Maximal Subrings of small $\mathbb{C}$-algebras

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Abstract

If k is a field, C is a k-algebra, then $A \subset C$ is called a maximal k-subring of C if there exists no k-subalgebra B of A such that $A \subsetneq B \subsetneq C$. We give several examples of maximal $\mathbb{C}$-subrings and we classify all the maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$ up to automorphisms of $\mathbb{C}[x]$. Later we classify a large number of maximal $\mathbb{C}$-subrings of the maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$.

1 Introduction

If A and B are rings, we say that B is a maximal subring of A if $B \subsetneq A$ and there is no ring C such that $B \subsetneq C \subsetneq A$. Even though there are good reasons to study the maximal subrings of a ring there are cases where it is very difficult to find all the maximal subrings. Such a case is $\mathbb{C}[x]$. For this reason we constrict ourselves only to those subrings that contain a copy of $\mathbb{C}$. There are several motivations to know the maximal subrings. It can be expected that rings and their maximal subrings share many properties. Second some extensions $B \subset A$ may be decomposed into a chain of consecutive maximal subrings $B \subset B_1 \subset \cdots \subset B_n \subset A$.

In this paper we classify all maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$ up to automorphisms. We also classify a big class of maximal $\mathbb{C}$-subrings of the maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$. That classification is not complete.

2 Basic Examples

Definition 1. *If $S \subseteq R$ is an extension of rings, then the conductor ideal of this extension is $\mathfrak{c} = \{x \in R | Rx \subseteq S\}$, that is the largest ideal of R that is contained in S.*

Definition 2. *If k is a field, C is a k-algebra, then $A \subset C$ is called a maximal k-subring of C if there exists no k-subalgebra B of A such that $A \subsetneq B \subsetneq C$.*

Lemma 2.1. *If $f : G \mapsto H$ is an epimorphism of groups, then the assignment $K \mapsto f(K)$ defines a one-to-one correspondence between the set $S_f(G)$ of all subgroups K of G which contain $\ker f$ and the set $S(H)$ of all sugroups of H.*

Proof. The assignment $K \mapsto f(K)$ defines a function $\phi : S_f(G) \mapsto S(H)$ and $f^{-1}(J)$ is a subgroup of G for every subgroup $J < H$. Since $f^{-1}(J) < H$ implies $\ker f < f^{-1}(J)$ and $f(f^{-1}(J)) = J$, ϕ is surjective. On the other hand $f^{-1}(f(K)) = K$ if and only if $\ker \pi < K$. It follows that ϕ is injective. With this the one-to-one correspondence is established. □

Corollary 2.2. *If N is a normal subgroup of a group G, then every subgroup of G/N is of the form K/N where K is a subgroup of G that contains N.*

Proof. Consider the canonical epimorphism $\pi : G \mapsto G/N$. Let $H < G/N$. From 2.1 we have that there exists $K < G$ such that $\pi(K) = H$. Clearly $N = \ker \pi < K$ and $H = \pi(K) = K/N$ as required. □

Corollary 2.3. *If I is an ideal of the ring R, then every subring of R/I is of the form S/I where S is a subring of R that contains I.*

Proof. There are three things to observe. Firstly, a ring is a commutative group with respect to each of its operations. Secondly, ring homomorphisms are group homomorphisms when considering either of the operations. Thirdly, a subset of a ring R that is a subgroup of both $(R, +)$ and $(R, \cdot)$ is a subring of R (and conversely).
Now consider the canonical epimorphism $\pi : R \mapsto R/I$ and a subring $S' < R/I$. From 2.1 the set $S = \pi^{-1}(S')$ is a subgroup of both $(R, +)$ and $(R, \cdot)$. Hence S is a subring of R. Furthermore, because $I \lhd (R, +)$ we have that $S' = S/I$ from 2.2. □

Proposition 2.4. *Let $\mathfrak{c}$ be the conductor of $S \subseteq R$. Then $S \in \mathfrak{M}(R)$ if and only if $S/\mathfrak{c} \in \mathfrak{M}(R/\mathfrak{c})$.*

Proof. Consider the canonical epimorphism $\pi : R \mapsto R/\mathfrak{c}$. 2.1 and 2.3 imply that the map $\pi : R \mapsto R/\mathfrak{c}$ induces an inclusion preserving bijection $\phi : S_{\mathfrak{c}}(R) \mapsto S(R/\mathfrak{c})$ between the set $S_{\mathfrak{c}}(R)$ of all subgroups H of R which contain $\mathfrak{c}(= Ker\ \pi)$ and the set $S(R/\mathfrak{c})$ of all subgroups of $R/\mathfrak{c}$. Hence:

$$S \subsetneq H \subsetneq R \text{ if and only if } \phi(S) \subsetneq \phi(H) \subsetneq \phi(R)$$

which is equivalent to:

$$S \subsetneq H \subsetneq R \text{ if and only if } S/\mathfrak{c} \subsetneq H/\mathfrak{c} \subsetneq R/\mathfrak{c}$$

This immediately implies that $S \in \mathfrak{M}(R)$ if and only if $S/\mathfrak{c} \in \mathfrak{M}(R/\mathfrak{c})$ as required. □

Remark 2.5. *In the proposition above the condition $\mathfrak{c}$ is the conductor ideal can be relaxed. Indeed the proof above only required that $\mathfrak{c}$ is an ideal of both S and R. So the statement of the proposition holds true for the more general case where I is any ideal of R which is contained in S.*

Let us give some examples of maximal subrings of polynomial rings over $\mathbb{C}$.

Example 1. $\mathbb{C}[x]$ *is a maximal subring of* $\mathbb{C}[x, x^{-1}]$.

Indeed if $\mathbb{C}[x]$ is not a maximal subring there exists a subring $S \subsetneq \mathbb{C}[x, x^{-1}]$ such that $S \setminus \mathbb{C}[x] \neq \varnothing$. So there exists an $n \in \mathbb{N}$ such that $x^{-n} \in S$. Consider the subring T which is generated by x^{-n} and notice that $T[x] \subseteq S$ and $T[x] = \mathbb{C}[x, x^{-1}]$ which is not possible.

Example 2. $A := \mathbb{C}[x, y] - \mathbb{C}x$ *is a maximal subring of* $\mathbb{C}[x, y]$.

The conductor obviously contains x^2 and y. In fact $\mathfrak{c} = (x^2, y)$ and we get $A/\mathfrak{c} = \mathbb{C}$ and $\mathbb{C}[x, y]/\mathfrak{c} = \mathbb{C}[x]/(x^2)$. Clearly $\mathbb{C} \in \mathfrak{M}(\mathbb{C}[x]/(x^2))$. Indeed suppose there is a subring R that properly contains $\mathbb{C}$, ie. $ax \in R$ for some $a \in \mathbb{C}$. Let $c_1 + c_2x \in \mathbb{C}[x]/(x^2)$. Then $c_1 + \frac{c_2}{a}ax \in \mathbb{C} + \mathbb{C}R \subset R$ which implies that $\mathbb{C}[x]/(x^2) \subset R$. Thus $\mathbb{C}[x]/(x^2) = R$. Now using proposition 2.4 it follows that $A \in \mathfrak{M}(\mathbb{C}[x, y])$.

Example 3. $A := \mathbb{C}[x] + (1 + xy)\mathbb{C}[x, y]$ *is a maximal subring of* $\mathbb{C}[x, y]$.

The conductor is $\mathfrak{c} = (1 + xy)$ and obviously we get $A/\mathfrak{c} \cong \mathbb{C}[x]$. On the other hand $\mathbb{C}[x, y]/\mathfrak{c} \cong \mathbb{C}[x, x^{-1}]$ because $1 + xy = 0 \Rightarrow y = -\frac{1}{x}$. But from example 1 it follows that $A/\mathfrak{c} \in \mathfrak{M}(\mathbb{C}[x, y]/\mathfrak{c})$. This result and proposition 2.4 give us $A \in \mathfrak{M}(\mathbb{C}[x, y])$.

Proposition 2.6. *All maximal subrings of $\mathbb{Q}$ are the localizations of $\mathbb{Z}$ at the prime ideals, ie.* $\mathfrak{M}(\mathbb{Q}) = \{\mathbb{Z}_{(p)} | p$ *is prime*$\}$.

Proof. We will prove that there is a an inclusion-reversing bijection between $\mathcal{P}$ the power set of the prime numbers and $S(\mathbb{Q})$ the set of the subrings of $\mathbb{Q}$. Define the map $\phi : \mathcal{P} \mapsto S(\mathbb{Q})$ so that $\phi(P) = S_P^{-1}\mathbb{Z}$ where $S_P = \{a \in \mathbb{Z} \mid a = \prod_{i=1}^n p_i \text{ where } p_i \in P, n \in \mathbb{N}\}$. Clearly ϕ is well defined.

Surjectivity:

Let $R \in S(\mathbb{Q})$. Consider $P_R = \{p | p^{-1} \in R\} \in \mathcal{P}$. Define S_R as follows:

$$S_R = \left\{ a \in \mathbb{Z} \mid a = \prod_{i=1}^n p_i \text{ where } p_i \in P_R, n \in \mathbb{N} \right\}$$

Clearly S_R and P_R are in a one-to-one correspondence. Let's show that $S_R^{-1}\mathbb{Z} = R$. By the definition of $S_R^{-1}\mathbb{Z}$ we can write:

$$S_R^{-1}\mathbb{Z} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid (a,b) = 1 \text{ and } b \in S_R \right\}$$

Take $\frac{a}{b} \in S_R^{-1}\mathbb{Z}$. We have that $\frac{1}{b} = \frac{1}{\prod_{i=1}^n p_i} = \prod \frac{1}{p_i} \in R$. So

$$\frac{a}{b} = \underbrace{a}_{\in R} \cdot \underbrace{\frac{1}{b}}_{\in R} \in R \Rightarrow S_R^{-1}\mathbb{Z} \subseteq R$$

Now we only have to prove $S_R^{-1}\mathbb{Z} \supseteq R$. Suppose by contradiction that we can find an element $\frac{a}{b} \in R \setminus S_R^{-1}\mathbb{Z}$ with $a, b \in \mathbb{Z}$. There must be some prime q such that $q \mid b$, $q \nmid a$ and $q \notin S$. Let $b = xq$ for some $x \in \mathbb{Z}$. Then we have:

$$\frac{a}{q} = \frac{xa}{xq} = x \cdot \frac{a}{xq} = \underbrace{x}_{\in R} \cdot \underbrace{\frac{a}{b}}_{\in R} \in R \tag{1}$$

Since $(a, q) = 1$ we can pick $u, v \in \mathbb{Z}$ such that $ua + vq = 1$. From (1) we have $\frac{a}{q} \in R$. Hence $q^{-1} = u \cdot \frac{a}{q} + v \in R$. We just deduced that $q \in S_R$, a contradiction. So $R \setminus S_R^{-1}\mathbb{Z} = \emptyset$ which implies that $R = S_R^{-1}\mathbb{Z}$ as required. We just proved the surjectivity of ϕ because we found P_R such that $\phi(P_R) = R$.

Injectivity is trivial. So the map $\phi : \mathcal{P} \mapsto S(\mathbb{Q})$ is bijective. Furthermore, by the way it is defined, ϕ is inclusion-reversing. Indeed:

$$P_{R_1} \subsetneq P_{R_2} \subsetneq P_{R_3} \Leftrightarrow S_{R_1}^{-1}\mathbb{Z} \supsetneq S_{R_2}^{-1}\mathbb{Z} \supsetneq S_{R_3}^{-1}\mathbb{Z} \Leftrightarrow R_1 \supsetneq R_2 \supsetneq R_3$$

Hence $R = S_R^{-1}\mathbb{Z} \in \mathfrak{M}(\mathbb{Q})$ if and only if P_R contains all but one prime number. But in this case $R = \mathbb{Z}_{(p)}$ for a prime number p. $\square$

3 Classification of the maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$

In this section we classify all the maximal subrings of $\mathbb{C}[x]$ up to automorphisms of $\mathbb{C}[x]$. We conclude that there are only two maximal subrings up to automorphisms of $\mathbb{C}[x]$ and they are $\mathbb{C}[x^2, x^3]$ and $\mathbb{C}[x^2, x^3 - x]$. At the beginning we find two special sets of maximal subrings and then prove that they are indeed the only ones.

Proposition 3.1. *Let A be a k-algebra and $B \subset A$ a sub-k-algebra such that* $\dim_k(A/B) = 1$. *Then B is a maximal k-subring.*

Proof. Suppose by contradiction that B is not a maximal k-subring of A. In this case we have that there exists V a k-subring of A such that $B \subsetneq V \subsetneq A$. But that means that $\dim_k(A/B) > \dim_k(V/B) \geq 1$ which implies that $\dim_k(A/B) > 1$ which is a contradiction. $\square$

Restricting 3.1 to our particular case $A = \mathbb{C}[x]$ we have:

Corollary 3.2. *If T is a $\mathbb{C}$-subring of $\mathbb{C}[x]$ such that* $\operatorname{codim} T = 1$ *then T is a maximal $\mathbb{C}$-subring of $\mathbb{C}[x]$.*

Lemma 3.3. *Let $a, b \in \mathbb{C}, a \neq b$ and define the sets $F_{a,b} = \{p \in \mathbb{C}[x] : p(a) = p(b)\}$. Then $F_{a,b}$ is a maximal $\mathbb{C}$-subring of $\mathbb{C}[x]$.*

Proof. Consider the linear functional $f : \mathbb{C}[x] \to \mathbb{C}$ so that $f(p) = p(a) - p(b)$. Clearly $F_{a,b} = \ker f$. Furthermore $\operatorname{Im} f = \mathbb{C}$. So:

$$\operatorname{codim}(\ker f) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\ker f) = \dim_{\mathbb{C}}(\operatorname{Im} f) = \dim_{\mathbb{C}} \mathbb{C} = 1$$

Since $\operatorname{codim} F_{a,b} = 1$, from 3.2 we have that $F_{a,b}$ is a maximal $\mathbb{C}$-subring of $\mathbb{C}[x]$. $\square$

Lemma 3.4. *Let $a, b \in \mathbb{C}$ and $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$. Then $\Phi(F_{a,b}) = F_{\Phi^{-1}(a),\Phi^{-1}(b)}$. In particular $F_{a,b} \cong F_{-1,1}$.*

Proof. The only polynomials in $\mathbb{C}[x]$ that have an inverse in the composition sense are the linear polynomials. In other words, the automorphism group of the polynomial ring in one variable is the group of affine maps $x \mapsto cx + d$. Let $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$ and consider $g \in \Phi(F_{a,b})$. There exists $p \in F_{a,b}$ such that $g = \Phi(p)$. Furthermore:

$$g(\Phi^{-1}(a)) = \Phi(p)(\Phi^{-1}(a)) = p(\Phi\Phi^{-1}(a)) = p(a) = p(b) = g(\Phi^{-1}(b))$$

Thus $g \in F_{\Phi^{-1}(a),\Phi^{-1}(b)} \implies \Phi(F_{a,b}) \subset F_{\Phi^{-1}(a),\Phi^{-1}(b)}$. But from lemma 3.3 both $\Phi(F_{a,b})$ and $F_{\Phi^{-1}(a),\Phi^{-1}(b)}$ are maximal $\mathbb{C}$-subrings (because Φ is an automorphism and isomorphisms preserve maximality) so the inclusion cannot be strict. Hence $\Phi(F_{a,b}) = F_{\Phi^{-1}(a),\Phi^{-1}(b)}$ which means that $F_{a,b} \cong F_{\Phi^{-1}(a),\Phi^{-1}(b)}$ as required. For the last bit $F_{a,b} \cong F_{-1,1}$, we just take $\Phi^*(x) = \frac{b-a}{2} \cdot x + \frac{a+b}{2}$ and verify that $\Phi^*(-1) = a$ and $\Phi^*(1) = b$. Indeed:

$$\Phi^*(-1) = \frac{b-a}{2} \cdot (-1) + \frac{a+b}{2} = \frac{a-b}{2} + \frac{a+b}{2} = a$$

$$\Phi^*(1) = \frac{b-a}{2} \cdot 1 + \frac{a+b}{2} = \frac{b-a}{2} + \frac{a+b}{2} = b$$

With this the proof is finished. □

Lemma 3.5. *Let $a \in \mathbb{C}$ and define the set $F_a = \{p \in \mathbb{C}[x] : p'(a) = 0\}$. Then F_a is a maximal $\mathbb{C}$-subring of $\mathbb{C}[x]$.*

Proof. Consider the linear functional $f : \mathbb{C}[x] \to \mathbb{C}$ so that $f(p) = p'(a)$. Clearly $F_a = \ker f$. Furthermore $\operatorname{Im} f = \mathbb{C}$. So:

$$\operatorname{codim}(\ker f) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\ker f) = \dim_{\mathbb{C}}(\operatorname{Im} f) = \dim_{\mathbb{C}} \mathbb{C} = 1$$

Since $\operatorname{codim} F_a = 1$, from 3.2 we have that F_a is a maximal $\mathbb{C}$-subring of $\mathbb{C}[x]$. □

Lemma 3.6. *Let $a \in \mathbb{C}$ and $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$. Then $\Phi(F_a) = F_{\Phi^{-1}(a)}$. In particular $F_a \cong F_0$.*

Proof. The proof is similar to the proof of lemma 3.4. Let $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$ and consider $g \in \Phi(F_a)$. There exists $p \in F_a$ such that $g = \Phi(p)$. Furthermore:

$$\frac{d}{dx}\Big|_{x=a} g(\Phi^{-1}(x)) = \frac{d}{dx}\Big|_{x=a} p(x) = 0$$

Thus $g \in F_{\Phi^{-1}(a)} \implies \Phi(F_a) \subset F_{\Phi^{-1}(a)}$. But from lemma 3.3 both $\Phi(F_a)$ and $F_{\Phi^{-1}(a)}$ are maximal $\mathbb{C}$-subrings so the inclusion cannot be strict. Thus $\Phi(F_a) \subset F_{\Phi^{-1}(a)}$. For $F_a \cong F_0$ take $\Phi^*(x) = x + a$ and we only need to check that $\Phi^*(0) = a$ which is true. □

Proposition 3.7. $F_{-1,1} = \mathbb{C}[x^2, x^3 - x]$ *and* $F_0 = \mathbb{C}[x^2, x^3]$.

Proof. Consider $F_{-1,1}$. We have $x^2 \in F_{-1,1}$ and $x^3 - x \in F_{-1,1}$. Thus $\mathbb{C}[x^2, x^3 - x] \subseteq F_{-1,1}$. But $\operatorname{codim}(\mathbb{C}[x^2, x^3 - x]) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\mathbb{C}[x^2, x^3 - x]) = \dim_{\mathbb{C}} \mathbb{C}\bar{x} = 1$. So $\mathbb{C}[x^2, x^3 - x] = F_{-1,1}$.

Now let's consider F_0. We have $x^2, x^3 \in F_0$ so $\mathbb{C}[x^2, x^3] \subseteq F_0$. Using again the codim argument we get $\operatorname{codim}(\mathbb{C}[x^2, x^3]) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\mathbb{C}[x^2, x^3]) = \dim_{\mathbb{C}}(\mathbb{C}\bar{x}) = 1 = \operatorname{codim}(F_0)$. Thus $\mathbb{C}[x^2, x^3] = F_0$ □

Corollary 3.8. *All the subrings $F_{a,b}$ and F_a are isomorphic (with respect to automorphisms of $\mathbb{C}[x]$) with $\mathbb{C}[x^2, x^3 - x]$ and $\mathbb{C}[x^2, x^3]$ respectively.*

Proof. This is a direct result of 3.4, 3.6 and 3.7. □

Remark 3.9. *In 3.8 we also have that $\mathbb{C}[x^2, x^3 - x]$ and $\mathbb{C}[x^2, x^3]$ are maximal $\mathbb{C}$-subrings.*

We have showed that $\mathbb{C}[x^2, x^3 - x]$ and $\mathbb{C}[x^2, x^3]$ are maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$. As claimed at the beginning of this section, these are the only maximal $\mathbb{C}$-subrings of $\mathbb{C}[x]$ up to automorphisms of $\mathbb{C}[x]$. We only need to show that there are no other.

Theorem 3.10. *All $\mathbb{C}$-subalgebras R of $\mathbb{C}[x]$ are finitely generated.*

Proof. Let $g_1 \in R$, $\deg(g_1) \geq 1$ such that $\deg(g_1)$ is as small as possible. Then find $g_2 \in R\,\mathbb{C}[g_1]$ such that $\deg(g_2)$ is as small as possible. We proceed inductively and find $g_n \in R \setminus \mathbb{C}[g_1, \dots, g_{n-1}]$ such that $\deg g_n$ is as small as possible. If we show that this process stops at some point, ie. $R \setminus \mathbb{C}[g_1, \dots, g_N] = \emptyset$,

we are done.
Denote $\deg(g_i) = d_i$ for simplicity. We claim that:

$$d_n \neq d_i (\text{mod } d_1) \text{ for all } i < n. \tag{2}$$

Indeed, if this were not true there would exist $k, l \in \mathbb{N}$ such that $d_k = d_l (\text{mod } d_1)$. So

$$d_k = d_l + m d_1 \text{ for some } m \in \mathbb{N}.$$

But this implies that $g_k = c \cdot g_l g_1^m + h$ for some $c \in \mathbb{C}$ and $h \in \mathbb{C}[x]$ such that $\deg(h) < d_k$. This means that we can substitute g_k with $h \in \mathbb{C}[g_1, \ldots, g_k]$ which contradicts our choice of generators with the smallest degrees possible. So (2) must be true. But (2) cannot hold for infinitely many generators because there are only finitely many (exactly d_1) residues mod d_1. This is equivalent to what we wanted. So R is finitely generated. $\square$

Definition 3. *Let R be a $\mathbb{C}$-subalgebra of $\mathbb{C}[x]$. We say that $\mathbb{C}[g_1, \ldots, g_n]$ is a minimal representation of R if $R = \mathbb{C}[g_1, \ldots, g_n]$ and the generators $g_1, \ldots, g_n$ are chosen with the smallest degrees possible as described in the proof of theorem 3.10.*

Corollary 3.11. *Let $R = \mathbb{C}[g_1, \ldots, g_n]$ be a minimal representation of R. Then it is true that* $\operatorname{codim} R \geq \deg(g_1) - 1 \geq n - 1$.

Proof. From the proof of theorem 3.10 we have that $n \leq \deg(g_1)$. On the other hand because $\mathbb{C}[g_1, \ldots, g_n]$ is a minimal representation or R we have that $x, x^2, \ldots, x^{\deg(g_1)-1} \notin R$. Thus $\mathbb{C}x, \mathbb{C}x^2, \ldots, \mathbb{C}x^{\deg(g_1)-1} \subset \mathbb{C}[x]/R$. This implies $\operatorname{codim} R = \dim_{\mathbb{C}}(\mathbb{C}[x]/R) \geq \deg(g_1) - 1$ which is what we wanted. $\square$

Let $R = \mathbb{C}[g_1, \ldots, g_n]$ be a minimal representation of the $\mathbb{C}$-subalgebra R. Define the polynomial map $\phi_{(g_1,\ldots,g_n)} : \mathbb{C} \mapsto \mathbb{C}^n$ so that $\phi_{(g_1,\ldots,g_n)}(t) = (g_1(t), \ldots, g_n(t))$. We call this map the corresponding polynomial map of R. Clearly R has many minimal representations. Nevertheles changing between minimal representations and therefore between polynomial maps leaves some properties of its polynomial curves intact, like injectivity and smoothness. The injectivity and smoothness of these polynomial maps is of particular interest to us because it seems to have the property that if the $\mathbb{C}$-subalgebra $\mathbb{C}[g_1, \ldots, g_n]$ is a maximal $\mathbb{C}$-subring then the curve $\phi_{(g_1,\ldots,g_n)}(t)$ has cusps and self-intersecting points, ie. $\phi_{(g_1,\ldots,g_n)}$ is not smooth or not injective. Indeed this is true in the case of $\mathbb{C}[x]$ as we will show below. This property is true even in the other cases in this paper but we will not prove it.

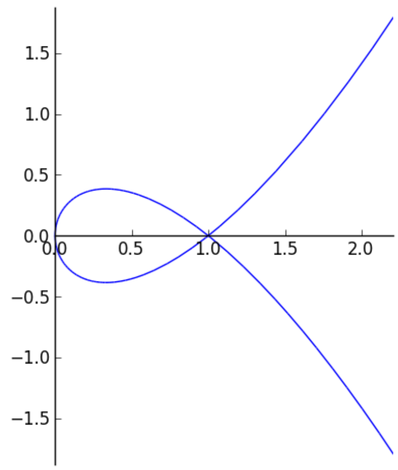

Figure 1: For the case of the maximal $\mathbb{C}$-subring $F_{-1,1} = \mathbb{C}[x^2, x^3 - x]$ the polynomial map $t \mapsto (t^2, t^3 - t)$ is not injective. The curve is self-intersecting at the points $t = -1$ and $t = 1$.

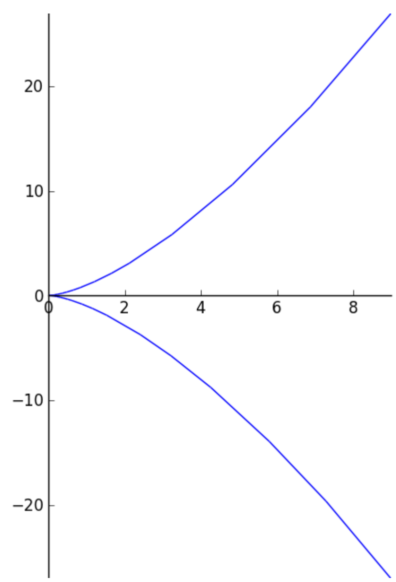

Figure 2: For the case of the maximal $\mathbb{C}$-subring $F_0 = \mathbb{C}[x^2, x^3]$ the polynomial map $t \mapsto (t^2, t^3)$ is not smooth. The curve has a cusp at the point $t = 0$.

Theorem 3.12. *Let R be a maximal $\mathbb{C}$-subring of $\mathbb{C}[x]$ and $\mathbb{C}[g_1, \ldots, g_n]$ be a minimal representation of R. Consider $t \xrightarrow{\phi} (g_1(t), \ldots, g_n(t))$. The following* *are true:*

1. *If ϕ is not injective then R is isomorphic to $\mathbb{C}[x^2, x^3 - x]$ up to automorphisms of $\mathbb{C}[x]$.*

2. *If ϕ is not smooth then R is isomorphic to $\mathbb{C}[x^2, x^3]$ up to automorphisms of $\mathbb{C}[x]$.*

Proof. Let's prove the first part of the theorem. Suppose ϕ is not injective. Then there exist $a, b \in \mathbb{C}$ such that $(g_1(a), \dots, g_n(a)) = (g_1(b), \dots, g_n(b))$. But this implies that $g_1, \dots, g_n \in F_{a,b}$. Thus $R = \mathbb{C}[g_1, \dots, g_n] \subset F_{a,b}$ and since both R and $F_{a,b}$ are maximal $\mathbb{C}$-subrings it follows that $R = F_{a,b}$. Now using corollary 3.8 we get what we want.

For the second part suppose that ϕ is not smooth. This means that $(g_1', \dots, g_n') = (1)$ which is equivalent to $g_1'(a) = \cdots = g_n'(a) = 0$ for some $a \in \mathbb{C}$. But this implies that $g_1, \dots, g_n \in F_a$. Proceeding as above we show that $R = F_a$ and from corollary 3.8 it follows that R is isomorphic to $\mathbb{C}[x^2, x^3]$ up to automorphisms of $\mathbb{C}[x]$ as required. $\square$

The only other possibility for the map ϕ, which is not covered in theorem 3.12, is the case when ϕ is both injective and smooth. The proof for this part is from Arno van den Essen.

Let k denote an algebraically closed field and $f := (f_1(t), ..., f_n(t)) : k \to k^n$ is a polynomial mapping. By f^* denote the induced ring homomorphism from $k[x_1, ..., x_n]$ to $k[t]$ given by $f^*(g) = g \circ f$. The image of f^* we denote by R. So $R = k[f_1(t), ..., f_n(t)]$.

Theorem 3.13. *If f is injective and $f'(c) \neq 0$ for all $c \in k$, then f^* is onto.*

Proof. Since $f_j(t) \notin k$ for some j, t is a zero of the non-zero polynomial $f_j(T) - f_j(t)$, hence t is integral over R. So $k[t]$ is finite over R and hence so is $M := k[t]/R$. Furthermore R is noetherian. This implies that its radical, denoted $\mathrm{rad}(R)$, equals $\bigcap_{i=1}^{s} \mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$ of R. In the lemma below we show that $M \subseteq \mathfrak{m}M$ for every maximal ideal $\mathfrak{m}$ of R. It follows that $M \subseteq \mathfrak{m}_1 ... \mathfrak{m}_s M$ and hence $M \subseteq rad(R)M$. So by Nakayama's lemma $M = 0$. i.e. $k[t] = R$, so f^* is onto. $\square$

Lemma 3.14. *$M/\mathfrak{m}M = 0$ for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Since, as observed above $R \subseteq k[t]$ is an integral extension, the going-up theorem implies that there exists a maximal ideal $\mathfrak{n}$ of $k[t]$ with $\mathfrak{n} \cap R = \mathfrak{m}$. Since k is algebraically closed, $\mathfrak{n}$ is of the form $k[t](t-c)$.

Claim: $\mathfrak{n} = \mathfrak{m}k[t]$.
It then follows that $k[t] \subseteq \mathfrak{n}k[t] + k \subseteq \mathfrak{m}k[t] + R$ i.e. $M \subseteq \mathfrak{m}M$ as desidered. To see the claim, observe that by hypothesis $f_i'(c)$ is nonzero for some i. Hence $f_i(t) - f_i(c) = (t-c)u(t)$ for some $u(t)$ in $k[t]$ and $u(c)$ is nonzero. Let $b_1, ..., b_r$ be the different zeros of $u(t)$, each of which is different from c. Since f is injective, for each j the vector $f(c)$ is different from $f(b_j)$. So for some i, $f_i(c)$ is different from $f_i(b_j)$. Hence $f_i(t) - f_i(c) = (t-c)v_j(t)$ for some $v_j(t)$ in $k[t]$ with $v_j(b_j)$ nonzero. Obviously $f_i(t) - f_i(c)$ is contained in R, hence $(t-c)v_j(t)$ belongs to $\mathfrak{n} \cap R = \mathfrak{m}$. By construction the polynomials $u(t), v_1(t), .., v_r(t)$ have no comon zero, hence they generate the unit ideal in $k[t]$. Then $(t-c)u(t), (t-c)v_1(t), ..., (t-c)v_r(t)$ generate the ideal $(t-c)k[t]\mathfrak{n}$. Since each of these generators belongs to $\mathfrak{m}$ we get that $\mathfrak{m}k[t] = \mathfrak{n}$, which completes the proof of the claim. ☐

4 Some maximal $\mathbb{C}$-subrings of F_a and $F_{a,b}$.

Proposition 4.1. *Let A be a k-algebra and $S, T \subset A$ are k-subalgebras such that* $\operatorname{codim} S = 1$ *and $S \neq T$. Then $S \cap T$ is a maximal k-subring of T.*

Proof. A is a vector space and S, T are vector subspaces of A. In other words A is a k-module and S, T are k-submodules of A. From the Second Isomorphism Theorem for modules we have that $(T+S)/S$ and $T/T \cap S$ are isomorphic. But from the assumptions that $\dim_k(A/S) = 1$ and $S \neq T$ it follows that $T+S = A$. Thus $A/S \cong T/T \cap S$ which implies that $\operatorname{codim}(T \cap S)$ in T is:

$$\dim_k(T/T \cap S) = \dim_k(A/S) = \operatorname{codim} S = 1$$

We only need to recall proposition 3.1 and we are done. ☐

Corollary 4.2. *The $\mathbb{C}$-subalgebras $F_a \cap F_b$, $F_a \cap F_{b,c}$ are maximal $\mathbb{C}$-subrings of F_a and $F_a \cap F_{b,c}$, $F_{a,b} \cap F_{c,d}$ are maximal $\mathbb{C}$-subrings of $F_{a,b}$.*

Proof. We only need to notice that $F_x, F_{y,z}$ are all different and $\operatorname{codim} F_x = \operatorname{codim} F_{y,z} = 1$ for all indices. Now the hypothesis is a direct corollary of proposition 4.1. ☐

Based on corollary 4.2 we will continue the classification of maximal $\mathbb{C}$-subrings as shown in the diagram. Clearly the $\mathbb{C}$-subalgebras connected by an arrow in the diagram are maximal $\mathbb{C}$-subrings (from the bottom up) of the corresponding $\mathbb{C}$-algebra. In fact from proposition 4.1 we can go even further by taking the intersections of the elements in the bottom row.

$$\begin{tikzcd}
 & & \mathbb{C}[x,y] \arrow[dl] \arrow[dr] & & \\
 & F_a \arrow[dl] \arrow[dr] & & F_{a,b} \arrow[dl] \arrow[dr] & \\
F_a \cap F_b & & F_a \cap F_{a,b} & & F_{a,b} \cap F_{c,d}
\end{tikzcd}$$

It is important to note that in this way we don't exhaust all the possibilites to find maximal $\mathbb{C}$-subrings. Indeed $\mathbb{C}[x^3, x^4, x^5]$ is not isomorphic (up to automorphisms of $\mathbb{C}[x]$) with any of the above.

4.1 The structure of $F_a \cap F_b$

Lemma 4.3. *$F_a \cap F_b = \mathbb{C}[h_1, h_2, h_3]$ where:*

1. $h_1(x) = (x-a)^2(2x + (a-3b))$
2. $h_2(x) = (x-a)^2(x^2 + (a-2b)b)$
3. $h_3(x) = (x-a)^2(2x^3 + (3a-5b)b^2)$

is a minimal representation of $F_a \cap F_b$.

Proof. Let $f(x) \in F_a \cap F_b = \{g \in \mathbb{C}[x] : g'(a) = g'(b) = 0\}$. Define $h(x) = f(x) - f(a)$. Clearly h satisfies $h(a) = h'(a) = 0$ and $h'(b) = 0$ and $h \in F_a \cap F_b$ is equivalent with $f \in F_a \cap F_b$. Thus $h(x) = (x-a)^2 q(x)$ where $q(x) \in \mathbb{C}[x]$ and $h'(x) = 2(x-a)q(x) + (x-a)^2 q'(x)$. Substituting $x = b$ and using the fact that $h(b) = 0$:

$2(b-a)q(b) + (b-a)^2 q^{'}(b) = 0$ and since $a \neq b$ we have $q(b) = \dfrac{1}{2}(a-b)q'(b)$

which is the condition $q(x)$ should satisfy for $h(x) \in F_a \cap F_b$. Hence $\deg(h_1(x)) = \deg((x-a)^2) + \underbrace{\deg(q(x))}_{\geq 1} \geq 2 + 1 = 3$. On the other hand from corollary 3.11 it follows that $\deg(h_1(x)) \leq \operatorname{codim}(F_a \cap F_b) + 1 = 2 + 1 = 3$.

Hence $\deg(h_1(x)) = 3$ and a minimal representation of $F_a \cap F_b$ has at most 3 generators. Thus we can just find three polynomials q_1, q_2, q_3 of degrees $1, 2$ and 3 respectively that satisfy the condition $q(b) = \frac{1}{2}(a-b)q'(b)$ and $h_1 = (x-a)^2 q_1, h_2 = (x-a)^2 q_2, h_3 = (x-a)^2 q_3$ will be a minimal set of generators of $F_a \cap F_b$. We search for q_k-s of the form $q_k(x) = x^k + c_k$. Then $q_k'(x) = kx^{k-1}$ and: $b^k + c_k = \frac{1}{2}(a-b)kb^{k-1} \Rightarrow c_k = \frac{1}{2}(ka - (k+2)b)b^{k-1}$ Substituting for k we find $c_1 = \frac{1}{2}(a-3b), c_2 = (a-2b)b, c_3 = \frac{1}{2}(3a-5b)b^2$ which is equivalent to what we wanted to prove. □

Lemma 4.4. *For all $a, b \in \mathbb{C}$ there exists $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$ such that $\Phi(F_a \cap F_b) = F_0 \cap F_{-1}$. In particular $F_a \cap F_b \cong F_0 \cap F_{-1}$.*

Proof. Define $\Phi(x) = (a-b)x + a \in Aut_{\mathbb{C}}\mathbb{C}[x]$. We have that $\Phi(0) = a$ and $\Phi(-1) = (a-b) \cdot (-1) + a = b$. Now we can use lemma 3.6 and get $\Phi(F_a) = F_{\Phi^{-1}(a)} = F_0$ and $\Phi(F_b) = F_{\Phi^{-1}(b)} = F_{-1}$. But we also have that $\Phi(F_a \cap F_b) = \Phi(F_a) \cap \Phi(F_b) = F_0 \cap F_{-1}$. So $F_a \cap F_b \cong F_0 \cap F_{-1}$ as required. □

Theorem 4.5. *$F_a \cap F_b \cong \mathbb{C}[x^2(2x+3), x^2(x^2-2), x^2(2x^3+5)]$ up to automorphisms of $\mathbb{C}[x]$.*

Proof. Direct result of lemma 4.4 and then lemma 4.3. □

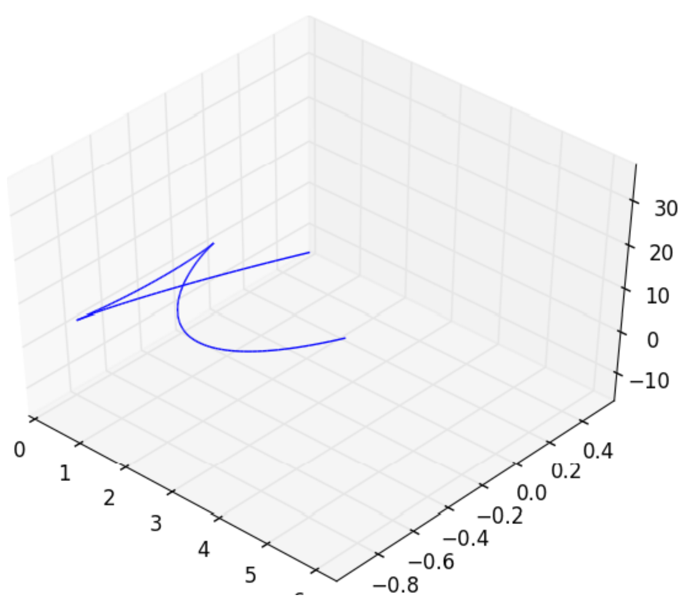

Figure 3: For $F_0 \cap F_{-1}$ the polynomial map $t \mapsto (t^2(2t+3), t^2(t^2-2), t^2(2t^3+5))$ is not smooth. The curve has two cusps at the points $t = 0, -1$.

4.2 The structure of $F_a \cap F_{b,c}$

Lemma 4.6. *Let $a, b, c \in \mathbb{C}$ such that $b \neq a, c$. The following are true:*

1. *If $2a = b + c$ then $F_a \cap F_{b,c} = \mathbb{C}[h_1, h_2]$ is a minimal representation where*

   (a) $h_1(x) = (x - a)^2$

   (b) $h_2(x) = x(x - a)^2(x - b)(x - c)$

2. *Denote $z = \frac{c-a}{b-a}$. If $2a \neq b+c$ then $F_a \cap F_{b,c} = \mathbb{C}[h_1, h_2, h_3]$ is a minimal representation where*

   (a) $h_1(x) = (x - a)^2 \left(x + \frac{z^2c-b}{1-z^2}\right)$

   (b) $h_2(x) = (x - a)^2 \left(x^2 + \frac{z^2c^2-b^2}{1-z^2}\right)$

   (c) $h_3(x) = (x - a)^2 \left(x^3 + \frac{z^2c^3-b^3}{1-z^2}\right)$

Proof. Let $f(x) \in F_a \cap F_{b,c} = \{g(x) \in \mathbb{C}[x] : g'(a) = 0 \text{ and } g(b) = g(c)\}$. Define $h(x) = f(x) - f(a)$. Clearly h satisfies $h'(a) = h(a) = 0$ and $h(b) = h(c)$ and $h \in F_a \cap F_{b,c}$ is equivalent with $f \in F_a \cap F_{b,c}$. Then we have that $h(x) = (x - a)^2 q(x)$ where $q(x) \in \mathbb{C}[x]$. The condition $h(b) = h(c)$ implies that q should satisfy $q(b) = \frac{(c-a)^2}{(b-a)^2} q(c) = z^2 q(c)$. Here there are 2 cases to consider:

1. $z^2 = 1$ (notice that only $z = -1$ is possible because $b \neq c$)
2. $z^2 \neq 1$

The first case is special because we can choose q_1 such that $\deg(q_1(x)) = 0 \Rightarrow \deg(h_1(x)) = 2$. Let $q_1(x) = 1$. From corollary 3.11 a minimal representation of $F_{a,b} \cap F_{c,d}$ will only have 2 generators. So we must have that $\deg(q_2(x)) = 3$ and $\deg(h_2(x)) = 5$. We can find easily $q_2(x) = x(x-b)(x-c)$ which satisfies the conditions and the corresponding polynomial $h_2(x) = x(x - a)^2(x - b)(x - c)$. With this we conclude that if a, b, c satisfy relation 1. we have $F_a \cap F_{b,c} = \mathbb{C}[(x - a)^2, x(x - a)^2(x - b)(x - c)]$.
If a, b, c satisfy condition 2. we proceed similarly to the proof of lemma 4.3

and find q_1, q_2 and q_3 polynomials of degree $1, 2$ and 3 respectively that satisfy $q(b) = z^2 q(c)$. Taking $q_1(x) = x + t_1$, $q_2(x) = x^2 + t_2$ and $q_3(x) = x^3 + t_3$ we find

$$t_1 = \frac{z^2c - b}{1 - z^2}, t_2 = \frac{z^2c^2 - b^2}{1 - z^2}, t_3 = \frac{z^2c^3 - b^3}{1 - z^2}$$

Thus:

$$h_1(x) = (x-a)^2(x+t_1), h_2(x) = (x-a)^2(x^2+t_2) \text{ and } h_3(x) = (x-a)^2(x^3+t_3)$$

are the generators of $F_a \cap F_{c,d}$. With this the proof is finished. $\square$

Definition 4. *If $R = F_a \cap F_{b,c}$ satisfies the condition $2a = b + c$ as in the first case of lemma 4.6, we call R a maximal $\mathbb{C}$-subring of the first kind. If $2a \neq b + c$ we call R a maximal $\mathbb{C}$-subring of the second kind.*

Lemma 4.7. *Let $a, b, c \in \mathbb{C}$ such that $b \neq a, c$ and denote $c^* = \frac{c-b}{b-a}$. Then there exists $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$ such that $\Phi(F_a \cap F_{b,c}) = F_{-1} \cap F_{0,c^*}$. In particular $F_a \cap F_{b,c} \cong F_{-1} \cap F_{0,c^*}$.*

Proof. Define $\Phi(x) = (b - a)x + b \in Aut_{\mathbb{C}}\mathbb{C}[x]$. We have that $\Phi(0) = b$ and $\Phi(-1) = (b - a) \cdot (-1) + b = a$. Now we can use lemma 3.4 and lemma 3.6 and get $\Phi(F_b) = F_{\Phi^{-1}(b)} = F_0$ and $\Phi(F_a) = F_{\Phi^{-1}(a)} = F_{-1}$. But we also have that $\Phi(F_a \cap F_{b,c}) = \Phi(F_a) \cap \Phi(F_{b,c}) = F_{-1} \cap F_{0,\Phi^{-1}(c)}$. We can check that $\Phi(c^*) = (b - a) \cdot \frac{c-b}{b-a} + b = c$. So $\Phi^{-1}(c) = c^*$ and $F_a \cap F_{b,c} \cong F_{-1} \cap F_{0,c^*}$ as required. $\square$

Remark 4.8. *Lemma 4.7 says that $F_a \cap F_{b,c} \cong F_{a'} \cap F_{b',c'}$ are isomorphic (up to automorphisms of $\mathbb{C}[x]$) if and only if $\frac{c-b}{b-a} = c^* = c'^* = \frac{c'-b'}{b'-a'}$.*

Because of lemma 4.7 and lemma 4.6 we can classify all the maximal $\mathbb{C}$-subrings of the first kind up to automorphisms of $\mathbb{C}[x]$. Indeed, notice that $c^* = \frac{c-a}{b-a} - 1$ which implies that if $2a = b + c$ then $c^* = -2$. Hence all the maximal $\mathbb{C}$-subrings of the first kind are isomorphic to $F_{-1} \cap F_{0,-2}$. Obviously it follows that all maximal $\mathbb{C}$-subrings of the second kind are isomorphic to $F_{-1} \cap F_{0,c^*}$ where $c^* \neq -2$.
Let's plot the graph of the polynomial map that we get for $F_{-1} \cap F_{0,-2}$. Applying lemma 4.6 we get

$$F_{-1} \cap F_{0,-2} = \mathbb{C}[(x+1)^2, x^2(x+1)^2(x+2)] = \mathbb{C}[(x+1)^2, (x+1)^2, x^2(x+1)^2(x+2)]$$

Denoting its polynomial map $\phi(t) = ((t+1)^2, (t+1)^2, t^2(t+1)^2(t+2))$ we get the graph:

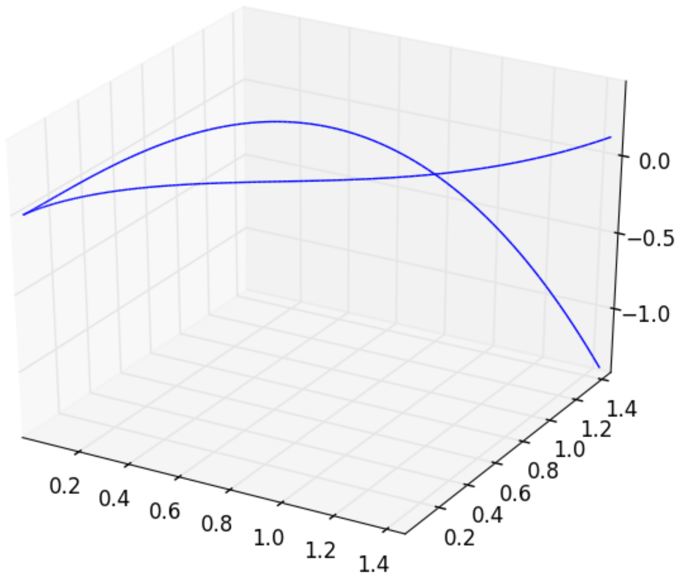

Figure 4: For the case of the maximal $\mathbb{C}$-subring $F_{-1} \cap F_{0,-2}$ the polynomial map $\phi(t) = ((t+1)^2, (t+1)^2, t^2(t+1)^2(t+2))$ is neither smooth nor injective. The curve has a cusp at the point $\phi(-1)$ and it is self intersecting at the points $\phi(0) = \phi(-2)$.

Next we look at the graphs of the two maximal $\mathbb{C}$-subrings of the second kind $F_{-1} \cap F_{0,1}$ and $F_{-1} \cap F_{0,-1}$. $F_{-1} \cap F_{0,1}$ is a more generic case which is a good representative of how the corresponding polynomial maps of the other maximal $\mathbb{C}$-subrings except $F_{-1} \cap F_{0,-2}$ and $F_{-1} \cap F_{0,-1}$. The reason why we look at $F_{-1} \cap F_{0,-1}$ independently is because in this case $a = c$. Applying lemma 4.6 for the maximal $\mathbb{C}$-subrings of the second kind we get:

1. $F_{-1} \cap F_{0,1} = \mathbb{C}[(x+1)^2(x - \frac{4}{3}), (x+1)^2(x^2 - \frac{4}{3}), (x+1)^2(x^3 - \frac{4}{3})]$
2. $F_{-1} \cap F_{0,-1} = \mathbb{C}[x(x+1)^2, x^2(x+1)^2, x^3(x+1)^2]$

Now we define their respective polynomial maps:

1. $\phi_1(t) = ((t+1)^2(t - \frac{4}{3}), (t+1)^2(t^2 - \frac{4}{3}), (t+1)^2(t^3 - \frac{4}{3}))$ and
2. $\phi_2(t) = (t(t+1)^2, t^2(t+1)^2, t^3(t+1)^2)$

We can now plot their graphs.

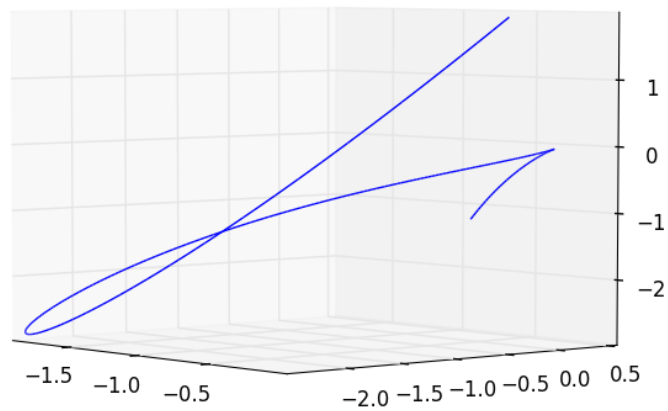

Figure 5: For the case of the maximal $\mathbb{C}$-subring $F_{-1} \cap F_{0,1}$ the polynomial map $\phi_1(t)$ is neither smooth nor injective. The curve has a cusp at the point $\phi_1(-1)$ and it is self intersecting at the points $\phi_1(0) = \phi(1)$.

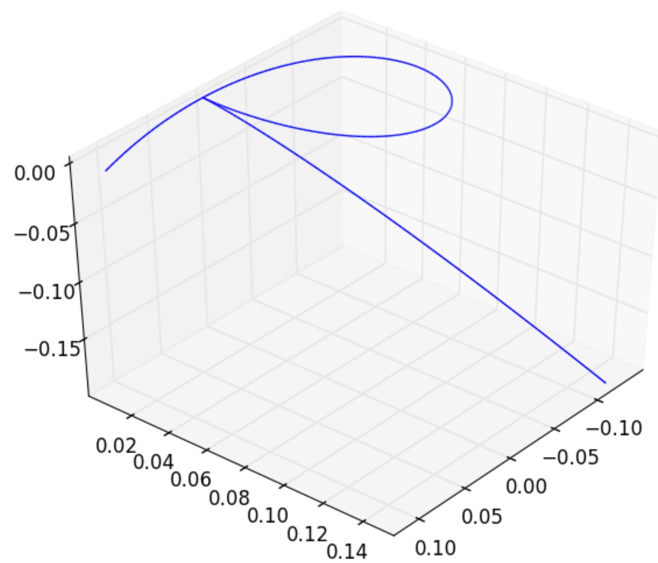

Figure 6: For the case of the maximal $\mathbb{C}$-subring $F_0 \cap F_{-1,0}$ the polynomial map $\phi_2(t)$ is neither smooth nor injective. The curve has a cusp at the point $\phi_2(0)$ and it is self intersecting at the points $\phi_2(0) = \phi_2(-1)$.

4.3 The structure of $F_{a,b} \cap F_{c,d}$

Lemma 4.9. *Let $a, b, c, d \in \mathbb{C}$ such that $a \neq b$ and $c \neq a, b, d$. The following are true:*

1. *If $a + b = c + d$ then $F_{a,b} \cap F_{c,d} = \mathbb{C}[h_1, h_2]$ is a minimal representation where*

   (a) $h_1(x) = (x - a)(x - b)$

   (b) $h_2(x) = x(x - a)(x - b)(x - c)(x - d)$

2. *Denote $z = \frac{(d-a)(d-b)}{(c-a)(c-b)}$. If $a + b \neq c + d$ then $F_{a,b} \cap F_{c,d} = \mathbb{C}[h_1, h_2, h_3]$ is a minimal representation where*

   (a) $h_1(x) = (x - a)(x - b)\left(x + \frac{c-zd}{z-1}\right)$

   (b) $h_2(x) = (x - a)(x - b)\left(x^2 + \frac{c^2-zd^2}{z-1}\right)$

   (c) $h_3(x) = (x - a)(x - b)\left(x^3 + \frac{c^3-zd^3}{z-1}\right)$

Proof. Let $f(x) \in F_{a,b} \cap F_{c,d} = \{g(x) \in \mathbb{C}[x] : g(a) = g(b) \text{ and } g(c) = g(d)\}$. Define $h(x) = f(x) - f(a)$. Clearly h satisfies $h(a) = h(b) = 0$ and $h(c) = h(d)$ and $h \in F_{a,b} \cap F_{c,d}$ is equivalent with $f \in F_{a,b} \cap F_{c,d}$. Then we have that $h(x) = (x - a)(x - b)q(x)$ where $q(x) \in \mathbb{C}[x]$. The condition $h(c) = h(d)$ implies that q should satisfy $q(c) = \frac{(d-a)(d-b)}{(c-a)(c-b)} q(d) = zq(d)$. Here there are 2 cases to consider:

1. $z = 1$ which is equivalent to $a + b = c + d$

2. $z \neq 1$ which is equivalent to $a + b \neq c + d$

The first case is special because we can choose q_1 such that $\deg(q_1(x))$ which implies $\deg(h_1(x)) = 2$. Let $q_1(x) = 1$. From corollary 3.11 a minimal representation of $F_{a,b} \cap F_{c,d}$ will only have 2 generators. So we must have that $\deg(q_2(x)) = 3$ and $\deg(h_2(x)) = 5$. We can easily find $q_2(x) = x(x-c)(x-d)$ which satisfies the condition. With this we conclude that in case a, b, c and d satisfy relation 1. we have

$$F_{a,b} \cap F_{c,d} = \mathbb{C}[(x - a)(x - b), x(x - a)(x - b)(x - c)(x - d)]$$

If a, b, c and d satisfy condition 2. we continue similarly to the proofs of the previous lemmas. We can find polynomials q_k-s of degrees $1, 2$ and 3

respectively that satisfy the condition $q(c) = zq(d)$. Then $h_1 = (x-a)(x-b)q_1, h_2 = (x-a)(x-b)q_2, h_3 = (x-a)(x-b)q_3$ will be a minimal set of generators of $F_{a,b} \cap F_{c,d}$. Let's search for polynomials of the form $q_k(x) = x^k + t_k$:

$$q_k(c) = zq_k(d) \Longrightarrow c^k + t_k = z(d^k + t_k) \Longrightarrow t_k = \frac{c^k - zd^k}{z-1}$$

Substituting for $k = 1,2,3$ we find $h_1(x) = (x-a)(x-b)\left(x + \frac{c-zd}{z-1}\right), h_2(x) = (x-a)(x-b)\left(x^2 + \frac{c^2-zd^2}{z-1}\right)$ and $h_3(x) = (x-a)(x-b)\left(x^3 + \frac{c^3-zd^3}{z-1}\right)$. With this the proof is finished. $\square$

Definition 5. *If $R = F_{a,b} \cap F_{c,d}$ satisfies the condition $a + b = c + d$ as in the first case of lemma 4.9, we call R a maximal $\mathbb{C}$-subring of the first kind. If $a + b \neq b + c$ we call R a maximal $\mathbb{C}$-subring of the second kind.*

Lemma 4.10. *Let $a, b, c, d \in \mathbb{C}$ such that $a \neq b$ and $c \neq a, b, d$ and denote $c^* = \frac{c-b}{b-a}$ and $d^* = \frac{d-b}{b-a}$. Then there exists $\Phi \in Aut_{\mathbb{C}}(\mathbb{C}[x])$ such that $\Phi(F_{a,b} \cap F_{c,d}) = F_{-1,0} \cap F_{c^*,d^*}$. In particular $F_{a,b} \cap F_{c,d} \cong F_{-1,0} \cap F_{c^*,d^*}$.*

Proof. Define $\Phi(x) = (b-a)x + b \in Aut_{\mathbb{C}}\mathbb{C}[x]$. We have that $\Phi(0) = b$ and $\Phi(-1) = (b-a)\cdot(-1) + b = a$. Now we can use lemma 3.4 and lemma 3.6 and get $\Phi(F_b) = F_{\Phi^{-1}(b)} = F_0$ and $\Phi(F_a) = F_{\Phi^{-1}(a)} = F_{-1}$. But we also have that $\Phi(F_{a,b} \cap F_{c,d}) = \Phi(F_{a,b}) \cap \Phi(F_{c,d}) = F_{-1,0} \cap F_{\Phi^{-1}(c),\Phi^{-1}(d)}$. We can check that $\Phi(c^*) = (b-a)\cdot\frac{c-b}{b-a} + b = c$ and $\Phi(d^*) = (b-a)\cdot\frac{d-b}{b-a} + b = d$. So $\Phi^{-1}(c) = c^*$ and $\Phi^{-1}(c) = c^*$. But this implies $\Phi(F_{a,b} \cap F_{c,d}) = F_{-1,0} \cap F_{c^*,d^*}$ as required and obviously $F_{a,b} \cap F_{c,d} \cong F_{-1,0} \cap F_{c^*,d^*}$. $\square$

Remark 4.11. *Lemma 4.10 says that $F_{a,b} \cap F_{c,d} \cong F_{a',b'} \cap F_{c',d'}$ are isomorphic (up to automorphisms of $\mathbb{C}[x]$) if and only if $c^* = c'^*$ and $d^* = d'^*$. We notice that the last equalities are true if and only if $\frac{c-b}{c'-b'} = \frac{d-b}{d'-b'}$. This equality can serve as a test to check if 2 subrings of the type $F_{a,b} \cap F_{c,d}$ are isomorphic.*

Because of lemma 4.10 and lemma 4.9 we can classify all the maximal $\mathbb{C}$-subrings of the first kind up to automorphisms of $\mathbb{C}[x]$. Notice that

$$c^* + d^* + 1 = \frac{(c-b) + (d-b) + (b-a)}{b-a} = \frac{(c+d) - (a+b)}{b-a}$$

So $(b-a)(c^* + d^* + 1) = (c+d) - (a+b)$ which is equivalent to $c^* + d^* = -1$ iff $a + b = c + d$. Hence the maximal $\mathbb{C}$-subring $F_{a,b} \cap F_{c,d}$ is of the first kind

if and only if $c^* + d^* = -1$. Thus all the maximal $\mathbb{C}$-subrings of the first kind are isomorphic to $F_{-1,0} \cap F_{c^*,-c^*-1}$. Obviously it follows that all the other maximal $\mathbb{C}$-subrings are of the second kind.
Let's plot the graph of the polynomial map that we get for $F_{-1,0} \cap F_{-2,1}$ which is a maximal $\mathbb{C}$-subring of the first kind. Applying lemma 4.9 we get

$$F_{-1,0} \cap F_{2,1} = \mathbb{C}[x(x+1), x^2(x+1)(x-1)(x+2)]$$

Denoting its polynomial map $\phi(t) = (t(t+1), t(t+1), t^2(t+1)(t-1)(t+2))$ we get the graph:

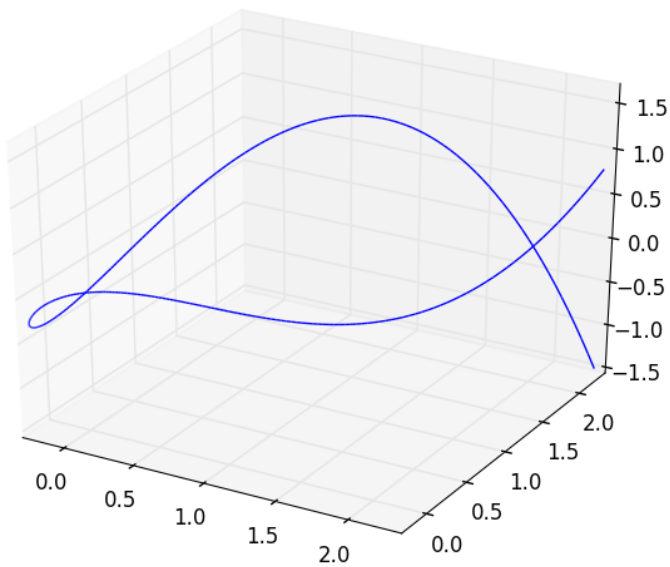

Figure 7: For the case of the maximal $\mathbb{C}$-subring $F_{-1,0} \cap F_{-2,1}$ the polynomial map $\phi(t)$ is not injective. The intersecting points are $\phi(-1) = \phi(0) = (0,0,0)$ and $\phi(-2) = \phi(1) = (2,2,0)$.

Next we look at the graphs of the two maximal $\mathbb{C}$-subrings of the second kind $F_{-1,0} \cap F_{1,2}$ and $F_{-1,0} \cap F_{1,0}$. $F_{-1,0} \cap F_{1,2}$ is a more generic case which is a good representative of how the corresponding polynomial maps of the other maximal $\mathbb{C}$-subrings except those of the first kind and $F_{-1,0} \cap F_{1,0}$. The reason why we look at $F_{-1} \cap F_{0,-1}$ independently is because in this case $b = c$ and the points on the graph where it is not injective coincide. Applying lemma 4.9 for the maximal $\mathbb{C}$-subrings of the second kind we get:

1. $F_{-1,0} \cap F_{1,2} = \mathbb{C}[x(x+1)(x-\frac{5}{2}), x(x+1)(x^2-\frac{11}{2}), x(x+1)(x^3-\frac{23}{2})]$
2. $F_{-1,0} \cap F_{1,0} = \mathbb{C}[x^2(x+1), x^3(x+1), x^4(x+1)]$

Now we define their respective polynomial maps:

1. $\phi_1(t) = (t(t+1)(t-\frac{5}{2}), t(t+1)(t^2-\frac{11}{2}), t(t+1)(t^3-\frac{23}{2}))$ and
2. $\phi_2(t) = (t^2(t+1), t^3(t+1), t^4(t+1))$

We can now plot their graphs.

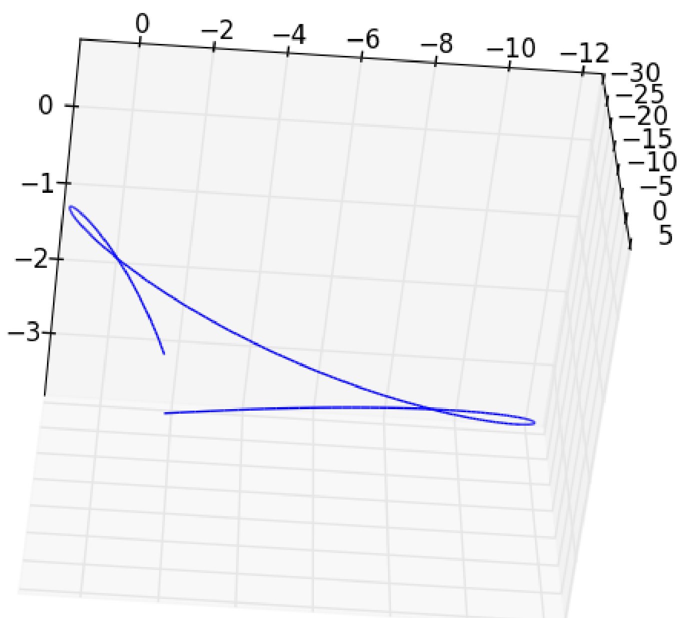

Figure 8: For the case of the maximal $\mathbb{C}$-subring $F_{-1,0} \cap F_{1,2}$ the polynomial map $\phi_1(t)$ is not injective. The intersecting points are $\phi_1(-1) = \phi_1(0) = (0,0,0)$ and $\phi_1(1) = \phi_1(2) = (-3,-9,21)$.

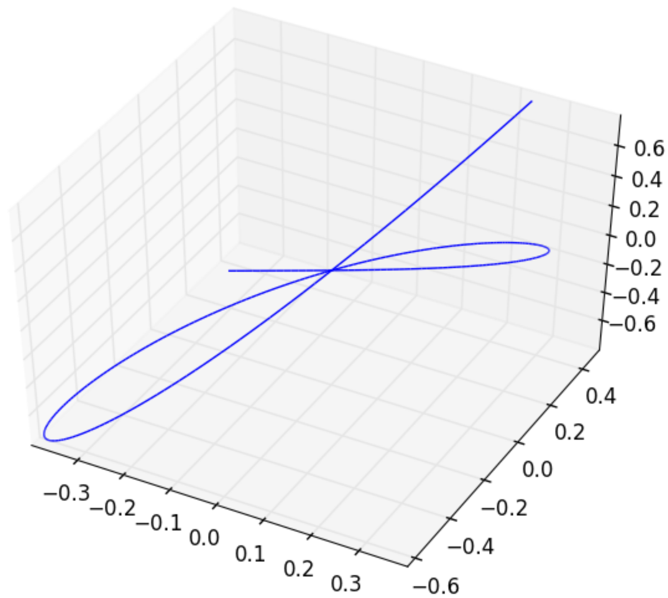

Figure 9: For the case of the maximal $\mathbb{C}$-subring $F_{-1,0} \cap F_{1,2}$ the polynomial map $\phi_2(t)$ is not injective. The intersecting points are $\phi_2(-1) = \phi(0) = \phi_2(1) = (2, 2, 0)$.

4.4 Ideas on how to prove that there are no other maximal $\mathbb{C}$-subrings.

Theorem 4.12. *Let B be a ring, $A < B$ and suppose $\pi : Spec(B) \mapsto Spec(A)$ given by $\mathfrak{p} \mapsto \mathfrak{p} \cap A$ is surjective. If $B_\mathfrak{p} \cong A_{\mathfrak{p}\cap A}$ for all $\mathfrak{p} \in Spec(B)$ then $A = B$.*

Proof. Let $b \in B$. We need to show that $b \in A$. Denote $I = \{s \in A | sb \in A\}$. Obviously I is an ideal of A. If $1 \in I$ we are done because $b = 1 \cdot b \in A$. Otherwise $I \ncong (1)$ so it's contained in some prime ideal $\mathfrak{q}$. From hypothesis, there is a prime ideal $\mathfrak{p} \in Spec(B)$ such that $\mathfrak{q} = \mathfrak{p} \cap A$. By assumption we have $B_\mathfrak{p} \cong A_\mathfrak{q}$. In particular, $b/1$ is in the image; say $b/1 = a/s$, with $s \notin \mathfrak{q}$. Therefore $tsb = ta$ for some $t \notin \mathfrak{q}$. Thus $ts \in I \subset \mathfrak{q}$. But $\mathfrak{q}$ is prime, so $s \in \mathfrak{q}$ or $t \in \mathfrak{q}$. This is a contradiction. □

We also have theorem 2.2 in [3] which can be very useful in our case.

Definition 6. *A ring homomorphism $f : K \mapsto A$ is called a minimal homomorphism if:*

1. *it is injective*

2. *non-surjective*

3. *for all the decomopositions $f = gh$ of f where g and h are injections, g or h is an isomorphism.*

Theorem 4.13. *Ferrand-Olivier. Let $f : A \mapsto B$ be a minimal homomorphism:*

1. *There exists a maximal ideal $\mathfrak{m}$ of A such that for all $\mathfrak{p} \in Spec(A)$, different from $\mathfrak{m}$, $A_{\mathfrak{p}} \mapsto B_{\mathfrak{p}}$ is an isomorphism.*

2. *The following conditions are equivalent:*

   (a) *the maximal ideal $\mathfrak{m}$ defined in part 1, can be "transferred" in B (this means that $f(\mathfrak{m})$ is an ideal in B).*

   (b) $\mathfrak{m}B = m$.

   (c) $A \to B \rightrightarrows B \otimes_A B$ *is an exact sequence.*

   (d) f *is finite.*

   (e) $Spec(A) \mapsto Spec(B)$ *is surjective.*

3. *If the conditions in part 2 are not satisfied, f is a flat epimorphism.*

The condition 2.e of the theorem is of interest to us because it is fulfilled by our problem. Indeed in our case subrings are also finitely generated $\mathbb{C}$-subalgebras and $\mathbb{C}[x]$ is a principal domain.

References

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[2] Harm Derksen; Stefan Maubach; *Maximal subrings of commutative rings.*

[3] D.Ferrand;J.-P. Olivier; *Homomorphismes Minimaux d'Anneaux* J. ALGEBRA 16, 461-471 (1970)