Polynomial automorphisms over finite fields

Stefan Maubach

March 2010
TOPIC: affine algebraic geometry

Objects of study: affine spaces:

\[ X \text{ is affine} \iff X = \text{spec } A \text{ for some ring } A. \]

Typical affine space:

\[ X = \mathbb{A}^n \text{ where } \mathbb{A} \text{ a field.} \]

"Generic" algebraic geometry dislikes \( \mathbb{A}^n \) because it is not compact (contrary to projective geometry).

But hey - \( \mathbb{A}^n \) is perhaps the most simple algebraic space there is!!

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Various ways of looking at polynomial maps:

- A map $k^n \rightarrow k^n$.
- A list of $n$ polynomials: $F \in (k[X_1, \ldots, X_n])^n$.
- A ring endomorphism of $k[X_1, \ldots, X_n]$ sending $g(X_1, \ldots, X_n)$ to $g(F_1, \ldots, F_n)$. 
A polynomial map $F$ is a polynomial automorphism if there is a polynomial map $G$ such that $F(G) = (X_1, \ldots, X_n)$. 

Example: $(X + Y^2, Y)$ has inverse $(X - Y^2, Y)$. 

$(X + Y^2, Y) \circ (X - Y^2, Y) = (X - Y^2 + Y^2, Y) = (X, Y)$.

$(X^p, Y^p): F^2p \to F^2p$ is not a polynomial automorphism, even though it induces a bijection of $F^p$. 

$(X^3, Y): \mathbb{R}^2 \to \mathbb{R}^2$ is not a polynomial automorphism, even though it induces a bijection of $\mathbb{R}^2$.

Remark: If $k$ is algebraically closed, then a polynomial endomorphism $k^n \to k^n$ which is a bijection, is an invertible polynomial map.
A polynomial map $F$ is a **polynomial automorphism** if there is a polynomial map $G$ such that $F(G) = (X_1, \ldots, X_n)$.

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**Remark:** If \( k \) is algebraically closed, then a polynomial endomorphism \( k^n \longrightarrow k^n \) which is a bijection, is an invertible polynomial map.
Polynomial automorphisms form a group, denoted by $\text{GA}_n(k)$.

Notations:

<table>
<thead>
<tr>
<th></th>
<th>Linear</th>
<th>Polynomial</th>
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<tbody>
<tr>
<td>All</td>
<td>$\text{ML}_n(k)$</td>
<td>$\text{MA}_n(k)$</td>
</tr>
<tr>
<td>Invertible</td>
<td>$\text{GL}_n(k)$</td>
<td>$\text{GA}_n(k)$</td>
</tr>
</tbody>
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Motivation: why over $\mathbb{F}_p$?

- Reduction-mod-$p$ techniques to (dis)prove things
  (Example: $F$ injective $\rightarrow$ $F$ surjective.)
  (Example: Belov-Kontsevich)

- Possible applications (cryptography etc.)

- Simply because it is interesting:
  1. Connections with discrete mathematics.
  2. Connections with finite group theory.
Jacobian Conjecture

If $F$ is invertible and $G$ inverse of $F$, then

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\det(Jac(F)) \in \mathbb{k}[X_1,\ldots,X_n]^* = \mathbb{k}^*.
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$$\text{Jac}(I) = \text{Jac}(G \circ (F)) = \text{Jac}(F) \cdot (\text{Jac}(G) \circ F)$$

Thus $\det(\text{Jac}(F)) \in k[\mathbf{X}_1, \ldots, \mathbf{X}_n]^* = k^*$. 

**Question:** If $F$ polynomial endomorphism, and $\det(\text{Jac}(F)) \in k^*$, is $F$ invertible?
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**QUESTION:** if \( F \) polynomial endomorphism, and \( \det(\text{Jac}(F)) \in k^* \), is \( F \) invertible?
LEMMA: If $F$ is invertible, then $\det(J(F)) \in k^*$. 

JACOBIAN CONJECTURE: $\text{char}(k) = 0$. If $F$ polynomial endomorphism, and $\det(\text{Jac}(F)) \in k^*$, is $F$ invertible?
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In $\text{char}(k) = p$: $F : X \rightarrow X - X^p$ has $\det(\text{Jac}(F)) = 1$ but $F(0) = F(1)$. 
Jacobian Conjecture

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(1) $\det(\text{Jac}(F)) \in k^*$
(2) $p \nmid [k(X_1, \ldots, X_n) : k(F_1, \ldots, F_n)]$,
is $F$ invertible?
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is $F$ invertible?

$$F : (X, Y) \longrightarrow (X + X^p, Y):$$
$$\left[ k(X, Y) : k(X + X^p, Y) \right] = p.$$
\[ \text{char}(k) = 0 : \]
\[ F = (X + a_1 X^2 + a_2 XY + a_3 Y^2, Y + b_1 X^2 + b_2 XY + b_3 Y^2) \]

\[ 1 = \det(\text{Jac}(F)) \]
\[ = 1 + \\
(2a_1 + b_2)X + \\
(a_2 + 2b_3)Y + \\
(2a_1 b_2 + 2a_2 b_1)X^2 + \\
(2b_2 a_2 + 4a_1 b_3 + 4a_3 b_1)XY + \\
(2a_2 b_3 + 2a_3 b_2)Y^2 \]

In char(k)=2 : (parts of) equations vanish. What are the right equations in char(k)=2(p)?
The Automorphism Group

(This whole talk: \( n \geq 2 \))

\( GL_n(k) \) is generated by
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- Permutations \( X_1 \leftrightarrow X_i \)
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- Map \( (aX_1 + bX_j, X_2, \ldots, X_n) \) \( (a \in k^*, b \in k) \)
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- Permutations $X_1 \leftrightarrow X_i$

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$GA_n(k)$ is generated by ???
**Elementary map:** \((X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)\), invertible with inverse

\((X_1 - f(X_2, \ldots, X_n), X_2, \ldots, X_n)\).
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**Triangular map:** \((X + f(Y, Z), Y + g(Z), Z + c)\)

\[= (X, Y, Z + c)(X, Y + g(Z), Z)(X + f(X, Y), Y, Z)\]
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\(J_n(k) := \text{set of triangular maps.}\)

\(\text{Aff}_n(k) := \text{set of compositions of invertible linear maps and translations.}\)
**Elementary map:** $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$, invertible with inverse

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$J_n(k):=$ set of triangular maps.

$Aff_n(k):=$ set of compositions of invertible linear maps and translations.

$TA_n(k) := < J_n(k), Aff_n(k) >$
In dimension 1: we understand the automorphism group. (They are linear.)
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In dimension 2: famous Jung-van der Kulk-theorem:

\[ \text{GA}_2(\mathbb{K}) = \text{TA}_2(\mathbb{K}) = \text{Aff}_2(\mathbb{K}) \rtimes J_2(\mathbb{K}) \]

Jung-van der Kulk is the reason that we can do a lot in dimension 2 !!!!
What about dimension 3?
What about dimension 3? Stupid idea: everything will be tame?

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N := (X - Y \Delta - Z \Delta^2, Y + Z \Delta, Z) \quad \text{where} \quad \Delta = XZ + Y^2.
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1972: Nagata: “I cannot tame the following map:”

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(Confusing and technical proof.) (2007 AMS Moore paper award.)
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$$(X - z^{-1}Y^2, Y)(X, Y + z^2X)(X + z^{-1}Y^2, Y)$$

$$= (X - 2(Xz + Y^2)Y - (Xz + Y^2)^2z, Y + (Xz + Y^2)z)$$
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Thus: $N$ is tame over $k[z, z^{-1}]$, i.e. $N$ in $TA_2(k[z, z^{-1}])$. 
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Thus: $N$ is tame over $k[z, z^{-1}]$, i.e. $N$ in $TA_2(k[z, z^{-1}])$.
Nagata proved: $N$ is NOT tame over $k[z]$, i.e. $N$ not in $TA_2(k[z])$. 
What about $TA_n(k) \subseteq GA_n(k)$ if $k = \mathbb{F}_q$ is a finite field?
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Denote $\text{Bij}_n(\mathbb{F}_q)$ as set of bijections on $\mathbb{F}_q^n$. We have a natural map

$\text{GA}_n(\mathbb{F}_q) \xrightarrow{\pi} \text{Bij}_n(\mathbb{F}_q)$. 

Simpler question: what is $\pi(\text{GA}_n(\mathbb{F}_q))$? Can we make every bijection on $\mathbb{F}_q^n$ as an invertible polynomial map?
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Simpler question: what is $\pi(\text{TA}_n(\mathbb{F}_q))$?

Why simpler? Because we have a set of generators!
Question: what is $\pi(\text{TA}_n(F_q))$?

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See $Bij_n(\mathbb{F}_q)$ as $\text{Sym}(q^n)$.

$TA_n(\mathbb{F}_q) = < GL_n(\mathbb{F}_q), \sigma_f >$ where $f$ runs over $\mathbb{F}_q[X_2, \ldots, X_n]$ and $\sigma_f := (X_1 + f, X_2, \ldots, X_n)$. 
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See $\text{Bij}_n(\mathbb{F}_q)$ as $\text{Sym}(q^n)$.

$\text{TA}_n(\mathbb{F}_q) = \langle \text{GL}_n(\mathbb{F}_q), \sigma_f \rangle$ where $f$ runs over $\mathbb{F}_q[X_2, \ldots, X_n]$ and $\sigma_f := (X_1 + f, X_2, \ldots, X_n)$.

We make finite subset $S \subset \mathbb{F}_q[X_2, \ldots, X_n]$ and define

$$\mathcal{G} := \langle \text{GL}_n(\mathbb{F}_q), \sigma_f ; f \in S \rangle$$

such that

$$\pi(\text{TA}_n(\mathbb{F}_q)) = \pi(\mathcal{G}).$$
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If $q = 2$ or $q$ odd, then indeed we find a 2-cycle!
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Hence if $q = 2$ or $q = \text{odd}$, then $\pi(T_n(\mathbb{F}_q)) = \text{Sym}(q^n)$. 

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If $q = 4, 8, 16, \ldots$ we don’t succeed to find a 2-cycle.
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If $q = 4, 8, 16, \ldots$ we don’t succeed to find a 2-cycle. In fact-all generators of $\text{TA}_n(\mathbb{F}_q)$ turn out to be even, i.e.

$\pi(\text{TA}_n(\mathbb{F}_q)) \subseteq \text{Alt}(q^n)$!

But: there’s another theorem:

**Theorem:** $H < \text{Sym}(m)$ Primitive $+ 3$-cycle $\longrightarrow H = \text{Alt}(m)$
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Suppose $F \in \text{GA}_n(\mathbb{F}_4)$ such that $\pi(F)$ odd permutation, then $\pi(F) \notin \pi(\text{TA}_n(\mathbb{F}_4))$, so $\text{GA}_n(\mathbb{F}_4) \neq \text{TA}_n(\mathbb{F}_4)$!
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So: Start looking for an odd automorphism!!! (Or prove they don’t exist)
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**Problem:** Do there exist “odd” polynomial automorphisms over $\mathbb{F}_4$?
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Problem: Do there exist “odd” polynomial automorphisms over $\mathbb{F}_4$? Exciting! Let’s try Nagata!

$$N = \left(\begin{array}{c}
X - 2(XZ + Y^2)Y - (XZ + Y^2)^2Z, \\
Y + (XZ + Y^2)Z, \\
Z
\end{array}\right)$$
Question: what is $\pi(T_n(\mathbb{F}_q))$?

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$= q(2q - 1)$. 

Hence, $N$ is even!
Question: what is $\pi(T_n(\mathbb{F}_q))$?

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$= q(2q - 1)$. Hence, $N$ exchanges $q^3 - q(2q - 1)$ elements - that means $\frac{q^3 - q(2q - 1)}{2}$ 2-cycles.
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$$\#\{(x, y, z) \mid z = 0 \text{ or } x = z^{-1}y^2\} = q^2 + (q - 1)q = q(2q - 1) \Rightarrow N \text{ exchanges } q^3 - q(2q - 1) \text{ elements -}

\text{that means } \frac{q^3-q(2q-1)}{2} \text{ 2-cycles. Which is odd } \iff q \text{ odd or } q = 2 \ldots$$
Question: what is \( \pi(T_n(\mathbb{F}_q)) \)?

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Hence, \( N \) exchanges \( q^3 - q(2q - 1) \) elements - that means \( q^3 - q(2q - 1) \) 2-cycles. Which is odd \( \iff \) \( q \) odd or \( q = 2 \ldots \) Hence, \( N \) is even!
Is there perhaps a combinatorial reason why $\pi(GA_n(\mathbb{F}_4))$ has only even permutations??
Another idea: study the bijections of $\mathbb{F}_9^n$ given by elements of $GA_n(\mathbb{F}_3)$. 

Then study the bijection of $\mathbb{F}_3^9$ given by Nagata - is this bijection in the group $\pi_9(TA_3(\mathbb{F}_3))$?

We put it all in the computer (joint work with R. Willems):. . .

Unfortunately, yes $\pi_9(N)$ is in $\pi_9(TA_3(\mathbb{F}_3))$.

In fact:

Corollary (of some theorem I proved) Let $F \in GA_2(\mathbb{F}_q[Z])$. Then $F$ is tamely mimickable.
Another idea: study the bijections of $\mathbb{F}_9^n$ given by elements of $\text{GA}_n(\mathbb{F}_3)$.

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$$
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$$

$$
\bigcup | \quad \bigcup |
$$

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**Corollary**

*(of some theorem I proved)* Let $F \in \text{GA}_2(\mathbb{F}_q[Z])$. Then $F$ is tamely mimickable.
Nagata can be mimicked by a tame map for every $q = p^m$ - i.e. exists $F \in TA_3(\mathbb{F}_p)$ such that $\pi_q N = \pi_q F$. 
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$$(X - z^{-1}Y^2, Y)(X, Y + z^2X), (X + z^{-1}Y^2, Y) = (X - 2\Delta Y - \Delta^2 z, Y + \Delta z)$$
Nagata can be mimicked by a tame map for every \( q = p^m \) - i.e. exists \( F \in TA_3(\mathbb{F}_p) \) such that \( \pi_q N = \pi_q F \). Proof is easy once you realize where to look... Remember Nagata’s way of making Nagata map?

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Do the Big Trick, since for \( z \in \mathbb{F}_q \) we have \( z^q = z \):
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$$(X - z^{q-2}Y^2, Y)(X, Y + z^2X), (X + z^{q-2}Y^2, Y) = (X - 2\Delta Y - \Delta^2 z, Y + \Delta z)$$

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$$= (X - 2\Delta Y - \Delta^2 z, Y + \Delta z)$$

Do the Big Trick, since for $z \in \mathbb{F}_q$ we have $z^q = z$:

This almost works - a bit more wiggling necessary (And for the general case, even more work.)
Another idea: define $MA_n^d(k) := \{ F \in MA_n(k) \mid \deg(F) \leq d \}$. If $k = \mathbb{F}_q$, then this is finite.
Another idea: define $MA^d_n(k) := \{ F \in MA_n(k) \mid \text{deg}(F) \leq d \}$. If $k = \mathbb{F}_q$, then this is finite. Now compute $GA^d_n(\mathbb{F}_q) := GA_n(\mathbb{F}_q) \cap MA^d_n(\mathbb{F}_q)$ by checking all $F \in MA^d_n(k)!$ We find ALL automorphisms of degree $\leq d$. Will we find new ones we didn’t know before?
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Let’s not be too ambitious: $n = 3$. And $q = 2, 3, 4, 5$.

Computable is (R. Willems):
$GA_3^2(\mathbb{F}_{2,3,4,5})$ and main part of $GA_3^3(\mathbb{F}_2)$. Surprisingly, results seem to be intersting!
Another idea: define $MA^d_n(k) := \{F \in MA_n(k) \mid \text{deg}(F) \leq d\}$. 
If $k = \mathbb{F}_q$, then this is finite. Now compute 
$GA^d_n(\mathbb{F}_q) := GA_n(\mathbb{F}_q) \cap MA^d_n(\mathbb{F}_q)$ by checking all $F \in MA^d_n(k)$! We find ALL automorphisms of degree $\leq d$. Will we find new ones we didn’t know before?

Let’s not be too ambitious: $n = 3$. And $q = 2, 3, 4, 5$.
Computable is (R. Willems):
$GA^2_3(\mathbb{F}_{2,3,4,5})$ and main part of $GA^3_3(\mathbb{F}_2)$. Surprisingly, results seem to be interesting!

**Observation:** $F \in GA^2_3(\mathbb{F}_q)$ seems to be $\in TA_3(\mathbb{F}_q)$, always. No idea why!
Also interesting: set of endomorphisms that induce bijections.
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I.e. computed: $\mathcal{B}(\mathbb{F}_2)_3^2 := \text{set of } F = X + H \in \text{MA}_3^2(\mathbb{F}_2) \text{ for which } F \text{ induces a bijection of } \mathbb{F}_2^3$. 
Also interesting: set of endomorphisms that induce bijections.
I.e. computed: \( \mathcal{B}(\mathbb{F}_2)^2_3 := \text{set of } F = X + H \in MA^2_3(\mathbb{F}_2) \text{ for which } F \text{ induces a bijection of } \mathbb{F}_2^3. \)
\#\mathcal{B}(\mathbb{F}_2)^2_3 = 336.
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We say \( B, B' \in \mathcal{B} \) are equivalent if exists \( F \in GA_3(\mathbb{F}_2) \) such that \( FB = B' \). It seems there are 4 such equivalence classes:
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\[
\begin{align*}
(X, Y, Z) & \quad 176, \text{ all tame!} \\
(X^8 + X^4 + X, Y, Z) & \quad 56 \\
(X^8 + X^2 + X, Y, Z) & \quad 56 \\
(X^4 + X^2 + X, Y, Z) & \quad 48
\end{align*}
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Everything is equivalent to 1-variable permutation polynomials.
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\end{align*}
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Everything is equivalent to 1-variable permutation polynomials. Degree 3: 1520 permutation polynomials, 400 equiv. to \((X, Y, Z)\) - again all tame. (In progress.)
Another “characteristic 2” anomaly: compare

\[ \text{GTAM}_n(k) := \text{normal closure of } \text{TA}_n(k) \text{ in } \text{GA}_n(k) \]
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$\text{GTAM}_n(k) := \text{normal closure of } \text{TA}_n(k) \text{ in } \text{GA}_n(k)$

$\cup$

$\text{GLIN}_n(k) := \text{normal closure of } \text{GL}_n(k) \text{ in } \text{GA}_n(k)$

\[\text{QUESTION 1: Is } \text{GLIN}_n(k) = \text{GTAM}_n(k)\]

\[\text{QUESTION 2: Is } \text{N(Nagata)} \text{ in } \text{GTAM}_n(k)\]

\[\text{ANSWER 1: YES if you can find invertible linear map } (aX_1, X_2, \ldots, X_n) \text{ where } a \neq 1.\]

\[\text{I.e. if } k \neq F_2\ldots\]

An intelligent computation yields: answer NO if $k = F_2\ldots$
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∪|
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ANSWER 1: YES if \(k \neq \mathbb{F}_2\), NO if \(k = \mathbb{F}_2\).

ANSWER 2:
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ANSWER 1: YES if \( k \neq \mathbb{F}_2 \), NO if \( k = \mathbb{F}_2 \).
ANSWER 2: Recent result: Nagata is shifted linearizable:
Exists linear map \( L \), and \( \varphi \in GA_n(k) \) such that \( \varphi^{-1}LN\varphi = L \).
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**ANSWER 2:** Recent result: Nagata is *shifted linearizable*:

Exists linear map \( L \), and \( \varphi \in \text{GA}_n(k) \) such that \( \varphi^{-1}LN\varphi = L \).

... If \( k \neq \mathbb{F}_2 \), of course...
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(NO would imply \( N \not\in \text{TA}_3(\mathbb{F}_2) \).)
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**THANK YOU** for enduring all those slides.
Theorem:

\[ \text{GLIN}_n(\mathbb{F}_2) \nsubseteq \text{GTAM}_n(\mathbb{F}_2). \]
Theorem:

$$\text{GLIN}_n(\mathbb{F}_2) \nsubseteq \text{GTAM}_n(\mathbb{F}_2).$$

Proof.
**Theorem:**

$$\text{GLIN}_n(\mathbb{F}_2) \not\subseteq \text{GTAM}_n(\mathbb{F}_2).$$

**Proof.** Remember, $\pi_2(\text{TA}_n(\mathbb{F}_2)) = \text{Sym}(2^n)$, as $\mathbb{F}_2$ was the exception to the exception.
Theorem:

$$\text{GLIN}_n(F_2) \not\subseteq \text{GTAM}_n(F_2).$$

Proof. Remember, $$\pi_2(TA_n(F_2)) = \text{Sym}(2^n),$$ as $$F_2$$ was the exception to the exception.

Now, notice that if $$n \geq 3,$$ then any element of $$\text{GL}_n(F_2)$$ is even.
Theorem:

\( \text{GLIN}_n(\mathbb{F}_2) \nsubseteq \text{GTAM}_n(\mathbb{F}_2) \).

**Proof.** Remember, \( \pi_2(TA_n(\mathbb{F}_2)) = \text{Sym}(2^n) \), as \( \mathbb{F}_2 \) was the exception to the exception.

Now, notice that if \( n \geq 3 \), then any element of \( \text{GL}_n(\mathbb{F}_2) \) is even. Hence \( \pi_2(\text{GLIN}_n(\mathbb{F}_2)) \subseteq \text{Alt}(2^n) \). If \( n = 2 \), then \( (X + Y, Y) \) is odd, unfortunately.
**Theorem:**

\[ \text{GLIN}_n(\mathbb{F}_2) \nsubseteq \text{GTAM}_n(\mathbb{F}_2). \]

**Proof.** Remember, \( \pi_2(TA_n(\mathbb{F}_2)) = \text{Sym}(2^n) \), as \( \mathbb{F}_2 \) was the exception to the exception.

Now, notice that if \( n \geq 3 \), then any element of \( \text{GL}_n(\mathbb{F}_2) \) is even. Hence \( \pi_2(\text{GLIN}_n(\mathbb{F}_2)) \subseteq \text{Alt}(2^n) \). If \( n = 2 \), then \((X + Y, Y)\) is odd, unfortunately. However, in dimension 2 we understand the automorphism group, and can do a computer calculation.
Theorem:

$$\text{GLIN}_n(F_2) \subseteq \text{GTAM}_n(F_2).$$

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Now, notice that if $$n \geq 3$$, then any element of $$\text{GL}_n(F_2)$$ is even. Hence $$\pi_2(\text{GLIN}_n(F_2)) \subseteq \text{Alt}(2^n).$$ If $$n = 2$$, then $$(X + Y, Y)$$ is odd, unfortunately. However, in dimension 2 we understand the automorphism group, and can do a computer calculation to see that

$$\frac{\#\pi_4(\text{GLIN}_2(F_2))}{\#\pi_4(\text{GTAM}_2(F_2))} = 2.$$ 

End proof.
Conclusions
Conclusions

* $\pi_q(\text{TA}_n(\mathbb{F}_q)) = \text{Sym}(q^n)$ if $q$ odd, $q = 2$.
* $\pi_q(\text{TA}_n(\mathbb{F}_q)) = \text{Alt}(q^n)$ if $q = 2^m$, $m \geq 2$. 
Conclusions

- $\pi_q(TA_n(\mathbb{F}_q)) = \text{Sym}(q^n)$ if $q$ odd, $q = 2$.
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  $\text{GLIN}_n(\mathbb{F}_2) \not\subseteq \text{GTAM}_n(\mathbb{F}_2)$... but
  $\text{GTAM}_n(\mathbb{F}_2) \subseteq \text{GLIN}_{n+1}(\mathbb{F}_2)$

*** THANK YOU ***
(for watching 175 slides... )
Conclusions

- \( \pi_q(TA_n(F_q)) = \text{Sym}(q^n) \) if \( q \) odd, \( q = 2 \).
- \( \pi_q(TA_n(F_q)) = \text{Alt}(q^n) \) if \( q = 2^m, \ m \geq 2 \).

- \( \text{GLIN}_n(F_q) = \text{GTAM}_n(F_q) \) if \( q \neq 2 \).
  - \( \text{GLIN}_n(F_2) \subset \subsetneq \text{GTAM}_n(F_2) \) . . . but
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- Nagata in \( \text{GTAM}_n(k) \) if \( k \neq F_2 \). If \( k = F_2 \) we don’t know. Yet.
Conclusions

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- More research is needed in \( \text{char}(k) = p \), which is a very unexplored topic for polynomial automorphisms - but apparently very very powerful! (Belov-Kontsjevich)
Conclusions

- $\pi_q(\text{TA}_n(\mathbb{F}_q)) = \text{Sym}(q^n)$ if $q$ odd, $q = 2$.
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- Nagata in $\text{GTAM}_n(k)$ if $k \neq \mathbb{F}_2$. If $k = \mathbb{F}_2$ we don't know. Yet.

- More research is needed in $\text{char}(k) = p$, which is a very unexplored topic for polynomial automorphisms - but apparently very powerful! (Belov-Kontsjevich)

*** THANK YOU ***
MOTIVATION:

Why study polynomial maps over finite fields, and not be a normal person and do the "C" thing?

REASON 1: Reduction-mod-


REASON 2: Polynomial maps over finite fields may have applications in discrete-mathematics like settings!
MOTIVATION:

Why study polynomial maps over finite fields, and not be a normal person and do the “$\mathbb{C}$” thing?
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REASON 1:
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Why study polynomial maps over finite fields, and not be a normal person and do the “\(\mathbb{C}\)” thing?

REASON 1: Reduction-mod-\(p\) techniques to solve problems over \(\mathbb{C}\).
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REASON 1: Reduction-mod-$p$ techniques to solve problems over $\mathbb{C}$. Classical example: an injective polynomial map is surjective.
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Why study polynomial maps over finite fields, and not be a normal person and do the “$\mathbb{C}$” thing?

REASON 1: Reduction-mod-$p$ techniques to solve problems over $\mathbb{C}$. Classical example: an injective polynomial map is surjective. Reason: an injective map on finite set is surjective.
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REASON 2: Polynomial maps over finite fields may have applications in discrete-mathematics like settings!
RE-MOTIVATION:

Why **NOT** study polynomial maps over finite fields! In fact, why didn’t anyone fill that **gaping hole** yet!

REASON 1: Reduction-mod-$p$ techniques to solve problems over $\mathbb{C}$. Classical example: an injective polynomial map is surjective. Reason: an injective map from a finite set to a finite. Very recent: Belov-Kontsjevich (yes, that guy) proved equivalence of two already long-standing conjectures: the Dixmier Conjecture (’68) and the Jacobian Conjecture (’39).

REASON 2: Polynomial maps over finite fields may have applications in discrete-mathematics like settings! (In fact, one of the reasons for this talk is the hope that there may be one or two of you in the audience who may see such a possible application!)
$GA_n(k)$

$TA_n(k)$
\[ \text{GA}_n(k) \]

\[ \cup \]

\[ \text{LF}_n(k) := \{ F \in \text{GA}_n(k) \mid \deg(F^m) \text{ bounded} \} \]

\[ \cup \]

\[ \text{ELFD}_n(k) := \{ \exp(D) \mid D \text{ locally finite derivation} \} \]

\[ \cup \]

\[ \text{TA}_n(k) \]
\[ \text{GA}_n(k) \cup \text{LF}_n(k) := \langle F \in \text{GA}_n(k) \mid \text{deg}(F^m) \text{ bounded} \rangle \]

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\[ ? \cup ? \ 	ext{not equal if } \text{char}(k) = 0. \]

\[ \text{TA}_n(k) \]
$\mathcal{G}_{n}(k)$
$\cup$
$\mathcal{L}_{n}(k) := \{ F \in \mathcal{G}_{n}(k) \mid \text{deg}(F^m) \text{ bounded} \}$
$\cup$
$\mathcal{E}_{n}(k) := \{ \exp(D) \mid D \text{ locally finite derivation} \}$
$\cup$
$\mathcal{G}_{n}(k) := \text{normalization of } \mathcal{G}_{n}(k)$
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$\cup$
$\mathcal{G}_{n}(k) := \text{not equal if } \text{char}(k) = 0.$
\[ GA_n(k) \]
\[ \cup \]
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\[ \cup \]
\[ GTAM_n(k) := \text{normalization of } TA_n(k) \]
\[ \cup \]
\[ GLIN_n(k) := \text{normalization of } GL_n(k) \]
\[ \not= \text{ if } \text{char}(k) = 0. \]
\[ TA_n(k) \]
Where in these groups is Nagata?
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No conjugate of Nagata is in $GL_n(k)$ for any field $k$!
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But: recent result: Nagata is *shifted linearizable*:
Where in these groups is Nagata?
No conjugate of Nagata is in $\text{GL}_n(k)$ for any field $k$.
But: recent result: Nagata is \textit{shifted linearizable}: choose $s \in k$ such that $s \neq 0, 1, -1$. 
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$$(s \exp(D))$$
Where in these groups is Nagata?

No conjugate of Nagata is in $\text{GL}_n(k)$ for any field $k$!

But: recent result: Nagata is *shifted linearizable*: choose $s \in k$ such that $s \neq 0, 1, -1$.

$$\exp\left(\frac{-s^2}{1 - s^2} D\right) \left( s \exp(D) \right) \exp\left(\frac{s^2}{1 - s^2} D\right)$$
Where in these groups is Nagata?
No conjugate of Nagata is in $\text{GL}_n(k)$ for any field $k$!
But: recent result: Nagata is \emph{shifted linearizable}: choose $s \in k$ such that $s \neq 0, 1, -1$.

$$\exp\left(\frac{-s^2}{1-s^2}D\right)\left(s \exp(D)\right)\exp\left(\frac{s^2}{1-s^2}D\right) = sl$$
Where in these groups is Nagata?

No conjugate of Nagata is in $GL_n(k)$ for any field $k$!

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\[
\exp\left(-\frac{s^2}{1-s^2}D\right)(s \exp(D)) \exp\left(\frac{s^2}{1-s^2}D\right) = sl
\]

Hence: Nagata map is in $GLIN_3(k)$!
Where in these groups is Nagata?
No conjugate of Nagata is in $\text{GL}_n(k)$ for any field $k$!
But: recent result: Nagata is \textit{shifted linearizable}: choose $s \in k$ such that $s \neq 0, 1, -1$.

$$\exp\left(-\frac{s^2}{1-s^2}D\right)(s \exp(D)) \exp\left(\frac{s^2}{1-s^2}D\right) = sl$$

Hence: Nagata map is in $\text{GLIN}_3(k)$! - If $k \neq \mathbb{F}_2, \mathbb{F}_3$, that is!!
How does $\text{GLIN}_n(k)$ compare to $\text{GTAM}_n(k)$?
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As soon as $(X_1 + f(X_2), X_2, \ldots, X_n) \in \text{GLIN}_n(k)$ for any $f \in k[X_2]$, then $\text{GLIN}_n(k) = \text{GTAM}_n(k)$. 
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\[(aX, Y)\]
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(X - bf(Y), Y)(aX, Y)(X + bf(Y), Y)
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f \in k[X_2], then \(\text{GLIN}_n(k) = \text{GTAM}_n(k)\). Choose some \(a \neq 0\): 

\[(a^{-1}X, Y)(X - bf(Y), Y)(aX, Y)(X + bf(Y), Y)\]
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\[(X + b(1 - a^{-1})f(Y), Y)\]
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Choose $b = (1 - a^{-1})^{-1}$. 

We will get back to that.
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\[Question:\text{How does }\text{GLIN}_n(F_2)\text{ and }\text{GTAM}_n(F_2)\text{ relate? We will Get Back To That.\ldots }\]
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$$(X + b(1 - a^{-1})f(Y), Y)$$

Choose $b = (1 - a^{-1})^{-1}$. Then $(X + f(Y), Y))$ in $\text{GLIN}_2(k)$!
How does $\text{GLIN}_n(k)$ compare to $\text{GTAM}_n(k)$?

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... if $k \neq \mathbb{F}_2$...
How does GLIN\(_n(k)\) compare to GTAM\(_n(k)\)?

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\[(X + b(1 - a^{-1})f(Y), Y)\]

Choose \(b = (1 - a^{-1})^{-1}\). Then \((X + f(Y), Y)\) in \(\text{GLIN}_2(k)\)!

...if \(k \neq \mathbb{F}_2\)...

**Question:** How does GLIN\(_n(\mathbb{F}_2)\) and GTAM\(_n(\mathbb{F}_2)\) relate?
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**Question:** How does $\text{GLIN}_n(\mathbb{F}_2)$ and $\text{GTAM}_n(\mathbb{F}_2)$ relate? We will Get Back To That...