Mathematics 1 (NWI-MOL004)

Prerequisites. 'Vwo Wiskunde B' with final grade 7 or higher is sufficient. Deficiencies can be repaired by means of the arithmetic booklet, which is strictly necessary if you score $\leq 74\%$ on the diagnostic test at the beginning of the course. At the second diagnostic test (after several weeks of mathematics education) you can demonstrate the progress that you made.

Contents. Calculation techniques (numbers, vectors, functions) and differential calculus. Mathematics is a cumulative art: you can only successfully complete a new chapter if you've mastered all preceding chapters. That's why there are several mid-term tests, so that you can't lag behind too much. Each week you'll study a new chapter from this book, consisting of

(1) theory with examples

(2) exercises with solutions to check your own work

This book contains all material that you need to master. Complicated examples usually have concise solutions with intermediate steps that you have to verify yourself on a piece of scrap paper. If you have trouble, don't hesitate to ask me, your teaching assistant or one of your clever-looking fellow students for additional explanation.

Procedure. An average week of mathematics consists of the following activities:

self-tuition		Go through the theory and examples of the new chapter and try to do at least 25% of the exercises.
lecture	Tue 08:45-10:30 / Thu 08:45-10:30	I explain the basic ideas by means of examples and applications while you try to follow me by pen and paper and catch my miscalculations.
self-tuition		Do another 50% of the exercises and verify your own work using my solutions. Write down inconsistencies and questions that arise.
exercise class	Tue 10:45-12:30 / Thu 10:45-12:30	Continue working on the exercises and discuss your inconsistencies and questions with your fellow students and/or teaching assistent.
self-tuition		Finish the remaining exercises and prepare for the test by going through the example tests.
test	Fri 08:30-10:30 (week 2, 4, 6 & 7)	Bring pen and paper and use this book, the arithmetic booklet and your own notes. Afterwards you can find my solutions on Blackboard.

Why so much mathematics? Central to your studies are the quantitative aspects of natural phenomena. You not only learn whether an insect (electron, ...) moves or not, but also how you can use mathematics to calculate in what direction and with what speed. You not only learn whether a plant (population, tumour, ...) grows or not, but also how you can use mathematical models and differential equations to predict how fast it grows and to what size. You not only learn whether a chemical mixture will react or not, but also what the quantitative behaviour of the reaction is and how much energy will be released. In addition, mathematics plays an important role in courses on classical or quantum mechanics, thermodynamics, electricity and magnetism, statistics, etc.

Graphing calculator. You are allowed to delegate stupid calculations and the drawing of complicated graphs to your pocket calculator or graphing calculator if you happen to possess one. That's why I will explain every now and then how you command the TI-83 (Plus) to do these jobs. If you purchased another brand, I guess you should take a look in the manual. Don't worry if the batteries of your calculator are running low, because you're allowed to present results such as

$$\frac{\sqrt{\sin 2}}{\arctan(\ln 7)}$$

at tests and exams without remarking that this is approximately equal to 0.87. However, I will become furious (penalty points!) if you present something which is easy to simplify as a solution, for example

$$\frac{2^{\ln 7}}{7^{\ln 2}}$$

Most calculators have two weaknesses:

1. They're not capable of providing exact results. For example, if you delegate the calculation of

   $$\int_0^1 \frac{1}{1+x^2}\, dx$$

   to your TI graphing calculator using the fnInt option from the MATH menu:

   $$\text{fnInt}\,(1/(1+X)^2\,,\, X\,,\, 0\,,\, 1)$$

   it dares to present the number 0.7853981634 to you. Unfortunately, this answer will be considered wrong at tests and exams because it is not perfect: after this course (or probably the next) you'll know that $\frac{\pi}{4}$ is the one and only correct answer.

2. They can't perform symbolic algebra. For example, if you let it calculate

   $$\int_0^a \frac{1}{1+x^2}\, dx$$

   it will go completely crazy. And symbolic algebra is indispensable for your studies: you'll often have to perform calculations with one or more unspecified constants. Such an a might for instance represent the gravitational constant at the moon Deimos, or the cholesterol level in a blood sample to be analysed.

1. Sequences and limits

Sequences. In this chapter we'll study sequences of real numbers. I can construct such a sequence for instance by counting the number of woodlice under my doormat:

day number	day 0	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10
woodlice	7	9	12	15	20	26	34	44	57	74	97

or by measuring Zompie's weight on his consecutive birthdays:

age	1 year	2 years	3 years	4 years	5 years	6 years	7 years	8 years	9 years	10 years
kilograms	3.8	4.4	5.0	5.4	5.7	6.0	6.2	6.3	6.5	6.6

In general, it is very difficult to describe these sequences mathematically with infinite precision. If you accept small errors in the description, however, you are able to find mathematical expressions for the terms of a sequence. For example, the number of woodlice will increase exponentially as long as there is sufficient space under my doormat. Hence, there exists a formula of the form $\text{woodlice}_n = \alpha \cdot \beta^n$ that approximates the number of woodlice on day n. Indeed,

$$\text{woodlice}_n = 7 \cdot 1.3^n$$

provides a fairly good description of the actual number of woodlice. Using this formula I can predict when the number of woodlice exceeds the value of 1001:

$$7 \cdot 1.3^n = 1001 \quad \Longrightarrow \quad 1.3^n = 143 \quad \Longrightarrow \quad \ln 1.3^n = \ln 143 \quad \Longrightarrow \quad n \ln 1.3 = \ln 143 \quad \Longrightarrow \quad n = \frac{\ln 143}{\ln 1.3}$$

which, according to my calculator, amounts to approximately 19 days; after 19 days there are more than 1000 woodlice. In summary, the procedure is as follows: perform counts or measurements to construct the beginning of the sequence, model the sequence using a suitable mathematical expression, and use this expression to make the desired prediction.
Let's use this procedure to predict Zompie's weight limit. Zompie's weight will usually belong to the category of 'bounded exponential growth', $\text{zompie}_n = \alpha - \beta \cdot \gamma^n$, and after a bit of trial and error I've found the following formula matching my measurements rather well:

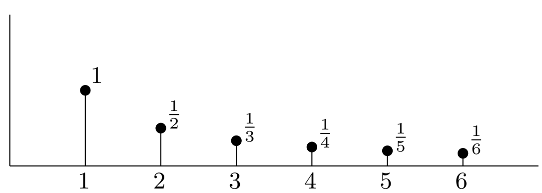
$$\text{zompie}_n = 7 - 4 \cdot 0.8^n$$

Using this model, I predict that his weight limit will be 7 kg. After studying the theory on limits on the next page, and standard limit (2) in particular, you'll know why.

Graphs of sequences. Infinite sequences of real numbers can be represented graphically. Some examples:

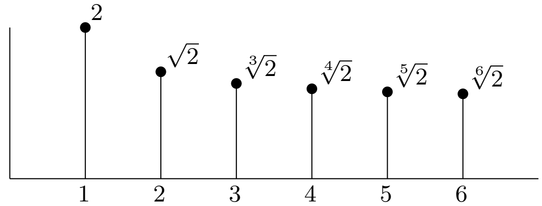
- the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

  put differently:
  the sequence $a_1, a_2, a_3, a_4, a_5, \ldots$ with $a_n = \frac{1}{n}$



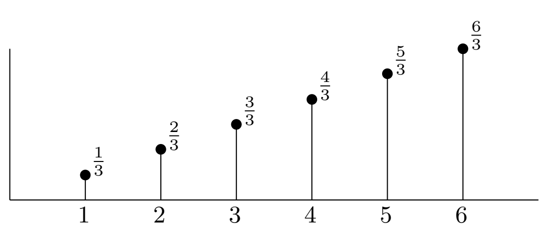
- the sequence $2\,, \sqrt{2}\,, \sqrt[3]{2}\,, \sqrt[4]{2}\,, \sqrt[5]{2}\,, \ldots$

  put differently:
  the sequence $b_1, b_2, b_3, b_4, b_5, \ldots$ with $b_n = \sqrt[n]{2}$



- the sequence $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$

  put differently:
  the sequence $c_1, c_2, c_3, c_4, c_5, \ldots$ with $c_n = \frac{n}{3}$



Limits. The 'limit' of a sequence $x_1, x_2, x_3, \ldots$ is the number that approximates best the value of x_n for very large values of n. In the graphical representation the limit is the height of the horizontal asymptote to the bullets. Obviously, such a limit does not always exist. If it does, the sequence is said to be 'convergent', and if it doesn't, 'divergent'. In our examples:

- The sequence $a_1, a_2, a_3, \ldots$ is convergent, its limit is 0. Notation: $\lim\limits_{n\to\infty} a_n = 0$ or simply $a_\infty = 0$.
- The sequence $b_1, b_2, b_3, \ldots$ converges, $\lim\limits_{n\to\infty} b_n = 1$.
- The sequence $c_1, c_2, c_3, \ldots$ is divergent, you can denote its behaviour as $\lim\limits_{n\to\infty} c_n = \infty$.
- The woodlice are doing well too: $\lim\limits_{n\to\infty} \text{woodlice}_n = \infty$.
- Fortunately, Zompie neatly sticks to his diet: $\lim\limits_{n\to\infty} \text{zompie}_n = 7$.

Standard limits. Four 'standard limits' (without proof):

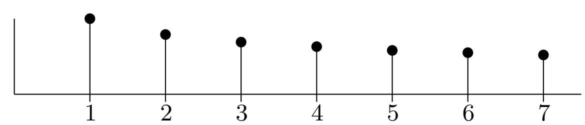
(1) $\lim\limits_{n\to\infty} \dfrac{1}{n^c} = 0 \quad \text{if } c > 0$

(2) $\lim\limits_{n\to\infty} c^n = 0 \quad \text{if } -1 < c < 1$

(3) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^c} = 0 \quad \text{if } c > 0$

(4) $\lim\limits_{n\to\infty} \dfrac{n^c}{d^n} = 0 \quad \text{if } c > 0 \text{ and } d > 1$

Example 1. Calculate $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt[3]{n}}$.

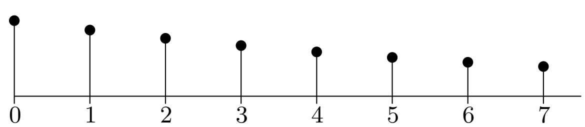


Solution. Use standard limit (1) with $c = \frac{1}{3}$:

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$$

The moral: if n becomes very large, $\frac{1}{\sqrt[3]{n}}$ is just about zero.

Example 2. Calculate $\lim\limits_{n\to\infty} \left(\dfrac{7}{8}\right)^n$.

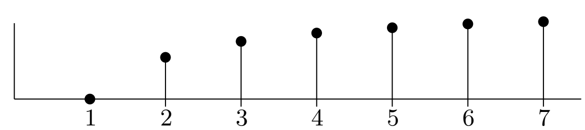


Solution. Standard limit (2) with $c = \frac{7}{8}$:

$$\lim_{n\to\infty} \left(\frac{7}{8}\right)^n = 0$$

The moral: $\frac{7}{8}$ to the power of a very big number is approximately zero.

Example 3. Calculate $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt[3]{n}}$.

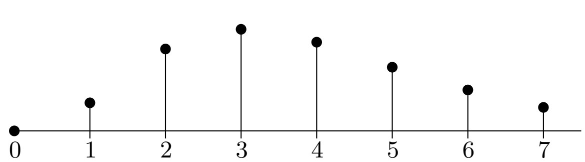


Solution. Standard limit (3) with $c = \frac{1}{3}$:

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt[3]{n}} = 0$$

Conclusion: for large values of n you can consider $\ln n$ to be negligible compared to $\sqrt[3]{n}$.

Example 4. Calculate $\lim\limits_{n\to\infty} \dfrac{n^3}{e^n}$.



Solution. Standard limit (4) with $c = 3$ and $d = e$:

$$\lim_{n\to\infty} \frac{n^3}{e^n} = 0$$

Hence, n^3 is much much smaller than e^n for large n.

Calculation rules for limits. The following neat rules enable you to calculate all sorts of complicated limits using the limits that you already know:

(1) If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$.

(2) If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $\lim\limits_{n\to\infty} (a_n - b_n) = a - b$.

(3) If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $\lim\limits_{n\to\infty} a_n \cdot b_n = a \cdot b$.

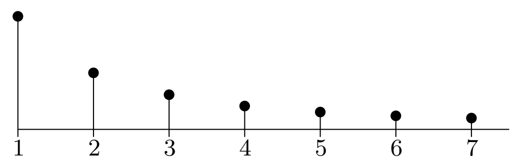
(4) If $\lim\limits_{n\to\infty} a_n = a$ and $c \in \mathbb{R}$, then $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot a$.

(5) If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$ and $b \neq 0$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

(6) If $\lim\limits_{n\to\infty} a_n = a$ and $c > 0$, then $\lim\limits_{n\to\infty} c^{a_n} = c^a$.

(7) If $\lim\limits_{n\to\infty} a_n = a$ and a and c are positive, then $\lim\limits_{n\to\infty} a_n^c = a^c$.

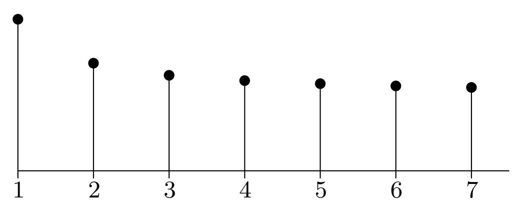
Example 5. Calculate $\lim\limits_{n\to\infty} \left(\dfrac{1}{n} + 2^{-n}\right)$.



Solution. I calculate the limits of $\frac{1}{n}$ and 2^{-n} separately and just add them (which is allowed according to rule 1):

$$\left.\begin{array}{ll} \lim\limits_{n\to\infty} \dfrac{1}{n} = 0 & \text{(standard limit 1 with } c = 1) \\ \lim\limits_{n\to\infty} 2^{-n} = 0 & \text{(standard limit 2 with } c = \frac{1}{2}) \end{array}\right\} \xRightarrow{\text{calculation rule 1}} \lim_{n\to\infty} \left(\frac{1}{n} + 2^{-n}\right) = 0 + 0 = 0$$

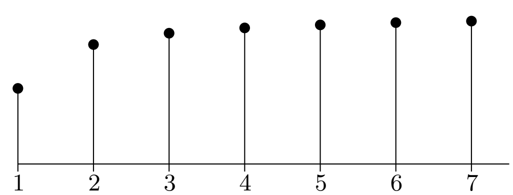
Example 6. Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2}$.



Solution. Apply rule (6) with $c = 2$ and $a_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1$$

Example 7. Calculate $\lim\limits_{n\to\infty} \sqrt{4 - \dfrac{3}{n}}$.



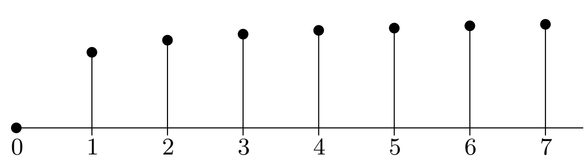
Solution. Use calculation rule (2):

$$\lim_{n\to\infty} \left(4 - \frac{3}{n}\right) = \lim_{n\to\infty} 4 - \lim_{n\to\infty} \frac{3}{n} = 4 - 0 = 4$$

Now, I apply rule (7) with $c = \frac{1}{2}$:

$$\lim_{n\to\infty} \sqrt{4 - \frac{3}{n}} = \sqrt{4} = 2$$

Example 8. Calculate $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + 3n} - n\right)$.



Solution. Multiplication by $\dfrac{\sqrt{n^2+3n}+n}{\sqrt{n^2+3n}+n}$ (the 'square root trick' from my arithmetic booklet) yields

$$\sqrt{n^2+3n} - n = \frac{\left(\sqrt{n^2+3n} - n\right)\left(\sqrt{n^2+3n} + n\right)}{\sqrt{n^2+3n}+n} = \frac{3n}{\sqrt{n^2+3n}+n} = \frac{3}{\sqrt{1 + \frac{3}{n}} + 1}$$

Then, I apply the calculation rules for limits to obtain $\lim\limits_{n\to\infty} \left(\sqrt{n^2+3n} - n\right) = \dfrac{3}{\sqrt{1+0}+1} = \dfrac{3}{2}$.

Example 9. Let n be the number of sand grains in the Sahara. Which of the following numbers do you think is bigger, $\ln n$ or $\sqrt{n}$?

Solution. By standard limit (3),

$$\lim_{n\to\infty} \frac{\ln n}{n^c} = 0 \quad \overset{c=\frac{1}{2}}{\Longrightarrow} \quad \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$$

Apparently, $\ln n$ is much smaller than $\sqrt{n}$ for very large n.

Linear sequence. A linear sequence is a sequence $s_0, s_1, s_2, \ldots$ of the kind

$$s_n = \alpha n + \beta$$

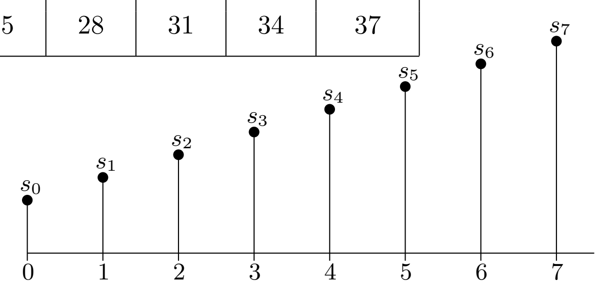
To put it differently: the bullets in the graphical representation of the sequence lie on a straight line. Instead of 'linear sequence' one can also say 'arithmetic sequence'. Such a sequence describes a process in which the increase per unit time is a constant:

$$s_{n+1} - s_n = \alpha$$

Example 10. On day 0 there are 7 spiders, and every new day their number increases by 3:

day 0	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10
7	10	13	16	19	22	25	28	31	34	37

The number of spiders on day n is $s_n = 7 + 3n$.

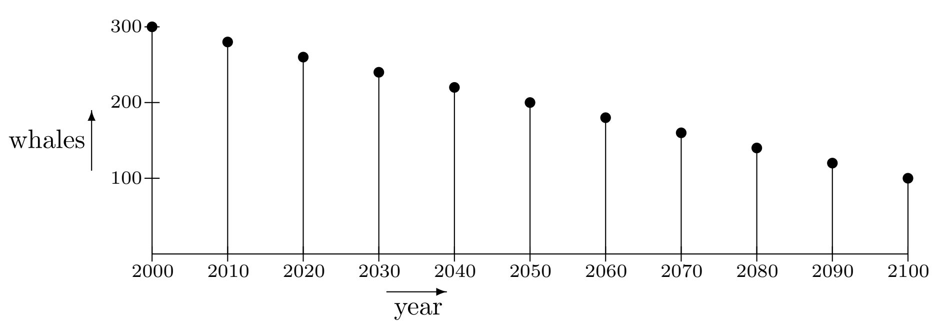


Example 11. The number of whales w_n in year n decreases linearly. Counting indicates that $w_{2000} = 300$ and $w_{2010} = 280$. Determine a formula for w_n and predict when the whales will be extinct.

Solution. I substitute the counted numbers in the model $w_n = \alpha n + \beta$:

$$\left.\begin{array}{l} w_{2000} = 300 \quad \Longrightarrow \quad 2000\,\alpha + \beta = 300 \\ w_{2010} = 280 \quad \Longrightarrow \quad 2010\,\alpha + \beta = 280 \end{array}\right\} \overset{\text{subtract}}{\Longrightarrow} \quad -10\,\alpha = 20 \quad \Longrightarrow \quad \alpha = -2 \quad \Longrightarrow \quad \beta = 4300$$

So $w_n = 4300 - 2n$, and if this linear model remains valid there will be no more whales left in the year 2150:



Exponential sequences. An exponential sequence is a sequence $s_0, s_1, s_2, \ldots$ of the kind

$$s_n = \alpha \cdot \beta^n$$

Instead of 'exponential sequence' one also says 'geometric sequence'. The constant β is the ratio between two subsequent terms of the sequence:

$$s_{n+1} = \beta \cdot s_n$$

This type of growth occurs extensively in nature. If, for example, there are A bunnies in year 0 ($b_0 = A$), each bunny gives birth to C babies each year and no bunnies die, the number of bunnies will increase exponentially with growth factor $C + 1$:

$$b_{n+1} - b_n = Ck_n \quad \Longrightarrow \quad b_{n+1} = (C+1)b_n \quad \Longrightarrow \quad b_n = A \cdot (C+1)^n$$

Example 12. On day 0 there are 768 spiders. Each day the number of spiders increases by fifty percent:

day 0	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9
768	1152	1728	2592	3888	5832	8748	13122	19683	29524.5

Hence, the number of spiders on day n increases exponentially:

$$s_n = 768 \cdot 1.5^n$$

By the way: don't let that half of a spider on day 9 bother you. This mathematical model is just a tool to represent reality as closely as possible.

Example 13. A students' kitchen is being visited by a slightly disturbing number of cockroaches:

day 0	day 1
5	6

Try to predict when the number of cockroaches will exceed 1000, assuming that students are not very eager to clean their kitchen, so the number of cockroaches will increase exponentially.

Solution. In the exponential model $c_n = \alpha \cdot \beta^n$ I substitute the given numbers of cockroaches:

$$\left.\begin{array}{lllll} c_0 = 5 & \Longrightarrow & \alpha \cdot \beta^0 = 5 & \Longrightarrow & \alpha = 5 \\ c_1 = 6 & \Longrightarrow & \alpha \cdot \beta^1 = 6 & \Longrightarrow & \beta = \frac{6}{5} \end{array}\right\} \quad \Longrightarrow \quad c_n = 5 \cdot 1.2^n$$

Now I just need to calculate when this equals 1000:

$$5 \cdot 1.2^n = 1000 \quad \Longrightarrow \quad 1.2^n = 200 \quad \overset{\text{ln-trick}}{\Longrightarrow} \quad n \ln 1.2 = \ln 200 \quad \Longrightarrow \quad n = \frac{\ln 200}{\ln 1.2} \approx 29$$

Example 14. This year (let's call it year 0) 100000 bridge players inhabit our country. Each year ten percent of them quit this dull sport as a consequence of death or demotivation, and there are no new bridge players being recruited:

$$\boxed{b_0 = 100000} \qquad \boxed{b_{n+1} = 0.9 \cdot b_n}$$

When will there be only four bridge players left?

Solution. From the data I deduce that $b_n = 100000 \cdot 0.9^n$, which satisfies our criteria for exponential growth, although 'exponential decay' would be a better name since the growth factor is below 1:

year 0	year 1	year 2	year 3	year 4	year 5	year 6	year 7	year 8	year 9
100000	90000	81000	72900	65610	59049	53144	47830	43047	38742

There are only four surviving bridge players when

$$100000 \cdot 0.9^n = 4 \quad \Longrightarrow \quad 0.9^n = 0.00004 \quad \Longrightarrow \quad n \ln 0.9 = \ln 0.00004 \quad \Longrightarrow \quad n = \frac{\ln 0.00004}{\ln 0.9} \approx 96$$

(so we can keep on playing bridge for almost a century).

Is exponential growth realistic? The exponential growth model is particularly suitable for describing uninterrupted iterative processes, such as the growth of a population under ideal circumstances without limitations. In practice, however, the proliferating woodlice have to deal with the limited space under my doormat. We need more sophisticated models to incorporate this, and I'll briefly discuss two of these models.

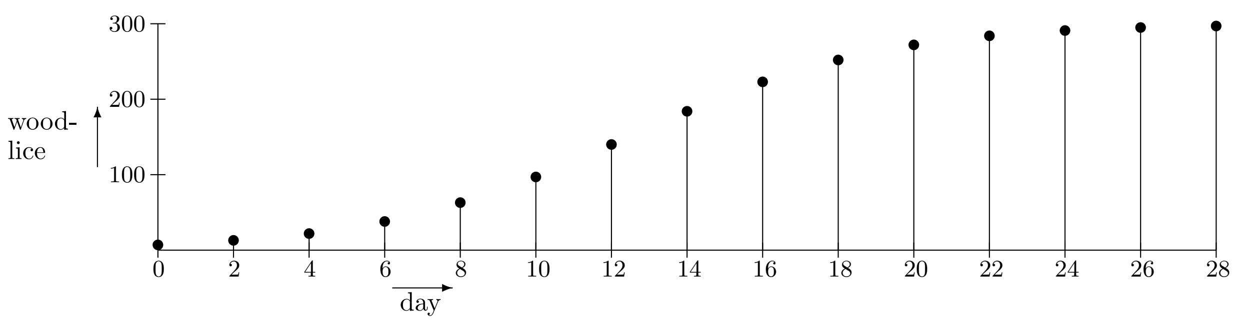
The logistic model. My doormat offers space for only 300 woodlice. Their initial exponential growth factor will then decrease as there is less space left. The space left under my doormat is $300 - p_n$. As long as there are few woodlice, their growth will be roughly proportional to their number, so $(p_{n+1} - p_n) \sim p_n$. However, when their number approaches the limit of 300, their growth will become proportional to the space left $(300 - p_n)$. This results in the 'logistic growth model':

$$p_{n+1} - p_n = \lambda \cdot p_n \cdot (300 - p_n)$$

If you count the number of woodlice twice (let's take $p_0 = 7$ and $p_1 = 9$), you can calculate the proportionality constant λ (which turns out to be approximately 0.000975):

$$p_{n+1} - p_n = 0.000975 \cdot p_n \cdot (300 - p_n)$$

and from this you should be able to predict (I mean calculate) the numbers using your graphing calculator:



Bounded exponential growth. In this model we assume that the population size p_n has limit α and that the increase is proportional to the available space $\alpha - p_n$:

$$p_{n+1} - p_n = \lambda(\alpha - p_n)$$

Then, the sequence p_n is of the kind $p_n = \alpha - \beta \cdot \gamma^n$ where γ is a number between 0 and 1, and the proportionality constant λ can be identified with $1 - \gamma$.

Example 15. I pour myself a glass of cold beer (3° C) and leave it untouched for ten minutes at an ambient temperature of 23° C. After one minute the temperature of the beer has risen to 5° C:

$$\boxed{T_0 = 3} \qquad \boxed{T_1 = 5} \qquad \boxed{T_\infty = 23}$$

What is the temperature of my beer after ten minutes?

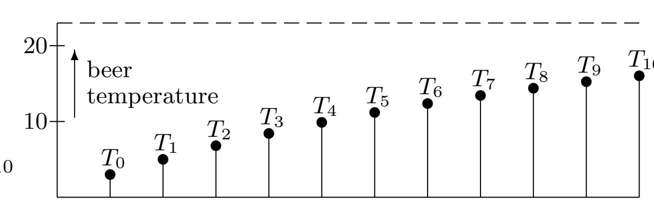
Solution. From thermodynamics you (hopefully) know that the increase of the beer temperature is, to good approximation, proportional to the temperature difference. Hence, the model 'bounded exponential growth' can be applied to describe the behaviour of the sequence $T_0, T_1, T_2, \ldots$:

$$T_n = 23 - \beta \cdot \gamma^n$$

Using the measurements I can calculate β and γ:

- $T_0 = 3$ implies that $23 - \beta = 3$, so $\beta = 20$
- $T_1 = 5$ implies that $23 - 20\gamma = 5$, so $\gamma = 0.9$

Therefore, $T_n = 23 - 20 \cdot 0.9^n$ so $T_{10} = 23 - 20 \cdot 0.9^{10}$ (which is about 16 degrees).



Summation of sequences. In order to sum (parts of) sequences we make use of a practical abbrevation, the sigma notation:

$$\boxed{\sum_{k=1}^{n} a_k \text{ means } a_1 + a_2 + \cdots + a_n}$$

Some examples:

$$\sum_{k=1}^{10} \sin k \quad \text{means} \quad \sin 1 + \sin 2 + \sin 3 + \sin 4 + \sin 5 + \sin 6 + \sin 7 + \sin 8 + \sin 9 + \sin 10$$

$$\sum_{k=3}^{13} \frac{k}{100} \quad \text{means} \quad \frac{3}{100} + \frac{4}{100} + \frac{5}{100} + \frac{6}{100} + \frac{7}{100} + \frac{8}{100} + \frac{9}{100} + \frac{10}{100} + \frac{11}{100} + \frac{12}{100} + \frac{13}{100}$$

$$\sum_{n=2}^{12} nx^{n-1} \quad \text{means} \quad 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 + 10x^9 + 11x^{10} + 12x^{11}$$

How to sum a linear sequence? A convenient method to sum a linear (or arithmetic) sequence:

1. count the number of terms
2. calculate their average value
3. the sum is then (number of terms) · (average value)

Example 16. Bontepoes managed to catch five mice on 1 November and two more mice on each subsequent day until 30 November. Thus, on n November she caught $3 + 2n$ mice, so the number of mice composes a linear sequence:

1 nov	2 nov	3 nov	4 nov	5 nov	6 nov	7 nov	8 nov	9 nov	....	29 nov	30 nov
5	7	9	11	13	15	17	19	21		61	63

Calculate the total number of mice caught by Bontepoes in November.

Solution. I use my convenient method:

1. the number of terms is 30
2. their average value is 34
3. so the total number is $30 \cdot 34 = \boxed{\text{1020 mice}}$

You might wonder how I found the average number of mice to be 34 that quickly. I use a trick for this: add the first term (5) to the last term (63) and divide this by 2. Evidently, this trick only works for linear sequences. In short:

$$\sum_{n=1}^{30} (3 + 2n) = 1020$$

How to sum an exponential sequence? Whenever I have to sum an exponential (or geometric) sequence, I use the formula

$$\boxed{1 + x + x^2 + x^3 + x^4 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}}$$

If you insist, I'll deliver a firm proof of this formula during the first lecture.

Example 17. Since 1967 I've made huge money by teaching at this university. My salary went up by 2% every year with a starting salary of (after conversion) 12500 euros:

1967	1968	1969	1970	1971	1972	....	2012	2013	2014
12500	12750	13005	13265	13530	13801		30473	31082	31704

How much did I earn between 1967 and 2014?

Solution. In the year $1967 + n$ I earned $12500 \cdot 1.02^n$ euros, so the total sum was

$$\begin{aligned} & 12500 + 12500 \cdot 1.02 + 12500 \cdot 1.02^2 + 12500 \cdot 1.02^3 + \cdots + \cdot 12500 \cdot 1.02^{47} \\ = \; & 12500 \cdot \left(1 + 1.02 + 1.02^2 + 1.02^3 + \cdots + 1.02^{47}\right) \\ = \; & 12500 \cdot \frac{1 - 1.02^{48}}{1 - 1.02} \; = \; 625000 \cdot \left(1.02^{48} - 1\right) \quad \text{which is no less than 991919 euros!!} \end{aligned}$$

The calculation that I made can be summarised as

$$\sum_{n=0}^{47} \left(12500 \cdot 1.02^n\right) = 625000 \cdot \left(1.02^{48} - 1\right)$$

Infinite sums. Sometimes it is possible to sum all terms of an infinite sequence $a_1, a_2, a_3, \ldots$. We agree on the following notation:

$$\boxed{a_1 + a_2 + a_3 + \cdots} \stackrel{\text{def}}{=} \lim_{n \to \infty} \left(a_1 + a_2 + a_3 + \cdots + a_n\right)$$

I use the symbol $\boxed{\stackrel{\text{def}}{=}}$ for: 'equals by definition' or 'means by agreement'. In $\sum$-notation:

$$\sum_{n=1}^{\infty} a_n \stackrel{\text{def}}{=} \lim_{n \to \infty} \sum_{k=1}^{n} a_k$$

Example 18. Zompie must lose weight. Today (day 0) I'm giving him 1 kg of pâté, but he'll get 20% less every day. How much pâté will he eat in total? Zompie lives forever, so I have to add an infinite number of terms:

day number	day 0	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	......
kg pâté	1.00	0.80	0.64	0.51	0.41	0.33	0.26	0.21	0.17	0.13	

Solution. On day n Zompie eats 0.8^n kg of pâté, so the total amount of pâté from day 0 until day n is

$$1 + 0.8 + 0.8^2 + 0.8^3 + \cdots + 0.8^n = \frac{1 - 0.8^{n+1}}{1 - 0.8}$$

I take the limit of this expression for $n \to \infty$ using standard limit 3 (with $c = 0.8$) and the calculation rules:

$$\lim_{n \to \infty} \left(1 + 0.8 + 0.8^2 + \cdots + 0.8^n\right) = \frac{1 - 0}{1 - 0.8} = \boxed{5 \text{ kg of paté}}$$

We write this calculation down as $1 + 0.8 + 0.8^2 + 0.8^3 + \cdots = 5$ or, in short, $\sum_{n=0}^{\infty} 0.8^n = 5$. In general, the following formula for the summation of geometric sequences applies:

$$\boxed{\text{If } -1 < x < 1 \text{ then } 1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}}$$

A brief note on terminology: $a_1, a_2, a_3, \ldots, a_n$ is called a sequence, whereas $a_1 + a_2 + a_3 + \cdots + a_n$ is called a series: a series is the sum of a sequence. Formally, 'sequence' and 'series' translate to 'rij' and 'reeks' in Dutch, respectively. However, 'reeks' can also (somewhat erroneously) be used to refer to a sequence. The Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$, for instance, is often called 'reeks van Fibonacci' instead of 'rij van Fibonacci'.

Example 19. Calculate $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\cdots$

Solution. Apply the foregoing formula with $x=-\frac{1}{2}$:

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\cdots=\frac{1}{1-(-\frac{1}{2})}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$$

Example 20. Calculate $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\frac{1}{7\cdot 8}+\frac{1}{8\cdot 9}+\cdots$

Solution. You can rewrite the terms of this series using partial fraction decomposition:

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

You can easily verify this by making the denominators equal and adding the fractions (you'll learn how to decompose fractions yourself in example 21). The sum of the sequence is then

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=\frac{1}{1}\ \text{(as the other terms cancel each other)}\quad=1$$

Example 21. Decompose $\frac{5x+1}{x^2+2x-15}$ into two simpler fractions.

Solution. This requires a few steps:

1. Recognise the product of two factors in the denominator: $x^2+2x-15=(x+5)(x-3)$.
2. Now, you expect to be able to decompose the fraction into two simpler fractions:

   $$\frac{5x+1}{(x+5)(x-3)}=\frac{A}{x+5}+\frac{B}{x-3}$$

   All we need is to find out what A and B should be in order to make this right.
3. Make the denominators the same and add the fractions:

   $$\frac{A}{x+5}+\frac{B}{x-3}=\frac{A(x-3)+B(x+5)}{(x+5)(x-3)}=\frac{(A+B)x+(-3A+5B)}{(x+5)(x-3)}$$

4. Now, determine A and B as follows: the numerator $(A+B)x+(-3A+5B)$ should equal $5x+1$, so

   $$\left.\begin{aligned}A+B=5\\-3A+5B=1\end{aligned}\right\}\implies\left\{\begin{aligned}A=3\\B=2\end{aligned}\right.$$

5. The desired partial fraction decomposition is therefore $\frac{5x+1}{x^2+2x-15}=\frac{3}{x+5}+\frac{2}{x-3}$.

Example 22. Calculate

a) $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$

b) $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$

Solution.

a) Give these fractions the same denominator and add them:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=\frac{840}{840}+\frac{420}{840}+\frac{280}{840}+\frac{210}{840}+\frac{168}{840}+\frac{140}{840}+\frac{120}{840}+\frac{105}{840}=\frac{2283}{840}=\frac{761}{280}$$

b) $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots=\infty$, I prove this as follows:

- $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$ is more than $\frac{1}{2}$

  (explanation: the sum of four terms greater than or equal to $\frac{1}{8}$ must be greater than $4\cdot\frac{1}{8}=\frac{1}{2}$)
- $\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}$ is more than $\frac{1}{2}$ as well

  (the sum of these eight terms, all $\geq\frac{1}{16}$, must be greater than $8\cdot\frac{1}{16}=\frac{1}{2}$)
- By the same token, $\frac{1}{17}+\cdots+\frac{1}{32}>\frac{1}{2}$

  (the sum of sixteen terms $\geq\frac{1}{32}$ is greater than $16\cdot\frac{1}{32}=\frac{1}{2}$
- and $\frac{1}{33}+\cdots+\frac{1}{64}>\frac{1}{2}$ as well

  (this is the sum of 32 terms $\geq\frac{1}{64}$)

Hence, I can divide the terms of the series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$ into infinitely many groups with each group $>\frac{1}{2}$, so I can exceed every number you can think of:

$$\boxed{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots=\infty}$$

Example 23.

a) What's your opinion on the statement $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\cdots=\infty$?

nonsense! might be true definitely true

b) What's your opinion on the statement $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\cdots=\frac{\pi^2}{6}$?

nonsense! might be true definitely true

Solution.

a) Nonsense! This series must be less than 2:

- $\frac{1}{n^2}$ is less than $\frac{1}{n(n-1)}$
- so $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots<1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=2$ (see example 20)

b) Might be true, since $\frac{\pi^2}{6}$ is less than 2. It turns out to be true indeed (but I can't prove this yet):

$$\boxed{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\cdots=\frac{\pi^2}{6}}$$

Summations using a calculator. To calculate $\sum_{n=7}^{35}(3+5n)$ you can choose between three methods:

1. You give your graphing calculator the command $\boxed{\text{sum}\,(\text{seq}\,(3+5n\,,\,n\,,\,7\,,\,35\,,\,1))}$ and in no time it comes up with the correct answer: 3132. How this command works:

sum	can be found in the menu LIST MATH
seq	can be found in the menu LIST OPS
$3+5n$	is the expression to be summed
n	is the variable
7	is the initial value of n
35	is the final value of n
1	is the step size by which n promenades from 7 to 35

2. You write out the full sum and after ten minutes of diligent labour you find
$$38+43+48+53+58+63+68+73+78+83+88+93+98+103+108+113$$
$$+118+123+128+133+138+143+148+153+158+163+168+173+178=3132$$

3. You brightly notice that the 29 terms of this series are at equal mutual distances: every term is 5 more than its predecessor. Then, the average value of all terms lies exactly in the middle between the first and the last term, it is $\frac{38+178}{2}=108$. Thus, the sum equals $29\cdot 108=3132$.

If you judge these methods on their merits, your conclusion will be that method 2 is the least appealing: it fails for even the brightest of humans when they are urged to calculate
$$\sum_{n=7}^{35000}(3+5n)$$
This calculation makes your calculator throw in the towel as well, by the way. Method 3, however, works perfectly:
$$\sum_{n=7}^{35000}(3+5n)=(\text{number of terms})\cdot(\text{average value})=34994\cdot\frac{38+175003}{2}=3062692377$$

Formal definition of limit. Serious students might not be satisfied with my intuitive definitions of the concepts of limit and convergence. They consider it twaddle and are eager to come to grips with the mathematical theory. That's why I present here both the formal and the intuitive definitions. If you happen to despise mathematics, I suggest you read only the intuitive definitions and be satisfied with a life in which everything remains somewhat foggy, but mathematics does not constitute an insurmountable obstacle.

$\boxed{\lim_{n\to\infty} a_n = a}$ $\stackrel{\text{def}}{=}$ (intuitively) if n is very large, a_n approximately equals a
(formally) for all $\varepsilon>0$ I can find an A such that for $n>A$ the following holds: $a-\varepsilon<a_n<a+\varepsilon$

$\boxed{\lim_{n\to\infty} a_n = \infty}$ $\stackrel{\text{def}}{=}$ (intuitively) if n is very large, a_n is very large as well
(formally) for every number B I can find an A such that for $n>A$ the following holds: $a_n>B$

$\boxed{\lim_{n\to\infty} a_n = -\infty}$ $\stackrel{\text{def}}{=}$ (intuitively) if n is very large, a_n is very negative
(formally) for every number B I can find an A such that for $n>A$ the following holds: $a_n<B$

$\boxed{a_1,a_2,a_3,\ldots \text{ is convergent}}$ $\stackrel{\text{def}}{=}$ (intuitively) the sequence approaches something
(formally) there exists an $a\in\mathbb{R}$ with $\lim_{n\to\infty} a_n=a$

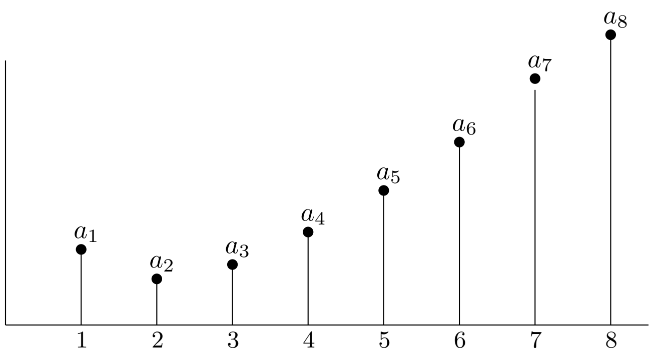
$\boxed{a_1,a_2,a_3,\ldots \text{ is divergent}}$ $\stackrel{\text{def}}{=}$ (intuitively) the sequence does not approach anything
(formally) $a_1,a_2,a_3,\ldots$ is not convergent

Exercises chapter 1

Exercise 1. Draw the beginning of the sequence $a_1, a_2, a_3, \ldots$ defined by

$$\boxed{a_n = \frac{1}{n}\sqrt{8 + n^2}}$$

and give your opinion on the limit of this sequence



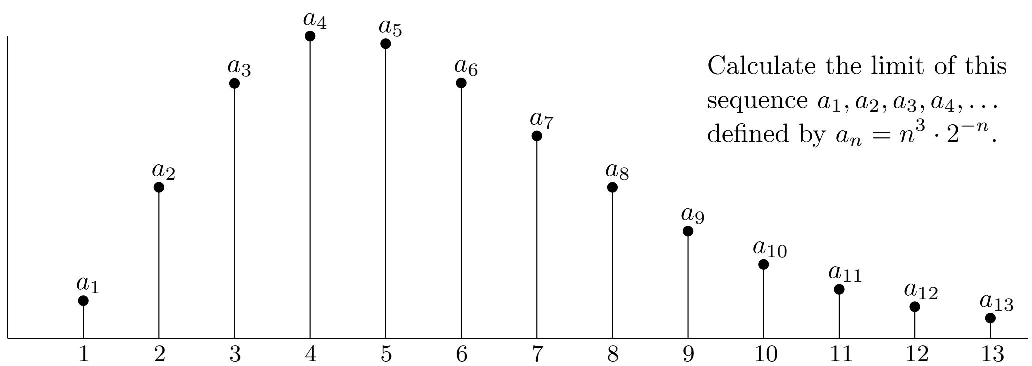
Exercise 2. This is the beginning of the sequence $a_1, a_2, a_3, \ldots$ defined by

$$\boxed{a_n = n - 2\ln n}$$

Does this sequence converge?

Exercise 3. Calculate $\lim_{n\to\infty}\left(\sqrt{n^2+1} - n\right)$.

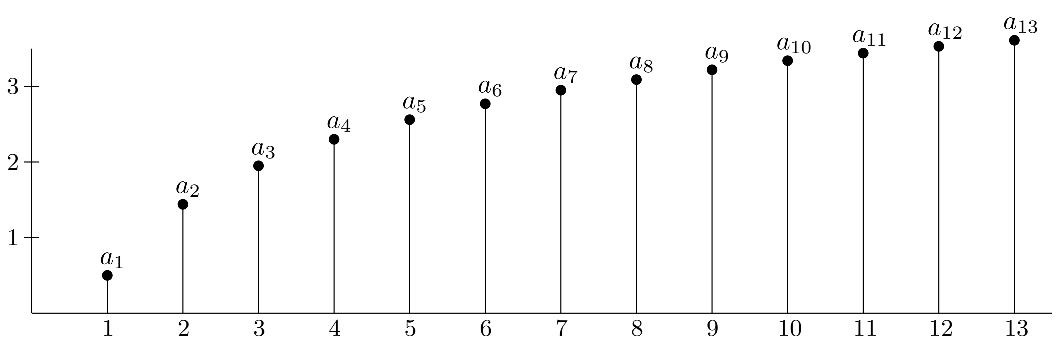
Exercise 4.



Calculate the limit of this sequence $a_1, a_2, a_3, a_4, \ldots$ defined by $a_n = n^3 \cdot 2^{-n}$.

Exercise 5. Jantje invests a euro in shares of a cat food manufacturer. The manufacturer distributes ten percent of profits every day and Jantje immediately invests these profits in even more cat food shares. Thus, his capital comprises 110 pennies after 1 day, 121 pennies after 2 days, et cetera. In short: after n days Jantje's wealth is $\left(\frac{11}{10}\right)^n$ euros. Pietje, on the other hand, tries to become rich by growing and selling melons. He is doing good business; he earns seven euros each day, so his bank account contains $7n$ euros on day n. Who is going to be the richer of the two in the long run?

Exercise 6.



Calculate the limit of this sequence $a_1, a_2, a_3, \ldots$ defined by $\boxed{a_n = \dfrac{7n-5}{3\sqrt{n}+n}}$.

Exercise 7. Scientific studies have shown that in the year n approximately

$$\sqrt{n + 6\sqrt{n}} \text{ thousands of cats}$$

will be living in Nijmegen. That is slightly more than in Arnhem, where only $\sqrt{n}$ thousands of cats are present. How many more cats will be living in Nijmegen than in Arnhem in the long run? Mathematically put: calculate

$$\lim_{n\to\infty} \left(\sqrt{n + 6\sqrt{n}} - \sqrt{n} \right)$$

Exercise 8. I've constructed a mathematical model for the number of birds b_n that a red cat hunts in his nth year of birth:

$$\boxed{\boxed{b_0 = 0} \qquad \boxed{b_{n+1} = \alpha b_n + \beta}}$$

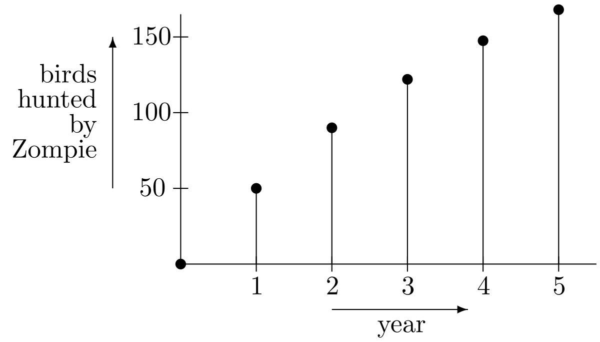
where the hunting constants α and β depend on the cat in question. Zompie's hunting constants are outrageously high: the numbers of birds hunted by Zompie in his early days were

$$\boxed{\begin{array}{l} b_1 = 50 \\ b_2 = 90 \end{array}}$$

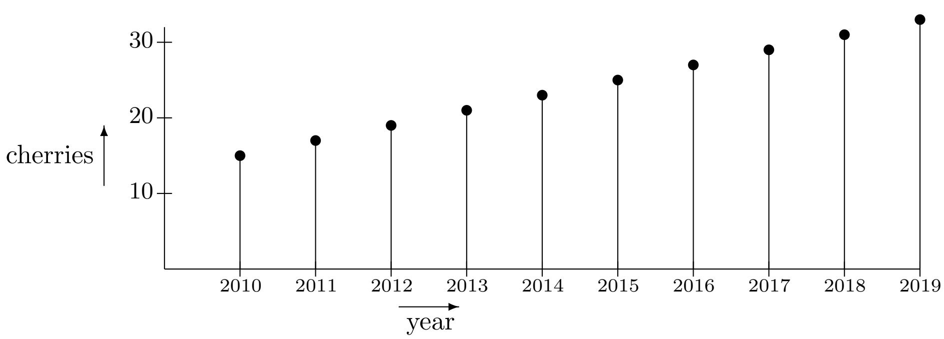
Calculate

a) Zompie's hunting constants α and β

b) Zompie's ultimate hunting achievement $\lim_{n\to\infty} b_n$



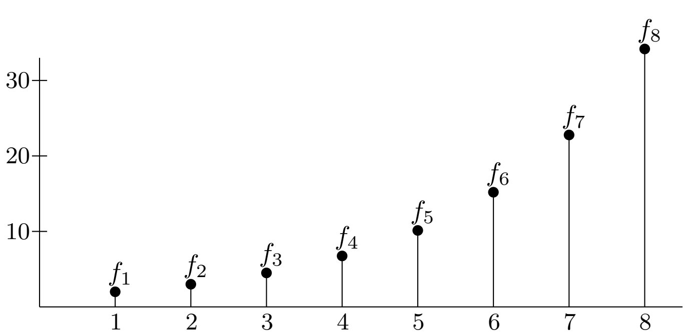
Exercise 9. The annual yield of my cherry tree increases linearly. In 2010 it produced only 15 cherries, but in 2014 I harvested no less than 23 cherries:



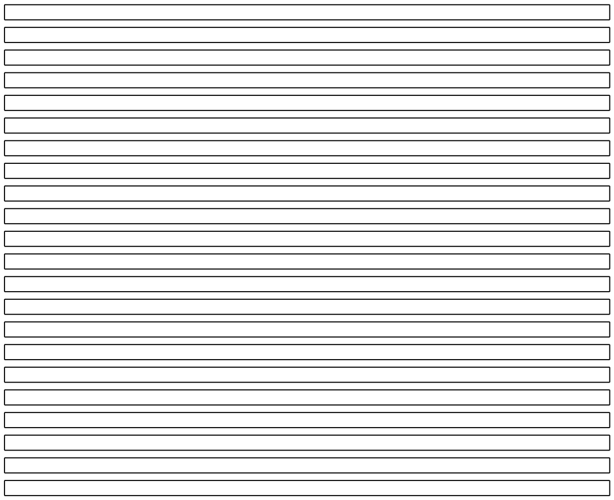
a) Find a formula for the yield in the year n

b) When will the annual yield exceed 100 cherries?

Exercise 10. A radioactive substance decays one percent a year. Find the half-life (i.e. the time after which only half of the original amount of substance is left).

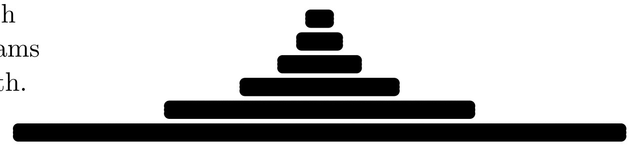


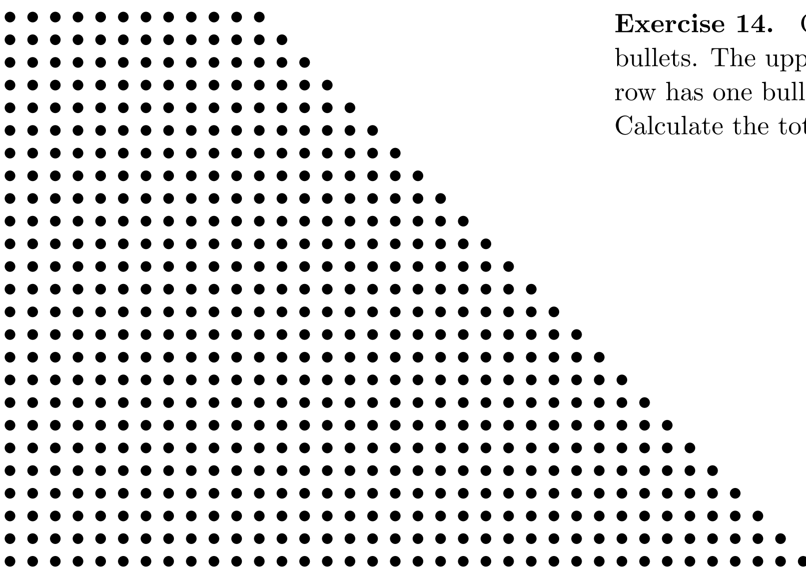
Exercise 11. The number of fleas in the living room increases exponentially. On day 1 I count two fleas ($f_1 = 2$) and the next day there are three ($f_2 = 3$). When will there be 10000 fleas?



Exercise 12. A 1 cm thick glass plate transmits 90% of the light. How many of these plates do I need to stack if I want them to transmit only 10% of the light?

Exercise 13. Six beams with a cumulative length of 9 metres are stacked. Each of the upper five beams is exactly half of the length of the beam underneath. What is the length of the bottom beam?





Exercise 14. On the left I've drawn 25 rows of bullets. The upper row contains 12 bullets and each row has one bullet more than the previous row. Calculate the total number of bullets.

Exercise 15.

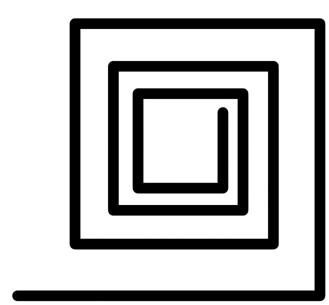
a) Calculate $1 + 5 + 9 + 13 + 17 + 21 + \cdots + 201$.

b) Rephrase the equality found in (a) using the $\sum$-notation

Exercise 16. This spiral consists of 14 segments. The length of each segment is 90% of the length of the previous segment. The first segment has a length of 4 cm.

a) What is the length of the spiral?

b) What would be the length of the spiral if I continued drawing segments for eternity?



Exercise 17. Calculate $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\frac{2}{243}+\frac{2}{729}+\frac{2}{2187}+\frac{2}{6561}+\frac{2}{19683}+\frac{2}{59049}+\frac{2}{177147}+\cdots$

Exercise 18. Calculate $\sum_{n=3}^{56}\frac{1}{n^2+n}$.

Exercise 19. Calculate, without using a calculator, the average to five decimal places of the numbers

$$2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}, 2^{-11}, 2^{-12}, 2^{-13}, 2^{-14}, 2^{-15}, 2^{-16}, 2^{-17}, 2^{-18}, 2^{-19}, 2^{-20}$$

Exercise 20. Simplify the expression $\frac{2^{\ln 7}}{7^{\ln 2}}$.

Exercise 21. Calculate

a) $\sum_{k=0}^{100}(3k+2)$ b) $\sum_{k=0}^{100}(3^k+2)$

Exercise 22. The ambient temperature is 20° C and the temperature T_n of my cup of tea after n minutes of waiting is initially way too high to drink:

$\boxed{T_0 = 90}$ $\boxed{T_1 = 76}$

That's why I let my tea cool down for a while. Calculate T_{10}, the tea temperature after ten minutes.

Exercise 23. Calculate $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\frac{1}{5\cdot 7}+\frac{1}{6\cdot 8}+\frac{1}{7\cdot 9}+\cdots+\frac{1}{999\cdot 1001}$.

Exercise 24.

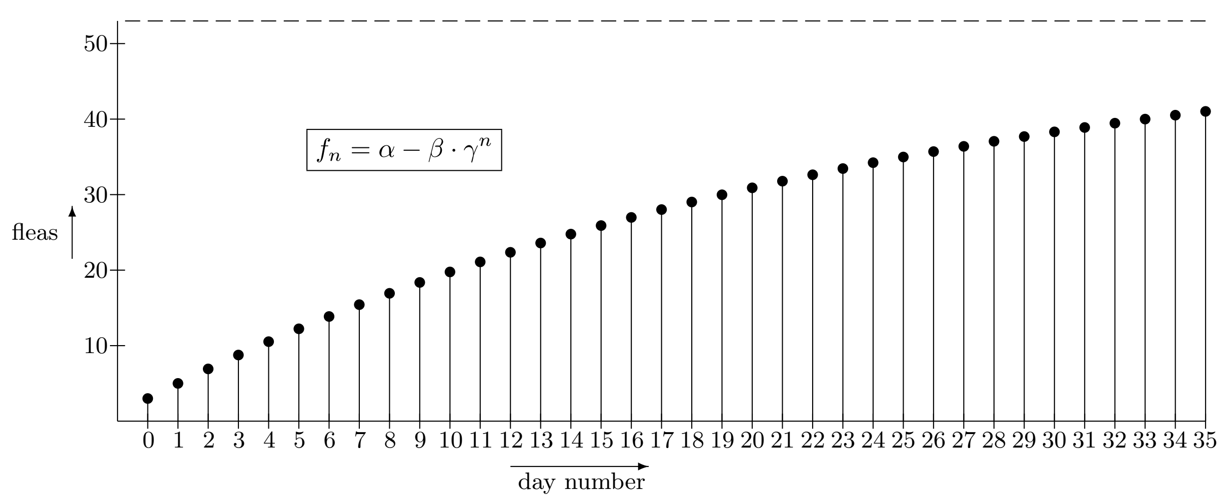
a) Rephrase $21+23+25+27+29+31+\cdots+99$ using the $\sum$-notation.

b) Calculate $21+23+25+27+29+31+\cdots+99$.

Exercise 25. On day 0 Wally had three fleas, on day 1 there were already five. The limit number of fleas on Wally is 53:

$\boxed{f_0 = 3}$ $\boxed{f_1 = 5}$ $\boxed{f_\infty = 53}$

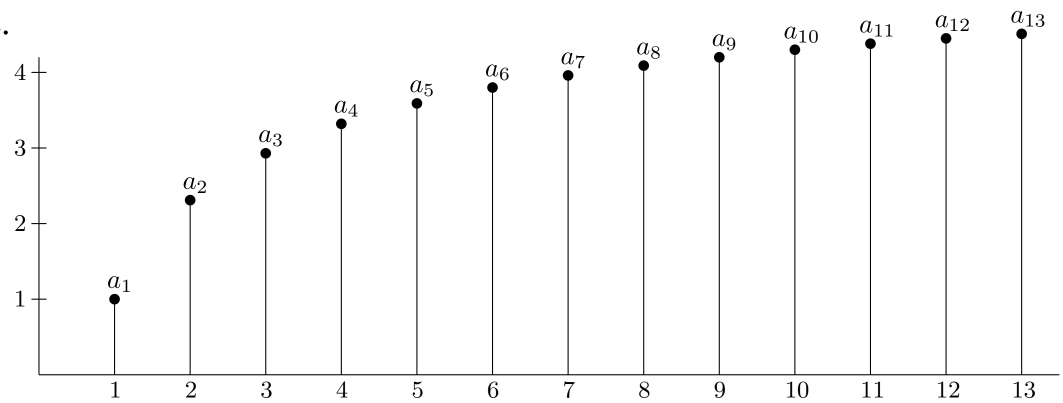
Estimate when Wally will have 40 fleas, assuming that the fleas follow the 'bounded exponential growth' model.



Exercise 26. Calculate $\sum_{n=3}^{\infty}\frac{4^n}{7^n}$.

Exercise 27. Calculate $\lim_{n\to\infty} \dfrac{1+2+3+\cdots+n}{n^2}$.

Exercise 28.



Calculate the limit of the sequence $a_1, a_2, a_3, \ldots$ defined by $\boxed{a_n = \dfrac{\sqrt[7]{n}+2\ln n}{\sqrt[5]{n}}}$.

Exercise 29.

a) How much is $\dfrac{1}{1}+\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+\dfrac{1}{9}+\cdots\cdots$?

b) How much is $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\dfrac{1}{9^2}+\cdots\cdots$?

c) Rephrase your conclusions from (a) and (b) using the $\sum$-notation.

Exercise 30. Calculate the following limits:

a) $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+\dfrac{1}{\sqrt{5}}+\dfrac{1}{\sqrt{6}}+\dfrac{1}{\sqrt{7}}+\dfrac{1}{\sqrt{8}}+\dfrac{1}{\sqrt{9}}+\dfrac{1}{\sqrt{10}}+\dfrac{1}{\sqrt{11}}+\dfrac{1}{\sqrt{12}}+\dfrac{1}{\sqrt{13}}+\cdots$

b) $1+\dfrac{3^1}{2^2}+\dfrac{3^2}{2^4}+\dfrac{3^3}{2^6}+\dfrac{3^4}{2^8}+\dfrac{3^5}{2^{10}}+\dfrac{3^6}{2^{12}}+\dfrac{3^7}{2^{14}}+\dfrac{3^8}{2^{16}}+\dfrac{3^9}{2^{18}}+\dfrac{3^{10}}{2^{20}}+\dfrac{3^{11}}{2^{22}}+\dfrac{3^{12}}{2^{24}}+\dfrac{3^{13}}{2^{26}}+\cdots$

Exercise 31. Decompose $\dfrac{5}{3+n-2n^2}$ into two simpler fractions.

Exercise 32. Calculate $\dfrac{1}{1\cdot 5}+\dfrac{1}{2\cdot 6}+\dfrac{1}{3\cdot 7}+\dfrac{1}{4\cdot 8}+\dfrac{1}{5\cdot 9}+\dfrac{1}{6\cdot 10}+\dfrac{1}{7\cdot 11}+\dfrac{1}{8\cdot 12}+\dfrac{1}{9\cdot 13}+\cdots$

Exercise 33. Calculate $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{1}{2\cdot 3\cdot 4}+\dfrac{1}{3\cdot 4\cdot 5}+\dfrac{1}{4\cdot 5\cdot 6}+\dfrac{1}{5\cdot 6\cdot 7}+\dfrac{1}{6\cdot 7\cdot 8}+\dfrac{1}{7\cdot 8\cdot 9}+\cdots$

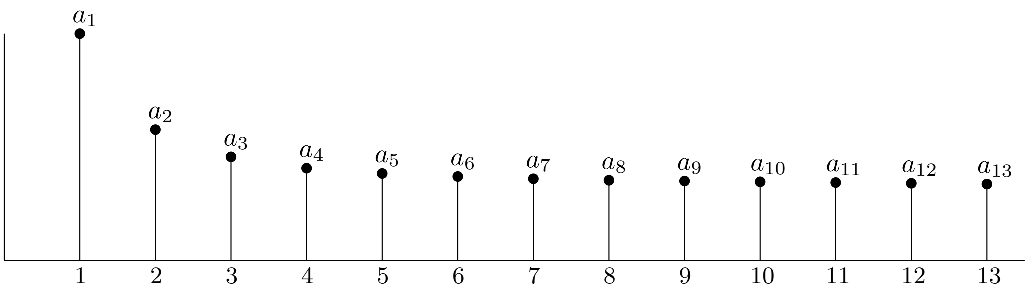
Exercise 34. Calculate the limit of the sequence

$$1\,,\ \sqrt{3}\,,\ \sqrt{2+\sqrt{3}}\,,\ \sqrt{2+\sqrt{2+\sqrt{3}}}\,,\cdots\cdots$$

(you may assume that this sequence actually has a limit).

Solutions chapter 1

Exercise 1.



Since $a_n = \frac{1}{n}\sqrt{8+n^2} = \sqrt{\frac{8}{n^2}+1}$ and $\lim_{n\to\infty} \frac{8}{n^2} = 0$, I believe that $\boxed{\lim_{n\to\infty} a_n = 1}$.

Exercise 2. Standard limit (3) says that $\ln n$ is negligible with respect to n, so

$$\lim_{n\to\infty} (n - 2\ln n) = \infty \qquad \text{(the sequence diverges)}$$

Exercise 3. $\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right) = 0$

I discovered this using the square root trick:

$$\sqrt{n^2+1} - n = \frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$$

This is less than $\frac{1}{n}$, so its limit must certainly be 0.

Exercise 4. $\lim_{n\to\infty} \frac{n^3}{2^n} = 0$

I immediately spotted this using standard limit (4) with $c = 3$ and $d = 2$.

Exercise 5. The standard limit $\lim_{n\to\infty} \frac{n^c}{d^n} = 0$ (if $c > 0$ and $d > 1$) implies that $\lim_{n\to\infty} \frac{n}{\left(\frac{11}{10}\right)^n} = 0$, so

$$\lim_{n\to\infty} \frac{7n}{\left(\frac{11}{10}\right)^n} = 0$$

The moral: cat food is much tastier than melons, so Jantje is going to be much wealthier than Pietje.

Exercise 6. The numerator will be dominated by $7n$ for very large n and the denominator by n. Hence, the limit will be 7. You can show this a little more formally by dividing numerator and denominator by n:

$$\lim_{n\to\infty} \frac{7n-5}{3\sqrt{n}+n} = \lim_{n\to\infty} \frac{7-\frac{5}{n}}{\frac{3}{\sqrt{n}}+1} = \frac{7-0}{0+1} = 7$$

Exercise 7. Let's apply the square root trick to the cats:

$$\sqrt{n+6\sqrt{n}} - \sqrt{n} = \frac{\left(\sqrt{n+6\sqrt{n}}-\sqrt{n}\right)\left(\sqrt{n+6\sqrt{n}}+\sqrt{n}\right)}{\sqrt{n+6\sqrt{n}}+\sqrt{n}} = \frac{6\sqrt{n}}{\sqrt{n+6\sqrt{n}}+\sqrt{n}} = \frac{6}{\sqrt{1+\frac{6}{\sqrt{n}}}+1}$$

The limit is then $\frac{6}{1+1} = 3$. Nijmegen will have 3000 more cats than Arnhem.

Exercise 8.

a) $b_1 = \alpha v_0 + \beta$ implies $\boxed{\beta = 50}$ and from $v_2 = \alpha v_1 + \beta$ I conclude that $\boxed{\alpha = 0.8}$.

b) Zompie's ultimate hunting achievement is $v_\infty = \lim\limits_{n\to\infty} v_n$ and I calculate this as follows:

$$v_\infty = \alpha v_\infty + \beta \quad\Longrightarrow\quad v_\infty = 0.8\, v_\infty + 50 \quad\Longrightarrow\quad 0.2\, v_\infty = 50 \quad\Longrightarrow\quad \boxed{v_\infty = 250}$$

Exercise 9.

a) The yield c_n in the year n is of the kind $c_n = \alpha n + \beta$. From the data I can calculate α and β:

$$\left.\begin{array}{lcl} c_{2010} = 15 & \Longrightarrow & 2010\,\alpha + \beta = 15 \\ c_{2014} = 23 & \Longrightarrow & 2014\,\alpha + \beta = 23 \end{array}\right\} \overset{-}{\Longrightarrow} \quad 4\alpha = 8 \quad\Longrightarrow\quad \alpha = 2 \quad\Longrightarrow\quad \beta = -4005$$

so $c_n = 2n - 4005$.

b) I calculate when c_n exceeds 100:

$$c_n > 100 \quad\Longrightarrow\quad 2n - 4005 > 100 \quad\Longrightarrow\quad 2n > 4105 \quad\Longrightarrow\quad n > 2052.5$$

In 2053 I expect to harvest more than 100 cherries provided that the tree manages to continue its linear behaviour.

Exercise 10. The amount of substance s_n after n years satisfies $s_n = s_0 \cdot 0.99^n$ and decreases exponentially. The original amount s_0 is halved when

$$s_n = \frac{1}{2} s_0 \quad\Longrightarrow\quad 0.99^n = 0.5 \quad\Longrightarrow\quad n = \frac{\ln 0.5}{\ln 0.99} \approx 69$$

Thus, the half-life is approximately 69 years.

Exercise 11. Let the number of fleas on day n be $f_n = \alpha \cdot \beta^n$. I substitute the data:

$$\left.\begin{array}{lcl} f_1 = 2 & \Longrightarrow & \alpha\beta = 2 \\ f_2 = 3 & \Longrightarrow & \alpha\beta^2 = 3 \end{array}\right\} \overset{\text{divide}}{\Longrightarrow} \quad \beta = \frac{3}{2} \quad\Longrightarrow\quad \alpha = \frac{4}{3} \quad\Longrightarrow\quad f_n = \frac{4}{3} \cdot \left(\frac{3}{2}\right)^n$$

which equals 10000 when

$$\left(\frac{3}{2}\right)^n = 7500 \quad \overset{\text{ln-trick}}{\Longrightarrow} \quad n = \frac{\ln 7500}{\ln 1.5} \approx 22$$

Exercise 12. The glass plate on top transmits 90%, the second plate transmits 90% of this 90%, which is 81%, and so forth. Hence, a construction of n stacked glass plates transmits a fraction of 0.9^n of the light. I have to calculate when this equals 0.1:

$$0.9^n = 0.1 \quad\Longrightarrow\quad \ln 0.9^n = \ln 0.1 \quad\Longrightarrow\quad n \ln 0.9 = \ln 0.1 \quad\Longrightarrow\quad n = \frac{\ln 0.1}{\ln 0.9}$$

If you happen to carry a calculator with you, you know that you have to buy 22 glass plates.

Exercise 13. Let's call the length of the bottom beam L. Then, the cumulative length is

$$L + \frac{L}{2} + \frac{L}{4} + \frac{L}{8} + \frac{L}{16} + \frac{L}{32} = L\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5\right) = L \cdot \frac{1 - (\frac{1}{2})^6}{1 - \frac{1}{2}} = \frac{63}{32} L$$

Since the cumulative length equals 9 metres, $L = \dfrac{32}{7}$ (which is about 4 metres and 57 cm).

Exercise 14. $12 + 13 + \cdots + 36$ is an arithmetic sequence because the difference between two subsequent terms is always the same. We have a beautiful method at our disposal to sum a sequence like this:

1. count the number of terms: there are 25 here
2. find their average value by taking the average of the first and last term
3. the sum of the sequence is (number of terms) · (average value) $= 25 \cdot \frac{12+36}{2} = 600$

Exercise 15.

a) $1 + 5 + 9 + 13 + 17 + 21 + \cdots + 201 =$ (number of terms) · (average value) $= 51 \cdot 101 = 5151$

b) $\displaystyle\sum_{n=0}^{50} (1 + 4n) = 5151$

Exercise 16.

a) The length of the nth segment is $4 \cdot (0.9)^{n-1}$, so the length of the spiral is

$$4\left(1 + 0.9 + (0.9)^2 + (0.9)^3 + \cdots + (0.9)^{13}\right) = 4 \cdot \frac{1 - (0.9)^{14}}{1 - 0.9} = 40\left(1 - (0.9)^{14}\right) \approx 30.85 \text{ cm}$$

b) The length of the infinite spiral is $4\left(1 + 0.9 + (0.9)^2 + (0.9)^3 + (0.9)^4 + \cdots\right) = 4 \cdot \dfrac{1}{1 - 0.9} = 40$ cm

Exercise 17. I hope that you've spotted the regularity: each term is 1/3 of its predecessor. This is called a geometric (or exponential) series, which can be calculated with the formula

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

by substituting $x = \frac{1}{3}$. Unfortunately, the first term is $\frac{2}{3}$ instead of 1, but this doesn't panic you: you just factor out $\frac{2}{3}$:

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots = \frac{2}{3} \cdot \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right) = \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 1$$

Exercise 18. This can be done using partial fraction decomposition: $\dfrac{1}{n^2+n} = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$ so

$$\sum_{n=3}^{56} \frac{1}{n^2+n} = \sum_{n=3}^{56}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{56} - \frac{1}{57}\right) = \frac{1}{3} - \frac{1}{57} = \frac{6}{19}$$

Exercise 19. The sum of these twenty numbers is

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{20} = \frac{1}{2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{19}\right) = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^{20}$$

so their average is precisely $0.05 - \dfrac{0.05}{2^{20}}$, and to five decimal places this equals 0.05000.

Exercise 20. In my arithmetic booklet I found the formula $p^q = e^{q \ln p}$:

$$\frac{2^{\ln 7}}{7^{\ln 2}} = \frac{e^{(\ln 7)\cdot(\ln 2)}}{e^{(\ln 2)\cdot(\ln 7)}} = 1$$

Exercise 21.

a) This is an arithmetic sequence, the difference between two subsequent terms is 3. You can calculate the sum of such a sequence by multiplying the number of terms (101 in this case) by their average value. You can find this average value by averaging the first and last term. Thus, the calculation is

$$\sum_{k=0}^{100}(3k+2) = 101 \cdot \frac{2+302}{2} = 15352$$

b) This series is slightly more difficult, because it is neither linear nor exponential. A good first step is to split the series into two simpler ones:

$$\sum_{k=0}^{100}(3^k+2) = \sum_{k=0}^{100} 3^k + \sum_{k=0}^{100} 2$$

You might have doubts about this step: is this allowed?? Should I understand that this is allowed or should I just take it for granted?? You should understand this: it's just repeated application of the ancient rule $x+y=y+x$:

$$\sum_{k=0}^{100}(3^k+2) = (3^0+2)+\cdots+(3^{100}+2) = (3^0+\cdots+3^{100})+(2+\cdots+2) = \sum_{k=0}^{100} 3^k + \sum_{k=0}^{100} 2$$

The rest is easy:

- $\displaystyle\sum_{k=0}^{100} 3^k = \frac{3^{101}-1}{2}$ (since this is a geometric series)
- $\displaystyle\sum_{k=0}^{100} 2 = \underbrace{2+2+2+2+2+2+2+2+\cdots+2}_{\text{101 twos}} = 202$
- So $\displaystyle\sum_{k=0}^{100}(3^k+2) = \frac{3^{101}-1}{2} + 202 = 773066281098016996554691694648431909053161283203$

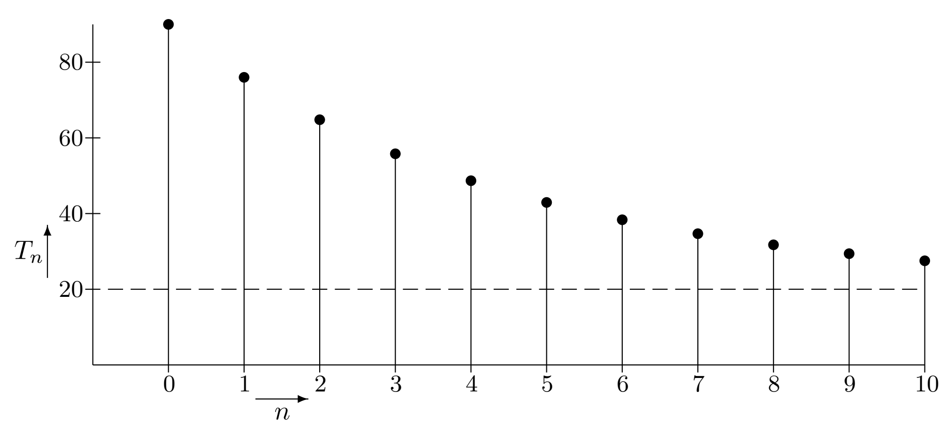
Exercise 22. The temperature decrease of the tea is proportional to the difference between the actual temperature and the ambient temperature. The sequence $T_0, T_1, T_2, T_3, \ldots$ behaves according to the 'bounded exponential growth' model (which should rather be called 'bounded exponential decay' in this context) with limit temperature 20:

$$T_n = 20 + \beta \cdot \gamma^n$$

I calculate the constants β and γ by substituting the data:

- $T_0 = 90$ implies that $20+\beta = 90 \implies \beta = 70 \implies T_n = 20 + 70 \cdot \gamma^n$
- $T_1 = 76$ implies that $20 + 70\gamma = 76 \implies \gamma = 0.8 \implies T_n = 20 + 70 \cdot 0.8^n$

After ten minutes of waiting the temperature of my tea is $T_{10} = 20 + 70 \cdot 0.8^{10} \approx 27.5$ degrees Celsius.



Exercise 23. We use partial fraction decomposition on $\dfrac{1}{n(n+2)}$ to obtain $\dfrac{A}{n}+\dfrac{B}{n+2}$ with

$$\frac{A}{n}+\frac{B}{n+2}=\frac{A(n+2)+Bn}{n(n+2)}=\frac{(A+B)n+2A}{n(n+2)} \quad\Longrightarrow\quad \begin{cases} A+B=0 \\ 2A=1 \end{cases} \quad\Longrightarrow\quad \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$$

The discovered partial fraction decomposition is $\dfrac{1}{n(n+2)}=\dfrac{\frac{1}{2}}{n}-\dfrac{\frac{1}{2}}{n+2}$ so

$$\begin{aligned}
&\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{999\cdot 1001}\\
&=\left(\frac{\frac{1}{2}}{1}-\frac{\frac{1}{2}}{3}\right)+\left(\frac{\frac{1}{2}}{2}-\frac{\frac{1}{2}}{4}\right)+\left(\frac{\frac{1}{2}}{3}-\frac{\frac{1}{2}}{5}\right)+\left(\frac{\frac{1}{2}}{4}-\frac{\frac{1}{2}}{6}\right)+\left(\frac{\frac{1}{2}}{5}-\frac{\frac{1}{2}}{7}\right)+\cdots+\left(\frac{\frac{1}{2}}{999}-\frac{\frac{1}{2}}{1001}\right)\\
&=\frac{\frac{1}{2}}{1}+\frac{\frac{1}{2}}{2}-\frac{\frac{1}{2}}{1000}-\frac{\frac{1}{2}}{1001}=\frac{1499499}{2002000}
\end{aligned}$$

Exercise 24. $\displaystyle\sum_{n=0}^{39}(2n+21)=(\text{number of terms})\cdot(\text{average value})=40\cdot 60=2400$

Exercise 25. α is the limit population:

$$f_n=\alpha-\beta\cdot\gamma^n \quad\overset{f_\infty=53}{\Longrightarrow}\quad f_n=53-\beta\cdot\gamma^n$$

Substitution of the data allows me to calculate the constants β and γ:

$$\left.\begin{array}{lllll} f_0=3 & \Longrightarrow & 53-\beta=3 & \Longrightarrow & \beta=50 \\ f_1=5 & \Longrightarrow & 53-\beta\gamma=5 & \Longrightarrow & \beta\gamma=48 \end{array}\right\} \overset{\text{divide}}{\Longrightarrow} \quad \gamma=0.96 \quad\Longrightarrow\quad f_n=53-50\cdot 0.96^n$$

and this equals 40 when $50\cdot 0.96^n=13$, which comes down to

$$0.96^n=0.26 \quad\Longrightarrow\quad \ln 0.96^n=\ln 0.26 \quad\Longrightarrow\quad n\ln 0.96=\ln 0.26 \quad\Longrightarrow\quad n=\frac{\ln 0.26}{\ln 0.96}$$

My calculator says that this is about 33, so after 33 days Wally can welcome flea number 40.

Exercise 26. Apply the formula for the summation of geometric series with $x=\frac{4}{7}$ and get rid of the first few terms:

$$\begin{aligned}
& 1+x+x^2+x^3+x^4+x^5+\cdots=\frac{1}{1-x}\\
\Longrightarrow\quad & 1+\frac{4}{7}+\left(\frac{4}{7}\right)^2+\left(\frac{4}{7}\right)^3+\left(\frac{4}{7}\right)^4+\left(\frac{4}{7}\right)^5+\left(\frac{4}{7}\right)^6+\left(\frac{4}{7}\right)^7+\cdots=\frac{7}{3}\\
\Longrightarrow\quad & \left(\frac{4}{7}\right)^3+\left(\frac{4}{7}\right)^4+\left(\frac{4}{7}\right)^5+\left(\frac{4}{7}\right)^6+\left(\frac{4}{7}\right)^7+\cdots=\boxed{\frac{64}{147}}
\end{aligned}$$

Exercise 27. First, we apply the method for the summation of linear series to calculate $1+2+3+\cdots+n$:

$$\begin{aligned}
1+2+3+\cdots+n &= (\text{number of terms})\cdot(\text{average value})\\
&= n\cdot\frac{(\text{first term})+(\text{last term})}{2}=n\cdot\frac{1+n}{2}=\frac{1}{2}n^2+\frac{1}{2}n
\end{aligned}$$

Now, the calculation of the desired limit is not very complicated:

$$\lim_{n\to\infty}\frac{1+2+3+\cdots+n}{n^2}=\lim_{n\to\infty}\frac{\frac{1}{2}n^2+\frac{1}{2}n}{n^2}=\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)=\frac{1}{2}$$

Exercise 28. I split this problem into two pieces:

$$\frac{\sqrt[7]{n}+2\ln n}{\sqrt[5]{n}} = \frac{\sqrt[7]{n}}{\sqrt[5]{n}} + 2\cdot\frac{\ln n}{\sqrt[5]{n}} = n^{\frac{1}{7}-\frac{1}{5}} + 2\cdot\frac{\ln n}{\sqrt[5]{n}} = n^{-\frac{2}{35}} + 2\cdot\frac{\ln n}{\sqrt[5]{n}}$$

so

$$\lim_{n\to\infty}\frac{\sqrt[7]{n}+2\ln n}{\sqrt[5]{n}} = \lim_{n\to\infty}\left(n^{-\frac{2}{35}} + 2\cdot\frac{\ln n}{\sqrt[5]{n}}\right) = 0+2\cdot 0 = 0$$

Exercise 29.

a) $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=\infty$. You can for instance prove this by contradiction:

Suppose $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=p$ with p a real number. (1)

Then, $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots<p$ since the nth term is less than the nth term in (1). (2)

So $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots<2p$ since this is the sum of (1) and (2).

However, in example 22 we found $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\infty$, so we have $\infty<2p:$ a contradiction!
Hence, our supposition (1) must be wrong, so $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots=\infty$.

b) I use a few neat tricks to calculate this:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6} \quad \text{(see example 23)} \qquad (1)$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots=\frac{\pi^2}{24} \quad \text{(which is (1) divided by 4)} \qquad (2)$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}-\frac{\pi^2}{24} \quad \text{(which is (1) minus (2))}$$

Thus, the solution is $\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}$.

c) $\sum_{n=0}^{\infty}\frac{1}{2n+1}=\infty$ and $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$

Exercise 30.

a) This can be done as follows:

- The fact that $\frac{1}{\sqrt{n}}\geq\frac{1}{n}$ implies that $1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\geq 1+\frac{1}{2}+\cdots+\frac{1}{n}$.
- Since $1+\frac{1}{2}+\frac{1}{3}+\cdots=\infty$ (see example 22), we can conclude that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots=\infty$.

b) $1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4+\left(\frac{3}{4}\right)^5+\cdots=\frac{1}{1-\frac{3}{4}}=4$

Exercise 31. Since I can write $3+n-2n^2$ as $(1+n)(3-2n)$, I expect this to be possible:

$$\frac{5}{(1+n)(3-2n)}=\frac{A}{1+n}+\frac{B}{3-2n}$$

To determine A and B I write the right-hand side as a single fraction:

$$\frac{A}{1+n}+\frac{B}{3-2n}=\frac{A(3-2n)+B(1+n)}{(1+n)(3-2n)}=\frac{(3A+B)+(-2A+B)n}{(1+n)(3-2n)}$$

The numerator should equal the number 5, which leads to two equations with two unknowns:

$$\left.\begin{aligned} 3A+B&=5\\ -2A+B&=0\end{aligned}\right\} \implies \left\{\begin{aligned} A&=1\\ B&=2\end{aligned}\right\} \implies \frac{5}{3+n-2n^2}=\frac{1}{1+n}+\frac{2}{3-2n}$$

(As you noticed, I presume that you have not the slightest difficulty in solving two equations with two unknowns; in case you do, be sure to panic in time and study chapter 7 of my arithmetic booklet carefully or ask your teaching assistant.)

Exercise 32. Using the procedure of example 21 you find a suitable partial fraction decomposition:

$$\frac{1}{n(n+4)}=\frac{1/4}{n}-\frac{1/4}{n+4}$$

and when you decompose all terms this way a lot of them will cancel; the result is

$$\left(\frac{1/4}{1}-\frac{1/4}{5}\right)+\left(\frac{1/4}{2}-\frac{1/4}{6}\right)+\left(\frac{1/4}{3}-\frac{1/4}{7}\right)+\left(\frac{1/4}{4}-\frac{1/4}{8}\right)+\left(\frac{1/4}{5}-\frac{1/4}{9}\right)+\left(\frac{1/4}{6}-\frac{1/4}{10}\right)+\cdots$$
$$=\frac{1/4}{1}+\frac{1/4}{2}+\frac{1/4}{3}+\frac{1/4}{4}=\boxed{\frac{25}{48}}$$

Exercise 33. I would love to decompose the monstrosity $\dfrac{1}{(n-1)\cdot n\cdot(n+1)}$ into simpler fractions:

$$\frac{1}{(n-1)\cdot n\cdot(n+1)}=\frac{A}{n-1}+\frac{B}{n}+\frac{C}{n+1}$$

To discover what A, B and C should be I give the whole kit and caboodle to the same denominator:

$$\frac{A}{n-1}+\frac{B}{n}+\frac{C}{n+1}=\frac{An(n+1)+B(n-1)(n+1)+Cn(n-1)}{(n-1)\cdot n\cdot(n+1)}=\frac{(A+B+C)n^2+(A-C)n-B}{(n-1)\cdot n\cdot(n+1)}$$

The numerator should be equal to 1, leading to three equations with three unknowns:

$$\left.\begin{aligned} A+B+C&=0\\ A-C&=0\\ -B&=1\end{aligned}\right\} \implies \left\{\begin{aligned} A&=\tfrac{1}{2}\\ B&=-1\\ C&=\tfrac{1}{2}\end{aligned}\right\} \implies \frac{1}{(n-1)\cdot n\cdot(n+1)}=\frac{1/2}{n-1}-\frac{1}{n}+\frac{1/2}{n+1}$$

Now, it's just a matter of substitution and praying that most of the mess will cancel:

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\frac{1}{4\cdot5\cdot6}+\frac{1}{5\cdot6\cdot7}+\frac{1}{6\cdot7\cdot8}+\frac{1}{7\cdot8\cdot9}+\frac{1}{8\cdot9\cdot10}+\cdots$$
$$=\left(\frac{1/2}{1}-\frac{1}{2}+\frac{1/2}{3}\right)+\left(\frac{1/2}{2}-\frac{1}{3}+\frac{1/2}{4}\right)+\left(\frac{1/2}{3}-\frac{1}{4}+\frac{1/2}{5}\right)+\left(\frac{1/2}{4}-\frac{1}{5}+\frac{1/2}{6}\right)+\left(\frac{1/2}{5}-\frac{1}{6}+\frac{1/2}{7}\right)+\cdots$$
$$=\frac{1/2}{1}-\frac{1}{2}+\frac{1/2}{2}=\boxed{\frac{1}{4}}$$

Exercise 34. We are dealing with the sequence $a_1, a_2, a_3, \ldots$ given by

$$\boxed{a_1=1} \qquad \boxed{a_{n+1}=\sqrt{2+a_n}}$$

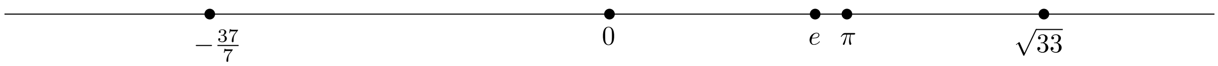
Let's call the desired limit a_∞. Then, $a_{n+1}=\sqrt{2+a_n}$ implies that

$$a_\infty=\sqrt{2+a_\infty} \implies a_\infty^2=2+a_\infty \implies a_\infty^2-a_\infty-2=0 \implies (a_\infty-2)(a_\infty+1)=0$$

Hence, the desired limit is 2 or -1. Obviously, it can't be negative (you see why?), so the limit is 2.

2. Numbers and vectors

In this chapter we are going to perform calculations using numbers, trigonometric functions and $\mathbb{R}^2$ vectors. In part this will be a recap of high school mathematics, but hopefully you will encounter some new material.



Real numbers. A real number is (by definition) a point on the line drawn above, which extends infinitely far to the left and to the right (due to the limited space on this page I've only drawn a small part). Creative minds have given beautiful names to some of these numbers, for example $\sqrt{33}$, π , e , $-\frac{37}{7}$ and 13. But most real numbers (still) live anonymous lives.

Intervals. The set of all real numbers is denoted by $\mathbb{R}$. By $x \in \mathbb{R}$ we mean: x is an element of $\mathbb{R}$. Or, simply put: x is a real number. In addition, we define some practical notation for 'intervals', i.e. uninterrupted subsets of $\mathbb{R}$:

$$[a, b] \stackrel{\text{def}}{=} \text{the set of all real numbers } x \text{ with } a \leq x \leq b$$

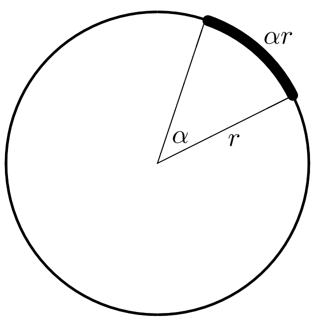
$$(a, b) \stackrel{\text{def}}{=} \text{the set of all real numbers } x \text{ with } a < x < b$$

$$[a, b) \stackrel{\text{def}}{=} \text{the set of all real numbers } x \text{ with } a \leq x < b$$

$$(a, b] \stackrel{\text{def}}{=} \text{the set of all real numbers } x \text{ with } a < x \leq b$$

$$[a, \infty) \stackrel{\text{def}}{=} \text{the set of all real numbers } x \text{ with } a \leq x$$

$$(-\infty, b) \stackrel{\text{def}}{=} \text{the set of all real numbers } x \text{ with } x < b$$



Radians and arc length. From the definition of 'radian' (arithmetic booklet, chapter 6) we can derive a nice formula for the length of a circular arc with radius r and angle α with the centre:

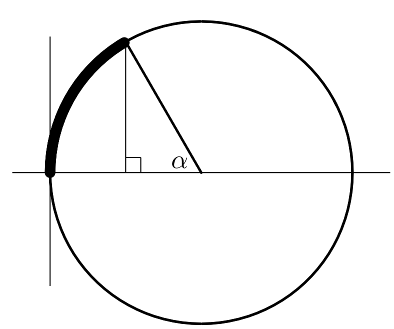
$$\boxed{\text{arc length} = \alpha r}$$

Example 1. Calculate the arc length of the circle $x^2 + y^2 = 4x$ between the points $(0, 0)$ and $(1, \sqrt{3})$.

Solution. Completing the square transforms the equation of the circle to its standard form (arithmetic booklet chapter 4):

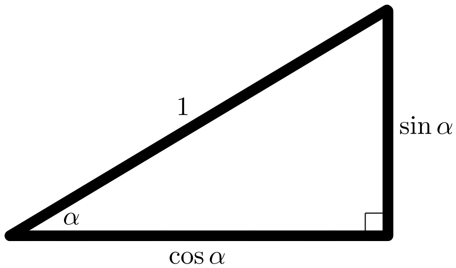
$$x^2 - 4x + y^2 = 0 \quad \Longrightarrow \quad x^2 - 4x + 4 + y^2 = 4 \quad \Longrightarrow \quad (x - 2)^2 + y^2 = 4$$

Thus, its centre is $(2, 0)$ and its radius is 2. Let's make a quick drawing:



The angle α in this drawing is $60°$, because the right-angled triangle has an adjacent side of 1 and a hypotenuse of 2, see arithmetic booklet. The arc length is the product of the radius (which is 2) and the angle α (which is $\frac{\pi}{3}$ radians):

$$\boxed{\text{arc length} = \frac{2}{3}\pi}$$



Sine and cosine. The definitions of $\sin\alpha$ and $\cos\alpha$ (refer to the arithmetic booklet) for $0 \le \alpha \le \frac{\pi}{2}$ are depicted in this figure. The following identities follow directly from the definitions:

$\cos(-\alpha) = \cos\alpha$	(cos is an even function)
$\sin(-\alpha) = -\sin\alpha$	(sin is an odd function)
$\cos(\alpha + 2\pi) = \cos\alpha$	(cos is periodic with period 2π)
$\sin(\alpha + 2\pi) = \sin\alpha$	(sin is periodic with period 2π)

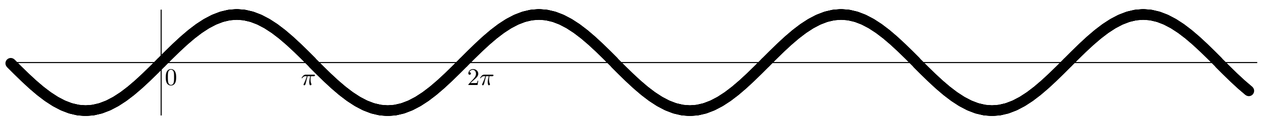
In order to perform calculations with cos and sin you will need the following identities as well:

$$\sin(\tfrac{\pi}{2} - x) = \cos x \qquad \cos(\tfrac{\pi}{2} - x) = \sin x \qquad \sin^2 x + \cos^2 x = 1$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \cos 2x = 2\cos^2 x - 1 \qquad \cos 2x = 1 - 2\sin^2 x$$

The graph of $y = \sin x$:



A few special values of the sine on the domain $[0, \frac{\pi}{2}]$ (see the special triangles in the arithmetic booklet):

$$\sin 0 = 0 \qquad \sin\tfrac{\pi}{6} = \tfrac{1}{2} \qquad \sin\tfrac{\pi}{4} = \tfrac{1}{2}\sqrt{2} \qquad \sin\tfrac{\pi}{3} = \tfrac{1}{2}\sqrt{3} \qquad \sin\tfrac{\pi}{2} = 1$$

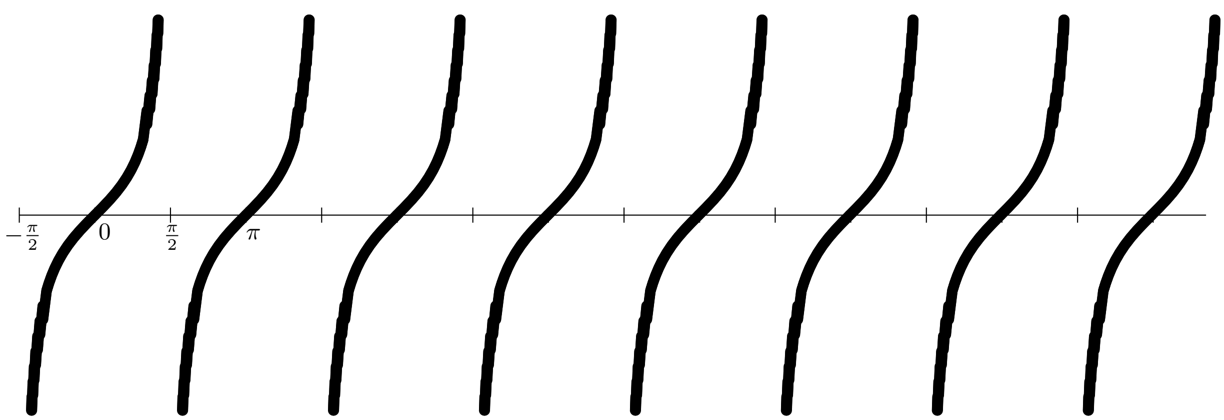
The graph of $y = \cos x$ is obtained by translating the sine graph a distance of $\frac{\pi}{2}$ to the left, because $\cos x = \sin(\frac{\pi}{2} + x)$. I will also give you some special cos values:

$$\cos 0 = 1 \qquad \cos\tfrac{\pi}{6} = \tfrac{1}{2}\sqrt{3} \qquad \cos\tfrac{\pi}{4} = \tfrac{1}{2}\sqrt{2} \qquad \cos\tfrac{\pi}{3} = \tfrac{1}{2} \qquad \cos\tfrac{\pi}{2} = 0$$

Tangent. We define $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$\tan x \stackrel{\text{def}}{=} \frac{\sin x}{\cos x}$$

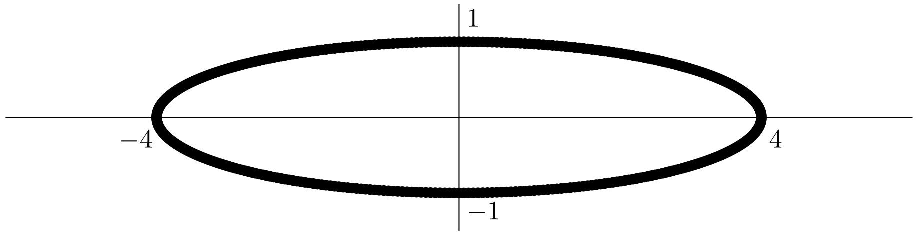
Evidently, this definition only makes sense when $\cos x \neq 0$, so the domain of tan is the set of all real numbers except the numbers $\cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$. The tangent function is periodic with period π:



Special tan values on $[0, \frac{\pi}{2})$:

$$\tan 0 = 0 \qquad \tan\tfrac{\pi}{6} = \tfrac{1}{3}\sqrt{3} \qquad \tan\tfrac{\pi}{4} = 1 \qquad \tan\tfrac{\pi}{3} = \sqrt{3}$$

Ellipses. If you stretch a circle in one dimension, you obtain an <u>ellipse</u>. Example: you stretch the unit circle by a factor of 4 in the x-direction:



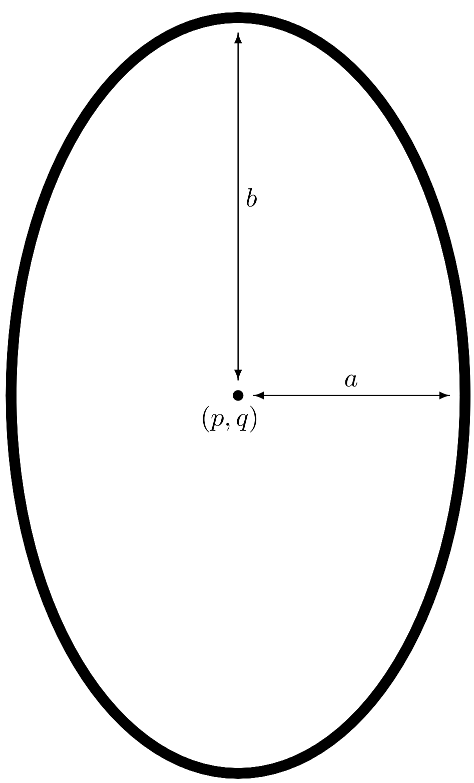
The equation of this ellipse (with semi-axes 4 and 1) can be found as follows:

- take the equation of the unit circle: $x^2 + y^2 = 1$,
- replace x by $\frac{x}{4}$ (that's what I mean by 'stretch by a factor of 4')
- and in no time you find $\left(\frac{x}{4}\right)^2 + y^2 = 1$ or, equivalently, $\frac{x^2}{16} + y^2 = 1$.

Obviously, the area of this ellipse is 4 times the area of the unit circle, so 4π. More generally: if you stretch the unit circle by a factor of a in the x-direction and by a factor of b in the y-direction, you obtain the ellipse with centre $(0, 0)$, semi-axis a and b, area πab and equation

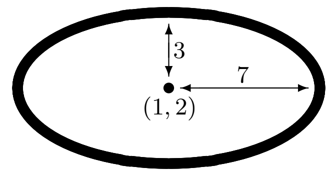
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Moreover, if I move the centre of this ellipse to (p, q), I obtain the ellipse



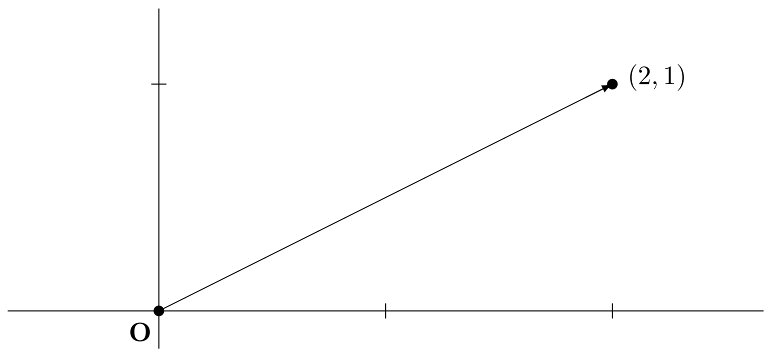
$$\boxed{\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1}$$

For example:



$$\frac{(x-1)^2}{49} + \frac{(y-2)^2}{9} = 1$$

Vectors in $\mathbb{R}^2$. A vector in $\mathbb{R}^2$ is a row (a_1, a_2) of two real numbers. The numbers a_1 and a_2 are called the coordinates of the vector. If you draw a system of axes and choose a unit length, you can represent such a vector as a point in the Cartesian plane. For example the vector $(2, 1)$:



Instead of the point $(2, 1)$ you can also draw the arrow from $(0, 0)$ to $(2, 1)$. Instead of $(0, 0)$ one also writes **O** (the origin). In this picture I've drawn the first coordinate axis horizontally and the second axis vertically, as is common practice in Nijmegen. The first coordinate axis is also called the x-axis or the x_1-axis, and the second axis the y-axis or the x_2-axis.

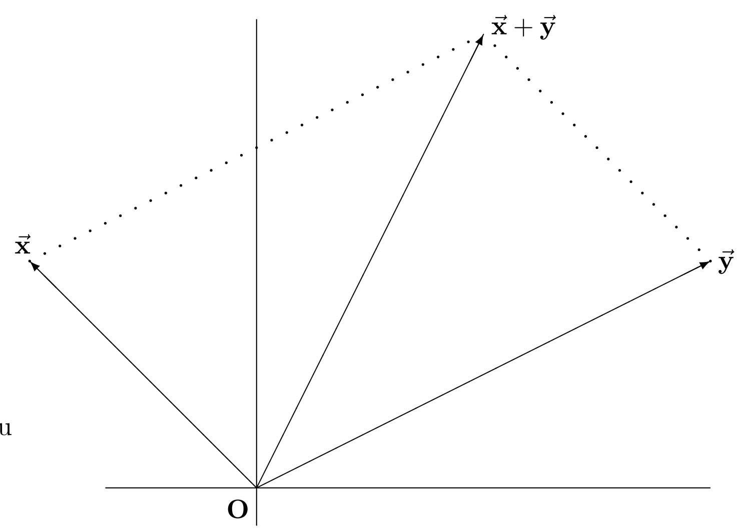
Addition of vectors. We define the sum of two vectors in $\mathbb{R}^2$ by

$$\boxed{(a_1, a_2) + (b_1, b_2) \stackrel{\text{def}}{=} (a_1 + b_1\, ,\, a_2 + b_2)} \qquad \text{(componentwise addition)}$$

Similarly, we define the difference of two vectors: $(a_1, a_2) - (b_1, b_2) \stackrel{\text{def}}{=} (a_1 - b_1, a_2 - b_2)$. Examples:

$$(2, 1) + (-1, 1) = (1, 2) \qquad (-\pi, \sqrt{7}) + (3, 0) = (3 - \pi, \sqrt{7}) \qquad \left(\frac{5}{2}, \frac{6}{7}\right) - \left(\frac{7}{3}, \frac{1}{7}\right) = \left(\frac{1}{6}, \frac{5}{7}\right)$$

Geometric interpretation of addition. Geometrically, addition of vectors amounts to drawing a parallelogram:



As you can see, I use boldface letters with an arrow on top ($\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$) as labels for vectors. This way I hope to ensure that you never ever confuse vectors with numbers.

Scalar multiplication. Let λ be a real number and let $\vec{\mathbf{x}}$ be the vector (x_1, x_2). We define:

$$\boxed{\lambda \cdot \vec{\mathbf{x}} \stackrel{\text{def}}{=} (\lambda x_1\, ,\, \lambda x_2)}$$

This kind of multiplication (number · vector) is called scalar multiplication and is 'componentwise' again. Some examples:

$$3 \cdot (2\, , -1) = (6\, , -3) \qquad \sqrt{2}\, (7\, , \sqrt{3}) = (7\sqrt{2}\, ,\, \sqrt{6}) \qquad \frac{2}{3}\, (-12, 5) = \left(-8\, ,\, \frac{10}{3}\right)$$

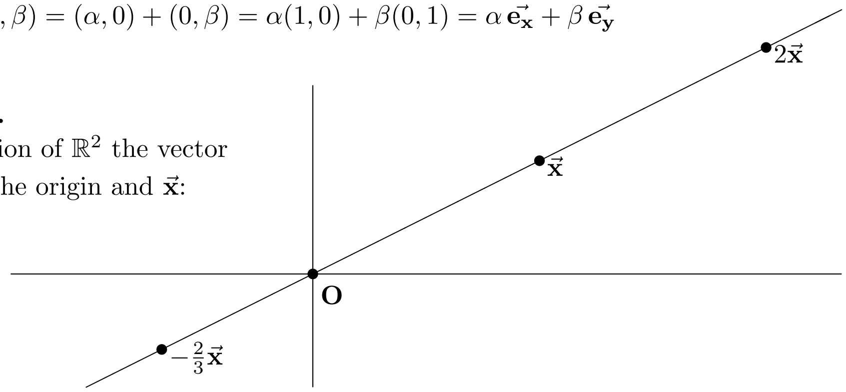
Natural basis for $\mathbb{R}^2$. The vectors $(1,0)$ and $(0,1)$ form the <u>natural basis</u> for $\mathbb{R}^2$. In physics courses the following notations are somewhat popular:

$$\boxed{\begin{aligned} \vec{\mathbf{e}}_{\mathbf{x}} &\stackrel{\text{def}}{=} (1,0) \\ \vec{\mathbf{e}}_{\mathbf{y}} &\stackrel{\text{def}}{=} (0,1) \end{aligned}} \quad \text{or sometimes} \quad \boxed{\begin{aligned} \vec{\mathbf{\imath}} &\stackrel{\text{def}}{=} (1,0) \\ \vec{\mathbf{\jmath}} &\stackrel{\text{def}}{=} (0,1) \end{aligned}}$$

Every vector from $\mathbb{R}^2$ can be expressed in terms of these basis vectors:

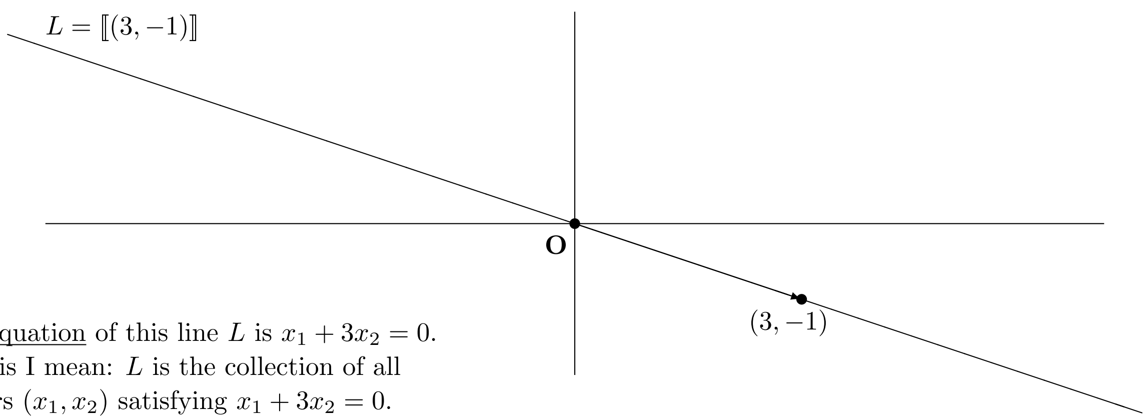
$$(\alpha, \beta) = (\alpha, 0) + (0, \beta) = \alpha(1,0) + \beta(0,1) = \alpha\, \vec{\mathbf{e}}_{\mathbf{x}} + \beta\, \vec{\mathbf{e}}_{\mathbf{y}}$$

Lines through the origin.
In our geometric interpretation of $\mathbb{R}^2$ the vector $\lambda\vec{\mathbf{x}}$ lies on the line through the origin and $\vec{\mathbf{x}}$:



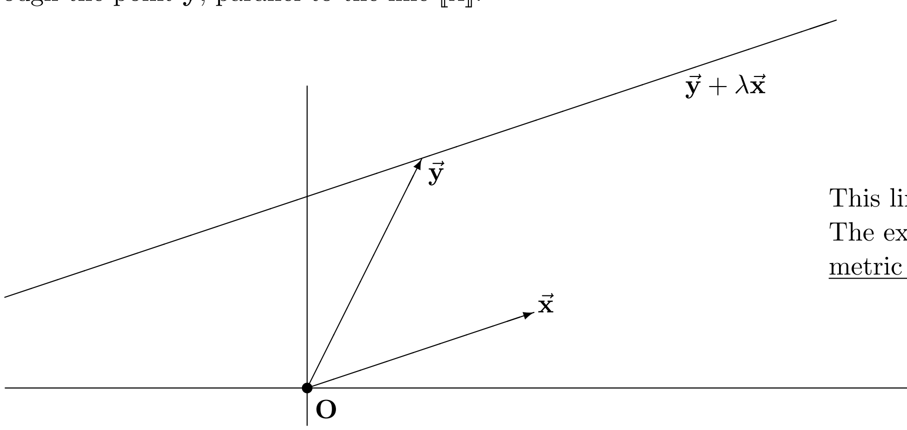
The line through $\mathbf{O}$ and $\vec{\mathbf{x}}$ is the collection of all multiples of $\vec{\mathbf{x}}$, i.e. all vectors $\lambda\vec{\mathbf{x}}$ with λ a real number. This line will be denoted by $\llbracket \vec{\mathbf{x}} \rrbracket$.

Example 2. The line $L = \llbracket (3,-1) \rrbracket$ is the collection of all multiples of the vector $(3,-1)$:



The <u>equation</u> of this line L is $x_1 + 3x_2 = 0$. By this I mean: L is the collection of all vectors (x_1, x_2) satisfying $x_1 + 3x_2 = 0$.

Other lines in $\mathbb{R}^2$. If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are vectors, then the collection of all vectors $\vec{\mathbf{y}} + \lambda\vec{\mathbf{x}}$ with $\lambda \in \mathbb{R}$ is a line through the point $\vec{\mathbf{y}}$, parallel to the line $\llbracket \vec{\mathbf{x}} \rrbracket$:

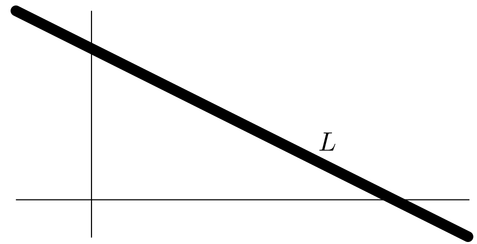


This line will be named $\vec{\mathbf{y}} + \llbracket \vec{\mathbf{x}} \rrbracket$. The expression $\vec{\mathbf{y}} + \lambda\vec{\mathbf{x}}$ is a <u>parametric representation</u> of the line.

Example 3. This line L is given by the equation

$$x_1 + 2x_2 = 2$$

Find a parametric representation of L.



Solution. From the equation $x_1 = 2 - 2x_2$ you conclude that a point (x_1, x_2) on L satisfies

$$(x_1, x_2) = (2 - 2x_2, x_2) = (2, 0) + (-2x_2, x_2) = (2, 0) + x_2 \cdot (-2, 1)$$

Now, you've found a parametric representation of L: $(2, 0) + \lambda(-2, 1)$.

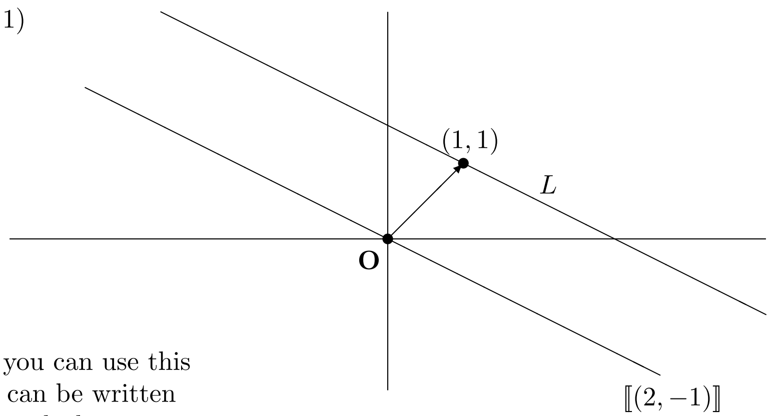
Example 4. Let L be the line $(1, 1) + [\![(2, -1)]\!]$.

a) Draw L.

b) Determine the equation of L.

Solution.

a) L is the line through the point $(1, 1)$ parallel to the line $[\![(2, -1)]\!]$:



b) In order to find the equation of L you can use this procedure: a point (x_1, x_2) on L can be written as $(1, 1) + \lambda(2, -1)$. Let's do some algebra:

$$(x_1, x_2) = (1, 1) + \lambda(2, -1) = (1, 1) + (2\lambda, -\lambda) = (1 + 2\lambda, 1 - \lambda) \implies \begin{cases} x_1 = 1 + 2\lambda \\ x_2 = 1 - \lambda \end{cases}$$

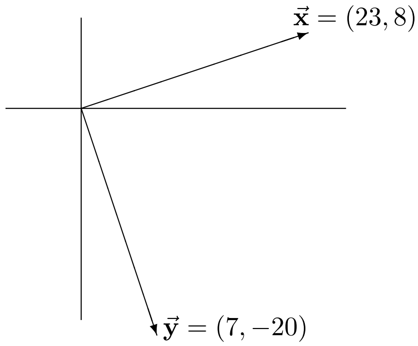
From these two equations you can eliminate λ: add the first equation ($x_1 = 1 + 2\lambda$) to the second equation multiplied by 2 ($2x_2 = 2 - 2\lambda$). This is how you find the equation of L: $x_1 + 2x_2 = 3$.

Norm and dot product. For vectors $\vec{\mathbf{x}} = (x_1, x_2)$ and $\vec{\mathbf{y}} = (y_1, y_2)$ we define:

$\|\vec{\mathbf{x}}\|$	$\stackrel{\text{def}}{=}$	$\sqrt{x_1^2 + x_2^2}$	(the <u>norm</u> of $\vec{\mathbf{x}}$)
$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}$	$\stackrel{\text{def}}{=}$	$x_1 y_1 + x_2 y_2$	(the <u>dot product</u> or <u>scalar product</u> or <u>inner product</u> of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$)
$\vec{\mathbf{x}} \perp \vec{\mathbf{y}}$	$\stackrel{\text{def}}{=}$	$(\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0)$	($\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are <u>orthogonal</u>)

From these definitions you can prove with little effort:

$$\begin{aligned} \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} &= \vec{\mathbf{y}} \bullet \vec{\mathbf{x}} \\ (\vec{\mathbf{x}} + \vec{\mathbf{y}}) \bullet \vec{\mathbf{z}} &= (\vec{\mathbf{x}} \bullet \vec{\mathbf{z}}) + (\vec{\mathbf{y}} \bullet \vec{\mathbf{z}}) \\ (\lambda\vec{\mathbf{x}}) \bullet \vec{\mathbf{y}} &= \lambda(\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}) \\ \|\vec{\mathbf{x}}\| &= \sqrt{\vec{\mathbf{x}} \bullet \vec{\mathbf{x}}} \end{aligned}$$



Example 5. For $\vec{\mathbf{x}} = (23, 8)$ and $\vec{\mathbf{y}} = (7, -20)$ we have:

$$\|\vec{\mathbf{x}}\| = \sqrt{23^2 + 8^2} = \sqrt{593}$$

$$\|\vec{\mathbf{y}}\| = \sqrt{7^2 + (-20)^2} = \sqrt{449}$$

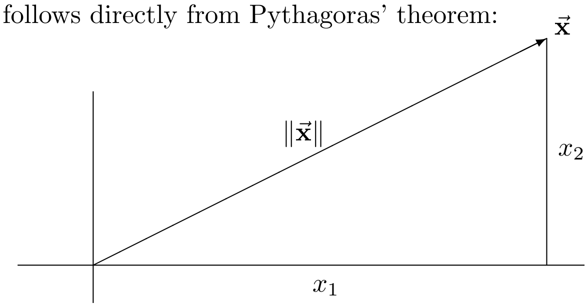
$$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 23 \cdot 7 + 8 \cdot (-20) = 1$$

$\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are not orthogonal

Length of a vector. The geometric interpretation of 'norm' is 'length':

$$\boxed{\|\vec{\mathbf{x}}\| \text{ is the length of the vector } \vec{\mathbf{x}}}$$

This follows directly from Pythagoras' theorem:



In this figure $\vec{\mathbf{x}} = (x_1, x_2)$. According to Pythagoras the hypotenuse is

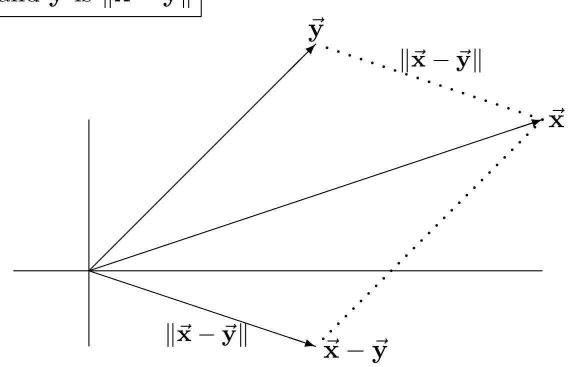
$$\sqrt{x_1^2 + x_2^2}$$

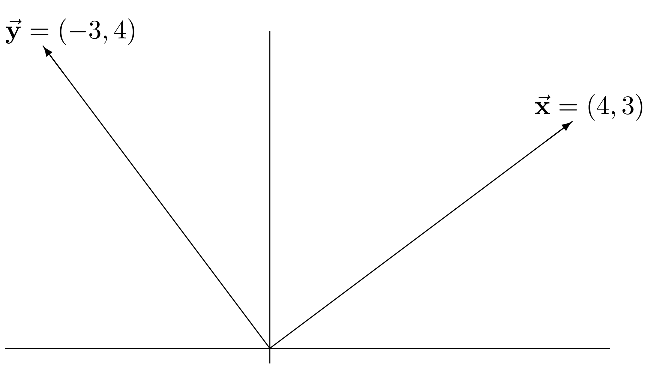
which matches our definition of $\|\vec{\mathbf{x}}\|$.

Distance between two points.

$$\boxed{\text{the distance between } \vec{\mathbf{x}} \text{ and } \vec{\mathbf{y}} \text{ is } \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|}$$

This follows from the fact that in a parallelogram two opposite sides have equal lengths:





Example 6. Let $\vec{\mathbf{x}} = (4, 3)$ and $\vec{\mathbf{y}} = (-3, 4)$.

a) Calculate the length of $\vec{\mathbf{x}}$.

b) Calculate the distance from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$.

Solution.

a) The length of $\vec{\mathbf{x}}$ is

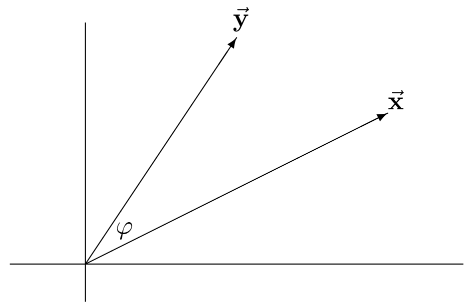
$$\|\vec{\mathbf{x}}\| = \|(4, 3)\| = \sqrt{4^2 + 3^2} = 5$$

b) The distance from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$ is

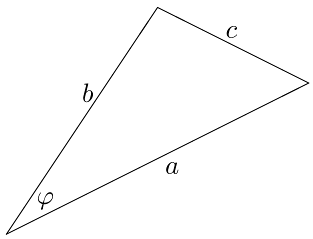
$$\|\vec{\mathbf{x}} - \vec{\mathbf{y}}\| = \|(7, -1)\| = \sqrt{50} = 5\sqrt{2}$$

Angle between two vectors.
The angle φ between two vectors $\|\vec{\mathbf{x}}\|$ and $\|\vec{\mathbf{y}}\|$ satisfies

$$\boxed{\cos\varphi = \frac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\|}}$$



Proof. This follows from the 'law of cosines' that you are going to prove yourself in exercise 20:



$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\varphi}$$

If you apply this law of cosines with $\begin{cases} a = \|\vec{\mathbf{x}}\| \\ b = \|\vec{\mathbf{y}}\| \\ c = \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\| \end{cases}$

you get: $\|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|^2 = \|\vec{\mathbf{x}}\|^2 + \|\vec{\mathbf{y}}\|^2 - 2\|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\| \cdot \cos\varphi$
and I'm going to do some editing to this formula:

- I rewrite the left-hand side using properties of the dot product:

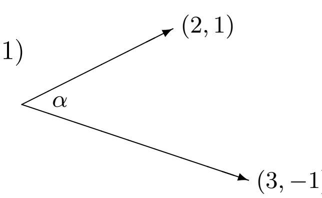
$$\begin{aligned} \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\|^2 &= (\vec{\mathbf{x}} - \vec{\mathbf{y}}) \bullet (\vec{\mathbf{x}} - \vec{\mathbf{y}}) \;=\; \vec{\mathbf{x}} \bullet (\vec{\mathbf{x}} - \vec{\mathbf{y}}) - \vec{\mathbf{y}} \bullet (\vec{\mathbf{x}} - \vec{\mathbf{y}}) \\ &= (\vec{\mathbf{x}} \bullet \vec{\mathbf{x}}) - (\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}) - (\vec{\mathbf{y}} \bullet \vec{\mathbf{x}}) + (\vec{\mathbf{y}} \bullet \vec{\mathbf{y}}) \;=\; \|\vec{\mathbf{x}}\|^2 - 2(\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}) + \|\vec{\mathbf{y}}\|^2 \end{aligned}$$

- which reduces the formula to $\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = \|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\| \cdot \cos\varphi$. □

Perpendicular. If the angle between two vectors is 90°, the cosine of this angle is 0. From the foregoing formula I can immediately conclude:

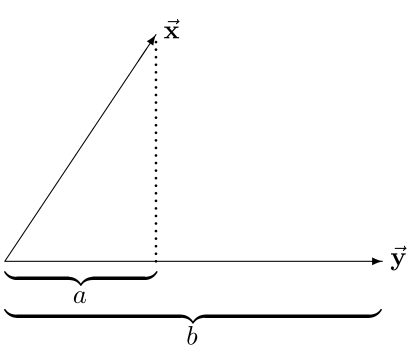
$$\boxed{\vec{\mathbf{x}} \perp \vec{\mathbf{y}} \quad \Longleftrightarrow \quad \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0}$$

Example 7. Calculate the angle α between the vectors $(2, 1)$ and $(3, -1)$



Solution. $\cos\alpha = \dfrac{(2,1) \bullet (3,-1)}{\|(2,1)\| \cdot \|(3,-1)\|} = \dfrac{5}{\sqrt{5} \cdot \sqrt{10}} = \dfrac{1}{\sqrt{2}} = \dfrac{1}{2}\sqrt{2}$
so $\alpha = \dfrac{\pi}{4}$ radians, which is 45 degrees.

Geometric interpretation of dot product. The dot product $\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}$ is the product of the length of $\vec{\mathbf{y}}$ and the length of the component of $\vec{\mathbf{x}}$ in the direction of $\vec{\mathbf{y}}$:



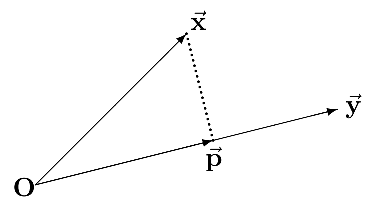
$$\boxed{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = ab}$$

Proof. If α is the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, then $a = \|\vec{\mathbf{x}}\|\cos\alpha$ and hence $\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = \|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\| \cdot \cos\alpha = ab$.

Remark. You are of course allowed to swap the roles of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$. Thus, the dot product is the product of the components in the direction of $\vec{\mathbf{x}}$ as well.

Projection on a line through the origin.

The <u>projection</u> of a vector $\vec{\mathbf{x}}$ on the line $[\![\vec{\mathbf{y}}]\!]$ is (by definition) the vector $\vec{\mathbf{p}} \in [\![\vec{\mathbf{y}}]\!]$ satisfying $\vec{\mathbf{x}} - \vec{\mathbf{p}} \perp \vec{\mathbf{y}}$.



Theorem. The projection of $\vec{\mathbf{x}}$ on $[\![\vec{\mathbf{y}}]\!]$ is the vector $\vec{\mathbf{p}} = \dfrac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\vec{\mathbf{y}} \bullet \vec{\mathbf{y}}} \cdot \vec{\mathbf{y}}$.

Proof. (in this proof λ is an abbreviation of the number $\frac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\vec{\mathbf{y}} \bullet \vec{\mathbf{y}}}$)

(1) $\vec{\mathbf{p}} \in [\![\vec{\mathbf{y}}]\!]$, because $\vec{\mathbf{p}} = \lambda \vec{\mathbf{y}}$.

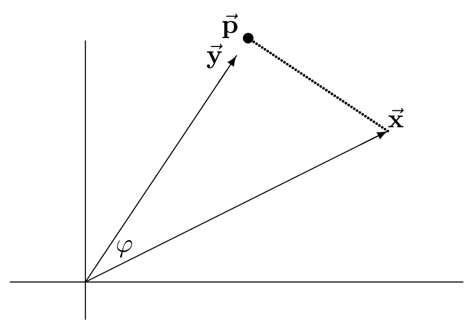
(2) $\vec{\mathbf{x}} - \vec{\mathbf{p}} \perp \vec{\mathbf{y}}$, because $(\vec{\mathbf{x}} - \vec{\mathbf{p}}) \bullet \vec{\mathbf{y}} = \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} - \vec{\mathbf{p}} \bullet \vec{\mathbf{y}} = \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} - \lambda \vec{\mathbf{y}} \bullet \vec{\mathbf{y}} = 0$. □

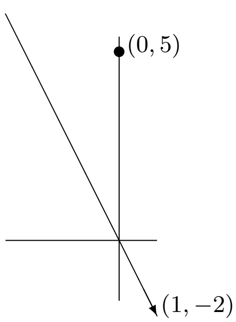
Remarks.

- You can also prove this theorem from our formula for the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.
- Another way of saying [projection on the line $[\![\vec{\mathbf{y}}]\!]$] is [projection on the vector $\vec{\mathbf{y}}$].
- $\vec{\mathbf{y}} \bullet \vec{\mathbf{y}}$ is the same as $\|\vec{\mathbf{y}}\|^2$, so you could also write the projection of $\vec{\mathbf{x}}$ on $\vec{\mathbf{y}}$ as $\dfrac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\|\vec{\mathbf{y}}\|^2} \cdot \vec{\mathbf{y}}$.

Example 8. Let $\vec{\mathbf{x}} = (4, 2)$ and $\vec{\mathbf{y}} = (2, 3)$.
Calculate the projection of $\vec{\mathbf{x}}$ on $\vec{\mathbf{y}}$.

Solution. $\vec{\mathbf{p}} = \dfrac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\vec{\mathbf{y}} \bullet \vec{\mathbf{y}}} \cdot \vec{\mathbf{y}} = \dfrac{14}{13} \cdot \vec{\mathbf{y}} = \left(\dfrac{28}{13}, \dfrac{42}{13} \right)$





Example 9. Which point on the line $[\![(1, -2)]\!]$ is closest to $(0, 5)$?

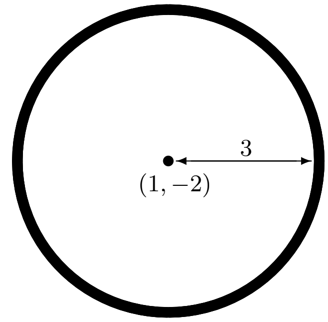
Solution. The desired point is the projection of the vector $(0, 5)$ on the line $[\![(1, -2)]\!]$, which is given by

$$\frac{(0,5) \bullet (1,-2)}{(1,-2) \bullet (1,-2)} \cdot (1,-2) = \frac{-10}{5} \cdot (1,-2) = (-2,4)$$

Different notations. In some books you may encounter notations different from the ones that I use in this chapter:

$\langle \vec{\mathbf{x}} \mid \vec{\mathbf{y}} \rangle$	instead of	$\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}$
$\lvert \vec{\mathbf{x}} \rvert$	instead of	$\|\vec{\mathbf{x}}\|$
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$	instead of	(x_1, x_2)

Exercises chapter 2



Exercise 1. Find the equation of the circle with centre $(1, -2)$ and radius 3.

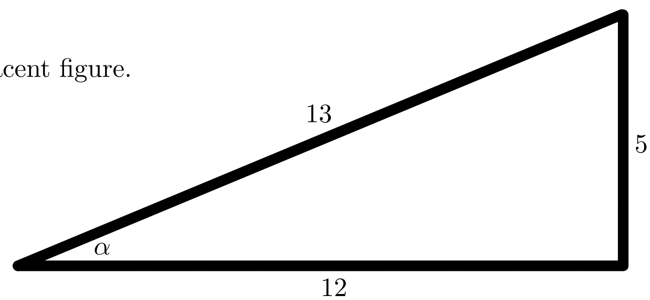
Exercise 2. Find all numbers $\varphi \in [0, 2\pi]$ that satisfy $\sin\varphi = \sin 2\varphi$.

Exercise 3.

a) Derive an expression for $\tan(x + y)$ in terms of $\tan x$ and $\tan y$.

b) Let x be a real number with $\tan x = 7$. Calculate $\tan 2x$ and $\tan 3x$.

Exercise 4. Let α be the angle depicted in the adjacent figure. Calculate $\sin 3\alpha$.

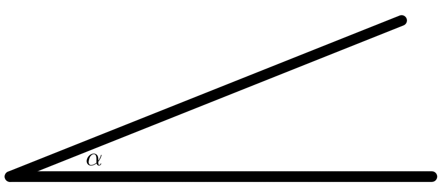


Exercise 5. Write $\sqrt{2 + 2\cos t}$ without any square roots.

Exercise 6. An ant and a beetle are walking along the x-axis during the time interval $0 \le t \le 2\pi$. Their positions at time t are given by

$$\begin{cases} \text{the ant is at the point } \cos 2t \\ \text{the beetle is at the point } \sin t \end{cases}$$

At which time(s) will these animals meet?



Exercise 7. This angle α satisfies:

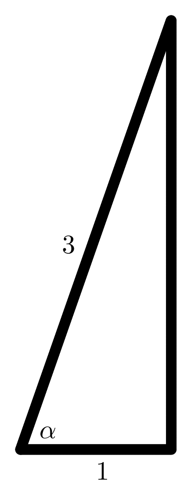
$$\boxed{\tan 2\alpha = \cos\alpha}$$

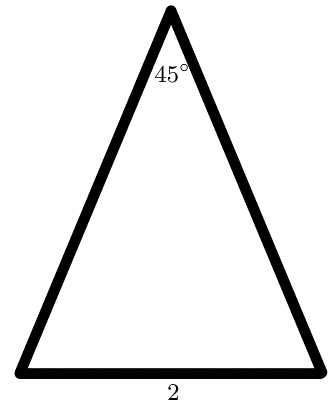
Calculate $\sin\alpha$.

Exercise 8. Let α be the angle in the adjacent figure.

a) Calculate $\tan\alpha$.

b) Calculate $\tan 2\alpha$.

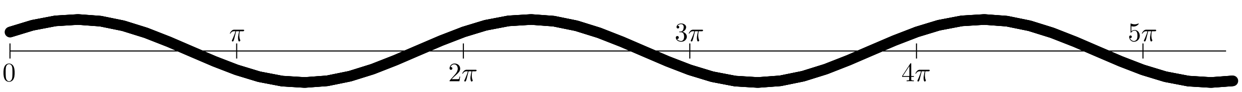




Exercise 9. This is an isosceles ('gelijkbenige') triangle with base 2 and vertex angle 45°. Calculate the length of the legs of the triangle.

Exercise 10. Prove the following identities:

$$\begin{aligned} \sin x \sin y &= \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y) \\ \cos x \cos y &= \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y) \\ \sin x \cos y &= \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y) \end{aligned}$$

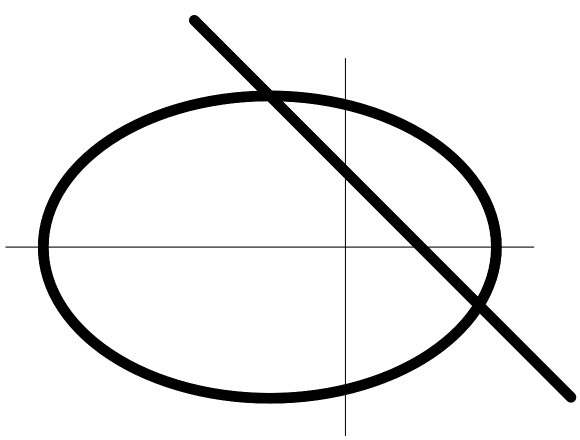


Exercise 11. Calculate the amplitude of the wave function $f(t) = \dfrac{\cos t}{4} + \dfrac{\sin t}{3}$.

Exercise 12. Find the minimum value of $x^2 - 3x + 7$ by completing the square.

Exercise 13. Draw a careful sketch of the curve with equation

$$x^2 + 9y^2 = 4x + 5$$



Exercise 14. This ellipse has equation

$$4(x+1)^2 + 9y^2 = 36$$

The line $x + y = 1$ intersects the ellipse in two points. Calculate the coordinates of these intersection points.

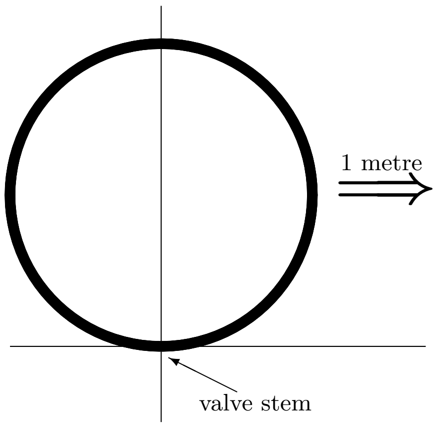
Exercise 15. Let E be the ellipse with equation

$$3x^2 + 2y^2 = 5 - x - 3y$$

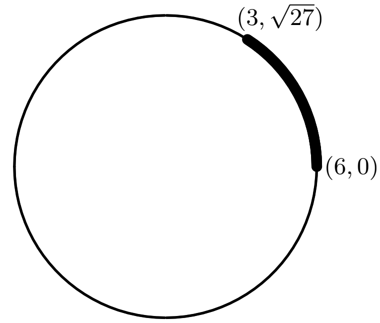
a) Find the centre of E.

b) Find the semi-axes of E.

c) Calculate the area of E.



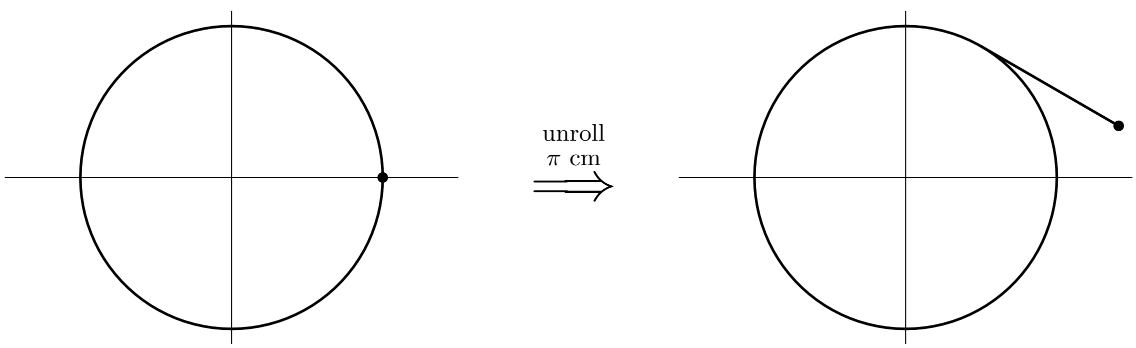
Exercise 16. (unit: metres)
The rear wheel of my bike has a radius of 0.5 metres, the valve stem is located at $(0, 0)$. Calculate the coordinates of the valve stem after cycling 1 metre in x-direction.

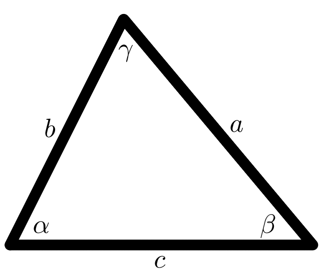


Exercise 17. Calculate the arc length between the points $(6, 0)$ and $(3, \sqrt{27})$ of the circle with equation

$$\boxed{x^2 + y^2 = 36}$$

Exercise 18. (unit: cm)
The end of a role of adhesive tape (radius: 3 cm) is initially located at the point $(3, 0)$. Where will it end up if I unroll π cm of adhesive tape while holding the centre of the roll fixed? (I pull the unrolled piece of tape in a neatly tight way, see figure.)



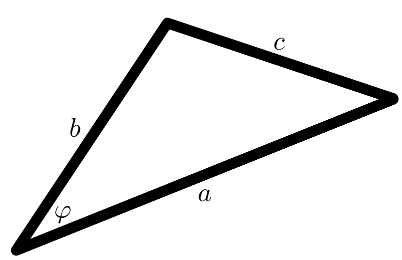


Exercise 19. Prove the law of sines:

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}$$

Exercise 20. Prove the law of cosines:

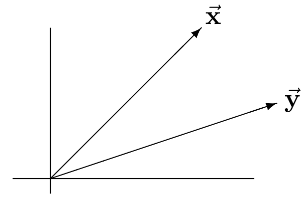
$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\varphi}$$



Exercise 21. Draw in $\mathbb{R}^2$ the lines $(2,3) + [\![(1,1)]\!]$ and $(-1,1) + [\![(3,-4)]\!]$ and find their intersection.

Exercise 22. Find a parametric representation of the line through $(1,2)$ and $(5,0)$.

Exercise 23. (geometric interpretation of the <u>difference</u> of two vectors)
Construct $\vec{\mathbf{x}} - \vec{\mathbf{y}}$ in the adjacent figure.



Exercise 24. Find the equation of the line with parametric representation $(1,3) + \lambda(5,2)$.

Exercise 25. Find a parametric representation of the line with equation $3x_1 + x_2 = 7$.

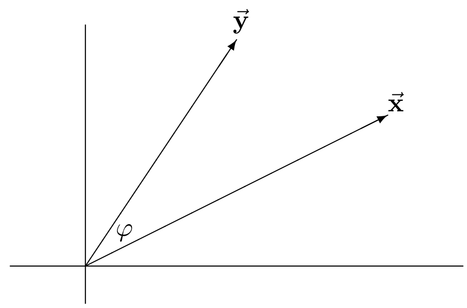
Exercise 26. Let $\vec{\mathbf{x}} = (4,2)$ and $\vec{\mathbf{y}} = (2,3)$.

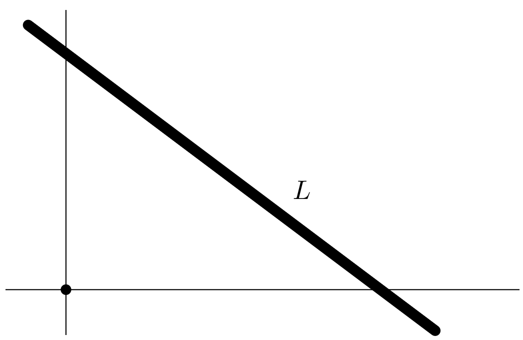
a) Calculate the dot product $\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}$.

b) Calculate $\|\vec{\mathbf{x}}\|$ and $\|\vec{\mathbf{y}}\|$.

c) Calculate the distance from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$.

d) Calculate $\cos\varphi$.





Exercise 27. Let L be the line $(7,1) + [\![(4,-3)]\!]$.

a) Find the equation of L.

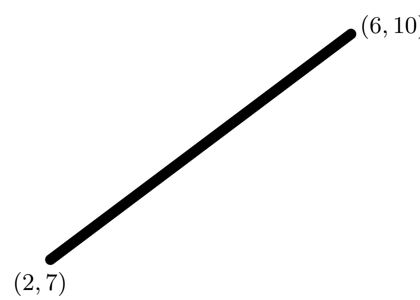
b) Calculate the distance from the origin to L.

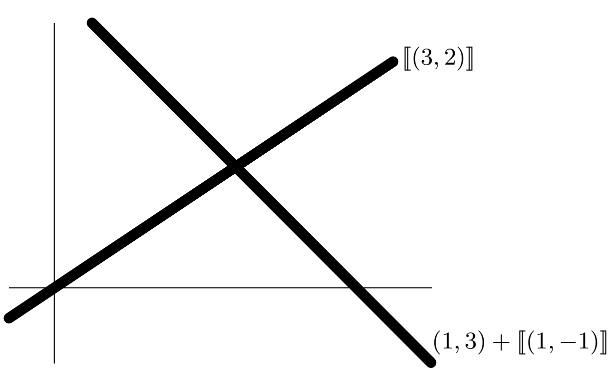
Exercise 28. (units: metres, seconds)
A little red pussycat runs with constant speed in precisely one second from the starting point $(2,7)$ to the end point $(6,10)$.

a) What is its speed?

b) Where is the pussycat after t seconds $(0 \le t \le 1)$?

c) What is its name?

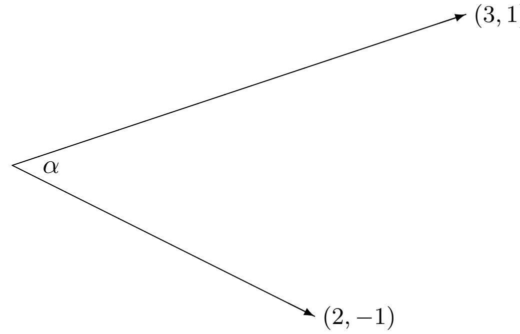


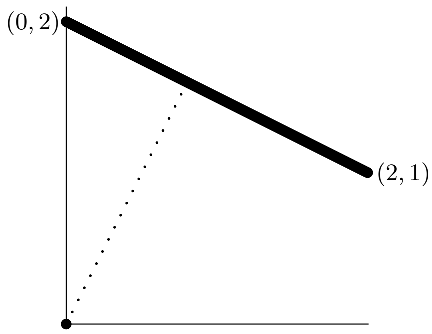


Exercise 29. Calculate the intersection point of the line $[\![(3,2)]\!]$ with the line $(1,3) + [\![(1,-1)]\!]$.

Exercise 30. Calculate the distance from the point $(6,0)$ to the line $(1,1) + [\![(3,2)]\!]$.

Exercise 31. Calculate the angle α between the vectors $(3,1)$ and $(2,-1)$.



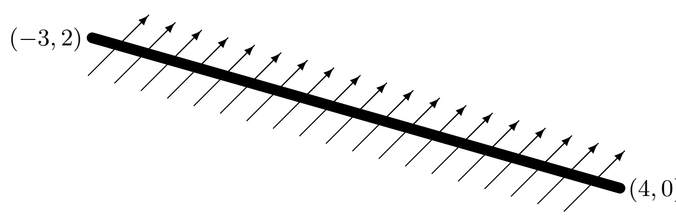


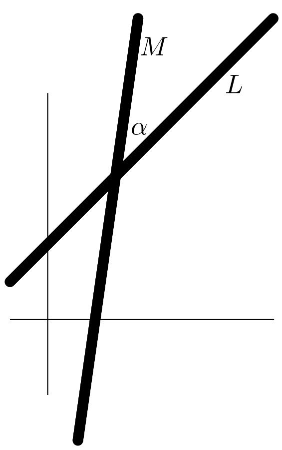
Exercise 32. Calculate the distance from the origin to the line through $(0,2)$ and $(2,1)$.

Exercise 33. (units: metres, newtons)
A poemie moves along a straight line from $(-3,2)$ to $(4,0)$ through the force field

$$\boxed{\mathbf{F}(x,y) = (1,1)}$$

Calculate the work done on the poemie by $\mathbf{F}$.

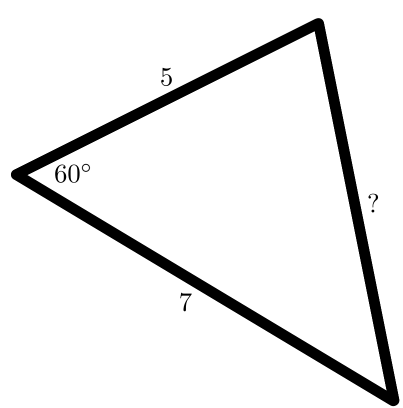




Exercise 34. Calculate $\tan\alpha$ where α is the angle between

L	$=$	the line through $(0,1)$ and $(1,2)$
M	$=$	the line through $(0,-3)$ and $(1,4)$

Exercise 35. Calculate the unknown edge of this triangle (I mean the edge with the question mark).



Solutions chapter 2

Exercise 1. The equation of a circle with centre (a, b) and radius r is $(x-a)^2+(y-b)^2=r^2$. For the given circle this becomes

$$(x-1)^2+(y+2)^2=9 \qquad \Longrightarrow \qquad \boxed{x^2-2x+y^2+4y=4}$$

Exercise 2. I use the identity $\sin 2\varphi = 2\sin\varphi\cos\varphi$:

$$\begin{aligned}\sin\varphi=\sin 2\varphi \quad &\Longleftrightarrow \quad \sin\varphi-\sin 2\varphi=0 &&\Longleftrightarrow \quad (\sin\varphi)\cdot(1-2\cos\varphi)=0\\ &\Longleftrightarrow \quad \sin\varphi=0 \ \text{ or } \ \cos\varphi=\tfrac{1}{2} &&\Longleftrightarrow \quad \varphi\in\{0,\pi,2\pi\} \ \text{ or } \ \varphi\in\{\tfrac{\pi}{3}\,,\,\tfrac{5\pi}{3}\}\end{aligned}$$

Hence, there are five solutions:

$$\varphi=0 \qquad \varphi=\pi \qquad \varphi=2\pi \qquad \varphi=\frac{\pi}{3} \qquad \varphi=\frac{5\pi}{3}$$

Exercise 3.

a) $\tan(x+y)=\dfrac{\sin(x+y)}{\cos(x+y)}=\dfrac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}$

If you divide both the numerator and the denominator by $\cos x\cos y$, you obtain:

$$\boxed{\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}}$$

b) If $\tan x = 7$, then

- $\tan 2x=\tan(x+x)=\dfrac{\tan x+\tan x}{1-\tan x\tan x}=\dfrac{14}{-48}=-\dfrac{7}{24}$
- $\tan 3x=\tan(2x+x)=\dfrac{\tan 2x+\tan x}{1-\tan 2x\tan x}=\dfrac{161}{73}$

Exercise 4. Unfortunately, our list of identities lacks an identity expressing $\sin 3\alpha$ in terms of $\sin\alpha$ and/or $\cos\alpha$, but what we can do is find this identity by ourselves: write $\sin 3\alpha$ as

$$\sin(2\alpha+\alpha)=\sin 2\alpha\cos\alpha+\cos 2\alpha\sin\alpha=(2\sin\alpha\cos\alpha)\cos\alpha+\big(2\cos^2\alpha-1\big)\sin\alpha=4\sin\alpha\cos^2\alpha-\sin\alpha$$

We did it! Now, we just substitute $\sin\alpha=\dfrac{5}{13}$ and $\cos\alpha=\dfrac{12}{13}$ to obtain $\sin 3\alpha=\dfrac{2035}{2197}$.

Exercise 5. Applying the famous formula $\cos t = 2\left(\cos\dfrac{t}{2}\right)^2-1$ yields

$$\sqrt{2+2\cos t}=\sqrt{4\left(\cos\frac{t}{2}\right)^2}=2\left|\cos\frac{t}{2}\right|$$

Exercise 6. You can transform the equation $\cos 2t=\sin t$ into a quadratic equation in $\sin t$ by first applying $\cos 2t = 1-2\sin^2 t$:

$$\cos 2t=\sin t \quad \Longleftrightarrow \quad 2\sin^2 t+\sin t\,-1=0 \quad \Longleftrightarrow \quad \sin t=-1 \ \text{ or } \ \sin t=\frac{1}{2}$$

which leads to three meeting times

$$t=\frac{1}{6}\pi \qquad t=\frac{5}{6}\pi \qquad t=\frac{3}{2}\pi$$

Exercise 7. Let's first rewrite $\tan 2\alpha$ a bit:

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{1-2\sin^2\alpha}$$

Now, the given equation becomes

$$\tan 2\alpha = \cos\alpha \quad\Longleftrightarrow\quad \frac{2\sin\alpha}{1-2\sin^2\alpha} = 1 \quad\Longleftrightarrow\quad 2\sin\alpha = 1-2\sin^2\alpha$$

This is a quadratic equation in $\sin\alpha$, which I can solve using the quadratic (ABC) formula:

$$2\sin^2\alpha + 2\sin\alpha - 1 = 0 \quad\Longleftrightarrow\quad \sin\alpha = \frac{-2\pm\sqrt{12}}{4} = -\frac{1}{2}\pm\frac{1}{2}\sqrt{3}$$

Obviously, a sine cannot equal $-\frac{1}{2}-\frac{1}{2}\sqrt{3}$, because this is less than -1. Conclusion:

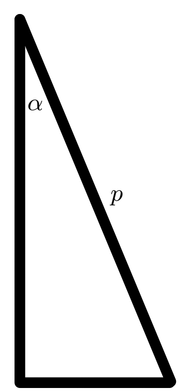
$$\boxed{\sin\alpha = -\frac{1}{2}+\frac{1}{2}\sqrt{3}}$$

Exercise 8.

a) The vertical side is $\sqrt{8}$ by Pythagoras' theorem, so $\tan\alpha = \sqrt{8}$.

b) From exercise 3, $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2\sqrt{8}}{1-8} = -\dfrac{2}{7}\sqrt{8} = -\dfrac{4}{7}\sqrt{2}$.

Exercise 9. I cut the triangle in halves to be able to calculate the desired length p as follows:



$$\alpha = \frac{\pi}{8} \quad\Longrightarrow\quad p = \frac{1}{\sin\frac{\pi}{8}}$$

Finally, I calculate $\sin\frac{\pi}{8}$ using the identity $\cos 2\alpha = 1-2\sin^2\alpha$:

$$\frac{1}{2}\sqrt{2} = 1-2\left(\sin\frac{\pi}{8}\right)^2 \quad\Longrightarrow\quad \left(\sin\frac{\pi}{8}\right)^2 = \frac{1}{2}-\frac{1}{4}\sqrt{2}$$

$$\Longrightarrow\quad \sin\frac{\pi}{8} = \sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}} \quad\Longrightarrow\quad p = \frac{1}{\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}} = \boxed{\sqrt{4+2\sqrt{2}}}$$

Remark. As you can see, I assume that you are capable of playing with square roots and fractions quite easily. For example, the final simplification step in my calculation of p requires two tricks:

1. the 'square root trick': multiply numerator and denominator by $\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}$
2. numerous convenient identities, such as $(p-q)(p+q) = p^2-q^2$

This simplification step is done as follows:

$$\frac{1}{\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}} = \frac{\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}}{\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}} = \frac{\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}}{\sqrt{\left(\frac{1}{2}-\frac{1}{4}\sqrt{2}\right)\left(\frac{1}{2}+\frac{1}{4}\sqrt{2}\right)}} = \frac{\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}}{\sqrt{\left(\frac{1}{2}\right)^2-\left(\frac{1}{4}\sqrt{2}\right)^2}}$$

$$= \frac{\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}}{\sqrt{\frac{1}{4}-\frac{1}{8}}} = \frac{\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}}}{\sqrt{\frac{1}{8}}} = \sqrt{8}\cdot\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2}} = \sqrt{8\cdot\left(\frac{1}{2}+\frac{1}{4}\sqrt{2}\right)} = \sqrt{4+2\sqrt{2}}$$

Exercise 10.

$$\begin{array}{lclcl}
\cos(x-y) & = & \cos x\cos(-y)-\sin x\sin(-y) & = & \cos x\cos y+\sin x\sin y \\
\cos(x+y) & = & & & \cos x\cos y-\sin x\sin y \\
\hline
\cos(x-y)+\cos(x+y) & & & = & 2\cos x\cos y
\end{array}$$

This is how we can prove the second identity. The other ones can be proven analogously.

Exercise 11. I can write $f(t)$ in the form $A\sin(t+B)$ where A is the desired amplitude:

$$f(t) = A\sin(t+B) = A\sin t\cos B + A\cos t\sin B \quad\Longrightarrow\quad \begin{cases} A\cos B = \frac{1}{3} \\ A\sin B = \frac{1}{4} \end{cases}$$

Let's square and add these equations:

$$\left.\begin{array}{l} A^2\cos^2 B = \frac{1}{9} \\ A^2\sin^2 B = \frac{1}{16} \end{array}\right\} \quad\Longrightarrow\quad A^2 = \frac{1}{9} + \frac{1}{16} \quad\Longrightarrow\quad A = \sqrt{\frac{1}{9} + \frac{1}{16}} = \boxed{\frac{5}{12}}$$

Is it really necessary to explain this step? Well, alright then:

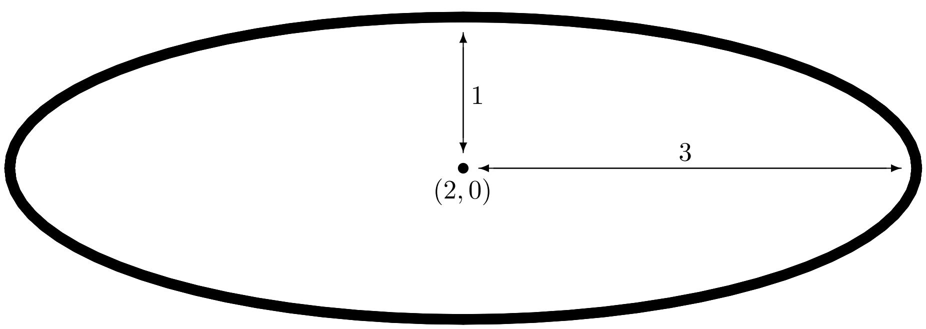
$$\sqrt{\frac{1}{9} + \frac{1}{16}} = \sqrt{\frac{16}{144} + \frac{9}{144}} = \sqrt{\frac{25}{144}} = \frac{5}{12}$$

Exercise 12. $x^2 - 3x + 7 = \left(x^2 - 3x + \frac{9}{4}\right) + \frac{19}{4} = \left(x - \frac{3}{2}\right)^2 + \frac{19}{4}$ with minimum $\frac{19}{4}$

Exercise 13. Write this as $x^2 - 4x + 9y^2 = 5$ and complete the square, which is to say: complete $x^2 - 4x$ to $x^2 - 4x + 4$, which is the square of $x - 2$. The equation of this mysterious figure then becomes

$$(x-2)^2 + 9y^2 = 9 \qquad \text{or} \qquad \frac{(x-2)^2}{9} + y^2 = 1$$

in which you immediately recognise an ellipse with centre $(2, 0)$ and semi-axes 3 and 1:



Exercise 14. We are looking for the solutions to the system of equations

$$\begin{cases} x + y = 1 \\ 4(x+1)^2 + 9y^2 = 36 \end{cases}$$

The first equation gives $x = 1 - y$, so we can eliminate x from the second equation by substitution:

$$4(2-y)^2 + 9y^2 = 36 \quad\Longleftrightarrow\quad 13y^2 - 16y - 20 = 0$$

We can solve this quadratic equation in y using the quadratic formula, but there is an easier way:

$$13y^2 - 16y - 20 = 0 \quad\Longleftrightarrow\quad (13y+10)(y-2) = 0 \quad\Longleftrightarrow\quad y = -\frac{10}{13} \ \text{ or } \ y = 2$$

so the intersection points are $\left(\frac{23}{13}\,,\, -\frac{10}{13}\right)$ and $(-1, 2)$.

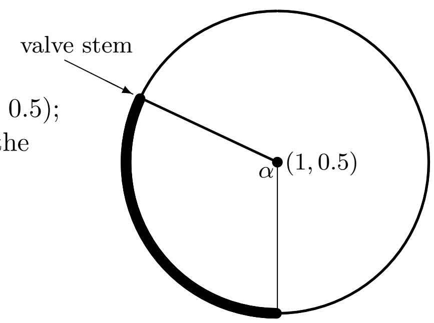
Exercise 15. This is a matter of sorting terms and completing a square:

$$\begin{aligned}
3x^2+x+2y^2+3y=5 &\iff 3\left(x^2+\frac{1}{3}x\right)+2\left(y^2+\frac{3}{2}y\right)=5\\
&\iff 3\left(x^2+\frac{1}{3}x+\frac{1}{36}\right)+2\left(y^2+\frac{3}{2}y+\frac{9}{16}\right)=\frac{149}{24}\\
&\iff 3\left(x+\frac{1}{6}\right)^2+2\left(y+\frac{3}{4}\right)^2=\frac{149}{24}\\
&\iff \frac{72}{149}\left(x+\frac{1}{6}\right)^2+\frac{48}{149}\left(y+\frac{3}{4}\right)^2=1\\
&\iff \frac{(x+\frac{1}{6})^2}{\left(\sqrt{\frac{149}{72}}\right)^2}+\frac{(y+\frac{3}{4})^2}{\left(\sqrt{\frac{149}{48}}\right)^2}=1
\end{aligned}$$

a) The centre is $\left(-\frac{1}{6}, -\frac{3}{4}\right)$.

b) The semi-axes are $\sqrt{\frac{149}{72}}$ and $\sqrt{\frac{149}{48}}$.

c) The area is $\pi\cdot\sqrt{\frac{149}{72}}\cdot\sqrt{\frac{149}{48}}=\boxed{\frac{149}{144}\pi\sqrt{6}}$.



Exercise 16. After cycling 1 metre, the centre of the wheel is at $(1\,,\,0.5)$; the bold arc has rolled over the ground, so its arc length is 1. Hence, the angle α is 2 radians, and the valve stem is at the point

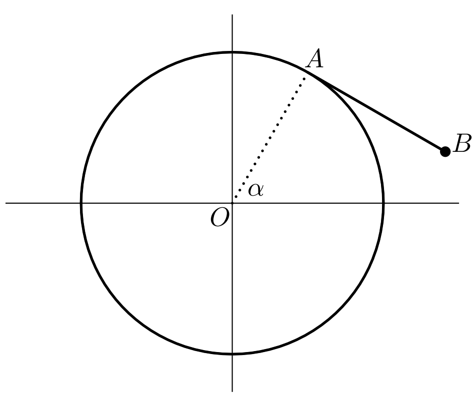
$$\boxed{(1-0.5\,\sin 2\,,\,0.5-0.5\,\cos 2)}$$

which is approximately $(0.545\,,\,0.708)$ according to my calculator.

Exercise 17. The arc corresponds to an angle of $\frac{\pi}{3}$ measured from the centre, because

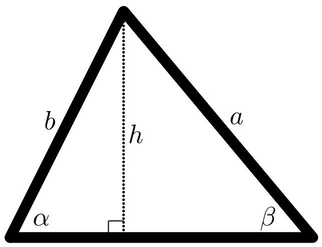
$$\left(3,\sqrt{27}\right)=\left(6\cos\frac{\pi}{3}\,,\,6\sin\frac{\pi}{3}\right)$$

Therefore, the arc length is $\boxed{2\pi}$.



Exercise 18. The angle α in the figure is $\frac{\pi}{3}$ radians, so $A=(3\cos\frac{\pi}{3}, 3\sin\frac{\pi}{3})=(\frac{3}{2}\,,\,\frac{3}{2}\sqrt{3})$. The vector AB is a multiple of $(\sqrt{3},-1)$ because is it perpendicular to OA. From $\|\text{AB}\|=\pi$ we have $\text{AB}=(\frac{\pi}{2}\sqrt{3},-\frac{\pi}{2})$ so

$$\boxed{\text{B}=\left(\frac{3}{2}+\frac{\pi}{2}\sqrt{3}\,,\,\frac{3}{2}\sqrt{3}-\frac{\pi}{2}\right)}$$

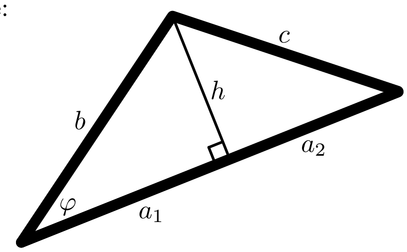


Exercise 19. I draw an altitude ('hoogtelijn') as auxiliary line:

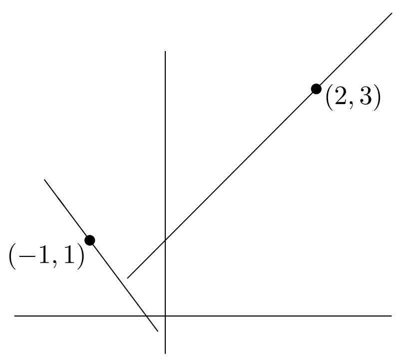
$$\left.\begin{aligned} h &= b\sin\alpha \\ h &= a\sin\beta \end{aligned}\right\} \implies b\sin\alpha = a\sin\beta \implies \frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

and I can prove the other equality analogously.

Exercise 20. Again, I draw an auxiliary line and use Pythagoras twice:



$$\left.\begin{aligned} c^2 &= h^2 + a_2^2 \\ h^2 &= b^2 - a_1^2 \end{aligned}\right\} \implies \begin{aligned} c^2 &= (b^2 - a_1^2) + a_2^2 \\ &= (b^2 - a_1^2) + (a - a_1)^2 \\ &= (b^2 - a_1^2) + (a^2 - 2aa_1 + a_1^2) \\ &= b^2 + a^2 - 2aa_1 \\ &= b^2 + a^2 - 2ab\cos\varphi \end{aligned}$$



Exercise 21. The intersection P lies on both lines, so

$$\begin{cases} P = (-1,1) + \lambda(3,-4) & = (-1+3\lambda, 1-4\lambda) \\ P = (2,3) + \mu(1,1) & = (2+\mu, 3+\mu) \end{cases}$$

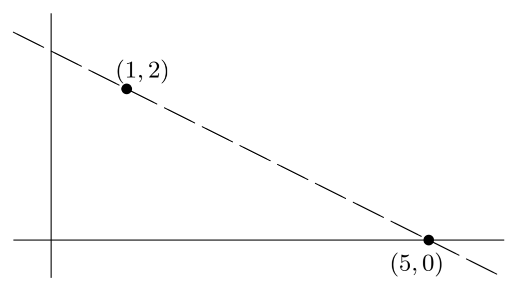
A little bit of algebra:

$$\left.\begin{aligned} -1+3\lambda = 2+\mu &\implies \mu = -3+3\lambda \\ 1-4\lambda = 3+\mu &\implies \mu = -2-4\lambda \end{aligned}\right\} \implies$$

$$-3+3\lambda = -2-4\lambda \implies \lambda = \tfrac{1}{7} \implies P = (-\tfrac{4}{7}\,,\ \tfrac{3}{7})$$

Exercise 22. A point on this line can be found by adding a multiple of the difference vector $(5,0)-(1,2)$ to $(1,2)$:

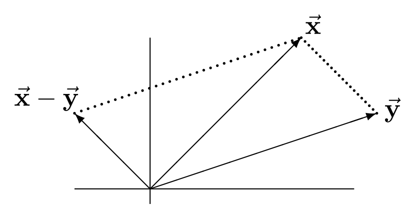
$$(1,2) + \lambda\left((5,0)-(1,2)\right) = \boxed{(1,2) + \lambda(4,-2)}$$



Remark. This line has infinitely many different parametric representations. Two examples:

$$\boxed{(1,2) + \lambda(2,-1)} \qquad \boxed{(3,1) + \lambda(-2,1)}$$

Exercise 23. If you add $\vec{\mathbf{y}}$ to the desired vector, the result must equal $\vec{\mathbf{x}}$, so you should draw a parallelogram:



Exercise 24. For every point (x_1, x_2) on the line there is a number λ with $(x_1, x_2) = (1,3) + \lambda(5,2) = (1+5\lambda, 3+2\lambda)$. Let's eliminate λ:

$$\left.\begin{array}{lll} x_1 = 1+5\lambda & \Longrightarrow & 2x_1 - 2 = 10\lambda \\ x_2 = 3+2\lambda & \Longrightarrow & 5x_2 - 15 = 10\lambda \end{array}\right\} \Longrightarrow 2x_1 - 2 = 5x_2 - 15 \Longrightarrow 2x_1 - 5x_2 = -13$$

Exercise 25. $3x_1 + x_2 = 7 \Longrightarrow x_2 = 7 - 3x_1 \Longrightarrow (x_1, x_2) = (x_1, 7-3x_1) = (0,7) + x_1(1,-3)$
This way we've found a parametric representation of the line: $(0,7) + \lambda(1,-3)$.

Exercise 26.

a) $\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = (4,2) \bullet (2,3) = 8 + 6 = 14$

b) $\|\vec{\mathbf{x}}\| = \|(4,2)\| = \sqrt{4^2+2^2} = 2\sqrt{5}$
$\|\vec{\mathbf{y}}\| = \|(2,3)\| = \sqrt{2^2+3^2} = \sqrt{13}$

c) The distance from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$ is $\|\vec{\mathbf{x}} - \vec{\mathbf{y}}\| = \|(2,-1)\| = \sqrt{5}$.

d) $\cos\varphi = \dfrac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\|} = \dfrac{7}{\sqrt{65}}$

Exercise 27.

a) A point (x,y) on L can be written as $(x,y) = (7,1) + \lambda(4,-3) = (7+4\lambda, 1-3\lambda)$. Now, we need to eliminate λ from the equations

$$\boxed{x = 7+4\lambda} \qquad \boxed{y = 1-3\lambda}$$

Multiply the first by 3, the second by 4, and add the resulting equations:

$$\boxed{3x + 4y = 25}$$

b) The desired distance is equal to the distance from $(7,1)$ to the line $[\![(4,-3)]\!]$. To calculate this distance let's first calculate the projection $\vec{\mathbf{p}}$ of $(7,1)$ on $[\![(4,-3)]\!]$:

$$\vec{\mathbf{p}} = \frac{(7,1) \bullet (4,-3)}{(4,-3) \bullet (4,-3)} \cdot (4,-3) = \frac{25}{25} \cdot (4,-3) = (4,-3)$$

Now, the desired distance is $\|(7,1) - \vec{\mathbf{p}}\| = \|(7,1) - (4,-3)\| = \|(3,4)\| = \sqrt{25} = \boxed{5}$.

Exercise 28.

a) The distance travelled was $\|(6,10) - (2,7)\| = \|(4,3)\| = \sqrt{4^2+3^2} = 5$
so the pussycat ran with a speed of 5 m/sec.

b) After t sec the pussycat was at starting point + t · (end point − starting point) $= (2,7) + t(4,3) = (2+4t, 7+3t)$.

c) Zompie

Exercise 29. The intersection can both be written as $\lambda(3,2)$ and as $(1,3) + \mu(1,-1)$. This turns out to be useful:

$$\lambda(3,2) = (1,3) + \mu(1,-1) \quad \Longrightarrow \quad (3\lambda, 2\lambda) = (1+\mu, 3-\mu) \quad \Longrightarrow \quad \begin{cases} 3\lambda = 1+\mu \\ 2\lambda = 3-\mu \end{cases}$$

This system of equations leads to $\lambda = \frac{4}{5}$ and $\mu = \frac{7}{5}$, so the desired point is $(\frac{12}{5}, \frac{8}{5})$.

Exercise 30. There are two methods to solve this problem:

Method 1. Translate the problem over $(-1,-1)$. The new problem is: calculate the distance from $(5,-1)$ to $[\![(3,2)]\!]$

- the projection of $(5,-1)$ on $[\![(3,2)]\!]$ is $(3,2)$
- so the desired distance is $\|(5,-1)-(3,2)\| = \sqrt{13}$

Method 2.

- a point on the line $(1,1)+[\![(3,2)]\!]$ can be written as $(1,1)+\lambda(3,2) = (3\lambda+1, 2\lambda+1)$
- the distance from this point to $(6,0)$ is

$$\begin{aligned}\|(3\lambda+1,2\lambda+1)-(6,0)\| &= \|(3\lambda-5,2\lambda+1)\| = \sqrt{(3\lambda-5)^2+(2\lambda+1)^2} \\ &= \sqrt{(9\lambda^2-30\lambda+25)+(4\lambda^2+4\lambda+1)} \\ &= \sqrt{13\lambda^2-26\lambda+26} = \sqrt{13(\lambda-1)^2+13}\end{aligned}$$

- and with the naked eye you can see that this has a minimum value of $\sqrt{13}$ (for $\lambda = 1$)

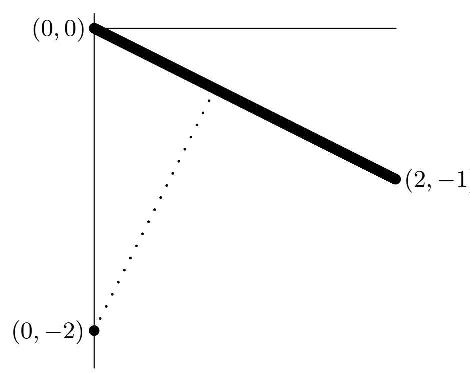
Exercise 31. $\cos\alpha = \dfrac{(3,1)\bullet(2,-1)}{\|(3,1)\|\cdot\|(2,-1)\|} = \dfrac{5}{\sqrt{10}\cdot\sqrt{5}} = \dfrac{1}{\sqrt{2}}$ implies that $\boxed{\alpha = 45^\circ}$.

Exercise 32. I translate the problem over the vector $(0,-2)$ (put differently: I move everything two metres downwards).
The problem reduces to:
calculate the distance from $(0,-2)$ to the line $[\![(2,-1)]\!]$.
The projection of $(0,-2)$ on $[\![(2,-1)]\!]$ is

$$\frac{(0,-2)\bullet(2,-1)}{(2,-1)\bullet(2,-1)}\cdot(2,-1) = \left(\frac{4}{5}, -\frac{2}{5}\right)$$

so the desired distance is

$$\left\|\left(\frac{4}{5}, -\frac{2}{5}\right)-(0,-2)\right\| = \left\|\left(\frac{4}{5}, \frac{8}{5}\right)\right\| = \frac{4}{5}\cdot\|(1,2)\| = \boxed{\frac{4}{5}\sqrt{5}}$$



Exercise 33. In the lecture I told you that work is the dot product (force vector) $\bullet$ (displacement vector):

$$\text{work} = (1,1)\bullet(7,-2) = \boxed{5 \text{ joules}}$$

Exercise 34. The direction of line L is $(1,1)$, because L can be written as $(0,1)+[\![(1,1)]\!]$. By the same token, the direction of M is $(1,7)$. Hence, we are interested in the angle α between the vectors $(1,1)$ and $(1,7)$:

$$\cos\alpha = \frac{(1,1)\bullet(1,7)}{\|(1,1)\|\cdot\|(1,7)\|} = \frac{8}{\sqrt{2}\cdot\sqrt{50}} = 0.8$$

Oh snap, I asked $\tan\alpha$ instead of $\cos\alpha$ to spice up this exercise a little bit. Fortunately, you can find a handy formula in chapter 6 of the arithmetic booklet:

$$1+\tan^2\alpha = \frac{1}{\cos^2\alpha} \quad\Longrightarrow\quad \boxed{\tan\alpha = 0.75}$$

Exercise 35. That's a job for the law of cosines:

$$?^2 = 5^2+7^2-70\cos 60^\circ = 25+49-35 = 39 \quad\Longrightarrow\quad \boxed{? = \sqrt{39}}$$

3. Matrices and $\mathbb{R}^3$

Matrices. A matrix is a rectangular arrangement of numbers. Examples:

$$\begin{pmatrix} 1 & 2 & 5 \\ 4 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 \\ 2 & 7 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -\frac{1}{2} & \sqrt{3} & -2 \\ -5 & \frac{3}{7} & -11 & 2\pi \\ \sqrt{2} & 0 & 7 & 0 \end{pmatrix}$$

Matrices are often used to collect large amounts of data or measurements.

Addition of matrices. Two matrices of equal size can be added 'componentwise': add the numbers on corresponding positions, for example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 0 & -1 \\ -2 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 2 \\ 2 & 10 & 6 \end{pmatrix}$$

Two straightforward properties:

- Addition of matrices is commutative, which is to say that $A + B = B + A$.
- Addition is associative as well: $(A + B) + C = A + (B + C)$.

Scalar multiplication. You can multiply a matrix by a real number. This too is done componentwise, for example:

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad 3A = \begin{pmatrix} 9 & -3 \\ 6 & 0 \\ -6 & 3 \end{pmatrix}$$

Transposed matrix. If you mirror a matrix A (rows become columns and vice versa, while the upper left corner remains fixed) you obtain the 'transposed' matrix A^{T}. For example:

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \qquad \Longrightarrow \qquad A^{\mathrm{T}} = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

Multiplication of matrices. Matrix multiplication is defined only if the number of columns of the first matrix equals the number of rows of the second matrix:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mk} \end{pmatrix} \text{ with } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In order to calculate c_{ij} you have to multiply the numbers from the ith row of the first matrix by those from the jth column of the second matrix and add the results, for example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 9 & 0 \\ 1 & 8 & 3 \\ 0 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 46 & 9 \\ 9 & 118 & 21 \end{pmatrix}$$

This multiplication amounts to six pieces of arithmetic. The 118, for instance, can be found by looking at the row and column that 'meet' at that position:

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \boxed{4 \quad 5 \quad 6} \end{pmatrix} \begin{pmatrix} \cdots & \boxed{\begin{matrix} 9 \\ 8 \\ 7 \end{matrix}} & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \boxed{118} & \cdots \end{pmatrix} \qquad (4 \cdot 9 + 5 \cdot 8 + 6 \cdot 7 = 118)$$

Matrix multiplication is not commutative (AB and BA are usually not equal), but it is associative: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

2×2 **matrices.** Some additional definitions for matrices of size 2 by 2:

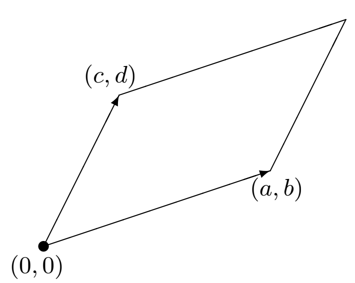
- the identity matrix (or unit matrix) is the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- the determinant of A (notation: $\det A$ or $|A|$) is the number $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\text{def}}{=} ad - bc$
- B is the inverse of A (notation: $B = A^{-1}$) if $AB = I$

There exist beautiful recipes for calculating inverses (see exercise 10, for example). The most important properties of the concepts introduced above are:

- $I \cdot A = A \cdot I = A$
- If B is the inverse of A, then A is the inverse of B.
- The determinant of I is 1.
- Determinant is a multiplicative concept, which is to say that

$$\boxed{\det AB = (\det A) \cdot (\det B)} \qquad \boxed{\det A^{-1} = \frac{1}{\det A}}$$

- The geometric interpretation of 'determinant':



The area of the parallelogram spanned by (a, b) and (c, d) equals the determinant of the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

(but if the orientation from (a, b) to (c, d) happens to be clockwise instead of anticlockwise, then the determinant equals determinant equals minus this area).

Example 1. For the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ we have: $A^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$ (check this for yourself).

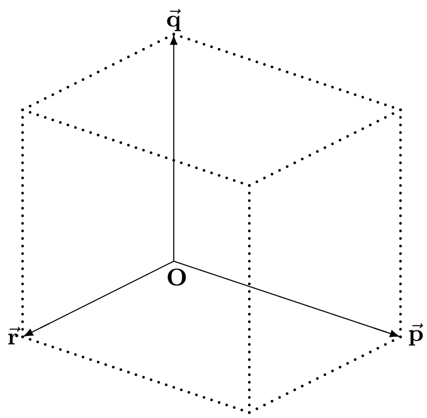
3×3 **matrices.** The determinant of a 3×3 matrix is the number

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \stackrel{\text{def}}{=} a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

This determinant is the sum of all possible products of three numbers taken from different rows and different columns. The products in \-direction get a +, whereas the products in /-direction get a −. Example:

$$\det \begin{pmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 4 & 3 \end{pmatrix} = 18 + 2 + 60 - 9 - 16 - 15 = 40$$

Concepts such as determinant and inverse have the same properties for 3×3 matrices as for 2×2.



Geometric interpretation of a 3×3 determinant.

$\det \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$ is the volume of the parallelepiped spanned by the vectors

$$\vec{\mathbf{p}} = (p_1, p_2, p_3)$$
$$\vec{\mathbf{q}} = (q_1, q_2, q_3)$$
$$\vec{\mathbf{r}} = (r_1, r_2, r_3)$$

(if the orientation $\vec{\mathbf{p}} \to \vec{\mathbf{q}} \to \vec{\mathbf{r}}$ is not $\vec{\mathbf{e_x}} \to \vec{\mathbf{e_y}} \to \vec{\mathbf{e_z}}$, the volume equals minus the determinant).

Systems of equations. If you are to solve for x_1 and x_2 given the system of equations

$$\begin{cases} 2x_1 + x_2 &= 7 \\ 5x_1 - 3x_2 &= 1 \end{cases}$$

you try to eliminate one of the 'unknowns', for instance by multiplying the first equation by 3 and adding this to the second equation:

$$\begin{array}{rcl} 2x_1 + x_2 = 7 \quad \Longrightarrow & & 6x_1 + 3x_2 = 21 \\ & & 5x_1 - 3x_2 = \ 1 \\ & + & \overline{\qquad\qquad\qquad} \\ & & 11x_1 \qquad = 22 \quad \Longrightarrow \quad x_1 = 2 \quad \Longrightarrow \quad \boxed{\begin{array}{c} x_1 = 2 \\ x_2 = 3 \end{array}} \end{array}$$

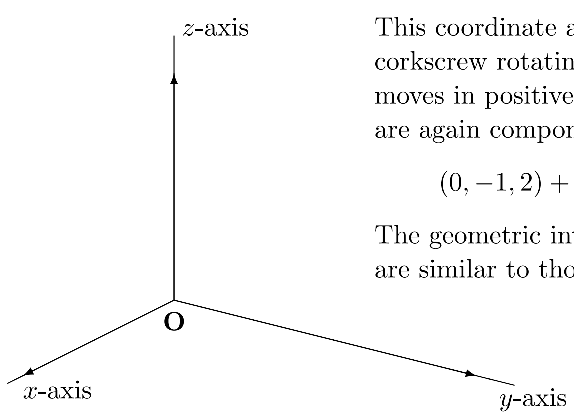
We can represent this system of equations using matrices as well:

$$\begin{pmatrix} 2 & 1 \\ 5 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

Now, a computer program capable of performing matrix calculations can easily solve this matrix equation:

$$\Longrightarrow \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & \frac{1}{11} \\ \frac{5}{11} & -\frac{2}{11} \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Vectors in $\mathbb{R}^3$. A <u>vector in $\mathbb{R}^3$</u> is a row (x_1, x_2, x_3) of three real numbers. You can draw such a vector in a figure with coordinate axes. In Nijmegen one usually draws the first coordinate axis (x_1-axis or x-axis) pointing to the front, the second (x_2-axis or y-axis) to the right, and the third (x_3-axis or z-axis) upwards:



This coordinate axes orientation is called 'right-handed': a right-handed corkscrew rotating from the (positive) x-axis to the (positive) y-axis moves in positive z-direction. Addition and scalar multiplication in $\mathbb{R}^3$ are again componentwise, for example:

$$(0, -1, 2) + (3, 4, 5) = (3, 3, 7) \qquad\qquad 3 \cdot (1, 2, -3) = (3, 6, -9)$$

The geometric interpretation of addition and scalar multiplication are similar to those in $\mathbb{R}^2$, as well as the following notations:

$[\![\vec{\mathbf{x}}]\!]$ is the set containing all multiples of $\vec{\mathbf{x}}$.
Geometric interpretation: the line through $\mathbf{O}$ and $\vec{\mathbf{x}}$.

$\vec{\mathbf{y}} + [\![\vec{\mathbf{x}}]\!]$ is the set containing all $\vec{\mathbf{y}} + \lambda\vec{\mathbf{x}}$ with $\lambda \in \mathbb{R}$.
Geometric interpretation: the line through the point $\vec{\mathbf{y}}$ parallel to the line $[\![\vec{\mathbf{x}}]\!]$.

Linear combinations. A linear combination of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ is (by definition) a vector of the form $\lambda\vec{\mathbf{x}} + \mu\vec{\mathbf{y}}$ with λ and μ real numbers. Examples:

- $(1, 4, 2)$ is a linear combination of $(3, -1, 6)$ and $(-1, 1, -2)$

  Proof: $(1, 4, 2) \;=\; \frac{5}{2}(3, -1, 6) \;+\; \frac{13}{2}(-1, 1, -2)$

- $(30, 20, 10)$ is a linear combination of $(1, 1, 1)$ and $(1, 2, 3)$

  Proof: $(30, 20, 10) = 40(1, 1, 1) - 10(1, 2, 3)$

- $(1, 2, 5)$ is not a linear combination of $(1, 0, 1)$ and $(1, 2, 0)$

  Proof: if $(1, 2, 5) = \lambda(1, 0, 1) + \mu(1, 2, 0)$, then $(1, 2, 5) = (\lambda + \mu, 2\mu, \lambda)$. From the equality of the second and third coordinates we have $\mu = 1$ and $\lambda = 5$, respectively. However, the equation then becomes $(1, 2, 5) = (6, 2, 5)$, which is nonsense.

Natural basis for $\mathbb{R}^3$. The vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ constitute the natural basis for $\mathbb{R}^3$. Popular notations for these basis vectors:

$$\boxed{\begin{array}{lcl} \vec{\mathbf{e_x}} & \stackrel{\text{def}}{=} & (1,0,0) \\ \vec{\mathbf{e_y}} & \stackrel{\text{def}}{=} & (0,1,0) \\ \vec{\mathbf{e_z}} & \stackrel{\text{def}}{=} & (0,0,1) \end{array}} \quad \text{and sometimes} \quad \boxed{\begin{array}{lcl} \vec{\mathbf{\imath}} & \stackrel{\text{def}}{=} & (1,0,0) \\ \vec{\mathbf{\jmath}} & \stackrel{\text{def}}{=} & (0,1,0) \\ \vec{\mathbf{k}} & \stackrel{\text{def}}{=} & (0,0,1) \end{array}}$$

Every vector from $\mathbb{R}^3$ is a linear combination of these basis vectors:

$$(\alpha, \beta, \gamma) = \alpha\,\vec{\mathbf{e_x}} + \beta\,\vec{\mathbf{e_y}} + \gamma\,\vec{\mathbf{e_z}}$$

Planes through the origin. The set containing all linear combinations of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ is (in general) the plane through $\mathbf{O}$, $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$. We denote this plane by $[\![\vec{\mathbf{x}}, \vec{\mathbf{y}}]\!]$ (the span of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$). Of course, a plane through the origin can also be described by an equation of the form $\alpha x_1 + \beta x_2 + \gamma x_3 = 0$.

Example 2. Let V be the plane $3x_1 + 5x_2 - 2x_3 = 0$. Write V as the span of two vectors.

Solution. I'll just grab two vectors satisfying this equation: $(5, -3, 0)$ and $(2, 0, 3)$. Fortunately, these vectors are not each other's multiples, so they span a plane. Problem solved:

$$V = [\![(5, -3, 0), (2, 0, 3)]\!]$$

Example 3. Let V be the plane $[\![(1, 2, -3), (-1, 1, 0)]\!]$. Find the equation of V.

Solution. I'll first find out which vectors (x_1, x_2, x_3) lie on this plane:

$$\begin{aligned} (x_1, x_2, x_3) \in V \quad &\Longrightarrow \quad (x_1, x_2, x_3) = \lambda(1, 2, -3) + \mu(-1, 1, 0) = (\lambda - \mu, 2\lambda + \mu, -3\lambda) \\ &\Longrightarrow \quad \begin{cases} x_1 = \lambda - \mu \\ x_2 = 2\lambda + \mu \\ x_3 = -3\lambda \end{cases} \end{aligned}$$

Let's do a little bit of algebra using these equations: the third equation gives $\lambda = -\frac{1}{3}x_3$, and the first $\mu = -\frac{1}{3}x_3 - x_1$. Substitution in the second equation yields $x_2 = -x_3 - x_1$, so

$$x_1 + x_2 + x_3 = 0$$

Planes not passing through the origin. If $\vec{\mathbf{x}}$, $\vec{\mathbf{y}}$ and $\vec{\mathbf{z}}$ are vectors from $\mathbb{R}^3$, the set containing all vectors $\vec{\mathbf{z}} + \lambda\vec{\mathbf{x}} + \mu\vec{\mathbf{y}}$, where λ and μ are real numbers, is the plane through the point $\vec{\mathbf{z}}$ parallel to the plane $[\![\vec{\mathbf{x}}, \vec{\mathbf{y}}]\!]$. We denote this plane by $\vec{\mathbf{z}} + [\![\vec{\mathbf{x}}, \vec{\mathbf{y}}]\!]$. The expression $\vec{\mathbf{z}} + \lambda\vec{\mathbf{x}} + \mu\vec{\mathbf{y}}$ is called the parametric representation of the plane. Such a plane can also be described by an equation of the form $\alpha x_1 + \beta x_2 + \gamma x_3 = \delta$.

Example 4. Find the equation of the plane with parametric representation

$$\vec{\mathbf{x}} = (1,1,1) + \lambda(1,-1,0) + \mu(0,1,-1)$$

Solution. Again, I'm going to find out which vectors (x_1, x_2, x_3) lie on this plane. I'll write my vectors vertically to improve readability:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \implies \begin{cases} x_1 = 1 + \lambda \\ x_2 = 1 - \lambda + \mu \\ x_3 = 1 - \mu \end{cases}$$

Now, you can easily see (or calculate) that the plane consists of all vectors (x_1, x_2, x_3) satisfying the equation $x_1 + x_2 + x_3 = 3$

Example 5. Find a parametric representation of the plane $x_1 + 2x_2 + x_3 = 4$.

Solution. Suppose $x_1 = \lambda$ and $x_2 = \mu$. Then $x_3 = 4 - \lambda - 2\mu$, so

$$\begin{cases} x_1 = \lambda \\ x_2 = \mu \\ x_3 = 4 - \lambda - 2\mu \end{cases} \quad \text{so} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

or, written horizontally: $\vec{\mathbf{x}} = (0,0,4) + \lambda(1,0,-1) + \mu(0,1,-2)$.

Norm and dot product in $\mathbb{R}^3$. For vectors $\vec{\mathbf{x}} = (x_1, x_2, x_3)$ and $\vec{\mathbf{y}} = (y_1, y_2, y_3)$ we define:

$$\begin{aligned} \|\vec{\mathbf{x}}\| &\overset{\text{def}}{=} \sqrt{x_1^2 + x_2^2 + x_3^2} && \text{(the \underline{norm} of } \vec{\mathbf{x}}\text{)} \\ \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} &\overset{\text{def}}{=} x_1y_1 + x_2y_2 + x_3y_3 && \text{(the \underline{dot product} of } \vec{\mathbf{x}} \text{ and } \vec{\mathbf{y}}\text{)} \\ \vec{\mathbf{x}} \perp \vec{\mathbf{y}} &\overset{\text{def}}{=} (\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0) && (\vec{\mathbf{x}} \text{ and } \vec{\mathbf{y}} \text{ are \underline{orthogonal}}) \end{aligned}$$

A couple of examples:

$$\|(3,2,6)\| = 7 \qquad (3,5,-7) \bullet (6,3,4) = 5 \qquad (3,5,-7) \perp (3,1,2)$$

Properties of norm and dot product that we encountered in $\mathbb{R}^2$ turn out to be equally valid in $\mathbb{R}^3$:

$$\begin{aligned} \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} &= \vec{\mathbf{y}} \bullet \vec{\mathbf{x}} \\ (\vec{\mathbf{x}} + \vec{\mathbf{y}}) \bullet \vec{\mathbf{z}} &= (\vec{\mathbf{x}} \bullet \vec{\mathbf{z}}) + (\vec{\mathbf{y}} \bullet \vec{\mathbf{z}}) \\ (\lambda\vec{\mathbf{x}}) \bullet \vec{\mathbf{y}} &= \lambda(\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}) \\ \|\vec{\mathbf{x}}\| &= \sqrt{\vec{\mathbf{x}} \bullet \vec{\mathbf{x}}} \\ \|\vec{\mathbf{x}}\| &= \text{the length of the vector } \vec{\mathbf{x}} \\ \|\vec{\mathbf{x}} - \vec{\mathbf{y}}\| &= \text{the distance from } \vec{\mathbf{x}} \text{ to } \vec{\mathbf{y}} \\ \vec{\mathbf{x}} \bullet \vec{\mathbf{y}} &= \|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\| \cdot \cos\varphi \;\; \text{with } \varphi \text{ the angle between } \vec{\mathbf{x}} \text{ and } \vec{\mathbf{y}} \\ \vec{\mathbf{x}} \perp \vec{\mathbf{y}} &\Leftrightarrow \vec{\mathbf{x}} \text{ is perpendicular to } \vec{\mathbf{y}} \end{aligned}$$

Example 6. What is the angle φ between the vector $\vec{\mathbf{x}} = (2,-1,2)$ and the vector $\vec{\mathbf{y}} = (4,1,1)$?

Solution. That is a matter of smartly applying the properties above:

- the norm of $\vec{\mathbf{x}}$ is $\|\vec{\mathbf{x}}\| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$
- the norm of $\vec{\mathbf{y}}$ is $\|\vec{\mathbf{y}}\| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$
- the cosine of the desired angle is $\cos\varphi = \frac{\vec{\mathbf{x}} \bullet \vec{\mathbf{y}}}{\|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\|} = \frac{9}{9\sqrt{2}} = \frac{1}{2}\sqrt{2}$
- so the angle itself is $\varphi = \frac{\pi}{4}$

Example 7. Calculate the distance from the point $(5,2,8)$ to the line $(1,2,3)+\lambda(1,1,1)$.

Solution. I first calculate the distance from $(5,2,8)$ to the point $(1,2,3)+\lambda(1,1,1)$ on this line:

$$\begin{aligned}\|(5,2,8)-(1+\lambda,2+\lambda,3+\lambda)\| &= \|(4-\lambda\,,\,-\lambda\,,\,5-\lambda)\| \\ &= \sqrt{(4-\lambda)^2+(-\lambda)^2+(5-\lambda)^2} \;=\; \sqrt{3\lambda^2-18\lambda+41} \\ &= \sqrt{3(\lambda^2-6\lambda+9)+14} \;=\; \sqrt{3(\lambda-3)^2+14}\end{aligned}$$

This expression has a minimum value of $\sqrt{14}$ for $\lambda=3$, so the distance from $(5,2,8)$ to the line is $\sqrt{14}$.

Example 8. Find a vector perpendicular to the plane $3x_1-2x_2+5x_3=0$.

Solution. Let me just mess around with the equation of the plane:

$$3x_1-2x_2+5x_3=0 \quad\Longleftrightarrow\quad (3,-2,5)\bullet(x_1,x_2,x_3)=0 \quad\Longleftrightarrow\quad (3,-2,5)\perp(x_1,x_2,x_3)$$

Evidently, the plane consists of all vectors perpendicular to $(3,-2,5)$. Thus, $(3,-2,5)$ is one of the vectors perpendicular to the plane.

Normal to a plane. A <u>normal</u> to a plane through the origin is a vector perpendicular to that plane. In example 7 we discovered a neat trick to find a normal to a plane:

$$\boxed{(\alpha,\beta,\gamma) \text{ is a normal to the plane } \alpha x_1+\beta x_2+\gamma x_3=0}$$

Example 9. Find a normal $\vec{\mathbf{n}}$ to the plane $V=[\![(0,1,3),(1,0,-5)]\!]$.

Solution. First, I determine the equation of V. The points (x_1,x_2,x_3) on V are linear combinations of $(0,1,3)$ and $(1,0,-5)$:

$$(x_1,x_2,x_3)=\lambda(0,1,3)+\mu(1,0,-5) \qquad\Longrightarrow\qquad \begin{cases} x_1=\mu \\ x_2=\lambda \\ x_3=3\lambda-5\mu \end{cases} \qquad\Longrightarrow\qquad x_3=3x_2-5x_1$$

Hence, V is the plane with equation $5x_1-3x_2+x_3=0$, so $(5,-3,1)$ is a normal to V.

Remark. Some people/books require a normal to be of length 1. In that case, you should divide the vector you found by its own length:

$$\|(5,-3,1)\|=\sqrt{5^2+(-3)^2+1^2}=\sqrt{35} \qquad\Longrightarrow\qquad \vec{\mathbf{n}}=\left(\frac{5}{\sqrt{35}}\,,\,\frac{-3}{\sqrt{35}}\,,\,\frac{1}{\sqrt{35}}\right)$$

The act of 'dividing by its own length' is called 'normalisation', and the resulting vector is said to be 'normalised'.

Projection on a line through the origin. For the projection of a vector on a line through the origin the same formula holds as in $\mathbb{R}^2$:

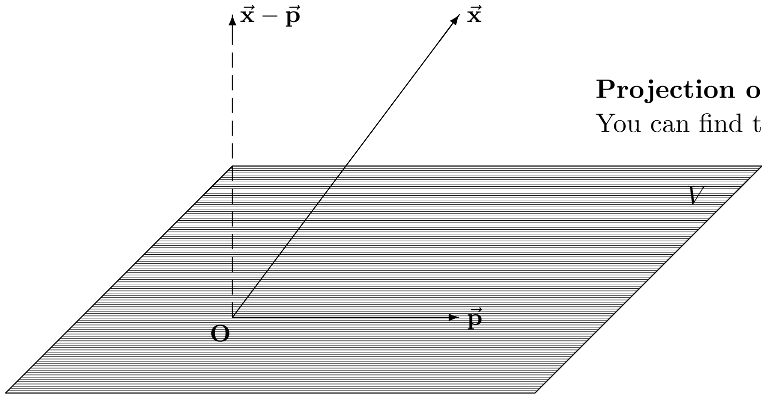
$$\boxed{\text{the projection of } \vec{\mathbf{x}} \text{ on } [\![\vec{\mathbf{y}}]\!] \text{ is } \frac{\vec{\mathbf{x}}\bullet\vec{\mathbf{y}}}{\vec{\mathbf{y}}\bullet\vec{\mathbf{y}}}\cdot\vec{\mathbf{y}}}$$

Example 10. Calculate the distance from the point $(3,-2,5)$ to the line $[\![(3,6,9)]\!]$.

Solution. The projection of $(3,-2,5)$ on $[\![(3,6,9)]\!]$ is

$$\frac{(3,-2,5)\bullet(3,6,9)}{(3,6,9)\bullet(3,6,9)}\cdot(3,6,9) \;=\; \frac{42}{126}\cdot(3,6,9) \;=\; (1,2,3)$$

so the desired distance is $\|(3,-2,5)-(1,2,3)\| \;=\; 2\sqrt{6}$.



Projection on a plane through the origin.
You can find the projection $\vec{\mathbf{p}}$ of $\vec{\mathbf{x}}$ on V as follows:

Step 1. Find a normal $\vec{\mathbf{n}}$ to V.

Step 2. Project $\vec{\mathbf{x}}$ on $\vec{\mathbf{n}}$, which yields the vector $\vec{\mathbf{x}} - \vec{\mathbf{p}}$.

Step 3. Subtract this vector from $\vec{\mathbf{x}}$.

Example 11. Find the projection of $(1, 0, 0)$ on the plane $[\![(1, 2, 0), (0, -1, 1)]\!]$.

Solution.

Step 1. The equation of this plane is $2x_1 - x_2 - x_3 = 0$, so $(2, -1, -1)$ is a normal to the plane.

Step 2. The projection of $(1, 0, 0)$ on this normal is $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$.

Step 3. The projection of $(1, 0, 0)$ on the plane is therefore $(1, 0, 0) - (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

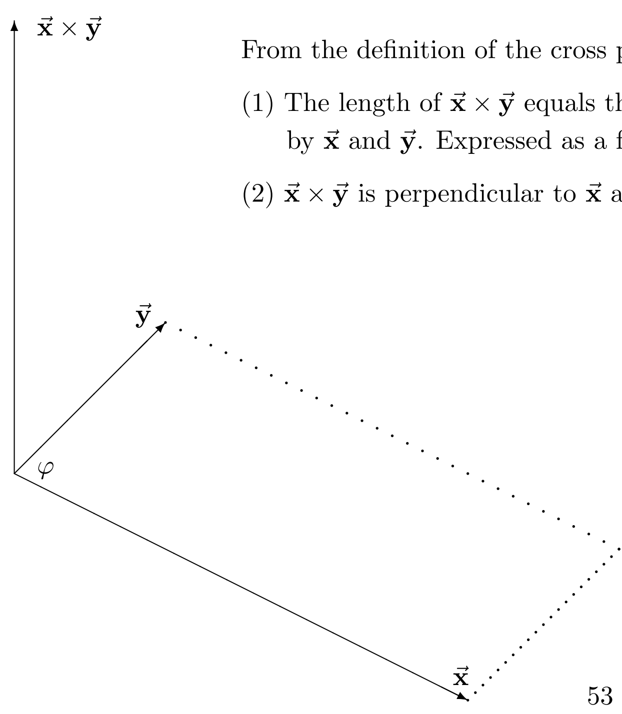
Cross product. If $\vec{\mathbf{x}} = (x_1, x_2, x_3)$ and $\vec{\mathbf{y}} = (y_1, y_2, y_3)$, we define the <u>cross product</u> $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ by

$$\boxed{\vec{\mathbf{x}} \times \vec{\mathbf{y}} \stackrel{\text{def}}{=} \det \begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}}$$

Example 12. The cross product of $(1, 5, 3)$ and $(2, 4, 6)$, pronounced '$(1, 5, 3)$ cross $(2, 4, 6)$', is

$$(1,5,3) \times (2,4,6) = \det \begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ 1 & 5 & 3 \\ 2 & 4 & 6 \end{pmatrix} = 30\vec{\mathbf{e_x}} + 6\vec{\mathbf{e_y}} + 4\vec{\mathbf{e_z}} - 10\vec{\mathbf{e_z}} - 6\vec{\mathbf{e_y}} - 12\vec{\mathbf{e_x}} = 18\vec{\mathbf{e_x}} - 6\vec{\mathbf{e_z}} = (18, 0, -6)$$

Geometric interpretation of the cross product.



From the definition of the cross product you can prove the following:

(1) The length of $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ equals the area of the parallelogram spanned by $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$. Expressed as a formula: $\|\vec{\mathbf{x}} \times \vec{\mathbf{y}}\| = \|\vec{\mathbf{x}}\| \cdot \|\vec{\mathbf{y}}\| \cdot \sin\varphi$.

(2) $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ is perpendicular to $\vec{\mathbf{x}}$ and to $\vec{\mathbf{y}}$.

Note that there always exist <u>two</u> vectors with this length being $\perp \vec{\mathbf{x}}$ and $\perp \vec{\mathbf{y}}$. The cross product equals one of these; you can find out which by rotating a corkscrew from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$: it will move in the direction of $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$, provided that the orientation of your corkscrew matches that of your coordinate system.

Remarks.

- The cross product is a determinant, but this determinant is a vector instead of a real number. This is because I put vectors instead of real numbers in the top row of the matrix.

- I can define the cross product equally well as follows:

$$\boxed{\vec{\mathbf{x}} \times \vec{\mathbf{y}} \overset{\text{def}}{=} (x_2y_3 - x_3y_2 \;,\; x_3y_1 - x_1y_3 \;,\; x_1y_2 - x_2y_1)}$$

- The cross product plays an important role in the laws of classical mechanics and electrodynamics. For example, the force exerted on a moving charged particle by a magnetic field is proportional to the cross product of the velocity vector of the particle and the magnetic field vector. In this chapter I will restrict myself to several simple geometric applications.

Example 13. Find a normal $\vec{\mathbf{n}}$ to the plane $[\![(2,5,1),(3,7,-1)]\!]$.

Solution. I could use the method of example 9, but a much easier way is the following:

$$\vec{\mathbf{n}} = (2,5,1) \times (3,7,-1) = \det\begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ 2 & 5 & 1 \\ 3 & 7 & -1 \end{pmatrix} = (-12,5,-1)$$

Example 14. Find the equation of the plane $[\![\vec{\mathbf{x}},\vec{\mathbf{y}}]\!]$ with $\vec{\mathbf{x}} = (1,2,0)$ and $\vec{\mathbf{y}} = (1,3,1)$.

Solution. I calculate the cross product of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$:

$$\vec{\mathbf{x}} \times \vec{\mathbf{y}} = \det\begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix} = (2,-1,1)$$

This cross product $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ is $\perp$ to the plane $[\![\vec{\mathbf{x}},\vec{\mathbf{y}}]\!]$, so

$$\begin{aligned} \vec{\mathbf{z}} \in [\![\vec{\mathbf{x}},\vec{\mathbf{y}}]\!] &\iff \vec{\mathbf{z}} \perp \vec{\mathbf{x}} \times \vec{\mathbf{y}} \\ &\iff \vec{\mathbf{z}} \bullet (\vec{\mathbf{x}} \times \vec{\mathbf{y}}) = 0 \\ &\iff \vec{\mathbf{z}} \bullet (2,-1,1) = 0 \\ &\iff 2z_1 - z_2 + z_3 = 0 \end{aligned}$$

Example 15. Which point on the plane $(2,-8,4) + [\![(3,5,1),(-2,0,1)]\!]$ is closest to the origin?

Solution. A vector perpendicular to the plane is

$$(3,5,1) \times (-2,0,1) = \det\begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ 3 & 5 & 1 \\ -2 & 0 & 1 \end{pmatrix} = (5,-5,10)$$

so the desired point must be a multiple of $(5,-5,10)$. All we need to find out is <u>which</u> multiple of $(5,-5,10)$ we're looking for. This requires quite some labour. A possible approach is the following:

- Find the equation of the plane: $x_1 - x_2 + 2x_3 = 18$.

- Substitute the point $\lambda(5,-5,10)$ in this equation to find λ: $\lambda = \dfrac{3}{5}$.

- Thus, the desired point is $(3,-3,6)$.

Exercises chapter 3

Exercise 1. Given the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, calculate:

a) $7A + B$ b) $\det A$ c) B^{-1} d) $AB - BA$

Exercise 2. Calculate (if possible) the products AB and BA if

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Exercise 3. Let A be the matrix $\begin{pmatrix} 3 & 8 \\ 5 & 7 \end{pmatrix}$.

a) Calculate the determinant of A.

b) Calculate the determinant of A^{T}.

c) Calculate the determinant of A^2 (by which I mean $A \cdot A$).

Exercise 4. Calculate the determinant of the matrix

$$\begin{pmatrix} 1 & -2 & 0 \\ -3 & 0 & -4 \\ 1 & 5 & 2 \end{pmatrix}$$

Exercise 5. Let A be the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

a) Calculate $A^{\mathrm{T}} \cdot A$.

b) Calculate $A \cdot A^{\mathrm{T}}$.

Exercise 6. Prove the following formula for the inverse of a 2×2 matrix:

$$\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \Longrightarrow \qquad A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

(but if $\det A = 0$, then A^{-1} does not exist).

Exercise 7.

a) Find x and y satisfying

$$\boxed{\begin{aligned} 3x + 2y &= 1 \\ 5x + 3y &= 4 \end{aligned}}$$

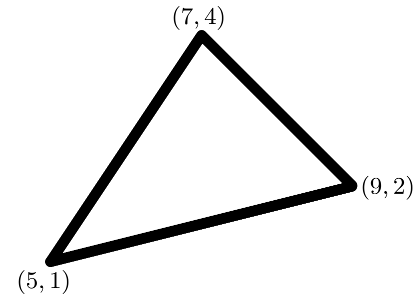
b) Write the system of equations in (a) using matrices.

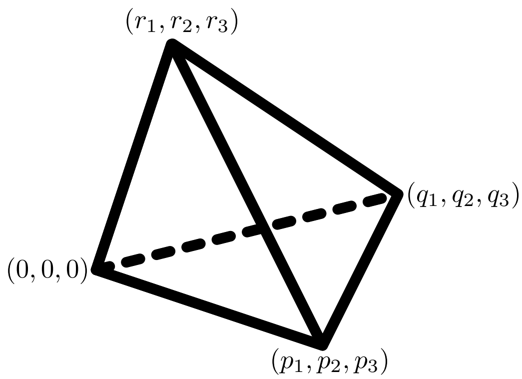
Exercise 8. Find the inverse of the matrix $A = \begin{pmatrix} 7 & 4 \\ 8 & 5 \end{pmatrix}$.

Exercise 9. Solve the following system of equations:

$$\begin{pmatrix} 2 & 5 & 3 \\ 3 & 7 & 2 \\ 1 & 8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix}$$

Exercise 10. Calculate the area of the triangle with vertices $(5,1)$, $(9,2)$ and $(7,4)$.





Exercise 11. Invent a handy formula for the volume of a tetrahedron with vertices $(0,0,0)$, (p_1,p_2,p_3), (q_1,q_2,q_3) and (r_1,r_2,r_3).

Exercise 12. Calculate the length of the vector $(3,6,-2)$.

Exercise 13. Calculate the dot product $(-3,5,1) \bullet (7,4,5)$.

Exercise 14. Calculate the distance from $(3,-1,7)$ to $(7,1,3)$.

Exercise 15. Let V be the collection of all vectors (x_1,x_2,x_3) satisfying

$$\begin{cases} 2x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$$

a) What kind of set is V?

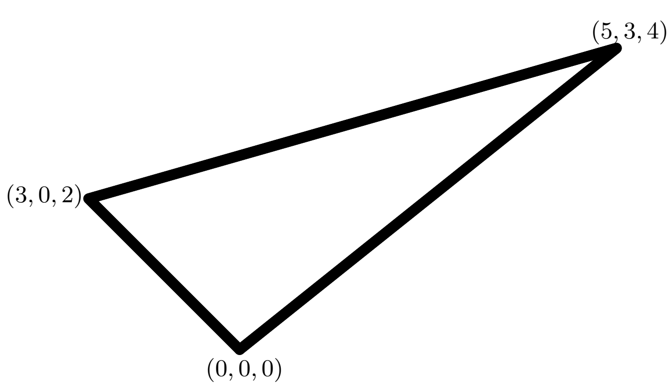
b) Give a parametric representation of V.

Exercise 16. Let $\vec{\mathbf{x}} = (5,1,-3)$ and $\vec{\mathbf{y}} = (1,-2,1)$.

a) Prove that $\vec{\mathbf{x}}$ is perpendicular to $\vec{\mathbf{y}}$.

b) Calculate the cross product of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

Exercise 17. Calculate the angle between the vectors $(1,0,0)$ and $(2,1,-\sqrt{3})$.

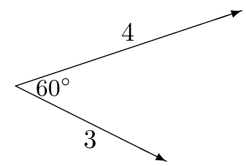


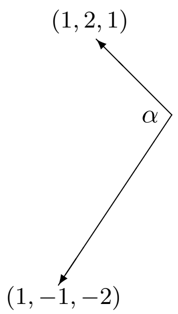
Exercise 18. Calculate the area of this triangle.

Exercise 19. Given two vectors in $\mathbb{R}^3$ with lengths 3 and 4 and an angle of $60°$ between them.

a) What is their dot product?

b) What is the length of their cross product?





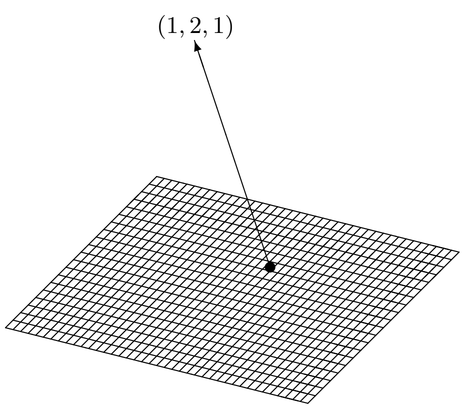
Exercise 20. Calculate the angle α between the vectors $(1, -1, -2)$ and $(1, 2, 1)$.

Exercise 21. Find the projection of $(6, -5, 1)$ on the line defined by the couple of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 3x_1 - x_2 + 5x_3 = 0 \end{cases}$$

Exercise 22. Which point on $(1, 2, 3) + [\![(-2, 1, 1)]\!]$ is closest to $(-2, 0, 5)$?

Exercise 23. Calculate the distance from $(5, 3, 1)$ to $(3, 3, 2) + [\![(1, 0, 0), (0, 4, 3)]\!]$.



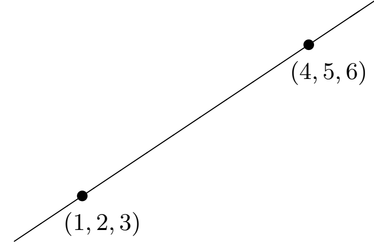
Exercise 24. Calculate the projection of the vector $(1, 2, 1)$ on the plane with equation

$$\boxed{2x_1 - x_2 + 3x_3 = 0}$$

Exercise 25. Calculate the distance from the line $(3, 2, 5) + [\![(1, 1, 2)]\!]$ to the line $(5, 0, 0) + [\![(-1, 0, 2)]\!]$.

Exercise 26. What is the angle α between the line $[\![(1, 1, 1)]\!]$ and the plane $[\![(3, -1, 0), (7, 1, 2)]\!]$? (If you're not carrying your calculator with you, just give me $\sin\alpha$ or $\cos\alpha$.)

Exercise 27. Find a parametric representation of the line in $\mathbb{R}^3$ through the points $(1, 2, 3)$ and $(4, 5, 6)$.



Exercise 28. Let V be the plane $(2,7,1)+[\![(0,1,0),(-3,0,1)]\!]$.

a) Determine the equation of V.

b) Which point on V is closest to the origin?

Exercise 29. Let L be the line in $\mathbb{R}^3$ consisting of all points (x_1,x_2,x_3) with

$$\begin{cases} 3x_1+5x_2-x_3=3 \\ x_1-2x_3=1 \end{cases}$$

a) Find a parametric representation of L.

b) Calculate the distance from L to the origin.

Exercise 30.

a) Determine the equation of the plane through $(0,0,0)$ perpendicular to $(3,2,5)$.

b) Determine the equation of the plane through $(7,8,-6)$ perpendicular to $(3,2,5)$.

Exercise 31. I define a plane V and a line L in $\mathbb{R}^3$ by means of their equations:

$$V: \quad x_1-3x_2+2x_3=0$$

$$L: \quad x_1=2x_2+2=3x_3+3$$

a) Calculate the sine of the angle between L and V.

b) Find the points on L that lie at a distance of 5 from V.

Exercise 32. Let L and M be lines in $\mathbb{R}^3$ defined by

$$\begin{cases} L \stackrel{\text{def}}{=} [\![(1,0,3)]\!] \\ M \stackrel{\text{def}}{=} (-1,1,0)+[\![(2,1,1)]\!] \end{cases}$$

Calculate the distance from L to M.

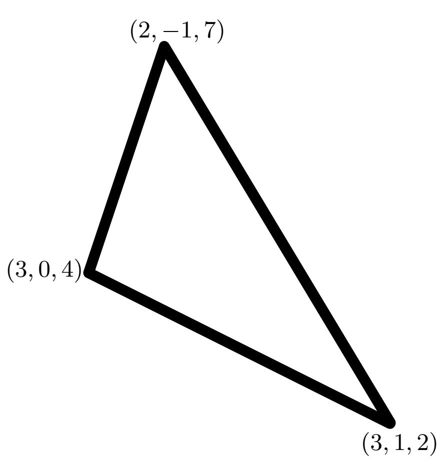
Exercise 33. Find the mirror image of $(1,0,-2)$ with respect to the plane $3x_1+x_2+x_3=0$.

Exercise 34. Let $\mathbf{V}$ be the plane through the origin and the points $(1,2,1)$ and $(3,7,4)$.

a) What is the equation of $\mathbf{V}$?

b) Find the projection of $(1,1,3)$ on $\mathbf{V}$.

c) Calculate the cosine of the angle φ between the line $[\![(1,1,3)]\!]$ and $\mathbf{V}$.



Exercise 35. Calculate the area of the triangle with vertices $(2,-1,7)$, $(3,0,4)$ and $(3,1,2)$.

Solutions chapter 3

Exercise 1.

a) $7A+B=\begin{pmatrix}7 & 14\\ 21 & 28\end{pmatrix}+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}7 & 15\\ 22 & 28\end{pmatrix}$

b) $\det A=-2$

c) $B^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (I'm sure you've managed to find this after some algebraic messing around.)

d) $AB-BA=\begin{pmatrix}2 & 1\\ 4 & 3\end{pmatrix}-\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix}=\begin{pmatrix}-1 & -3\\ 3 & 1\end{pmatrix}$

Exercise 2. $A\cdot B=\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}$ and $B\cdot A=\begin{pmatrix}3 & 3 & 2\\ 1 & 2 & 1\\ 2 & -2 & 0\end{pmatrix}$

Exercise 3.

a) $\det A=\det\begin{pmatrix}3 & 8\\ 5 & 7\end{pmatrix}=3\cdot 7-8\cdot 5=-19$

b) $\det A^{\mathrm{T}}=\det\begin{pmatrix}3 & 5\\ 8 & 7\end{pmatrix}=3\cdot 7-5\cdot 8=-19$

c) This is easy if you remembered that 'determinant' is multiplicative: $\det A^2=(\det A)^2=361$.

Exercise 4. Just use the definition:

$$\det\begin{pmatrix}1 & -2 & 0\\ -3 & 0 & -4\\ 1 & 5 & 2\end{pmatrix}=16$$

Exercise 5.

a) $A^{\mathrm{T}}\cdot A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix}=\begin{pmatrix}2 & 1 & 1 & 1\\ 1 & 2 & 1 & 1\\ 1 & 1 & 2 & 0\\ 1 & 1 & 0 & 1\end{pmatrix}$

b) $A\cdot A^{\mathrm{T}}=\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 3\end{pmatrix}$

Exercise 6. You can just verify this:

$$\frac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\frac{1}{\det A}\begin{pmatrix}da-bc & 0\\ 0 & -bc+ad\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

Exercise 7.

a) I eliminate y by multiplying the first equation by 3, the second by 2, and subtracting the second from the first:

$$\begin{aligned} 3x+2y=1 &\implies & 9x+6y&=3 \\ 5x+3y=4 &\implies & 10x+6y&=8 \\ && - \quad \overline{-x \qquad} &\overline{= -5} \end{aligned}$$

which yields the solution $\boxed{\begin{aligned} x&=5 \\ y&=-7 \end{aligned}}$.

b) $\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Exercise 8. I use the formula from exercise 6:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 5 & -4 \\ -8 & 7 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5 & -4 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & -\frac{4}{3} \\ -\frac{8}{3} & \frac{7}{3} \end{pmatrix}$$

Exercise 9. The equations to be solved are:

$$\boxed{\begin{aligned} 2x+5y+3z&=8 \\ 3x+7y+2z&=5 \\ x+8y+5z&=9 \end{aligned}}$$

I'll use the third equation to eliminate x from the others:

$$\begin{aligned} &[\text{1st equation}] - 2\cdot[\text{3rd equation}] \implies -11y - 7z = -10 \implies 11y + 7z = 10 \qquad (*) \\ &[\text{2nd equation}] - 3\cdot[\text{3rd equation}] \implies -17y - 13z = -22 \implies 17y + 13z = 22 \qquad (**) \end{aligned}$$

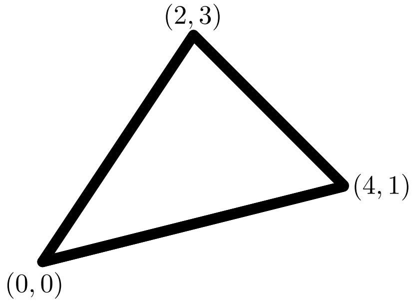
Next, I can eliminate z:

$$13\cdot(*) - 7\cdot(**) \implies 24y = -24 \implies y = -1$$

and the rest is easy:

$$\boxed{\begin{aligned} x&=2 \\ y&=-1 \\ z&=3 \end{aligned}}$$

Exercise 10. I translate the triangle five metres to the left and one metre down:



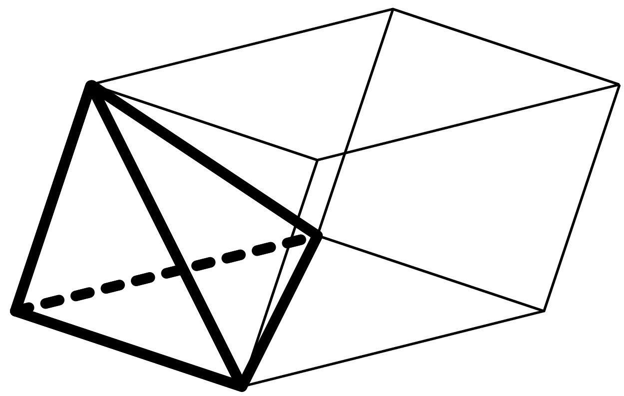
The area of this triangle equals half of the area of the parallelogram spanned by $(4,1)$ and $(2,3)$, so

$$\text{area} = \frac{1}{2}\det \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = 5$$

Exercise 11. The volume of this tetrahedron is

$$\frac{1}{6}\det\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$$

You can see this as follows:



The volume of a tetrahedron (more generally: of a pyramid) is

$$\frac{1}{3} \cdot (\text{area of the tetrahedron base}) \cdot \text{height}$$

and the volume of the parallelepiped spanned by (p_1, p_2, p_3), (q_1, q_2, q_3) and (r_1, r_2, r_3) is

$$(\text{area of the parallelepiped base}) \cdot \text{height}$$

However, the volume of this parallelepiped is also given by

$$\det\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$$

Now, the formula on top of this page follows from the fact that the base of the tetrahedron is half as large as the base of the parallelepiped.

Exercise 12. The length of $(3, 6, -2)$ is $\|(3, 6, -2)\| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{49} = 7$.

Exercise 13. $(-3, 5, 1) \bullet (7, 4, 5) = -21 + 20 + 5 = 4$

Exercise 14. The distance from $(3, -1, 7)$ to $(7, 1, 3)$ is

$$\|(3, -1, 7) - (7, 1, 3)\| = \|(-4, -2, 4)\| = \sqrt{(-4)^2 + (-2)^2 + 4^2} = \sqrt{36} = 6$$

Exercise 15.

a) This set is a line through the origin, because it is the intersection line of the planes $2x_1 + x_2 - x_3 = 0$ and $x_1 - x_2 + 2x_3 = 0$, both passing through the origin.

b) A parametric representation can for instance be found as follows:

$$\left.\begin{array}{lcl} 2x_1 + x_2 - x_3 = 0 & \Longrightarrow & x_2 = x_3 - 2x_1 \\ x_1 - x_2 + 2x_3 = 0 & \Longrightarrow & x_2 = x_1 + 2x_3 \end{array}\right\} \Longrightarrow x_3 - 2x_1 = x_1 + 2x_3 \Longrightarrow x_3 = -3x_1$$

x_2 can be expressed in terms of x_1 as well, because $x_2 = x_3 - 2x_1 = -5x_1$. Hence, a vector on this line can be written as $(x_1, -5x_1, -3x_1)$, which equals $x_1(1, -5, -3)$. Thus, a suitable parametric representation is $\lambda(1, -5, -3)$.

Exercise 16.

a) $\vec{\mathbf{x}} \bullet \vec{\mathbf{y}} = 0$ so $\vec{\mathbf{x}}$ is perpendicular to $\vec{\mathbf{y}}$.

b) The cross product is

$$\vec{\mathbf{x}} \times \vec{\mathbf{y}} = \det \begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ 5 & 1 & -3 \\ 1 & -2 & 1 \end{pmatrix} = (-5, -8, -11)$$

Exercise 17. The angle φ between $(1, 0, 0)$ and $(2, 1, -\sqrt{3})$ is given by

$$\cos\varphi = \frac{(1,0,0) \bullet (2,1,-\sqrt{3})}{\|(1,0,0)\| \cdot \|(2,1,-\sqrt{3})\|} = \frac{2}{1 \cdot \sqrt{8}} = \frac{1}{2}\sqrt{2} \implies \varphi = \frac{\pi}{4}$$

Exercise 18. This area equals half of the area of the parallelogram spanned by $(3, 0, 2)$ and $(5, 3, 4)$, which I calculate using the cross product

$$(3,0,2) \times (5,3,4) = \det \begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ 3 & 0 & 2 \\ 5 & 3 & 4 \end{pmatrix} = (-6, -2, 9)$$

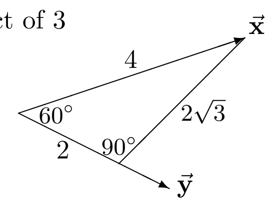
The area of the parallelogram is

$$\|(-6,-2,9)\| = \sqrt{(-6)^2 + (-2)^2 + 9^2} = \sqrt{121} = 11$$

so the area of the triangle equals $\dfrac{11}{2}$.

Exercise 19.

a) I use the geometric interpretation of the dot product, which equals the product of 3 (the length of $\vec{\mathbf{y}}$) and 2 (the length of the projection of $\vec{\mathbf{x}}$ on $\vec{\mathbf{y}}$), so $\boxed{6}$.



b) The length of their cross product equals the area of the spanned parallelogram with base 3 and height $2\sqrt{3}$:

$$\|\vec{\mathbf{x}} \times \vec{\mathbf{y}}\| = 3 \cdot 2\sqrt{3} = \boxed{6\sqrt{3}}$$

Exercise 20. The cosine of this angle is

$$\cos\alpha = \frac{(1,-1,-2) \bullet (1,2,1)}{\|(1,-1,-2)\| \cdot \|(1,2,1)\|} = \frac{-3}{\sqrt{6} \cdot \sqrt{6}} = -\frac{1}{2}$$

so $\alpha = \dfrac{2}{3}\pi$ (which is 120 degrees).

Exercise 21. Let's find a parametric representation of the line first:

$$\left.\begin{array}{ll} (*) & x_1 + 2x_2 + x_3 = 0 \\ (**) & 3x_1 - x_2 + 5x_3 = 0 \end{array}\right\} \implies \left\{\begin{array}{lllll} (*) + 2 \cdot (**) & \implies & 7x_1 + 11x_3 = 0 & \implies & x_1 = -\frac{11}{7}x_3 \\ 3 \cdot (*) - (**) & \implies & 7x_2 - 2x_3 = 0 & \implies & x_2 = \frac{2}{7}x_3 \end{array}\right.$$

Thus, a vector on this line can be written as $(-\frac{11}{7}x_3, \frac{2}{7}x_3, x_3) = \frac{1}{7}x_3(-11, 2, 7)$, so we are dealing with the line $[\![(-11, 2, 7)]\!]$. The desired projection is the projection of $(6, -5, 1)$ on $(-11, 2, 7)$, which can be calculated as follows:

$$\frac{(6,-5,1) \bullet (-11,2,7)}{(-11,2,7) \bullet (-11,2,7)} \cdot (-11,2,7) = \left(\frac{253}{58}, -\frac{23}{29}, -\frac{161}{58}\right)$$

Exercise 22. Let's first solve the problem translated over the vector $(-1,-2,-3)$: which point on $[\![(-2,1,1)]\!]$ is closest to $(-3,-2,2)$? That happens to be the projection of $(-3,-2,2)$ on $[\![(-2,1,1)]\!]$, which is

$$\frac{(-3,-2,2)\bullet(-2,1,1)}{(-2,1,1)\bullet(-2,1,1)}\cdot(-2,1,1)=(-2,1,1)$$

Now, I translate this back over $(1,2,3)$ to obtain the answer to the original problem: $(-1,3,4)$.

Exercise 23. It's always a good idea to simplify a complex problem like this, for instance by translating everything over the vector $(-3,-3,-2)$. Distances remain unchanged (invariant) under translations, so the new problem (with the same answer as the original problem) becomes:

Calculate the distance from the point $(2,0,-1)$ to the plane $[\![(1,0,0),(0,4,3)]\!]$.

This job can be done in many ways, for instance as follows:

Step 1. A normal to the plane is $(1,0,0)\times(0,4,3)=(0,-3,4)$.

Step 2. The projection of $(2,0,-1)$ on this normal is

$$\frac{(2,0,-1)\bullet(0,-3,4)}{(0,-3,4)\bullet(0,-3,4)}\cdot(0,-3,4)=\left(0\,,\,\frac{12}{25}\,,\,-\frac{16}{25}\right)$$

Step 3. The desired distance equals the length of this projection:

$$\left\|\left(0\,,\,\frac{12}{25}\,,\,-\frac{16}{25}\right)\right\|=\frac{4}{5}$$

Exercise 24.

Step 1. A normal to this plane is $(2,-1,3)$.

Step 2. The projection of $(1,2,1)$ on this normal is

$$\frac{(1,2,1)\bullet(2,-1,3)}{(2,-1,3)\bullet(2,-1,3)}\cdot(2,-1,3)=\left(\frac{3}{7}\,,\,-\frac{3}{14}\,,\,\frac{9}{14}\right)$$

Step 3. Hence, the projection of $(1,2,1)$ on the plane is

$$(1,2,1)-\left(\frac{3}{7}\,,\,-\frac{3}{14}\,,\,\frac{9}{14}\right)=\left(\frac{4}{7}\,,\,\frac{31}{14}\,,\,\frac{5}{14}\right)$$

Exercise 25. This can for instance be done as follows:

- a vector perpendicular to both lines is $(1,1,2)\times(-1,0,2)=(2,-4,1)$
- I normalise this vector to length 1: $\vec{\mathbf{n}}=\frac{(2,-4,1)}{\|(2,-4,1)\|}=\frac{(2,-4,1)}{\sqrt{21}}$
- now, I just grab a vector pointing from one line to the other: $(5,0,0)-(3,2,5)=(2,-2,-5)$
- the desired distance equals the (absolute value of) the dot product $(2,-2,-5)\bullet\vec{\mathbf{n}}=\frac{7}{\sqrt{21}}=\frac{1}{3}\sqrt{21}$

Exercise 26. A vector perpendicular to this plane is $(-1,-3,5)$ (which you can find using the cross product: $(3,-1,0)\times(7,1,2)=(-2,-6,10)$). The angle φ between $(1,1,1)$ and $(-1,-3,5)$ then satisfies

$$\cos\varphi=\frac{(1,1,1)\bullet(-1,-3,5)}{\|(1,1,1)\|\cdot\|(-1,-3,5)\|}=\frac{1}{\sqrt{105}}$$

The desired angle α is $\frac{\pi}{2}-\varphi$, so

$$\sin\alpha=\frac{1}{\sqrt{105}}\qquad\overset{\text{calculator}}{\Longrightarrow}\qquad\alpha\text{ equals approximately 5.6 degrees}$$

Exercise 27. This is the line $(1,2,3)+[\![(4,5,6)-(1,2,3)]\!]=(1,2,3)+[\![(3,3,3)]\!]=(1,2,3)+[\![(1,1,1)]\!]$. Therefore, a parametric representation is $(1,2,3)+\lambda(1,1,1)$.

Exercise 28.

a) $x_1+3x_3=5$

b) Let W be the plane through $\mathbf{O}$ parallel to V. The equation of W is $x_1+3x_3=0$, so a vector perpendicular to W is $(1,0,3)$. Then, the desired point must be a multiple of $(1,0,3)$, say $(\lambda,0,3\lambda)$. Substitution in the equation of V yields $\lambda=\frac{1}{2}$. Hence, the point is $(\frac{1}{2},0,\frac{3}{2})$.

Exercise 29.

a) Suppose $x_3=\lambda$. Then $\vec{\mathbf{x}}\in L \iff \begin{cases} 3x_1+5x_2-\lambda=3 \\ x_1-2\lambda=1 \end{cases} \iff \begin{cases} x_1=1+2\lambda \\ x_2=-\lambda \end{cases}$

$\iff \vec{\mathbf{x}}=(1,0,0)+\lambda(2,-1,1)$ So $L=(1,0,0)+[\![(2,-1,1)]\!]$.

b) This problem can be solved in many many ways. Three examples:

(1) The desired distance equals the distance of a random vector on L to its projection on $[\![(2,-1,1)]\!]$. The projection of $(1,0,0)$ on $(2,-1,1)$ is $(\frac{2}{3},-\frac{1}{3},\frac{1}{3})$, so the distance from L to $\vec{\mathbf{O}}$ is $\|(1,0,0)-(\frac{2}{3},-\frac{1}{3},\frac{1}{3})\|=\frac{1}{3}\sqrt{3}$.

(2) The point $\vec{\mathbf{x}}$ on L closest to $\vec{\mathbf{O}}$ can be found by intersecting L with the plane V through $\vec{\mathbf{O}}$ and perpendicular to L. This plane V has equation $2x_1-x_2+x_3=0$. Substitution of $\vec{\mathbf{x}}=(1+2\lambda,-\lambda,\lambda)$ yields $\lambda=-\frac{1}{3}$, so $\vec{\mathbf{x}}=(\frac{1}{3},\frac{1}{3},-\frac{1}{3})$ and $\|\vec{\mathbf{x}}\|=\frac{1}{3}\sqrt{3}$.

(3) The distance from $(1+2\lambda,-\lambda,\lambda)$ to $\vec{\mathbf{O}}$ is $\|(1+2\lambda,-\lambda,\lambda)\|=\sqrt{6\lambda^2+4\lambda+1}=\sqrt{6\big(\lambda+\frac{1}{3}\big)^2+\frac{1}{3}}$, which has a minimum value of $\sqrt{\frac{1}{3}}$.

Exercise 30.

a) That's the plane $\boxed{3x_1+2x_2+5x_3=0}$ because you can write this equation as $(3,2,5)\bullet(x_1,x_2,x_3)=0$ or, in other words, $(x_1,x_2,x_3)\perp(3,2,5)$.

b) This plane must be parallel to the plane found in (a), so its equation is of the form $3x_1+2x_2+5x_3=c$. You can find the value of the constant c easily by substituting the point $(7,8,-6)$, yielding $\boxed{3x_1+2x_2+5x_3=7}$.

Exercise 31.

a) L is the line $(0,-1,-1)+[\![(6,3,2)]\!]$. The desired angle φ equals the angle between $[\![(6,3,2)]\!]$ and V. Since the vector $(1,-3,2)$ is perpendicular to V, $\frac{\pi}{2}-\varphi$ is the angle between $(6,3,2)$ and $(1,-3,2)$. Therefore,

$$\sin\varphi=\cos\Big(\frac{\pi}{2}-\varphi\Big)=\frac{(6,3,2)\bullet(1,-3,2)}{\|(6,3,2)\|\cdot\|(1,-3,2)\|}=\frac{1}{7\sqrt{14}}$$

b) Such a point lies on L and is of the form $(0,-1,-1)+\lambda(6,3,2)$. Its projection on the vector $(1,-3,2)$, which is $\perp V$, is then

$$\vec{\mathbf{p}}=\frac{(6\lambda,3\lambda-1,2\lambda-1)\bullet(1,-3,2)}{(1,-3,2)\bullet(1,-3,2)}\cdot(1,-3,2)=\frac{\lambda+1}{14}\cdot(1,-3,2)$$

The length of this $\vec{\mathbf{p}}$ must be 5, so

$$\frac{|\lambda+1|}{14}\cdot\|(1,-3,2)\|=5 \implies |\lambda+1|=5\sqrt{14} \implies \lambda=-1\pm5\sqrt{14}$$

Thus, the desired points are

$$\boxed{\begin{array}{c}\big(-6+30\sqrt{14}\,,\ -4+15\sqrt{14}\,,\ -3+10\sqrt{14}\big)\\ \big(-6-30\sqrt{14}\,,\ -4-15\sqrt{14}\,,\ -3-10\sqrt{14}\big)\end{array}}$$

Exercise 32. You can do this analoguously to exercise 25, but this time I choose otherwise:

- $V = [\![(1,0,3),(2,1,1)]\!]$ is the plane through L parallel to M.
- The desired distance equals the distance from $(-1,1,0)$ to V.
- A normal to V is $(1,0,3)\times(2,1,1) = (-3,5,1)$.
- The projection of $(-1,1,0)$ on this normal $(-3,5,1)$ is $(-\frac{24}{35}, \frac{8}{7}, \frac{8}{35})$.
- The desired distance equals the norm of this projection, which is

$$\left\|\left(-\frac{24}{35}, \frac{8}{7}, \frac{8}{35}\right)\right\| = \left\|\frac{8}{35}\cdot(-3,5,1)\right\| = \frac{8}{35}\,\|(-3,5,1)\| = \frac{8}{35}\sqrt{35}$$

Exercise 33. A method saving laborious calculations is the following:

- a vector perpendicular to the plane is $(3,1,1)$
- the projection of $(1,0,-2)$ on $(3,1,1)$ is $\left(\frac{3}{11}, \frac{1}{11}, \frac{1}{11}\right)$
- so the desired mirror image is $(1,0,-2) - \left(\frac{6}{11}, \frac{2}{11}, \frac{2}{11}\right) = \left(\frac{5}{11}, -\frac{2}{11}, -\frac{24}{11}\right)$

Exercise 34.

a) $x_1 - x_2 + x_3 = 0$

b) A vector perpendicular to $\mathbf{V}$ is $(1,-1,1)$. The projection of $(1,1,3)$ on $(1,-1,1)$ is

$$\frac{(1,1,3)\bullet(1,-1,1)}{(1,-1,1)\bullet(1,-1,1)}\cdot(1,-1,1) = (1,-1,1)$$

so the projection of $(1,1,3)$ on $\mathbf{V}$ is $(1,1,3)-(1,-1,1) = (0,2,2)$.

c) This angle φ equals the angle between $(1,1,3)$ and $(0,2,2)$, so

$$\cos\varphi = \frac{(1,1,3)\bullet(0,2,2)}{\|(1,1,3)\|\cdot\|(0,2,2)\|} = \sqrt{\frac{8}{11}}$$

(my calculator says $\varphi \approx 0.55$ radians $\approx 31\frac{1}{2}$ degrees)

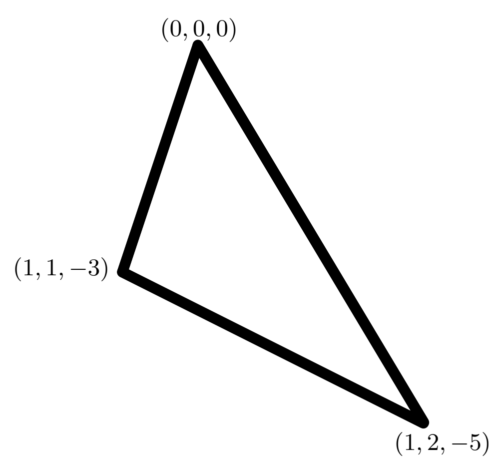
Exercise 35. Translate the triangle over $(-2,1,-7)$. This translation leaves the area invariant, so the problem reduces to:

Calculate the area of the triangle with vertices $(0,0,0)$, $(1,1,-3)$ and $(1,2,-5)$.

Solution. The area of the parallelogram spanned by $(1,1,-3)$ and $(1,2,-5)$ is

$$\|(1,1,-3)\times(1,2,-5)\| = \|(1,2,1)\| = \sqrt{6}$$

The area of the triangle is half of this area: $\frac{1}{2}\sqrt{6}$.



4. Functions

Definition. By $f : A \to B$ (pronunciation: f is a function from A to B) we mean: f is a rule that associates precisely one element $f(x)$ from B to every element x from A.

Domain and range. If $f : A \to B$, we call A the domain of f. The collection of all function values $f(x)$ is called the range of f.

Example 1. sin is a function from $\mathbb{R}$ to $\mathbb{R}$. The domain of sin is $\mathbb{R}$ and its range is $[-1, 1]$.

Example 2. I can interpret 'square' as a function from $\mathbb{N}$ (the set containing all natural numbers $0, 1, 2, 3, \ldots$) to $\mathbb{N}$. Its domain is then $\mathbb{N}$, and its range is $\{0, 1, 4, 9, \ldots\}$. A (small part of an infinite) table:

0	1	2	3	4	5	6	7	8	9	$\cdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
0	1	4	9	16	25	36	49	64	81	$\cdots$

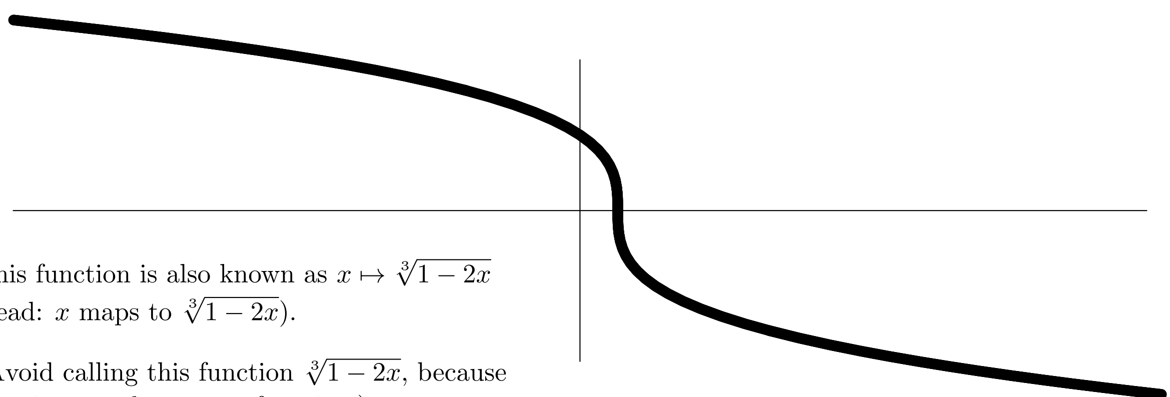
Example 3. The function 'age' with as domain the collection of my cats is defined by the table on the right. The range is the set $\{7, 8, 10, 12, 16, 20\}$.

Alpje	$\mapsto$	12
Bontepoes	$\mapsto$	16
Fafner	$\mapsto$	12
Kaasje	$\mapsto$	7
Pommetje	$\mapsto$	8
Wally	$\mapsto$	10
Wobbeltje	$\mapsto$	20
Zompie	$\mapsto$	8

Example 4. The function rule (also called 'rule of correspondence')

$$\boxed{f(x) = \sqrt[3]{1 - 2x}}$$

defines a function f from $\mathbb{R}$ to $\mathbb{R}$. The domain is $\mathbb{R}$ and the range is $\mathbb{R}$ as well. The collection of all points $(x, f(x))$ is called the graph of f:



This function is also known as $x \mapsto \sqrt[3]{1 - 2x}$ (read: x maps to $\sqrt[3]{1 - 2x}$).

(Avoid calling this function $\sqrt[3]{1 - 2x}$, because that is a number, not a function.)

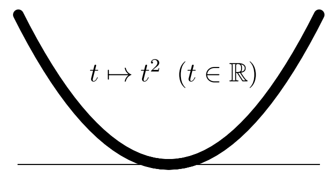
Increase and decrease. For functions with domain and range consisting of numbers we define

$$f \text{ is increasing} \stackrel{\text{def}}{=} \text{every pair } x, y \text{ with } x < y \text{ satisfies } f(x) < f(y)$$

$$f \text{ is decreasing} \stackrel{\text{def}}{=} \text{every pair } x, y \text{ with } x < y \text{ satisfies } f(x) > f(y)$$

Thus, the function sin from example 1 is neither increasing nor decreasing, the function from example 2 is increasing and that from example 4 is decreasing. For the function from example 3 these concepts are undefined (because cats are not numbers).

Example 5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by



$$\boxed{f(t) = t^2}$$

This function is not increasing, because $f(-7) > f(-6)$.
It is neither decreasing, because $f(3) < f(\pi)$.

Addition of functions. Two functions f and g with equal domains and with ranges consisting of numbers can be added. Their sum function $f + g$ is defined by the function rule

$$\boxed{(f + g)(x) \stackrel{\text{def}}{=} f(x) + g(x)}$$

This is called 'pointwise addition'. In similar ways you can define different operations on functions (difference, product, quotient, scalar product).

Example 6. If f and g, both with domain $\mathbb{R}$, are given by

$$f(x) = 7x + 5 \qquad\qquad g(x) = 2x - 3$$

then their sum $f + g$ and their products $f \cdot g$ have the following function rules:

$$(f + g)(x) = 9x + 2 \qquad\qquad (f \cdot g)(x) = 14x^2 - 11x - 15$$

Inverse. If every element of the range occurs precisely once as a function value, we define

$$\boxed{\text{the inverse of } f} \stackrel{\text{def}}{=} \boxed{\text{the function } \overleftarrow{f} \text{ with } \overleftarrow{f}\big(f(x)\big) = x}$$

The function sin from example 1 has no inverse (because the numbers from the range $[-1, 1]$ occur multiple times as function values), but the function from example 2 does: its inverse is the function $n \mapsto \sqrt{n}\;\; (n \in \mathbb{N})$.
The function from example 4 has inverse

$$\overleftarrow{f}(x) = \frac{1 - x^3}{2}$$

(wonder how I discovered this?), but the functions from examples 3 and 5 are not invertible: if I tell you something funny about the eight-year-old cat, you have no clue about which I'm talking.

Composition of functions. If $f : A \to B$ and $g : B \to C$, we define the composition $g \circ f : A \to C$ by

$$\boxed{(g \circ f)(x) \stackrel{\text{def}}{=} g(f(x))}$$

Composing g with f amounts to: do f first and then do g.

Example 7. Let f be the function $x \mapsto x^3$. You can compose f with the sine function in two different ways:

$$f \circ \sin \quad \text{is the function} \quad x \mapsto (\sin x)^3$$
$$\sin \circ f \quad \text{is the function} \quad x \mapsto \sin(x^3)$$

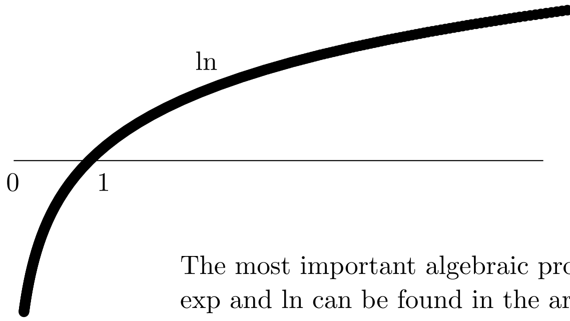
Example 8. Write $f(x) = 3^{\sqrt{5x+7}}$ as a composition of simple functions.

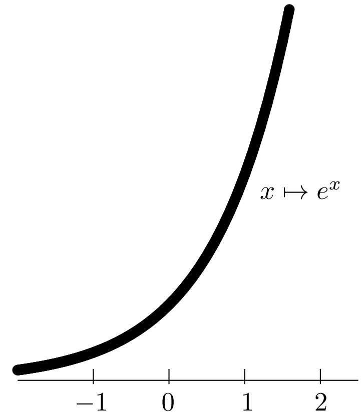
Solution. I take the simple functions

$$p(x) = 3^x \qquad\qquad q(x) = \sqrt{x} \qquad\qquad r(x) = 5x + 7$$

Then, the function f is $p \circ q \circ r$, because $f(x) = p\left(q\left(r(x)\right)\right)$.

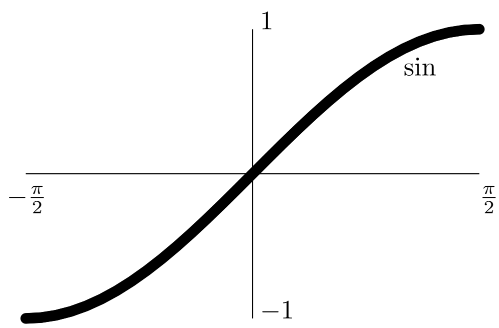
Exp and ln. The function $\exp : \mathbb{R} \to (0, \infty)$ is defined by $\exp(x) = e^x$, see the graph on the right. Its inverse is $\ln : (0, \infty) \to \mathbb{R}$, see below.





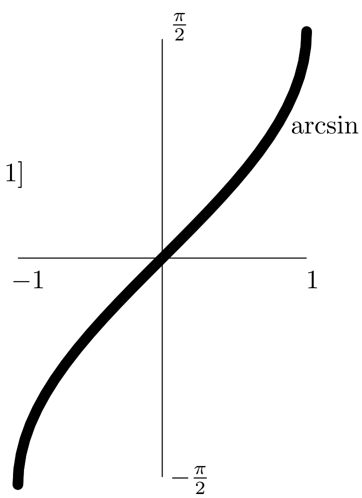
The most important algebraic properties of exp and ln can be found in the arithmetic booklet.

Arcsine. The function sin is neither increasing nor decreasing. Therefore, it has no inverse. That's a pity for most students, because they would love to be able to use a function that inverts the sine. That's why smart mathematicians invented a beautiful trick. They took a small piece of the sine function which does increase, namely the piece of the sine with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$:



Now, we define the <u>arcsine</u> as the inverse of the function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ drawn up here. The domain of this new function is $[-1, 1]$ and the graph of arcsin is the reflection of the graph above across the line $y = x$.

Instead of $\overleftarrow{f}$ one sometimes writes f^{-1}. This is why the arcsine is probably called $\sin^{-1}$ on your calculator.



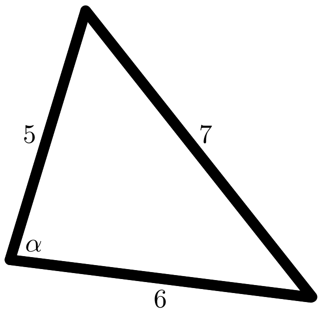
Arccosine and arctangent. Similarly, cos and tan are neither inreasing nor decreasing, so they do not have inverses, but by reducing their domains we can consider small pieces of these functions which do have inverses. We do this as follows:

$$\arccos : [-1, 1] \to [0, \pi] \quad \text{is the inverse of} \quad \cos : [0, \pi] \to [-1, 1]$$

$$\arctan : \mathbb{R} \to \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \quad \text{is the inverse of} \quad \tan : \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \to \mathbb{R}$$

The following properties follow from these definitions:

$$\begin{array}{llll} \arcsin(\sin x) = x & \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & \sin(\arcsin x) = x & \text{if } x \in [-1, 1] \\ \arccos(\cos x) = x & \text{if } x \in [0, \pi] & \cos(\arccos x) = x & \text{if } x \in [-1, 1] \\ \arctan(\tan x) = x & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & \tan(\arctan x) = x & \text{for all } x \in \mathbb{R} \end{array}$$



Example 9. Calculate the angle α in this figure.

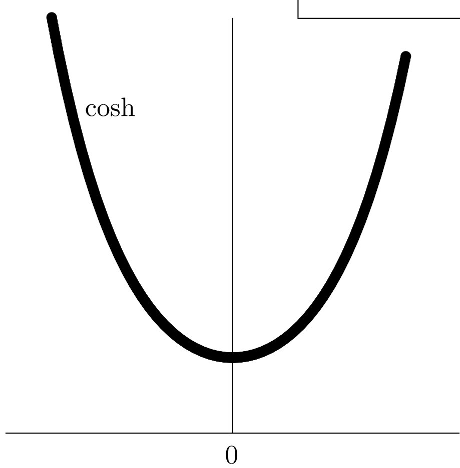
Solution. According to the law of cosines,

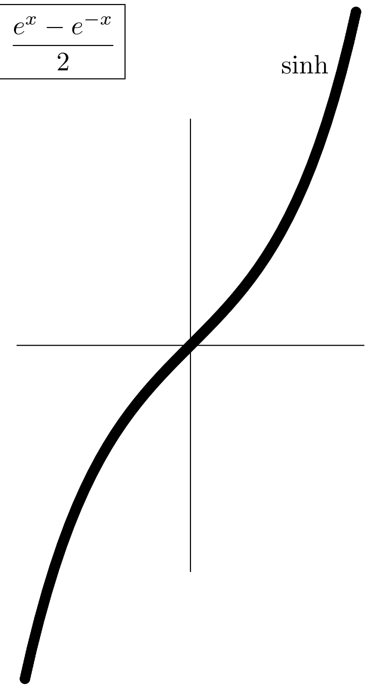
$$7^2 = 5^2 + 6^2 - 60\cos\alpha \quad\Longrightarrow\quad \cos\alpha = \frac{1}{5} \quad\Longrightarrow\quad \alpha = \arccos\frac{1}{5}$$

(my calculator says approximately 1.37 radians or 78.5 degrees)

Hyperbolic functions. The functions cosh (hyperbolic cosine) and sinh are defined by

$$\boxed{\cosh x \overset{\text{def}}{=} \frac{e^x + e^{-x}}{2}} \qquad \boxed{\sinh x \overset{\text{def}}{=} \frac{e^x - e^{-x}}{2}}$$



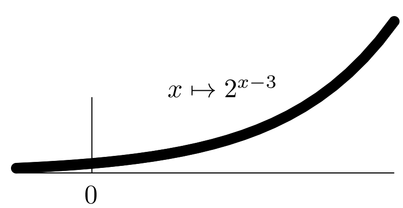


The graph of cosh reminds me of a hanging cord. Indeed, using elementary mechanics you can prove that a uniform cord always hangs along the graph of

$$x \mapsto a + b\cosh(cx + d)$$

How to calculate the inverse? Sometimes it is possible to explicitly calculate the inverse of a given function f: start with $f(x) = y$ and express x in terms of y. Unfortunately, this is not always possible. The graph of $\overleftarrow{f}$, however, can always be drawn, since it is just the graph of f with the roles of x and y interchanged. Put differently: the graph of $\overleftarrow{f}$ is the reflection of the graph of f across the line $x = y$.

Example 10. The function $f : \mathbb{R} \to (0,\infty)$ given by $f(x) = 2^{x-3}$ is increasing and therefore has an inverse $\overleftarrow{f} : (0,\infty) \to \mathbb{R}$. Find a function rule for this inverse.

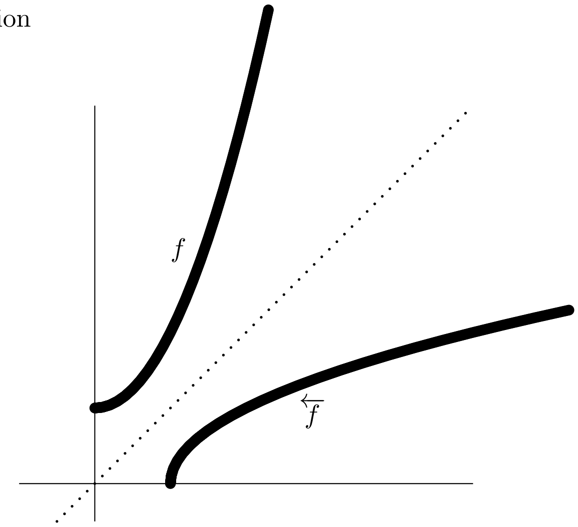


Solution. I use y as an abbreviation of $f(x)$:

$$y = 2^{x-3} \quad\Longleftrightarrow\quad \ln y = \ln 2^{x-3} = (x-3)\ln 2 \quad\Longleftrightarrow\quad x - 3 = \frac{\ln y}{\ln 2} \quad\Longleftrightarrow\quad x = 3 + \frac{\ln y}{\ln 2}$$

Conclusion: $\overleftarrow{f}(y) = 3 + \dfrac{\ln y}{\ln 2}$ or, equivalently, $\overleftarrow{f}(x) = 3 + \dfrac{\ln x}{\ln 2}$.

Example 11. Draw and calculate the inverse of the function $f : (0, \infty) \to (1, \infty)$ with $f(x) = 1 + x^2$.



Solution. $y = 1 + x^2 \iff y - 1 = x^2 \iff x = \sqrt{y-1}$
We've found a function rule for $\overleftarrow{f}$:

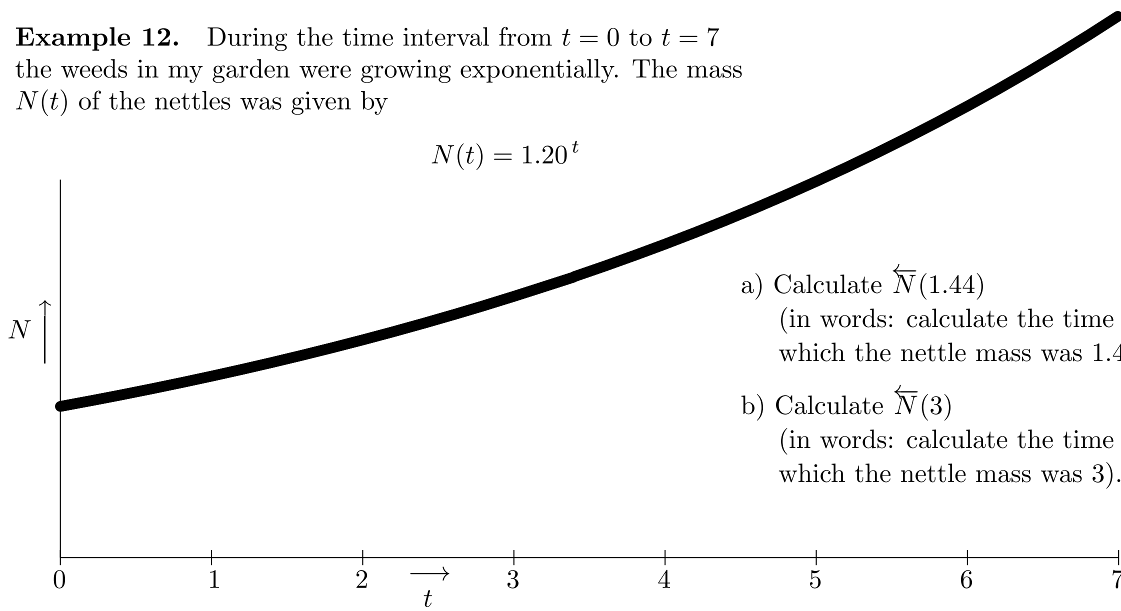
$$\overleftarrow{f}(y) = \sqrt{y-1}$$

or, equivalently,

$$\overleftarrow{f}(x) = \sqrt{x-1}$$

Example 12. During the time interval from $t = 0$ to $t = 7$ the weeds in my garden were growing exponentially. The mass $N(t)$ of the nettles was given by

$$N(t) = 1.20^{\,t}$$



a) Calculate $\overleftarrow{N}(1.44)$
(in words: calculate the time at which the nettle mass was 1.44).

b) Calculate $\overleftarrow{N}(3)$
(in words: calculate the time at which the nettle mass was 3).

Solution.

a) $\overleftarrow{N}(1.44) = 2$, because $N(2) = 1.44$.

b) $1.20^t = 3 \implies e^{t \ln 1.20} = 3 \implies t \ln 1.20 = \ln 3 \implies t = \dfrac{\ln 3}{\ln 1.20}$

Conclusion: $\overleftarrow{N}(3) = \dfrac{\ln 3}{\ln 1.20} \approx 6.026.$

Exponential functions. An <u>exponential function</u> is a function of the kind $x \mapsto \alpha \cdot \beta^x$ with α and β constants. If $\beta > 1$ you are dealing with exponential growth, if $\beta < 1$ it is called exponential decay. If you encounter a function in the wild, and you suspect it to be exponential based on theoretical considerations, two measurements are required to calculate α and β in order to find a function rule for f.

Example 13. The amount of CO_2 in the atmosphere, measured in Hawaii, increased from 320 ppm in 1970 to 400 ppm in 2010. How much CO_2 do you expect in the year 2050

a) assuming that the CO_2 concentration increases linearly, that is to say according to a function of the kind

$$CO_2(t) = \alpha t + \beta$$

b) assuming that the CO_2 concentration increases exponentially, that is to say according to a function of the kind

$$CO_2(t) = \alpha \cdot \beta^t$$

Solution.

a) I calculate α and β by substituting the two measurements in the linear model and subtracting the resulting equations:

$$\left.\begin{array}{lll} CO_2(1970) = 320 & \Longrightarrow & 1970\,\alpha + \beta = 320 \\ CO_2(2010) = 400 & \Longrightarrow & 2010\,\alpha + \beta = 400 \end{array}\right\} \Longrightarrow \quad 40\,\alpha = 80 \quad \Longrightarrow \quad \left\{\begin{array}{l} \alpha = 2 \\ \beta = -3620 \end{array}\right.$$

Conclusion: $CO_2(t) = 2t - 3620$ so $CO_2(2050) = \boxed{480 \text{ ppm}}$.

b) In the exponential model I divide the second equation by the first to find α and β:

$$\left.\begin{array}{lll} CO_2(1970) = 320 & \Longrightarrow & \alpha \cdot \beta^{1970} = 320 \\ CO_2(2010) = 400 & \Longrightarrow & \alpha \cdot \beta^{2010} = 400 \end{array}\right\} \Longrightarrow \quad \beta^{40} = \frac{400}{320} = 1.25 \quad \Longrightarrow \quad \left\{\begin{array}{l} \beta = 1.25^{\frac{1}{40}} \\ \alpha = \frac{400}{1.25^{\frac{201}{4}}} \end{array}\right.$$

Conclusion: $CO_2(t) = \frac{400}{1.25^{\frac{201}{4}}} \cdot 1.25^{\frac{t}{40}}$ so $CO_2(2050) = \boxed{500 \text{ ppm}}$.

Limits of functions. Sometimes you're interested in the approximate value of $f(x)$ if x is close to a. This value is denoted by

$$\lim_{x\to a} f(x)$$

If you're only interested in the case where x approaches a from the right, we write

$$\lim_{x\downarrow a} f(x) \qquad \text{(the right-sided limit)}$$

Below, I've collected a few different flavours of these limits. For those interested I'll include the precise definitions (in which D is the domain of f) as well, but in most cases the intuitive definitions are sufficient:

$\boxed{\lim_{x\to a} f(x) = p}$ $\stackrel{\text{def}}{=}$ (intuitively) if x approaches a, $f(x)$ is approximately p
(formally) for every positive real number ε I can find a positive real number δ such that: if $x \in D$ and $a - \delta < x < a + \delta$, then $p - \varepsilon < f(x) < p + \varepsilon$

$\boxed{\lim_{x\to \infty} f(x) = p}$ $\stackrel{\text{def}}{=}$ (intuitively) if x is very large, $f(x)$ approximately equals p
(formally) for every positive real number ε I can find a real number A such that: if $x \in D$ and $x > A$, then $p - \varepsilon < f(x) < p + \varepsilon$

$\boxed{\lim_{x\to a} f(x) = \infty}$ $\stackrel{\text{def}}{=}$ (intuitively) if x approaches a, $f(x)$ becomes very large
(formally) for every real number B I can find a positive real number δ such that: if $x \in D$ and $a - \delta < x < a + \delta$, then $f(x) > B$

$\boxed{\lim_{x\uparrow a} f(x) = p}$ $\stackrel{\text{def}}{=}$ (intuitively) if x is just below a, $f(x)$ is approximately p
(formally) for every positive real number ε I can find a positive real number δ such that: if $x \in D$ and $a - \delta < x < a$, then $p - \varepsilon < f(x) < p + \varepsilon$

$\boxed{\lim_{x\downarrow a} f(x) = p}$ $\stackrel{\text{def}}{=}$ (intuitively) if x is just above a, $f(x)$ is approximately p
(formally) for every positive real number ε I can find a positive real number δ such that: if $x \in D$ and $a < x < a + \delta$, then $p - \varepsilon < f(x) < p + \varepsilon$

For all these different concepts of limits similar rules to those for $\lim_{n\to\infty}$ apply. In addition:

$$\lim_{x\to a} f(x) = p \iff \left(\lim_{x\uparrow a} f(x) = p \text{ en } \lim_{x\downarrow a} f(x) = p\right)$$

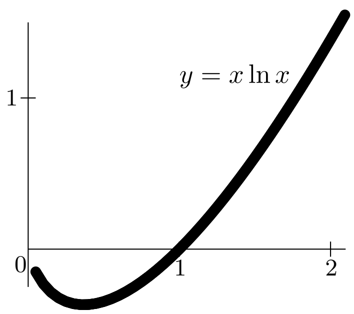
Example 14. Calculate $\lim_{x\to 5} \frac{1}{x^2}$.

Solution. In simple cases like this you just substitute the value that x approaches: $\lim_{x\to 5} \frac{1}{x^2} = \frac{1}{25}$.

Example 15. Calculate $\lim_{x\to 1} \frac{x^7-1}{x-1}$.

Solution. Substitutition of $x = 1$ won't work. Fortunately, you recognise the sum of a geometric sequence:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = \frac{1-x^7}{1-x} = \frac{x^7-1}{x-1} \implies \lim_{x\to 1} \frac{x^7-1}{x-1} = 7$$



Example 16. Have a guess: what is $\lim_{x\to 0} x \ln x$?

Solution. Get your calculator out and try some tiny positive values for x. Soon, you'll discover that

$$\boxed{\lim_{x\to 0} x \ln x = 0}$$

(Obviously, this is not a formal proof, but my guess is that very few students will lose sleep over it.)

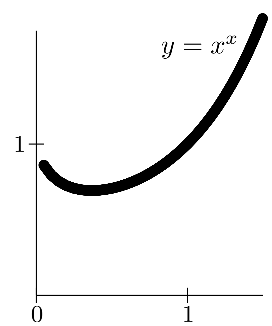
Example 17. Calculate $\lim_{x\to 0} x^x$.

Solution. You can go the experimental way again, your calculator for example will produce

$$0.001^{0.001} = 0.9931160484$$

which makes it easy to guess the exact answer:

$$\boxed{\lim_{x\to 0} x^x = 1}$$



An enthousiastic mathematician prefers a more formal proof, she might neatly use example 16:

$$\lim_{x\to 0} x^x = \lim_{x\to 0} \left(e^{\ln x}\right)^x = \lim_{x\to 0} e^{x \ln x} = e^0 = 1$$

Example 18. Calculate $\lim_{x\to\infty} \arctan x$.

Solution. In this case your calculator terribly fails, producing results such as

$$\tan^{-1}(1000) = 1.569796327$$

which are (still) unrecognisable to a freshman's eyes. Or maybe you did recognise this fancy number? Looking at the graph of arctan with the naked eye, you clearly see that

$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$

Real math fanatics of course don't trust their eyes and are only satisfied with a formal proof using the exact definition of the concept of limit:

Suppose $\varepsilon > 0$. Choose $A = \tan\left(\frac{\pi}{2} - \varepsilon\right)$. If $x > A$, then $\arctan x > \arctan A = \frac{\pi}{2} - \varepsilon$. □

***n*!.** The product of the numbers $1, \ldots, n$ is denoted by $n!$ (pronunciation: n factorial), and by $0!$ we mean the number 1. Examples:

$$\begin{array}{lll lll lll lll} 0! &=& 1 & 3! &=& 6 & 6! &=& 720 & 9! &=& 362880 \\ 1! &=& 1 & 4! &=& 24 & 7! &=& 5040 & 10! &=& 3628800 \\ 2! &=& 2 & 5! &=& 120 & 8! &=& 40320 & 11! &=& 39916800 \end{array}$$

Taylor series for exp. You can write e^x as an infinite series (called a power series or Taylor series):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Example 19. Calculate $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$

Solution. Take the Taylor series for e^x and substitute $x = -1$:

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots = e^{-1} = \frac{1}{e}$$

Example 20. Expand $\cosh x$ in a Taylor series.

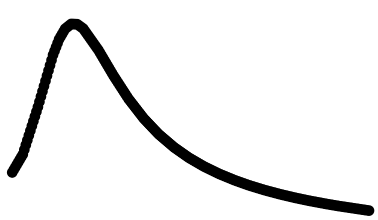
Solution. We use $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and add the power series for e^x and e^{-x}:

$$\begin{array}{rcl} e^x &=& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \\ e^{-x} &=& 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots \\ \hline e^x + e^{-x} &=& 2 \quad + 2\cdot\frac{x^2}{2!} \quad + 2\cdot\frac{x^4}{4!} \quad + 2\cdot\frac{x^6}{6!} \; + \; \cdots \end{array} \implies \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Continuity. Let f be a function with domain D, and let $a \in D$. Then, we define the following concepts:

$\boxed{f \text{ is continuous in } a}$	$\stackrel{\text{def}}{=}$	(intuitively)	the graph of f does not show a jump at a
		(formally)	$\lim\limits_{x \to a} f(x) = f(a)$
$\boxed{f \text{ is continuous}}$	$\stackrel{\text{def}}{=}$	(intuitively)	the graph of f does not show any jumps on its domain
		(formally)	f is continuous in a for all $a \in D$

In this course we study continuous functions (almost) exclusively. For example, the functions sin, cos, tan, ln, arcsin, arccos, arctan, sinh, cosh, ... are all continuous. Sums, products, quotients and compositions of continuous functions are continuous as well.



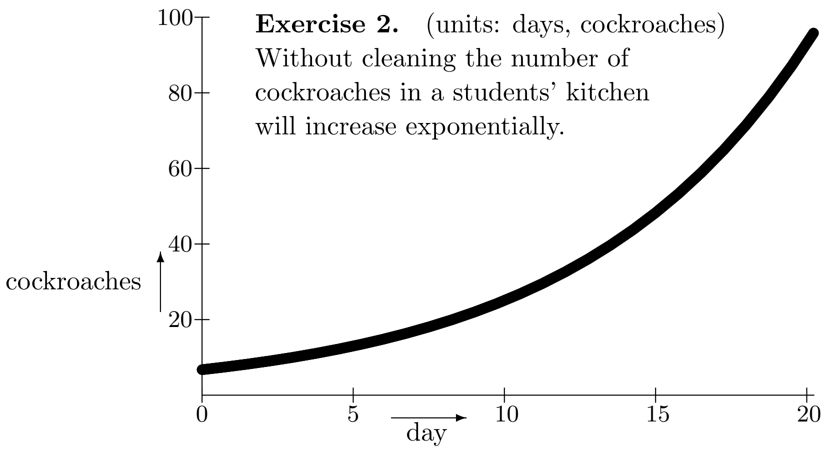
continuous

not continuous (discontinuous)

Exercises chapter 4

Exercise 1. Solve for x in the equation $3\sqrt{\arctan x} = 2 + \arctan x$.

Exercise 2. (units: days, cockroaches)
Without cleaning the number of cockroaches in a students' kitchen will increase exponentially.



Cockroach counts have revealed that

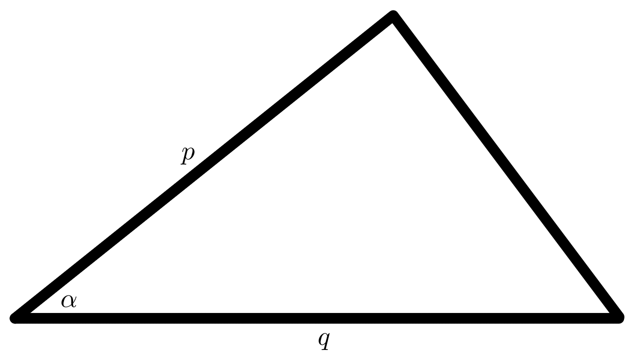
$\boxed{\text{Coc}(3) = 10}$ $\boxed{\text{Coc}(5) = 13}$

The students decide to start cleaning as soon as the number of cockroaches exceeds 100. When will this happen?

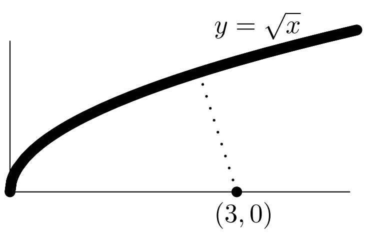
Exercise 3. Calculate $\lim_{x\to 3} \arctan\sqrt{x}$.

Exercise 4. Calculate $\lim_{x\to 0} x^{3x}$.

Exercise 5. Express the area of this triangle in terms of α, p and q.



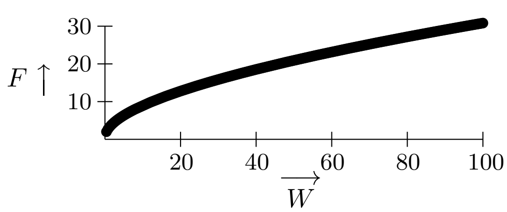
Exercise 6. Give me a function $f : \mathbb{R} \to \mathbb{R}$ that is not continuous at 5.



Exercise 7. Calculate the distance from the point $(3, 0)$ to the graph of the square root function.

Exercise 8. (units: kg, cm)
Homo erectus is supported by its feet. A human being of weight W requires a minimum foot length F:

$$F = W^{\frac{2}{3}} + 2W^{\frac{1}{3}}$$



a) How much weight should Porky (70 kg, foot length 24 cm) lose?

b) Find a function rule and draw a graph for the maximum weight as a function of foot length.

Exercise 9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\boxed{f(x) = \frac{x}{1 + x^4}}$$

a) Is f increasing?

b) Is f decreasing?

Exercise 10. Find a function rule for the inverse of the function $t \mapsto 3 + e^{2+t}$.

Exercise 11. We define functions f and g from $\mathbb{R}$ to $\mathbb{R}$ by

$$\boxed{f(x) = \ln\left(x^2 + 1\right)} \qquad \boxed{g(x) = e^{3x}}$$

a) Determine a function rule for $g \circ f$.

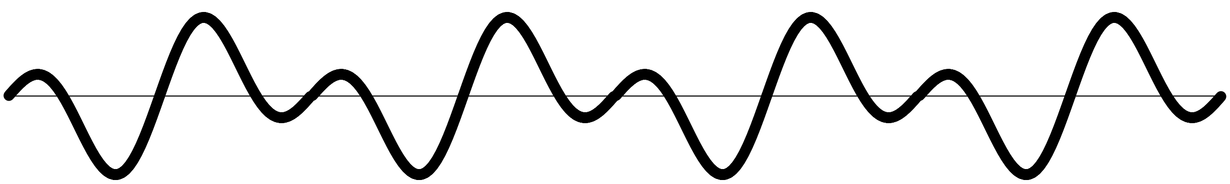
b) Determine a function rule for $f \circ g$.

Exercise 12.

a) Calculate $\arctan \sqrt{3}$.

b) Calculate $\arctan(-1)$.

Exercise 13. I define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x + \sin 2x$:



Find all zeros of f (I mean: find all values of x for which $f(x) = 0$).

Exercise 14.

a) Give a function rule for a function

$$\text{bonus} : [7, 10] \to \mathbb{R}$$

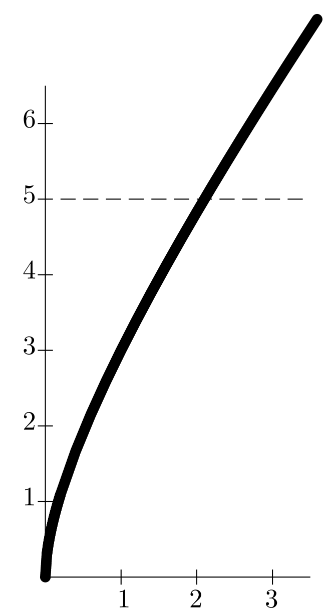
which maps every 'happy grade' to the rounded 'repair bonus'.

b) Draw the graph of your function bonus.

Exercise 15. This is a piece of the graph of the function f with function rule

$$\boxed{f(x) = x + 2\sqrt{x}}$$

Determine the point where f assumes a value of 5 (please provide an exact answer, I don't like approximations such as 2.101020514).



Exercise 16. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Exercise 17. Calculate $\lim_{n\to\infty} \arctan \sqrt{1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{2^n}{3^n}}$.

Exercise 18. Calculate $\lim_{t\to 0} t(\ln t)^2$.

Exercise 19. We define a new function *hyperbolic tangent* by $\boxed{\tanh x \stackrel{\text{def}}{=} \frac{\sinh x}{\cosh x}}$.

a) Calculate the limit of the sequence $\tanh 1\,,\ \tanh 2\,,\ \tanh 3\,,\ \tanh 4\,,\ \tanh 5\,,\ \tanh 6\,,\ \tanh 7\,,\ \tanh 8\,,\ \ldots$

b) Sketch the graph of the hyperbolic tangent.

Exercise 20. Prove that $\sin\left(\arctan \frac{5}{3}\right) = \frac{5}{\sqrt{34}}$.

Exercise 21. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 7 + \sqrt[5]{3x - 2}$$

Calculate the inverse of f.

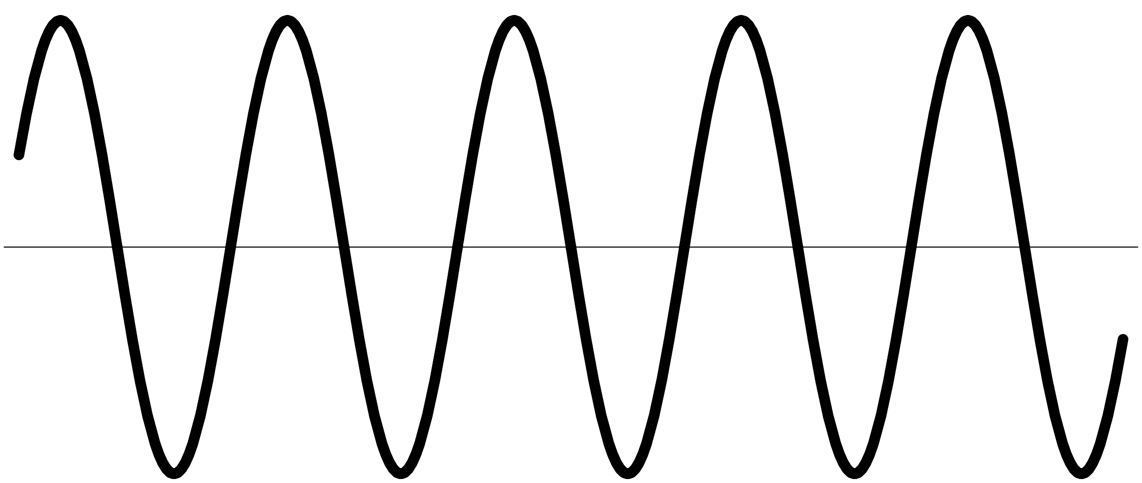
Exercise 22. Draw the graph of the function arccos.

Exercise 23. Let $f : [0, \infty) \to [1, \infty)$ be given by

$$f(x) = \frac{e^x + e^{-x}}{2}$$

(so f is the function cosh with its domain restricted to $[0, \infty)$).
Since f is increasing, it has an inverse. Determine a function rule for $\overleftarrow{f}$.

Exercise 24. If you draw the graph of the function $f(x) = 3\sin x + 5\cos x$ (or let your calculator make a plot), the result looks like a sinusoid:



In fact, this nicely agrees with your experience that, for example, adding two sound waves with different amplitudes but equal frequencies yields another sound wave. Prove that f is indeed a function of the kind

$$f(x) = A\sin(x + B)$$

and calculate the constants A and B.

Exercise 25. Sketch the graph of the function

$$\boxed{x \mapsto (\cosh x)^2 - (\sinh x)^2}$$

and explain why this graph is rather dull.

Exercise 26. When I pet my sweet little Bontepoes, she can't stop purring. More precisely: a petting intensity P generates a purring intensity $2^{\ln P}$. Find a function rule that allows me to calculate how intense I should pet Bontepoes in order to generate a desired purring intensity S. Put differently: find the inverse of the function $S = 2^{\ln P}$.

Exercise 27. We define the function f by

$$\boxed{f(x) = \arctan x + \arctan \frac{1}{x}}$$

a) What is the domain of f ?

b) Sketch the graph of f.

Exercise 28. Which of the following functions has function rule $x \mapsto \sin^2 x$?

$$\sin \circ \sin \qquad \sin \circ \text{square} \qquad \text{square} \circ \sin$$

(by square I mean the function $x \mapsto x^2$)

Exercise 29. (units: kg, days)
The amount of junk in a student's room increases exponentially from $t = 0$ to $t = 7$. Given are two measurements:

$$\boxed{\text{junk}\,(0) = 2} \qquad \boxed{\text{junk}\,(1) = 3}$$

Sketch the junk function and calculate junk (7).

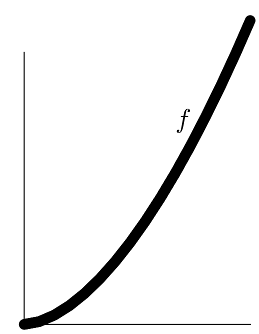
Exercise 30. Draw the graph of the function arctan.

Exercise 31. We define the function f with domain $[0, \infty)$ by

$$f(x) = x^{\sqrt{3}}$$

a) Find a function rule for $f \circ f$, and simplify it as much as possible.

b) Do the same for $\overleftarrow{f} \circ \overleftarrow{f}$.



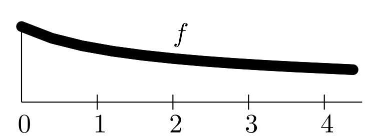
Exercise 32. Calculate $\displaystyle\lim_{x \to 1} \frac{1 - \sqrt[7]{x}}{1 - x}$.

Exercise 33. We define the function f with domain $[0, \infty)$ by

$$f(x) = \cos\left(\arctan \sqrt{x}\right)$$

a) Calculate $f(3)$.

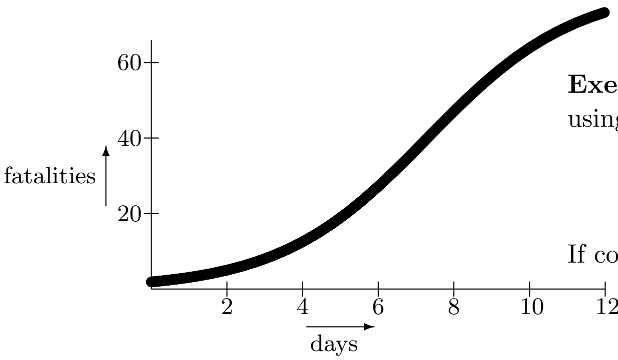
b) Calculate $\displaystyle\lim_{x \to \infty} f(x)$.



Exercise 34. Write the Taylor series for e^x using the $\sum$-notation.

Exercise 35. Find a power series expansion for $\sinh x$.

Exercise 36. Calculate $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \frac{x^5}{6!} + \frac{x^6}{7!} + \frac{x^7}{8!} + \frac{x^8}{9!} + \frac{x^9}{10!} + \cdots$



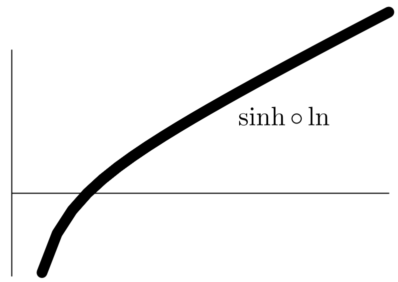
Exercise 37. I reconstruct a plague outbreak (80 fatalities) using the logistic model: the number of fatalities after t days is

$$F(t) = \frac{80}{1 + \beta \cdot \gamma^t}$$

If counts indicate that $F(2) = 5$ en $F(3) = 8$, at what moment does my model predict a number of 60 fatalities?

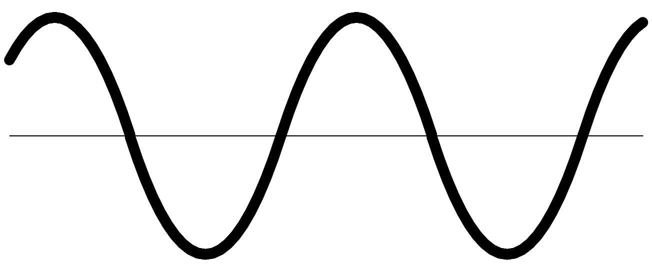
Exercise 38. Determine a function rule for the inverse of the function

$$\sinh \circ \ln : (0, \infty) \to \mathbb{R}$$



Exercise 39. Calculate $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(2n+1)!}$.

Exercise 40.



Calculate the maximum value of the function f defined by

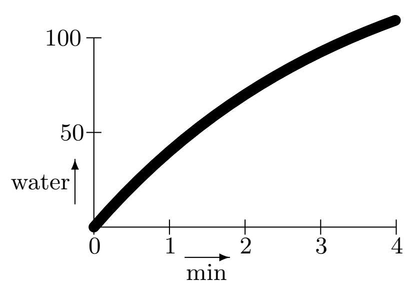
$$f(x) = \cos x + 2\cos\left(x + \frac{\pi}{3}\right)$$

(exact calculations please, a calculator-produced result gives you no more than a single point).

Exercise 41. (units: minutes, grams)
Bontepoes is having a walk when suddenly at time $t = 0$ it starts raining heavily. The amount of water W in her fur behaves according to the model 'bounded exponential growth' with

$W(0) = 0$ $\quad$ $W(1) = 40$ $\quad$ $W(2) = 70$

When will Bontepoesje carry precisely 100 grams of water in her fur?



Solutions chapter 4

Exercise 1. In $3\sqrt{\arctan x} = 2 + \arctan x$ you should recognise a quadratic equation in $\sqrt{\arctan x}$:

$$3\sqrt{\arctan x} = 2 + \arctan x \iff (\sqrt{\arctan x})^2 - 3\sqrt{\arctan x} + 2 = 0$$
$$\iff \sqrt{\arctan x} = \frac{3 \pm \sqrt{1}}{2} = 1 \text{ or } 2 \iff \arctan x = 1 \text{ or } 4$$

Obviously, $\arctan x$ can impossibly equal 4. Therefore, $\arctan x = 1$ so $x = \tan 1$.

Exercise 2. Without cleaning the cockroach function remains of the form $\mathrm{Coc}\,(t) = \alpha \cdot \beta^t$. I calculate the constants α and β by substitution of the counts:

$$\left.\begin{array}{l} \mathrm{Coc}\,(3) = 10 \implies \alpha \cdot \beta^3 = 10 \\ \mathrm{Coc}\,(5) = 13 \implies \alpha \cdot \beta^5 = 13 \end{array}\right\} \overset{\text{divide}}{\implies} \beta^2 = 1.3 \implies \left\{\begin{array}{l} \beta = 1.3^{\frac{1}{2}} \\ \alpha = 10 \cdot 1.3^{-\frac{3}{2}} \end{array}\right.$$

and now I can delight the cockroaches giving them a function rule for their number:

$$\mathrm{Coc}\,(t) = 10 \cdot 1.3^{\frac{t-3}{2}}$$

Finally, I calculate when the students need to take up a broom:

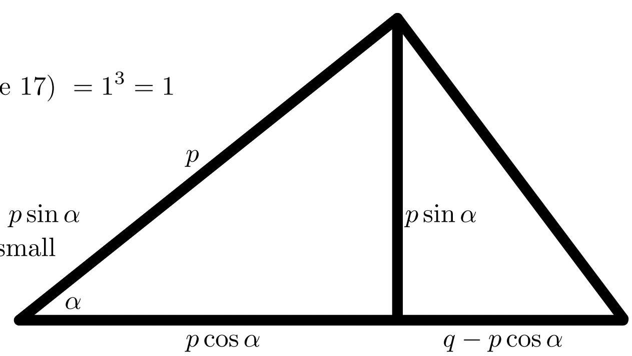
$$\mathrm{Coc}\,(t) = 100 \implies 1.3^{\frac{t-3}{2}} = 10 \overset{\text{ln-trick}}{\implies} \frac{t-3}{2}\ln 1.3 = \ln 10 \implies \boxed{t = 3 + \frac{\ln 100}{\ln 1.3}}$$

It is of paramount importance that you be able to smoothly perform calculations with ln, which must be the case because you've learned the ln-trick in the arithmetic booklet. Brooms are necessary after approximately three weeks.

Exercise 3. $\lim_{x\to 3} \arctan \sqrt{x} = \arctan \sqrt{3} = \frac{\pi}{3}$

Exercise 4. $\lim_{x\to 0} x^{3x} = \lim_{x\to 0} (x^x)^3 = \text{(see example 17)} = 1^3 = 1$

Exercise 5. I draw an auxiliary line with length $p \sin \alpha$ which allows me to calculate the areas of the two small triangles without effort:



$$\left\{\begin{array}{lclcl} \text{area left triangle} & = & \frac{1}{2} \cdot \text{base} \cdot \text{height} & = & \frac{1}{2} \cdot p\cos\alpha \cdot p\sin\alpha \\ \text{area right triangle} & = & \frac{1}{2} \cdot \text{base} \cdot \text{height} & = & \frac{1}{2} \cdot (q - p\cos\alpha) \cdot p\sin\alpha \end{array}\right.$$

Adding these expressions yields: total area = $\boxed{\frac{1}{2}pq\sin\alpha}$.

Remark. An entirely different approach using the cross product: the desired area is half of the area of the parallelogram spanned by the vectors $(q, 0, 0)$ and $(p\cos\alpha, p\sin\alpha, 0)$. The cross product of these vectors is:

$$(q, 0, 0) \times (p\cos\alpha, p\sin\alpha, 0) = \det\begin{pmatrix} \vec{\mathbf{e_x}} & \vec{\mathbf{e_y}} & \vec{\mathbf{e_z}} \\ q & 0 & 0 \\ p\cos\alpha & p\sin\alpha & 0 \end{pmatrix} = (0, 0, pq\sin\alpha)$$

Hence, the parallelogram has an area $\|(0, 0, pq\sin\alpha)\| = pq\sin\alpha$, so the area of the triangle is $\frac{1}{2}pq\sin\alpha$.

Exercise 6. I take a function with a jump at $x = 5$, for example:

$$f(x) = \begin{cases} 2 & \text{if } x \leq 5 \\ 3 & \text{if } x > 5 \end{cases}$$

Exercise 7. The distance from $(3, 0)$ to $(x, \sqrt{x})$ is

$$\|(3,0) - (x, \sqrt{x})\| = \|(3 - x, -\sqrt{x})\| = \sqrt{(3-x)^2 + (-\sqrt{x})^2} = \sqrt{(x^2 - 6x + 9) + x} = \sqrt{x^2 - 5x + 9}$$

The desired distance is the minimum value of this expression, which I can calculate by completing a square:

$$\sqrt{x^2 - 5x + 9} = \sqrt{\left(x - \frac{5}{2}\right)^2 + \frac{11}{4}}$$

This is miminum when $x = \frac{5}{2}$, so the distance is $\sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}$.

Exercise 8.

a) Porky's weight goal W satisfies $W^{\frac{2}{3}} + 2W^{\frac{1}{3}} - 24 = 0$. The left-hand side of this equation can be factorised:

$$\left(W^{\frac{1}{3}} + 6\right)\left(W^{\frac{1}{3}} - 4\right) = 0$$

Conclusion: $W^{\frac{1}{3}} = 4$ so $W = 64$. Porky should lose 6 kg.

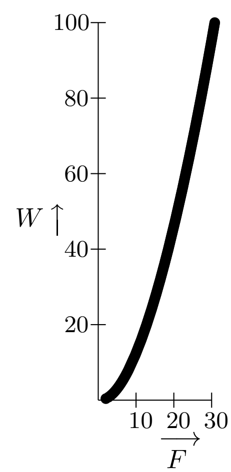
b) $W^{\frac{2}{3}} + 2W^{\frac{1}{3}} - F = 0$ is a quadratic equation in $W^{\frac{1}{3}}$ with solutions

$$W^{\frac{1}{3}} = -1 \pm \sqrt{1 + V}$$

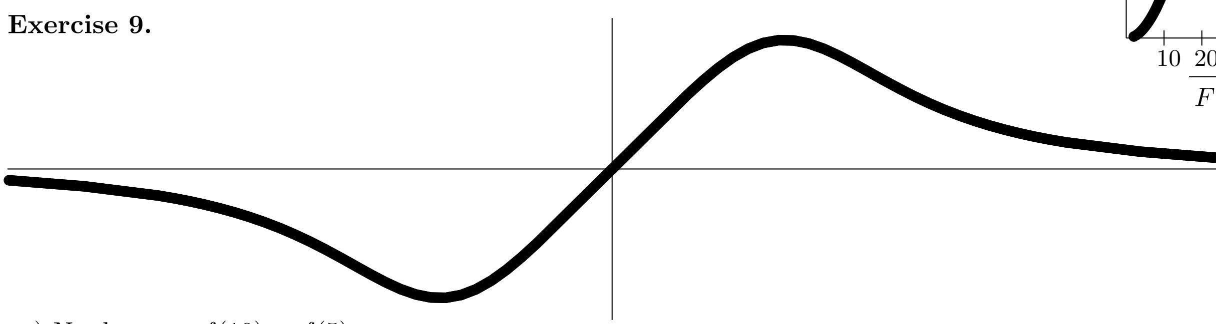
The minus sign only applies to creatures such as birds and vampires, so

$$W^{\frac{1}{3}} = -1 + \sqrt{1 + F} \quad \Longrightarrow \quad W = \left(-1 + \sqrt{1 + F}\right)^3$$

The graph of the maximum weight W as a function of foot length F can be drawn easily by reflecting the graph given in the exercise across the diagonal:



Exercise 9.



a) No, because $f(10) < f(5)$.

b) No, because $f(0) < f(1)$.

Exercise 10. I do this as follows:

$$y = 3 + e^{2+t} \quad \Longleftrightarrow \quad y - 3 = e^{2+t} \quad \Longleftrightarrow \quad \ln(y - 3) = 2 + t \quad \Longleftrightarrow \quad t = \ln(y - 3) - 2$$

Thus, the inverse is the function $t \mapsto \ln(t - 3) - 2$.

Exercise 11.

a) $(g \circ f)(x) = g(f(x)) = g\left(\ln(x^2+1)\right) = e^{3\ln(x^2+1)} = e^{\ln(x^2+1)^3} = (x^2+1)^3$

b) $(f \circ g)(x) = f(g(x)) = f\left(e^{3x}\right) = \ln\left((e^{3x})^2+1\right) = \ln(e^{6x}+1)$

Exercise 12.

a) $\tan\dfrac{\pi}{3} = \sqrt{3} \quad \Longrightarrow \quad \arctan\sqrt{3} = \dfrac{\pi}{3}$

b) $\tan\left(-\dfrac{\pi}{4}\right) = -1 \quad \Longrightarrow \quad \arctan(-1) = -\dfrac{\pi}{4}$

Exercise 13. This is easy using the identity $\sin 2x = 2\sin x\cos x$:

$$\sin x + \sin 2x = 0 \quad \Longleftrightarrow \quad (\sin x)\cdot(1+2\cos x) = 0 \quad \Longleftrightarrow \quad \sin x = 0 \ \text{ or } \ \cos x = -\frac{1}{2}$$

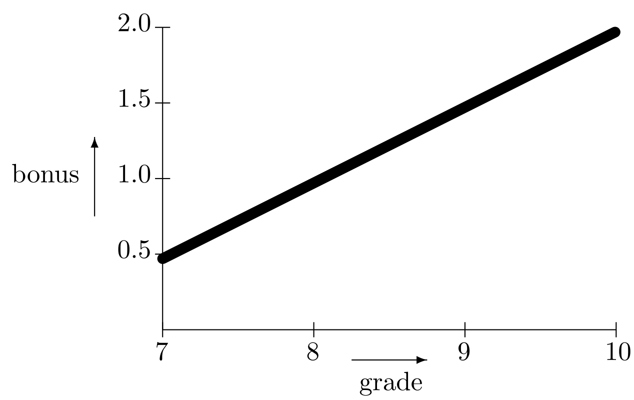
Hence, the zeros of f are

- the points x where $\sin x = 0$, i.e. all points $n\pi$ with n an integer
- the points where $\cos x = -\frac{1}{2}$, i.e. all points $\frac{2}{3}\pi + 2n\pi$ and $\frac{4}{3}\pi + 2n\pi$ with n an integer

Exercise 14. Many different solutions are possible. I take the function

$$\boxed{\text{bonus}\,(x) = \frac{x}{2} - 3.03}$$

with its graph depicted on the right.



Exercise 15. I solve this quadratic equation in $\sqrt{x}$ using the quadratic formula:

$$x + 2\sqrt{x} = 5 \quad \Longleftrightarrow \quad (\sqrt{x})^2 + 2\sqrt{x} - 5 = 0 \quad \Longleftrightarrow \quad \sqrt{x} = \frac{-2 \pm \sqrt{24}}{2} = -1 \pm \sqrt{6}$$

Since roots cannot be negative, we have $\sqrt{x} = -1 + \sqrt{6}$. so $x = (-1+\sqrt{6})^2 = 7 - 2\sqrt{6}$.

Exercise 16. You can for example do this as follows:

- $\lim\limits_{x\to 0} x^x = 1$ (from example 17)
- Substituting $x = \frac{1}{n}$ in this limit yields $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$ or $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$.
- Hence, $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Exercise 17. In chapter 1 you've learned how to sum a geometric series:

$$1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{2^n}{3^n} = \frac{1 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}}$$

Taking the limit $n \to \infty$ of the left-hand side and the right-hand side yields

$$\lim_{n\to\infty}\left(1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{2^n}{3^n}\right) = \frac{1}{1-\frac{2}{3}} = 3$$

so

$$\lim_{n\to\infty} \arctan\sqrt{1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{2^n}{3^n}} = \arctan\sqrt{3} = \frac{\pi}{3}$$

Exercise 18. This is a nasty one. Of course, you can get an impression of this limit by substituting a very small value of t, for example 0.000001, and let your calculator do the work. However, mathematics masterminds won't be satisfied until they've constructed a formal proof. For example:

- $\lim\limits_{x\to 0} x \ln x = 0$ (from example 16)
- Substituting $x = \sqrt{t}$ yields $\lim\limits_{t\to 0} \sqrt{t}\,\ln\sqrt{t} = 0$.
- Because $\ln\sqrt{t} = \dfrac{1}{2}\ln t$, we have $\lim\limits_{t\to 0} \dfrac{1}{2}\sqrt{t}\,\ln t = 0$ and $\lim\limits_{t\to 0} \sqrt{t}\,\ln t = 0$ as well.
- Hence, $\lim\limits_{t\to 0} \left(\sqrt{t}\,\ln t\right)^2 = 0^2 = 0$, so $\lim\limits_{t\to 0} t(\ln t)^2 = 0$.

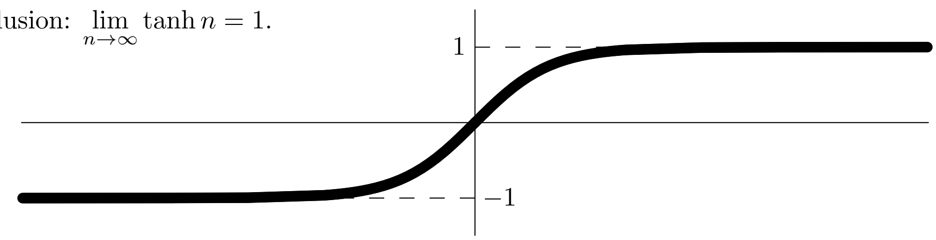
Exercise 19.

a) $\tanh n = \dfrac{\sinh n}{\cosh n} = \dfrac{e^n - e^{-n}}{e^n + e^{-n}} = \dfrac{1 - e^{-2n}}{1 + e^{-2n}}$

Both the numerator and the denominator approach 1, because $\lim\limits_{n\to\infty} e^{-2n} = 0$.

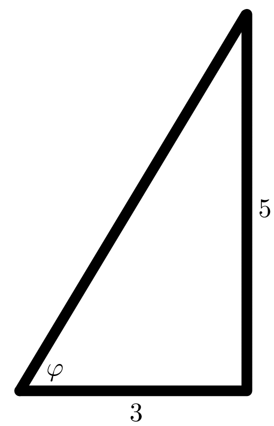
Conclusion: $\lim\limits_{n\to\infty} \tanh n = 1$.

b)



Exercise 20. $\arctan\frac{5}{3}$ is the angle φ in this triangle.
According to Pythagoras the hypotenuse is $\sqrt{34}$, so

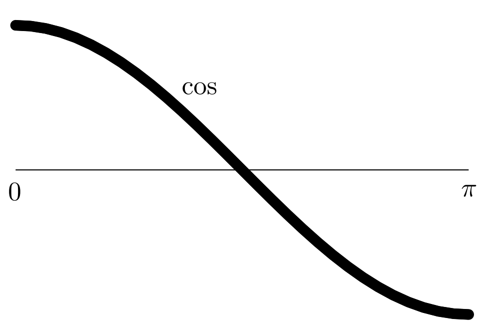
$$\sin\left(\arctan\frac{5}{3}\right) = \sin\varphi = \frac{5}{\sqrt{34}}$$



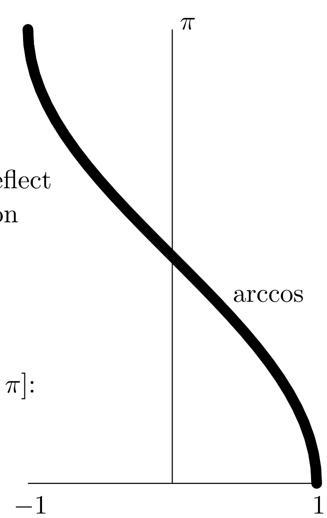
Exercise 21. I abbreviate $f(x)$ to y and try to express x in terms of y:

$$y = 7 + \sqrt[5]{3x-2} \implies y - 7 = \sqrt[5]{3x-2} \implies (y-7)^5 = 3x - 2 \implies x = \frac{(y-7)^5 + 2}{3}$$

Conclusion: $\overleftarrow{f}(y) = \dfrac{(y-7)^5 + 2}{3}$.



Exercise 22. If you reflect the graph of the function $\cos : [0, \pi] \to [-1, 1]$ across the line $y = x$, you obtain the graph of the function $\arccos : [-1, 1] \to [0, \pi]$:



Exercise 23. Let's call $f(x)$ y for the moment and try to express x in terms of y. You'll only succeed if you've mastered sufficient algebra:

$$\begin{aligned}
y = \cosh x \iff & y = \frac{e^x + e^{-x}}{2} && \iff 2y = e^x + e^{-x} \\
\iff & e^x - 2y + e^{-x} = 0 && \iff e^{2x} - 2ye^x + 1 = 0 \\
\iff & (e^x)^2 - 2ye^x + 1 = 0 && \iff e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1} \\
\iff & e^x = y + \sqrt{y^2 - 1} \ \ (\pm \text{ must be } + \text{ because } e^x \text{ always exceeds 1 on the given domain}) \\
\iff & x = \ln\left(y + \sqrt{y^2 - 1}\right)
\end{aligned}$$

The desired function rule is $\overleftarrow{f}(t) = \ln\left(t + \sqrt{t^2 - 1}\right)$.

Exercise 24. You can rewrite $A\sin(x + B)$ (using the formula for $\sin(p + q)$ from the arithmetic booklet) to

$$A\sin(x + B) = A\sin x \cos B + A\cos x \sin B$$

For this to be equal to $3\sin x + 5\cos x$ the constants A and B should satisfy

$$\begin{cases} A\cos B = 3 & (*) \\ A\sin B = 5 & (**) \end{cases}$$

You can find A by squaring and adding these equations:

$$(A\cos B)^2 + (A\sin B)^2 = 3^2 + 5^2 = 34 \implies A^2 = 34 \implies A = \pm\sqrt{34}$$

and B is most easily calculated by dividing $(**)$ by $(*)$:

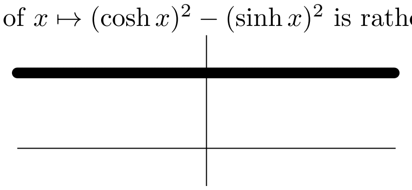
$$\tan B = \frac{5}{3} \implies B = \arctan\frac{5}{3}$$

Final question: what's the sign of A? You can for example find this out by substituting $x = 0$:

$$\pm\sqrt{34}\sin\left(0 + \arctan\frac{5}{3}\right) = \pm\sqrt{34} \cdot \frac{5}{\sqrt{34}} = \pm 5$$

which should equal $3\sin 0 + 5\cos 0 = 5$. Apparently, the plus sign is correct: $A = \sqrt{34}$.

Exercise 25. Indeed, the graph of $x \mapsto (\cosh x)^2 - (\sinh x)^2$ is rather dull:



This can be easily explained:

$$\begin{aligned}
(\cosh x)^2 - (\sinh x)^2 &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1
\end{aligned}$$

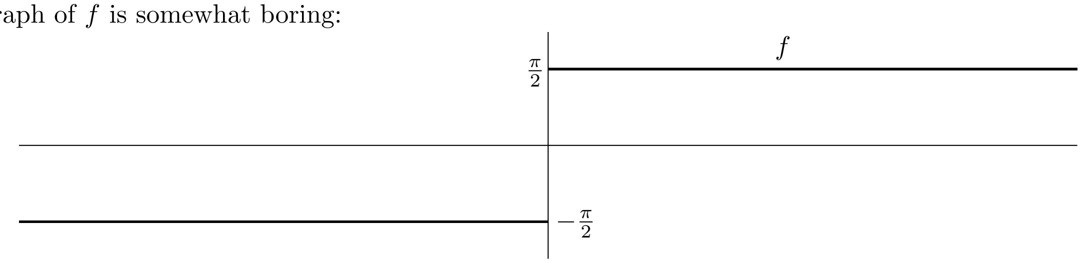
Exercise 26. I try to express P in terms of S:

$$\begin{aligned}
S = 2^{\ln P} &\implies \ln S = \ln\left(2^{\ln P}\right) \implies \ln S = (\ln P) \cdot (\ln 2) \implies \ln P = \frac{\ln S}{\ln 2} \\
&\implies P = e^{\frac{\ln S}{\ln 2}} \implies P = \left(e^{\ln S}\right)^{\frac{1}{\ln 2}} \implies P = S^{\frac{1}{\ln 2}}
\end{aligned}$$

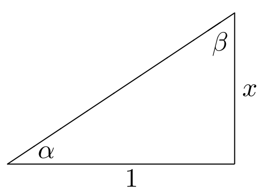
Exercise 27.

a) For $x = 0$ the definition of $f(x)$ is meaningless (because $\frac{1}{0}$ is nonsense), but for all other real numbers the definition makes sense. Put differently: the domain is the collection of all real numbers except 0.

b) The graph of f is somewhat boring:



This graph consists of two straight line segments. That the function value is $\frac{\pi}{2}$ for all $x > 0$ can for example be demonstrated as follows:



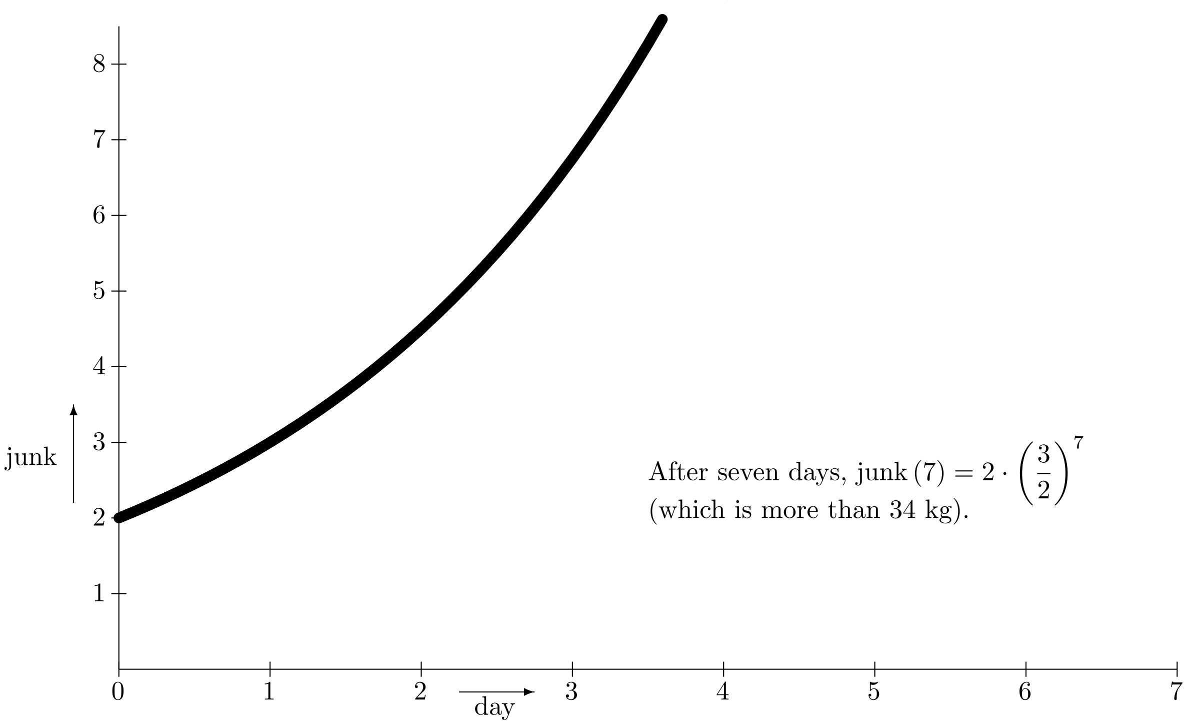
For this rectangular triangle we have $\arctan x = \alpha$ and $\arctan \frac{1}{x} = \beta$.
Hopefully, you remember that $\alpha + \beta = \frac{\pi}{2}$, so $f(x) = \frac{\pi}{2}$.
So the right half of the graph of f is a straight line at height $\frac{\pi}{2}$.
Because f is rotationally symmetric about the origin, the left half of the graph is a straight line at height $-\frac{\pi}{2}$.

Exercise 28. That must be square $\circ$ sin, because

- $(\text{square} \circ \sin)(x) = \text{square}(\sin x) = (\sin x)^2 = \sin^2 x$
- the function $\sin \circ \sin$ has function rule $x \mapsto \sin(\sin x)$
- and $\sin \circ\, \text{square}$ is the function $x \mapsto \sin(x^2)$

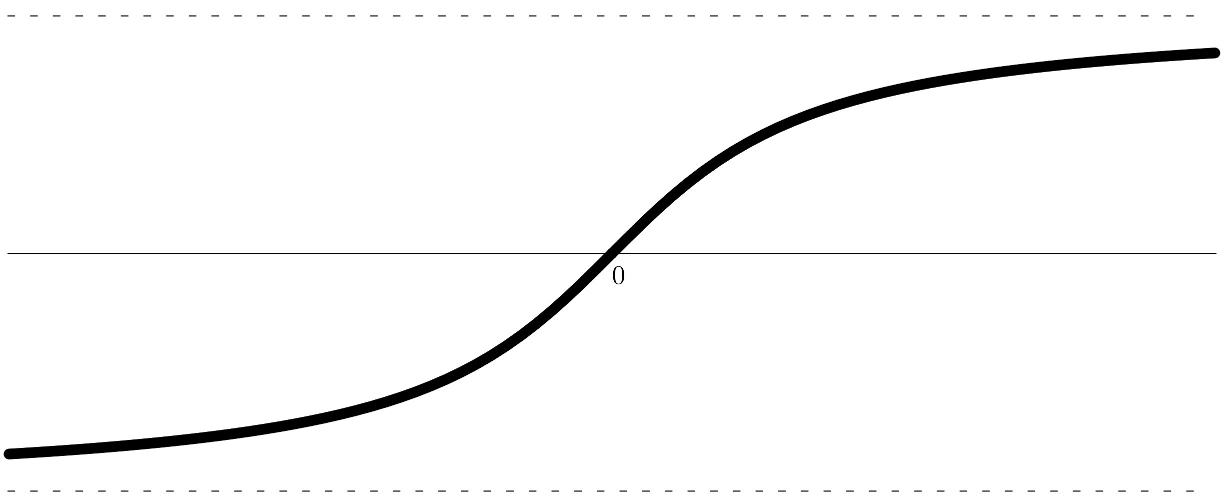
Exercise 29. I substitute the measurements in the exponential model $\text{junk}\,(t) = \alpha \cdot \beta^t$:

$$\left.\begin{array}{l} \text{junk}\,(0) = 2 \implies \alpha \cdot \beta^0 = 2 \implies \alpha = 2 \\ \text{junk}\,(1) = 3 \implies \alpha \cdot \beta^1 = 3 \implies \beta = \frac{3}{2} \end{array}\right\} \implies \text{junk}\,(t) = 2 \cdot \left(\frac{3}{2}\right)^t$$



After seven days, $\text{junk}\,(7) = 2 \cdot \left(\frac{3}{2}\right)^7$ (which is more than 34 kg).

Exercise 30. arctan is the inverse of the function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, so the graph of arctan is the reflection of the graph of $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ across the line $y = x$:



(the dashed lines are the asymptotes at height $-\frac{\pi}{2}$ and $\frac{\pi}{2}$)

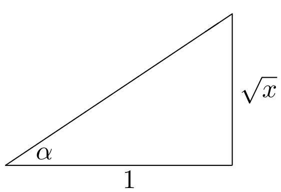
Exercise 31.

a) $(f \circ f)(x) = f(f(x)) = f(x^{\sqrt{3}}) = \left(x^{\sqrt{3}}\right)^{\sqrt{3}} = x^3$

b) $\overleftarrow{f}(x) = x^{\frac{1}{\sqrt{3}}}$, so $(\overleftarrow{f} \circ \overleftarrow{f})(x) = \overleftarrow{f}\left(x^{\frac{1}{\sqrt{3}}}\right) = \left(x^{\frac{1}{\sqrt{3}}}\right)^{\frac{1}{\sqrt{3}}} = x^{\frac{1}{3}} = \sqrt[3]{x}$

Exercise 32. Apply the notorious formula $1 + p + p^2 + p^3 + p^4 + p^5 + p^6 = \dfrac{1 - p^7}{1 - p}$ with $p = \sqrt[7]{x}$ to find

$$\lim_{x \to 1} \frac{1 - \sqrt[7]{x}}{1 - x} = \lim_{x \to 1} \frac{1}{1 + \sqrt[7]{x} + (\sqrt[7]{x})^2 + (\sqrt[7]{x})^3 + (\sqrt[7]{x})^4 + (\sqrt[7]{x})^5 + (\sqrt[7]{x})^6} = \boxed{\frac{1}{7}}$$



Exercise 33. It is useful to quickly draw a rectangular triangle. The angle α is then $\arctan \sqrt{x}$ and the hypotenuse is $\sqrt{1 + x}$ (according to Pythagoras). Now, $f(x) = \cos \alpha = \dfrac{1}{\sqrt{1 + x}}$, so $f(3) = \dfrac{1}{2}$ and $\lim\limits_{x \to \infty} f(x) = 0$.

Exercise 34. $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Exercise 35. In $\sinh x = \dfrac{e^x - e^{-x}}{2}$ I substitute the Taylor series for e^x and e^{-x}, resulting in

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots$$

Exercise 36. After multiplying this series by x you hopefully recognise the result being equal to $e^x - 1$. Then, your conclusion is:

$$1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \frac{x^5}{6!} + \frac{x^6}{7!} + \frac{x^7}{8!} + \frac{x^8}{9!} + \cdots = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Exercise 37. I substitute the counts:

$$\left.\begin{array}{l} F(2)=5 \implies \beta\gamma^2=15 \\ F(3)=8 \implies \beta\gamma^3=9 \end{array}\right\} \implies \gamma=\frac{3}{5} \text{ and } \beta=\frac{125}{3}$$

There are 60 fatalities when $\dfrac{80}{1+\frac{125}{3}\cdot\left(\frac{3}{5}\right)^t}=60 \implies \left(\dfrac{3}{5}\right)^t=\dfrac{1}{125} \implies \boxed{t=\dfrac{\ln 125}{\ln 5-\ln 3}} \approx 9.45$

Exercise 38. From $y=\sinh(\ln x)$ you can derive that (using the definition of sinh)

$$y=\frac{x-\frac{1}{x}}{2} \implies 2y=x-\frac{1}{x} \implies x^2-2yx-1=0 \implies x=\frac{2y\pm\sqrt{4y^2+4}}{2}=y\pm\sqrt{y^2+1}$$

Because x is positive, $x=y+\sqrt{y^2+1}$, so the desired function rule is $\boxed{y\mapsto y+\sqrt{y^2+1}}$.

Exercise 39. I hope you've recognised the power series of the function sinh:

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=\sinh x \overset{x=1}{\implies} \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}=\sinh 1 \implies \sum_{n=0}^{\infty}\frac{2}{(2n+1)!}=2\sinh 1=\boxed{e-\frac{1}{e}}$$

Exercise 40. Using $\cos(x+y)=\cos x\cos y-\sin x\sin y$ the function rule can be rewritten:

$$f(x)=2\cdot\cos x-\sqrt{3}\cdot\sin x$$

Using the method of exercise 24 you can see that the function $x\mapsto A\cos x+B\sin x$ has a maximum value of $\sqrt{A^2+B^2}$, which is $\boxed{\sqrt{7}}$ here.

Exercise 41. 'Bounded exponential growth' implies that W is given by

$$W=\alpha-\beta\cdot\gamma^t \overset{W(0)=0}{\implies} \beta=\alpha \implies W=\alpha-\alpha\cdot\gamma^t$$

I substitute the other measurements:

$$\left.\begin{array}{l} W(1)=40 \implies \alpha-\alpha\gamma=40 \\ W(2)=70 \implies \alpha-\alpha\gamma^2=70 \end{array}\right\} \implies \frac{\alpha-\alpha\gamma^2}{\alpha-\alpha\gamma}=\frac{7}{4} \implies 1+\gamma=\frac{7}{4} \implies \gamma=\frac{3}{4}$$

So $\alpha=160$ and $W=160-160\cdot 0.75^t$, which equals 100 when

$$160\cdot 0.75^t=60 \implies 0.75^t=0.375 \implies \boxed{t=\frac{\ln 0.375}{\ln 0.75}} \quad \text{(after approximately 3 min 25 sec)}$$

5. Differentiation

The most important mathematical technique is differentiation. In this chapter the emphasis is on its algebraic aspects, but from next week (until the end of your life) you will be kept busy with the numerous applications of differential calculus.

Derivative. We abbreviate $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ to $f'(x)$.

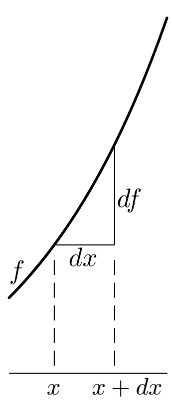
f is differentiable at a	$\overset{\text{def}}{=}$	$f'(a)$ exists
f is differentiable	$\overset{\text{def}}{=}$	f is differentiable at each point in its domain
the derivative of f	$\overset{\text{def}}{=}$	the function f'
f is twice differentiable	$\overset{\text{def}}{=}$	f is differentiable and f' is differentiable
the second derivative of f	$\overset{\text{def}}{=}$	the function $\left(f'\right)'$ (shorthand notation: f'')
the third derivative of f	$\overset{\text{def}}{=}$	the function f'''
$\vdots$		$\vdots$
the 37th derivative of f	$\overset{\text{def}}{=}$	the function $f^{'''''''''''''''''''''''''''''''''''''}$ (shorthand notation: $f^{(37)}$)

Remark. You might have learned different definitions of the derivative in the past. Hopefully, these are all equivalent:

$$f'(a) \;=\; \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \;=\; \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

In addition, we will occasionally use different notations:

- By $\dfrac{df}{dx}$ we mean f'.
- By $\dfrac{df(x)}{dx}$ or $\dfrac{d}{dx}f(x)$ we mean $f'(x)$.
- By $\left[\dfrac{df(x)}{dx}\right]_{x=a}$ or $\left[\dfrac{df(x)}{dx}\right]_{a}$ we mean $f'(a)$.
- By $\dfrac{d^2f(x)}{dx^2}$ we mean $f''(x)$.



These notations reflect the following intuitive interpretation of the derivative:

- df represents an incredibly tiny increase of f
- dx represents an incredibly tiny increase of x

The number $\dfrac{df}{dx}$ is the ratio between these increases:

$$\frac{df}{dx} = \text{the slope of the graph of } f \text{ at the point } (x, f(x))$$

Standard derivatives.

$\sin'(x) = \cos x$	$\cos'(x) = -\sin x$	$\tan'(x) = \dfrac{1}{\cos^2 x}$
$\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$	$\arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$	$\arctan'(x) = \dfrac{1}{1+x^2}$
$\ln'(x) = \dfrac{1}{x}$	$\dfrac{d}{dx}\, c = 0$	$\dfrac{d}{dx}\, x^c = cx^{c-1}$
$\dfrac{d}{dx}\, e^x = e^x$	$\dfrac{d}{dx}\, c^x = c^x \ln c$	

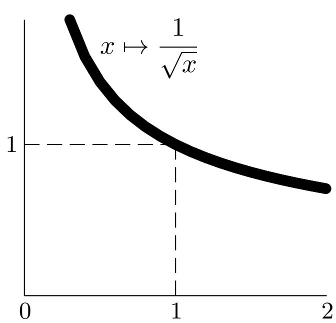
Example 1. Find the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$.

Solution. Use the standard derivative $\dfrac{d}{dx} x^c = cx^{c-1}$ with $c = -\dfrac{1}{2}$:

$$f(x) = x^{-\frac{1}{2}} \quad \Longrightarrow \quad f'(x) = -\frac{1}{2} x^{-\frac{3}{2}}$$

Example 2. Calculate $f'(1)$ given that $f(x) = \dfrac{1}{\sqrt{x}}$.

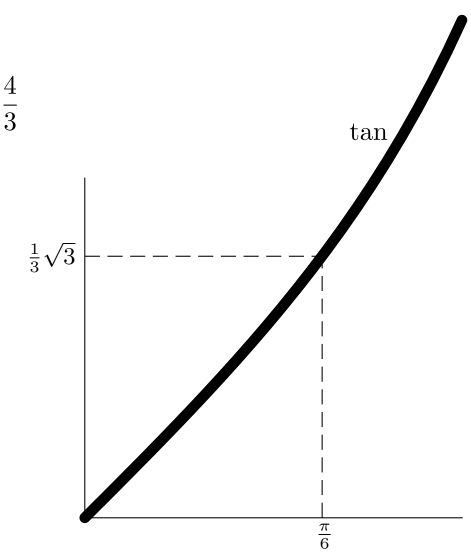
Solution. Substitute $x = 1$ in the solution to example 1: $f'(1) = -\dfrac{1}{2} \cdot 1^{-\frac{3}{2}} = -\dfrac{1}{2}$.



The geometric interpretation of this result: at the point $(1, 1)$ of the graph its slope equals $-\frac{1}{2}$, which is to say that if I move, starting from this point, two nanometre to the right, I'll go one nanometre downwards.

Example 3. Calculate $\left[\dfrac{d}{dx} \tan x\right]_{x=\frac{\pi}{6}}$.

Solution. $\dfrac{d}{dx} \tan x = \dfrac{1}{\cos^2 x} \quad \Longrightarrow \quad \left[\dfrac{d}{dx} \tan x\right]_{x=\frac{\pi}{6}} = \dfrac{1}{\cos^2 \frac{\pi}{6}} = \dfrac{4}{3}$



This result again has implications for the slope of the graph of the tangent function at the point $\left(\frac{\pi}{6}, \frac{1}{3}\sqrt{3}\right)$: three picometres to the right amounts to four picometres upwards.

Example 4. Calculate $\left[\dfrac{d^2 \sin x}{dx^2}\right]_{\frac{\pi}{6}}$.

Solution. $\sin''(x) = \dfrac{d}{dx}\cos x = -\sin x \quad \Longrightarrow \quad \sin''\left(\dfrac{\pi}{6}\right) = -\sin\dfrac{\pi}{6} = -\dfrac{1}{2}$

Example 5. Calculate $\ln''(7)$.

Solution. $\ln'(x) = \dfrac{1}{x} \quad \Longrightarrow \quad \ln''(x) = -\dfrac{1}{x^2} \quad \Longrightarrow \quad \ln''(7) = -\dfrac{1}{49}$

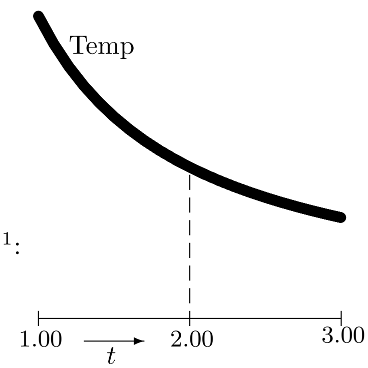
(In the second step of this calculation I used $\dfrac{d}{dx}x^c = cx^{c-1}$ with $c = -1$.)

Rules for differentiation.

$(cf)' = c \cdot f'$	$(f+g)' = f' + g'$ (sum rule)
$(fg)' = f'g + g'f$ (product rule)	$\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$ (quotient rule)
$(g \circ f)'(x) = g'\big(f(x)\big) \cdot f'(x)$ (chain rule)	$\big(\overleftarrow{f}\big)'(x) = \dfrac{1}{f'\big(\overleftarrow{f}(x)\big)}$

Example 6. (units: hours, degrees Celsius)
Between 1:00 and 3:00 AM the temperature was given by

$$\boxed{\text{Temp}\,(t) = \frac{2}{t}}$$



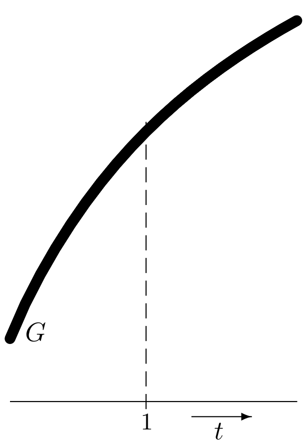
How fast was the temperature decreasing at 2:00 AM?

Solution. Use the differentiation rule $(cf)' = c \cdot f'$ with $c = 2$ and $f(t) = t^{-1}$:

$$\frac{d\,\text{Temp}}{dt} = 2 \cdot \left(-t^{-2}\right) = \frac{-2}{t^2} \quad \Longrightarrow \quad \left[\frac{d\,\text{Temp}}{dt}\right]_{t=2} = -\frac{1}{2}$$

At $t = 2$ the temperature was decreasing by 0.5 degrees/hour.

Remark. Soft science students sometimes use the quotient rule to differentiate $\frac{2}{t}$, but you know better.



Example 7. (units: years, kg)
Wobbeltje's weight at age t was

$$\boxed{W(t) = \sqrt{t} + \arctan t}$$

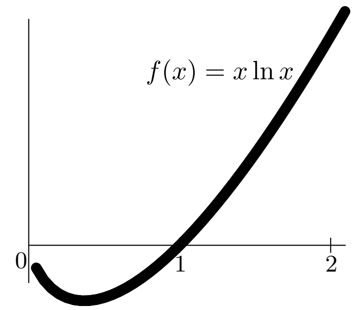
How fast was Wobbeltje gaining weight at her first birthday?

Solution. Use the sum rule $(f+g)' = f' + g'$ with $f(t) = \sqrt{t}$ and $g(t) = \arctan t$:

$$\frac{dW}{dt} = \frac{1}{2\sqrt{t}} + \frac{1}{1+t^2} \quad \Longrightarrow \quad \left[\frac{dW}{dt}\right]_{t=1} = 1$$

so at $t = 1$ Wobbeltje was gaining 1 kg/year.

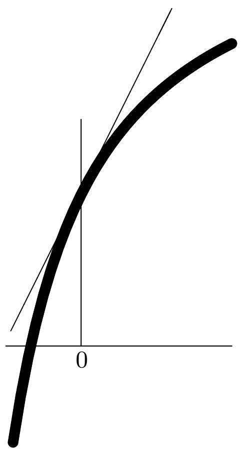
Example 8. Calculate the angle between the graph of $f(x) = x \ln x$ and the x-axis at the point $(1, 0)$.



Solution. I calculate the derivative using the product rule:

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = 1 + \ln x \quad \Longrightarrow \quad f'(1) = 1$$

so the desired angle is 45°.



Example 9. Determine the equation of the tangent line at $(0, 1)$ to

$$\boxed{y = \frac{1+3x}{1+x}}$$

Solution. I let the quotient rule do its job:

$$\frac{dy}{dx} = \frac{3(1+x) - 1(1+3x)}{(1+x)^2} = \frac{2}{(1+x)^2} \qquad \Longrightarrow \qquad \left[\frac{dy}{dx}\right]_{x=0} = 2$$

The tangent line has slope 2 and passes through $(0, 1)$, so its name is

$$\boxed{y = 2x + 1}$$

Example 10. Find the derivative of $\sin \sqrt{x}$.

Solution. The function $x \mapsto \sin \sqrt{x}$ is the composition $g \circ f$ with g the sine function and f the square root function, so you can find the derivative using the chain rule:

$$\frac{d}{dx} \sin \sqrt{x} = g'(f(x)) \cdot f'(x) = (\cos \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

Remark. I'll explain how the chain rule works in this example:

Step 1. Calculate $\frac{d}{dx} \sin x$. Result: $\cos x$.

Step 2. Replace in the result of step 1 the symbol x with $\sqrt{x}$. This yields $\cos \sqrt{x}$.

Step 3. Multiply this by $\frac{d}{dx}\sqrt{x}$ and you're finished: $\frac{d}{dx} \sin \sqrt{x} = (\cos \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$.

Example 11. Calculate the derivative of $(x^2 + \cos x)^7$.

Solution. The function $x \mapsto (x^2 + \cos x)^7$ is the composition $g \circ f$ of $g(x) = x^7$ and $f(x) = x^2 + \cos x$, so we apply the chain rule:

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) = 7(x^2 + \cos x)^6 \cdot (2x - \sin x)$$

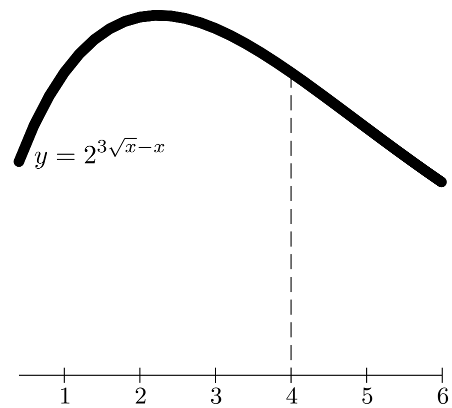
Remark. Here, the chain rule worked as follows:

Step 1. Calculate $\frac{d}{dx} x^7$. That's an easy one: $7x^6$.

Step 2. Replace in the result of step 1 the symbol x with $x^2 + \cos x$. You obtain $7(x^2 + \cos x)^6$.

Step 3. Multiply the result of step 2 by $\frac{d}{dx}(x^2 + \cos x)$.

Example 12. How steep is the adjacent curve at $(4,4)$?



Solution. By the chain rule,

$$\frac{dy}{dx} = \left(2^{3\sqrt{x}-x}\ln 2\right)\cdot\left(\frac{3}{2\sqrt{x}}-1\right) \quad\Longrightarrow\quad \left[\frac{dy}{dx}\right]_{x=4} = -\ln 2$$

so the slope at $(4,4)$ is $-\ln 2$. Hence, the angle between the curve and the 'horizontal' is $\arctan(\ln 2) \approx 34.7$ degrees.

Example 13. Determine the derivative of x^x.

Solution. $x^x = e^{x\ln x} = (g\circ f)(x)$ with $g(x) = e^x$ and $f(x) = x\ln x$. The product rule dictates that $f'(x) = 1 + \ln x$ so, according to the chain rule,

$$\frac{d}{dx}x^x = \left(g\circ f\right)'(x) = g'\big(f(x)\big)\cdot f'(x) = e^{f(x)}\cdot(1+\ln x) = x^x(1+\ln x)$$

Example 14. Determine the second derivative of $f(x) = 7^x$ at the point 1.

Solution. We successively calculate:

- $f'(x) = \dfrac{d}{dx}\left(e^{x\ln 7}\right) = e^{x\ln 7}\cdot\ln 7 = 7^x\cdot\ln 7$
- $f''(x) = \dfrac{d}{dx}\left(7^x\cdot\ln 7\right) = 7^x\cdot(\ln 7)^2$
- $f''(1) = 7\cdot(\ln 7)^2$

Example 15. Calculate $\left[\dfrac{d}{dx}\arcsin\sqrt{x}\right]_{x=\frac{1}{5}}$.

Solution. Apply the chain rule:

$$\frac{d}{dx}\arcsin\sqrt{x} = \frac{1}{\sqrt{1-x}}\cdot\frac{d}{dx}\sqrt{x} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} \quad\Longrightarrow\quad \left[\frac{d}{dx}\arcsin\sqrt{x}\right]_{x=\frac{1}{5}} = \frac{5}{4}$$

Example 16. Let $f(x) = x^3 + x + 5$.

a) Calculate $f'(x)$.

b) Calculate $\overleftarrow{f}(3)$.

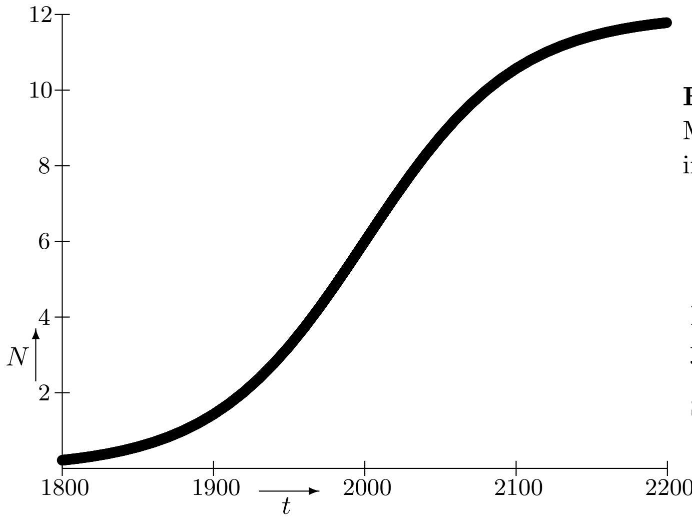
c) Calculate $\big(\overleftarrow{f}\big)'(3)$.

Solution.

a) $f'(x) = 3x^2 + 1$

b) You are asked to find the number x satisfying $f(x) = 3$. After a few minutes of haphazardly trying, I found $x = -1$, so $\overleftarrow{f}(3) = -1$.

c) $\big(\overleftarrow{f}\big)'(3) = \dfrac{1}{f'\big(\overleftarrow{f}(3)\big)} = \dfrac{1}{f'(-1)} = \dfrac{1}{4}$



Example 17. (units: years, billion people)
My model for the size N of the world population in the year t is

$$\boxed{N(t) = \frac{12}{1 + e^{40-0.02\,t}}}$$

How fast was the world population growing in the year 2000 according to my model?

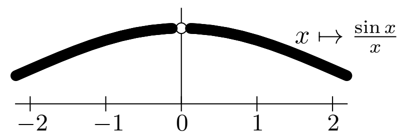
Solution. N is the composition $g \circ f$ with

$$\boxed{g(x) = 12\,x^{-1}} \qquad \boxed{f(t) = 1 + e^{40-0.02\,t}}$$

$$\Longrightarrow \quad \frac{dN}{dt} = -12\left(1 + e^{40-0.02\,t}\right)^{-2} \cdot e^{40-0.02\,t} \cdot (-0.02) \quad \Longrightarrow \quad \left[\frac{dN}{dt}\right]_{t=2000} = 0.06$$

so in the year 2000 the world population was growing by 60 million people per year.

Example 18. Calculate $\lim_{x\to 0} \frac{\sin x}{x}$.



Solution. You can use the definition of $\sin'(0)$:

$$\lim_{x\to 0} \frac{\sin x}{x} \;=\; \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} \;=\; \sin'(0) \;= \cos 0 \;=\; 1$$

Example 19. Calculate $\lim_{n\to\infty} n \sin \frac{1}{n}$.

Solution. Substitute in the preceding example $x = \frac{1}{n}$:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} n \sin \frac{1}{n} = 1$$

Example 20. Calculate $\lim_{t\to 0} \frac{\tan t}{t}$.

Solution. $\lim_{t\to 0} \frac{\tan t}{t} = \lim_{t\to 0} \frac{\tan t - \tan 0}{t - 0} = \left[\frac{d\,\tan t}{dt}\right]_{t=0} = \left[\frac{1}{\cos^2 t}\right]_{t=0} = \frac{1}{\cos^2 0} = 1$

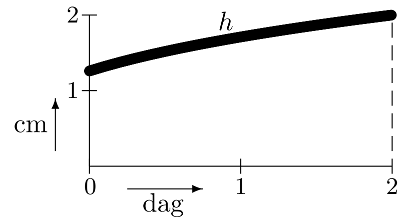
Example 21. Calculate $\lim_{n\to\infty} n \arctan \frac{1}{n}$.

Solution. This limit has something to do with the behaviour of the arctan function around 0:

- $\arctan'(0) = \frac{1}{1+0^2} = 1$
- which means that $\lim_{h\to 0} \frac{\arctan(0+h) - \arctan 0}{h} = 1$
- implying $\lim_{h\to 0} \frac{\arctan h}{h} = 1$
- so (substitute $h = \frac{1}{n}$) $\lim_{n\to\infty} n \arctan \frac{1}{n} = 1$

Example 22. (units: days, cm)

The height of my houseplant after t days is $\boxed{h(t) = \sqrt[3]{2+3t}}$.



a) What is the height of my houseplant after two days?

b) How fast is it growing after two days?

Solution.

a) After two days its height is $h(2) = 2$ cm.

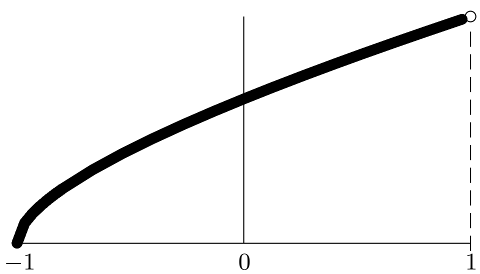
b) Because $h(t) = \sqrt[3]{u}$ with $u = 2+3t$ I use the chain rule:

$$\frac{dh}{dt} = \frac{dh}{du}\cdot\frac{du}{dt} = \frac{1}{3}u^{-\frac{2}{3}}\cdot 3 = u^{-\frac{2}{3}} = (2+3t)^{-\frac{2}{3}} \quad\Longrightarrow\quad \left[\frac{dh}{dt}\right]_{t=2} = 8^{-\frac{2}{3}} = \frac{1}{4}$$

At $t=2$ the houseplant is growing by 0.25 cm/day.

L'Hôpital's rule. If f and g are differentiable in all points $\neq a$ and $f(a) = g(a) = 0$, then

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

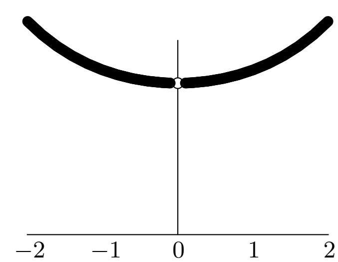


Example 23. Calculate $\displaystyle\lim_{x\to 1}\frac{\sqrt{1-x^2}}{\arccos x}$.

Solution. Unfortunately, at the point $x=1$ the quotient is undefined, because both numerator and denominator are zero there. In cases like this l'Hôpital's rule offers help. It allows you to differentiate numerator and denominator for free, which hopefully leads to an easier limit:

$$\lim_{x\to 1}\frac{\sqrt{1-x^2}}{\arccos x} = \lim_{x\to 1}\frac{\frac{-2x}{2\sqrt{1-x^2}}}{\frac{-1}{\sqrt{1-x^2}}} = \lim_{x\to 1} x = 1$$

Example 24. Calculate $\displaystyle\lim_{x\to 0}\frac{x^2}{1-\cos x}$.



Solution. Again an atrocity, for substituting $x=0$ yields $\frac{0}{0}$ which makes you very sad. Fortunately, l'Hôpital is helpful again:

$$\lim_{x\to 0}\frac{x^2}{1-\cos x} = \lim_{x\to 0}\frac{2x}{\sin x} = 2 \text{ (see example 18)}$$

Example 25. Calculate $\displaystyle\lim_{x\to 0}\frac{x^2}{3-\cos x}$.

Solution. l'Hôpital won't work here, because the requirements for his rule are not met here (you see why?). However, this limit happens to be really easy:

$$\lim_{x\to 0}\frac{x^2}{3-\cos x} = \frac{0^2}{3-\cos 0} = \frac{0}{2} = 0$$

Velocity. The velocity or speed $v(t)$ of a moving object at time t is (by definition) the extent to which the distance $s(t)$ travelled by the object increases. Expressed as a formula:

$$\boxed{v(t) = \frac{d}{dt}\, s(t)} \qquad \text{or} \qquad \boxed{v = \frac{ds}{dt}}$$

Physicists make a sharp distinction between velocity (the rate at which the position vector of an object changes) and speed (the rate at which the travelled distance changes). This distinction won't prevent me from using both words interchangeably, for we are concerned with mathematics instead of physics.

Example 26. (units: hours, km)
I took a three hour walk. After t hours I'd walked a distance

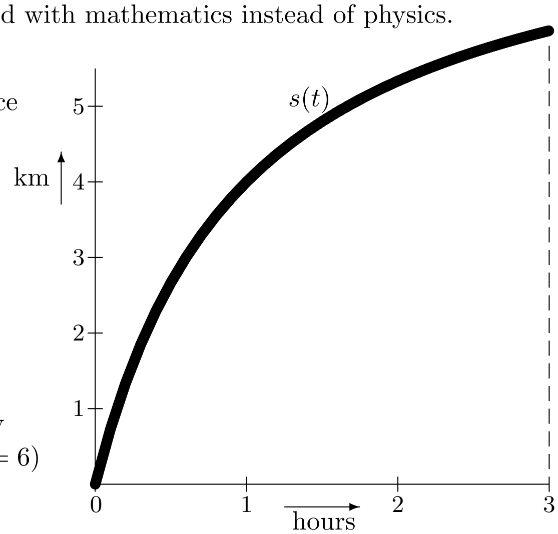
$$\boxed{s(t) = \frac{8t}{t+1}}$$

Calculate my initial, final and average velocity.

Solution. By the quotient rule,

$$v(t) = \frac{ds}{dt} = \frac{8(t+1) - 8t}{(t+1)^2} = \frac{8}{(t+1)^2}$$

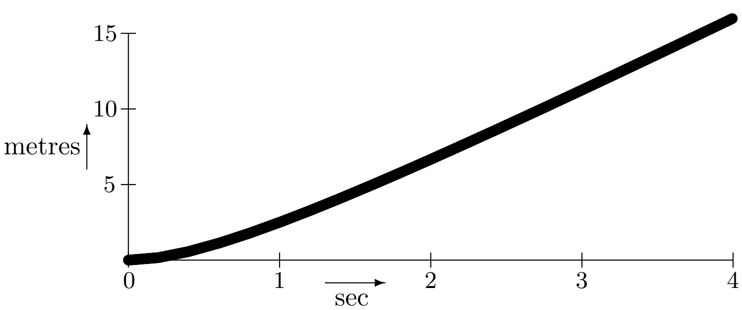
My initial velocity was $v(0) = 8$ km/h and my final velocity $v(3) = 0.5$ km/h. In three hours I walked six km (for $s(3) = 6$) so my average velocity was 2 km/h.



Acceleration. The acceleration $a(t)$ of a moving object at time t is (by definition) the rate at which its velocity $v(t)$ increases:

$$\boxed{a(t) = \frac{d}{dt}\, v(t)} \qquad \text{or} \qquad \boxed{a = \frac{dv}{dt}}$$

Hence, the acceleration is the second derivative of the travelled distance.



Example 27. After t seconds Kaasje has travelled a distance of

$$s(t) = \frac{5t^2}{1+t} \text{ metres.}$$

What is his velocity at $t = 4$?

Solution. Differentiate $s(t)$ twice:

$$v(t) = \frac{10t + 5t^2}{(1+t)^2} \implies a(t) = \frac{10}{(1+t)^3}$$
$$\implies a(4) = 0.08 \text{ m/sec}^2$$

Example 28. Pommetje's velocity after t seconds is

$$\boxed{v(t) = \frac{8}{1 + 8\,e^{-t}} \text{ metres per second}}$$

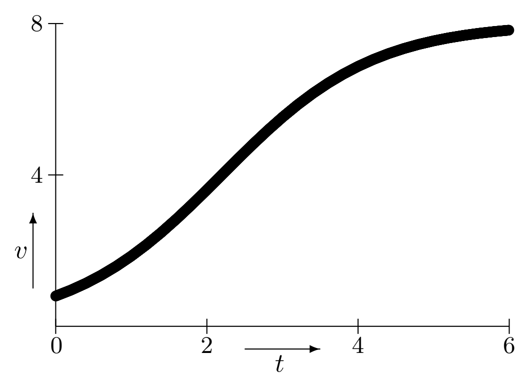
When does his acceleration equal exactly 2 m/sec^2?

Solution. Pommetje's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{64\,e^{-t}}{\left(1 + 8\,e^{-t}\right)^2}$$

which equals 2 when

$$\left(1 + 8\,e^{-t}\right)^2 = 32\,e^{-t} \implies 64\left(e^{-t}\right)^2 - 16\,e^{-t} + 1 = 0 \implies \left(8\,e^{-t} - 1\right)^2 = 0 \implies \boxed{t = \ln 8}$$



Taylor series for arctan and ln. Using our theory you can prove beautiful Taylor expansions:

$$\begin{array}{|lcll|}\hline \frac{1}{1-x} & = & 1+x+x^2+x^3+x^4+x^5+x^6+\cdots & \text{if } -1<x<1 \\ \arctan x & = & x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\cdots & \text{if } -1\le x\le 1 \\ \ln(1+x) & = & x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots & \text{if } -1<x\le 1 \\ \hline\end{array}$$

The first formula was proven in chapter 1. I'll prove the second as follows: the derivative of the right-hand side is

$$1-x^2+x^4-x^6+\cdots = (\text{use the 1st formula with } -x^2 \text{ instead of } x) = \frac{1}{1+x^2}$$

It turns out that $\arctan x$ and the right-hand side of the second formula have the same derivative, which means that they differ only a constant. And this constant is obviously zero (which you discover by substituting $x=0$). You can prove the Taylor series for $\ln(1+x)$ analogously.

Tangent line. The tangent line to f at a is the line l which has the same value and slope as f at $x=a$. The function rule for this tangent line is:

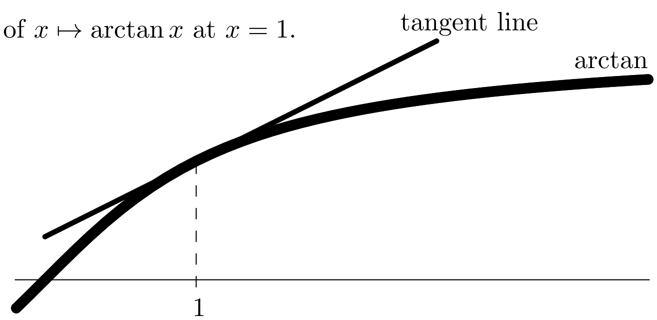
$$\boxed{l(x) = f(a) + f'(a)\cdot(x-a)}$$

Verify for yourself that this l satisfies:

(1) $l(a) = f(a)$ (l and f have the same value at a)

(2) $l'(a) = f'(a)$ (l and f have the same slope at a)

Example 29. Find the tangent line to the graph of $x \mapsto \arctan x$ at $x=1$.



Solution. This tangent line l has function rule

$$l(x) = \arctan 1 + \arctan'(1)\cdot(x-1)$$

Because $\arctan 1 = \frac{\pi}{4}$ and $\arctan'(1) = \frac{1}{2}$,

$$l(x) = \frac{\pi}{4} + \frac{x-1}{2}$$

Differentiation using a graphing calculator. Unfortunately, my graphing calculator can't differentiate functions. Even at simple tasks, such as calculating the derivative of $\sqrt{x}$, it will throw in the towel. What I can do is command my GR to calculate

$$\left[\frac{d}{dx}\sqrt{x}\right]_{x=4}$$

using the nDeriv function from the MATH menu. If I give my calculator the instructions

$$\boxed{\text{nDeriv}\,(\sqrt{x}\,,\,x\,,\,4)}$$

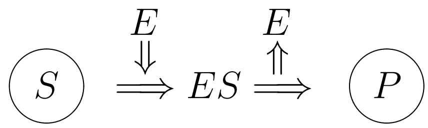
it will answer 0.250000002, which is indeed nearly correct. However, the toy really tries to fool me at the command

$$\boxed{\text{nDeriv}\,(1/x^2\,,\,x\,,\,0)}$$

where it dares to present the answer 0, which is a tragedy every student should feel truly ashamed of.

Michaelis-Menten. *This section is intended for chemistry and MLS students in preparation of their biochemistry project; science students are allowed to skip it. For more information on the biochemical aspects I refer to http://www.wiley.com/college/pratt/0471393878/student/animations/enzyme_kinetics/index.html*

Most biological reactions in cells of living organisms are catalysed by enzymes. A substrate S reacts with an enzym E to form an enzyme-substrate complex ES which subsequently dissociates into enzyme E and a product P:



The conversation rate V is the rate at which the amount (or concentration) of the product P increases. Or, equivalently, V is the rate at which the substrate concentration S decreases:

$$\boxed{V = \frac{dP}{dt}} \qquad \boxed{V = -\frac{dS}{dt}}$$

In 1913 Leonor Michaelis and Maud Menten proposed a model describing how the reaction rate V depends on the substrate concentration S:

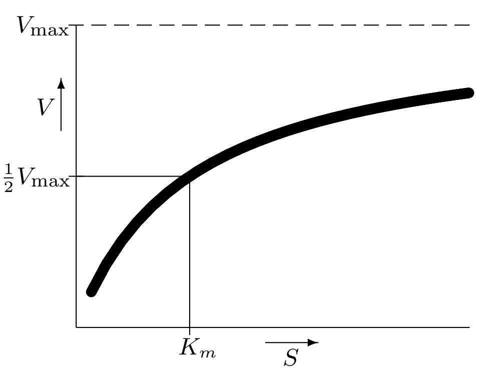
$$V = \frac{\alpha S}{\beta + S}$$

where the 'reaction constants' α and β depend on the specific reaction conditions and the amount of enzyme. In exercise 32 you are asked to prove the following interpretations of α and β:

$$\begin{aligned} \alpha &= \text{the theoretical maximum of the conversion rate} \\ &= \text{the conversion rate in case that there is unlimited substrate available} \\ &= \lim_{S \to \infty} V \qquad \text{(notation: } V_{\max}\text{)} \end{aligned}$$

$$\begin{aligned} \beta &= \text{the substrate concentration at which the conversation rate is 'half-maximal'} \\ &= \text{the substrate concentration at which } V = \tfrac{1}{2} V_{\max} \end{aligned}$$

This β is usually called K_m, the Michaelis constant. The Michaelis-Menten equation then becomes



$$\boxed{V = \frac{V_{\max} \cdot S}{K_m + S}}$$

The graph of V as a function of S is called the Michaelis-Menten curve, which is part of a hyperbola with a horizontal asymptote at height $V_{\max}$.

An important skill for life scientists is to study processes which you suspect (or know) to belong to the category 'Michaelis-Menten':

1. Perform a few (at least two, preferably more) experiments in which you measure the reaction rate at different values of S.

2. Take a piece of scrap paper and calculate the reaction constants $V_{\max}$ and K_m from your measurements.

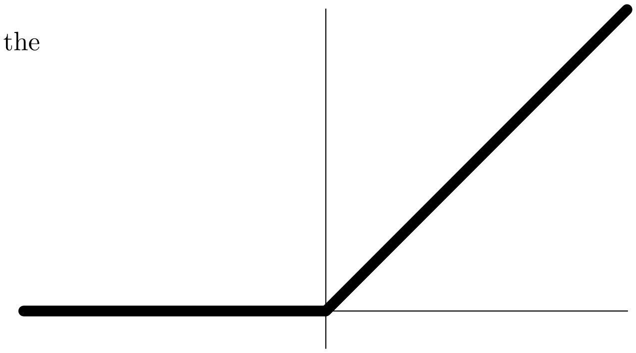
3. Now you know 'everything' about the process in question and you can, for instance, predict reaction rates at different concentrations without effort.

Exercises chapter 5

Exercise 1. The adjacent graph is the graph of the function f with domain $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Is this function differentiable?

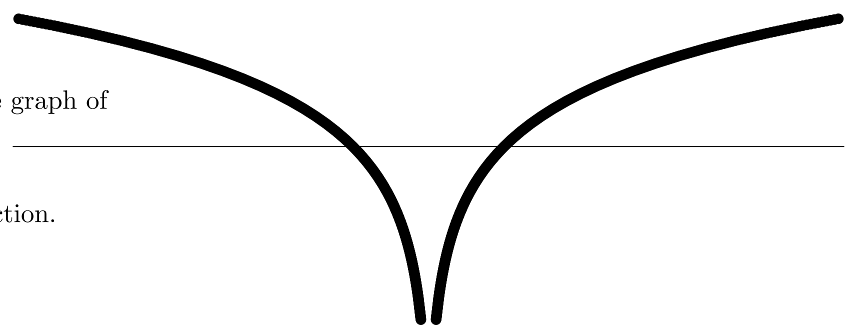


Exercise 2. Determine $\dfrac{dy}{dx}$ if

a) $y = \ln x^3$ b) $y = \ln 3x$ c) $y = \log x$ d) $y = 3^x$ e) $y = 3^{-x}$

Exercise 3. Calculate $\left[\dfrac{dy}{dx}\right]_{x=5}$ if

a) $y = (1+3x)^3$ b) $y = \sqrt{1+3x}$ c) $y = \dfrac{1}{1+3x}$ d) $y = \dfrac{1}{\sqrt{1+3x}}$



Exercise 4. This is (part of) the graph of

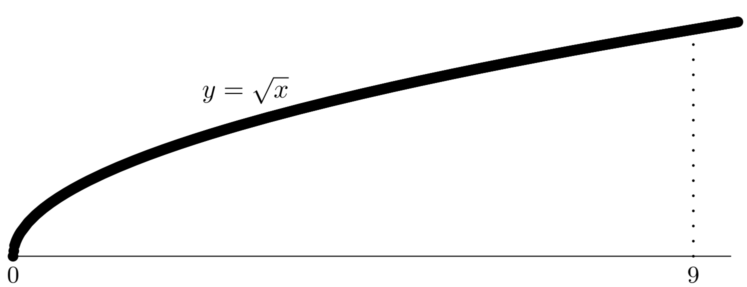
$$\boxed{y = \ln|x|}$$

Find the derivative $\dfrac{dy}{dx}$ of this function.

Exercise 5. Calculate

a) $\dfrac{d}{dx}\left(x^3 \sin x^2\right)$ b) $\dfrac{d}{dx}\left(x^3 \sin^2 x\right)$

Exercise 6. Calculate $\left[\dfrac{d}{dx}\, x^{\sqrt{x}}\right]_{x=4}$.



Exercise 7.

a) Calculate the derivative of $y = \sqrt{x}$ at the point $x = 9$.

b) Calculate the second derivative of $y = \sqrt{x}$ at $x = 9$.

Exercise 8. Let f be defined by $f(t) = \dfrac{9+t^3}{t}$.

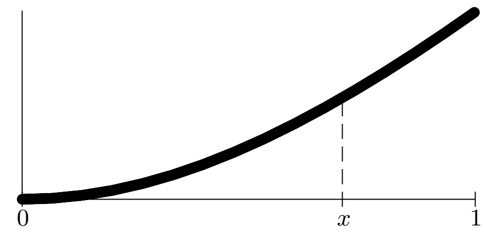
a) Calculate $\left[\dfrac{df}{dt}\right]_{t=3}$.

b) Calculate $\left[\dfrac{d^2 f}{dt^2}\right]_{t=3}$.

Exercise 9. (unit: kilometres)
An aircraft leaves the ground at $x = 0$, its height y above the point x for $0 \leq x \leq 1$ is given by

$$\boxed{y = \sqrt{1 + x^2} - 1}$$

At what point does the airplane rise under an angle of 30° ?



Exercise 10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sqrt[3]{x^5 + 7x}$$

a) Calculate $f'(1)$.

b) Calculate $\left(\overleftarrow{f}\right)'(2)$.

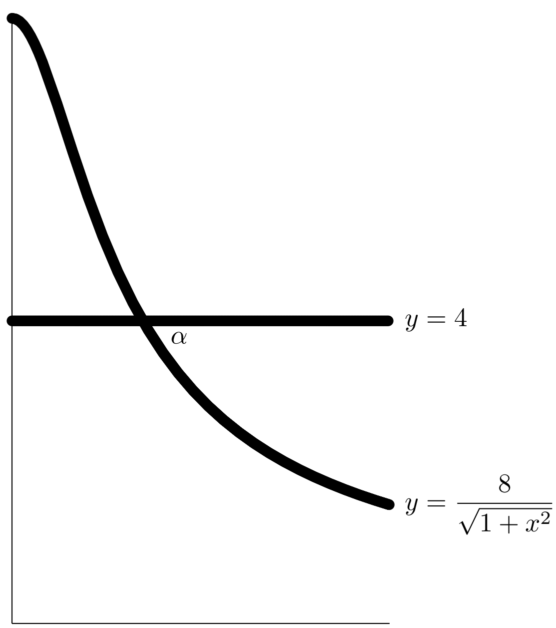
Exercise 11.

a) Calculate $\dfrac{d}{dx} \sqrt[5]{1 + x}$

b) and use this to calculate $\lim\limits_{n \to \infty} n \left(\sqrt[5]{1 + \dfrac{1}{n}} - 1 \right)$.

Exercise 12. Calculate the following limits:

a) $\lim\limits_{x \to 3} \dfrac{x^x - 25}{3^x - 25}$ b) $\lim\limits_{x \to 3} \dfrac{x^x - 27}{3^x - 27}$ c) $\lim\limits_{x \to 0} \dfrac{\ln\left(1 + x^2\right)}{1 - \cos x}$



Exercise 13. Calculate the angle α between the line $y = 4$ and the graph of

$$\boxed{y = \frac{8}{\sqrt{1 + x^2}}}$$

Exercise 14.

a) Calculate the derivative of ln at 1.

b) Calculate the second derivative of ln at 1.

c) Calculate the third derivative of ln at 1.

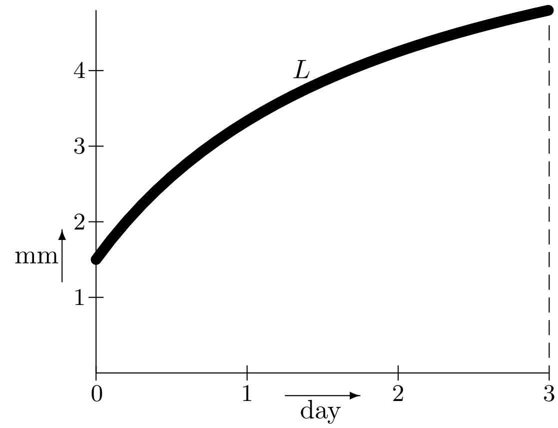
d) Calculate the tenth derivative of ln at 1.

Exercise 15. (units: days, mm)
My guppy is growing fast, its length after t days is

$$\boxed{L(t) = \frac{3+7t}{2+t}}$$

a) How tall is my guppy after three days?

b) How fast is it growing after three days?



Exercise 16. (units: metres, seconds)
I started at time $t = 0$ and sprinted for ten seconds. The distance I'd travelled after t seconds was

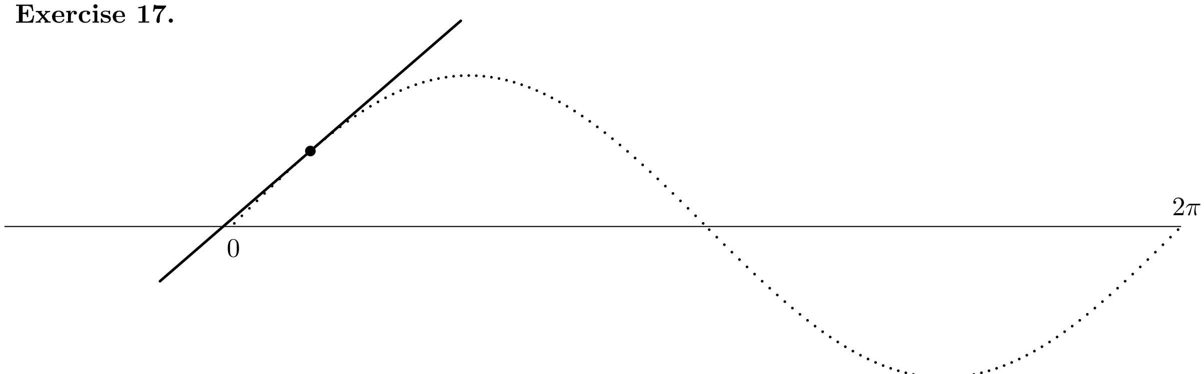
$$\boxed{s(t) = t^2 - \frac{t^3}{21}}$$

a) What was my velocity at time t?

b) What was my maximum velocity?

c) What was my acceleration after $3\frac{1}{2}$ seconds?

Exercise 17.

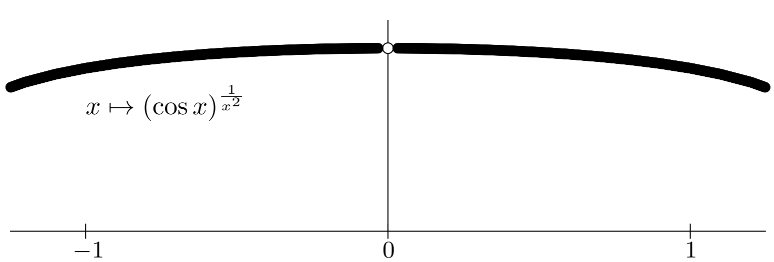


This is the graph of the function sin at domain $[0, 2\pi]$.

a) Determine the equation of the tangent line to this graph at the point $(\frac{\pi}{6}, \frac{1}{2})$.

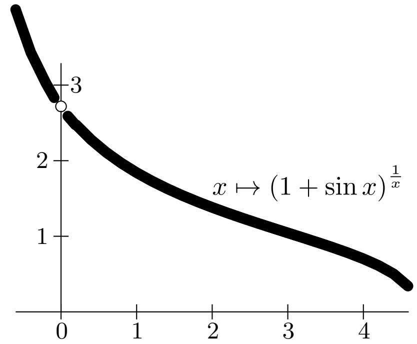
b) Calculate the intersection point of this tangent line and the x-axis.

c) At what points of the sine graph is the slope $-\frac{1}{2}$?



Exercise 18. Calculate $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$.

Exercise 19. Calculate $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.



Exercise 20.

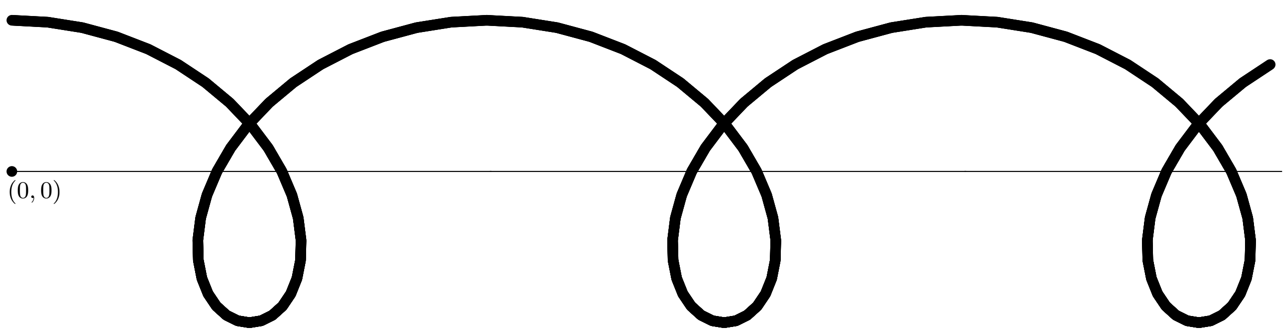
a) Calculate the derivative at $x = 0$ of the function $f(x) = \sqrt[3]{1+x}$.

b) Calculate $\lim_{x\to 0}\dfrac{\sqrt[3]{1+x}-1}{x}$.

c) Calculate $\lim_{n\to\infty}\left(\sqrt[3]{n^3+n^2}-n\right)$.

Exercise 21. (units: metres, seconds)
Fido is running through the Cartesian plane while its owner remains at the origin. At time t the neurotic beast is at the point

$$\boxed{\text{fido}(t) = (t+2\sin t\,,\ 2\cos t)}$$



a) Give a function rule for the distance between fido and its owner at time t.

b) How fast is this distance increasing at time $t = \dfrac{\pi}{2}$?

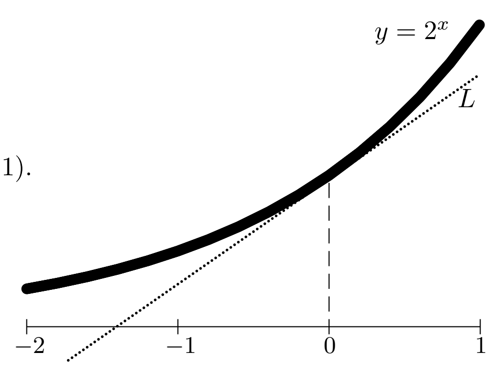
Exercise 22. Calculate

a) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\cdots$

b) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\cdots$

c) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots$

Exercise 23. L is the tangent line to the graph of $y = 2^x$ at $(0, 1)$.
Find the intersection point of L and the x-axis.



Exercise 24. We are all familiar with the appearance of balloons left hanging a few days after a party. The air inside the balloons leaks out and what used to be joyful decoration slowly becomes a pile of wizened pieces of latex. We study a balloon with initially 5 litres of air in it, 7% of which leaks out each day.

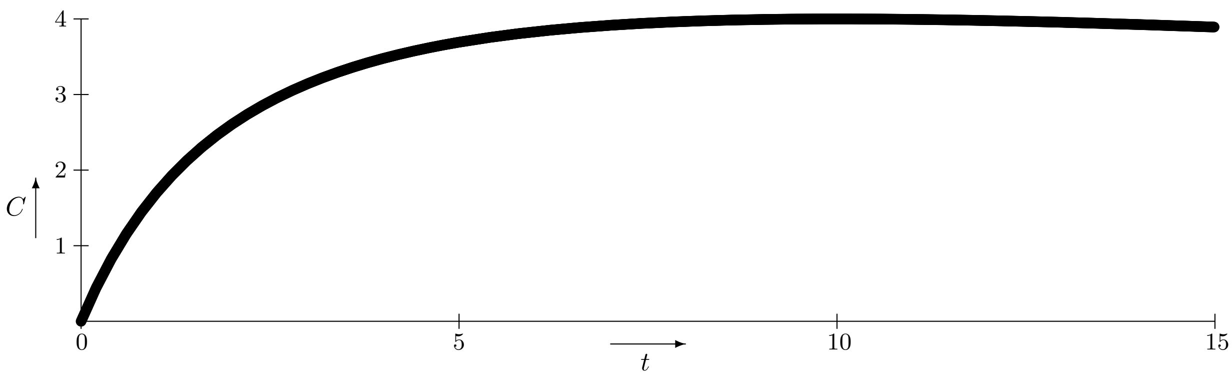
a) Give a formula for the amount of air L in the balloon (in litres) after t days.

b) With what rate does the amount of air decrease after precisely 4 days?

c) After how many days will there be only 2 litres of air left in the balloon?

Exercise 25. A drug is administered by injections to the blood. The concentration C (in weight percents) after t minutes is

$$\boxed{C = \frac{240\,t}{t^2 + 40\,t + 100}}$$



a) Calculate the derivative of C at $t = 0$. What does this tell you about the concentration immediately after the injection?

b) Calculate the maximum concentration. When is this maximum concentration established?

c) A patient is allowed to leave the hospital as soon as the concentration has dropped to 1%. How many minutes after the injection will this be the case?

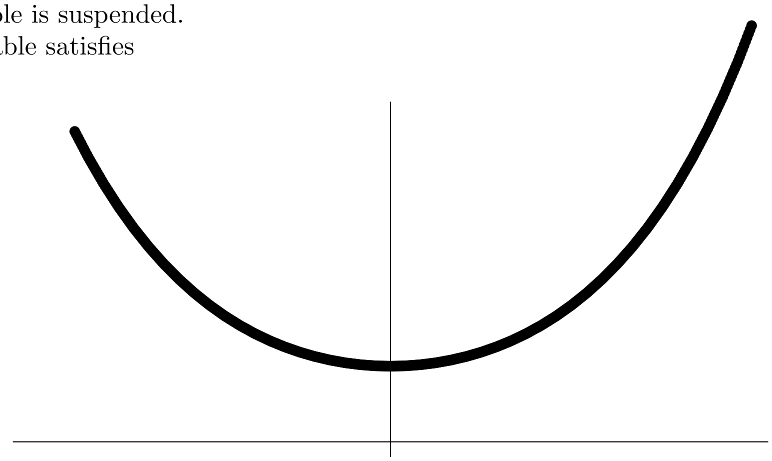
If you study MLS or chemistry, you're allowed to skip exercises 26-31, continue with exercise 32.

Exercise 26. Calculate the derivatives of the functions cosh and sinh.

Exercise 27. We investigate how a cable is suspended.
The shape $y = f(x)$ of a homogenuous cable satisfies

$$\boxed{\frac{d^2y}{dx^2} = C\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

The constant C in this equation is $\frac{\rho g}{H}$ with ρ the mass per unit length, g the gravitational acceleration and H the cable tension at the lowest point.



a) Prove that $y = \left(\frac{1}{C}\cosh Cx\right) + D$ (with D an arbitrary constant) satisfies this equation.

b) Check on your way home whether you encounter cosh-like functions in the wild.

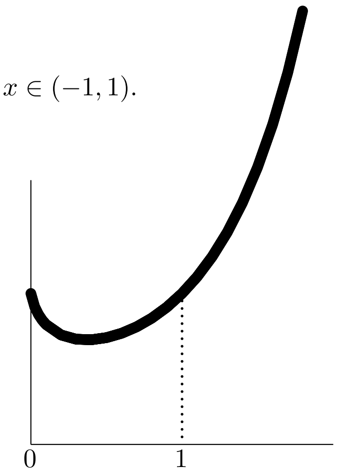
Exercise 28. Calculate $\sum_{n=0}^{\infty} nx^n$ for $x \in (-1, 1)$.

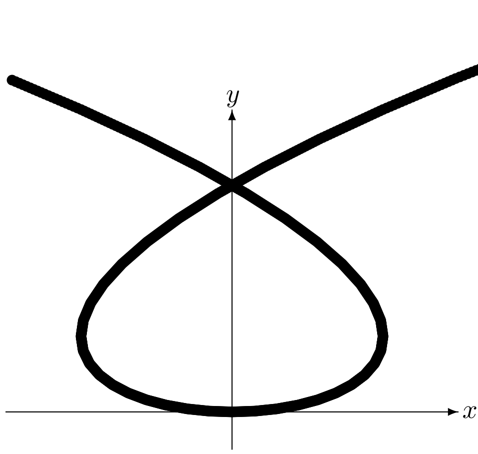
Exercise 29. Calculate $1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5 + 49x^6 + \cdots$ for $x \in (-1, 1)$.

Exercise 30. Let f be the function $x \mapsto x^x$ with as its domain the collection of all positive real numbers. The adjacent figure is a sketch of the graph of f.

a) Determine the derivative of f.

b) Determine the tangent line to f at $x = 1$.





Exercise 31. The adjacent curve has equation

$$\boxed{y^3 - 6y^2 + 9y = x^2}$$

a) Prove that the point $P = (2, 4)$ lies on the curve.

b) Calculate the derivative of y with respect to x at P.

c) Find the equation of the tangent line at P to the curve.

Exercise 32-43 are intended for chemistry and MLS students, science students are allowed to skip them.

Exercise 32. A substrate S is converted by an enzyme E into a product P with the conversion rate V depending on the concentration of S according to

$$\boxed{V = \frac{3S}{2 + S}}$$

a) Calculate $\dfrac{dV}{dS}$ and prove the theorem: the higher S, the higher V.

b) Calculate $V_{\max}$, i.e. the maximum conversion rate. At what value of S is this conversion rate realised?

c) Calculate the Michaelis constant K_m, which is the concentration of S for which the conversion rate is half-maximal.

d) What is remarkable about the results of (b) and (c)?

Exercise 33. Assume that the enzymatic reaction $S \to P$ satisfies the Michaelis-Menten equation. Calculate the slope $\frac{dV}{dS}$ of the Michaelis-Menten curve

a) for very low substrate concentrations, I mean for $S \ll K_m$

b) for the medium substrate concentration $S = K_m$

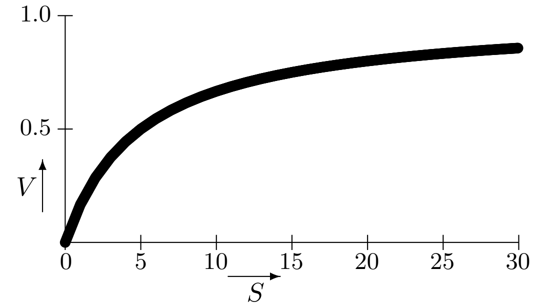
c) for the rather high substrate concentration S with $V = 0.9\, V_{\max}$

(Express your answers in terms of the reaction constants $V_{\max}$ and K_m.)

Exercise 34. (units: S in mM, V in mM/min)
Given a Michaelis-Menten reaction with

$$\boxed{V = \frac{S}{5+S}}$$

Calculate the slope $\frac{dV}{dS}$ of the Michaelis-Menten curve at the point where the conversion rate is half-maximal.

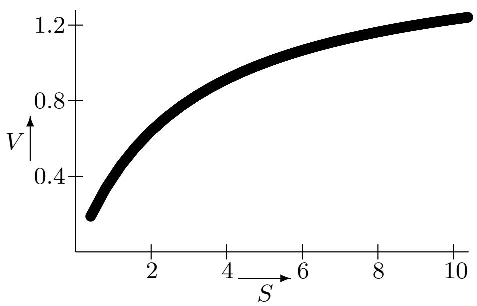


Exercise 35. For a Michaelis-Menten reaction I measure the reaction rate at two substrate concentrations:

S	V
10 mM	1.0 mM/min
20 mM	1.5 mM/min

a) Calculate the reaction constants K_m and $V_{\max}$.

b) Calculate the reaction rate to be expected at $S = 30$ mM.



Exercise 36. A certain Michaelis-Menten conversion has reaction constants

$$\boxed{V_{\max} = 1.6 \text{ mM/min}} \qquad \boxed{K_m = 3 \text{ mM}}$$

At what substrate concentration S is $V = 1.2$ mM/min ?

Exercise 37. Given a Michaelis-Menten reaction $S \xrightarrow{E} P$ with $V_{\max} = 2$ en $K_m = 1$.

a) Express V in terms of S and sketch the Michaelis-Menten plot. What kind of curve is this?

b) Express $\frac{1}{V}$ in terms of $\frac{1}{S}$ and draw the corresponding graph (the Lineweaver-Burk plot). Any comments?

c) Would you remark the same for different values of $V_{\max}$ and K_m?

Exercise 38. A Michaelis-Menten reaction is experimentally studied by measuring the reaction rate at two different substrate concentrations:

S	V
20 mM	2.8 mM/min
40 mM	4.0 mM/min

Calculate the reaction constants $V_{\max}$ and K_m from these measurements.

Exercise 39. I investigate an enzymatic reaction by measuring the initial conversion rate at three different substrate concentrations:

S	V
1.0	1.11
2.0	1.76
5.0	2.94

a) Could this possibly be a Michaelis-Menten reaction?

b) If so, estimate the reaction constants $V_{\max}$ and K_m.

Exercise 40. For a given Michaelis-Menten enzyme reaction the substrate concentration S and the conversion rate V are related according to

$$S = \frac{V}{5 - 2V}$$

a) Write this relation as a standard Michaelis-Menten equation $V = \dfrac{\cdots \cdot S}{\cdots + S}$.
What is $V_{\max}$ and what is K_m?

b) Sketch the Michaelis-Menten curve.

c) Sketch the Lineweaver-Burk plot.

d) Sketch the Eadie-Hofstee plot as well, which is the graph of V versus $\frac{V}{S}$.

e) In (d) you joyfully discovered that the Eady-Hofstee plot is a straight line. Is this always true for a Michaelis-Menten reaction?

f) How can you deduce directly from the Eadie-Hofstee plot what K_m and $V_{\max}$ are?

g) Name an advantage of using the EH plot instead of the LB plot.

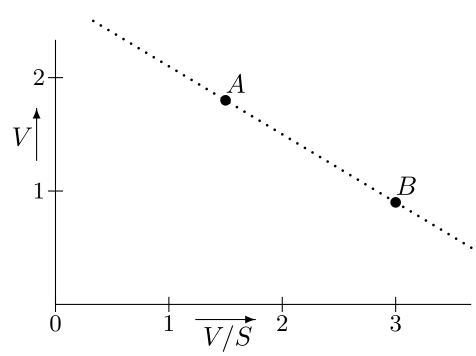
Exercise 41. Give a sketch and an equation for the Eady-Hofstee plot of a Michaelis-Menten reaction satisfying

$$V = \frac{3S}{1 + 2S}$$

Exercise 42. Experimentally I found two points of the Eadie-Hofstee plot of a Michaelis-Menten reaction:

$A = (1.5\,,\,1.8)$ $B = (3.0\,,\,0.9)$

Calculate the reaction constants K_m and $V_{\max}$.



Exercise 43. For a certain Michaelis-Menten reaction the conversion rate depends on the substrate concentration according to

$$V = \frac{4S}{1 + 2S}$$

Find the equation of the corresponding Eady-Hofstee plot and draw this line.

Solutions chapter 5

Exercise 1. No, because

- $\lim_{x \uparrow 0} \frac{f(x) - f(0)}{x - 0} = 0$
- $\lim_{x \downarrow 0} \frac{f(x) - f(0)}{x - 0} = 1$
- so $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

Exercise 2.

a) You can choose between the chain rule (with $y = \ln u$ and $u = x^3$) and the trick $\ln x^3 = 3 \ln x$, both yielding

$$\frac{dy}{dx} = \frac{3}{x}$$

b) Use either the chain rule (with $y = \ln u$ and $u = 3x$) or the trick $\ln 3x = \ln 3 + \ln x$, both resulting in

$$\frac{dy}{dx} = \frac{1}{x}$$

c) Express $\log x$ (the logarithm of x with base 10) in terms of ln (see arithmetic booklet chapter 5):

$$y = \log x = \frac{\ln x}{\ln 10} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{x \ln 10}$$

d) That's a standard derivative:

$$\frac{dy}{dx} = 3^x \ln 3$$

e) Using the chain rule (with $y = 3^u$ and $u = -x$) you find

$$\frac{dy}{dx} = -3^{-x} \ln 3$$

Exercise 3.

a) By the chain rule (with $y = u^3$ and $u = 1 + 3x$),

$$\frac{dy}{dx} = 9(1 + 3x)^2 \quad \Longrightarrow \quad \left[\frac{dy}{dx}\right]_{x=5} = 2304$$

b) By the chain rule (with $y = \sqrt{u}$ and $u = 1 + 3x$),

$$\frac{dy}{dx} = \frac{3}{2\sqrt{1 + 3x}} \quad \Longrightarrow \quad \left[\frac{dy}{dx}\right]_{x=5} = \frac{3}{8}$$

c) By the chain rule (with $y = u^{-1}$ and $u = 1 + 3x$),

$$\frac{dy}{dx} = \frac{-3}{(1 + 3x)^2} \quad \Longrightarrow \quad \left[\frac{dy}{dx}\right]_{x=5} = -\frac{3}{256}$$

d) By the chain rule (with $y = u^{-\frac{1}{2}}$ and $u = 1 + 3x$),

$$\frac{dy}{dx} = -\frac{3}{2}(1 + 3x)^{-\frac{3}{2}} \quad \Longrightarrow \quad \left[\frac{dy}{dx}\right]_{x=5} = -\frac{3}{128}$$

Exercise 4. You cannot apply our differentiation rules directly on expressions with absolute values. I guess the only way to solve this exercise is to consider the two cases $x > 0$ and $x < 0$ separately:

- $x > 0 \quad \Longrightarrow \quad \dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln x = \dfrac{1}{x}$
- $x < 0 \quad \Longrightarrow \quad \dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln(-x) = \dfrac{1}{-x}\cdot(-1) = \dfrac{1}{x}$

Coincidently, we find the same solution in both cases. Put together:

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \text{for all } x \neq 0$$

Exercise 5. I apply the product rule:

a) $\dfrac{d}{dx}\left(x^3 \sin x^2\right) = 3x^2 \sin x^2 + 2x^4 \cos x^2$

b) $\dfrac{d}{dx}\left(x^3 \sin^2 x\right) = 3x^2 \sin^2 x + 2x^3 \sin x \cos x$

Exercise 6. $x^{\sqrt{x}} = e^{\sqrt{x}\ln x}$, which I differentiate using the chain rule followed by the product rule:

$$\frac{d}{dx}x^{\sqrt{x}} = x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) \qquad \Longrightarrow \qquad \left[\frac{d}{dx}x^{\sqrt{x}}\right]_{x=4} = 8 + 8\ln 2$$

Exercise 7.

a) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} \quad \Longrightarrow \quad \left[\dfrac{dy}{dx}\right]_{x=9} = \dfrac{1}{6}$

b) $\dfrac{d^2y}{dx^2} = \dfrac{-1}{4x\sqrt{x}} \quad \Longrightarrow \quad \left[\dfrac{d^2y}{dx^2}\right]_{x=9} = -\dfrac{1}{108}$

Exercise 8.

a) $f(t) = \dfrac{9}{t} + t^2 \quad \Longrightarrow \quad \dfrac{df}{dt} = -\dfrac{9}{t^2} + 2t \quad \Longrightarrow \quad \left[\dfrac{df}{dt}\right]_{t=3} = 5$

b) $\dfrac{d^2f}{dt^2} = \dfrac{d}{dt}\left(-\dfrac{9}{t^2} + 2t\right) = \dfrac{18}{t^3} + 2 \quad \Longrightarrow \quad \left[\dfrac{d^2f}{dt^2}\right]_{t=3} = \dfrac{8}{3}$

Exercise 9. I calculate the slope of the flying route above the point x:

$$y = \left(1 + x^2\right)^{\frac{1}{2}} - 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}$$

An angle of $30°$ corresponds to slope $\tan 30° = \dfrac{1}{\sqrt{3}}$, so the desired point x satisfies

$$\frac{x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{3}} \quad \Longrightarrow \quad \frac{x^2}{1+x^2} = \frac{1}{3} \quad \Longrightarrow \quad 3x^2 = 1 + x^2 \quad \Longrightarrow \quad 2x^2 = 1 \quad \Longrightarrow \quad \boxed{x = \frac{1}{\sqrt{2}}}$$

Exercise 10.

a) $f'(x) = \dfrac{1}{3}(x^5 + 7x)^{-\frac{2}{3}} \cdot (5x^4 + 7) \quad \Longrightarrow \quad f'(1) = 1$

b) $\left(\overleftarrow{f}\right)'(2) = \dfrac{1}{f'(\overleftarrow{f}(2))} = \dfrac{1}{f'(1)} = 1$

Exercise 11. Let's give $\sqrt[5]{1+x}$ the name $f(x)$ to facilitate talking:

a) The derivative is

$$f'(x) = \frac{1}{5}(1+x)^{-\frac{4}{5}}$$

b) From (a) we have $f'(0) = \frac{1}{5}$ or, put differently,

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \frac{1}{5}$$

which amounts to

$$\lim_{h\to 0} \frac{\sqrt[5]{1+h} - 1}{h} = \frac{1}{5}$$

and if I substitute $h = \frac{1}{n}$, I finally obtain

$$\lim_{n\to\infty} n\left(\sqrt[5]{1+\frac{1}{n}} - 1\right) = \frac{1}{5}$$

Exercise 12.

a) This one is a bit dull, because you can just substitute $x = 3$:

$$\lim_{x\to 3} \frac{x^x - 25}{3^x - 25} = \frac{3^3 - 25}{3^3 - 25} = 1$$

b) This time, however, bluntly substituting $x = 3$ leads to the catastrophe $\frac{0}{0}$. Fortunately, we possess a beautiful aid to prevent these kind of disasters, namely l'Hôpital's rule:

$$\lim_{x\to 3} \frac{x^x - 27}{3^x - 27} = \lim_{x\to 3} \frac{x^x(1+\ln x)}{3^x \ln 3} = \frac{1+\ln 3}{\ln 3}$$

c) Again, substitution of $x = 0$ yields $\frac{0}{0}$, so again we employ l'Hôpital:

$$\lim_{x\to 0} \frac{\ln\left(1+x^2\right)}{1-\cos x} = \lim_{x\to 0} \frac{\frac{2x}{1+x^2}}{\sin x}$$

That doesn't seem to have helped much, because substitution of $x = 0$ once again leads to a disaster. There are two ways to proceed:

- applying l'Hôpital a second time will help you crack the limit
- or, somewhat niftier:

$$\cdots = \lim_{x\to 0}\left(\frac{x}{\sin x} \cdot \frac{2}{1+x^2}\right) = 1 \cdot 2 = 2$$

Exercise 13. The intersection point is $\left(\sqrt{3}\,,\,4\right)$. I calculate the slope of $y = \dfrac{8}{\sqrt{1+x^2}}$ at this point:

$$y = 8\left(1+x^2\right)^{-\frac{1}{2}} \quad\Longrightarrow\quad \frac{dy}{dx} = -4\left(1+x^2\right)^{-\frac{3}{2}} \cdot 2x \quad\Longrightarrow\quad \left[\frac{dy}{dx}\right]_{x=\sqrt{3}} = -\sqrt{3} \quad\Longrightarrow\quad \boxed{\alpha = 60^\circ}$$

Exercise 14.

a) $\ln'(x) = \dfrac{1}{x} \Longrightarrow \ln'(1) = 1$

b) $\ln''(x) = -\dfrac{1}{x^2} \Longrightarrow \ln''(1) = -1$

c) $\ln'''(x) = \dfrac{2}{x^3} \Longrightarrow \ln'''(1) = 2$

d) $\ln^{(10)}(x) = -\dfrac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{x^{10}} \Longrightarrow \ln^{(10)}(1) = -9! = -362880$

Exercise 15.

a) Its length after three days is $L(3) = \dfrac{24}{5} = 4.8$ mm.

b) According to the quotient rule,

$$\frac{dL}{dt} = \frac{7 \cdot (2+t) - 1 \cdot (3+7t)}{(2+t)^2} = \frac{11}{(2+t)^2} \quad \Longrightarrow \quad \left[\frac{dL}{dt}\right]_{t=3} = \frac{11}{25} = 0.44$$

so at $t = 3$ my guppy is growing by 0.44 mm/day.

Exercise 16.

a) My velocity at time t was $v(t) = \dfrac{d}{dt}\, s(t) = 2t - \dfrac{t^2}{7}$ m/sec.

b) This can be done by completing the square: $v(t) = -\dfrac{1}{7}(t-7)^2 + 7$ with maximum 7 at $t = 7$, so my maximum velocity was 7 m/sec.

c) $a(t) = \dfrac{d}{dt}\, v(t) = 2 - \dfrac{2t}{7}$ so $a(3\frac{1}{2}) = 1$ m/sec^2

Exercise 17.

a) The tangent line to sin at $\frac{\pi}{6}$ is

$$l(x) = \sin\frac{\pi}{6} + \left(\sin'\left(\frac{\pi}{6}\right)\right) \cdot \left(x - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \left(\cos\left(\frac{\pi}{6}\right)\right) \cdot \left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \left(\frac{1}{2}\sqrt{3}\right) \cdot \left(x - \frac{\pi}{6}\right)$$

b) I have to calculate when $l(x) = 0$:

$$l(x) = 0 \quad \Longleftrightarrow \quad x = \frac{\pi}{6} - \frac{1}{\sqrt{3}}$$

so the desired intersection point is the point

$$\left(\frac{\pi}{6} - \frac{1}{\sqrt{3}} \,,\, 0\right)$$

c) The slope is $-\frac{1}{2}$ when

$$\sin'(x) = -\frac{1}{2} \quad \Longleftrightarrow \quad \cos x = -\frac{1}{2} \quad \Longleftrightarrow \quad x = \frac{2}{3}\pi \ \text{ or } \ x = \frac{4}{3}\pi$$

so the desired points are $\left(\dfrac{2}{3}\pi \,,\, \dfrac{1}{2}\sqrt{3}\right)$ and $\left(\dfrac{4}{3}\pi \,,\, -\dfrac{1}{2}\sqrt{3}\right)$.

Exercise 18. This exercise is screaming for the ln-trick:

$$\lim_{x\to 0} \ln\left((\cos x)^{\frac{1}{x^2}}\right) = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \text{ (using l'Hôpital) } -\frac{1}{2}$$

so $\displaystyle\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$.

Exercise 19. Same trick, I use the function $f(x) = \ln(1 + \sin x)$:

$$\lim_{x\to 0} \ln\left((1+\sin x)^{\frac{1}{x}}\right) = \lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = f'(0) = 1$$

so $\displaystyle\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} = e$.

Exercise 20.

a) $f(x) = (1+x)^{\frac{1}{3}}$, so $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$ which yields $f'(0) = \frac{1}{3}$.

b) Same question as (a), because $f'(0)$ is equivalent to $\lim_{x\to 0} \frac{f(x)-f(0)}{x}$.

c) In (b) we discovered that $\lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x} = \frac{1}{3}$.

If you substitute $x = \frac{1}{n}$, you find: $\lim_{n\to\infty} \left(\sqrt[3]{n^3+n^2} - n\right) = \frac{1}{3}$.

Exercise 21.

a) The distance from fido to its owner at time t is

$$\text{distance}\,(t) = \|\text{fido}\,(t)\| = \sqrt{(t+2\sin t)^2 + (2\cos t)^2} = \sqrt{t^2 + 4t\sin t + 4}$$

b) The rate of increase of this distance at time t is

$$\frac{d\,\text{distance}}{dt} = \frac{1}{2\sqrt{t^2+4t\sin t+4}} \cdot (2t + 4\sin t + 4t\cos t) = \frac{t + 2\sin t + 2t\cos t}{\sqrt{t^2+4t\sin t+4}}$$

and if I substitute $t = \frac{\pi}{2}$ I find

$$\left[\frac{d\,\text{distance}}{dt}\right]_{t=\frac{\pi}{2}} = \frac{\frac{\pi}{2}+2}{\sqrt{\frac{\pi^2}{4}+2\pi+4}} = \frac{\frac{\pi}{2}+2}{\sqrt{\left(\frac{\pi}{2}+2\right)^2}} = \boxed{1 \text{ metre per second}}$$

Exercise 22.

a) Use the power series for $\frac{1}{1-x}$ and substitute $x = -\frac{1}{2}$:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \boxed{\frac{2}{3}}$$

b) Use the Taylor series for $\ln(1+x)$ and substitute $x = 1$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \boxed{\ln 2}$$

c) Use the Taylor series for arctan and substitute $x = 1$:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots = \arctan 1 = \boxed{\frac{\pi}{4}}$$

Exercise 23. I calculate the slope of $y = 2^x$ at the point $(0,1)$ first:

$$\frac{dy}{dx} = 2^x \ln 2 \quad \Longrightarrow \quad \left[\frac{dy}{dx}\right]_{x=0} = \ln 2$$

Thus, the equation of L is $y = 1 + x\ln 2$, so the intersection point of L and the x-axis is $\boxed{\left(\frac{-1}{\ln 2}, 0\right)}$.

Exercise 24.

a) The amount of air in the balloon after t days is $\boxed{L = 5 \cdot 0.93^t}$.

b) $\frac{dL}{dt} = 5 \cdot 0.93^t \cdot \ln 0.93 \quad \Longrightarrow \quad \left[\frac{dL}{dt}\right]_{t=4} = \boxed{5 \cdot 0.93^4 \cdot \ln 0.93}$ (approximately -0.27 litres/day)

c) $L = 2 \quad \Longrightarrow \quad 0.93^t = 0.4 \quad \Longrightarrow \quad \ln 0.93^t = \ln 0.4 \quad \Longrightarrow \quad t\ln 0.93 = \ln 0.4 \quad \Longrightarrow \quad \boxed{t = \frac{\ln 0.4}{\ln 0.93}}$

(after approximately 12.6 days)

Exercise 25.

a) I use the quotient rule:

$$\frac{dC}{dt} = 240 \cdot \frac{100 - t^2}{(t^2 + 40t + 100)^2} \quad \Longrightarrow \quad \boxed{\left[\frac{dC}{dt}\right]_{t=0} = 2.4}$$

Immediately after the injection the concentration increases by 2.4 percent/minute.

b) When the concentration is maximum we have $\dfrac{dC}{dt} = 0$ so $t = 10$ and $\boxed{C(10) = 4}$.

c) I try to find the times at which the concentration equals 1%:

$$C(t) = 1 \quad \Longrightarrow \quad 240t = t^2 + 40t + 100 \quad \Longrightarrow \quad t^2 - 200t + 100 = 0 \quad \overset{\text{quadratic formula}}{\Longrightarrow} \quad t = 100 \pm 30\sqrt{11}$$

The patient is allowed to go home when $\boxed{t = 100 + 30\sqrt{11}}$
(after almost 200 minutes, don't make the terrible mistake to send him home after $100 - 30\sqrt{11}$ min).

Exercise 26. $\cosh' = \sinh$ and $\sinh' = \cosh$. You discover this by straightforward calculations:

$$\cosh'(x) = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$
$$\sinh'(x) = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Exercise 27.

a) We consider the left-hand side and the right-hand side consecutively:

- $\dfrac{dy}{dx} = \sinh Cx$ so $\dfrac{d^2y}{dx^2} = C\cosh Cx$

  (I used the results from the previous exercise.)

- $C\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = C\sqrt{1 + (\sinh Cx)^2} = C\cosh Cx$

  (I used the well-known formula $(\cosh p)^2 - (\sinh p)^2 = 1$.)

b) Evidently, cables hang along the graph of cosh-functions. On your way home you'll probably encounter some high-voltage cables or spider silks. And maybe you're even lucky to have your bicycle chain falling off.

Exercise 28. Differentiating the left-hand side and right-hand side of $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$ yields

$$0 + 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \quad \text{so} \quad \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Exercise 29. From exercise 28 we have $x + 2x^2 + 3x^3 + \cdots = \dfrac{x}{(1-x)^2}$ which, upon differentiation, implies

$$1 + 4x + 9x^2 + \cdots = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{1+x}{(1-x)^3}$$

Exercise 30.

a) That's an easy one for those who understand the chain rule and the product rule:

$$f(x) = x^x = e^{x\ln x} \quad \Longrightarrow \quad f'(x) = e^{x\ln x}(1 + \ln x) = x^x(1 + \ln x)$$

b) From (a) we have $f'(1) = 1$, so the tangent line l is the function $l(x) = f(1) + f'(1)(x - 1) = x$.

Exercise 31.

a) Simple substitution: $64 - 96 + 36 = 4$.

b) You're not able to write y explicitly as a function of x in the neighbourhood of this point. That's why I use a trick: take the equation of the curve and differentiate both sides with respect to x:

$$y^3 - 6y^2 + 9y = x^2 \quad \Longrightarrow \quad 3y^2 \cdot \frac{dy}{dx} - 12y \cdot \frac{dy}{dx} + 9 \cdot \frac{dy}{dx} = 2x$$

This might have been very hard to you (at least, if you have difficulties with the chain rule), but the rest is surprisingly easy:

$$(3y^2 - 12y + 9) \cdot \frac{dy}{dx} = 2x \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2x}{3y^2 - 12y + 9} \qquad \Longrightarrow \qquad \left[\frac{dy}{dx}\right]_P = \frac{4}{9}$$

c) I apply our formula for the tangent line to the (unknown) function $y = f(x)$ that describes the curve in the neighbourhood of P:

$$l(x) = f(2) + f'(2) \cdot (x - 2) = 4 + \frac{4}{9} \cdot (x - 2)$$

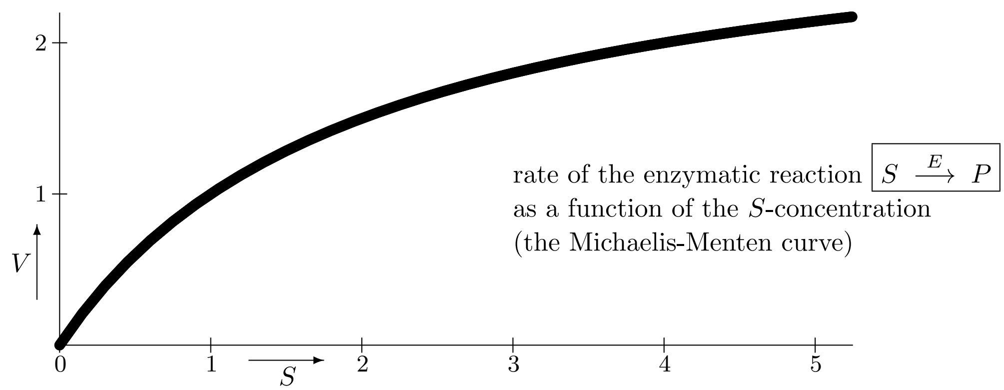
so the equation of this tangent line is $y = 4 + \frac{4}{9} \cdot (x - 2) \quad \Longrightarrow \quad \boxed{4x - 9y + 28 = 0}$

Exercise 32.

a) According to the quotient rule,

$$\frac{dV}{dS} = \frac{3(2 + S) - 3S}{(2 + S)^2} = \frac{6}{(2 + S)^2}$$

This expression is always positive, so if you plot V against S you obtain an increasing function:



Hence: the higher S, the higher V.

b) $V_{\max}$ is the conversion rate at very high S, I mean the horizontal asymptote to the Michaelis Menten hyperbola:

$$V_{\max} = \lim_{S\to\infty} V = \lim_{S\to\infty} \frac{3S}{2 + S} = \lim_{S\to\infty} \frac{3}{\frac{2}{S} + 1} = \frac{3}{0 + 1} = \boxed{3}$$

This theoretical maximum rate is realised at $S = \infty$, which means that it will $\boxed{\text{never}}$ be reached.

c) The conversion rate is half-maximal when

$$V = \frac{1}{2} V_{\max} = \frac{3}{2} \quad \Longrightarrow \quad \frac{3S}{2 + S} = \frac{3}{2} \quad \Longrightarrow \quad 3S \cdot 2 = (2 + S) \cdot 3 \quad \Longrightarrow \quad 6S = 6 + 3S \quad \Longrightarrow \quad \boxed{S = 2}$$

d) These results are precisely the constants in the given Michaelis-Menten equation. Evidently, that is not a coincidence; you can use these findings to prove that the conversion rate of whatever Michaelis-Menten process always obeys

$$\boxed{V = \frac{V_{\max} \cdot S}{K_m + S}}$$

Exercise 33. I differentiate V with respect to S using the quotient rule:

$$V = \frac{V_{\max} \cdot S}{K_m + S} \quad \Longrightarrow \quad \frac{dV}{dS} = \frac{V_{\max} \cdot (K_m + S) - V_{\max} \cdot S}{(K_m + S)^2} = \frac{V_{\max} \cdot K_m}{(K_m + S)^2}$$

a) If I substitute $S = 0$, I find $\left[\frac{dV}{dS}\right]_{S=0} = \frac{V_{\max} \cdot K_m}{(K_m + 0)^2} = \boxed{\frac{V_{\max}}{K_m}}$.

b) Here, I should substitute $S = K_m$: $\left[\frac{dV}{dS}\right]_{S=K_m} = \frac{V_{\max} \cdot K_m}{(K_m + K_m)^2} = \boxed{\frac{V_{\max}}{4\,K_m}}$.

c) That is a little bit more involved, let's first calculate at what substrate concentration this happens:

$$V = 0.9\,V_{\max} \quad \Longrightarrow \quad \frac{V_{\max} \cdot S}{K_m + S} = 0.9\,V_{\max} \quad \Longrightarrow \quad \frac{S}{K_m + S} = 0.9 \quad \Longrightarrow \quad S = 0.9\,K_m + 0.9\,S$$

$$\Longrightarrow \quad 0.1\,S = 0.9\,K_m \quad \Longrightarrow \quad S = 9\,K_m$$

Substitution of this S-value yields $\left[\frac{dV}{dS}\right]_{S=9K_m} = \frac{V_{\max} \cdot K_m}{(K_m + 9\,K_m)^2} = \boxed{\frac{V_{\max}}{100\,K_m}}$.

Exercise 34. That is the point at which the substrate concentration S equals $K_m = 5$:

$$V = \frac{S}{5 + S} \quad \Longrightarrow \quad \frac{dV}{dS} = \frac{5}{(5+S)^2} \quad \Longrightarrow \quad \left[\frac{dV}{dS}\right]_{S=5} = \boxed{0.05}$$

Exercise 35.

a) I calculate the reaction constants via the formula $V = \frac{V_{\max} \cdot S}{K_m + S}$:

$$\left.\begin{array}{lll} 1.0 = \frac{V_{\max} \cdot 10}{K_m + 10} & \Longrightarrow & K_m + 10 = 10\,V_{\max} \\ 1.5 = \frac{V_{\max} \cdot 20}{K_m + 20} & \Longrightarrow & 1.5\,K_m + 30 = 20\,V_{\max} \end{array}\right\} \Longrightarrow 1.5\,K_m + 30 = 2(K_m + 10) \Longrightarrow K_m = 20$$

and then $K_m + 10 = 10\,V_{\max}$ implies that $V_{\max} = 3$.

b) The relation between V and S is

$$V = \frac{3S}{20 + S}$$

so at a substrate concentration of 30 mM I expect a reaction rate of $\boxed{1.8 \text{ mM/min}}$.

Exercise 36. The relation between the conversion rate V and the substrate concentration S is

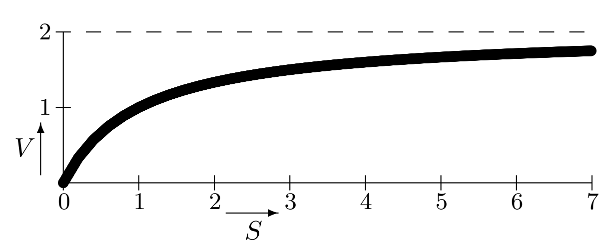
$$V = \frac{V_{\max} \cdot S}{K_m + S} = \frac{1.6\,S}{3 + S}$$

which equals 1.2 if $1.6\,S = 1.2 \cdot (3 + S) \quad \Longrightarrow \quad 1.6\,S = 3.6 + 1.2\,S \quad \Longrightarrow \quad 0.4\,S = 3.6 \quad \Longrightarrow \quad S = \boxed{9 \text{ mM}}$

Exercise 37.

a) You probably remember how the conversion rate depends on the amount of substrate:

$$V = \frac{V_{\max} \cdot S}{K_m + S} \quad \Longrightarrow \quad \boxed{V = \frac{2S}{1 + S}} \quad \Longrightarrow$$

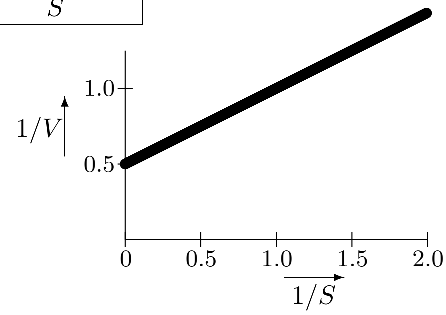


This is a hyperbola with an asymptote at height $V_{\max} = 2$.

b) I try to find the relation between $\frac{1}{V}$ and $\frac{1}{S}$:

$$V = \frac{2S}{1+S} \quad \Longrightarrow \quad \frac{1}{V} = \frac{1+S}{2S} = \frac{1}{2S} + \frac{S}{2S} \quad \Longrightarrow \quad \boxed{\frac{1}{V} = 0.5 \cdot \frac{1}{S} + 0.5}$$

so the graph of $\frac{1}{V}$ versus $\frac{1}{S}$ is remarkably simple:
it is a straight line!!
(the line $y = 0.5\,x + 0.5$)



c) Let's check whether the Lineweaver-Burk is as well a straight line at different values of $V_{\max}$ and K_m:

$$V = \frac{V_{\max} \cdot S}{K_m + S} \quad \Longrightarrow \quad \frac{1}{V} = \frac{K_m + S}{V_{\max} \cdot S} \quad \Longrightarrow \quad \boxed{\frac{1}{V} = \frac{K_m}{V_{\max}} \cdot \frac{1}{S} + \frac{1}{V_{\max}}}$$

and this is indeed a straight line with slope $\frac{K_m}{V_{\max}}$ and y-intercept (the piece cut off the y-axis by the line) $\frac{1}{V_{\max}}$. In practice the Lineweaver-Burk plot is more useful than the Michaelis-Menten plot if you are to determine the reaction constants $V_{\max}$ and K_m. You need only two (but preferably three or four) measurements to construct the Lineweaver-Burk line, determine its slope and y-intercept and use a piece of scrap paper to calculate $V_{\max}$ and K_m in no time.

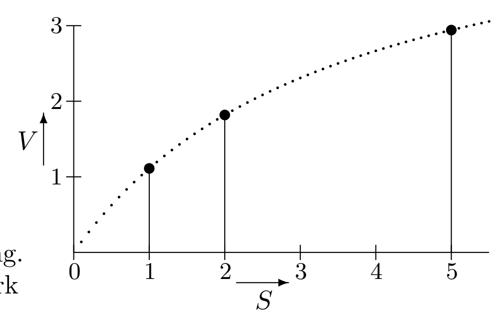
Exercise 38. Use $V = \dfrac{V_{\max} \cdot S}{K_m + S}$ and substitute the measurements:

$$\left.\begin{array}{lll} 2.8 = \dfrac{V_{\max} \cdot 20}{K_m + 20} & \Longrightarrow & 2.8\,K_m + 56 = 20\,V_{\max} \\ 4.0 = \dfrac{V_{\max} \cdot 40}{K_m + 40} & \Longrightarrow & 4.0\,K_m + 160 = 40\,V_{\max} \end{array}\right\} \Longrightarrow 4.0\,K_m + 160 = 2 \cdot (2.8\,K_m + 56) \Longrightarrow K_m = 30$$

and the rest is peanuts, you'll find $\boxed{K_m = 30 \text{ mM}}$ and $\boxed{V_{\max} = 7 \text{ mM/min}}$.

Exercise 39.

a) If you put the three measurements in a graph of V versus S, you get the vague impression that the points might lie on a Michaelis-Menten hyperbola:



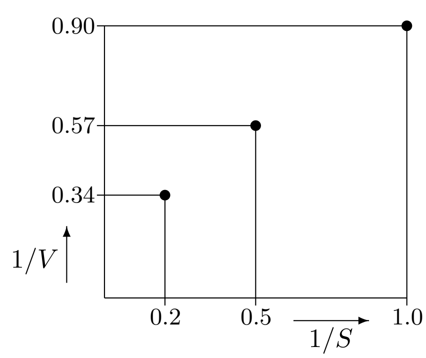
This, however, does not strike me as particularly convincing. Inspired by exercise 39b, I prefer using the Lineweaver-Burk plot, so I plot $\frac{1}{V}$ versus $\frac{1}{S}$:

S	V
1.0	1.11
2.0	1.76
5.0	2.94

$\Longrightarrow$

$\frac{1}{S}$	$\frac{1}{V}$
1.0	0.90
0.5	0.57
0.2	0.34

$\Longrightarrow$



Now, I can safely and happily conclude that these three points lie on a single line, which you can check with your ruler. This does convince me of the Michaelis-Menten character of this reaction.

b) I try to find the equation of this line. To good approximation it is the line through the points (0.2 , 0.34) and (1.0 , 0.90), which has

- slope $\frac{0.90-0.34}{1.0-0.2} = \frac{0.56}{0.8} = 0.7$
- hence equation $y = 0.7\,x + b \overset{(1,0.9)}{\Longrightarrow} y = 0.7\,x + 0.2$

Using exercise 37 I conclude: $\boxed{V_{\max} = 5}$ and $\boxed{K_m = 3.5}$.

Remark. In my calculations, I crudely took the line through two of three measurements. You can certainly do better after taking your Statistics course, where you will be taught how to find the best fitting line through multiple points.

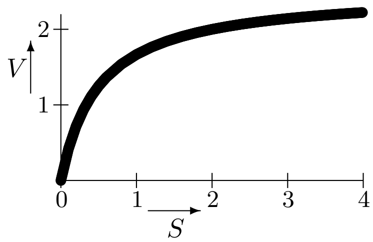
Exercise 40.

a) The given relation implies that $S \cdot (5 - 2V) = V$, so

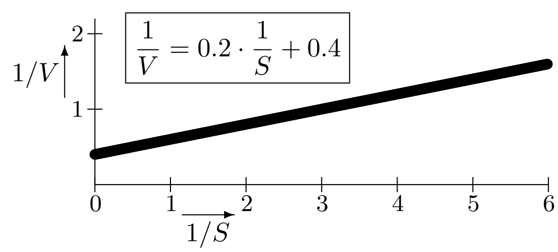
$$5S - 2VS = V \quad \Longrightarrow \quad V \cdot (1 + 2S) = 5S \quad \Longrightarrow \quad V = \frac{5S}{1+2S} \quad \Longrightarrow \quad \boxed{V = \frac{2.5\,S}{0.5 + S}}$$

and now you can see that $V_{\max} = 2.5$ and $K_m = 0.5$.

b)

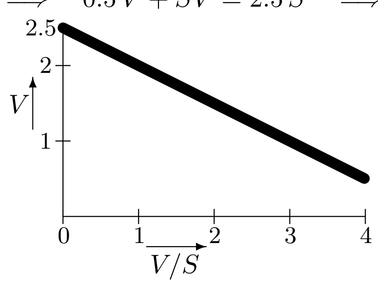


c)



d) Starting with the Michaelis-Menten equation I try to write V as a function of $\frac{V}{S}$:

$$V = \frac{2.5\,S}{0.5+S} \quad \Longrightarrow \quad 0.5\,V + SV = 2.5\,S \quad \overset{:S}{\Longrightarrow} \quad 0.5 \cdot \frac{V}{S} + V = 2.5 \quad \Longrightarrow \quad \boxed{V = 2.5 - 0.5 \cdot \frac{V}{S}}$$



If I put $\frac{V}{S}$ on the x-axis and V on the y-axis, it is the line $y = 2.5 - 0.5\,x$.

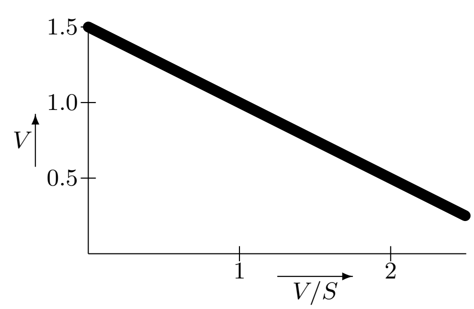
e) It is always a straight line: $V = \dfrac{V_{\max} \cdot S}{K_m + S} \quad \Longrightarrow \quad \boxed{V = V_{\max} - K_m \cdot \frac{V}{S}}$

f) K_m is minus the slope of this line and $V_{\max}$ is the height of the intersection with the y-axis.

g) If you perform experiments with $S = 1, 2, 3, 4$, then the obtained points will be nicely spread over the EH plot, which allows for accurate determination of the best EH line and hence $V_{\max}$ and K_m. In the LB plot, however, these points will cluster to the left of the plot.

Exercise 41. I have to write V as a function of V/S:

$$V = \frac{3S}{1+2S} \quad \Longrightarrow \quad V + 2VS = 3S$$
$$\Longrightarrow \quad \frac{V}{S} + 2V = 3$$
$$\Longrightarrow \quad \boxed{V = 1.5 - 0.5 \cdot \frac{V}{S}}$$



Exercise 42.

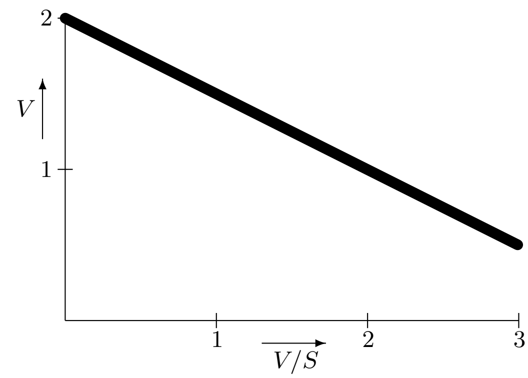
- The slope of the Eadie-Hofstee line is $\dfrac{0.9-1.8}{3.0-1.5} = -0.6$, so $\boxed{K_m = 0.6}$.
- The equation of the line is $y = -0.6\,x + 2.7$, so $\boxed{V_{\max} = 2.7}$.

Exercise 43. I express V in terms of V/S:

$$V = \frac{4S}{1+2S} \quad\Longrightarrow\quad V(1+2S) = 4S$$

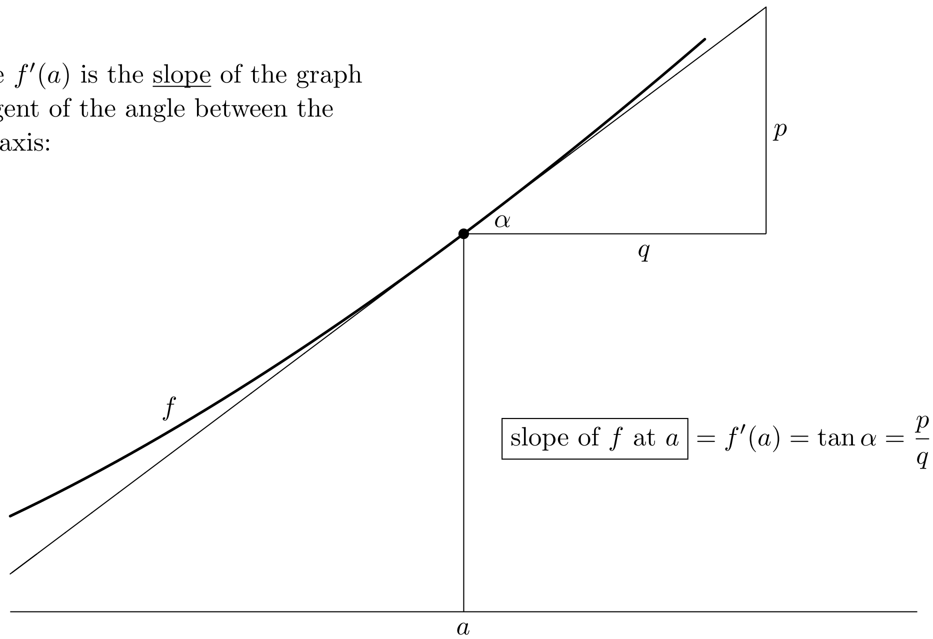
$$\Longrightarrow\quad V + 2VS = 4S \quad\Longrightarrow\quad \frac{V}{S} + 2V = 4$$

$$\Longrightarrow\quad \boxed{V = 2 - 0.5\cdot\frac{V}{S}}$$



6. Maxima and minima

Slope. The derivative $f'(a)$ is the slope of the graph of f at a. It is the tangent of the angle between the tangent line and the x-axis:



$$\boxed{\text{slope of } f \text{ at } a} = f'(a) = \tan\alpha = \frac{p}{q}$$

Increase and decrease. For a differentiable function f on a closed domain:

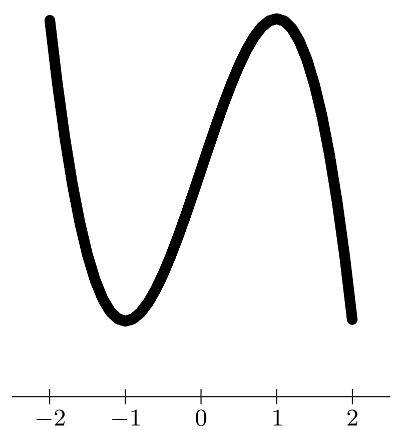
$$\begin{aligned} f'(x) > 0 \ \text{ for all } x &\implies f \text{ is increasing} \\ f'(x) < 0 \ \text{ for all } x &\implies f \text{ is decreasing} \\ f'(x) = 0 \ \text{ for all } x &\iff f \text{ is constant} \end{aligned}$$

Example 1. Investigate on what part of $\mathbb{R}$ the function $x \mapsto 3 + 3x - x^3$ is increasing.

Solution. The derivative of this function is

$$\frac{d}{dx}(3 + 3x - x^3) = 3 - 3x^2$$

which is positive if x is between -1 and 1, so the function is increasing from $x = -1$ to $x = 1$.



Example 2. Prove that all $x \in (-1, 1)$ satisfy: $\arcsin x = \arctan \dfrac{x}{\sqrt{1-x^2}}$.

Proof. Define the 'difference function' $f : (-1, 1) \to \mathbb{R}$ by

$$f(x) = \arcsin x - \arctan \frac{x}{\sqrt{1-x^2}}$$

The derivative of f is

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{1 + \frac{x^2}{1-x^2}} \cdot \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - (1-x^2) \cdot \frac{\frac{1}{\sqrt{1-x^2}}}{1-x^2} = 0$$

so f is a constant function. Finally, $f(0) = 0$ implies that f is the zero function.

Extreme values. The relation between the concepts of increasing/decreasing functions and derivative allows us to find the maximum and minimum values of functions easily. In order to find those points at which f assumes an extreme value we only need to investigate the following points:

(1) the boundaries of the domain of f

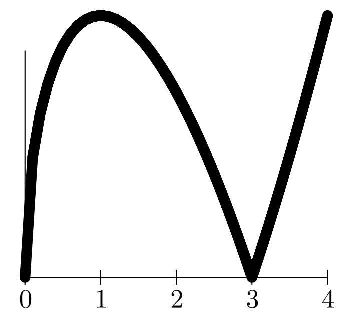
(2) the points at which f is not differentiable

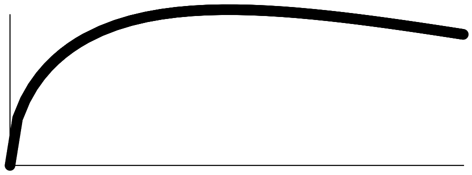
(3) the zeros of f'

Example 3. Find the extrema of the function $f : [0,4] \to \mathbb{R}$ with function rule

$$\boxed{f(x) = |3-x|\sqrt{3x}}$$

Solution. We investigate the boundary points of the domain (0 and 4), the points at which f is not differentiable (that's 0 and 3) and the zeros of f' (that's 1, which you can check for yourself if you don't believe me). Thus, we find maxima at $x = 1$ and $x = 4$ and minima at $x = 0$ and $x = 3$.





Example 4. Calculate the maximum of the function

$$\boxed{f(t) = \frac{\sqrt{t}}{2^t}}$$

Solution. I differentiate $f(t) = \sqrt{t} \cdot 2^{-t}$ using the product rule:

$$\frac{df}{dt} = \frac{1}{2\sqrt{t}} \cdot 2^{-t} + \sqrt{t} \cdot \left(-2^{-t}\ln 2\right)$$

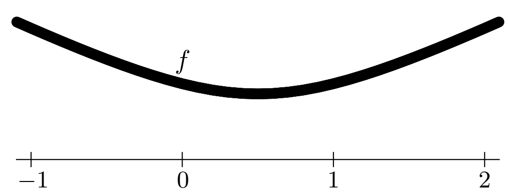
which equals zero if $t = \dfrac{1}{2\ln 2}$, so the maximum is found: $f\left(\dfrac{1}{2\ln 2}\right) = \boxed{\dfrac{1}{\sqrt{2e\ln 2}}}$.

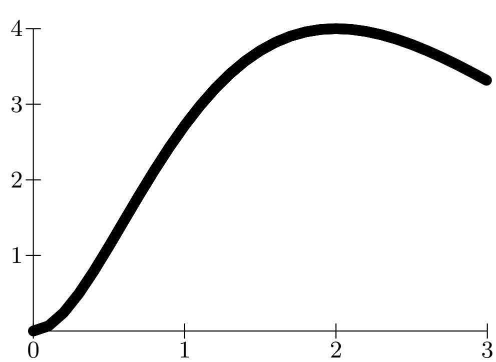
Example 5. Where is $f(x) = \sqrt{1 + x^2 - x}$ minimum?

Solution. The derivative is

$$\frac{df}{dx} = \frac{2x-1}{2\sqrt{1+x^2-x}} \quad \text{which is} \begin{cases} \text{negative} & \text{if } x < 0.5 \\ \text{zero} & \text{if } x = 0.5 \\ \text{positive} & \text{if } x > 0.5 \end{cases}$$

so f is minimum at $x = 0.5$.





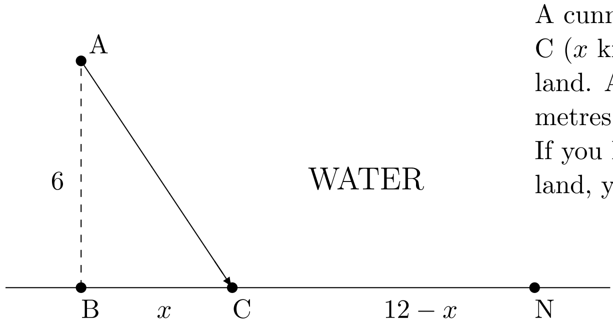
Example 6. Calculate the maximum of x^2e^{2-x} for $0 < x < 3$.

Solution. I take the derivative using the product rule:

$$\frac{d}{dx}x^2e^{2-x} = 2xe^{2-x} - x^2e^{2-x}$$

which is zero if $x = 2$, so the maximum is $\boxed{4}$.

Example 7. Ornithologists have discovered that certain birds avoid flying over large areas of water; they prefer to take a detour over land. This has probably nothing to do with their fear of falling in the water. Instead, flying over water requires significantly more energy, because air rises over land but falls over water. Let's assume that flying over water requires 25% more energy than flying over land. Suppose you are a birdie that wants to fly from island A (6 kilometres away from the nearest point B at the coast) to your nest N (located 12 kilometres away from B along the coastline), thereby minimising the amount of energy required for your flight. How should you do this?



A cunning plan: fly along a straight line over water to point C (x kilometres away from B) and continue your flight over land. According to Pythagoras you've flown $\sqrt{36+x^2}$ kilometres over water and $12-x$ kilometres over land.
If you let k be your energy consumption per kilometre over land, your total energy consumption is given by

$$E(x) = \frac{5}{4}k\sqrt{36+x^2} + k(12-x)$$

Now, you should find an x such that $E(x)$ is minimum. Calculate the derivative:

$$E'(x) = \frac{5}{4}k \cdot \frac{x}{\sqrt{36+x^2}} - k\,, \text{ which is } \begin{cases} \text{negative} & \text{if } x < 8 \\ \text{zero} & \text{if } x = 8 \\ \text{positive} & \text{if } x > 8 \end{cases}$$

Hence, you fly from the island to point C located at eight kilometres from B and then straight to your nest.

Example 8. The water level $W(t)$ at a certain location at sea exhibits a periodic behaviour according to the formula

$$W(t) = \frac{10}{4+\cos t}$$

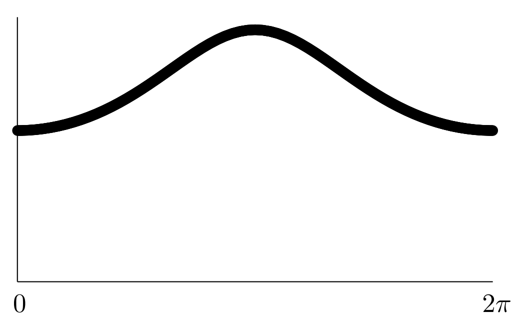
a) Sketch the graph of W during one period, for example on the domain $[0, 2\pi]$.

b) At what time $t \in [0, 2\pi]$ is the water level highest?

c) At what time $t \in [0, 2\pi]$ is the water level increasing fastest?

Solution.

a)



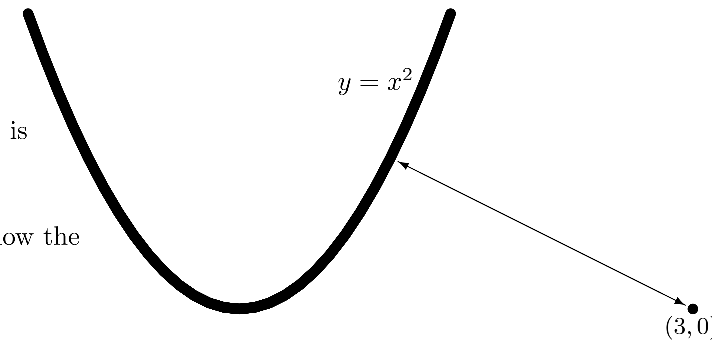
b) $W(t)$ is maximum if the denominator is minimum, hence if $\cos t = -1$, so $t = \pi$.

c) The question is: when is $W'(t)$ maximum?
The easiest way to find this out is by setting the derivative of $W'(t)$ equal to zero.
This remains quite a hell of a job, but of course ambitious students won't withdraw from this:

$$W''(t) = \frac{d}{dt}\left(\frac{10\sin t}{(4+\cos t)^2}\right) = -10\frac{\cos^2 t - 4\cos t - 2}{(4+\cos t)^3}$$

which equals zero if $\cos^2 t - 4\cos t - 2 = 0$. For such a quadratic equation in $\cos t$ you might have a tool like the quadratic formula in your toolbox. If you don't, you'll as well be fine by completing a square. The result: $\cos t = 2 - \sqrt{6}$. By the way, it might be possible that your quadratic formula yielded $\cos t = 2 \pm \sqrt{6}$, but this plus sign is obviously madness. From the graph you can see that the desired time is somewhere between 0 and π, so it must be $t = \arccos(2-\sqrt{6})$, which is approximately 2.037.

Example 9. Calculate the distance from the point $(3,0)$ to the parabola $y = x^2$.



Solution. The distance from $(3,0)$ to (x, x^2) is

$$\|(x, x^2) - (3,0)\| = \sqrt{(x-3)^2 + x^4}$$

This distance has a minimum when the lot below the square root sign is minimum, so I differentiate

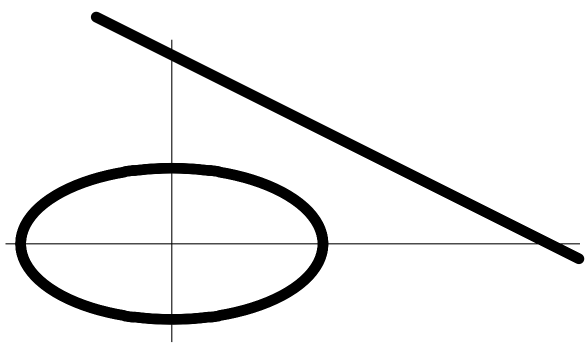
$$\operatorname{lot}(x) \stackrel{\text{def}}{=} (x-3)^2 + x^4$$

and investigate the usual way when this lot is minimum:

$$\frac{d\operatorname{lot}}{dx} = 0 \quad\Longrightarrow\quad 2x - 6 + 4x^3 = 0 \quad\Longrightarrow\quad 2x^3 + x - 3 = 0 \quad\Longrightarrow\quad (x-1)(2x^2+2x+3) = 0 \quad\Longrightarrow\quad x = 1$$

so the point closest to the parabola is $(1,1)$ and the desired distance is

$$\|(3,0) - (1,1)\| = \|(2,-1)\| = \boxed{\sqrt{5}}$$



Example 10. What point on the ellipse $x^2 + 4y^2 = 4$ is closest to the line $x + 2y = 5$?

Solution. This point lies somewhere on the upper right part of the ellipse, where

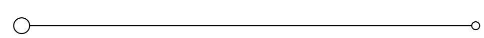
$$y = \frac{1}{2}\sqrt{4 - x^2}$$

Evidently, the tangent line to the ellipse at the desired point is parallel to the line $x + 2y = 5$, so it has slope $-\frac{1}{2}$:

$$\frac{dy}{dx} = \frac{-x}{2\sqrt{4-x^2}} = -\frac{1}{2} \quad\Longrightarrow\quad x = \sqrt{4-x^2} \quad\Longrightarrow\quad x^2 = 4 - x^2 \quad\Longrightarrow\quad x^2 = 2 \quad\Longrightarrow\quad x = \sqrt{2}$$

Thus, the point closest to the ellipse is $\left(\sqrt{2}\,,\, \dfrac{1}{\sqrt{2}}\right)$.

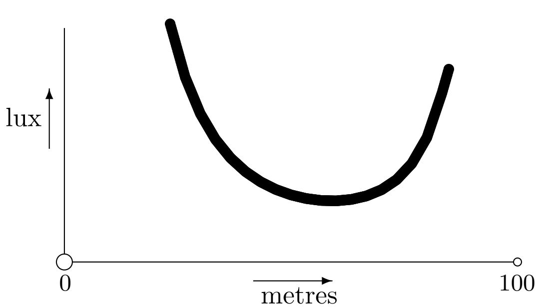
Example 11. (units: metres, candelas, lux)
A 100 metre long dark alley is only lit by a large lamp (300 candelas) at the beginning and a somewhat smaller lamp (100 candelas) at the end of the alley.



In order to secretly commit a crime I try to find the darkest spot in the alley. Where is that?

Solution. The illuminance by a point source of light is inversely proportional to the square of the distance. This implies that after x metres, neglecting a multiplicative constant,

$$\boxed{\operatorname{lux}(x) = \frac{3}{x^2} + \frac{1}{(100-x)^2}}$$



I try to find the spot where this is minimum by setting the derivative equal to zero:

$$\frac{d\operatorname{lux}}{dx} = \frac{-6}{x^3} + \frac{2}{(100-x)^3} = 0 \quad\Longrightarrow\quad x^3 = 3(100-x)^3$$

$$\Longrightarrow\quad x = \sqrt[3]{3}\,(100-x) \quad\Longrightarrow\quad \boxed{x = \frac{100\sqrt[3]{3}}{1+\sqrt[3]{3}}}$$

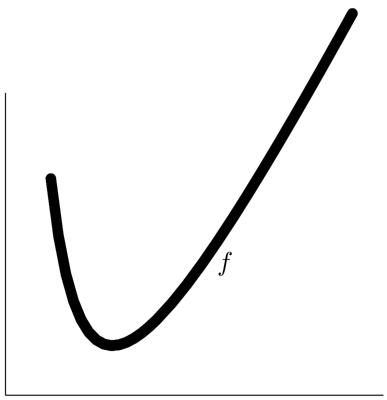
(my calculator recommends $x \approx 59$ for the crime scene)

Exercises chapter 6

Exercise 1. (units: days, bodies)
On its first day, a cholera epidemic took 90 victims, and on its second day there were 162 more:

$$\boxed{\text{dead}\,(1) = 90} \qquad \boxed{\text{dead}\,(2) = 162}$$

Assume that the number of deaths on day t satisfies the well-known epidemic model $\boxed{\text{dead} = \alpha\, t\, \beta^t}$ and predict when the climax of the epidemic occurs.



Exercise 2. Calculate the minimum on the domain $(0, \infty)$ of

$$\boxed{f(x) = \frac{2x^2 - 5x + 4}{x}}$$

Exercise 3. (units: metres, seconds)
John starts at time $t = 0$. His distance travelled after t seconds is given in metres by

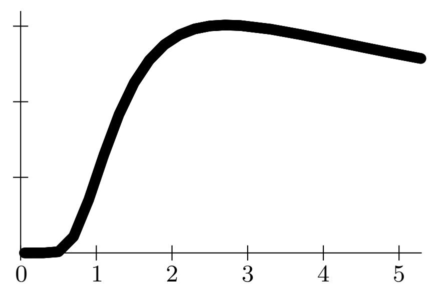
$$\boxed{s(t) = t \arctan t}$$

a) What was his velocity after t seconds?

b) What was his acceleration after t seconds?

Exercise 4. Find the maximum on the domain $(0, \infty)$ of the function

$$\boxed{f(x) = x^{\frac{3}{x}}}$$

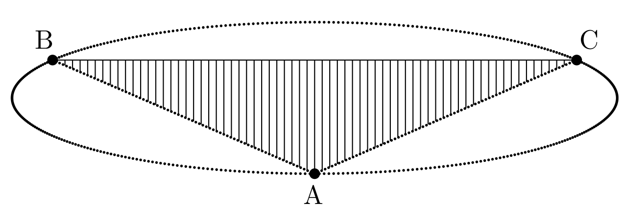


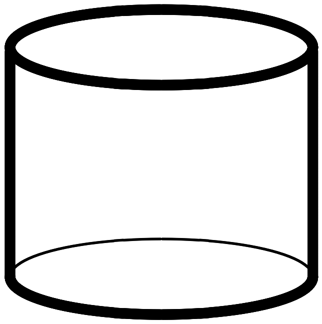
Exercise 5.

a) Determine the derivative of $\arctan \dfrac{1+x}{1-x}$.

b) Prove for $x < 1$ the formula $\boxed{\arctan \dfrac{1+x}{1-x} = \dfrac{\pi}{4} + \arctan x}$.

Exercise 6. The adjacent dotted ellipse has equation $x^2 + 16y^2 = 16$, the points B and C lie on height t above x-axis, A is the point $(0, -1)$. For which t is the area of the triangle ABC maximum?





Exercise 7. I'm the manager of a factory producing cans of cat food with a volume of precisely 330 cm^3. I'd like to choose the dimensions such that the total tin area (top plus bottom plus shell) is minimum. That will spare the environment (tin waste) and my bank account (material costs). Calculate this minimum area (but be sure to keep the can cylindrical or the cats won't like its content).

Exercise 8. The males of the frog species *Eleutherodactylus coqui* have the task to defend their eggs. They do this during a fraction τ of the time (the so-called *defence fraction*). If they spent all time defending their eggs ($\tau = 1$), there wouldn't be time left to find new reproduction partners. If they neglect their duty altogether, however, their eggs wouldn't have any chance of survival. Therefore, the size $F(\tau)$ of their offspring per unit of time depends in a rather complicated way on the defence fraction τ:

$$F(\tau) = \frac{g(\tau)}{3 + 5\tau}$$

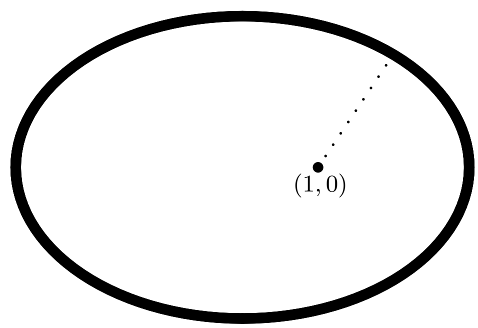
with $g(\tau)$ the chance of survival for their eggs at a defence fraction τ.

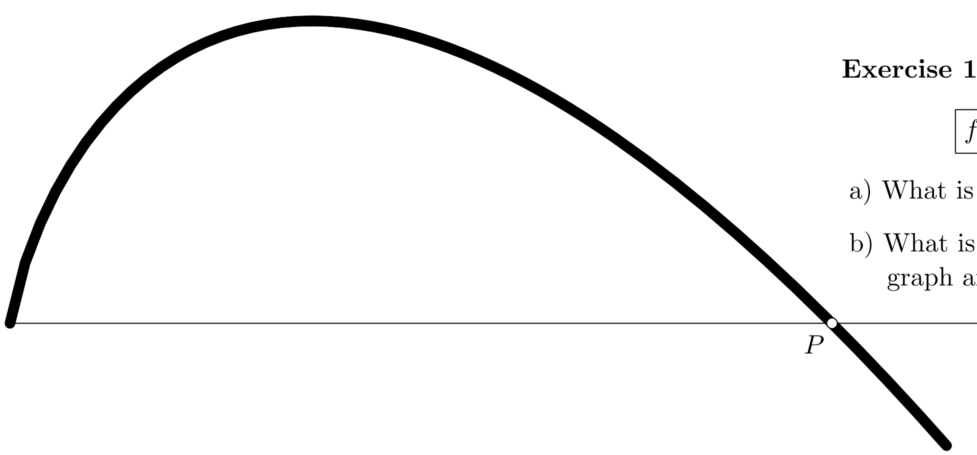
a) Express $\dfrac{dF(\tau)}{d\tau}$ in terms of the function g and (if necessary) its derivative.

b) What defence fraction τ would you recommend to a frog male in order to maximise its offspring in the case that g is the function $\tau \mapsto \sqrt{\tau}$?

Exercise 9. Calculate the distance from $(1, 0)$ to the ellipse

$$\boxed{x^2 + 2y^2 = 8}$$





Exercise 10. This is the graph of

$$\boxed{f(x) = x - x \ln x}$$

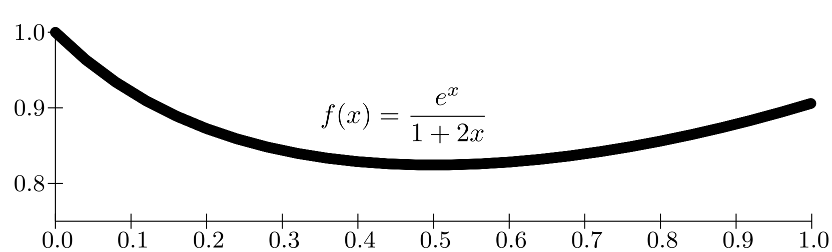
a) What is the maximum of f?

b) What is the angle between the graph and the x-axis at P?

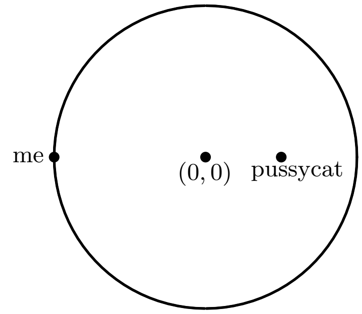
Exercise 11. Find the minimum value of the function

$$\boxed{f(x) = \frac{e^x}{1 + 2x}}$$

on the domain $0 \leq x \leq 1$.

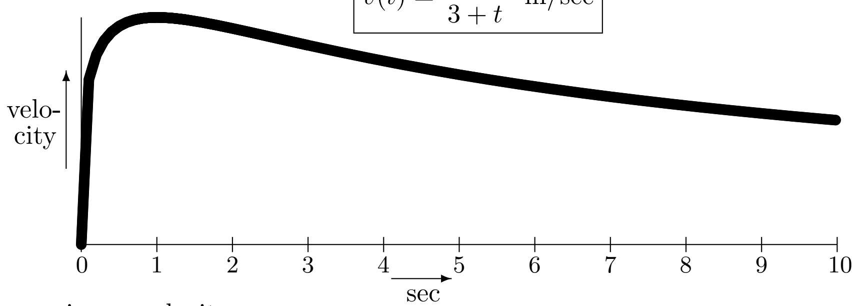


Exercise 12. A circularly shaped pool has radius 1 and centre $(0, 0)$. I'm standing on the edge of the pool, at the point $(-1, 0)$, and want to go as quickly as possible to the drowning pussycat at $(\frac{1}{2}, 0)$. I can walk a piece along the edge, then jump into the water and swim towards the floundering darling. My walking speed is twice as large as my swimming speed. Where should I jump into the water to reach the pussycat as quickly as possible?



Exercise 13. (units: metres, seconds)
I sprint for ten seconds with my velocity after t seconds given by

$$\boxed{v(t) = \frac{12\sqrt[4]{t}}{3+t} \quad \text{m/sec}}$$



Calculate my maximum velocity.

Exercise 14. Racing cyclists have to climb from a height of approximately 222 metres to 936 metres. The profile of the mountain (unit of length: km) is the graph of

$$\boxed{f(x) = \frac{2+x^2}{9+x^2}} \quad (0 \leq x \leq 10)$$

a) Calculate the slope at $x = 1$.

b) Calculate the slope at $x = 2$.

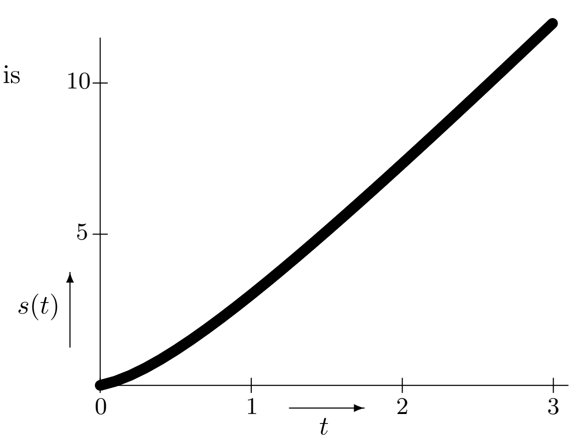
c) If you watch the mountain profile closely, you might suspect that there is a point somewhere between $x = 1$ and $x = 2$ where the slope is maximum. What point is this?

Exercise 15. (units: km, hours)
Suppose the travelled distance after t hours of walking is

$$\boxed{s(t) = \frac{5t^2+t}{1+t}}$$

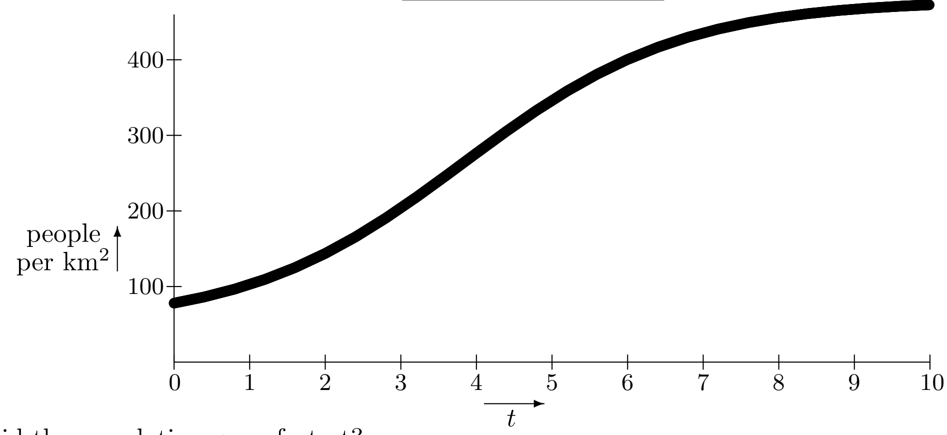
a) Calculate the velocity at time $t = 1$.

b) Calculate the acceleration at time $t = 1$.



Exercise 16. (units: 20 years, people per km^2)
The population density in the Netherlands is given from $t = 0$ (which is the year 1870) until $t = 10$ (the year 2070) by

$$\boxed{N = 50 + \frac{30}{0.07 + 2^{-t}}}$$



When did the population grow fastest?

Exercise 17.

a) Sketch the graph of the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{4}{\sqrt{x}}$$

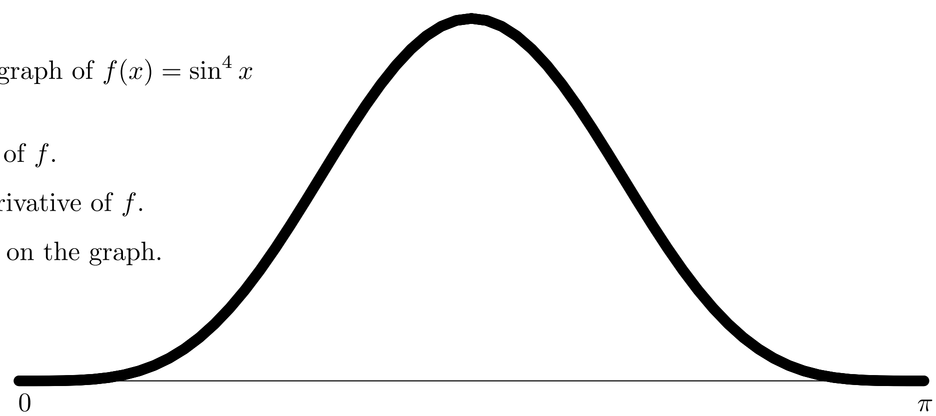
b) Which point on this graph is closest to the origin?

Exercise 18. This is the graph of $f(x) = \sin^4 x$ on the domain $(0, \pi)$.

a) Calculate the derivative of f.

b) Calculate the second derivative of f.

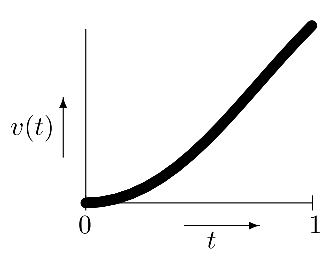
c) Find the steepest points on the graph.



Exercise 19. A beetle is creeping along the x-axis. Its position $x(t)$ as a function of time t is given by

$$x(t) = \sqrt{1 + t^2}$$

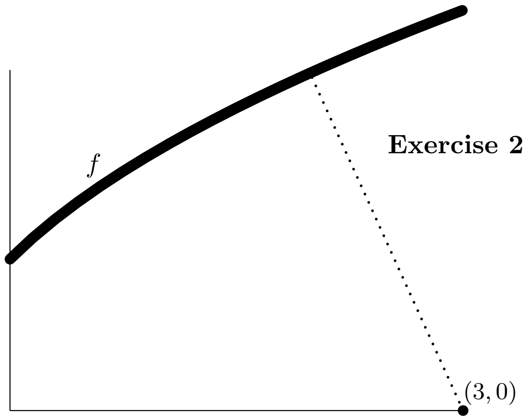
Calculate its velocity $\dot{x}$ and its acceleration $\ddot{x}$.



Exercise 20. (units: metres, sec)
Starting at rest I take a sprint of precisely one second, from $t = 0$ until $t = 1$. My velocity during this second is

$$\boxed{v(t) = \arctan(t^2)}$$

When is my acceleration maximum?



Exercise 21. Calculate the distance from $(3,0)$ to the graph of

$$\boxed{f(x) = \sqrt{1+2x}}$$

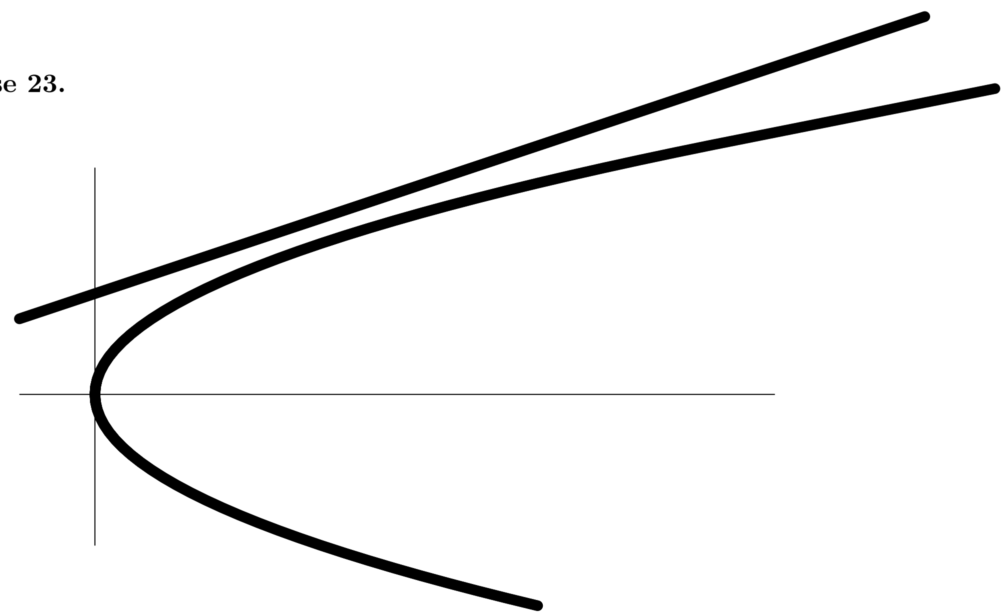
Exercise 22. (units: metres, seconds)
An ostrich walks along the real axis. Its position at time t is given (from $t = 0$) by

$$\boxed{x(t) = 3\ln(t^2+9)}$$

a) Calculate its speed $\dot{x}$.

b) Calculate its maximum speed.

Exercise 23.



The line and parabola drawn up here have equations

$$\boxed{x + 4 = 3y} \qquad \boxed{4x = 3y^2}$$

From my drawing you can clearly see that the line and parabola never intersect, but... maybe you're hesitant to trust my drawing skills, so a couple of questions for you:

a) Prove that there is no intersection point.

b) Which point on the parabola is closest to the line?

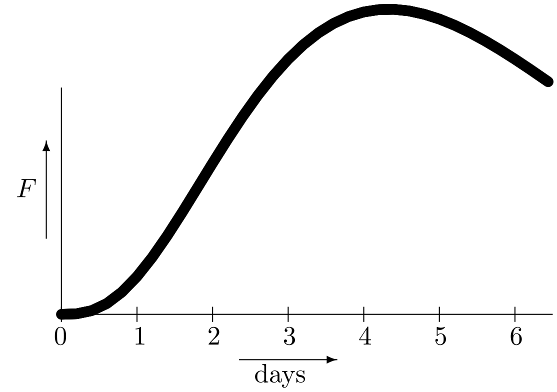
c) What is the distance between this point and the line?

Exercise 24. (units: days, thousand fleas)
A flea plague F behaves from $t = 0$ according to

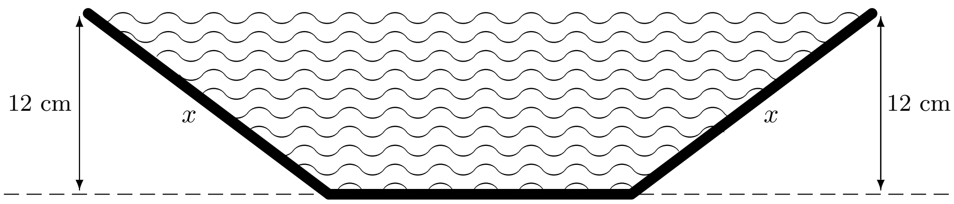
$$\boxed{F = \frac{t^3}{2^t}}$$

a) Calculate $\dfrac{dF}{dt}$.

b) When was the number of fleas largest?



Exercise 25. I'm going to fold a symmetric roof gutter from a 50 cm wide rectangular metal plate. The height of the gutter is required to be 12 cm, and both diagonal parts are x cm in length, see the cross-section below. The capacity of the gutter is the area of that part of its cross-section that can contain water.



a) Express the capacity C of the gutter in terms of x.

b) I want to fold the plate such that the gutter has maximum capacity. Find the value of x for which the capacity of the gutter is maximum.

c) What is the maximum capacity of the gutter?

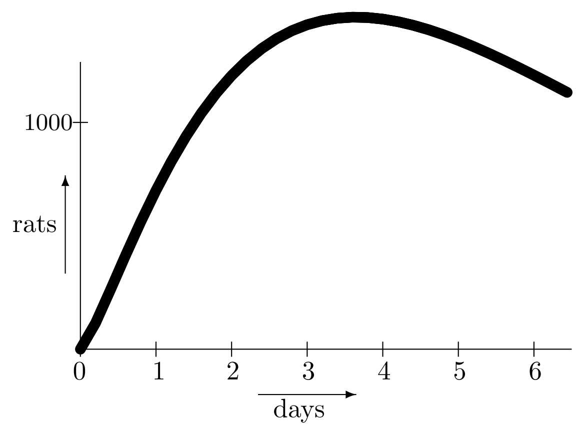
Exercise 26. (units: days, rats)
A rat plague behaves from $t = 0$ according to

$$\boxed{\text{rats}\,(t) = 1000 \cdot t^{1.3} \cdot 0.7^t}$$

a) When is the number of rats largest?

b) When does the number of rats increase fastest?

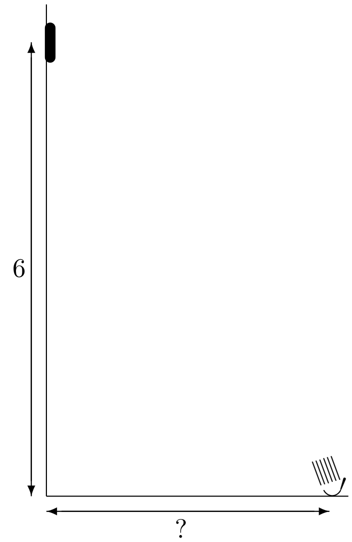
c) When does the number of rats decrease fastest?



Exercise 27. Supermarket chain Massmart sells top-up cards for prepaid mobile phones (which goes well) and telephone cards for phone boxes (which is declining business). The weekly sales (at all locations together) satisfy the following model:

$$\boxed{A_{\text{tuc}} = 800 \cdot 1.015^t} \qquad \boxed{A_{\text{box}} = 5500 \cdot 1.012^{-t}} \qquad \boxed{A_{\text{tot}} = A_{\text{tuc}} + A_{\text{box}}}$$

(t is the time in weeks from 1 January 1996, A_{tuc} and A_{box} are the weekly sales in euros of top-up cards and telephone cards, all figures have been converted into euros)
When (year and month) are the total sales A_{tot} minimum?

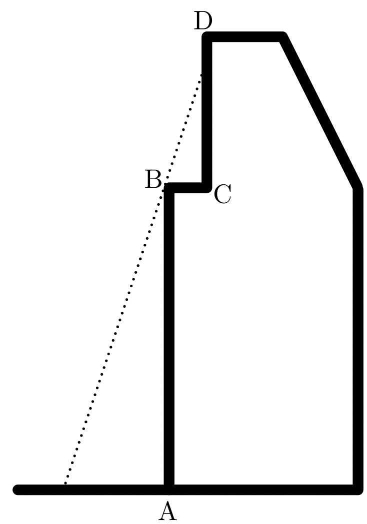


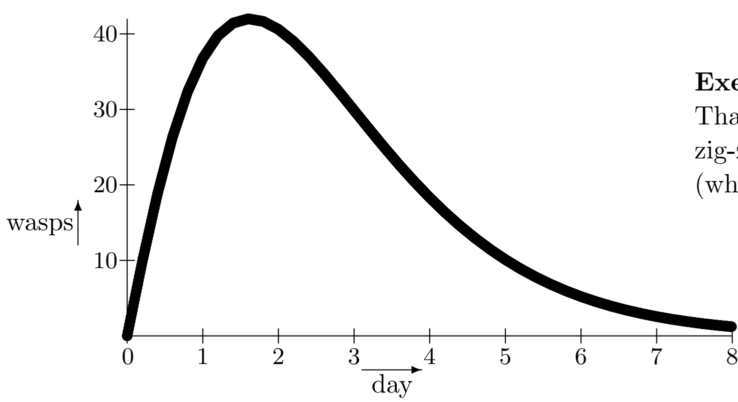
Exercise 28. The façade of my mansion is decorated by a painting at 6 m height that presents a purring pussycat. Using a spotlight on the ground I wish to illuminate the pussycat painting as brightly as possible. At what position should I mount the spotlight?

Exercise 29. The relevant dimensions of this tall house are:

AB	=	8 metres
BC	=	1 metres
CD	=	4 metres

I'd like to put a ladder against wall CD. Obviously, this ladder must pass across the weird protruding part at B. What is the minimum length of the ladder?





Exercise 30. (units: months, wasps)
Thanks to wasp control the average number of zig-zag elm sawflies per elm tree since $t = 0$ (which was 1 April 2015) has been given by

$$\text{wasp}(t) = 50 \cdot \frac{t + t^2}{e^t}$$

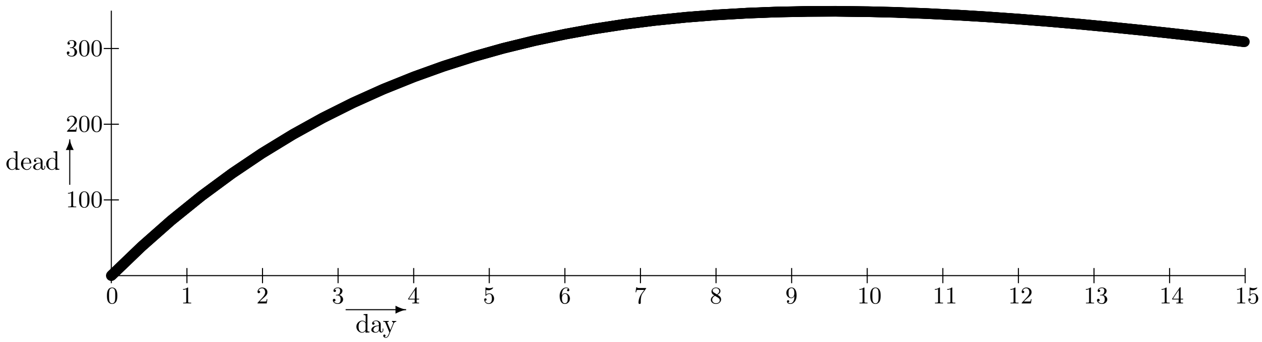
When did the plague decrease fastest?

Solutions chapter 6

Exercise 1. I calculate the epidemic coefficients α and β first:

$$\left.\begin{array}{lll} \text{dead}\,(1) = 90 & \Longrightarrow & \alpha\beta = 90 \\ \text{dead}\,(2) = 162 & \Longrightarrow & 2\alpha\beta^2 = 162 \end{array}\right\} \overset{\text{divide}}{\Longrightarrow} \quad 2\beta = \frac{162}{90} \quad \Longrightarrow \quad \beta = 0.9 \quad \Longrightarrow \quad \alpha = 100$$

so the number of deaths on day t is $\boxed{\text{dead}\,(t) = 100\,t \cdot 0.9^t}$.



By the product rule, the derivative of this function is

$$\frac{d\,\text{dead}}{dt} = 100 \cdot 0.9^t + 100t \cdot 0.9^t \cdot \ln 0.9$$

which equals zero if $t \cdot \ln 0.9 = -1$, so $\boxed{t = \dfrac{-1}{\ln 0.9}}$ (after approximately 9.5 days).

Exercise 2. I differentiate f using the quotient rule and investigate the behaviour of the derivative sign:

$$f'(x) = \frac{2x^2 - 4}{x^2} \quad \Longrightarrow \quad f'(x) \text{ is } \begin{cases} \text{negative} & \text{if } x < \sqrt{2} \\ \text{zero} & \text{if } x = \sqrt{2} \\ \text{positive} & \text{if } x > \sqrt{2} \end{cases}$$

Thus, $f(x)$ is minimum when $x = \sqrt{2}$, and its minimum is $f(\sqrt{2}) = 4\sqrt{2} - 5$.

Exercise 3.

a) $v(t) = s'(t) = \dfrac{t}{1+t^2} + \arctan t$ m/sec

b) $a(t) = v'(t) = \dfrac{2}{(1+t^2)^2}$ m/sec^2

Exercise 4. The derivative of this function is

$$f'(x) = \frac{d}{dx}\, x^{\frac{3}{x}} = \frac{d}{dx}\, e^{\frac{3}{x}\cdot\ln x} = e^{\frac{3}{x}\cdot\ln x} \cdot \left(\frac{-3}{x^2}\ln x + \frac{3}{x^2}\right) = x^{\frac{3}{x}} \cdot \frac{3}{x^2} \cdot (1 - \ln x)$$

Let's check where this equals zero:

$$f'(x) = 0 \quad \Longleftrightarrow \quad \ln x = 1 \quad \Longleftrightarrow \quad x = e$$

so the maximum is $f(e) = e^{\frac{3}{e}}$.

Exercise 5.

a) $\dfrac{d}{dx} \arctan \dfrac{1+x}{1-x} = \dfrac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \cdot \dfrac{2}{(1-x)^2} = \dfrac{2}{2+2x^2} = \dfrac{1}{1+x^2}$

b) This derivative is equal to the derivative of $\arctan x$, so these two functions differ only a constant on their common domain $(-\infty, 0)$. By substituting $x = 0$ you discover that this difference constant is $\frac{\pi}{4}$.

Exercise 6. Calculate the area of the triangle:

$$\left.\begin{aligned} \text{height} &= t+1 \\ \text{base BC} &= 8\sqrt{1-t^2} \end{aligned}\right\} \implies \text{area} = f(t) = 4(t+1)\sqrt{1-t^2}$$

This area is maximum if $f'(t) = 0$, which (after some simple calculations) amounts to $t = \dfrac{1}{2}$.

Exercise 7. If the radius is r and the height is h, then the volume is $\pi r^2 h$. This should equal 330, so

$$h = \frac{330}{\pi r^2}$$

Therefore, the total tin area $f(r)$ is

$$f(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{660}{r}$$

I am to find the minimum of this function, so I investigate where its derivative equals zero:

$$f'(r) = 4\pi r - \frac{660}{r^2} = 0 \quad \implies \quad r = \sqrt[3]{\frac{165}{\pi}}$$

The behaviour of the sign of f' shows that this is indeed a minimum, so the minimum area is

$$f\left(\sqrt[3]{\frac{165}{\pi}}\right) = 2\pi\left(\frac{165}{\pi}\right)^{\frac{2}{3}} + 660\left(\frac{165}{\pi}\right)^{-\frac{1}{3}} \approx 264 \text{ cm}^2$$

Exercise 8.

a) $F'(\tau) = \dfrac{g'(\tau)\cdot(3+5\tau) - 5g(\tau)}{(3+5\tau)^2}$

b) If $g(\tau) = \sqrt{\tau}$, then $F'(\tau) = \dfrac{\frac{3+5\tau}{2\sqrt{\tau}} - 5\sqrt{\tau}}{(3+5\tau)^2}$ which is $\begin{cases} \text{positive} & \text{if } \tau < \frac{3}{5} \\ \text{zero} & \text{if } \tau = \frac{3}{5} \\ \text{negative} & \text{if } \tau > \frac{3}{5} \end{cases}$

A good recommendation to the frog male: $\boxed{\text{defend your eggs 60\% of your time}}$.

Exercise 9. The distance $f(x)$ from $(1,0)$ to the point $\left(x\,,\ \sqrt{4-\dfrac{x^2}{2}}\right)$ on the ellipse is

$$f(x) = \left\| (1,0) - \left(x\,,\ \sqrt{4-\frac{x^2}{2}}\right)\right\| = \sqrt{(1-x)^2 + 4 - \frac{x^2}{2}} = \sqrt{\frac{x^2}{2} - 2x + 5}$$

We are asked to find the minimum value of f. You probably found this by differentiation, but I prefer completing a square:

$$f(x) = \sqrt{\frac{1}{2}(x-2)^2 + 3} \quad \implies \quad \text{the minimum of } f \text{ is } \boxed{\sqrt{3}}$$

Exercise 10.

a) The derivative of f is $-\ln x$, which is

$$\begin{cases} \text{positive} & \text{if } 0 < x < 1 \\ \text{zero} & \text{if } x = 1 \\ \text{negative} & \text{if } x > 1 \end{cases}$$

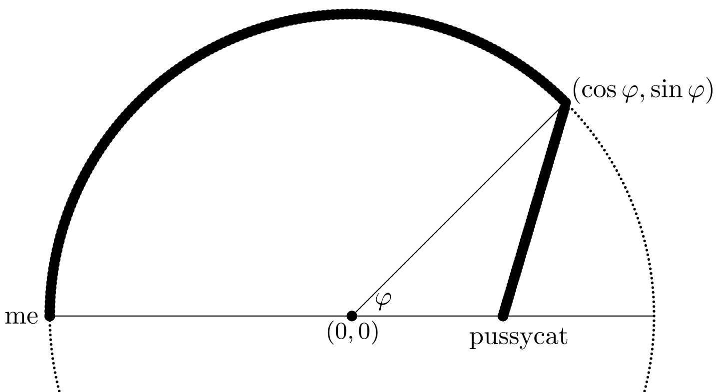
so $f(x)$ is maximum if $x = 1$, and the maximum value of f is $f(1) = 1$.

b) P is the intersection point of the graph and the x-axis. After some algebra, I found the x-coordinate of P to be $x = e$, and the derivative at that point is $-\ln e = -1$, so the angle with the x-axis is 45°.

Exercise 11. I try to find the zero of the derivative of f:

$$f'(x) = 0 \quad \Longrightarrow \quad \frac{e^x(1+2x) - 2e^x}{(1+2x)^2} = 0 \quad \Longrightarrow \quad e^x(2x-1) = 0 \quad \Longrightarrow \quad x = \frac{1}{2}$$

so the minimum of f is $f(0.5) = \boxed{\frac{\sqrt{e}}{2}}$.



Exercise 12. If I jump into the water at $(\cos\varphi, \sin\varphi)$, the required walking time is $\frac{1}{s}(\pi - \varphi)$ (with s my walking speed), and the required swimming time

$$\tfrac{2}{s}\sqrt{(\tfrac{1}{2} - \cos\varphi)^2 + \sin^2\varphi} = \tfrac{1}{s}\sqrt{5 - 4\cos\varphi}$$

Thus, we must find the minimum of the function $f : [0, \pi] \to \mathbb{R}$ given by

$$\boxed{f(\varphi) = \pi - \varphi + \sqrt{5 - 4\cos\varphi}}$$

Let's first calculate f':

$$f'(\varphi) = -1 + \frac{2\sin\varphi}{\sqrt{5 - 4\cos\varphi}}$$

Now, let's find out where this equals zero:

$$2\sin\varphi = \sqrt{5 - 4\cos\varphi} \quad \Longrightarrow \quad 4\cos^2\varphi - 4\cos\varphi + 1 = 0 \quad \Longrightarrow \quad \cos\varphi = \frac{1}{2} \quad \Longrightarrow \quad \varphi = \frac{\pi}{3}$$

Can I conclude that I should jump into the water at $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$? Of course not, because

$$f'(\varphi) \text{ is } \begin{cases} \text{negative} & \text{if } 0 \le \varphi < \frac{\pi}{3} \\ \text{zero} & \text{if } \varphi = \frac{\pi}{3} \\ \text{negative} & \text{if } \frac{\pi}{3} < \varphi \le \pi \end{cases}$$

Conclusion: f is a decreasing function, so it has a minimum at $\varphi = \pi$. I immediately jump into the water at $(-1, 0)$.

Remark. This was a terribly hard problem, and my only hope is that the little pussycat did not drown during our calculations. Evidently, the most difficult part was the smart choice of the jump angle φ as variable. In some cases you can solve practical problems only by translating them into mathematics via smart choices.

Exercise 13. Calculate when $v'(t) = 0$:

$$v'(t) = 0 \quad \Longrightarrow \quad \frac{3t^{-\frac{3}{4}}(3+t) - 12t^{\frac{1}{4}}}{(3+t)^2} = 0 \quad \Longrightarrow \quad 3t^{-\frac{3}{4}}(3+t) = 12t^{\frac{1}{4}} \quad \Longrightarrow \quad 3 + t = 4t \quad \Longrightarrow \quad t = 1$$

so I reach my maximum velocity, which is $v(1) = 3$ m/sec, already after a single second.

Exercise 14. Using the quotient rule I calculate the derivative of the mountain function:

$$f(x) = \frac{2 + x^2}{9 + x^2} \quad \Longrightarrow \quad f'(x) = \frac{14x}{(9 + x^2)^2}$$

a) The slope at $x = 1$ is $f'(1) = \frac{14}{100} = 0.14$.

b) The slope at $x = 2$ is $f'(2) = \frac{28}{169} \approx 0.17$.

c) At the desired point the derivative of the slope should equal zero. The derivative of the slope at x is

$$f''(x) = \frac{126 - 42x^2}{(9 + x^2)^3}$$

which equals zero if $126 = 42x^2$, amounting to $x = \sqrt{3}$.

Exercise 15.

a) Calculation of the velocity:

$$v(t) = s'(t) = \frac{5t^2 + 10t + 1}{(1+t)^2} \quad \Longrightarrow \quad v(1) = 4 \text{ km/u}$$

b) Calculation of the acceleration:

$$a(t) = v'(t) = \frac{8}{(1+t)^3} \quad \Longrightarrow \quad a(1) = 1 \text{ km/u}^2$$

Exercise 16. The population density growth is

$$\frac{dN}{dt} = \frac{d}{dt} 30 \left(0.07 + 2^{-t}\right)^{-1} = -30 \left(0.07 + 2^{-t}\right)^{-2} \cdot \left(-2^{-t} \ln 2\right) = 30 \ln 2 \cdot 2^{-t} \cdot \left(0.07 + 2^{-t}\right)^{-2}$$

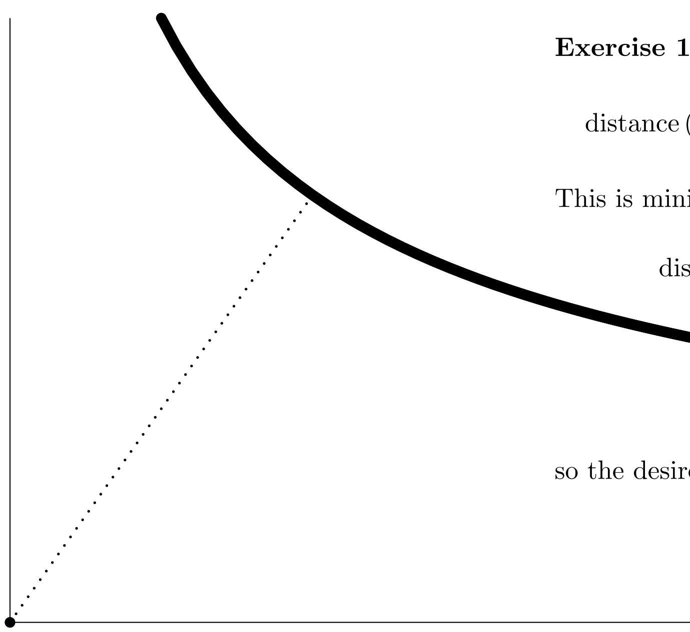
I find out when this is maximum by differentiating again:

$$\frac{d^2N}{dt^2} = 30 \ln 2 \cdot \left(-2^{-t} \ln 2 \cdot \left(0.07 + 2^{-t}\right)^{-2} + 2^{-t} \cdot \left(-2 \left(0.07 + 2^{-t}\right)^{-3} \cdot \left(-2^{-t} \ln 2\right)\right)\right)$$

At the desired time this should equal zero, yielding an equation that you can easily solve by first dividing by $30 \left(\ln 2\right)^2 \cdot 2^{-t}$ and then multiplying by $\left(0.07 + 2^{-t}\right)^3$:

$$-\left(0.07 + 2^{-t}\right) + 2 \cdot 2^{-t} = 0 \quad \Longrightarrow \quad 2^{-t} = 0.07 \quad \overset{\text{ln-trick}}{\Longrightarrow} \quad \boxed{t = -\frac{\ln 0.07}{\ln 2}}$$

That's approximately 3.8365, which corresponds to 23 September 1946.



Exercise 17. The distance from $(x, f(x))$ to the origin is

$$\text{distance}\,(x) = \|(x, f(x))\| = \sqrt{x^2 + (f(x))^2} = \sqrt{x^2 + \frac{16}{x}}$$

This is minimum if its derivative vanishes:

$$\begin{aligned} \text{distance}'(x) = 0 \quad &\Longrightarrow \quad \frac{2x - \frac{16}{x^2}}{2\sqrt{x^2 + \frac{16}{x}}} = 0 \\ &\Longrightarrow \quad 2x = \frac{16}{x^2} \\ &\Longrightarrow \quad x^3 = 8 \quad \Longrightarrow x = 2 \end{aligned}$$

so the desired point is $\left(2\,,\, 2\sqrt{2}\right)$.

Exercise 18.

a) $f'(x) = 4\sin^3 x \cos x$

b) $f''(x) = 4\left(3\sin^2 x \cos^2 x - \sin^4 x\right) = 4\sin^2 x\left(4\cos^2 x - 1\right)$

c) Finding the points x on $(0, \pi)$ where the slope f' has an extreme value amounts to setting the derivative of that slope equal to zero:

$$f''(x) = 0 \quad \Longleftrightarrow \quad \cos^2 x = \frac{1}{4} \quad \Longleftrightarrow \quad x = \frac{1}{3}\pi \ \text{ or } x = \frac{2}{3}\pi$$

At $x = \frac{1}{3}\pi$ the function f increases most, and at $x = \frac{2}{3}\pi$ it decreases most.

Exercise 19. Did I mention this dot notation before? Physicists often write $\dot{x}$ instead of $x'(t)$ in order to keep their formulas short and clean. And by $\ddot{x}$ they obviously mean $x''(t)$.

- You calculate the velocity using the chain rule:

$$\dot{x} = \frac{1}{2\sqrt{1+t^2}} \cdot 2t = \frac{t}{\sqrt{1+t^2}}$$

- and for the acceleration you need the quotient rule:

$$\ddot{x} = \frac{\sqrt{1+t^2} - \frac{t}{\sqrt{1+t^2}} \cdot t}{1+t^2} = \left(1+t^2\right)^{-\frac{3}{2}}$$

Exercise 20. My acceleration at time t is

$$a(t) = v'(t) = \frac{2t}{1+t^4}$$

which is maximum if $a'(t) = 0 \quad \Longrightarrow \quad \dfrac{2-6t^4}{\left(1+t^4\right)^2} = 0 \quad \Longrightarrow \quad 6t^4 = 2 \quad \Longrightarrow \quad \boxed{t = \dfrac{1}{\sqrt[4]{3}}}$

Exercise 21. The distance from $\left(x\,,\, \sqrt{1+2x}\right)$ to $(3, 0)$ is

$$\left\|\left(x, \sqrt{1+2x}\right) - (3,0)\right\| = \left\|\left(x-3, \sqrt{1+2x}\right)\right\| = \sqrt{(x-3)^2 + (1+2x)} = \sqrt{x^2 - 4x + 10}$$

I have to find the minimum of this, which I can do either by differentiation or by completing a square:

$$\sqrt{x^2-4x+10} = \sqrt{(x-2)^2+6} \quad \Longrightarrow \quad \text{distance} = \boxed{\sqrt{6}}$$

Exercise 22.

a) Its speed is $\dot{x} = \dfrac{6t}{t^2+9}$.

b) Its acceleration is $\ddot{x} = \dfrac{54-6t^2}{(t^2+9)^2}$, which is

$$\begin{cases} \text{positive} & \text{if } 0 \le t < 3 \\ \text{zero} & \text{if } t = 3 \\ \text{negative} & \text{if } t > 3 \end{cases}$$

Hence, the ostrich reaches its maximum speed after 3 second, and this maximum speed is 1 m/sec.

Exercise 23.

a) I have to prove that these two equations don't have any common solution, which I do by eliminating x:

$$\left.\begin{array}{l} x+4=3y \implies x=3y-4 \\ 4x=3y^2 \end{array}\right\} \implies 4(3y-4)=3y^2 \implies 3y^2-12y+16=0$$

The left-hand side of this quadratic equation is $3(y-2)^2+4$, which of course can never be zero.

b) I can treat the line and the upper part of the parabola as functions with function rules

$$\boxed{\text{line:} \quad l(x)=\frac{1}{3}x+\frac{4}{3}} \qquad \boxed{\text{parabola:} \quad p(x)=2\sqrt{\frac{x}{3}}}$$

At the desired point, p should have the same direction as l, so their slopes should be equal. Because l has slope $\frac{1}{3}$ everywhere, p should have slope $\frac{1}{3}$ at this point as well:

$$p'(x)=\frac{1}{3} \implies \frac{1}{\sqrt{3x}}=\frac{1}{3} \implies x=3 \implies \text{that's the point } \boxed{(3,2)}$$

c) This can be done in many ways, for example:

- the line through $(3,2)$ perpendicular to the line $x=3y-4$ has equation $3x+y=11$
- this line intersects the line $x=3y-4$ at the point $(\frac{29}{10}, \frac{23}{10})$
- so the desired distance is $\left\|(3,2)-\left(\frac{29}{10}, \frac{23}{10}\right)\right\| = \left\|\left(\frac{1}{10}, -\frac{3}{10}\right)\right\| = \frac{1}{10}\cdot\|(1,-3)\| = \boxed{\frac{1}{10}\sqrt{10}}$

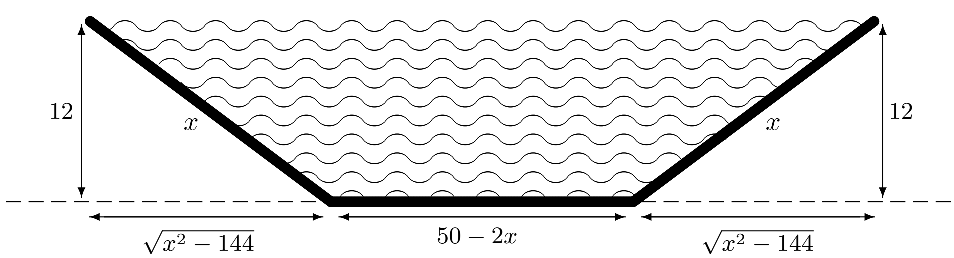
Exercise 24.

a) By the quotient rule (or product rule if you've written F as $t^3\cdot 2^{-t}$) you'll find $\boxed{\frac{dF}{dt}=\frac{3t^2-t^3\ln 2}{2^t}}$.

b) At the flea climax this derivative equals zero, which is the case if $t=0$ (not interesting) or $\boxed{t=\frac{3}{\ln 2}}$ (after approximately 4.33 days).

Exercise 25.

a) Let me just put the rest of the dimensions in the figure (many thanks to Pythagoras):



The capacity is the area of a $50-2x+2\sqrt{x^2-144}$ by 12 cm rectangle minus the area of the two little triangles:

$$C=12\cdot\left(50-2x+2\sqrt{x^2-144}\right)-2\cdot 6\cdot\sqrt{x^2-144}=\boxed{600-24x+12\sqrt{x^2-144}}$$

b) I try to find the x where the derivative of C equals zero:

$$\frac{dC}{dx}=-24+\frac{12x}{\sqrt{x^2-144}}=0 \implies x=2\sqrt{x^2-144} \implies x^2=4(x^2-144) \implies \boxed{x=8\sqrt{3}}$$

c) The maximum capacity of the gutter is $C(8\sqrt{3})=\boxed{600-144\sqrt{3}}$ (approximately 350 cm^2).

Exercise 26.

a) I calculate the derivative of the rats function using the product rule:

$$\frac{d\,\text{rats}}{dt} = 1000 \cdot \left(1.3\,t^{0.3} \cdot 0.7^t + t^{1.3} \cdot 0.7^t \,\ln 0.7\right) \quad \text{which equals zero if } \boxed{t = \frac{-1.3}{\ln 0.7}}$$

(after approximately 3.64 days).

b) The second derivative of the rats function is

$$\frac{d^2\,\text{rats}}{dt^2} = 1000 \cdot \left(0.39\,t^{-0.7} \cdot 0.7^t + 2.6\,t^{0.3} \cdot 0.7^t \,\ln 0.7 + t^{1.3} \cdot 0.7^t \,(\ln 0.7)^2\right)$$

which is zero if $0.39\,t^{-0.7} + 2.6\,t^{0.3}\,\ln 0.7 + t^{1.3}\,(\ln 0.7)^2 = 0 \quad \overset{\cdot t^{0.7}}{\Longrightarrow} \quad (\ln 0.7)^2\,t^2 + (2.6\,\ln 0.7)\,t + 0.39 = 0$

$$\overset{\text{quadratic formula}}{\Longrightarrow} \quad t = \frac{-2.6\,\ln 0.7 \pm \sqrt{6.76 - 1.56}\,\ln 0.7}{2(\ln 0.7)^2} = \frac{-1.3 \pm \sqrt{1.3}}{\ln 0.7}$$

so maximum increase of the number of rats occurs at $\boxed{t = \frac{-1.3 + \sqrt{1.3}}{\ln 0.7}}$ (after almost 0.45 days).

c) For maximum decrease you must wait a little longer: $\boxed{t = \frac{-1.3 - \sqrt{1.3}}{\ln 0.7}}$ (after approximately 6.84 days).

Exercise 27. The derivative of A_{tot} is

$$\frac{d\,A_{\text{tot}}}{dt} = \frac{d\,A_{\text{tuc}}}{dt} + \frac{d\,A_{\text{box}}}{dt} = \boxed{800 \cdot 1.015^t \ln 1.015 - 5500 \cdot 1.012^{-t} \ln 1.012}$$

At the time t at which A_{tot} is minimum, this derivative equals zero:

$$\frac{d\,A_{\text{tot}}}{dt} = 0 \implies 800 \cdot 1.015^t \ln 1.015 = 5500 \cdot 1.012^{-t} \ln 1.012 \implies (1.015 \cdot 1.012)^t = \frac{55}{8} \cdot \frac{\ln 1.012}{\ln 1.015}$$

$$\implies t = \frac{\ln\left(\frac{55}{8} \cdot \frac{\ln 1.012}{\ln 1.015}\right)}{\ln\left(1.015 \cdot 1.012\right)} \approx 63.62 \quad \text{(which is in March 1997)}$$

Exercise 28. Suppose I mound the spotlight at x metres. The distance spotlight–pussycat is then (by Pythagoras)

$$r = \sqrt{36 + x^2}$$

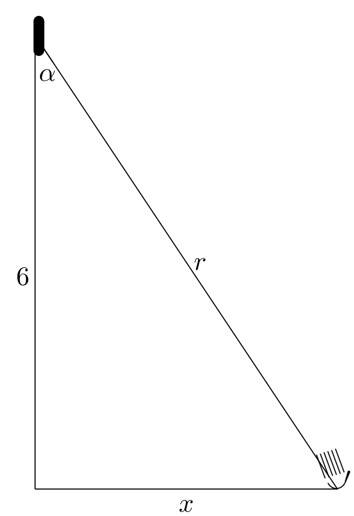
The intensity of a light beam is inversely proportional to r^2, and only the horizontal component of the beam illuminates the painting, which is proportional to

$$f(x) = \frac{1}{r^2} \cdot \sin\alpha = \frac{1}{r^2} \cdot \frac{x}{r} = \frac{x}{(36 + x^2)^{\frac{3}{2}}}$$

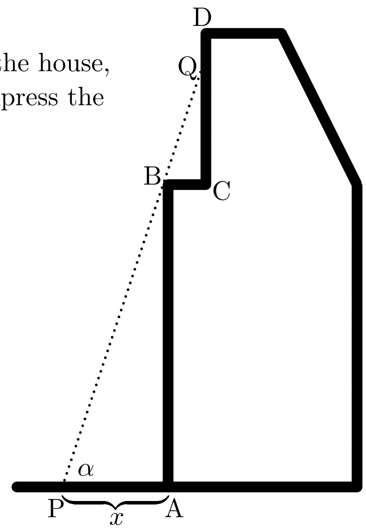
This function f is maximum if its derivative equals zero, so

$$\frac{\left(36 + x^2\right)^{\frac{3}{2}} - 3x^2\left(36 + x^2\right)^{\frac{1}{2}}}{\left(36 + x^2\right)^3} = 0 \implies \boxed{x = \sqrt{18}}$$

(which is approximately 4.24 metres).



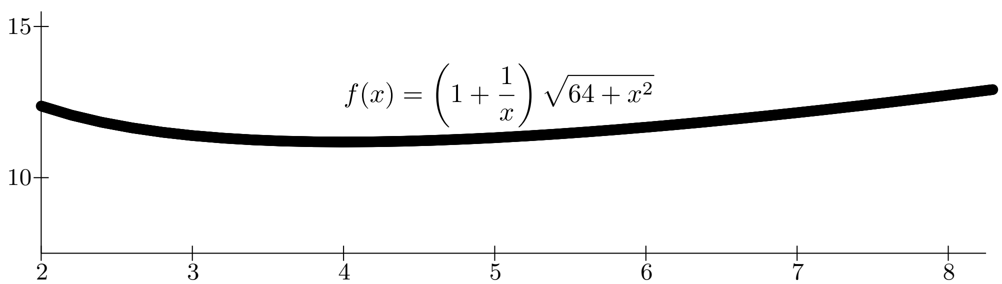
Exercise 29. The foot of the ladder (P) is at x metres from the house, and the ladder makes an angle α with the ground. I'll try to express the length of the ladder in terms of x:



- Pythagoras $\Longrightarrow$ $\mathrm{PB} = \sqrt{64 + x^2}$
- angle CBQ equals α as well
- $\cos\alpha = \dfrac{\mathrm{AP}}{\mathrm{PB}} = \dfrac{\mathrm{BC}}{\mathrm{BQ}} \Longrightarrow \mathrm{BQ} = \dfrac{1}{x}\sqrt{64+x^2}$
- The length of the ladder is $\mathrm{PB} + \mathrm{BQ}$, which is

$$\boxed{f(x) = \left(1 + \frac{1}{x}\right)\sqrt{64 + x^2}}$$

Now, the question is: what is the minimum value of this function f on the domain $[2, \infty)$?
(x must be less than 2, otherwise Q would end up above D, which makes climbing rather complicated)
I'll sketch you the graph of f; maybe you'll be able to guess the minimum with the naked eye:



Too bad, I suppose that's not going to work. That means that we have to differentiate f (the product rule will do) and set its derivative equal to zero:

$$f'(x) = 0 \quad \Longrightarrow \quad -\frac{1}{x^2}\sqrt{64+x^2} + \left(1 + \frac{1}{x}\right)\cdot\frac{x}{\sqrt{64+x^2}} = 0 \quad \Longrightarrow \quad x^3 = 64 \quad \Longrightarrow \quad x = 4$$

That calculation could have been worse. We conclude that the minimum ladder length is

$$f(4) = 5\sqrt{5} \qquad \text{(which is approximately 11 metres and 18 centimetres)}$$

Exercise 30. We are looking for the point where the graph declines steepest. I took a piece of scrap paper and calculated the first and second derivative:

$$\frac{d\,\text{wasp}}{dt} = 50 \cdot \frac{1 + t - t^2}{e^t} \qquad\qquad \frac{d^2\,\text{wasp}}{dt^2} = 50 \cdot \frac{t^2 - 3t}{e^t}$$

Would you please be so kind as to check my calculations carefully? You're allowed (even morally obliged) to scoff at me if I made a mistake. I didn't use the quotient rule (since I've never managed to memorise it), but the product rule saved me with $\text{wasp}\,(t) = 50(t^2 - 3t)e^{-t}$. The second derivative vanishes at $t = 3$, so at that time the number of zig-zag elm sawflies decreased fastest, which was at 1 July 2015.

7. Partial derivatives

Functions of two variables. A function of two variables x and y is a function $f : D \to \mathbb{R}$ with D a subset of the xy-plane. The graph of f is the collection of all points $(x, y, f(x, y))$ with $(x, y) \in D$.

Partial derivative. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ of two variables we define:

- $\dfrac{\partial f}{\partial x}$ is the derivative of the function $x \mapsto f(x, y)$, called the partial derivative of f with respect to x.
- The value of $\dfrac{\partial f}{\partial x}$ at the point (x, y) is denoted by $\dfrac{\partial f(x,y)}{\partial x}$ or $\dfrac{\partial}{\partial x} f(x, y)$ or $\left[\dfrac{\partial f}{\partial x}\right]_{(x,y)}$.
- $\dfrac{\partial f}{\partial y}$ is the derivative of the function $y \mapsto f(x, y)$.

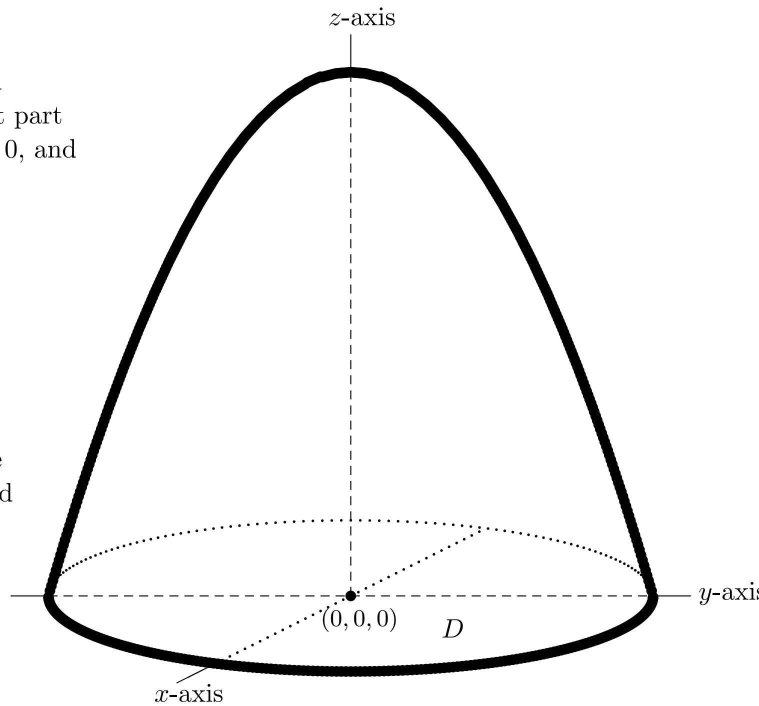
Example 1. D is the disk $x^2 + y^2 \leq 3$ and $f(x, y) = 3 - x^2 - y^2$. The graph of f is that part of the paraboloid $z = 3 - x^2 - y^2$ where $z \geq 0$, and

- $\dfrac{\partial f(x,y)}{\partial y} = -2y$

  (which you can find by differentiating $3 - x^2 - y^2$ with respect to the variable y, treating x as a constant)

- $\left[\dfrac{\partial f}{\partial y}\right]_{(1,1)} = -2$

  (this tells you something about the slope at $(1, 1, 1)$: if you intersect the paraboloid with the plane $x = 1$, the intersection curve at $(1, 1, 1)$ has slope -2; put differently, the slope of the paraboloid at the point $(1, 1, 1)$ in the y-direction equals -2)



Example 2. Calculate the partial derivatives of $f(x, y) = (x^2 + y) \sin xy$.

Solution.

- If I am to differentiate f partially with respect to x, I consider y to be a constant:

$$\frac{\partial f(x,y)}{\partial x} = 2x \sin xy + (x^2 y + y^2) \cos xy$$

- and if I am to differentiate f partially with respect to y, I take x to be a constant:

$$\frac{\partial f(x,y)}{\partial y} = \sin xy + (x^3 + xy) \cos xy$$

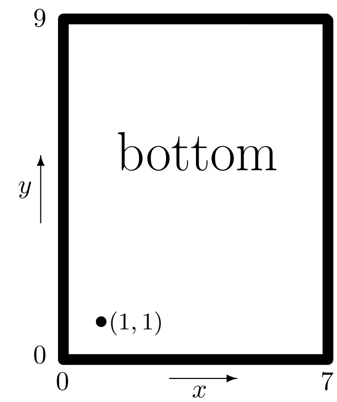
Example 3. The bottom of Wally's kennel is the rectangle $[0,7] \times [0,9]$. The roof of the kennel is nicely vaulted: its height $h(x,y)$ above (x,y) is given by the formula

$$\boxed{h(x,y) = 3\sqrt{x} + 2\sqrt{y}}$$

An ant is sitting on the roof straight above $(1,1)$ at the point $(1,1,5)$.

a) How steep should the ant climb if it wants to creep in y-direction?

b) How steep should the ant climb if it wants to creep in x-direction?



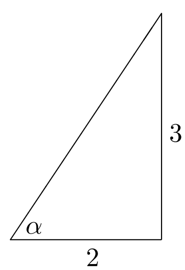
Solution.

a) The slope of the roof in y-direction is $\dfrac{\partial h(x,y)}{\partial y} = \dfrac{1}{\sqrt{y}}$ and above $(1,1)$ this equals $\left[\dfrac{\partial h(x,y)}{\partial y}\right]_{(1,1)} = 1$

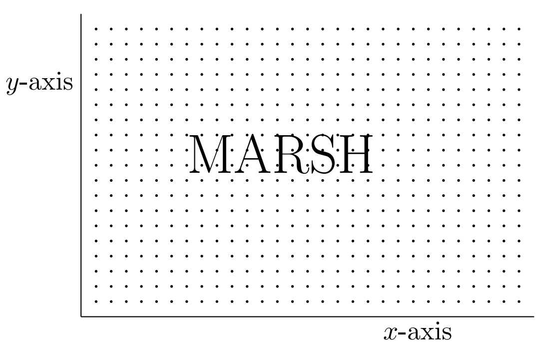
The ant should creep upwards under an angle of 45°.

b) The slope in x-direction is $\dfrac{\partial h(x,y)}{\partial x} = \dfrac{3}{2\sqrt{x}} \quad \Longrightarrow \quad \left[\dfrac{\partial h}{\partial x}\right]_{(1,1)} = \dfrac{3}{2}$

Hence, the angle α between the creeping route of the ant and the horizontal is



$$\alpha = \arctan \frac{3}{2} \qquad \text{(approximately } 56°\text{)}$$



Example 4. A marshy area consists of the points (x,y) with $x > 0$ and $y > 0$. The humidity of the mud at position (x,y) is given by the formula

$$\boxed{f(x,y) = \frac{2x+y}{1+y+x^2}}$$

What is the most humid spot of the marsh?

Solution. At the points where f has a maximum the derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ both equal zero:

$$\begin{aligned}
\frac{\partial f}{\partial x} = 0 \quad &\Longleftrightarrow \quad \text{(thinking, thinking)} \quad \Longleftrightarrow \quad x^2 + xy - y - 1 = 0 \quad \Longleftrightarrow \quad (x-1)(x+y+1) = 0 \\
&\Longleftrightarrow \quad x = 1 \text{ or } x + y = -1 \\
\frac{\partial f}{\partial y} = 0 \quad &\Longleftrightarrow \quad \text{(thinking, thinking)} \quad \Longleftrightarrow \quad x = 1
\end{aligned}$$

Conclusion: only the points on the line $x = 1$ are eligible candidates. Indeed, these points are incredibly humid (check for yourself).

Higher derivatives.

by	$\dfrac{\partial^2 f}{\partial x\, \partial y}$	we mean	$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$
by	$\dfrac{\partial^2 f}{\partial y\, \partial x}$	we mean	$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$
by	$\dfrac{\partial^2 f}{\partial x^2}$	we mean	$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$
by	$\dfrac{\partial^2 f}{\partial y^2}$	we mean	$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$

Example 5. Calculate the mixed partial derivatives of the function

$$f(x, y) = (x^2 + y) \sin xy$$

Solution. I mean those derivatives that are taken 'first with respect to y and then to x' and 'first with respect to x and then to y':

$$\frac{\partial^2 f(x,y)}{\partial x\, \partial y} = \frac{\partial}{\partial x}\big(\sin xy + (x^3 + xy)\cos xy\big) = (3x^2 + 2y)\cos xy - (x^3 y + xy^2)\sin xy$$

$$\frac{\partial^2 f(x,y)}{\partial y\, \partial x} = \frac{\partial}{\partial y}\big(2x \sin xy + (x^2 y + y^2)\cos xy\big) = (3x^2 + 2y)\cos xy - (x^3 y + xy^2)\sin xy$$

You might have discovered that these results are equal. Of course, this is not a coincidence. For functions with continuous partial derivatives one can prove that:

$$\boxed{\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}}$$

Example 6. The ant on top of the roof of Wally's kennel is sitting at the point $(1, 4, 7)$ and decides to creep a tiny piece in x-direction, namely to the point above $(1 + dx\,,\, 4)$ with $dx = \frac{1}{1000}$. What is the increase of the height of the ant?

Solution. A bit of algebra reveals that $\left[\dfrac{\partial h}{\partial x}\right]_{(1,4)} = \dfrac{3}{2}$.

The height increase of the ant is $dh = \dfrac{\partial h}{\partial x} \cdot dx = \dfrac{3}{2} dx = \dfrac{3}{2000}$.

Example 7. Again, the ant starts at $(1, 4, 7)$ and creeps to the point above $(1 + dx, 4 + dy)$ with $dx = \frac{1}{1000}$ and $dy = \frac{3}{1000}$. What is its height increase? And how steep is its climbing route?

Solution. The height increase due to the increase of the x-coordinate (from 1 to $1 + dx$) should be added to the height increase due to the increase of the y-coordinate:

$$dh = \frac{\partial h}{\partial x} \cdot dx + \frac{\partial h}{\partial y} \cdot dy = \frac{3}{2} dx + \frac{1}{2} dy = \frac{3}{1000}$$

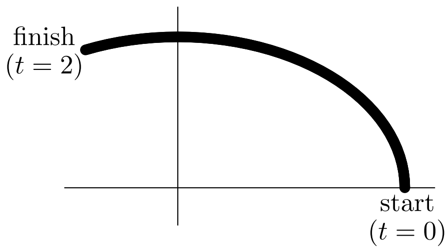
Since the traversed distance in the horizontal plane equals $\sqrt{(dx)^2 + (dy)^2} = \dfrac{\sqrt{10}}{1000}$, the ant climbs under an angle of $\arctan \dfrac{3}{\sqrt{10}} \approx 0.759$ radians ≈ 43.5 degrees.

Chain rule. The bottom line of this example, namely that the height increase is the sum of the components in x- and y-direction, is sometimes called the chain rule for functions of two variables:

$$\boxed{df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy}$$

Sometimes the variables x and y are dependent on another variable t. Example: at time t I am located at the point (x, y). The chain rule can then be formulated as follows:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$



Example 8. The temperature at (x, y) is given by $f(x, y) = 10xy$ degrees Celsius. From $t = 0$ to $t = 2$ I take an elliptically shaped walk, where my position at time t is given by $(x(t), y(t))$ with

$$\boxed{\begin{array}{c} x(t) = 3\cos t \\ y(t) = 2\sin t \end{array}}$$

Find the hottest spot on my walk.

Solution. The chain rule states that

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 10y \cdot (-3\sin t) + 10x \cdot 2\cos t = -60\sin^2 t + 60\cos^2 t = 60\cos 2t$$

This equals zero when $t = \dfrac{\pi}{4}$, so this is the hottest moment in time: my position is then $\left(\dfrac{3}{2}\sqrt{2}\,,\ \sqrt{2}\right)$ where the temperature is no less than thirty degrees Celsius.

Example 9. Just like in example 3 an ant is located at the point $(1, 1, 5)$ on the roof of Wally's kennel. The ant decides to climb in the direction $(3, 4)$, which is to say that it starts creeping from $(1, 1, 5)$ to $(1 + 3\varepsilon, 1 + 4\varepsilon, \cdots)$. How steep is its climb?

Solution. The method from example 7 amounts to the following: first, we normalise the vector $(3, 4)$, which means that we divide it by its own length (which is 5):

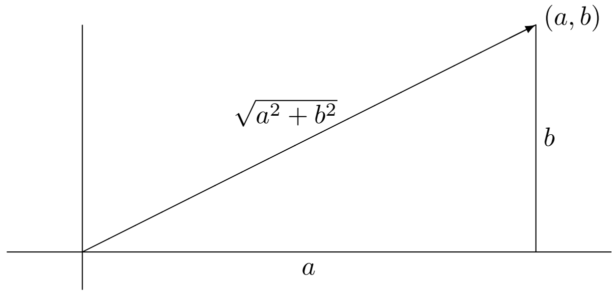
$$\text{the normalised creeping direction is } \left(\frac{3}{5}\,,\ \frac{4}{5}\right)$$

The slope in this direction is

$$\frac{3}{5} \cdot \left[\frac{\partial h}{\partial x}\right]_{(1,1)} + \frac{4}{5} \cdot \left[\frac{\partial h}{\partial y}\right]_{(1,1)} = \frac{3}{5} \cdot \frac{3}{2} + \frac{4}{5} \cdot 1 = \frac{17}{10}$$

This slope is called the derivative of h at the point $(1, 1)$ in the direction $(3, 4)$.

Direction vector. The length of a vector (a, b) is $\|(a, b)\| = \sqrt{a^2 + b^2}$. Hopefully, you remember this from chapter 2. In case you don't, you can easily prove it using Pythagoras:



If you are to calculate the derivative in the direction (a, b), you should first normalise this direction vector, which means dividing the vector by its own length. What you obtain is a normalised direction vector with length 1. Then, you continue as follows:

Directional derivative. Let (a, b) be a normalised vector (i.e. $a^2 + b^2 = 1$). The derivative of f in the direction (a, b), called the directional derivative of f along (a, b), is

$$\boxed{a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y}}$$

Example 10.

a) Calculate the partial derivatives of the function $(x, y) \mapsto e^{xy}$ at the point $(3, 7)$.

b) Calculate the derivative of the function $(x, y) \mapsto e^{xy}$ in the direction $(1, 2)$ at the point $(3, 7)$.

Solution.

a) $\dfrac{\partial e^{xy}}{\partial x} = ye^{xy}$, so $\left[\dfrac{\partial e^{xy}}{\partial x}\right]_{(3,7)} = 7e^{21}$

$\dfrac{\partial e^{xy}}{\partial y} = xe^{xy}$, so $\left[\dfrac{\partial e^{xy}}{\partial y}\right]_{(3,7)} = 3e^{21}$

b) The length of the direction vector $(1, 2)$ is $\sqrt{5}$, so the normalised direction vector is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$. Thus, the derivative of $(x, y) \mapsto e^{xy}$ in the direction $(1, 2)$ at the point $(3, 7)$ is

$$\frac{1}{\sqrt{5}} \cdot \left[\frac{\partial e^{xy}}{\partial x}\right]_{(3,7)} + \frac{2}{\sqrt{5}} \cdot \left[\frac{\partial e^{xy}}{\partial y}\right]_{(3,7)} = \frac{13\, e^{21}}{\sqrt{5}}$$

Steepest slope and steepest direction. From the theory above two more interesting formulas follow:

- the slope of $z = f(x, y)$ at the point $(x, y, f(x, y))$ is

$$\boxed{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

  (which is the slope in the direction in which z increases most)

- this steepest direction is

$$\boxed{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

Gradient. This 'steepest direction' vector is also called the gradient of f and denoted by $\operatorname{grad} f$ or ∇f:

$$\boxed{\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

The norm of this gradient is the steepest slope.

Example 11. At the point $(1, 1, 5)$ on the roof of Wally's kennel I put a marble. In what direction will the marble start to roll?

Solution. I calculate the gradient of the roof height function h at $(1, 1)$:

$$\left.\begin{aligned} \left[\frac{\partial h}{\partial x}\right]_{(1,1)} &= \frac{3}{2} \\ \left[\frac{\partial h}{\partial y}\right]_{(1,1)} &= 1 \end{aligned}\right\} \quad \Longrightarrow \quad (\nabla h)_{(1,1)} = \left(\frac{3}{2}, 1\right)$$

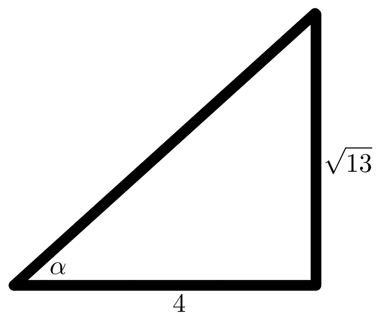
so the direction of the steepest slope is $\left(\frac{3}{2}, 1\right)$ or, equivalently, $(3, 2)$. Marbles usually roll downhill, so the marble starts rolling in the direction $(-3, -2)$. By the way, the marble will soon reach the ground, since the slope at the point $(1, 1)$ in the direction $(-3, -2)$ is no less than

$$-\| (\nabla h)_{(1,1)} \| = -\frac{1}{2}\sqrt{13}$$

Example 12. How steep is the roof of Wally's kennel at the point $(4, 4, 10)$?

Solution.

- The slope at $(x, y, h(x, y))$ is $\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} = \sqrt{\frac{9}{4x} + \frac{1}{y}}$.
- At the point $(4, 4, 10)$ the slope becomes $\frac{\sqrt{13}}{4}$.
- The angle α between the plane tangent to the roof at this point and the xy-plane is then



$$\alpha = \arctan \frac{\sqrt{13}}{4} \approx 42^\circ$$

Example 13. Let $f(x, y) = xy^2$.

a) Determine the gradient of f.

b) Determine the gradient of f at the point $(2, 3)$.

c) In what direction from the point $(2, 3)$ does f increase most?

Solution.

a) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(y^2, 2xy\right)$

b) $(\nabla f)_{(2,3)} = \left(y^2, 2xy\right)_{(2,3)} = (9, 12)$

c) In the direction $(9, 12)$ or, equivalently, in the direction $(3, 4)$.

Tangent plane. The equation of the tangent plane to f at the point $(a, b, f(a, b))$ is

$$\boxed{z = f(a, b) + \left[\frac{\partial f}{\partial x}\right]_{(a,b)} (x - a) + \left[\frac{\partial f}{\partial y}\right]_{(a,b)} (y - b)}$$

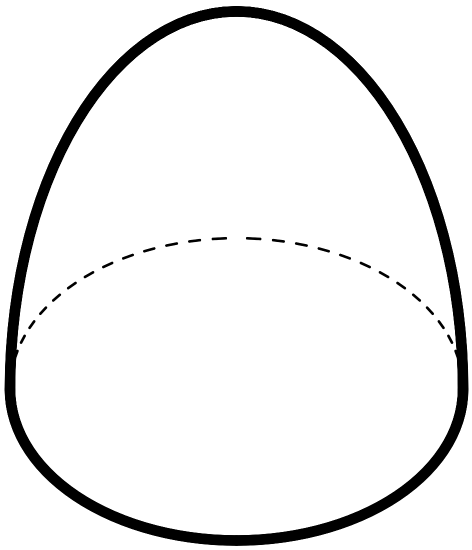
This follows from the fact that the tangent plane is a plane (a species of the form $z = \lambda + \mu x + \nu y$) and that the value and directions (partial derivatives) of this plane at the point under consideration should match those of f.

Example 14. Find the equation of the plane tangent to the curved surface $z = x^3 + 2xy$ at the point $(1, 2, 5)$.

Solution. The partial derivatives at this point are

$$\left[\frac{\partial z}{\partial x}\right]_{(1,2)} = \left[3x^2 + 2y\right]_{(1,2)} = 7 \qquad \left[\frac{\partial z}{\partial y}\right]_{(1,2)} = \left[2x\right]_{(1,2)} = 2$$

so the equation of the tangent plane is $z = 5 + 7(x - 1) + 2(y - 2)$, which can as well be written as $7x + 2y - z = 6$.



Example 15. Find the tangent plane at $(1, 1, 1)$ to the ellipsoid

$$3x^2 + 2y^2 + z^2 = 6$$

Solution. This ellipsoid is the graph of the function

$$z = \sqrt{6 - 3x^2 - 2y^2}$$

which has its partial derivatives at this point given by

$$\frac{\partial z}{\partial x} = \frac{-3x}{\sqrt{6 - 3x^2 - 2y^2}} \quad \Longrightarrow \quad \left[\frac{\partial z}{\partial x}\right]_{(1,1)} = -3$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{\sqrt{6 - 3x^2 - 2y^2}} \quad \Longrightarrow \quad \left[\frac{\partial z}{\partial y}\right]_{(1,1)} = -2$$

Therefore, the equation of the tangent plane is
$z = 1 - 3(x - 1) - 2(y - 1)$ or, equivalently, $3x + 2y + z = 6$.

Total derivative. The equation of the tangent plane nicely shows what increase df of the f-value is caused by small increases dx and dy of the coordinates:

$$\boxed{df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy}$$

This df is called the total derivative or total differential of f.

Example 16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^2 + y^3$.

a) What is the total derivative of f?

b) What is the total derivative of f at $(5, 2)$?

c) What is the equation of the tangent plane to f at the point $(5, 2, 33)$?

Solution.

a) The total differential of f is $df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 2x\, dx + 3y^2\, dy$.

b) At the points $(5, 2)$ this becomes $(df)_{(5,2)} = 10\, dx + 12\, dy$.

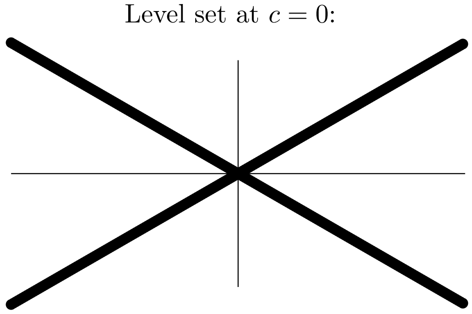
c) The tangent plane is $z - 33 = 10(x - 5) + 12(y - 2) \quad \Longrightarrow \quad 10x + 12y - z = 41$

Level set. Given a function f and a constant c, the collection of all points (x, y) satisfying $f(x, y) = c$ is called a level set (also called level curve or fiber in some books) of f.

Example 17. Draw two level sets of the function $f(x, y) = x^2 - 3y^2$.

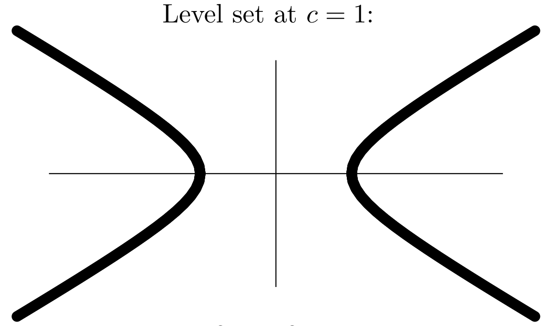
Solution. I'll sketch the level sets at level 0 and at level 1:

Level set at $c = 0$:



$x^2 - 3y^2 = 0$

Level set at $c = 1$:



$x^2 - 3y^2 = 1$

Functions of three variables. All concepts in this chapter can be generalised straightforwardly to functions of three or more variables. For example, for a function f with as its domain a piece of $\mathbb{R}^3$:

- the gradient of f is $\nabla f = \left(\dfrac{\partial f}{\partial x}\,,\,\dfrac{\partial f}{\partial y}\,,\,\dfrac{\partial f}{\partial z}\right)$
- the total derivative of f is $df = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy + \dfrac{\partial f}{\partial z}\,dz$

Example 18. (units: degrees Celsius, metres)
The temperature at a point (x, y, z) is given by

$$\text{Temp}\,(x, y, z) = \frac{100}{1 + x^2 + y^2 + z^2}$$

I am located at the point $P = (1, -2, 2)$.

a) What is the temperature at P?

b) In what direction from P does the temperature increase most?

c) How much does the temperature increase in this direction?

d) How much does the temperature increase from P in the direction $(-1, 4, -8)$?

e) Find the best linear approximation (i.e. a function of the type $(x, y, z) \mapsto \alpha + \beta x + \gamma y + \delta z$) to the temperature function in the neighbourhood of P.

f) What level set contains P?

Solution.

a) $\text{Temp}\,(P) = 10$ degrees

b) $\nabla\text{Temp} = \left(\dfrac{\partial\,\text{Temp}}{\partial x}\,,\,\dfrac{\partial\,\text{Temp}}{\partial y}\,,\,\dfrac{\partial\,\text{Temp}}{\partial z}\right)$

$$= \left(\frac{-200x}{(1 + x^2 + y^2 + z^2)^2}\,,\,\frac{-200y}{(1 + x^2 + y^2 + z^2)^2}\,,\,\frac{-200z}{(1 + x^2 + y^2 + z^2)^2}\right)$$

$\Longrightarrow$ $(\nabla\text{Temp})_P = (-2, 4, -4)$, so the direction from P in which the temperature increases most is the direction $(-2, 4, -4)$ or, equivalently, $(-1, 2, -2)$. This happens to be exactly the direction pointing straight to the origin. I suppose there is a little heater located at the origin.

c) The temperature increase in this direction is $\|\nabla\text{Temp}\| = \|(-2, 4, -4)\| = 6$ degrees per metre.

d) Normalise the direction vector: $\|(-1, 4, -8)\| = 9$, so the normalised direction vector is $\left(-\dfrac{1}{9}\,,\,\dfrac{4}{9}\,,\,-\dfrac{8}{9}\right)$

The increase in this direction is $-\dfrac{1}{9}\cdot(-2) + \dfrac{4}{9}\cdot 4 - \dfrac{8}{9}\cdot(-4) = \dfrac{50}{9}$ $^\circ$C/m.

e) The total derivative at P is $(d\,\text{Temp})_P = -2\,dx + 4\,dy - 4\,dz$, so the best linear approximation T to Temp at P is $T(x, y, z) = 10 - 2(x - 1) + 4(y + 2) - 4(z - 2)$.

f) This level set consists of all points with a temperature of 10 degrees, which is the sphere with the little heater at the origin as its centre and a radius of 3 metres
(in this context this level set can be called an isothermal surface).

Exercises chapter 7

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^3 + 5xy$.

a) Calculate $\dfrac{\partial f(x,y)}{\partial x}$.

b) Calculate $\left[\dfrac{\partial f(x,y)}{\partial x}\right]_{(2,3)}$.

Exercise 2. Let $F(M,E)$ be the number of mice caught (and devoured) by a Nijmegian cat per day given a mouse density of M mice/acre and a devouring duration of E hours/mouse (the fact that this devouring duration differs among cats is due to the fact that some cats prefer to play with their prey for hours before eating it, while others get straight to the point). I discovered the following formula for $F(M,E)$:

$$F(M,E) = \frac{\sqrt{M}}{7+E}$$

Currently, the mouse density equals $M = 100$ and Pommetje's devouring duration equals $E = 3$, so the given formula implies that Pommetje catches 1 mouse per day. Now, answer the following questions by calculating the partial derivatives:

a) How much does the number of mice caught by Pommetje increase when the mouse density increases a tiny little bit, i.e. from 100 to $100 + dM$?

b) How much does the number of mice caught by Pommetje increase when her devouring duration decreases from 3 to $3 - dE$?

Exercise 3. Let $f(x,y) = \sqrt{6x - x^2 - y^2}$.

a) What is the domain of f ?

b) Draw a sketch of the graph of f.

c) Calculate $\left[\dfrac{\partial f}{\partial x}\right]_{(1,2)}$.

d) What is the geometric interpretation of the result of (c)?

Exercise 4.

a) Calculate $\dfrac{\partial}{\partial x}\, \dfrac{x}{\sqrt{x^2+y^2}}$.

b) Calculate $\dfrac{\partial}{\partial x}\, \arcsin \dfrac{x}{\sqrt{x^2+y^2}}$.

c) Calculate $\dfrac{\partial}{\partial x}\, \arctan \dfrac{x}{y}$.

Exercise 5. The temperature at the point (x,y) is given by

$$T(x,y) = \frac{100}{3+x^2+y^2} \quad \text{degrees Celsius}$$

Calculate the temperature decrease (in degrees per unit length) in the case that you start walking from the point $(1,1)$ in the y-direction.

Exercise 6. Calculate the slope of the tangent line at the point $(1,1,2)$ to the intersection curve of the plane $y = 1$ and the paraboloid $z = x^2 + y^2$.

Exercise 7.

a) Draw that part of the sphere $x^2 + y^2 + z^2 = 36$ where the x-, y- and z-coordinates are all positive and draw the intersection curve with the plane $x = 4$.

b) Calculate the slope of this curve at the point $(4, 4, 2)$.

Exercise 8. Let $f(x, y) = x^3 y$.

a) Calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

b) Calculate $\dfrac{\partial^2 f}{\partial y\, \partial x}$ and $\dfrac{\partial^2 f}{\partial x\, \partial y}$.

c) Calculate $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$.

Exercise 9. An ant is sitting on top of the roof of Wally's kennel above the point $(1, 4)$, i.e. at the point $(1, 4, 7)$.

a) How steep should the ant climb if it wants to creep in x-direction?

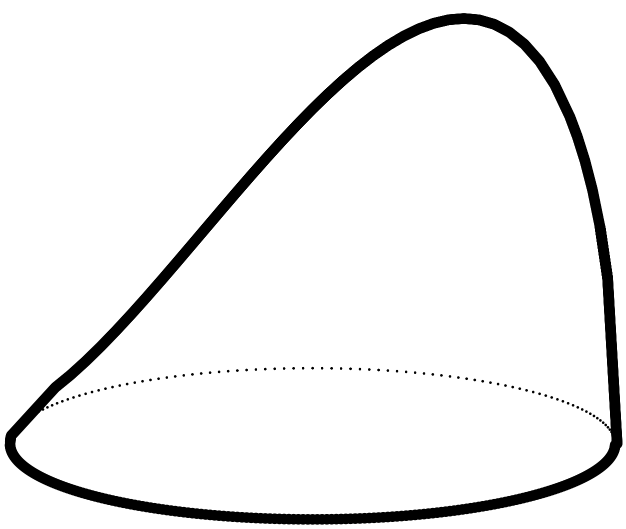
b) How steep should the ant climb if it wants to creep in y-direction?

Exercise 10. Check whether the following functions f satisfy $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$:

a) $f(x, y) = x \arctan y$

b) $f(x, y) = x^y$ with as its domain the collection of all points (x, y) with $x > 0$ and $y > 0$

Exercise 11. Calculate $\dfrac{\partial^2 z}{\partial y \partial x}$ if $z = \sin(x^2 y)$.



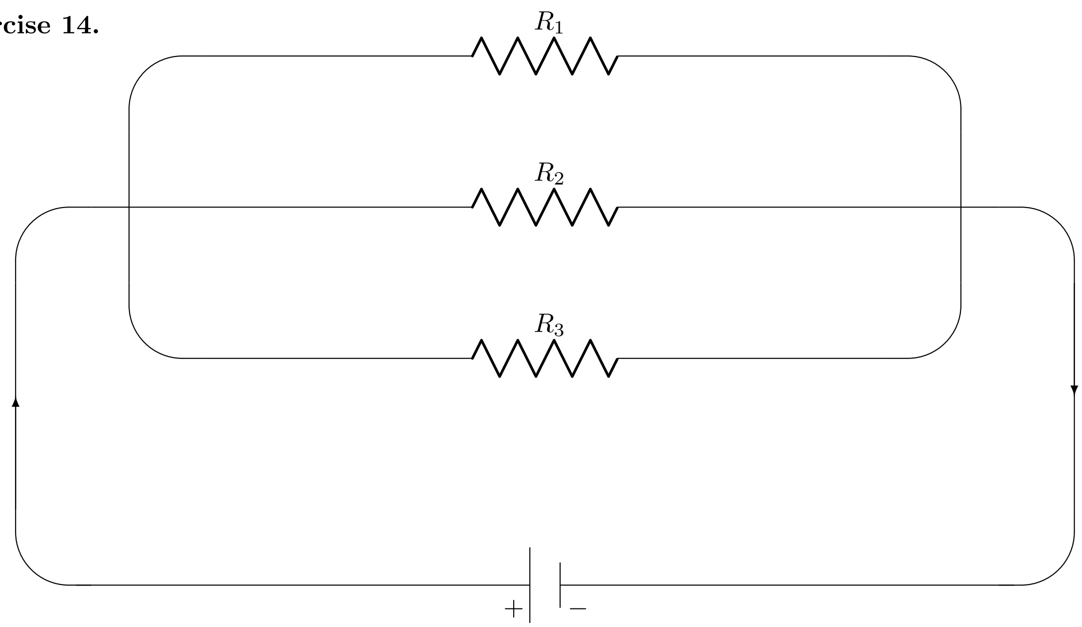
Exercise 12. This hill is the graph of the function

$$f(x, y) = \left(1 + \frac{y}{2}\right) \sqrt{4 - x^2 - y^2}$$

with as its domain the disk $x^2 + y^2 \leq 4$. Find the highest point on the hill.

Exercise 13. The ant is sitting at the point $(1, 4, 7)$ on the roof of Wally's kennel and wishes to climb in the direction $(1, 2)$, heading for $(1 + \varepsilon, 4 + 2\varepsilon, \cdots)$. How steep is its climb?

Exercise 14.



This circuit consists of three parallel resistors R_1, R_2 and R_3. You might remember from your very first lessons on electrical circuits that the value of the equivalent resistance R can be calculated using the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

The values of the individual resistors (in ohms) are given by

R_1	=	30
R_2	=	45
R_3	=	90

a) Calculate R.

b) Calculate $\frac{\partial R}{\partial R_2}$.

c) What does the result of (b) mean?

Exercise 15. An 'ideal gas' obeys the ideal gas law

$$\frac{PV}{T} = C$$

with P the pressure, V the volume and T the temperature (in kelvins) of the gas. The constant C is the product of the number of moles of gas and a physical constant. Assume that for a particlar ideal gas

P	=	3
V	=	5
T	=	300

Derive a formula for the change of gas pressure caused by small changes in V and T.

Exercise 16. The bottom of my garden house is the rectangle $[-1,3]\times[-1,3]$ in the xy-plane. The height of the zinc roof above the point (x,y) is given by

$$\boxed{z = 13 + x^3 - 3xy + y^3}$$

It has rained, so there is a small puddle on top of the roof. Find its coordinates.

Exercise 17. For which $x \geq 0$ and $y \geq 0$ is $4x + y - (x+y)^2$ maximum?

Suggestion. First, draw the area in which you should search for the maximum. Then, investigate whether there are points in the interior of this area where the function $(x,y) \mapsto 4x + y - (x+y)^2$ is maximum by equating the partial derivatives to zero. Does this job drive you to pure despair? Don't forget to check the boundary of the area as well.

Exercise 18. I put a marble on the roof of Wally's kennel above $(1,4)$. In what direction will it roll?

Exercise 19. How steep is the zinc roof of my garden house at the point $(1,2,16)$?

Exercise 20. Calculate the directional derivative along $\left(1,\sqrt{2}\right)$ of

a) $\ln\left(x^2+y^2\right)$ at the point $(1,2)$

b) e^{x+y} at the point $(0,0)$

Exercise 21.

a) Calculate the slope of the roof of Wally's kennel at the point $(x,y,f(x,y))$.

b) Complete the sentence: at the point $(\cdots,1,\cdots)$ the roof of Wally's kennel makes an angle of exactly $60°$ with the horizontal.

Exercise 22. The concentration of a cyclic AMP molecule at points (x,y) in the neighbourhood of the point $P=(3,1)$ is given by

$$\boxed{\text{AMP}\,(x,y) \;=\; \frac{16}{xy+1} \;\text{ percent}}$$

A *Dictyostelium discoideum* amoebe at the point P, driven by chemotaxis, moves in the direction in which the AMP concentration increases most.

a) Calculate the increase of the AMP concentration at P in the direction $(4,-3)$.

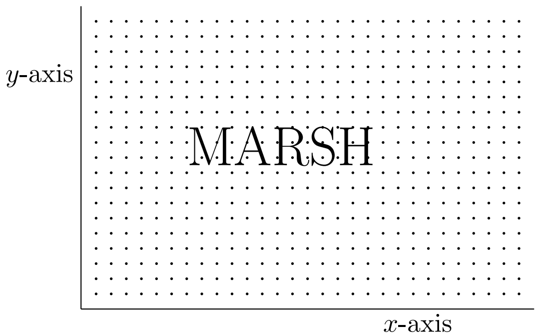
b) In what direction does the amoebe move?

Exercise 23. A room has as its floor the rectangle $[0,2]\times[0,3]$. The height of the vaulted ceiling above the point (x,y) is

$$\boxed{h(x,y) = 2x + 3y + 7 - x^2 - y^2}$$

a) Find the heighest point of the ceiling.

b) Calculate the slope of the ceiling above the point $(0,0)$ in the direction $(1,2)$.



Exercise 24. In the marsh the noise disturbance at the point (x, y) due to unstoppable croacking by raucous frogs is approximately given by

$$\frac{8xy}{x^2 + y^2} \quad \text{decibels}$$

At what spots is the noise loudest?

Exercise 25. We define the function $f : \mathbb{R}^2 \to [-1, 1]$ by

$$f(x, y) = \sin xy$$

Let G be the graph of f, which is to say that G consists of all points $(x, y, \sin xy)$, and let P be the point

$$P = \left(\frac{\pi}{3}\,,\, -1\,,\, -\frac{\sqrt{3}}{2} \right)$$

a) Determine the slope of G at the point P.

b) Determine the equation of the tangent plane to G at the point P.

Exercise 26. (units: km, tonnes)
Due to an accident at the point $(0, 0)$ the environment becomes polluted with oil. The oil density at the accident site is no less than 1 tonne/km^2, but further away it could have been worse: at the point (x, y) the oil density is only

$$\text{oil}\,(x, y) = \frac{1}{\sqrt{1 + x^2 + 2y^2}}$$

How much does the pollution increase if I swim a tiny distance from the point $(1, 1)$ towards the accident site?

Exercise 27. Let S be the curved surface with equation

$$z = x^2 - 4xy - 2y^2 + 12x - 12y - 1$$

a) What horizontal plane is tangent to S? (by 'horizontal' I mean: perpendicular to the z-axis)

b) What is the point of tangency?

Exercise 28.

a) Calculate the maximum value of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x + y}{1 + x^2 + y^2}$$

b) Calculate the maximum value of the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = xe^{-(x^2+y^2)}$$

Exercise 29. We define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 4 \arcsin \frac{x}{\sqrt{x^2 + y^2}}$$

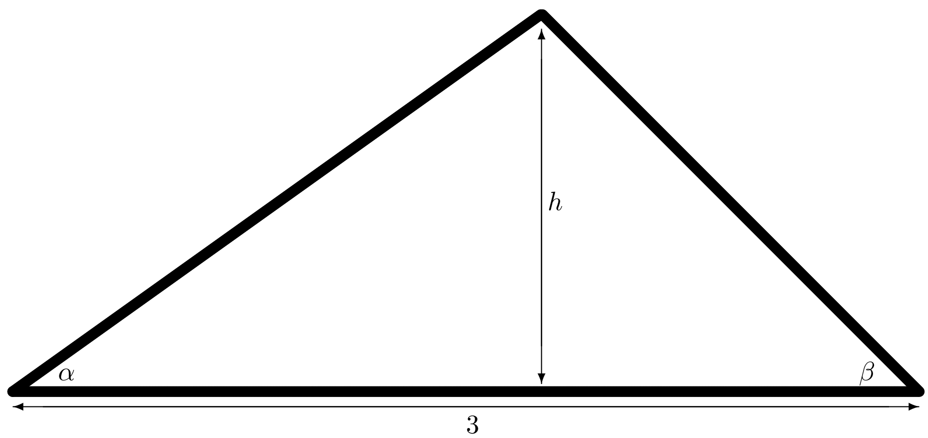
a) Calculate $f(1, 1)$.

b) Calculate $\lim_{x \to 1} \frac{f(x, 1) - \pi}{x - 1}$.

c) Calculate $\lim_{y \to 1} \frac{f(1, y) - \pi}{y - 1}$.

d) Calculate the slope of f at $(1, 1)$.

Exercise 30. (units: metres, radians)
I build a tent with base 3 and base angles α and β. Its height is h:



a) Express h in terms of the angles α and β.

b) Derive a formula for the change of height h as a function of small changes in α and β.

Exercise 31. Let D be the collection of all points (x, y) with $x^2 + y^2 < 1$ (put differently: D is the interior of the unit disk in the x-y-plane). We define the function f with domain D by

$$f(x, y) = \frac{4x + 3y}{\sqrt{1 - x^2 - y^2}}$$

a) Calculate the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(0, 0)$.

b) Calculate the derivative of f at $(0, 0)$ in the direction $(3, 4)$.

c) Calculate the slope of f at $(0, 0)$.

Exercise 32. Determine the equation of the tangent plane to the graph of

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

at the point above $P = (3, 4)$.

Exercise 33. Let f be the function $f(x, y) = x^3 - xy^2$.

a) Determine the total differential df at $(3, 2)$.

b) Determine the equation of the tangent plane to f at the point $(3, 2, f(3, 2))$.

Exercise 34. Find the level curves of $z = 7 - x^2 - y^2$.

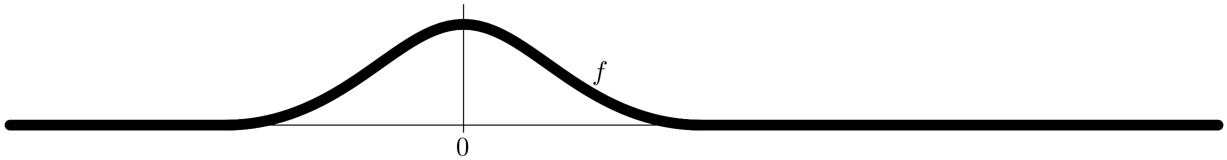
Exercise 35. We define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^2 + y^2 - 2x - 4y$$

a) Find a point where $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$.

b) Check whether f at the point found in (a) is maximum or minimum.

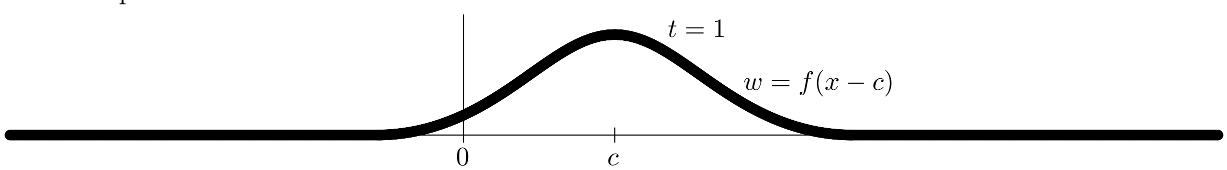
Exercise 36. (units: metres, seconds)
Let f be the function drawn below (I won't spoil its function rule):



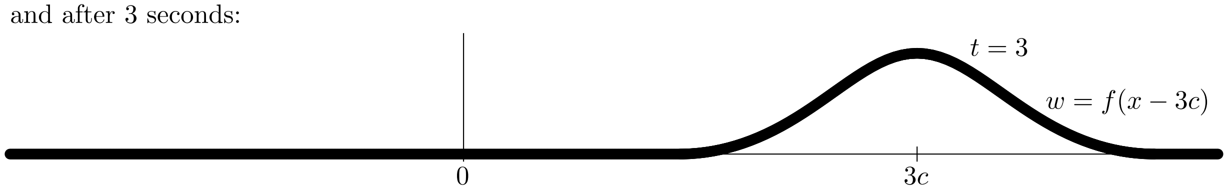
If I move this graph to the right with speed c, the resulting function after t seconds is

$$\boxed{w(x, t) = f(x - ct)}$$

For example after 1 second:



and after 3 seconds:



Hence, this function w describes a wave moving to the right along the x-axis. Prove that w satisfies the

one-dimensional wave equation $\boxed{\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}}$

Exercise 37. (units: kilometres, degrees Celsius)
Fafner is located at $(0, 0)$ and wants to walk in a direction in which the temperature increases. The temperature is given by

$$T(x, y) = 2x + \sin y$$

a) How much (in degrees/km) does the temperature increase if Fafner starts walking from $(0, 0)$ in the direction $(1, 2)$?

b) What direction would you recommend to Fafner if he desires to experience maximum temperature increase?

Solutions chapter 7

Exercise 1.

a) Differentiate $f(x,y)$ with respect to x treating y as a constant: $\dfrac{\partial f(x,y)}{\partial x} = 3x^2 + 5y$.

b) $\left[\dfrac{\partial f(x,y)}{\partial x}\right]_{(2,3)} = \left[3x^2 + 5y\right]_{(2,3)} = 27$

Exercise 2.

a) $\left[\dfrac{\partial F(M,E)}{\partial M}\right]_{(100,3)} = \left[\dfrac{1}{(14+2E)\sqrt{M}}\right]_{(100,3)} = \dfrac{1}{200}$

A mouse density increase dM leads to an increase in the number of mice caught of $\dfrac{dM}{200}$.

b) $\left[\dfrac{\partial F(M,E)}{\partial E}\right]_{(100,3)} = \left[\dfrac{-\sqrt{M}}{(7+E)^2}\right]_{(100,3)} = -\dfrac{1}{10}$

A devouring duration decrease $-dE$ leads to an increase in the number of mice caught of $\dfrac{dE}{10}$.

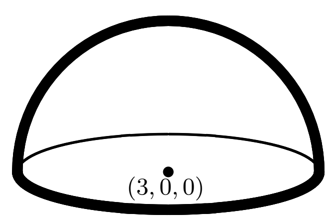
Exercise 3.

a) Write $6x - x^2 - y^2$ as $9 - (x-3)^2 - y^2$. The domain of f is the collection of all (x,y) for which this expression is ≥ 0, which is the disk $(x-3)^2 + y^2 \leq 9$ with centre $(3,0)$ and radius 3.

b) I abbreviate $f(x,y)$ to z:

$$z = \sqrt{9 - (x-3)^2 - y^2} \implies z^2 = 9 - (x-3)^2 - y^2 \implies (x-3)^2 + y^2 + z^2 = 9$$

This is the sphere with centre $(3,0,0)$ and radius, so the graph of f is the upper half of this sphere:



c) $\left[\dfrac{\partial f}{\partial x}\right]_{(1,2)} = \left[\dfrac{3-x}{\sqrt{6x - x^2 - y^2}}\right]_{(1,2)} = 2$

d) The result of (c) is the slope of the intersection curve of this sphere and the plane $y = 2$ at the point $(1,2,1)$.

Exercise 4.

a) Quotient rule $\implies$ $\dfrac{\partial}{\partial x}\dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \dfrac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$

b) Chain rule $\implies$ $\dfrac{\partial}{\partial x}\arcsin\dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{1}{\sqrt{1 - \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}} \cdot \dfrac{\partial}{\partial x}\dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{\sqrt{y^2}}{x^2+y^2} = \dfrac{|y|}{x^2+y^2}$

c) Chain rule $\implies$ $\dfrac{\partial}{\partial x}\arctan\dfrac{x}{y} = \dfrac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \dfrac{1}{y} = \dfrac{y}{x^2+y^2}$

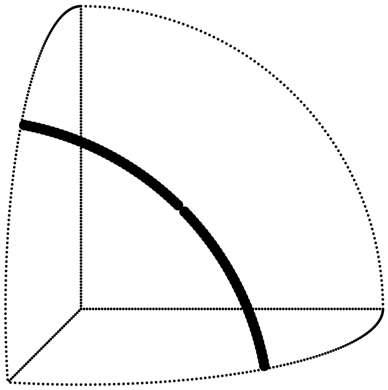
Exercise 5. If I walk in y-direction, x remains constant, so I have to calculate the partial derivative with respect to y:

$$\frac{\partial T(x,y)}{\partial y} = \frac{-200y}{(3+x^2+y^2)^2}$$

In $(1,1)$ this equals -8, so the temperature initially decreases 8 degrees per unit length.

Exercise 6. On this curve y remains 1, so the desired slope is the partial derivative of z with respect to x at the point $(1,1)$, which is

$$\left[\frac{\partial z}{\partial x}\right]_{(1,1)} = \Big[2x\Big]_{(1,1)} = 2$$



Exercise 7.

b) The slope of this curve at the point $(4,4,2)$ is

$$\left[\frac{\partial z}{\partial y}\right]_{(4,4)} = \left[\frac{\partial\sqrt{36-x^2-y^2}}{\partial y}\right]_{(4,4)} = \left[\frac{-y}{\sqrt{36-x^2-y^2}}\right]_{(4,4)} = -2$$

Exercise 8.

$$\frac{\partial f}{\partial x} = 3x^2y \qquad \frac{\partial f}{\partial y} = x^3 \qquad \frac{\partial^2 f}{\partial y\,\partial x} = 3x^2 \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 3x^2 \qquad \frac{\partial^2 f}{\partial x^2} = 6xy \qquad \frac{\partial^2 f}{\partial y^2} = 0$$

Exercise 9. The roof has equation $z = 3\sqrt{x} + 2\sqrt{y}$

a) The slope at the point $(1,4,7)$ in x-direction is $\left[\frac{\partial z}{\partial x}\right]_{(1,4)} = \left[\frac{3}{2\sqrt{x}}\right]_{(1,4)} = \frac{3}{2}$.

b) The slope at the point $(1,4,7)$ in y-direction is $\left[\frac{\partial z}{\partial y}\right]_{(1,4)} = \left[\frac{1}{\sqrt{y}}\right]_{(1,4)} = \frac{1}{2}$.

Exercise 10.

a) $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{1}{1+y^2}$

b) $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x} = x^{y-1}\,(1 + y\ln x)$

Exercise 11. I use two steps for this:

$$\frac{\partial z}{\partial x} = \big(\cos(x^2y)\big)\cdot 2xy \quad\Longrightarrow\quad \frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\big(2xy\cos(x^2y)\big) = 2x\cos(x^2y) - 2x^3y\sin(x^2y)$$

Exercise 12. On top of the hill the partial derivatives both equal zero. Let's first consider the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \left(1 + \frac{y}{2}\right)\cdot\frac{-x}{\sqrt{4-x^2-y^2}}$$

which equals zero when $y=-2$ or $x=0$. You can immediately throw away the option $y=-2$, because there the height of the hill is 0. Therefore, we only have to consider the points with $x=0$. This greatly simplifies the problem, since the height above a point $(0,y)$ is

$$h(y) = \left(1+\frac{y}{2}\right)\sqrt{4-y^2}$$

and finding the maximum of h is easy using the method of chapter 6:

$$h'(y) = \frac{1}{2}\sqrt{4-y^2} + \left(1+\frac{y}{2}\right)\cdot\frac{-y}{\sqrt{4-y^2}} = \frac{2-y-y^2}{\sqrt{4-y^2}} = \frac{(2+y)(1-y)}{\sqrt{4-y^2}}$$

which equals zero when $y=-2$ or $y=1$. Because you've already thrown away $y=-2$, we now know that $y=1$. Hence, the top of the hill is the point

$$\left(0\,,\,1\,,\,\frac{3}{2}\sqrt{3}\right)$$

Exercise 13. For this roof $f(x,y) = 3\sqrt{x} + 2\sqrt{y}$ we have:

$$\left[\frac{\partial f}{\partial x}\right]_{(1,4)} = \left[\frac{3}{2\sqrt{x}}\right]_{(1,4)} = \frac{3}{2} \qquad \left[\frac{\partial f}{\partial y}\right]_{(1,4)} = \left[\frac{1}{\sqrt{y}}\right]_{(1,4)} = \frac{1}{2}$$

and I calculate the slope above $(1,4)$ in the direction $(1,2)$ in two steps:

1. the normalised direction vector is $\dfrac{(1,2)}{\|(1,2)\|} = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$
2. and the slope in that direction thus equals $\dfrac{1}{\sqrt{5}} \cdot \left[\dfrac{\partial f}{\partial x}\right]_{(1,4)} + \dfrac{2}{\sqrt{5}} \cdot \left[\dfrac{\partial f}{\partial y}\right]_{(1,4)} = \dfrac{\sqrt{5}}{2}$

Exercise 14.

a) $\dfrac{1}{R} = \dfrac{1}{30} + \dfrac{1}{45} + \dfrac{1}{90} = \dfrac{1}{15} \quad \Longrightarrow \quad R = 15$ ohms

b) I'll use two different methods:

1. (a crude method): write R explicitly as a function of the three variables R_1, R_2 and R_3:

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \quad \Longrightarrow \quad \frac{\partial R}{\partial R_2} = \frac{-1}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2} \cdot \frac{-1}{{R_2}^2} = \frac{1}{9}$$

2. (a smart method): differentiate the left-hand side and right-hand side of $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ partially with respect to R_2:

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_2} = -\frac{1}{{R_2}^2} \quad \Longrightarrow \quad \frac{\partial R}{\partial R_2} = \frac{R^2}{{R_2}^2} = \frac{1}{9}$$

(I prefer this method because it works equally well for equations that cannot be explicitly 'solved'.)

c) Given the values of the resistors, a small change in the value of R_2 leads to a change in the value of the equivalent resistance R that is approximately $\frac{1}{9}$ times smaller.

Exercise 15. You can calculate the constant C from the given values: $C = \dfrac{PV}{T} = \dfrac{1}{20} \quad \Longrightarrow \quad P = \dfrac{T}{20V}$

Now, I take the well-known formula for the total differential dP:

$$dP = \frac{\partial P}{\partial V}\,dV + \frac{\partial P}{\partial T}\,dT = -\frac{T}{20V^2}\,dV + \frac{1}{20V}\,dT = -\frac{3}{5}\,dV + \frac{1}{100}\,dT$$

Exercise 16. The partial derivatives are

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \qquad \frac{\partial z}{\partial y} = -3x + 3y^2$$

Water will leak away unless both partial derivatives equal zero:

$$\left.\begin{array}{lll} 3x^2 - 3y = 0 & \Longrightarrow & x^2 = y \\ -3x + 3y^2 = 0 & \Longrightarrow & x = y^2 \end{array}\right\} \quad \Longrightarrow \quad \text{either } x = y = 0 \text{ or } x = y = 1$$

Hence, only the spots straight above $(0,0)$ and $(1,1)$ are under consideration. The point $(0,0)$ can be discarded, because on the line $y = 0$ we have

$$\frac{\partial z}{\partial x} = 3x^2$$

which is rather positive both for $x < 0$ and for $x > 0$, so the water will be able to leak away. Thus, the only point left is $(1,1)$. Straight above this point the water will indeed form a puddle, because

- on the line $y = 1$ we have $\dfrac{\partial z}{\partial x} = 3x^2 - 3$, which is negative if x is just below 1, and positive for $x > 1$
- on the line $x = 1$ we have $\dfrac{\partial z}{\partial y} = -3 + 3y^2$, which is negative for $y < 1$ and positive for $y > 1$

Conclusion: the only puddle formed is located at the point $(1,1,12)$.

Exercise 17. First, I try to find points where the partial derivatives of $f(x,y) = 4x + y - (x+y)^2$ both equal zero:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 4 - 2(x+y) = 0 \quad \Longrightarrow \quad x + y = 2 \\ \frac{\partial f}{\partial y} &= 1 - 2(x+y) = 0 \quad \Longrightarrow \quad x + y = \tfrac{1}{2} \end{aligned}\right\rangle \Longrightarrow \text{ panic!!}$$

Apparently, there is no interior maximum, so I check the boundary of the domain:

Boundary $x = 0$: $f(0,y) = y - y^2 = \frac{1}{4} - (y - \frac{1}{2})^2$ with maximum $\frac{1}{4}$ at the point $(0, \frac{1}{2})$

Boundary $y = 0$: $f(x,0) = 4x - x^2 = 4 - (x-2)^2$ with maximum 4 at the point $(2,0)$

$(2,0)$ turns out to be the desired point.

Exercise 18. The steepest direction of the roof above the point $(1,4)$ is

$$\left(\left[\frac{\partial f}{\partial x}\right]_{(1,4)} , \left[\frac{\partial f}{\partial y}\right]_{(1,4)} \right) = \left(\frac{3}{2} , \frac{1}{2} \right)$$

or, equivalently, the direction $(3,1)$. Because marbles have a tendency to roll downhill, the marble will strongly prefer the direction

$$(-3,-1)$$

Exercise 19. The required partial derivatives are

$$\left[\frac{\partial z}{\partial x}\right]_{(1,2)} = \left[3x^2 - 3y\right]_{(1,2)} = -3 \qquad \left[\frac{\partial z}{\partial y}\right]_{(1,2)} = \left[-3x + 3y^2\right]_{(1,2)} = 9$$

so the slope of the roof above the point $(1,2)$ is

$$\sqrt{\left[\frac{\partial z}{\partial x}\right]_{(1,2)}^2 + \left[\frac{\partial z}{\partial y}\right]_{(1,2)}^2} = \sqrt{(-3)^2 + 9^2} = 3\sqrt{10}$$

Exercise 20. The normalised direction vector is $\dfrac{(1\,,\,\sqrt{2})}{\|(1\,,\,\sqrt{2})\|} = \left(\sqrt{\dfrac{1}{3}}\,,\,\sqrt{\dfrac{2}{3}}\right)$.

a) The partial derivatives are

$$\left[\frac{\partial \ln\left(x^2+y^2\right)}{\partial x}\right]_{(1,2)} = \left[\frac{2x}{x^2+y^2}\right]_{(1,2)} = \frac{2}{5} \qquad \left[\frac{\partial \ln\left(x^2+y^2\right)}{\partial y}\right]_{(1,2)} = \left[\frac{2y}{x^2+y^2}\right]_{(1,2)} = \frac{4}{5}$$

so the directional derivative along $\left(1\,,\,\sqrt{2}\right)$ is

$$\sqrt{\frac{1}{3}} \cdot \left[\frac{\partial \ln\left(x^2+y^2\right)}{\partial x}\right]_{(1,2)} + \sqrt{\frac{2}{3}} \cdot \left[\frac{\partial \ln\left(x^2+y^2\right)}{\partial y}\right]_{(1,2)} = \frac{2}{15}\sqrt{3} + \frac{4}{15}\sqrt{6}$$

b) The partial derivatives are

$$\left[\frac{\partial\, e^{x+y}}{\partial x}\right]_{(0,0)} = \left[e^{x+y}\right]_{(0,0)} = 1 \qquad \left[\frac{\partial\, e^{x+y}}{\partial y}\right]_{(0,0)} = \left[e^{x+y}\right]_{(0,0)} = 1$$

so the directional derivative at $(0,0)$ along $\left(1\,,\,\sqrt{2}\right)$ is

$$\sqrt{\frac{1}{3}} \cdot \left[\frac{\partial\, e^{x+y}}{\partial x}\right]_{(0,0)} + \sqrt{\frac{2}{3}} \cdot \left[\frac{\partial\, e^{x+y}}{\partial y}\right]_{(0,0)} = \sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}}$$

Exercise 21.

a) The slope at $(x, y, f(x,y))$ is $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{\left(\frac{3}{2\sqrt{x}}\right)^2 + \left(\frac{1}{\sqrt{y}}\right)^2} = \sqrt{\frac{9}{4x} + \frac{1}{y}}$.

b) Above the point $(x, 1)$ the slope equals $h(x) = \sqrt{\frac{9}{4x} + 1}$

. Because $60°$ amounts to a slope of $\sqrt{3}$, I am looking for the point where $h(x)$ equals $\sqrt{3}$:

$$h(x) = \sqrt{3} \quad \Longrightarrow \quad \frac{9}{4x} + 1 = 3 \quad \Longrightarrow \quad x = \frac{9}{8} \quad \Longrightarrow \quad \text{the desired point is } \left(\frac{9}{8}\,,\,1\,,\,2 + \frac{9}{4}\sqrt{2}\right)$$

Exercise 22. The partial derivatives at P are

$$\left[\frac{\partial\,\text{AMP}}{\partial x}\right]_P = \left[\frac{-16y}{(xy+1)^2}\right]_P = -1 \qquad\qquad \left[\frac{\partial\,\text{AMP}}{\partial y}\right]_P = \left[\frac{-16x}{(xy+1)^2}\right]_P = -3$$

a) The normalised direction vector is $\frac{(4,-3)}{\|(4,-3)\|} = \left(\frac{4}{5}\,,\,\frac{-3}{5}\right)$ so the AMP increase in the direction $(4, -3)$ is

$$\frac{4}{5} \cdot (-1) + \frac{-3}{5} \cdot (-3) = 1 \text{ percent per unit length}$$

b) The direction of motion of the amoebe is the gradient of AMP:

$$(\nabla\,\text{AMP})_P = \left(\left[\frac{\partial\,\text{AMP}}{\partial x}\right]_P, \left[\frac{\partial\,\text{AMP}}{\partial y}\right]_P\right) = (-1, -3)$$

(in that direction the AMP increase is no less than $\sqrt{10}$ percent)

Exercise 23.

a) The partial derivatives are $\frac{\partial h}{\partial x} = 2 - 2x$ and $\frac{\partial h}{\partial y} = 3 - 2y$. I investigate the behaviour of their signs:

$$\frac{\partial h}{\partial x} \text{ is } \begin{cases} \text{positive} & \text{if } x < 1 \\ \text{zero} & \text{if } x = 1 \\ \text{negative} & \text{if } x > 1 \end{cases} \qquad\qquad \frac{\partial h}{\partial y} \text{ is } \begin{cases} \text{positive} & \text{if } y < \frac{3}{2} \\ \text{zero} & \text{if } y = \frac{3}{2} \\ \text{negative} & \text{if } y > \frac{3}{2} \end{cases}$$

Thus, the heighest point is located above $\left(1\,,\,\frac{3}{2}\right)$ and there the ceiling height is $\frac{41}{4}$.

b) The partial derivatives at $(0, 0)$ are $\left[\frac{\partial h}{\partial x}\right]_{(0,0)} = 2$ and $\left[\frac{\partial h}{\partial y}\right]_{(0,0)} = 3$ and the normalised direction vector is

$$\frac{(1,2)}{\|(1,2)\|} = \left(\frac{1}{\sqrt{5}}\,,\,\frac{2}{\sqrt{5}}\right)$$

so the derivative of h at $(0, 0)$ in this direction is $\frac{1}{\sqrt{5}} \cdot 2 + \frac{2}{\sqrt{5}} \cdot 3 = \frac{8}{\sqrt{5}}$.

Exercise 24. I calculate the partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{8y^3 - 8x^2y}{(x^2+y^2)^2} \qquad\qquad \frac{\partial f}{\partial y} = \frac{8x^3 - 8y^2x}{(x^2+y^2)^2}$$

and find out where they both equal zero:

$$\left.\begin{array}{lcl} 8y^3 - 8x^2y = 0 & \Longleftrightarrow & y = 0 \text{ or } x = y \\ 8x^3 - 8y^2x = 0 & \Longleftrightarrow & x = 0 \text{ or } x = y \end{array}\right\} \quad \Longleftrightarrow \quad x = y$$

Thus, the partial derivatives both equal zero on the line $y = x$. The noise disturbance is 4 decibels everywhere on this line and there are no points where it exceeds 4 decibels, because

$$\frac{\partial f}{\partial x} \text{ is } \begin{cases} \text{positive} & \text{if } x < y \\ \text{negative} & \text{if } x > y \end{cases}$$

Exercise 25. The partial derivatives of f at P are

$$\left[\frac{\partial f}{\partial x}\right]_P = \Big[y\cos xy\Big]_P = -\frac{1}{2} \qquad \left[\frac{\partial f}{\partial y}\right]_P = \Big[x\cos xy\Big]_P = \frac{\pi}{6}$$

a) The slope at P is $\sqrt{\left(\left[\frac{\partial f}{\partial x}\right]_P\right)^2 + \left(\left[\frac{\partial f}{\partial y}\right]_P\right)^2} = \frac{1}{6}\sqrt{\pi^2+9}.$

b) The tangent plane at P has equation $z = f\left(\frac{\pi}{3}, -1\right) + \left(-\frac{1}{2}\right)\cdot\left(x-\frac{\pi}{3}\right) + \frac{\pi}{6}\cdot(y-(-1))$

which amounts to $z = \frac{-\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{3}\right) + \frac{\pi}{6}(y+1).$

Exercise 26. I calculate the partial derivatives of the oil at $(1,1)$:

$$\frac{\partial\,\text{olie}}{\partial x} = -x(1+x^2+2y^2)^{-\frac{3}{2}} \implies \left[\frac{\partial\,\text{olie}}{\partial x}\right]_{(1,1)} = -\frac{1}{8}$$
$$\frac{\partial\,\text{olie}}{\partial y} = -2y(1+x^2+2y^2)^{-\frac{3}{2}} \implies \left[\frac{\partial\,\text{olie}}{\partial y}\right]_{(1,1)} = -\frac{1}{4}$$

My swimming direction is $(-1,-1)$, which should be normalised to $\frac{(-1,-1)}{\|(-1,-1)\|} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
Now, the increase of oil pollution in my swimming direction is

$$-\frac{1}{\sqrt{2}}\cdot\left[\frac{\partial\,\text{olie}}{\partial x}\right]_{(1,1)} - \frac{1}{\sqrt{2}}\cdot\left[\frac{\partial\,\text{olie}}{\partial y}\right]_{(1,1)} = -\frac{1}{\sqrt{2}}\cdot\left(-\frac{1}{8}\right) - \frac{1}{\sqrt{2}}\cdot\left(-\frac{1}{4}\right) = \frac{3}{8\sqrt{2}} \text{ tonnes/km}^2 \text{ per km}$$

Exercise 27.

a) At the point of tangency the partial derivatives of z with respect to x and y equal zero:

$$\begin{cases} \frac{\partial z}{\partial x} = 0 \implies 2x-4y+12=0 \implies x-2y+6=0 \\ \frac{\partial z}{\partial y} = 0 \implies -4x-4y-12=0 \implies x+y+3=0 \end{cases}$$

After some algebraic messing around with these two equations, you discover that $x=-4$ and $y=1$ so $z=-31$. Therefore, the desired plane has equation $z=-31$

b) and the point of tangency is $(-4,1,-31)$.

Exercise 28.

a) This maximum can be found by equating both partial derivatives to zero:

$$\frac{\partial f(x,y)}{\partial x} = 0 \implies 1-x^2+y^2-2xy=0 \qquad \frac{\partial f(x,y)}{\partial y} = 0 \implies 1+x^2-y^2-2xy=0$$

Subtraction of the left equation from the right equation yields $x^2=y^2$, so $x=y$ (because the maximum can only be located in the area with positive x and y). Addition of the two equations yields $2xy=1$, so the maximum is located at

$$P = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \implies \text{maximum} = f(P) = \frac{1}{2}\sqrt{2}$$

b) Note that $g(x,y) \leq g(x,0)$ and abbreviate $g(x,0)$ to $h(x)$. Then,

$$\frac{dh(x)}{dx} = (1-2x^2)e^{-x^2}$$

so $h(x)$ is maximum if $x = \frac{1}{\sqrt{2}}$ and the maximum value of g is $g\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2e}}$.

Exercise 29.

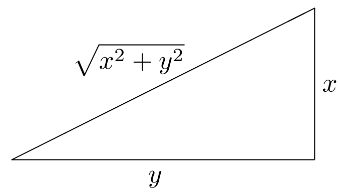
a) $f(1,1) = \pi$

b) $\lim_{x\to 1} \frac{f(x,1)-\pi}{x-1} = \left[\frac{\partial f}{\partial x}\right]_{(1,1)} = \left[\frac{4y}{x^2+y^2}\right]_{(1,1)} = 2$

c) $\lim_{y\to 1} \frac{f(1,y)-\pi}{y-1} = \left[\frac{\partial f}{\partial y}\right]_{(1,1)} = \left[\frac{-4x}{x^2+y^2}\right]_{(1,1)} = -2$

d) The slope of f at $(1,1)$ is $\sqrt{\left[\frac{\partial f}{\partial x}\right]_{(1,1)}^2 + \left[\frac{\partial f}{\partial y}\right]_{(1,1)}^2} = \sqrt{2^2+(-2)^2} = \sqrt{8}.$

Remark. The algebra in this exercise must have been hard, unless you discovered that $\arcsin \frac{x}{\sqrt{x^2+y^2}} = \arctan \frac{x}{y}$ (for instance by drawing the adjacent triangle).



Exercise 30.

a) A simple high-school exercise:

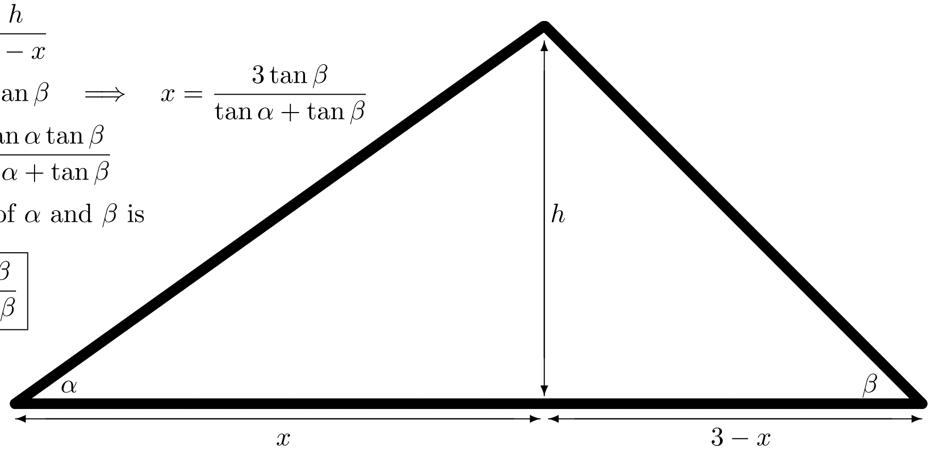
$\tan\alpha = \frac{h}{x} \qquad \tan\beta = \frac{h}{3-x}$

$\implies \quad x\tan\alpha = (3-x)\tan\beta \quad \implies \quad x = \frac{3\tan\beta}{\tan\alpha+\tan\beta}$

$\implies \quad h = x\tan\alpha = \frac{3\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

so h expressed in terms of α and β is

$$\boxed{h(\alpha,\beta) = \frac{3\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}}$$



b) The change of height dh due to small changes in the angles $d\alpha$ and $d\beta$ is

$$dh = \frac{\partial h}{\partial \alpha}\,d\alpha + \frac{\partial h}{\partial \beta}\,d\beta = \frac{3\tan^2\beta}{(\tan\alpha+\tan\beta)^2\cos^2\alpha}\,d\alpha + \frac{3\tan^2\alpha}{(\tan\alpha+\tan\beta)^2\cos^2\beta}\,d\beta$$

Exercise 31.

a)

$$\frac{\partial f}{\partial x} = \frac{4\sqrt{1-x^2-y^2} + \frac{x}{\sqrt{1-x^2-y^2}}(4x+3y)}{1-x^2-y^2} \implies \left[\frac{\partial f}{\partial x}\right]_{(0,0)} = 4$$

$$\frac{\partial f}{\partial y} = \frac{3\sqrt{1-x^2-y^2} + \frac{y}{\sqrt{1-x^2-y^2}}(4x+3y)}{1-x^2-y^2} \implies \left[\frac{\partial f}{\partial y}\right]_{(0,0)} = 3$$

b) First, normalise the direction vector $(3,4)$ yielding $\frac{(3,4)}{\|(3,4)\|} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

Then, the derivative at $(0,0)$ in this direction is $\frac{3}{5}\cdot 4 + \frac{4}{5}\cdot 3 = \frac{24}{5}$.

c) The slope of f at $(0,0)$ is $\sqrt{4^2+3^2} = 5$.

Exercise 32. I calculate the partial derivatives of f at P:

- $\dfrac{\partial f}{\partial x} = \dfrac{-x}{(x^2+y^2)^{\frac{3}{2}}} \qquad \Longrightarrow \qquad \left[\dfrac{\partial f}{\partial x}\right]_P = -\dfrac{3}{125}$
- $\dfrac{\partial f}{\partial y} = \dfrac{-y}{(x^2+y^2)^{\frac{3}{2}}} \qquad \Longrightarrow \qquad \left[\dfrac{\partial f}{\partial y}\right]_P = -\dfrac{4}{125}$

so the tangent plane is the function $(x, y) \mapsto \dfrac{1}{5} - \dfrac{3}{125}(x-3) - \dfrac{4}{125}(y-4)$.

Exercise 33.

a) $df = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy = (3x^2 - y^2)\,dx - 2xy\,dy \quad \Longrightarrow \quad df_{(3,2)} = 23\,dx - 12\,dy$

b) The plane tangent to f at the point $(3, 2, 15)$ is the plane $z - 15 = 23(x-3) - 12(y-2)$ which reduces to $23x - 12y - z = 30$.

Exercise 34. The level curve at level c is

$$7 - x^2 - y^2 = c \qquad \Longleftrightarrow \qquad x^2 + y^2 = 7 - c$$

so the level curves are the sets of points (x, y) for which $x^2 + y^2$ is a constant, which are the circles with centre $(0, 0)$.

Exercise 35.

a) $\dfrac{\partial f}{\partial x} = 2x - 2 = 0$ at the points $(1, \dots)$ and $\dfrac{\partial f}{\partial y} = 2y - 4 = 0$ at the points $(\dots, 2)$
so the partial derivatives are both zero at the point $(1, 2)$.

b) f has a minimum at the point $(1, 2)$, which you can see by investigating the behaviour of the signs of the partial derivatives, or (easier) by noting that $f(x, y) = (x-1)^2 + (y-2)^2 - 5$.

Exercise 36. I use the simple chain rule from chapter 5:

$$\frac{\partial w}{\partial t} = -cf'(x - ct) \quad \Longrightarrow \quad \frac{\partial^2 w}{\partial t^2} = c^2 f''(x - ct)$$
$$\frac{\partial w}{\partial x} = f'(x - ct) \quad \Longrightarrow \quad \frac{\partial^2 w}{\partial x^2} = f''(x - ct)$$

This immediately implies that w satisfies the wave equation.

Exercise 37. $\left[\dfrac{\partial T}{\partial x}\right]_{(0,0)} = 2$ en $\left[\dfrac{\partial T}{\partial y}\right]_{(0,0)} = 1$

a) After normalisation this is the direction $\left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$
so the temperature increase in this direction is $\dfrac{1}{\sqrt{5}} \cdot 2 + \dfrac{2}{\sqrt{5}} \cdot 1 = \dfrac{4}{\sqrt{5}}$ degrees/km.

b) Fafner should start walking in the direction $(2, 1)$!

Example test chapter 1

including chapters 1 and 2 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n-5$ points.

1. $\alpha = \dfrac{\beta}{1-\beta}$ implies that $\beta =$

   a. $\dfrac{\alpha}{1+\alpha}$ b. $\dfrac{\alpha}{1-\alpha}$ c. $\dfrac{1+\alpha}{\alpha}$ d. $\dfrac{1-\alpha}{\alpha}$

2. The quotient $\dfrac{8^{n+2}}{4^{n+3}}$ is equal to

   a. 2^{n-1} b. 2^n c. 2^{n+1} d. 2^{n+2}

3. You can simplify $(3x+1)^2 - (3x-1)^2$ to obtain

   a. $3x$ b. $6x$ c. $12x$ d. $24x$

4. When does $(5-x)^4$ equal $(x-5)^4$?

   a. never b. always c. almost never d. almost always

5. If $x = \dfrac{y+1}{y-1}$ and $y = \dfrac{z-1}{z+1}$, then $x =$

   a. z b. z^{-1} c. $-z$ d. $-z^{-1}$

Exercise 2. On day n Bhèta ate a_n bonbons:

day	0	1	2	3	4	5	6	7	············	99	100
bonbons	17	11	25	10	14	15	31	6	············	18	??

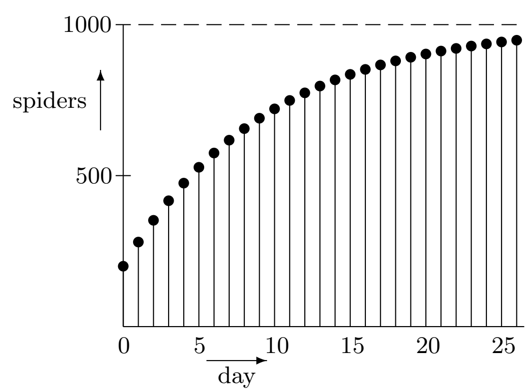
Calculate a_{100} from these data given the fact that $\sum_{n=0}^{99} (a_{n+1} - a_n) = 23$.

Exercise 3. Decompose $\dfrac{1-2x}{6+5x+x^2}$ into two simpler fractions.

Exercise 4. The number of spiders in my cellar increases from 200 (on day 0) via 280 (on day 1) to the limit 1000:

$s_0 = 200$ $s_1 = 280$ $s_\infty = 1000$

When can I expect for the first time 900 or more spiders in my cellar if the spider population satisfies the model 'bounded exponential growth'?



Exercise 5. (units: sec, cm)
Starting from $t = 0$ I take one step forward every second. The step size at $t = 0$ is 100 cm, but every next step is only 80% of the size of the previous step:

t	0	1	2	3	4	···
step size	100.00	80.00	64.00	51.20	40.96	···

Calculate the limit of the total travelled distance $100.00 + 80.00 + 64.00 + 51.20 + 40.96 + \cdots$

Solutions

Exercise 1. $\boxed{\text{1a, 2b, 3c, 4b, 5c}}$

1. $\alpha = \dfrac{\beta}{1-\beta} \implies \alpha(1-\beta) = \beta \implies \alpha - \alpha\beta = \beta \implies \alpha = \beta + \alpha\beta = \beta(1+\alpha) \implies \beta = \dfrac{\alpha}{1+\alpha}$
2. $\dfrac{8^{n+2}}{4^{n+3}} = \dfrac{2^{3n+6}}{2^{2n+6}} = 2^{(3n+6)-(2n+6)} = 2^n$
3. $(3x+1)^2 - (3x-1)^2 = (9x^2+6x+1) - (9x^2-6x+1) = 12x$
4. Always, because $(5-x)^4 = (-(x-5))^4 = (-1)^4 \cdot (x-5)^4 = 1 \cdot (x-5)^4 = (x-5)^4$.
5. $x = \dfrac{\frac{z-1}{z+1}+1}{\frac{z-1}{z+1}-1} = \langle\text{multiply numerator and denominator by } (z+1)\rangle = \dfrac{(z-1)+(z+1)}{(z-1)-(z+1)} = \dfrac{2z}{-2} = -z$

Exercise 2. On day 100 Bhèta ate no less than $\boxed{\text{40 bonbons}}$, because

$$a_{100} = a_0 + (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{100} - a_{99}) = a_0 + \sum_{n=0}^{99}(a_{n+1} - a_n) = 17 + 23 = 40$$

Exercise 3. Because the denominator equals $(2+x)(3+x)$, I except to be able to decompose it into

$$\frac{A}{2+x} + \frac{B}{3+x} = \frac{A(3+x)+B(2+x)}{(2+x)(3+x)} = \frac{(3A+2B)+(A+B)x}{(2+x)(3+x)} \implies \begin{cases} 3A+2B = 1 \\ A+B = -2 \end{cases}$$

These equations yield $A = 5$ and $B = -7$, so the desired partial fraction decomposition is $\boxed{\dfrac{5}{2+x} - \dfrac{7}{3+x}}$.

Exercise 4. In this model $s_n = \alpha - \beta \cdot \gamma^n$ the constant α represents the limit, so $s_n = 1000 - \beta \cdot \gamma^n$. $s_0 = 200$ implies $\beta = 800$:

$$s_n = 1000 - 800 \cdot \gamma^n \quad \stackrel{s_1=280}{\implies} \quad 800\gamma = 720 \quad \implies \quad \gamma = 0.9 \quad \implies \quad s_n = 1000 - 800 \cdot 0.9^n$$

which equals 900 when $800 \cdot 0.9^n = 100$, so

$$\left(\frac{9}{10}\right)^n = \frac{1}{8} \quad \implies \quad \left(\frac{10}{9}\right)^n = 8 \quad \stackrel{\text{ln-trick}}{\implies} \quad n \ln\frac{10}{9} = \ln 8 \quad \implies \quad \boxed{n = \frac{\ln 8}{\ln 10 - \ln 9}}$$

Since this is approximately 19.74, I expect more than 900 spiders $\boxed{\text{on day 20}}$. You too?

Exercise 5. The limit distance is $100\left(1 + 0.8 + 0.8^2 + 0.8^3 + \cdots\right) = 100 \cdot \dfrac{1}{1-0.8} = \dfrac{100}{0.2} = \boxed{\text{500 cm}}$.

Example test chapter 2

including chapters 4 and 6 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n - 5$ points.

1. How many real numbers x satisfy $\cos(3 + x) = 3 + \cos x$?

   a. none b. one c. two d. three

2. The curve with equation $(3 + y)^2 = y^2 + x$ is a

   a. line b. circle c. parabola d. ellipse

3. For all real numbers x the expression $\sqrt{1 - \cos 2x}$ equals

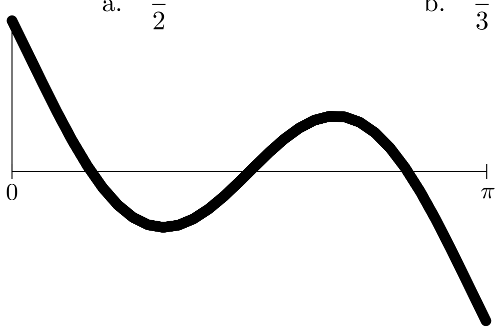
   a. $\sqrt{2} \cdot \sin x$ b. $\sqrt{2} \cdot \cos x$ c. $\sqrt{2} \cdot |\sin x|$ d. $\sqrt{2} \cdot |\cos x|$

4. The circle $x^2 + y^2 = 2x + 2$ has area

   a. π b. 2π c. 3π d. 4π

5. The line through the points $(-1, -2)$ and $(3, 4)$ has slope

   a. $\frac{3}{2}$ b. $\frac{4}{3}$ c. $\frac{5}{4}$ d. $\frac{6}{5}$



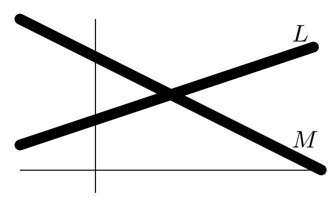
Exercise 2. Find the three zeros of the function

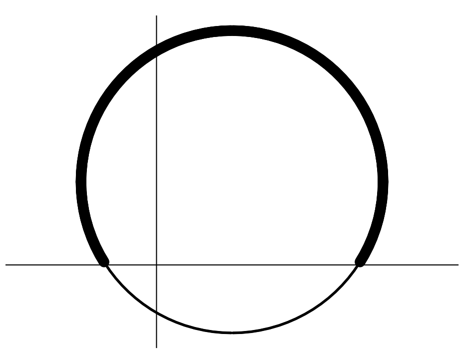
$$f(x) = \cos x - \sin 2x$$

on the domain $[0, \pi]$.

Exercise 3. Calculate the angle between the lines

$$L: \quad 3y = 2 + x$$
$$M: \quad 2y = 3 - x$$





Exercise 4. The equation of this circle is

$$x^2 + y^2 = 2(x + y + 1)$$

Calculate the arc length of the part of the circle above the x-axis.

Exercise 5.

a) Draw the set V of all points (x, y) satisfying $\|(1, -2) - (x, y)\| = 2$.

b) Draw the set W of all points (x, y) satisfying $(1, -2) \bullet (x, y) = 0$.

c) Calculate the distance from V to W.

Solutions

Exercise 1. $\boxed{\text{1a, 2a, 3c, 4c, 5a}}$

1. None, because both cosine values are in the range $[-1, 1]$, so they can't differ 3.
2. $(3+y)^2 = y^2 + x \quad\Longrightarrow\quad 9 + 6y + y^2 = y^2 + x \quad\Longrightarrow\quad 9 + 6y = x$ (which is a line)
3. $\cos 2x = 1 - 2\sin^2 x$ implies $\sqrt{1-\cos 2x} = \sqrt{2\sin^2 x} = \sqrt{2}\cdot\sqrt{\sin^2 x} = \sqrt{2}\cdot|\sin x|$
4. That is the circle $(x-1)^2 + y^2 = 3$, so its radius is $\sqrt{3}$.
5. Walk from $(-1,-2)$ to $(3,4)$, the slope is $\dfrac{y\text{-increase}}{x\text{-increase}} = \dfrac{6}{4} = \dfrac{3}{2}$.

Exercise 2. I calculate the zeros as follows:

$$\cos x - \sin 2x = 0 \iff \cos x - 2\sin x\cos x = 0 \iff \cos x\cdot(1 - 2\sin x) = 0$$

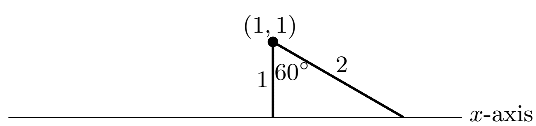
$$\iff \cos x = 0 \text{ or } \sin x = \frac{1}{2} \iff \boxed{x = \frac{\pi}{2}} \text{ or } \boxed{x = \frac{\pi}{6}} \text{ or } \boxed{x = \frac{5\pi}{6}}$$

Exercise 3. You can for instance do this as follows:

- Perpendicular to L is the vector $(-1, 3)$, because L is parallel to the line $3y = x$ which you can read as $(x, y) \perp (-1, 3)$.
- Perpendicular to M is the vector $(1, 2)$.
- Hence, the desired angle φ equals the angle between $(-1, 3)$ and $(1, 2)$:

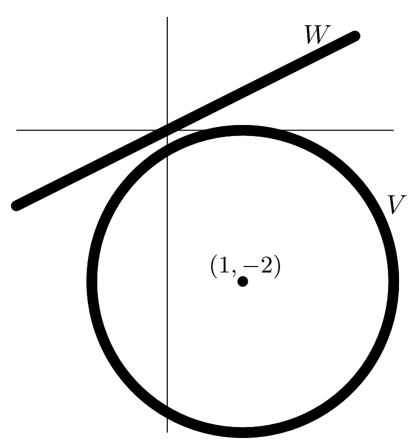
$$\cos\varphi = \frac{(-1,3)\bullet(1,2)}{\|(-1,3)\|\cdot\|(1,2)\|} = \frac{1}{\sqrt{2}} \quad\Longrightarrow\quad \varphi = \boxed{45 \text{ degrees}}$$

Exercise 4. Write the circle equation as $(x-1)^2 + (y-1)^2 = 4$, so you can see that the centre is $(1, 1)$ and the radius is 2. With elementary geometry you can show that you see the bold arc under an angle of 240° from the centre, for example like this:



240 degrees equals $\frac{4}{3}\pi$ radians, so the arc length is $\boxed{\dfrac{8}{3}\pi}$.

Exercise 5.



a) V is the set of all (x, y) at a distance 2 from $(1, -2)$, which is the circle with centre $(1, -2)$ and radius 2.

b) W is the line with equation $x - 2y = 0$.

c) Because the vector $(1, -2)$ is perpendicular to W, $(0, 0)$ is the point on W closest to V. The distance from $(0, 0)$ to the centre is $\sqrt{5}$, so the distance to V is

$$\boxed{\sqrt{5} - 2}$$

Example test chapter 3

including chapters 3 and 5 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n - 5$ points.

1. For how many real numbers x does ${}^2\log(3x) = 1 + {}^2\log x$ hold?

   a. none b. one c. two d. three

2. You can simplify $\dfrac{x - \sqrt{x}}{1 - \sqrt{x}}$ to obtain

   a. $x - 1$ b. $1 - \sqrt{x}$ c. $-\sqrt{x}$ d. $1 - x$

3. For all positive real numbers x the expression $2^{\ln x}$ equals

   a. $x \ln 2$ b. $x^{\ln 2}$ c. x^2 d. $\sqrt{x}$

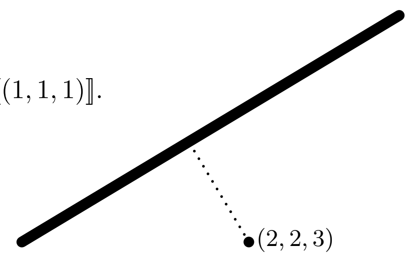
4. The number of real numbers x satisfying $|1 - x^2| = |2 - x^2|$ is

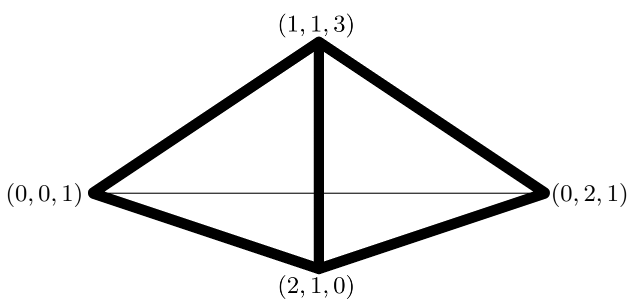
   a. zero b. one c. two d. four

5. How many real solutions to the equation $e^{2x} = 20 + e^x$ exist?

   a. none b. one c. two d. three

Exercise 2. Calculate the distance from $(2, 2, 3)$ to the line $(1, 0, 0) + [\![(1, 1, 1)]\!]$.

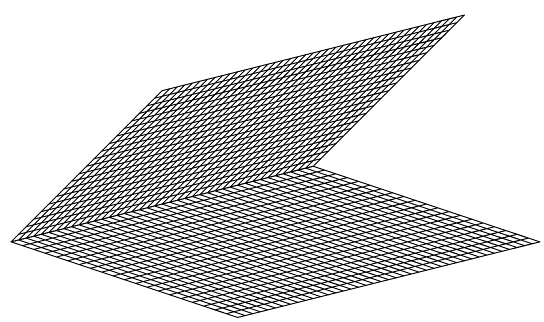




Exercise 3. Calculate the volume of the tetrahedron with vertices $(0, 0, 1)$, $(2, 1, 0)$ $(0, 2, 1)$ and $(1, 1, 3)$.

Exercise 4. Calculate the angle between the planes

$$V \stackrel{\text{def}}{=} \text{the plane } 2x + 3y + z = 5$$
$$W \stackrel{\text{def}}{=} \text{the plane } [\![(2, 1, 0), (3, 0, 1)]\!]$$



Exercise 5. Find the matrix A satisfying $A \cdot \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 9 \\ 0 & 1 \end{pmatrix}$.

Solutions

Exercise 1. $\boxed{\text{1a, 2c, 3b, 4c, 5b}}$

1. I use the trick $2^{\text{left-hand side}} = 2^{\text{right-hand side}}$:

$$^2\!\log(3x) = 1 + {}^2\!\log x \quad \Longrightarrow \quad 3x = 2\cdot x \quad \Longrightarrow \quad x = 0 \quad \Longrightarrow \quad \text{impossible because } {}^2\!\log 0 \text{ does not exist}$$

2. Factor out a common factor $\sqrt{x}$ from the numerator: $\dfrac{x-\sqrt{x}}{1-\sqrt{x}} = \dfrac{\sqrt{x}\cdot(\sqrt{x}-1)}{1-\sqrt{x}} = -\sqrt{x}$.

3. You can for example see this as follows: $2^{\ln x} = e^{\ln 2^{\ln x}} = e^{(\ln x)(\ln 2)} = \left(e^{\ln x}\right)^{\ln 2} = x^{\ln 2}$.

4. $|1-x^2| = |2-x^2|$ implies $1-x^2 = x^2-2$, so $2x^2 = 3$. The two solutions are $\pm\sqrt{1.5}$.

5. Only $x = \ln 5$ is a solution to this equation, because

$$e^{2x} - e^x - 20 = 0 \quad \Longrightarrow \quad (e^x-5)\,(e^x+4) = 0 \quad \Longrightarrow \quad e^x - 5 = 0 \quad \Longrightarrow \quad x = \ln 5$$

Exercise 2. I translate the problem over $(-1,0,0)$ and calculate the distance from $(1,2,3)$ to $[\![(1,1,1)]\!]$:

- the projection of $(1,2,3)$ on $(1,1,1)$ is $\dfrac{(1,2,3)\bullet(1,1,1)}{(1,1,1)\bullet(1,1,1)}\cdot(1,1,1) = 2\cdot(1,1,1) = (2,2,2)$
- so the distance is $\|(1,2,3)-(2,2,2)\| = \|(-1,0,1)\| = \boxed{\sqrt{2}}$

Exercise 3. I lower the tetrahedron one metre, so the vertices become $(0,0,0)$, $(2,1,-1)$, $(0,2,0)$ and $(1,1,2)$. Now, I can use the formula that you invented in exercise 11 from chapter 3: the volume is

$$\frac{1}{6}\det\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \frac{1}{6}\cdot 10 = \boxed{\frac{5}{3}}$$

Exercise 4. I can for instance calculate this angle α as follows:

- perpendicular to V is the vector $(2,3,1)$, because V is parallel to the plane $2x+3y+z=0$
- perpendicular to W is the vector $(2,1,0)\times(3,0,1) = (1,-2,-3)$
- so $\cos\alpha = \dfrac{(2,3,1)\bullet(1,-2,-3)}{\|(2,3,1)\|\cdot\|(1,-2,-3)\|} = \dfrac{-7}{\sqrt{14}\cdot\sqrt{14}} = -\dfrac{1}{2}$
- which gives $\alpha = 120°$, so the (acute) angle between the planes V and W is $\boxed{60°}$

Exercise 5. You probably reduced this exercise to four equations in four unknowns, but you can as well use chap 3 ex 6:

$$A = \begin{pmatrix} 2 & 9 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 9 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -41 & 25 \\ -5 & 3 \end{pmatrix}$$

Example test chapter 4

including chapters 7 and 8 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n - 5$ points.

1. The tangent line at $(1, 2)$ to the graph of $f(x) = \dfrac{3 - x^2}{2 - x^2}$ has equation

   a. $x + y = 3$ b. $y = 1 + x$ c. $y = 2x$ d. $x + 2y = 5$

2. If $f(x) = \sin(\arctan x)$ then $\overleftarrow{f}(x) =$

   a. $\arcsin(\tan x)$ b. $\tan(\arcsin x)$ c. $\arcsin(\arctan x)$ d. $\arctan(\sin x)$

3. The derivative of $\sqrt{3 - \dfrac{3}{4x}}$ at $x = 1$ is

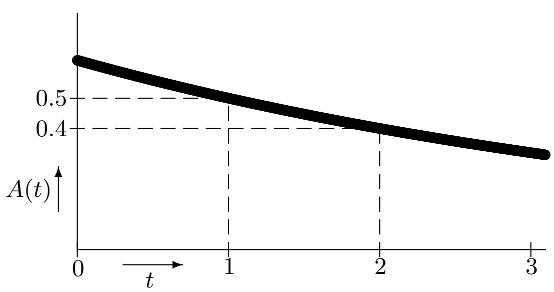
   a. -0.75 b. -0.25 c. 0.25 d. 0.75

4. The number of real solutions to the equation $\sqrt{x} = 3 - x$ is

   a. zero b. one c. two d. three

5. How much jenever (32% alcohol) should be mixed with 0.4 litres of beer (5% alcohol) to obtain a mix with precisely 8% of alcohol?

   a. 3 cl b. 4 cl c. 5 cl d. 6 cl



Exercise 2. (units: hours, promilles)
My blood alcohol content $A(t)$ satisfies:

$$\boxed{A(1) = 0.5} \qquad \boxed{A(2) = 0.4}$$

$$\boxed{A \text{ decreases exponentially from } t = 0}$$

At what time t is $A(t) = 0.1$?

Exercise 3. I define two functions f and g from $[0, \infty)$ to $[0, \infty)$ by

$$\boxed{f(x) = \ln(1 + x)} \qquad \boxed{g(x) = \ln(1 + x^2)}$$

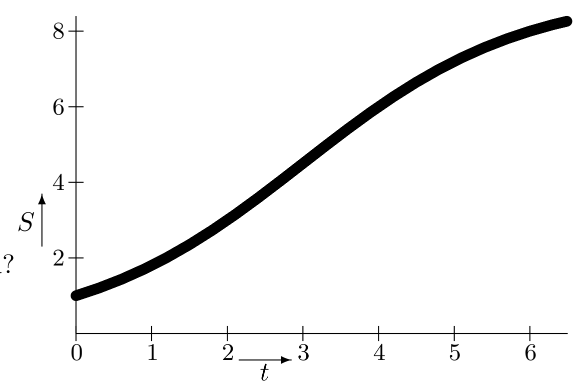
a) Determine a simplest possible function rule for the function $\overleftarrow{g} \circ f$.

b) Determine a simplest possible function rule for the function $\overleftarrow{f} \circ g$.

Exercise 4. (units: days, grams)
The amount of mould S in my lunch box is logistically increasing with a limit of 9 grams. Given is

$$\boxed{S(0) = 1.0} \qquad \boxed{S(1) = 1.8} \qquad \boxed{S(\infty) = 9}$$

When will my lunch box contain exactly 8 grams of mould?



Exercise 5. Check whether the following formula holds for all real numbers x (give either a proof or a counterexample):

$$\boxed{\cosh(2x) = (\cosh x)^2 + (\sinh x)^2}$$

Solutions

Exercise 1. $\boxed{\text{1c, 2b, 3c, 4b, 5c}}$

1. $f(x) = 1 + \dfrac{1}{2-x^2} \quad\Longrightarrow\quad f'(x) = \dfrac{-1}{(2-x^2)^2}\cdot(-2x) \quad\Longrightarrow\quad f'(1) = 2 \quad\Longrightarrow\quad$ tangent line $y = 2x$
2. $y = \sin(\arctan x) \quad\Longrightarrow\quad \arcsin y = \arctan x \quad\Longrightarrow\quad \tan(\arcsin y) = x$
3. Chain rule $\quad\Longrightarrow\quad \dfrac{d}{dx}\sqrt{3-\dfrac{3}{4x}} = \dfrac{1}{2\sqrt{3-\frac{3}{4x}}}\cdot\dfrac{3}{4x^2} \quad\Longrightarrow\quad \left[\dfrac{d}{dx}\sqrt{3-\dfrac{3}{4x}}\right]_{x=1} = \dfrac{1}{3}\cdot\dfrac{3}{4} = \dfrac{1}{4}$
4. $x + \sqrt{x} - 3 = 0 \quad\Longrightarrow\quad \sqrt{x} = \dfrac{-1\pm\sqrt{13}}{2} \quad\Longrightarrow\quad \sqrt{x} = \dfrac{-1+\sqrt{13}}{2} \quad\Longrightarrow\quad x = \left(\dfrac{-1+\sqrt{13}}{2}\right)^2$
5. Suppose you add x litres of jenever. Because the beer contains 0.02 litres of alcohol, we have

$$\frac{0.32\,x + 0.02}{x+0.4} = 0.08 \quad\Longrightarrow\quad 0.32\,x + 0.02 = 0.08\,x + 0.032 \quad\Longrightarrow\quad 0.24\,x = 0.012 \quad\Longrightarrow\quad x = 0.05$$

Exercise 2. I calculate the constants α and β in the exponential model $A(t) = \alpha\cdot\beta^t$:

$$\left.\begin{array}{l} A(1) = 0.5 \quad\Longrightarrow\quad \alpha\beta = 0.5 \\ A(2) = 0.4 \quad\Longrightarrow\quad \alpha\beta^2 = 0.4 \end{array}\right\} \overset{\text{divide}}{\Longrightarrow} \quad \beta = \frac{4}{5} \quad\Longrightarrow\quad \alpha = \frac{5}{8} \quad\Longrightarrow\quad A(t) = \frac{5}{8}\cdot\left(\frac{4}{5}\right)^t$$

which equals 0.1 when $\left(\dfrac{4}{5}\right)^t = 0.16 \overset{\text{ln-trick}}{\Longrightarrow} \boxed{t = \dfrac{\ln 0.16}{\ln 0.8}}$ (after approximately 8 hours and 13 minutes)

Exercise 3.

a) I determine a function rule for $\overleftarrow{g}$:

$$y = g(x) \quad\Longrightarrow\quad y = \ln(1{+}x^2) \quad\Longrightarrow\quad e^y = 1{+}x^2 \quad\Longrightarrow\quad x^2 = e^y{-}1 \quad\Longrightarrow\quad x = \overleftarrow{g}(y) = \sqrt{e^y-1}$$

Hence, the composition $\overleftarrow{g}\circ f$ is $(\overleftarrow{g}\circ f)(x) = \overleftarrow{g}(f(x)) = \sqrt{e^{f(x)}-1} = \sqrt{(1+x)-1} = \boxed{\sqrt{x}}$.

b) I determine a function rule for $\overleftarrow{f}$:

$$y = f(x) \quad\Longrightarrow\quad y = \ln(1+x) \quad\Longrightarrow\quad e^y = 1+x \quad\Longrightarrow\quad x = \overleftarrow{f}(y) = e^y - 1$$

Hence, the composition $\overleftarrow{f}\circ g$ is $\left(\overleftarrow{f}\circ g\right)(x) = \overleftarrow{f}(g(x)) = e^{g(x)} - 1 = 1 + x^2 - 1 = \boxed{x^2}$.

Exercise 4. In the standard form of a logistic process with limit 9 I substitute the measurements:

$$S = \frac{9}{1+\beta\cdot\gamma^t} \overset{S(0)=1.0}{\Longrightarrow} \beta = 8 \quad\Longrightarrow\quad S = \frac{9}{1+8\cdot\gamma^t} \overset{S(1)=1.8}{\Longrightarrow} \gamma = 0.5 \quad\Longrightarrow\quad S = \frac{9}{1+8\cdot 0.5^t}$$

Now, I figure out when this equals 8:

$$\frac{9}{1+8\cdot 0.5^t} = 8 \quad\Longrightarrow\quad 0.5^t = \frac{1}{64} \quad\Longrightarrow\quad 2^t = 64 \quad\Longrightarrow\quad \boxed{t = 6} \quad \text{(so after precisely 6 days)}$$

Exercise 5. This formula holds for all real x. The proof is just a matter of substitution:

$$(\cosh x)^2 + (\sinh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^x+e^{-x})^2 + (e^x - e^{-x})^2}{4}$$
$$= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4} = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh(2x)$$

Example test chapter 5

including chapters 1-4 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n-5$ points.

1. The line through $(3,5)$ and $(5,2)$ passes through the point

   a. $(10,-7)$ b. $(11,-7)$ c. $(12,-7)$ d. $(13,-7)$

2. For all $x>2$ the expression $\dfrac{\sqrt{x^2-4}}{\sqrt{x-2}}$ equals

   a. $\sqrt{x-2}$ b. $\sqrt{x+2}$ c. $\sqrt{x}-2$ d. $\sqrt{x}+2$

3. $p=\dfrac{2t+1}{1-t}$ implies $t=$

   a. $\dfrac{1-p}{2-p}$ b. $\dfrac{p-1}{2+p}$ c. $\dfrac{p+1}{2-p}$ d. $\dfrac{1-p}{p+2}$

4. Complete the sentence: the point $(5,3)$ lies $\cdots\cdots$ the circle with equation $x^2+y^2=3+6x$

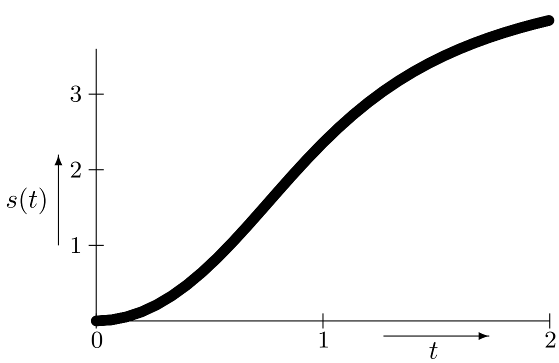
   a. inside b. outside c. on the edge of

5. When does the square of $\left(\sqrt{5}\right)^x$ equal 5^x ?

   a. never b. sometimes c. often d. always

Exercise 2. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $\boxed{f(x)=\sqrt{5+x^2}}$.

a) Calculate $\left[\dfrac{df}{dx}\right]_{x=2}$. b) Calculate $\left[\dfrac{d^2f}{dx^2}\right]_{x=2}$. c) Calculate $\left[\dfrac{d^3f}{dx^3}\right]_{x=2}$.



Exercise 3. (units: metres, seconds)
Zompie runs from its basket to its food bowl in two seconds. The distance travelled by Zompie after t seconds for $0\le t\le 2$ is given by

$$\boxed{s(t)=3\arctan(t^2)}$$

a) Calculate Zompie's velocity at $t=1$.

b) Calculate Zompie's acceleration at $t=1$.

Exercise 4. Calculate the following limits:

a) $\displaystyle\lim_{x\to1}\frac{3^x-3}{x-1}$ b) $\displaystyle\lim_{x\to0}\frac{\sin x}{1-\sqrt{1-x}}$

Do one of the exercises 5_{sc} (recommended for science) and $5_{mls,ch}$ (intended for mls and chemistry).

Exercise $5_{\mathbf{sc}}$ Determine the equation of the tangent line at the point $(1,3)$ to the curve

$$\boxed{y^3=8xy+3x^3}$$

Exercise $5_{\mathbf{mls,ch}}$ The Lineweaver-Burk plot of a Michaelis-Menten reaction passes through $(1,3)$ and $(2,5)$. Find the reaction constants $V_{\max}$ and K_m.

Solutions

Exercise 1. $\boxed{\text{1b, 2b, 3b, 4b, 5d}}$

1. The slope is $\frac{2-5}{5-3} = -\frac{3}{2}$, so the equation of the line is $y = -\frac{3}{2}x + \frac{19}{2}$ and it passes through $(11, -7)$.

2. $\dfrac{\sqrt{x^2-4}}{\sqrt{x-2}} = \dfrac{\sqrt{(x+2)(x-2)}}{\sqrt{x-2}} = \dfrac{\sqrt{x+2}\cdot\sqrt{x-2}}{\sqrt{x-2}} = \sqrt{x+2}$

3. $p(1-t) = 2t+1 \implies p - pt = 2t+1 \implies p-1 = 2t+pt = t(2+p) \implies t = \dfrac{p-1}{2+p}$

4. Write the circle equation as $(x-3)^2 + y^2 = 12$. The distance from $(5,3)$ to the centre $(3,0)$ is $\sqrt{13}$ which is greater than the radius $\sqrt{12}$

5. The calculation rule $(p^q)^r = p^{qr}$ implies $\left(\left(\sqrt{5}\right)^x\right)^2 = \left(\sqrt{5}\right)^{x\cdot 2} = \left(\sqrt{5}\right)^{2\cdot x} = \left(\left(\sqrt{5}\right)^2\right)^x = 5^x$.

Exercise 2.

a) $f'(x) = \dfrac{x}{\sqrt{5+x^2}} \implies f'(2) = \boxed{\dfrac{2}{3}}$

b) $f''(x) = \dfrac{5}{(5+x^2)^{\frac{3}{2}}} \implies f''(2) = \boxed{\dfrac{5}{27}}$

c) $f'''(x) = \dfrac{-15x}{(5+x^2)^{\frac{5}{2}}} \implies f'''(2) = \boxed{-\dfrac{10}{81}}$

Exercise 3.

a) $v(t) = s'(t) = \dfrac{6t}{1+t^4} \implies v(1) = \boxed{3 \text{ m/sec}}$

b) $a(t) = v'(t) = \dfrac{6-18t^4}{(1+t^4)^2} \implies a(1) = \boxed{-3 \text{ m/sec}^2}$

Exercise 4.

a) That is (by definition) the derivative of $x \mapsto 3^x$ at $x = 1$, which is $\boxed{3\ln 3}$.

b) Because substitution of $x = 0$ leads to $\frac{0}{0}$ I use l'Hôpital: $\displaystyle\lim_{x\to 0} \frac{\sin x}{1-\sqrt{1-x}} = \lim_{x\to 0} \frac{\cos x}{\frac{1}{2\sqrt{1-x}}} = \frac{1}{1/2} = \boxed{2}$.

Exercise 5$_{\mathbf{sc}}$ I differentiate the left-hand side and the right-hand side with respect to x:

$$y^3 = 8xy + 3x^3 \overset{d/dx}{\implies} 3y^2\cdot\frac{dy}{dx} = 8y + 8x\cdot\frac{dy}{dx} + 9x^2 \implies \frac{dy}{dx} = \frac{8y+9x^2}{3y^2-8x} \implies \left[\frac{dy}{dx}\right]_{(1,3)} = \frac{33}{19}$$

Hence, the tangent line is $\boxed{y = \dfrac{33}{19}x + \dfrac{24}{19}}$.

Exercise 5$_{\mathbf{mls,ch}}$ The equation of the LB line is $y = 2x+1$ so

$$\left.\begin{array}{l} \frac{1}{V_{\max}} = 1 \\ \frac{K_m}{V_{\max}} = 2 \end{array}\right\} \implies \boxed{V_{\max} = 1} \text{ and } \boxed{K_m = 2}$$

Example test chapter 6

including chapters 5 and 6 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n-5$ points.

1. If $e^x = 2^{x+2}$ then $x =$

   a. $\dfrac{\ln 4}{1+\ln 2}$ b. $\dfrac{1+\ln 2}{\ln 4}$ c. $\dfrac{\ln 4}{1-\ln 2}$ d. $\dfrac{1-\ln 2}{\ln 4}$

2. You can write $\cos x - \sin x$ as

   a. $\sqrt{2}\,\sin\left(x+\frac{\pi}{4}\right)$ b. $\sqrt{2}\,\cos\left(x+\frac{\pi}{4}\right)$ c. $\sqrt{2}\,\sin\left(x-\frac{\pi}{4}\right)$ d. $\sqrt{2}\,\cos\left(x-\frac{\pi}{4}\right)$

3. ${}^{9}\log x$ is equivalent to

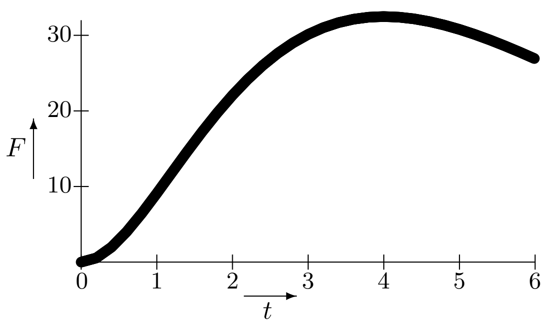
   a. $\left({}^{3}\log x\right)^2$ b. $\sqrt{{}^{3}\log x}$ c. $\frac{1}{2}\cdot {}^{3}\log x$ d. $2\cdot {}^{3}\log x$

4. If α is between 0 and π and $\cos\alpha = -0.8$, then $\tan\alpha =$

   a. 0.75 b. −0.75 c. 1.33 d. −1.33

5. You can simplify $\dfrac{1-e^x}{1-e^{-x}}$ to obtain

   a. e^x b. $-e^x$ c. e^{-x} d. $-e^{-x}$



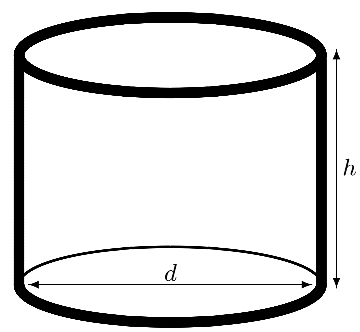
Exercise 2. (units: day, chickens)
The number of chickens per day infected by bird flu for $0 \le t \le 6$ is given by

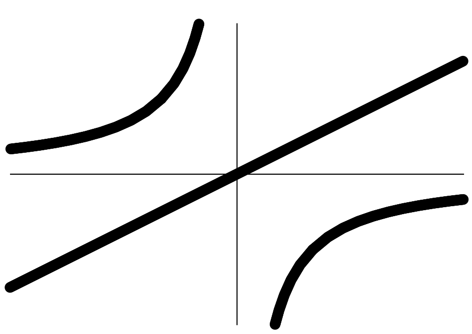
$$\boxed{F = 15t^2e^{-t/2}}$$

a) At what time is F largest?

b) At what time does F increase fastest?

Exercise 3. (unit: dm)
I would like to design a cylindrically shaped tomato paste can with $d+h=1$ (by d I mean the diameter of the bottom and h is the height). How should I choose d and h if I want a maximal tomate paste concent?



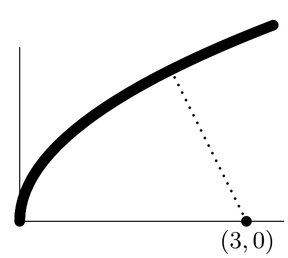


Exercise 4. The equation of these two curves are

line:	$x = 2y$
hyperbola:	$xy = -2$

What points on the hyperbola are closest to the line?

Exercise 5. What point on the graph of the function $x \mapsto \sqrt{2x}$ is closest to the point $(3,0)$?



Solutions

Exercise 1. $\boxed{\text{1c, 2b, 3c, 4b, 5b}}$

1. $x = \ln\left(2^{x+2}\right) = (x+2)\ln 2 = x\ln 2 + 2\ln 2 \quad\Longrightarrow\quad x(1-\ln 2) = 2\ln 2 \quad\Longrightarrow\quad x = \dfrac{2\ln 2}{1-\ln 2}$
2. This follows from $\cos\left(x+\dfrac{\pi}{4}\right) = \cos x\cos\dfrac{\pi}{4} - \sin x\sin\dfrac{\pi}{4} = \dfrac{\cos x - \sin x}{\sqrt{2}}$.
3. This can for example be shown as follows: ${}^{9}\log x = \dfrac{\ln x}{\ln 9} = \dfrac{\ln x}{2\ln 3} = \dfrac{1}{2}\cdot\dfrac{\ln x}{\ln 3} = \dfrac{1}{2}\cdot {}^{3}\log x$.
4. $\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-0.64} = \sqrt{0.36} = 0.6$ en dus $\tan\alpha = \frac{0.6}{-0.8} = -0.75$
5. The numerator is $-e^x$ times the denominator.

Exercise 2. I calculate the first and second derivative of F using the product rule:

$$F = 15t^2e^{-t/2} \quad\Longrightarrow\quad \frac{dF}{dt} = 30\,t\,e^{-t/2} - 7.5\,t^2\,e^{-t/2} \quad\Longrightarrow\quad \frac{d^2F}{dt^2} = \left(3.75\,t^2 - 30t + 30\right)e^{-t/2}$$

a) Where F is largest, its derivative equals zero, which is the case at $t = 0$ (not interesting) or $\boxed{t = 4}$.

b) Where F increases fastest, its second derivative equals zero:

$$3.75\,t^2 - 30t + 30 = 0 \quad\overset{\text{quadratic formula}}{\Longrightarrow}\quad t = 4 + 2\sqrt{2}\ \text{(not interesting) or}\ \boxed{t = 4 - 2\sqrt{2}} \approx 1.1716$$

Exercise 3. Given d, we have

$$\left.\begin{array}{lcl} \text{area (bottom)} & = & \frac{1}{4}\pi d^2 \\ \text{height} & = & 1-d \end{array}\right\} \quad\Longrightarrow\quad \text{content} = \frac{1}{4}\pi d^2(1-d)$$

Now, the question is: for what choice of d is $f(d)$, defined by $f(d) = d^2(1-d) = d^2 - d^3$, maximum? Set $f'(d) = 0$ equal to zero:

$$f'(d) = 0 \quad\Longleftrightarrow\quad 2d - 3d^2 = 0 \quad\Longleftrightarrow\quad d = 0 \ \text{or}\ d = \frac{2}{3}$$

The solution $d = 0$ would amount to a ridiculous can, so I choose $\boxed{d = \dfrac{2}{3}\ \text{and}\ h = \dfrac{1}{3}}$.

Exercise 4. Express y in terms of x, and set the derivative $\frac{dy}{dx}$ at the desired points equal to $\frac{1}{2}$:

$$y = \frac{-2}{x} \quad\Longrightarrow\quad \frac{dy}{dx} = \frac{2}{x^2} = \frac{1}{2} \quad\Longrightarrow\quad x^2 = 4 \quad\Longrightarrow\quad x = \pm 2$$

so the desired points are $\boxed{(2,-1)\ \text{and}\ (-2,1)}$.

Exercise 5. The distance from $\left(x, \sqrt{2x}\right)$ to $(3,0)$ is

$$\left\|\left(x,\sqrt{2x}\right) - (3,0)\right\| = \left\|\left(x-3,\sqrt{2x}\right)\right\| = \sqrt{(x-3)^2 + 2x} = \sqrt{x^2 - 4x + 9} = \sqrt{(x-2)^2 + 5}$$

which is minimum if $x = 2$, so the desired point is $\boxed{(2,2)}$.

Example test chapter 7

including chapters 7 and 8 from the arithmetic booklet

Exercise 1. Multiple choice, no explanations needed, n correct answers corresponds to $3n-5$ points.

1. The derivative of $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}$ at $x=0$ is

   a. -1 b. 0 c. 1 d. 2

2. How much whisky (37% alcohol) should be mixed with wine (12% alcohol) to obtain one litre of a mix with 18 percent of alcohol?

   a. 24 cl b. 25 cl c. 26 cl d. 27 cl

3. If $f(x)=\ln(1+x^2)$ then $f''(0)=$

   a. -1 b. 0 c. 1 d. 2

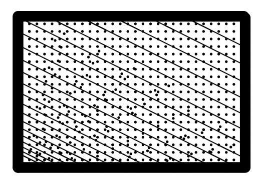
4. $\alpha=\beta+4$ and $\alpha^2=\beta^2-8$ imply $\alpha=$

   a. -1 b. 0 c. 1 d. 2

5. The tangent line at $(1,1)$ to the hyperbola $y=\dfrac{3x-1}{x+1}$ has equation

   a. $y=x$ b. $y=2x-1$ c. $y=3x-2$ d. $y=4x-3$

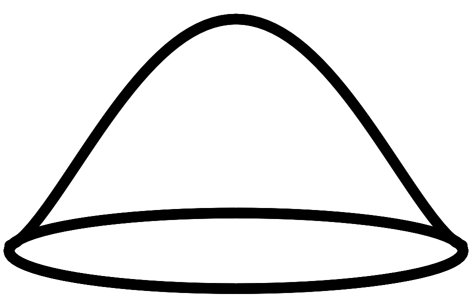
Exercise 2. (units: metres, percent)
In the mud pool $[0,3]\times[0,2]$ the humidity is given by



$$\boxed{H(x,y)=2^x\cdot 3^y}$$

a) Determine the gradient of H at the point $(2,1)$.

b) Sketch the level curve of H passing through $(2,1)$, and determine its equation.



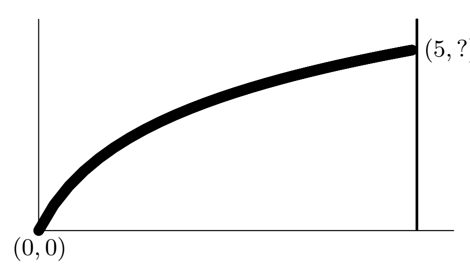
Exercise 3. The point $P=(1,1,1)$ lies on the curved surface S with equation

$$\boxed{z=\sqrt{2\,(2-y^2)^3-x^2}}$$

Determine the equation of the tangent plane at P to S.

Exercise 4. (units: km, degrees Celsius)
From the starting point $(0,0)$ I walk constantly in the direction in which the temperature increases most, until I arrive at the line $x=5$. At what point on this line will I arrive if the temperature is given by

$$\boxed{\text{Temp}\,(x,y)=x+x^2+2y}$$



Solutions

Exercise 1. $\boxed{\text{1c, 2a, 3d, 4c, 5a}}$

1. According to the quotient rule the derivative is $\dfrac{\frac{1}{2\sqrt{1+x}}\cdot\sqrt{1-x}-\frac{-1}{2\sqrt{1-x}}\cdot\sqrt{1+x}}{1-x} \overset{x=0}{\Longrightarrow} 1$

2. The mix of x litres of whisky and $1-x$ litres of wine should contain 0.18 litres of alcohol, so
$$0.37\,x+0.12\cdot(1-x)=0.18 \quad\Longrightarrow\quad 0.25\,x=0.06 \quad\Longrightarrow\quad x=0.24$$

3. $f'(x)=\dfrac{2x}{1+x^2} \quad\Longrightarrow\quad f''(x)=\dfrac{2-2x^2}{\left(1+x^2\right)^2} \quad\Longrightarrow\quad f''(0)=2$

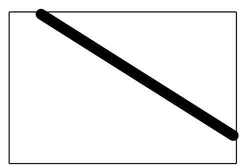
4. Substitute $\alpha-4$ for β in the second equation: $\alpha^2=(\alpha-4)^2-8 \quad\Longrightarrow\quad \alpha^2=\alpha^2-8\alpha+8 \quad\Longrightarrow\quad \alpha=1$

5. $\left[\dfrac{dy}{dx}\right]_{x=1}=\left[\dfrac{4}{(x+1)^2}\right]_{x=1}=1 \quad\Longrightarrow\quad$ the tangent line is $y=x$

Exercise 2.

a) $\nabla V=\left(\dfrac{\partial V}{\partial x}\,,\,\dfrac{\partial V}{\partial y}\right)=\left(2^x\cdot3^y\cdot\ln 2\,,\,2^x\cdot3^y\cdot\ln 3\right) \quad\Longrightarrow\quad (\nabla V)_{(2,1)}=\boxed{(12\ln 2\,,\,12\ln 3)}$

b) $V(x,y)=12 \quad\overset{\text{ln-trick}}{\Longrightarrow}\quad \boxed{x\ln 2+y\ln 3=\ln 12}$



(which is a straight line segment)

Exercise 3. I calculate the partial derivatives and the total differential of z at P:

- $\dfrac{\partial z}{\partial x}=\dfrac{-2x}{2\sqrt{2\left(2-y^2\right)^3-x^2}} \quad\Longrightarrow\quad \left[\dfrac{\partial z}{\partial x}\right]_P=-1$
- $\dfrac{\partial z}{\partial y}=\dfrac{-12y\left(2-y^2\right)^2}{2\sqrt{2\left(2-y^2\right)^3-x^2}} \quad\Longrightarrow\quad \left[\dfrac{\partial z}{\partial y}\right]_P=-6$
- so the total differential is $dz=-dx-6dy$
- and hence the tangent plane is $(z-1)=-(x-1)-6(y-1)$ which reduces to $\boxed{x+6y+z=8}$.

Exercise 4. My walking direction from (x,y) is $\nabla\text{Temp}=(1+2x\,,\,2)$, so my route satisfies
$$\frac{dy}{dx}=\frac{2}{1+2x} \quad\Longrightarrow\quad y=\ln(1+2x)+C \quad\overset{(0,0)}{\Longrightarrow}\quad y=\ln(1+2x)$$

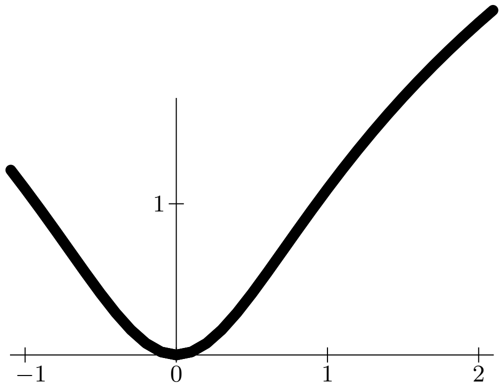
Thus, I'll arrive at the point $\boxed{(5\,,\,\ln 11)}$.

Example exam 1

Scores per exercise:

ex 1	ex 2	ex 3	ex 4	ex 5	ex 6	ex 7	ex 8	ex 9
10	12	12	10	12	10	12	10	12

$$\text{grade} = \frac{\text{total score}}{10}$$



Exercise 1. Find the points on the curve

$$\boxed{y = \ln\left(1 + 2x^2\right)}$$

where the slope is precisely 1.

Exercise 2. (units: hours, metres)
A snail is creeping in $\mathbb{R}^2$ along a straight line from $(-1, 1)$ to $(1, 0)$, its position at time t between $t = 0$ and $t = 1$ is given by the formula

$$\boxed{\operatorname{snail}(t) = \left(-1 + 2\sqrt{t}\,,\, 1 - \sqrt{t}\right)}$$

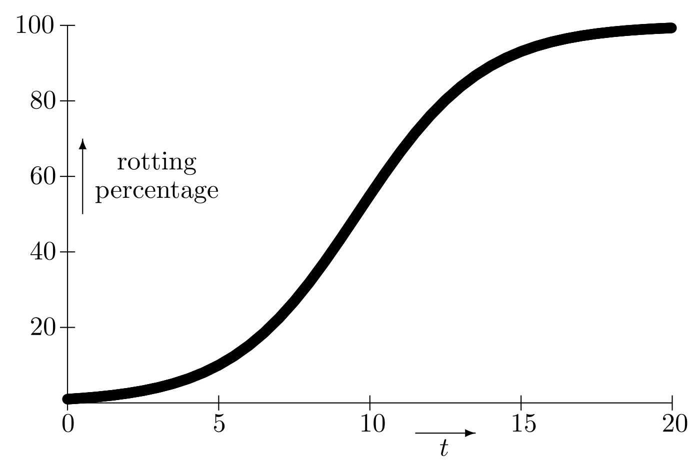
At what time t is the snail closest to the point $(0, 0)$?

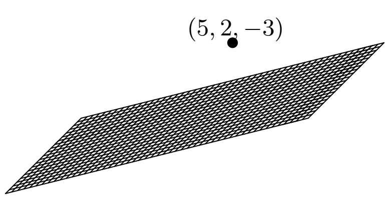
Exercise 3. (units: days, percent)
At time $t = 0$ only one percent of a set of raspberries is rotten, but at $t = 5$ this has already increased to ten percent:

$$\boxed{P(0) = 1} \qquad \boxed{P(5) = 10}$$

Find the time at which exactly fifty percent of the set of raspberries is rotten. You can assume that the rotting process follows the logistic model

$$\boxed{P = \frac{100}{1 + \beta \cdot \gamma^t}}$$

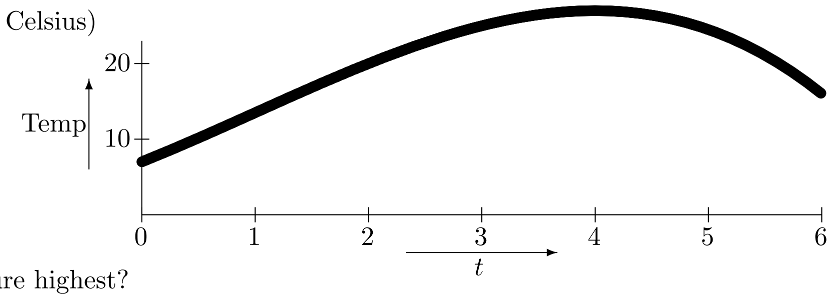




Exercise 4. Calculate the distance from the point $(5, 2, -3)$ to the plane with equation $2x + y = 2z$.

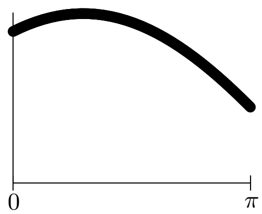
Exercise 5. (units: hours, degrees Celsius)
In the period from $t = 0$ to $t = 6$ the temperature was given by the formula

$$\text{Temp}(t) = 7 + 6t + \frac{3}{4}t^2 - \frac{1}{4}t^3$$



a) At what time was the temperature highest?

b) At what time did the temperature increase fastest?



Exercise 6. Calculate the maximum value on the domain $[0, \pi]$ of the function f given by

$$f(x) = \sin\frac{x}{2} + 2\cos\frac{x}{2}$$

Exercise 7. Simplify the following expressions:

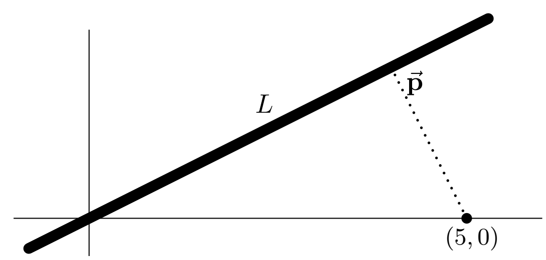
a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(1+x^2)^n}$

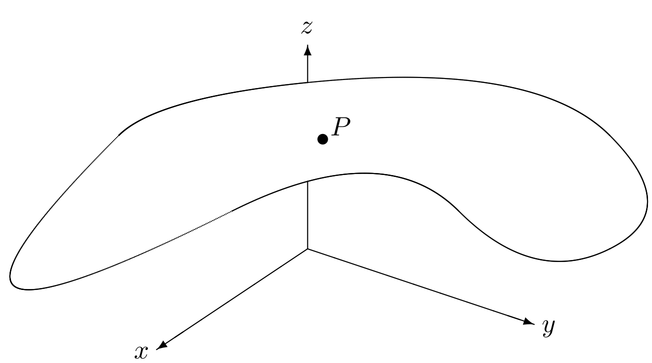
b) $\displaystyle\tan\left(\arcsin\frac{x}{\sqrt{1+x^2}}\right)$

Exercise 8. L is the line with equation $x = 2y$.

a) Calculate the projection of $(5, 0)$ on L.

b) Calculate the distance from $(5, 0)$ to L.





Exercise 9. This curved surface is part of the roof that is given by the equation

$$z = \frac{1 + x + y}{1 + x^2 + y^2}$$

Let P be the point $(1, 1, 1)$ on the roof.

a) Calculate the slope of the plane at P.

b) Find the tangent plane at P to the roof.

Solutions

Exercise 1. The derivative of y with respect to x equals 1 if

$$\frac{dy}{dx} = \frac{4x}{1+2x^2} = 1 \implies 1+2x^2 = 4x \implies 2x^2-4x+1=0 \implies x = \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{1}{2}\sqrt{2}$$

so the desired points are $\boxed{\left(1 \pm \frac{1}{2}\sqrt{2}\,,\, \ln\left(4 \pm 2\sqrt{2}\right)\right)}$.

Exercise 2. There are many ways to tackle this problem. For example:

- At time t its distance to the origin is

$$\|\text{snail}\,(t)\| = \sqrt{\left(-1+2\sqrt{t}\right)^2 + \left(1-\sqrt{t}\right)^2} = \sqrt{5t - 6\sqrt{t}+2} = \sqrt{5} \cdot \sqrt{t - 1.2\sqrt{t} + 0.4}$$

- I complete a square: $t - 1.2\sqrt{t} + 0.4 = \left(\sqrt{t} - 0.6\right)^2 + 0.04$.
- This is minimum if $\sqrt{t} = 0.6$ which amounts to $\boxed{t = 0.36}$.

Exercise 3. I substitute the data:

$$P(0) = 1 \implies \beta = 99 \implies P = \frac{100}{1 + 99 \cdot \gamma^t}$$

$$P(5) = 10 \implies 99 \cdot \gamma^5 = 9 \implies \gamma^5 = \frac{1}{11} \implies P = \frac{100}{1 + 99 \cdot 11^{-0.2\,t}}$$

Now, I need to calculate when this equals 50:

$$P = 50 \implies 11^{-0.2\,t} = \frac{1}{99} \implies 11^{0.2\,t} = 99 \overset{\text{ln-trick}}{\implies} \boxed{t = \frac{5 \ln 99}{\ln 11}} \quad \text{(after approximately 9.58 days)}$$

Exercise 4. You can do this very neatly as follows:

- Perpendicular to this plane is the vector $(2, 1, -2)$.
- The projection of $(5, 2, -3)$ on $(2, 1, -2)$ is $\dfrac{(5,2,-3) \bullet (2,1,-2)}{(2,1,-2) \bullet (2,1,-2)} \cdot (2,1,-2) = (4,2,-4)$.
- Hence, the desired distance is $\|(4,2,-4)\| = \boxed{6}$.

Exercise 5.

a) $\text{Temp}'(t) = 6 + \frac{3}{2}t - \frac{3}{4}t^2 = \frac{3}{4}(2+t)(4-t)$. This is positive from $t = 0$ to $t = 4$ (the temperature increases) and negative if $t > 4$ (it gets colder there). Thus, at $\boxed{t = 4}$ the temperature is highest.

b) $\text{Temp}''(t) = \frac{3}{2} - \frac{3}{2}t$. This is positive if $t < 1$ (the graph becomes steeper) and negative if $t > 1$ (the graph becomes less steep). Thus, the temperature increases fastest at $\boxed{t = 1}$.

Exercise 6. Write this function rule as $\alpha \sin(\frac{x}{2} + \beta)$, then the number $|\alpha|$ is the desired maximum:

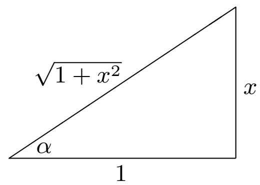
$$\alpha \sin\left(\frac{x}{2} + \beta\right) = \alpha \sin\frac{x}{2}\cos\beta + \alpha\cos\frac{x}{2}\sin\beta \implies \begin{cases} \alpha\cos\beta = 1 \implies \alpha^2\cos^2\beta = 1 \\ \alpha\sin\beta = 2 \implies \alpha^2\sin^2\beta = 4 \end{cases} \quad + \quad \alpha^2 = 5$$

so the maximum value of f is $\boxed{\sqrt{5}}$.

Exercise 7.

a) This is a geometric sequence: $\sum_{n=0}^{\infty}\left(\frac{x^2}{1+x^2}\right)^n = \frac{1}{1-\frac{x^2}{1+x^2}} = \frac{1}{\frac{1}{1+x^2}} = \boxed{1+x^2}$.

b) This can most easily be done geometrically, because $\arcsin\frac{x}{\sqrt{1+x^2}}$ is the angle α in this picture:



$\Longrightarrow \quad \tan\alpha = \boxed{x}$

Exercise 8. $L = [\![(2,1)]\!]$, because the vector $(2,1)$ satisfies the equation of L.

a) The projection of $(5,0)$ on $(2,1)$ is $\frac{(5,0)\bullet(2,1)}{(2,1)\bullet(2,1)}\cdot(2,1) = \boxed{(4,2)}$.

b) This distance is $\|(5,0)-(4,2)\| = \|(1,-2)\| = \boxed{\sqrt{5}}$.

Exercise 9. First, I calculate the partial derivatives at P:

$$\frac{\partial z}{\partial x} = \frac{1+x^2+y^2-(1+x+y)\cdot 2x}{(1+x^2+y^2)^2} \quad \Longrightarrow \quad \left[\frac{\partial z}{\partial x}\right]_{(1,1)} = -\frac{1}{3} \quad \text{and analoguously:} \quad \left[\frac{\partial z}{\partial y}\right]_{(1,1)} = -\frac{1}{3}$$

a) The slope at P is $\sqrt{\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^2} = \boxed{\frac{1}{3}\sqrt{2}}$.

b) The tangent plane is $z-1 = -\frac{1}{3}(x-1)-\frac{1}{3}(y-1) \quad \Longrightarrow \quad \boxed{x+y+3z=5}$

Example exam 2

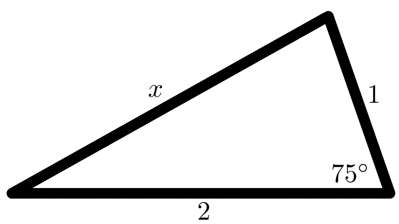
Scores per exercise:

ex 1	ex 2	ex 3	ex 4	ex 5	ex 6	ex 7	ex 8
15	10	15	10	10	15	10	15

$\text{grade} = \dfrac{\text{total score}}{10}$

Exercise 1. Simplify the following expressions:

a) $\dfrac{x+3\sqrt{x}}{3+\sqrt{x}}$ b) $\dfrac{2^{3\ln x}}{x^{\ln 8}}$ c) $\dfrac{1}{\sin^2 x+\tan^2 x+\cos^2 x}+\sin^2 x$

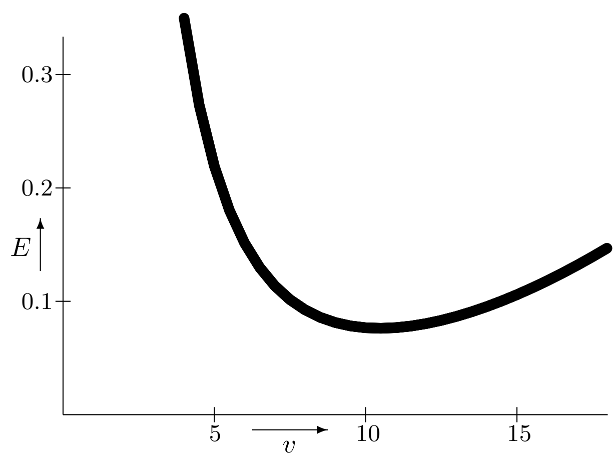


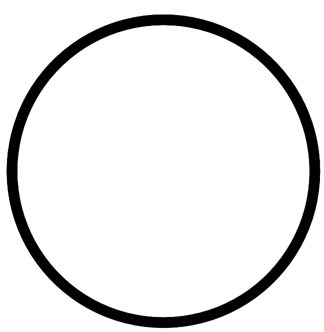
Exercise 2. Calculate in the adjacent triangle the length of the side x opposite to the angle of 75 degrees if the other sides have lengths 1 and 2.

Exercise 3. (units: metres, seconds, joules)
According to a study by Vance Tucker the energy consumption E (in joules/metre) of a flying parakeet depends on its velocity v (in m/sec) as follows:

$$\boxed{E = \frac{6}{v^2}+\frac{v^2}{2000}-\frac{1}{30}}$$

Calculate (using differentiation) the exact optimum cruise speed of a parakeet, i.e. the velocity that requires the least energy per metre.



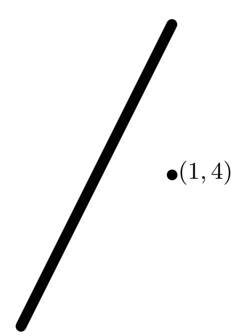


Exercise 4. This circle has equation $x^2+y^2=2x$.

a) Determine the centre and radius of the circle.

b) Calculate the arc length of that part of the circle where $y \geq x\sqrt{3}$.

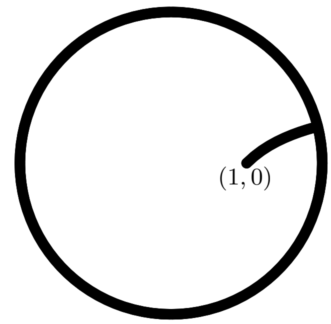
Exercise 5. Calculate the distance from the point $(1,4)$ to the line with equation $y=2x+3$.

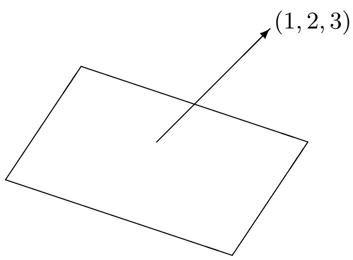


Exercise 6. (units: km, degrees Celsius)
On the disk $x^2 + y^2 \leq 4$ the temperature is given by

$$\boxed{\text{Temp}\,(x, y) = 3y + x^3}$$

A pussycat starts at $(1, 0)$ and walks constantly in the direction in which the temperature increases most, until it arrives at the edge of the circle. Find the equation of the route of the pussycat.



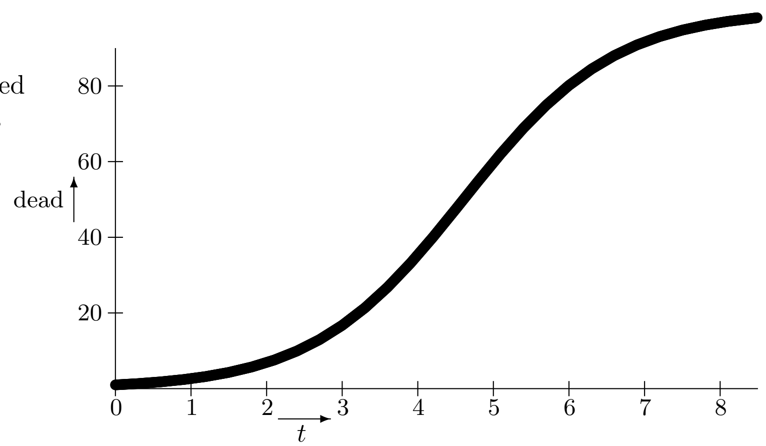


Exercise 7. Calculate the angle between the vector $(1, 2, 3)$ and the plane $[\![(4, 0, 1), (5, 1, 0)]\!]$.

Exercise 8. An H5N1 epidemic caused a total number of victims after t weeks

$$\boxed{\text{dead}\,(t) = \frac{1}{0.01 + e^{-t}}}$$

Calculate the climax of this logistic epidemic, which is to say the exact time t at which the function dead (t) increases fastest.



Solutions

Exercise 1.

a) $\dfrac{x+3\sqrt{x}}{3+\sqrt{x}} = \dfrac{\sqrt{x}\cdot(\sqrt{x}+3)}{3+\sqrt{x}} = \boxed{\sqrt{x}}$

b) $\dfrac{2^{3\ln x}}{x^{\ln 8}} = \dfrac{8^{\ln x}}{x^{\ln 8}} = \dfrac{e^{\ln x\cdot\ln 8}}{e^{\ln 8\cdot\ln x}} = \boxed{1}$

c) $\dfrac{1}{\sin^2 x+\tan^2 x+\cos^2 x}+\sin^2 x = \dfrac{1}{1+\tan^2 x}+\sin^2 x = \cos^2 x+\sin^2 x = \boxed{1}$

Exercise 2. According to the law of cosines,

$$x^2 = 5-4\cos 75^\circ \quad\Longrightarrow\quad x=\sqrt{5-4\cos 75^\circ}$$

This answers will be rewarded with 7 points, but for 10 points you should calculate $\cos 75^\circ$:

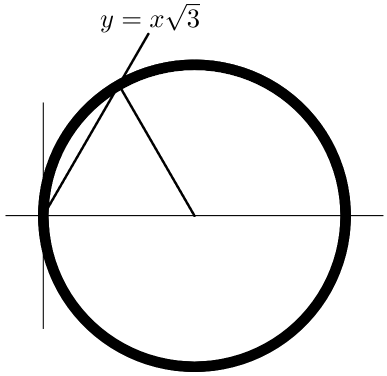
$$\cos 75^\circ = \cos(45^\circ+30^\circ) = \cos 45^\circ\cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2} \quad\Longrightarrow\quad \boxed{x=\sqrt{5-\sqrt{6}+\sqrt{2}}}$$

(approximately 1.99).

Exercise 3. At the optimum speed the derivative of E equals zero:

$$\frac{dE}{dt} = -\frac{12}{v^3}+\frac{v}{1000}=0 \quad\Longrightarrow\quad \frac{12}{v^3}=\frac{v}{1000} \quad\Longrightarrow\quad v^4=12000 \quad\Longrightarrow\quad \boxed{v=\sqrt[4]{12000}}$$

(which is approximately 10.47 m/sec).



Exercise 4.

a) Rewrite the equation to $(x-1)^2+y^2=1$, so that you can see that the centre is $(1,0)$ and the radius is 1.

b) The angle between the line $y=x\sqrt{3}$ and the x-axis is 60°. The triangle in my figure is equilateral, so the arc in question can be seen under an angle of 60° from the centre

$$\Longrightarrow \quad \boxed{\text{arc length} = \frac{\pi}{3}}$$

Exercise 5. The distance from $(1,4)$ to the point (x,y) on this line is

$$\|(x,2x+3)-(1,4)\| = \|(x-1,2x-1)\| = \sqrt{(x-1)^2+(2x-1)^2} = \sqrt{5x^2-6x+2} = \sqrt{5\left(x-\frac{3}{5}\right)^2+\frac{1}{5}}$$

and the minimum value of this is $\sqrt{0+\dfrac{1}{5}} = \boxed{\dfrac{1}{\sqrt{5}}}$.

Exercise 6. The walking direction from (x,y) is $\nabla\,\text{Temp} = (3x^2,3)$ so the route satisfies

$$\frac{dy}{dx}=\frac{3}{3x^2}=\frac{1}{x^2} \quad\Longrightarrow\quad y=-\frac{1}{x}+C$$

Substitution of $(1,0)$ yields $C=1$, so the equation of the route of the pussycat is $\boxed{y=1-\dfrac{1}{x}}$.

Exercise 7. For example:

- A normal to this plane is $(4,0,1) \times (5,1,0) = (-1,5,4)$.
- The angle α between $(1,2,3)$ and this normal satisfies

$$\cos\alpha = \frac{(1,2,3) \bullet (-1,5,4)}{\|(1,2,3)\| \cdot \|(-1,5,4)\|} = \frac{21}{\sqrt{14} \cdot \sqrt{42}} = \frac{1}{2}\sqrt{3} \quad \Longrightarrow \quad \alpha = 30^\circ$$

- Hence, the angle between $(1,2,3)$ and the plane is $\boxed{60^\circ}$.

Exercise 8. The extent to which this function increases can be found by differentiation:

$$\text{dead}\,(t) = \left(0.01 + e^{-t}\right)^{-1} \quad \Longrightarrow \quad \frac{d\,\text{dead}}{dt} = -\left(0.01 + e^{-t}\right)^{-2} \cdot (-e^{-t}) = \frac{e^{-t}}{(0.01 + e^{-t})^2}$$

If this is maximum, its derivative equals zero, so I have to calculate the second derivative of dead:

$$\frac{d^2\text{dead}}{dt^2} = \frac{-e^{-t} \cdot (0.01 + e^{-t})^2 - e^{-t} \cdot 2\,(0.01 + e^{-t}) \cdot (-e^{-t})}{(0.01 + e^{-t})^4} = \frac{-0.01\,e^{-t} + e^{-2t}}{(0.01 + e^{-t})^3}$$

which equals zero if the numerator equals zero, so if

$$0.01\,e^{-t} = e^{-2t} \quad \overset{\cdot e^t}{\Longrightarrow} \quad 0.01 = e^{-t} \quad \overset{\frac{1}{\cdots}}{\Longrightarrow} \quad 100 = e^t \quad \Longrightarrow \quad \boxed{t = \ln 100} \quad \text{(after approximately 4.6 weeks)}$$