The mysteries of
Affine Algebraic Geometry

Stefan Maubach

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What is Affine Algebraic Geometry?

It is a subfield of Algebraic Geometry! (Duh!)

Typical objects:

\[ k^n \leftrightarrow k[ X_1, \ldots, X_n ] \]

\[ V \leftrightarrow \mathcal{O}(V) := k[ X_1, \ldots, X_n ] / \mathcal{I}(V) \]

Geometrically sometimes "more difficult" than projective geometry (affine spaces are rarely compact).

Algebraically, more simple! (There's always a ring.)

Subtopic - but of fundamental importance to the whole of Algebraic geometry.
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Objects, hence morphisms!

$$F : k^n \longrightarrow k^n$$

polynomial map if $F = (F_1, \ldots, F_n), F_i \in k[X_1, \ldots, X_n]$.

Example: $F = (X + Y^2, Y)$ is polynomial map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$. 
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Example: \( F = (X + Y^2, Y) \) is polynomial map \( \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \).

Any linear map is a polynomial map.
Understanding polynomial automorphisms

A map $F : \mathbb{k}^n \rightarrow \mathbb{k}^n$ given by $n$ polynomials:

$$F = (F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n)).$$

Example:

$$F = (X + Y^2, Y).$$

Various ways of looking at polynomial maps:

$\Rightarrow A$ map $\mathbb{k}^n \rightarrow \mathbb{k}^n$.

$\Rightarrow A$ list of $n$ polynomials: $F \in (\mathbb{k}[X_1, \ldots, X_n])^n$.

$\Rightarrow A$ ring automorphism of $\mathbb{k}[X_1, \ldots, X_n]$ sending $g(X_1, \ldots, X_n)$ to $g(F_1, \ldots, F_n)$. 

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A polynomial map $F$ is a polynomial automorphism if there is a polynomial map $G$ such that $F(G) = (X_1, \ldots, X_n)$.
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Example: $(X + Y^2, Y)$ has inverse $(X - Y^2, Y)$. 
Understanding polynomial automorphisms

A polynomial map $F$ is a **polynomial automorphism** if there is a polynomial map $G$ such that $F(G) = (X_1, \ldots, X_n)$.

Example: $(X + Y^2, Y)$ has inverse $(X - Y^2, Y)$.

\[
(X + Y^2, Y) \circ (X - Y^2, Y) = ([X - Y^2] + [Y]^2, [Y]) \\
= (X - Y^2 + Y^2, Y) \\
= (X, Y).
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$(X^p, Y) : \mathbb{F}_p^2 \longrightarrow \mathbb{F}_p^2$ is not a polynomial automorphism, even though it induces a bijection of $\mathbb{F}_p$!
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$(X^p, Y) : \mathbb{F}_p^2 \longrightarrow \mathbb{F}_p^2$ is not a polynomial automorphism, even though it induces a bijection of $\mathbb{F}_p$!

$(X^3, Y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is not a polynomial automorphism, even though it induces a bijection of $\mathbb{R}$!
Remark: If $k$ is algebraically closed, $\text{char}(k) = 0$, then a polynomial endomorphism $k^n \rightarrow k^n$ which is a bijection, is an invertible polynomial map.

$(X^p, Y) : \mathbb{F}_p^2 \rightarrow \mathbb{F}_p^2$ is not a polynomial automorphism, even though it induces a bijection of $\mathbb{F}_p$!

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Understanding polynomial automorphisms

Group of polynomial automorphisms with coefficients in a ring $R$ is denoted by $GA_n(R)$ (similarly to $GL_n(R)$).
A topic is defined by its problems.

Many problems in AAG: inspired by linear algebra!
(In some sense: AAG most "natural generalization of linear algebra"… )
Problems in AAG: Jacobian Conjecture

char($k$) = 0

$L$ linear map;

$L \in \text{GL}_n(k)$ invertible $\iff$ det($L$) = det(Jac($L$)) $\in k^*$
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$G \circ F = (X_1, \ldots, X_n)$. 
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$F$ invertible, i.e.

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\[ \text{det(Jac}(F)) \in k[X_1, \ldots, X_n]^* = k^*. \]
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\[ F \in \text{GA}_n(k) \text{ invertible} \Rightarrow \det(\text{Jac}(F)) \in k^* \]

**Jacobian Conjecture:**

\[ F \in \text{GA}_n(k) \text{ invertible} \iff \det(\text{Jac}(F)) \in k^* \]
History of the Jacobian Conjecture

B. Segre proved the general case in 1956. And again in 1960. All wrong! — but it took about till 1970 that it was clear that the problem was open.

By the way, many, many, wrong proofs followed...
History of the Jacobian Conjecture

Formulated in 1939 By O. H. Keller.
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History of the Jacobian Conjecture

J.C. was advertised by Abhyankar, Bass, and others
“Visual” version of Jacobian Conjecture

Volume-preserving polynomial maps are invertible.

Figure: Image of raster under \((X + \frac{1}{2}Y^2, Y + \frac{1}{6}(X + \frac{1}{2}Y^2)^2)\).
Jacobian Conjecture very particular for polynomials:

\[
F : (x, y) \rightarrow (e^x, ye^{-x})
\]

\[
\text{Jac}(F) = \begin{pmatrix} e^x & 0 \\ -ye^{-x} & e^{-x} \end{pmatrix}
\]

\[
\det(\text{Jac}(F)) = 1
\]
Jacobian Conjecture in \( \text{char}(k) = p \):

- \( L \) linear map;
  - \( L \in \text{GL}_n(k) \) invertible \iff \( \det(L) = \det(\text{Jac}(L)) \in k^* \)
  - \( F \in \text{GA}_n(k) \) invertible \implies \( \det(\text{Jac}(F)) \in k^* \)
Jacobian Conjecture in char\((k) = p\):

\(L\) linear map;
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\(F \in \text{GA}_n(k)\text{ invertible} \implies \det(\text{Jac}(F)) \in k^*\)

\[F : k^1 \longrightarrow k^1\]
\[X \longrightarrow X - X^p\]

\(\text{Jac}(F) = 1\) but \(F(0) = F(1) = 0\).
Jacobian Conjecture in char$(k) = p$:

$L$ linear map;
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$F \in \text{GA}_n(k)$ invertible $\Rightarrow$ $\det(\text{Jac}(F)) \in k^*$

$$F : k^1 \longrightarrow k^1$$
$$X \longrightarrow X - X^p$$

$\text{Jac}(F) = 1$ but $F(0) = F(1) = 0$.

**Jacobian Conjecture in char$(k) = p$:** Suppose $\det(\text{Jac}(F)) = 1$ and $p \nmid [k(X_1, \ldots, X_n) : k(F_1, \ldots, F_n)]$. Then $F$ is an automorphism.
Jacobian Conjecture in char($k$) = $p$:

char($k$) = 0:

$$F = (X + a_1 X^2 + a_2 XY + a_3 Y^2, Y + b_1 X^2 + b_2 XY + b_3 Y^2)$$

$$1 = \det(\text{Jac}(F))$$

$$= 1 +$$

$$(2a_1 + b_2)X +$$

$$(a_2 + 2b_3)Y +$$

$$(2a_1 b_2 + 2a_2 b_1)X^2 +$$

$$(2b_2 a_2 + 4a_1 b_3 + 4a_3 b_1)XY +$$

$$(2a_2 b_3 + 2a_3 b_2)Y^2$$

In char($k$) = 2: (parts of) equations vanish. **Question:** What are the right equations in char($k$) = 2? (or $p$?)
Enough about the Jacobian Problem! Another problem:

Cancellation problem
Cancellation problem: introduction

$V, W$ vector spaces, if $V \times k \cong W \times k$ then $V \cong W$.

$V$ vector space, then $V \times k \cong k^{n+1}$ implies $V \cong k^n$. 
Cancellation problem: introduction

\( V, W \) vector spaces, if \( V \times k \cong W \times k \) then \( V \cong W \).

\( V \) vector space, then \( V \times k \cong k^{n+1} \) implies \( V \cong k^n \).

\( V, W \) varieties, if \( V \times k \cong W \times k \) then \( V \cong W ? \)
Cancellation problem: introduction

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$V, W$ varieties, if $V \times k \cong W \times k$ then $V \cong W$?

**Cancellation problem:** $V$ variety. $V \times k \cong k^{n+1}$, is $V \cong k^n$?
Cancellation $V \times k \cong W \times k$

counterexamples

1972(?): Hoechster: over $\mathbb{R}$
Cancellation $V \times k \cong W \times k$

counterexamples

1972(?): Hoechster: over $\mathbb{R}$
1986(?): Danielewski: $V : xz + y^2 + 1 = 0$, $W : x^2z + y^2 + 1$
   (over $\mathbb{C}$)
   (Not a UFD)
Cancellation $V \times k \cong W \times k$

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2008: Finston & M.: “Best” counterexamples so far (UFD, over $\mathbb{C}$, lowest possible dimension):

$V_{n,m} := \{(x, y, z, u, v) \mid x^2 + y^3 + z^7 = 0, x^m u - y^n v - 1 = 0\}$
Cancellation $V \times k \cong W \times k$

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$V_{n,m} := \{(x, y, z, u, v) | x^2 + y^3 + z^7 = 0, x^mu - y^n v - 1 = 0\}$

2010: better examples by Dubouloz/Moser/Poloni...
Cancellation $V \times k \cong W \times k$

counterexamples

Still looking for an example where $V = k^n$!
Abhyankar-Sataye conjecture/ coordinates

Denote $\mathbb{C}[X_1, \ldots, X_n]$ as $\mathbb{C}^n$. 
Denote $\mathbb{C}[X_1, \ldots, X_n]$ as $\mathbb{C}^n$. $f \in \mathbb{C}^n$ is called a *coordinate* if there exist $f_2, \ldots, f_n$ such that

$$\mathbb{C}[f, f_2, f_3, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_n]$$
Abhyankar-Sataye conjecture/coordinates

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$$\mathbb{C}[f, f_2, f_3, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_n]$$

Or equivalently: $(f, f_2, \ldots, f_n)$ is a polynomial automorphism.
Abhyankar-Sataye conjecture/coordinates

\[ \mathbb{C}[X_1, \ldots, X_n] =: \mathbb{C}^n. \]  
\( f \) Coordinate means \((f, f_2, \ldots, f_n)\)  
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\[ \mathbb{C}[X_1, \ldots, X_n] =: \mathbb{C}^n. \text{ } f \text{ Coordinate means } (f, f_2, \ldots, f_n) \text{ automorphism.} \]

If \( f \) is a coordinate, then \( \mathbb{C}^n/(f) \cong \mathbb{C}^{n-1} \) (just take \( f = X_1 \)).
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**Abhyankar-Sathaye conjecture** (AS(n)): If \( \mathbb{C}^n/(f) \cong \mathbb{C}^{n-1} \) then \( f \) is a coordinate.
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**Abhyankar-Sathaye conjecture** (AS\((n))\): If \(\mathbb{C}^n/(f) \cong \mathbb{C}^{n-1}\) then \(f\) is a coordinate.

**Unnamed problem:** How to recognise if \(f \in \mathbb{C}^n\) is a coordinate? Is \(x + xz^2 + zy^2\) a coordinate?
Abhyankar-Sataye conjecture/coordinates

\[ \mathbb{C}[X_1, \ldots, X_n] =: \mathbb{C}^{[n]} \]. \( f \) Coordinate means \((f, f_2, \ldots, f_n)\) automorphism.

If \( f \) is a coordinate, then \( \mathbb{C}^{[n]}/(f) \cong \mathbb{C}^{[n-1]} \) (just take \( f = X_1 \)).

**Abhyankar-Sathaye conjecture** (AS(n)): If \( \mathbb{C}^{[n]}/(f) \cong \mathbb{C}^{[n-1]} \) then \( f \) is a coordinate.

**Unnamed problem:** How to recognise if \( f \in \mathbb{C}^{[n]} \) is a coordinate? Is \( x + xz^2 + zy^2 \) a coordinate?

AS(2) is true.
Linearization problem

Let $F \in \text{GA}_n(k)$. 

Important to know if there exists $\varphi \in \text{GA}_n(k)$ such that $\varphi^{-1}F\varphi \in \text{GL}_n(k)$.

Needed: $F$ has a fixed point $p$. (i.e. $(X+1, Y)$ is not linearizable.)


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Main question here:

**Linearization Problem:** Let $F^s = I$ some $s$. Is $F$ linearizable?
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Main question here:

**Linearization Problem:** Let $F^s = I$ some $s$. Is $F$ linearizable?

Proven for $n = 2$. 
The Automorphism Group

(This whole talk: \( n \geq 2 \))

\( \text{GL}_n(k) \) is generated by...
The Automorphism Group

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- Permutations \( X_1 \leftrightarrow X_i \)
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\( GA_n(k) \) is generated by ???
**Elementary map:** \((X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)\), invertible with inverse

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**Triangular map:** \((X + f(Y, Z), Y + g(Z), Z + c)\)

\[= (X, Y, Z + c)(X, Y + g(Z), Z)(X + f(X, Y), Y, Z)\]
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\(J_n(k) := \) set of triangular maps.
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\(J_n(k) := \text{set of triangular maps.}\)

\(\text{Aff}_n(k) := \text{set of compositions of invertible linear maps and translations.}\)
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\(J_n(k) :=\) set of triangular maps.

\(\text{Aff}_n(k) :=\) set of compositions of invertible linear maps and translations.

\(\text{TA}_n(k) := \langle J_n(k), \text{Aff}_n(k) \rangle\)
In dimension 1: we understand the automorphism group. (They are linear.)
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In dimension 2: famous Jung-van der Kulk-theorem:

\[
\text{GA}_2(\mathbb{K}) = \text{TA}_2(\mathbb{K}) = \text{Aff}_2(\mathbb{K}) \times \text{J}_2(\mathbb{K})
\]

Jung-van der Kulk is the reason that we can do a lot in dimension 2!
What about dimension 3?

Stupid idea: everything will be tame?

1972: Nagata: “I cannot tame the following map:
\[ N := (X - 2Y \Delta - Z \Delta, Y + Z \Delta, Z) \text{ where } \Delta = XZ + Y^2. \]

Nagata’s map is the historically most important map for polynomial automorphisms. It is a very elegant but complicated map.

AMAZING result: Umirbaev-Shestakov (2004) \( N \) is not tame!! . . . in characteristic ZERO. . . (Difficult and technical proof. ) (2007 AMS Moore paper award.)
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AMS E.H. Moore Research Article Prize

Ivan Shestakov (center) and Ualbai Umirbaev (right) with Jim Arthur.
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(X - z^{-1}Y^2, Y)(X, Y + z^2X)(X + z^{-1}Y^2, Y) \\
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How did Nagata make Nagata’s map? Study maps over $k[z, z^{-1}]$: 

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Thus: $N$ is tame over $k[z, z^{-1}]$, i.e. $N$ in $TA_2(k[z, z^{-1}])$. 

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Thus: $N$ is tame over $k[z, z^{-1}]$, i.e. $N$ in $TA_2(k[z, z^{-1}])$.
Nagata proved: $N$ is NOT tame over $k[z]$, i.e. $N$ not in $TA_2(k[z])$. 
Stably tameness

$N$ tame in one dimension higher:

$N := (X - 2Y\Delta - Z\Delta^2, Y + Z\Delta, Z, W)$ where $\Delta = XZ + Y^2$. 
Stably tameness

$N$ tame in one dimension higher:

$N := (X - 2Y \Delta - Z \Delta^2, Y + Z \Delta, Z, W)$ where $\Delta = XZ + Y^2$.

$$(X + 2YW - ZW^2, Y - ZW, Z, W) \circ$$

$$(X, Y, Z, W - \frac{1}{2} \Delta) \circ$$

$$(X - 2YW - ZW^2, Y + ZW, Z, W) \circ$$

$$(X, Y, Z, W + \frac{1}{2} \Delta)$$

$= N$
Linearizing Nagata?

(Bass, '84?) $N$ is not linearizable.
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Linearizing Nagata?

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However: \( 2N \) \((= 2I \circ N)\) is linearizable. \(-N\) is not linearizable. \(iN\) is linearizable.
Linearizing Nagata?

(Bass, ’84?) \( N \) is not linearizable.
However: \( 2N \) (\( = 2I \circ N \)) is linearizable. \( -N \) is not linearizable.
\( iN \) is linearizable.

**Theorem:** (Maubach, Poloni, ’09) \( sN \) is linearizable unless \( s = 1, -1 \).

(Part of a deeper theorem - on a Lie algebra...)
Over finite fields

What about $\text{TA}_n(k) \subseteq \text{GA}_n(k)$ if $k = \mathbb{F}_q$ is a finite field?
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Denote $\text{Bij}_n(\mathbb{F}_q)$ as set of bijections on $\mathbb{F}_q^n$. We have a natural map

$$GA_n(\mathbb{F}_q) \xrightarrow{\pi_q} \text{Bij}_n(\mathbb{F}_q).$$
Over finite fields

What about $T \mathbb{A}_n(k) \subseteq \mathbb{G} \mathbb{A}_n(k)$ if $k = \mathbb{F}_q$ is a finite field?

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What is $\pi_q(\mathbb{G} \mathbb{A}_n(\mathbb{F}_q))$? Can we make every bijection on $\mathbb{F}_q^n$ as an *invertible* polynomial map?
\[ F_1 = (x + y^2, y) \]
What about $\text{TA}_n(k) \subseteq \text{GA}_n(k)$ if $k = \mathbb{F}_q$ is a finite field?

Denote $\text{Bij}_n(\mathbb{F}_q)$ as set of bijections on $\mathbb{F}_q^n$. We have a natural map

$$\text{GA}_n(\mathbb{F}_q) \xrightarrow{\pi_q} \text{Bij}_n(\mathbb{F}_q).$$

What is $\pi_q(\text{GA}_n(\mathbb{F}_q))$? Can we make every bijection on $\mathbb{F}_q^n$ as an invertible polynomial map?
What about TA\(_n(k) \subseteq GA\(_n(k)\) if \(k = \mathbb{F}_q\) is a finite field? Denote Bij\(_n(\mathbb{F}_q)\) as set of bijections on \(\mathbb{F}_q^n\). We have a natural map
\[ GA\(_n(\mathbb{F}_q) \xrightarrow{\pi_q} \text{Bij}\(_n(\mathbb{F}_q). \]
What is \(\pi_q(GA\(_n(\mathbb{F}_q))\)? Can we make every bijection on \(\mathbb{F}_q^n\) as an *invertible* polynomial map?
Simpler question: what is \(\pi_q(TA\(_n(\mathbb{F}_q))\)?
Why simpler? Because we have a set of generators!
Question: what is $\pi_q(TA_n(\mathbb{F}_q))$?
See $Bij_n(\mathbb{F}_q)$ as $\text{Sym}(q^n)$. 
Question: what is $\pi_q(\text{TA}_n(\mathbb{F}_q))$?

See $\text{Bij}_n(\mathbb{F}_q)$ as $\text{Sym}(q^n)$.

$\text{TA}_n(\mathbb{F}_q) = \langle \text{GL}_n(\mathbb{F}_q), \sigma_f \rangle$ where $f$ runs over $\mathbb{F}_q[X_2, \ldots, X_n]$ and $\sigma_f := (X_1 + f, X_2, \ldots, X_n)$. 

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$\text{TA}_n(\mathbb{F}_q) = \langle \text{GL}_n(\mathbb{F}_q), \sigma_f \rangle$ where $f$ runs over $\mathbb{F}_q[X_2, \ldots, X_n]$ and $\sigma_f := (X_1 + f, X_2, \ldots, X_n)$.

We make finite subset $S \subset \mathbb{F}_q[X_2, \ldots, X_n]$ and define

$$\mathcal{G} := \langle \text{GL}_n(\mathbb{F}_q), \sigma_f ; f \in S \rangle$$

such that

$$\pi_q(\text{TA}_n(\mathbb{F}_q)) = \pi_q(\mathcal{G}).$$
$F_1 = (x + y^2, y)$

$F_3 = (x + 1, y)$

$F_4 = (y, x)$

$F_5 = (2x, y)$

$F_2 = (x + y, y)$
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?
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Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

(1) $\pi_q(T_n(\mathbb{F}_q)) = \pi_q(\mathcal{G})$ is 2-transitive, hence primitive.
You might know: if $H < \text{Sym}(m)$ is primitive + a 2-cycle then $H = \text{Sym}(m)$. 
Question: what is $\pi_q(T_n(F_q))$?

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If $q = 2$ or $q$ odd, then indeed we find a 2-cycle!
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If $q = 2$ or $q$ odd, then indeed we find a 2-cycle!
Hence if $q = 2$ or $q$ odd, then $\pi_q(T_n(\mathbb{F}_q)) = \text{Sym}(q^n)$. 
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q = \text{odd}$, then $\pi_q(TA_n(\mathbb{F}_q)) = \text{Sym}(q^n)$.
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q = \text{odd}$, then $\pi_q(TA_n(\mathbb{F}_q)) = \text{Sym}(q^n)$. If $q = 4, 8, 16, \ldots$ we don't succeed to find a 2-cycle.
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q = \text{odd}$, then $\pi_q(TA_n(\mathbb{F}_q)) = \text{Sym}(q^n)$. If $q = 4, 8, 16, \ldots$ we don’t succeed to find a 2-cycle. In fact-all generators of $TA_n(\mathbb{F}_q)$ turn out to be even, i.e.

$\pi_q(TA_n(\mathbb{F}_q)) \subseteq \text{Alt}(q^n)$!

But: there’s another theorem:

**Theorem:** $H < \text{Sym}(m)$ Primitive $+$ 3-cycle $\longrightarrow H = \text{Alt}(m)$ or $H = \text{Sym}(m)$. 
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

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Answer: if $q = 2$ or $q$ = odd, then $\pi_q(TA_n(\mathbb{F}_q)) = \text{Sym}(q^n)$. If $q = 4, 8, 16, \ldots$ we don’t succeed to find a 2-cycle. In fact-all generators of $TA_n(\mathbb{F}_q)$ turn out to be even, i.e.

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Hence, if $q = 4, 8, 16, \ldots$ then $\pi_q(T_n(\mathbb{F}_q)) = \text{Alt}(m)$!
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q = \text{odd}$, then $\pi_q(TA_n(\mathbb{F}_q)) = \text{Sym}(q^n)$.

Answer: if $q = 4, 8, 16, 32, \ldots$ then $\pi_q(TA_n(\mathbb{F}_q)) = \text{Alt}(q^n)$.

Suppose $F \in \text{GA}_n(\mathbb{F}_4)$ such that $\pi(F)$ odd permutation, then $\pi(F) \not\in \pi(TA_n(\mathbb{F}_4))$, so $\text{GA}_n(\mathbb{F}_4) \neq TA_n(\mathbb{F}_4)$!
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q = \text{odd}$, then $\pi_q(\text{TA}_n(\mathbb{F}_q)) = \text{Sym}(q^n)$.

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Suppose $F \in \text{GA}_n(\mathbb{F}_4)$ such that $\pi(F)$ odd permutation, then $\pi(F) \notin \pi(\text{TA}_n(\mathbb{F}_4))$, so $\text{GA}_n(\mathbb{F}_4) \neq \text{TA}_n(\mathbb{F}_4)$!

So: Start looking for an odd automorphism!!! (Or prove they don’t exist)
Question: what is $\pi_q(T_n(F_q))$?
Answer: if $q = 2$ or $q$ = odd, then $\pi_q(T_n(F_q)) = \text{Sym}(q^n)$.
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**Problem:** Do there exist “odd” polynomial automorphisms over $\mathbb{F}_4$?
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$$N = \begin{pmatrix} X - 2(XZ + Y^2)Y - (XZ + Y^2)^2Z, \\ Y + (XZ + Y^2)Z, \\ Z \end{pmatrix}$$
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Y + (XZ + Y^2)Z, \\
Z
\end{pmatrix}
\]

\ldots drumroll\ldots
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$$N = \begin{pmatrix} X - 2(XZ + Y^2)Y - (XZ + Y^2)^2Z, \\ Y + (XZ + Y^2)Z, \\ Z \end{pmatrix}$$

…drumroll… Nagata is EVEN if and only if $q = 4, 8, 16, \ldots$ and ODD otherwise…
Question: what is $\pi_q(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q = \text{odd}$, then $\pi_q(T_n(\mathbb{F}_q)) = \text{Sym}(q^n)$.

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$$N = \begin{pmatrix} X - 2(XZ + Y^2)Y - (XZ + Y^2)^2Z, \\ Y + (XZ + Y^2)Z, \\ Z \end{pmatrix}$$

...drumroll... Nagata is EVEN if and only if $q = 4, 8, 16, \ldots$ and ODD otherwise... so far: no odd example found!
Different approach?

Is there perhaps a combinatorial reason why \( \pi(GA_n(\mathbb{F}_4)) \) has only even permutations??
Losing less information: embedding $\mathbb{F}_q$ into $\mathbb{F}_{q^m}$.

$$GA_n(\mathbb{F}_q) \subset GA_n(\mathbb{F}_{q^m}) \xrightarrow{\pi_{q^m}} \text{sym}(q^{mn}).$$
Losing less information: embedding $\mathbb{F}_q$ into $\mathbb{F}_{q^m}$.

$$\text{GA}_n(\mathbb{F}_q) \subset \text{GA}_n(\mathbb{F}_{q^m}) \xrightarrow{\pi_{q^m}} \text{sym}(q^{mn}).$$
Losing less information: embedding $\mathbb{F}_q$ into $\mathbb{F}_{q^m}$.

$$\begin{align*}
\text{GA}_n(\mathbb{F}_q) & \subset \text{GA}_n(\mathbb{F}_{q^m}) \xrightarrow{\pi_{q^m}} \text{sym}(q^{mn}). \\
\text{GA}_n(\mathbb{F}_q) & \rightarrow \pi_{q^m}(\text{GA}_n(\mathbb{F}_q)) \subset \text{sym}(q^{mn}) \\
\bigcup & \bigcup \\
\text{TA}_n(\mathbb{F}_q) & \rightarrow \pi_{q^m}(\text{TA}_n(\mathbb{F}_q)) \subset \text{sym}(q^{mn})
\end{align*}$$

However:
Losing less information: embedding $\mathbb{F}_q$ into $\mathbb{F}_{q^m}$.

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\end{align*}
\]

(1) Compute $\pi_{q^m}(\text{TA}_n(\mathbb{F}_q))$, 

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$$\bigcup \bigcup$$

$$\text{TA}_n(\mathbb{F}_q) \rightarrow \pi_{q^m}(\text{TA}_n(\mathbb{F}_q)) \subset \text{sym}(q^{mn})$$

1. Compute $\pi_{q^m}(\text{TA}_n(\mathbb{F}_q))$,
2. check if $\pi_{q^m}(N) \notin \pi_{q^m}(\text{TA}_n(\mathbb{F}_q))$,
Losing less information: embedding \( \mathbb{F}_q \) into \( \mathbb{F}_{q^m} \).

\[
\begin{align*}
\text{GA}_n(\mathbb{F}_q) & \subset \text{GA}_n(\mathbb{F}_{q^m}) & \xrightarrow{\pi_{q^m}} & \text{sym}(q^{mn}) \quad \text{(1)} \\
\text{GA}_n(\mathbb{F}_q) & \longrightarrow \pi_{q^m}(\text{GA}_n(\mathbb{F}_q)) & \subset & \text{sym}(q^{mn})
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\]

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\]

(1) Compute \( \pi_{q^m}(\text{TA}_n(\mathbb{F}_q)) \),

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and hop, (3) \( \text{TA}_n(\mathbb{F}_q) \neq \text{GA}_n(\mathbb{F}_q) \) and immortal fame!
Losing less information: embedding $\mathbb{F}_q$ into $\mathbb{F}_{q^m}$.

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\text{GA}_n(\mathbb{F}_q) \subset \text{GA}_n(\mathbb{F}_{q^m}) \xrightarrow{\pi_{q^m}} \text{sym}(q^{mn}).
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However:
Mimicking Nagata’s map:

**Theorem:** (M) [- general stuff - ]

**Corollary:** For every extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$, there exists $T_m \in TA_3(\mathbb{F}_{q^m})$ such that $T_m$ “mimicks” $N$, i.e.

$$\pi_{q^m}(T_m) = \pi_{q^m}(N).$$
Mimicking Nagata’s map:

**Theorem:** (M) [ - general stuff - ]

**Corollary:** For every extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$, there exists $T_m \in TA_3(\mathbb{F}_{q^m})$ such that $T_m$ “mimicks” $N$, i.e.

$$\pi_{q^m}(T_m) = \pi_{q^m}(N).$$

Theorem states: for *practical* purposes, tame is almost always enough!
Nagata can be mimicked by a tame map for every $q = p^m$ - i.e. exists $F \in TA_3(\mathbb{F}_p)$ such that $\pi_q N = \pi_q F$. 

Proof is easy once you realize where to look. Remember Nagata's way of making Nagata map?

$$(X - z - 1 Y^2, \, Y^2) (X, \, Y + z^2 X) = (X - 2\Delta Y - \Delta z^2, \, Y + \Delta z).$$
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(X - z^{-1}Y^2, Y)(X, Y + z^2X), (X + z^{-1}Y^2, Y) = (X - 2\Delta Y - \Delta^2 z, Y + \Delta z)
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Do the Big Trick, since for $z \in \mathbb{F}_q$ we have $z^q = z$: 
Nagata can be mimicked by a tame map for every \( q = p^m \) - i.e. exists \( F \in TA_3(\mathbb{F}_q) \) such that \( \pi_q N = \pi_q F \). Proof is easy once you realize where to look... Remember Nagata’s way of making Nagata map?

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Do the Big Trick, since for $z \in \mathbb{F}_q$ we have $z^q = z$:

This almost works - a bit more wiggling necessary (And for the general case, even more work.)
Another topic: additive group actions

$G$ group, acting on $\mathbb{C}^n$ means:

$$\phi \in G \times \mathbb{C}^n \quad \text{such that} \quad \phi_{gh} = \phi_g + h \quad \text{(in a “continuous way”).}$$

Special example: $G = <C, +>$. Denoted by $G_a$.

Example:

$t \in G_a \rightarrow \phi_t := (X_1 + t, X_2, \ldots, X_n)$.

Define $D: \mathbb{C}[X_1, \ldots, X_n] \rightarrow \mathbb{C}[X_1, \ldots, X_n]$ as the ‘log’ of the action:

$$D(P) := \frac{\partial}{\partial t} \phi_t(P) \bigg|_{t=0}$$

Example:

$$\frac{\partial}{\partial t} P \bigg|_{t=0} = \frac{\partial P}{\partial X_1}$$
Another topic: additive group actions

$G$ group, acting on $\mathbb{C}^n$ means:

$\varphi_g \in G\mathcal{A}_n(\mathbb{C})$ such that $\varphi_g \varphi_h = \varphi_{g+h}$ (in a “continuous way”).
Another topic: additive group actions

$G$ group, acting on $\mathbb{C}^n$ means:

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Define \( D : \mathbb{C}[X_1, \ldots, X_n] \rightarrow \mathbb{C}[X_1, \ldots, X_n] \) as the ‘log’ of the action:

\[
D(P) := \left. \frac{\partial}{\partial t} \varphi_t(P) \right|_{t=0}
\]

**Example:**

\[
\left. \frac{\partial}{\partial t} P(X_1 + t, X_2, \ldots, X_n) \right|_{t=0} = \frac{\partial P}{\partial X_1}(X_1, X_2, \ldots, X_n)
\]
Additive group actions

Define $D : \mathbb{C}[X_1, \ldots, X_n] \rightarrow \mathbb{C}[X_1, \ldots, X_n]$ as the ‘log’ of the action:

$$D(P) := \frac{\partial}{\partial t} \varphi_t(P)|_{t=0}$$

Example:

$$\frac{\partial}{\partial t} P(X_1 + t, X_2, \ldots, X_n)|_{t=0} = \frac{\partial P}{\partial X_1}(X_1, X_2, \ldots, X_n)$$
Additive group actions

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$$\frac{\partial}{\partial t} P(X_1 + t, X_2, \ldots, X_n) \bigg|_{t=0} = \frac{\partial}{\partial X_1}(X_1, X_2, \ldots, X_n)$$

$$D := \frac{\partial}{\partial X_1}$$

and indeed:

$$\exp(tD)(P) = P(X_1 + t, X_2, \ldots, X_n)$$
Additive group actions

$D$ is a locally nilpotent derivation:

$D(fg) = fD(g) + D(f)g, \quad D(f + g) = D(f) + D(g)$

(derivation)

For all $f$, there exists an $m_f$ such that $D^{m_f}(f) = 0$. (locally nilpotent)

Example:

$$\frac{\partial}{\partial t} P(X_1 + t, X_2, \ldots, X_n)|_{t=0} = \frac{\partial P}{\partial X_1}(X_1, X_2, \ldots, X_n)$$

$D := \frac{\partial}{\partial X_1}$

and indeed:

$$\exp(tD)(P) = P(X_1 + t, X_2, \ldots, X_n)$$
Another example:

\[ \delta := -2Y \frac{\partial}{\partial X} + Z \frac{\partial}{\partial Y} \]

is locally nilpotent derivation.
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is locally nilpotent derivation.

\[ \delta(XZ) = \delta(X)Z + X\delta(Z) = -2Y \cdot Z. \]
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\[ \delta(XZ) = \delta(X)Z + X\delta(Z) = -2Y \cdot Z. \]

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Thus,

\[ \delta(XZ + Y^2) = 0 \]

\[ \delta(\Delta) = 0 \text{ where } \Delta = XZ + Y^2. \]
Another example:

\[ \delta := -2Y \frac{\partial}{\partial X} + Z \frac{\partial}{\partial Y} \]

is locally nilpotent derivation.

\[
\begin{align*}
\delta(XZ) &= \delta(X)Z + X\delta(Z) = -2Y \cdot Z. \\
\delta(Y^2) &= 2Y\delta(Y) = 2Y \cdot Z.
\end{align*}
\]

Hence,

\[
\delta(XZ + Y^2) = 0
\]

\[
\delta(\Delta) = 0 \text{ where } \Delta = XZ + Y^2.
\]

Hence: \( D := \Delta \delta \) is also an LND:

\[
D^3(X) = D^2(\Delta \cdot -2Y) = \Delta \cdot -2 \cdot D^2(Y) = \Delta \cdot -2 \cdot D(Z) = 0
\]

etc.
\[\delta := -2Y \frac{\partial}{\partial X} + Z \frac{\partial}{\partial Y},\]

\[D := \Delta \delta, \quad \Delta := XZ + Y^2\]
\[ \delta := -2Y \frac{\partial}{\partial X} + Z \frac{\partial}{\partial Y}, \]
\[ D := \Delta \delta, \quad \Delta := XZ + Y^2 \]

Now compute:

\[ \varphi_t := \exp(tD) := (\exp(tD)(X), \exp(tD)(Y), \exp(tD)(Z)) \]
\[
\delta := -2Y \frac{\partial}{\partial X} + Z \frac{\partial}{\partial Y},
\]
\[
D := \Delta \delta, \quad \Delta := XZ + Y^2
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Now compute:

\[
\varphi_t := \exp(tD) := (\exp(tD)(X), \exp(tD)(Y), \exp(tD)(Z))
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\[
\exp(tD)(X) = X + tD(X) + \frac{1}{2} t^2 D^2(X)
\]
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Now compute:

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Examine \( t = 1 \):
\begin{align*}
\exp(D)(X) &= X - 2\Delta Y - \Delta^2 Z) \\
\exp(D)(Y) &= Y + \Delta Z \\
\exp(D)(Z) &= Z
\end{align*}

Examine \( t = 1 \):
\[ \exp(D)(X) = X - 2\Delta Y - \Delta^2 Z \]
\[ \exp(D)(Y) = Y + \Delta Z \]
\[ \exp(D)(Z) = Z \]

Examine \( t = 1 \): Nagata’s automorphism!
Just one more slide:

I hope you got an impression of the beauty of Affine Algebraic Geometry!

THANK YOU
(for enduring 177 slides. . . )
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