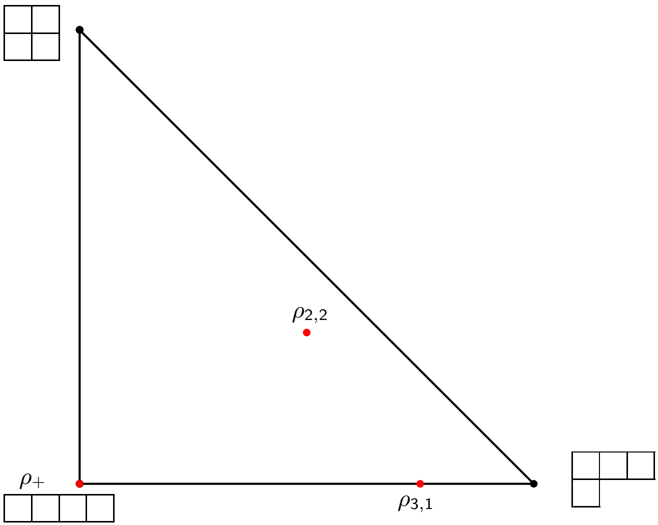

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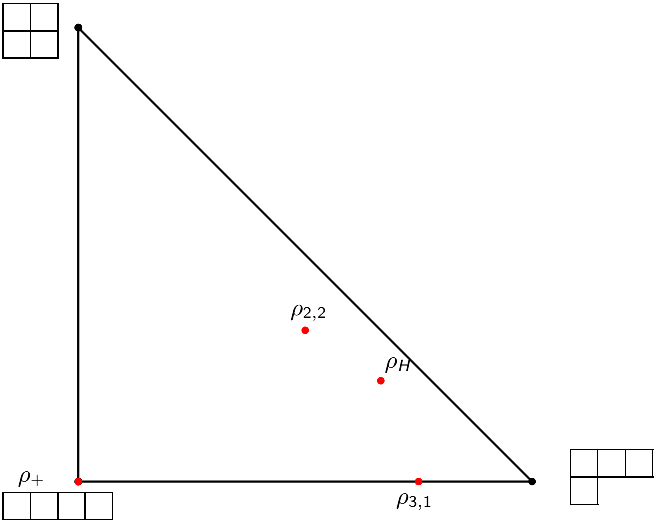

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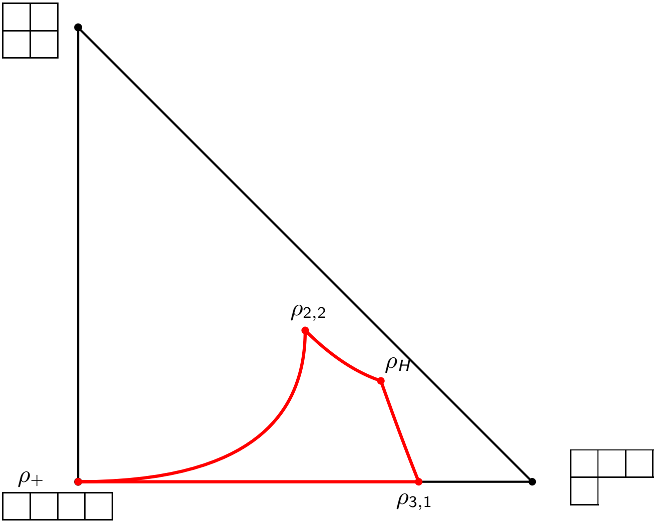

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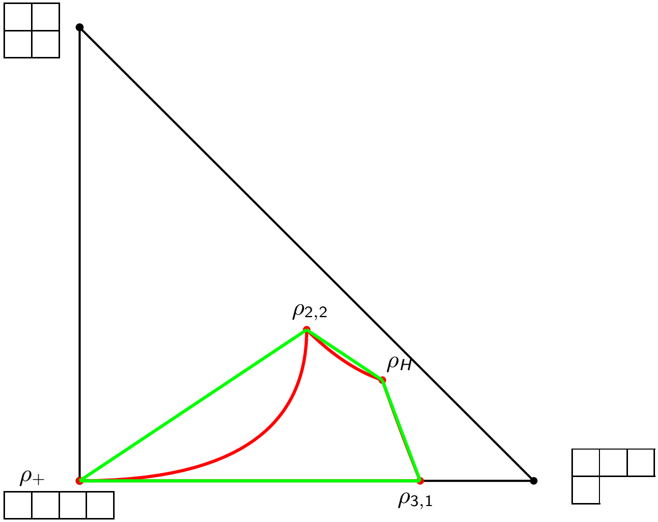

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In 1999 they showed that, already in the five qubit situation, the set of separable states on the center possesses a part that is bulging outward.

A measure for multiparticle entanglement

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Then a norm on the dual of $\mathcal{B}(\mathcal{H})$, is define by

$$\|\omega\|^V := \inf \left\{ \sum_{i=1}^k \lambda_i \,\middle|\, \omega = \sum_{i=1}^k \lambda_i \nu_i \,, k \in \mathbb{N}, \lambda_1, \lambda_2, \ldots, \lambda_k > 0, \nu_1, \nu_2, \ldots, \nu_k \in V \right\};$$

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When ρ is a state on $\mathcal{B}(\mathcal{H})$, we define its entanglement $E(\rho)$ by

$$E(\rho) := \|\rho\|^V \ .$$

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  (Here we would actually prefer equality!)

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Let $n, d \in \mathbb{N}$, and let Y denote an n-block Young frame with height $\leq d$. The entanglement of the completely symmetric state ρ_Y satisfies

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In particular, the antisymmetric state has entanglement

$$E(\rho_-) = n! \, .$$