An expedition through a hierarchy

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Prologue

Through several courses I took in my Bachelor I was introduced to the concept of freeness. Since I also took a course in lattice theory, it made sense to combine these two topics and make this the subject of this Bachelor thesis.
More specifically, I am considering a special hierarchy of completions, the one-sided, free completions of lattices. In this hierarchy two very special, and very well known, structures arise: the MacNeille completion, discovered in 1937 by H.M. MacNeille (1907-1973) and the canonical extension, discovered in 1951 by B. Jónsson (1920-*) and A. Tarski (1901-1983). Although these two structures have been examined extensively, not much is known about the rest of the hierarchy. My goal in this thesis is therefore to gain some intuition and knowledge about the properties of this hierarchy.

Chapter 1

Non-existence of free complete Boolean algebras

1.1 Prerequisites

I will start with an introduction to make the rest of this thesis understandable for people with no background in this area. If you are familiar with order theory I advice you to skip this not so interesting list of definitions.

1.1.1 Posets, lattices and Boolean algebra's

One of the most general object occurring in order-theory is the partially ordered set, a set with a partial order on it. This structure leads to definitions of algebra's such as the lattice and the Boolean algebra.

Definition 1.1 (Poset) *A* partially ordered set*, or poset, is a structure* $\mathbb{P} = (P, \leq)$ *with a set* P *and a binary relation* $\leq$*, the partial order, which is reflexive, anti-symmetric and transitive, i.e.* $\forall a, b, c \in P$ *we have:*

O1 $a \leq a$*;*

O2 *if* $a \leq b$ *and* $b \leq a$ *then* $a = b$*;*

O3 *if* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$*.*

In a partial order, it is not always possible to compare two elements. A poset in which every two elements are incomparable is called an anti-chain. However, sometimes the least upper bound (sup) or greatest lower bound (inf) of two elements exists. If the sup exists for every two elements in the poset, it is called a $\vee$-semilattice and it is called a $\wedge$-semilattice if the inf exists for every two elements in the poset. A poset that is both a $\vee$-semilattice and a $\wedge$-semilattice is called a lattice.

Definition 1.2 (Lattice) *Let* $\mathbb{L}$ *be a poset and* $x \vee y := \sup\{x, y\}$ *and* $x \wedge y := \inf\{x, y\}$. *Then the structure* $\mathbb{L} = (L, \vee, \wedge, 0, 1)$ *is defined to be a* lattice *if and only if for all* $x, y \in L$ $x \vee y$ *and* $x \wedge y$ *exist in* L *and* 0 *is the least and* 1 *is the largest element.*

The concept lattice can again be extended into the concept Boolean Algebra, denoted as BA.

Definition 1.3 (Boolean algebra) *A* Boolean algebra *is a structure* $\mathbb{A} = (A, \vee, \wedge, \prime, 0, 1)$ *with a set* A*, two binary operations, join* ($\vee$) *and meet* ($\wedge$)*, a unary operation complement* ($\prime$) *and two constants 0 and 1, such that* $\forall x, y, z \in A$ *we have that:*

B1 $x \vee (y \vee z) = (x \vee y) \vee z$;

B1' $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

B2 $x \vee y = y \vee x$;

B2' $x \wedge y = y \wedge x$;

B3 $x \vee (x \wedge y) = x$;

B3' $x \wedge (x \vee y) = x$;

B4 $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

B4' $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

B5 $x \vee x\prime = 1$;

B5' $x \wedge x\prime = 0$.

Subsets of a power set $\wp(X)$ closed under union, intersection, and complement which contain $\emptyset$ and X are Boolean algebras. Here union plays the role of $\vee$, intersection plays the role of $\wedge$ and complement plays the role of $\prime$. $\emptyset$ can be seen as 0 and X is the greatest element 1.

The properties **B1, B1', B2, B2', B3, B3', B5** and **B5'** form the algebraic definition of a lattice. In this definition a lattice $\mathbb{L}$, and thus also a BA, can again be seen as a poset by defining a partial order $\leq$ on the set L by for $a, b \in L$:

$$a \leq b \Longleftrightarrow a \vee b = b.$$

A subset of a lattice or a BA is called a subalgebra if it is closed under all the operations. In the case of a BA the definition would look like this:

Definition 1.4 (Subalgebra) *A structure* $\mathbb{A} = (A, \vee_A, \wedge_A, \prime_A, 1_A, 0_A)$ *is a* subalgebra *of a Boolean algebra* $\mathbb{B} = (B, \vee_B, \wedge_B, \prime_B, 1_B, 0_B)$ *if for all* $a, b \in A$ *we have*

S1 *$A \subseteq B$;*

S2 $1_A = 1_B$*;*

S2' $0_A = 0_B$*;*

S3 $a \vee_A b = a \vee_B b$*;*

S3' $a \wedge_A b = a \wedge_B b$*.*

Because of **B1** and **B1'** in the definition of BA it is clear that you can join, or meet, any finite number of elements of A using induction. However, not necessarily do the infinite join, $\bigvee C := \sup C$, and infinite meet, $\bigwedge C := \inf C$, exist for any subset C of A.

Definition 1.5 (Complete) *Let $\mathbb{L}$ be a lattice. We say that $\mathbb{L}$ is* complete *if and only if for any $C \subseteq L$, $\bigvee C$ and $\bigwedge C$ exist.*

A lattice that is join-complete, $\bigvee C$ exists for any $C \subseteq L$, is always also meet-complete. Later on we will distinguish the two by considering different structure preserving functions between them.
The properties **B4** and **B4'** of the definition of the Boolean algebra are called the *distributive laws*. There also exist stronger distributive properties.

Definition 1.6 *Let $\mathbb{L}$ be a $\bigvee$-complete lattice. If for any subset $\{y_j\}_{j\in J}$ of $\mathbb{L}$ and any $x \in \mathbb{L}$ we have*

$$x \wedge (\bigvee_{j\in J} y_j) = \bigvee_{j\in J} (x \wedge y_j)$$

we say $\mathbb{L}$ satisfies the join infinite distributive law*.*
The dual condition, where the order is turned upside-down down, is the meet infinite distributive law*.*

The strongest sense of distributivity is stated in the *infinite distributive law*:

Definition 1.7 (Completely distributive) *Let $\mathbb{K}$ be a complete lattice. We say that K is* completely distributive *if and only if for any doubly indexed subset $\{x_{ij}\}_{i\in I, j\in J}$ of $\mathbb{K}$ we have*

$$\bigwedge_{i\in I}(\bigvee_{j\in J_i} x_{ij}) = \bigvee_{h:I\to J_i} (\bigwedge_{i\in I} x_{ih(i)}),$$

where $h : I \to J_i$ ranges over all choice functions.

Complete distributivity is such a strong property that it almost only holds for sets of sets closed under arbitrary intersection and arbitrary union, but it at least does hold for these.

A way to get new algebra's is to look which smaller ones are sitting in the ones you already have, the subalgebra's.

Definition 1.8 (Generating a subalgebra) *Let X be a subset of a lattice $\mathbb{L}$. Then $\langle X \rangle := \bigcap\{K \subseteq L \mid X \subseteq K$ and K is a subalgebra of $\mathbb{L}\}$ is the* subalgebra generated by *X in $\mathbb{L}$.*
I will say that $\mathbb{L}$ is generated *by X if $\mathbb{L} = \langle X \rangle$.*

In the same way you can make the subalgebra of a BA generated by a subset. If a lattice has a subset such that every element in the lattice can be reached by an arbitrary meet or by an arbitrary join that subset is said to be lying dense in the lattice. But this is just one of the ways to define denseness.

Definition 1.9 (Dense) *Let $\mathbb{L}$ be a lattice, $Q \subseteq L$. Then we have six cases: we say Q is*

1. *$\bigvee$-dense if $\forall a \in L\ \exists A \subseteq Q$ with $\bigvee A = a$ (D1),*
2. *$\bigwedge$-dense if $\forall b \in L\ \exists B \subseteq Q$ with $\bigwedge B = b$ (D2),*
3. *both $\bigvee$- and $\bigwedge$-dense if both D1 and D2 hold.*
4. *$\bigwedge\bigvee$-dense if $\forall a \in L\ \exists A_i \subseteq Q$, $i \in I$ with $\bigwedge_{i \in I} \bigvee A_i = a$ (D3),*
5. *$\bigvee\bigwedge$-dense if $\forall b \in L\ \exists B_j \subseteq Q$, $j \in J$ with $\bigvee_{j \in J} \bigwedge B_j = b$ (D4) and*
6. *both $\bigwedge\bigvee$- and $\bigvee\bigwedge$-dense if both D3 and D4 hold.*

I will sometimes use a simple notation for lattices, L instead of $\mathbb{L}$.
There are some special subsets of posets and subalgebra's of lattices and BA's that I will pay special attention to. Filters and ideals represent possibly non-existing elements of a poset, an independent subset is a special antichain and directed sets make sure that there is always a common upper bound of two elements.

Definition 1.10 (Filter) *Let $\mathbb{L}$ be a lattice and $0 \neq F \subseteq L$ a subset. F is called a filter if both of the following conditions old:*

- *for every $x, y \in F$ we have $x \wedge y \in F$;*
- *if for every $x \in F$ and $y \in P$, $x \leq y$ implies that $y \in F$.*

A consequence of this definition is that in every filter F of a lattice $\mathbb{L}$ we have $x \wedge y \in F$ for any $x, y \in F$ and for every filter F of a BA we have $1 \in F$.
The smallest filter containing an element l of a lattice $\mathbb{L}$ is $\uparrow l = \{x \in L \mid x \geq l\}$ which we call the *principal upset*. A filter F of a BA $\mathbb{B}$ is called *proper* if $0 \notin F$ and it is called an *ultrafilter* if for every $b \in B$ either $b \in F$ or $b' \in F$ but not both.
The set of filters can be seen as a poset, with the dual inclusion order:

$$F \leq F' \iff F' \subseteq F.$$

The dual definition of a filter leads to the definition of an ideal. The set of ideals is equipped with the inclusion order:

$$I \leq I' \Longleftrightarrow I \subseteq I'.$$

The set of ideals of a lattice $\mathbb{L}$ is denoted by $Id(\mathbb{L})$, the set of filters by $Filt(\mathbb{L})$ and the set of ultrafilters of a BA $\mathbb{B}$ by $Ult(\mathbb{B})$.
We will use ultrafilters in section 3.1, therefore the next theorem will be useful.

Theorem 1.11 (Ultrafilter theorem) *Let X be a filter of a Boolean algebra $\mathbb{B}$. Then there exists an ultrafilter U of $\mathbb{B}$ with $X \subseteq U$.*

Proof: See [3, page 237]. □

1.1.2 Morphisms

Of course there are also structure preserving functions between the described algebra's above.

Definition 1.12 (Lattice homomorphism) *A* (lattice) homomorphism *from $\mathbb{K}$ into $\mathbb{L}$ is a function $h : \mathbb{K} \longrightarrow \mathbb{L}$ with for all $a, b \in K$:*

H1 $h(a \vee_{\mathbb{K}} b) = h(a) \vee_{\mathbb{L}} h(b)$;

H2 $h(a \wedge_{\mathbb{K}} b) = h(a) \wedge_{\mathbb{L}} h(b)$;

H3 $h(0_{\mathbb{K}}) = 0_{\mathbb{L}}, \quad h(1_{\mathbb{K}}) = 1_{\mathbb{L}}$;

If h is injective it is called a (lattice) embedding, *if it is bijective then h is called a* (lattice) isomorphism.

Note that the structure preserving functions between $\vee$-semilattices, or $\wedge$-semilattices, can be given by only the property H1, or H2. The structure preserving functions between join-complete lattices are given by an extra property. Let $f : \mathbb{L} \longrightarrow \mathbb{K}$ be a lattice homomorphism between two join-complete lattices. We say f is $\bigvee$-preserving if and only if:

C1 for any $A \subseteq L$ we have $f(\bigvee_{\mathbb{L}} A) = \bigvee_{\mathbb{K}}\{f(a) \mid a \in A\}$.

Dually we can define $\bigwedge$-preserving homomorphisms using C2. A lattice homomorphism $g : \mathbb{L} \longrightarrow \mathbb{K}$ between two meet-complete lattices is said to be $\bigwedge$-preserving if

C2 for any $A \subseteq L$ we have $g(\bigwedge_{\mathbb{L}} A) = \bigwedge_{\mathbb{K}}\{g(a) \mid a \in A\}$.

A function with both property C1 and C2 is called complete.

Definition 1.13 (Homomorphism of Boolean algebras) *A* homomorphism (of Boolean algebras) *from a BA $\mathbb{A}$ into a BA $\mathbb{B}$ is a function $h : \mathbb{A} \longrightarrow \mathbb{B}$ with for all $a, b \in A$:*

H1 $h(a \vee_{\mathbb{A}} b) = h(a) \vee_{\mathbb{B}} h(b)$*;*

H2 $h(a \wedge_{\mathbb{A}} b) = h(a) \wedge_{\mathbb{B}} h(b)$*;*

H3 $h(0_{\mathbb{A}}) = 0, h(1_{\mathbb{A}}) = 1$*;*

H4 $h(a\prime_{\mathbb{A}}) = h(a)\prime_{\mathbb{B}}$.

As in the lattice case, if h *is injective it is called an* embedding (of Boolean algebras)*, if it is bijective then* h *is called an* isomorphism (of Boolean algebras)*.*

Note that the image of a homomorphism from Boolean algebra's $\mathbb{A}$ into $\mathbb{B}$ is a subalgebra of $\mathbb{B}$. A homomorphism can give you an completion of a BA if the homomorphism embeds into a complete algebra.

Definition 1.14 (Completion) *Let* $\mathbb{K}$ *be a lattice and* $e : \mathbb{K} \hookrightarrow \mathbb{L}$ *a lattice embedding from* $\mathbb{K}$ *into a complete lattice* $\mathbb{L}$*. Then* $(e, \mathbb{L})$ *is called a* completion *of* $\mathbb{K}$*.*

Some algebras have, just as some topologies, a compactness property, a sense of finiteness.

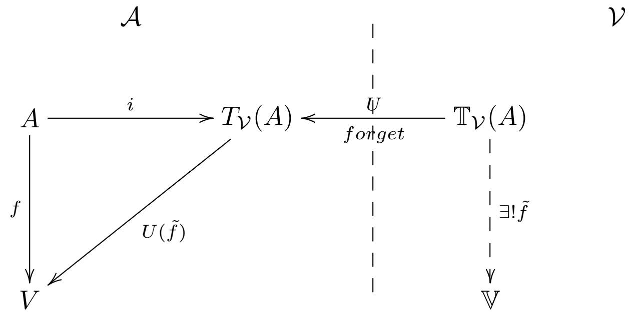
Definition 1.15 (Compact) *A lattice embedding* $e : L \hookrightarrow C$ *from a lattice* L *into a complete lattice is* compact *provided for all* $T, S \subseteq L$*, if* $\bigwedge e(S) \leq \bigvee e(T)$ *in* C *then there exist* $T' \subseteq_{fin} T$*,* $S' \subseteq_{fin} S$ *such that* $\bigwedge T' \leq \bigvee S'$ *in* L*.*

1.1.3 Freeness

It is nice to have a notion of a most complicated object in a class of algebra's, a free object. You could demand that every element of this class is a homomorphic image of the free object. To define this free object I will use some category theory which I won't explain further but can be found in chapter 2, section 10, of A course in Universal Algebra [1].

We start with a class of algebras $\mathcal{V}$ and a class of 'simpler' algebras $\mathcal{A}$, in the sense that algebras in $\mathcal{V}$ have more structure or more properties. Let U be the forgetful functor from $\mathcal{V}$ into $\mathcal{A}$ forgetting the extra structure of properties. Then a free object of $\mathcal{V}$ over an object A in $\mathcal{A}$ is a tuple $(\mathbb{T}_{\mathcal{V}}(A), i)$ consisting of an object of $\mathcal{V}$ and a $\mathcal{A}$-homomorphism from A into $T_{\mathcal{V}}(A) := U(\mathbb{T}_{\mathcal{V}}(A))$ with the following universal mapping property:

For any element $\mathbb{V}$ in $\mathcal{V}$ and any $\mathcal{A}$-homomorphism f from A into $V := U(\mathbb{V})$ there exists a unique $\mathcal{V}$-homomorphism $\tilde{f}$ from $\mathbb{T}_{\mathcal{V}}(A)$ in $\mathbb{V}$ such that $U(\tilde{f}) \circ i = f$. In other words, the following diagram commutes:



Not always does every algebra of a class have such free objects over another class. I will give two explicit examples, one of a class that has a free object over another class, and one that does not.

1.2 The class of complete Boolean algebra's

An example of a forgetful functor without free objects is the class of complete Boolean algebra's over sets, where the homomorphisms are both $\bigvee-$ and $\bigwedge-$preserving. That such a forgetful functor does not exist was proved first, and independently, by both Gaifman and Hales. Since then a new proof is available. It uses another theorem, proved later by Solovay.

In this section I will try to give a little bit of insight in how Gaifman's and Hales' result follows from Solovay's theorem. I won't explain the proof of Solovay's theorem since it is too complicated. You can find the proof in chapter section 13 of the Handbook of Boolean Algebra's [9, page 191].

To understand the content of the theorem you will need some knowledge of topology. I will assume that you are familiar with the concepts topological space, interior (*int*), closure (*cl*), the discrete topology and the product topology [11]. The proof also uses the regular open algebra of a topological space X, which is defined in the following way.

Definition 1.16 *Let X be a topological space.*
For a subset A of X the regularization *of A is defined by $r(A) := int(cl(A))$.*
$U \subset X$ is regular open *if $r(U) = U$.*
$RO(X) = \{U \subset X : U$ regular open $\}$ is the regular open algebra *of X.*

Theorem 1.17 *$RO(X)$ is a complete Boolean algebra under set-theoretical inclusion. The distinguished elements and the operations of $RO(X)$ are given by*

$0 = \emptyset, \quad 1 = X,$

$U \vee V = r(U \cup V),$

$U \wedge V = U \cap V,$

$U\prime = int(X \backslash U),$

$\bigvee A = r(\bigcup A),$

$\bigwedge A = r(\bigcap M).$

Solovay proved that if you apply this construction to the product space ${}^{\omega}\kappa$ of any infinite cardinal κ, you get an algebra which is the size of this cardinal, but that is still countably generated.

Theorem 1.18 (Solovay) *Let κ, an infinite cardinal, have the discrete topology and let ${}^{\omega}\kappa$ have the product topology. Then the collapsing algebra, that is $RO({}^{\omega}\kappa)$, is countably completely generated and has cardinality at least κ.*

The result of Gaifman and Hales follows almost directly. If there is a countably generated complete Boolean algebra of every size, then there could not be a free (set-sized) object with the universal mapping property over infinite sets.

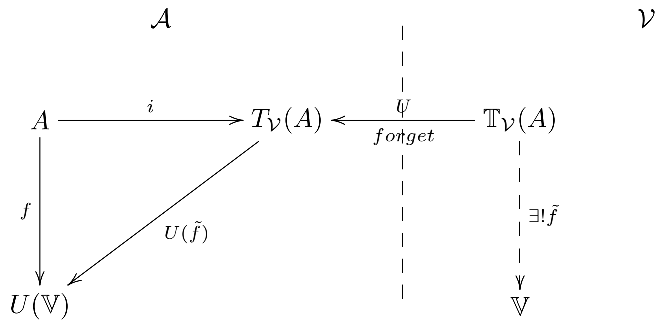
Corollary 1.19 (Gaifman, Hales) *There are no free complete Boolean algebras over infinite sets.*

Proof: Assume that I is an infinite set and (F, e) is a free complete Boolean algebra over I. Let κ be any cardinal greater than $|F|$ and consider the collapsing algebra $B = RO({}^{\omega}\kappa)$. By Solovay's theorem, B has a countable set X of complete generators; let $f : I \longrightarrow X$ be onto. Now if $g : F \longrightarrow B$ is complete and satisfies $g \circ e = f$, then $Im(g)$ is a complete subalgebra of B including X. So g is onto and $\kappa \leq |B| \leq |F| < \kappa$, a contradiction. □

To make the situation in this proof a little more clear I will compare the result we just obtained with the definition of freeness I gave in the previous section.

Let $\mathcal{V}$ be the class of complete Boolean algebras (CBA) where the CBA-homomorphisms are BA-homomorphisms that preserve arbitrary joins and meets. Let $\mathcal{A}$ be the class of sets (Sets) where the homomorphisms are simply functions.
The forgetful functor U from $\mathcal{V}$ to $\mathcal{A}$ sends Boolean algebras to their underlying sets and CBA-homomorphisms to their underlying functions.

We are looking for a free object $\mathbb{T}_{\mathcal{V}}(A)$ in $\mathcal{V}$ over a set A in $\mathcal{A}$ and a function i (in Sets) from A to $\mathbb{T}_{\mathcal{V}}(A)$. The free object $(\mathbb{T}_{\mathcal{V}}(A), i)$ has to make the following diagram commute for every $\mathbb{V}$ in $\mathcal{V}$ and any function f in Sets from A to $U(\mathbb{V})$.



If the set A is infinite and we choose V to be a collapsing algebra the argument in corollary 1.19 shows that we have to conclude that $\mathbb{T}_{\mathcal{V}}(A)$ would have to be bigger than any cardinal, which is of course impossible.

Note that we just considered complete Boolean algebras with homomorphisms that are both $\bigvee$- and $\bigwedge$-preserving. In the next section I will show that if we only consider homomorphisms that preserve one or the other there is no problem.

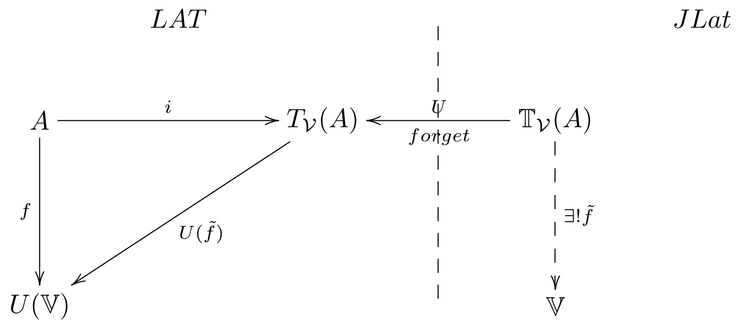
Chapter 2

The hierarchy of one-sided completions

2.1 The first introduction

The theorem due to Gaifman and Hales states that there exists no free complete BA over an infinitely generated BA in ZF(C). However, starting with a poset or a lattice, there does exist a free join-completion and a free meet-completion. I will concentrate at the free join- and meet-completion over a lattice. These completions are the complete lattices consisting of respectively all the ideals and all the filters of a lattice.

Lets sketch the situation of a join-completion using the definition of a free object. Let $\mathcal{V}$ be the class of join-complete lattices (JLat) where the homomorphisms are lattice-homomorphisms that preserve arbitrary joins. Let $\mathcal{A}$ be the class of lattices (LAT) with lattice-homomorphisms. The forgetful functor U from $\mathcal{V}$ to $\mathcal{A}$ sends a join-complete lattice to the underlying lattice and a JLat-homomorphisms to the underlying lattice-homomorphism.

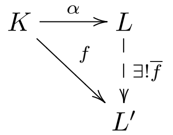


We are looking for a free object over a lattice A consisting of an object $\mathbb{T}_{\mathcal{V}}(A)$ in $\mathcal{V}$ and a lattice-homomorphism $i : A \longrightarrow \mathbb{T}_{\mathcal{V}}(A)$. Further we want for every object $\mathbb{V}$ in $\mathcal{V}$ and every lattice-homomorphism f from A to $U(\mathbb{V})$ that there exists a JLat-homomorphism $\tilde{f}$ from $\mathbb{T}_{\mathcal{V}}(A)$ to $\mathbb{V}$ with $U(\tilde{f}) \circ i = f$.

Let's take a look at a proposition and some lemmas.

Proposition 2.1 *Assume that (L, α) is a $\bigvee$-completion of a lattice K. Then the following are equivalent:*

- *(L, α) is a compact completion;*
- *whenever L' is a complete lattice and $f : K \longrightarrow L'$ is a lattice-homomorphism, then there exists a unique lattice homomorphism $\overline{f} : L \longrightarrow L'$ preserving arbitrary joins such that $\overline{f} \circ \alpha = f$.*



This map is given by

$$\overline{f}(x) := \bigvee \{f(p) \mid x \geq \alpha(p)\};$$

- *there exists an isomorphism $\eta : L \cong Id(K)$ with $\eta(\alpha(p)) = \downarrow(p)$ for all $p \in K$.*

Proof: This is originally proved in [7], but can also be found in [5, Prop. 2.1]. □

In other words there is only one, up to isomorphism, compact $\bigvee$-completion of a lattice K which we will denote by $(\mathbb{F}_{\bigvee}(K), \alpha_K)$. Further we know that $(\mathbb{F}_{\bigvee}(K), \alpha_K)$ has the universal mapping property we saw in section 1.1.3. We can conclude that the class of join-complete lattices over a lattice K has a free object, namely the free join-completion $(\alpha_K, \mathbb{F}_{\bigvee}(K))$.

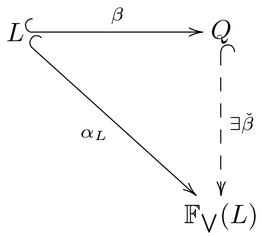
Lemma 2.2 *Let K be a lattice. Then $\alpha_K : K \longrightarrow \mathbb{F}_{\bigvee}(K)$ has the following properties:*

1. *α_K preserves all existing meets;*
2. *α_K preserves finite joins;*
3. *$\mathbb{F}_{\bigvee}(K)$ is a $\wedge$-semilattice;*

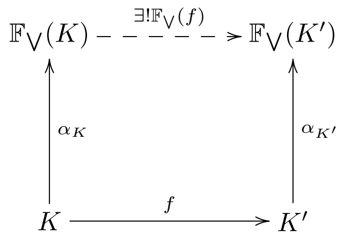
Proof: See [5, Lemma 2.2]. □

There is another property of the free join-completion that I will use later on.

Lemma 2.3 *Let L be a lattice and $\beta : L \hookrightarrow Q$ an order-embedding such that L is $\bigvee$-dense in Q. Then there exists an $\bigvee$-preserving embedding $\breve{\beta} : Q \hookrightarrow \mathbb{F}_{\bigvee}(L)$ that makes the following diagram commute:*



The free join-completion is in fact a functor. Given two lattices K and K' and a lattice-homomorphism map $f : K \longrightarrow K'$ there is a unique join preserving map $\mathbb{F}_{\bigvee}(f) : \mathbb{F}_{\bigvee}(K) \longrightarrow \mathbb{F}_{\bigvee}(K')$ such that the following diagram commutes.



If I consider K as a sublattice of $\mathbb{F}_{\bigvee}(K)$ then the definition of $\mathbb{F}_{\bigvee}(f)$ is

$$\mathbb{F}_{\bigvee}(f)(y) = \bigvee\{f(p) \mid y \geq p\} \text{ for } y \in \mathbb{F}_{\bigvee}(K).$$

The following lemma describes some properties of this 'lifted' map $\mathbb{F}_{\bigvee}(f)$.

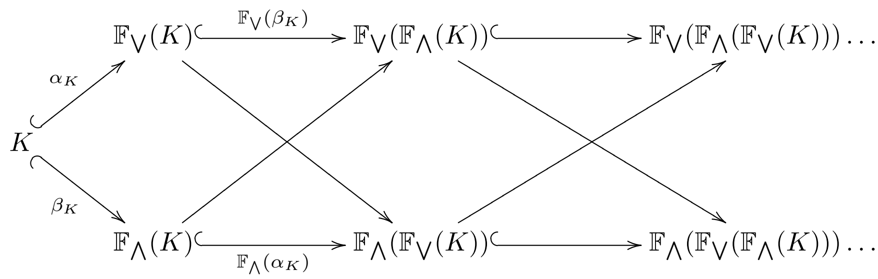
Lemma 2.4 *Let K and K' be lattices and let $f : K \to K'$ be a lattice-homomorphism.*

1. *(a) $\mathbb{F}_{\bigvee}(f)$ is the unique extension of f to $\mathbb{F}_{\bigvee}(K)$ that preserves $\bigvee$.*
   (b) $\mathbb{F}_{\bigvee}(f)$ preserves $\wedge$.
2. *If f is a lattice-embedding, then $\mathbb{F}_{\bigvee}(f)$ is a lattice-embedding.*

Proof: See [5, Lemma 2.4]. □

Of course all the constructions described have dual versions, which give rise to the free meet-completion $(\mathbb{F}_{\bigwedge}(K), \beta_K)$, a $\bigwedge$-preserving function $\underline{f} : \mathbb{F}_{\bigwedge}(K) \longrightarrow L$ for any function f from K into a complete lattice L and the lifted function, the unique meet preserving map $\mathbb{F}_{\bigwedge}(f) : \mathbb{F}_{\bigwedge}(K) \longrightarrow \mathbb{F}_{\bigwedge}(K')$ of a lattice-homomorphism $f : K \longrightarrow K'$. And for any embedding $\beta : K \hookrightarrow Q$ such that K is $\bigwedge$-dense in Q, there exists an embedding $\hat{\beta} : Q \hookrightarrow \mathbb{F}_{\bigwedge}(K)$.

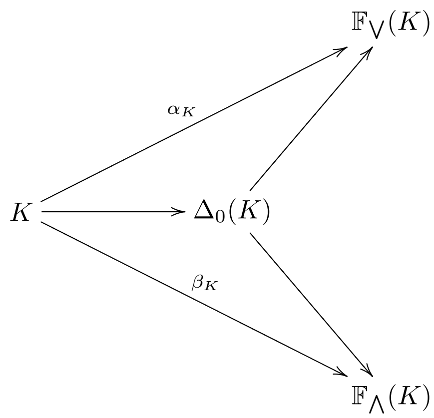
Iterating the above constructions, thus making $\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(K))$ and $\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(K))$ and so on, gives us an infinite hierarchy.



2.2 Two-sided completions

We already have infinitely many join- and meet-completions. However, it would be nice to also have an object, with a more symmetric behavior relative to $\bigvee$ and $\bigwedge$.
We will represent $\mathbb{F}_{\bigvee}(K)$ with Σ_0, $\mathbb{F}_{\bigwedge}(K)$ with Π_0, $\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(K))$ with Σ_1, $\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(K))$ with Π_1 and so on. We will denote a two-sided completion with Δ.

On the first level, for example, we will be looking for the 'common part' of the Σ_0 and Π_0 completion, Δ_0 . That is, we are looking for a complete lattice $\Delta_0(K)$ such that:
$\Delta_0(K)$ embeds in $\mathbb{F}_{\bigvee}(K)$ with a $\bigvee$-preserving homomorphism and into $\mathbb{F}_{\bigwedge}(K)$ with a $\bigwedge$-preserving homomorphism. Furthermore we want these homomorphisms to make the following diagram commute:



In fact, we can look for such an interpolating structure at every step of the hierarchy of one-sided completions. An interesting question is now: Is there a largest interpolation structure at every step, Δ_i, that has these properties?
It turns out that Δ_0 and Δ_1 exist, although they are not free objects. I will give their definitions and explain some of their properties.

2.2.1 The MacNeille completion

The MacNeille completion is an interpolating structure that can be embedded in $\mathbb{F}_{\bigvee}(K)$ and $\mathbb{F}_{\bigwedge}(K)$ for a lattice K.

Definition 2.5 (MacNeille completion) *Let K be a lattice. For $A \subseteq K$ I define the sets of upper and lower bounds of A to be*

$$A^u := \{y \in K \mid (\forall x \in A)\ x \leq y\},$$

$$A^l := \{y \in K \mid (\forall x \in A)\ y \leq x\}.$$

Further we define the MacNeille completion of K to be

$$\mathcal{N}(K) := \{A \subseteq K \mid A^{ul} = A\}.$$

Traditionally a set A for which $A = A^{ul}$ is called a (lower) cut. It is also known as a normal ideal.

It can be proved that the MacNeille completion $\mathcal{N}(K)$ of a lattice K is indeed complete. Also for all $x \in K$ we have $\downarrow x \in \mathcal{N}(K)$. Therefore we can define the lattice-embedding $\phi : K \hookrightarrow \mathcal{N}(K)$ by $\phi(x) = \downarrow(x)$. In other words, $(\phi, \mathcal{N}(K))$ is a completion of K.

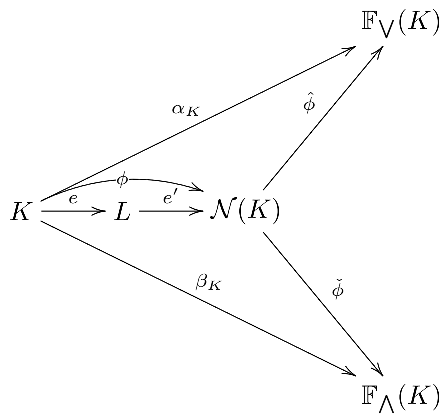
For the embedding ϕ the following properties hold:

Lemma 2.6 *Let K be a lattice, and $\phi : K \hookrightarrow \mathcal{N}(K)$ with $\phi(x) = \downarrow(x)$.*

1. *$\phi(K)$ is both $\bigvee$- and $\bigwedge$-dense in $\mathcal{N}(K)$, thus $(\phi, \mathcal{N}(K))$ is a $\bigvee$- and $\bigwedge$-completion of K. Also ϕ preserves arbitrary existing joins and meets.*
2. *Let L be a complete lattice and let K be a $\bigvee$- and $\bigwedge$-dense subset of L. Then $L \cong \mathcal{N}(K)$.*
3. *Let $e : K \hookrightarrow L$ be an embedding with $e(K)$ both $\bigvee$- and $\bigwedge$-dense in L, then there is a lattice-embedding f from L into $\mathcal{N}(K)$* such that $f \circ e = \phi$.

Proof: See [3, Theorem 7.40 and 7.41]. □

Since $\mathcal{N}(K)$ is both a $\bigvee$- and $\bigwedge$-completion of K there are functions $\hat{\phi}$ and $\check{\phi}$ from the MacNeille completion into respectively $\mathbb{F}_{\bigvee}(K)$ and $\mathbb{F}_{\bigwedge}(K)$, see lemma 2.3. Combining this with lemma 2.6 I know that $\mathcal{N}(K)$ is the largest structure such that for any e and L as in lemma 2.6-3, there is an $e' : L \longrightarrow \mathcal{N}(K)$ which makes the following diagram commute:



In other words, the MacNeille completion is indeed the structure Δ_0 we were looking for.

2.2.2 The canonical extension

Now, let's start looking for the interpolating structure Δ_1, the largest interpolating structure that can be embedded in $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(K))$ and $\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(K))$.

Theorem 2.7 *For a lattice K, the order induced on*

$$Int(K) := \mathbb{F}_\bigvee(K) \cup \mathbb{F}_\bigwedge(K)$$

in $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(K))$ is the same as the order induced on $Int(K)$ in $\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(K))$.

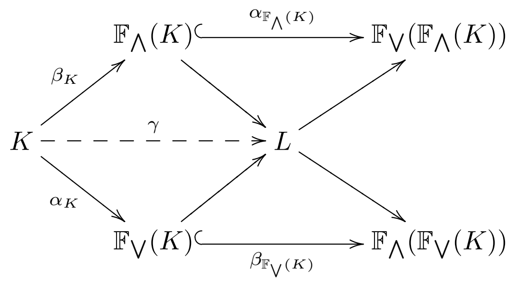
Proof: See [4, Theorem 3.1]. □

From this lemma we know that $Int(K)$ is an interpolating structure. Gehrke and Priestley [5, Prop. 3.5] proved that every possible intermediate structure can be embedded in the MacNeille completion of $Int(K)$, so the next definition is natural.

Definition 2.8 *For a lattice K, define the canonical extension, K^δ, by*

$$K^\delta = \mathcal{N}(Int(K)).$$

Lemma 2.9 *For a lattice K the canonical extension K^δ is the greatest possible interpolant L that commutes in the following diagram:*



Proof: See [5, Proposition 3.5] □

It turns out that the canonical extension is the unique two-sided completion (K, γ) characterized by the following properties. This is originally proved in [8], but for looking up see [4, Definition 2.2]:

1. K is complete and γ is an embedding;
2. γ is compact;
3. $Im(\gamma)$ lies both $\bigwedge\bigvee$- and $\bigvee\bigwedge$-dense in K.

Further we know that if K is distributive then K^δ is completely distributive.

Another theorem known but not proved here, says that the canonical extension of a BA has a connection with the set of ultrafilters.

Theorem 2.10 *Let $\mathbb{B}$ be a Boolean algebra, then $\mathbb{B}^\delta$ is isomorphic to $\wp(Ult(B))$.*

Proof: See [8]. □

Chapter 3

First explorations of the hierarchy

This far, we have only seen constructions, lemmas and theorems proved by other people. What we found out this far is that the hierarchy of one-sided completions gives rise to two Δ-objects. But what else is there to discover about this hierarchy? In this chapter I will try to give some answers to this question.

3.1 A property of growth

A first thing to wonder about, when exploring the hierarchy, is how much the one-sided completions grow at each step.
For example: what is the difference in size between a BA $\mathbb{B}$ and its free $\bigwedge$-completion $\mathbb{F}_{\bigwedge}(\mathbb{B})$? It turns out that $\mathbb{F}_{\bigwedge}(\mathbb{B})$ has the same size as the power set of $\mathbb{B}$. To prove this I first claim a few lemmas.

Definition 3.1 (Independent) *A subset A of a BA is called* independent *if for arbitrary disjoint finite subsets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ of A it holds that:*

$$u_1 \wedge \ldots \wedge u_n \wedge v_n' \wedge \ldots \wedge v_m' > 0.$$

Definition 3.2 (FIP) *A subset $S \subseteq \mathbb{B}$ has the* finite intersection property (FIP) *if*

$$\forall n \in \mathbb{N}\ \forall a_1, \ldots, a_n \in S\ :\ a_1 \wedge \ldots \wedge a_n > 0.$$

Theorem 3.3 (Balcar, Franěk) *Let $\mathbb{B}$ be an infinite BA, then there exists an independent subposet $\mathcal{A} \subseteq \mathbb{B}$ with $|\mathcal{A}| = |\mathbb{B}|$.*

The theorem can be found in section 13, page 196, of the Handbook of Boolean Algebras [9], you will find the proof on page 202.

Proposition 3.4 *Let $\mathbb{B}$ be an infinite BA and $\mathcal{A} \leq \mathbb{B}$ an independent subposet with $|\mathcal{A}| = |\mathbb{B}|$. Then for any $S \subseteq \mathcal{A}$ there is a ultrafilter $P_S \in \mathbb{F}_{\bigwedge}(\mathbb{B})$ with the following properties.*

1. $S \subseteq P_S$,
2. $\neg(\mathcal{A} \backslash S) := \{\neg b \mid b \in \mathcal{A}, b \notin S\} \subseteq P_S$.

Proof: We will show that $T = S \cup \neg(\mathcal{A}\backslash S)$ has the finite intersection property (FIP). Recall that this implies that T is contained in an ultrafilter, which follows from the Boolean prime ideal theorem [9, section 2, page 33].
Let $a_1, \ldots, a_n \in S \cup \neg(\mathcal{A}\backslash S)$. Then for every $1 \leq i \leq n$ we have $a_i \in S$ or $a_i \in \neg(\mathcal{A}\backslash S)$. Thus in the latter case $\neg a_1 \in \mathcal{A}\backslash S$. If we rename the a_i as $b_1, \ldots, b_k$ for elements in S and $c_1, \ldots, c_l$ for elements in $\mathcal{A}\backslash S$ we have by independence

$$a_1 \wedge \ldots \wedge a_n = b_1 \wedge \ldots \wedge b_k \wedge \neg c_1 \wedge \ldots \wedge \neg c_l > 0.$$

We can conclude that $S \cup \neg(\mathcal{A}\backslash S)$ has indeed the FIP and therefore there is an ultrafilter P_S with $S \subseteq P_S$ and $\neg(\mathcal{A}\backslash S) \subseteq P_S$ as needed. □

This proposition leads to the proof of the following theorem.

Theorem 3.5 *Let $\mathbb{B}$ be a BA with cardinality κ. Then $|\mathbb{F}_{\bigwedge}(\mathbb{B})| = 2^\kappa$.*

Proof: By proposition 3.3 we can find an independent subposet $\mathcal{A}$ of $\mathbb{B}$ with $|\mathcal{A}| = |\mathbb{B}|$. Let $S, T \subset \mathcal{A}$ with $S \neq T$. Then, without loss of generality, we can say that there is a $t \in T$ such that $t \notin S$. Note that if $t \in T$ then also $t \in P_T$. Furthermore we know that $t \in \mathcal{A}\backslash S$ thus $\neg t \in \neg(\mathcal{A}\backslash S)$. So $\neg t \in P_S$ and by the prime filter properties of P_T we have that $\neg t \notin P_T$. And therefore $P_T \neq P_S$. By this discussion we now know

$$|\{P_S \mid S \subseteq \mathcal{A}\}| = |\{S \mid S \subseteq \mathcal{A}\}| = 2^{|\mathcal{A}|}.$$

Also we know that $|\mathcal{A}| = |\mathbb{B}| = \kappa$ and that $\{P_S \mid S \subseteq \mathcal{A}\}$ is a subset of $\mathbb{F}_{\bigwedge}(\mathbb{B})$. So I can conclude

$$|\mathbb{F}_{\bigwedge}(\mathbb{B})| \geq 2^{\mathcal{A}} = 2^\kappa.$$

On the other hand we have $\mathbb{F}_{\bigwedge}(\mathbb{B}) = Filt(\mathbb{B}) \subseteq \wp(\mathbb{B})$. This gives

$$|\mathbb{F}_{\bigwedge}(\mathbb{B})| \leq |\wp(\mathbb{B})| = 2^\kappa.$$

□

3.2 Generating steps

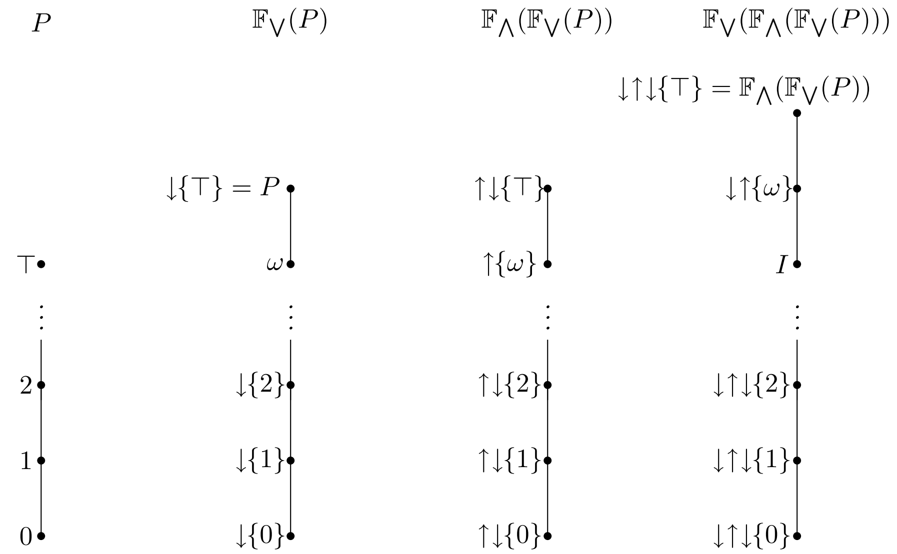
As we saw in the previous section, the one-sided completions grow very fast when you start with a Boolean algebra. The next question worth asking is: are these big structures still generated by the lattice we started with?

In general, we will say that a subset K of a lattice L generates L if all elements of L can be obtained by closing K under $\bigvee$ and $\bigwedge$. The free $\bigvee$-completion of a lattice is always $\bigvee$-generated by the image of this lattice. However, by taking a look at a simple example, we will find out that once we iterate the one-sided completions in the hierarchy, the acquired complete lattices are not always generated by the image of the original lattice.

Example 3.6 *Let P be the lattice $\omega+1$ where we will represent the extra element with $\top$. The free $\bigvee$-completion consists of all the ideals of P, i.e.*

$$\mathbb{F}_{\bigvee}(P) = \{\downarrow\{n\} \mid n \in \omega\} \cup \{\bigcup\{n \in \omega\} = \omega\} \cup \{\downarrow\top\}.$$

The free $\bigwedge$-completion of $\mathbb{F}_{\bigvee}(P)$ consists only of the principle upsets, but taking the $\bigvee$-completion again adds a new ideal, namely $I = \bigcup\{\uparrow\downarrow n \mid n \in \omega\}$.



Note that $\mathbb{F}_{\bigvee}(P) \cong \mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(P))$. In fact the structure of an acquired lattice is only different when taking the join-completion since the lattices in this example have no infinite descending chains.
To get back to our question, let's explore this difference. We will inspect the function $\alpha_P : P \longrightarrow \mathbb{F}_{\bigvee}(P)$. We see that:

$\downarrow\{n\}$ *is the image of* $n \in P$,

$\omega = \bigvee\{\downarrow\{n\} \mid n \in \omega\}$ *and*

$P = \downarrow\{\top\}$ *is the image of* $\top \in P$.

In other words, $\mathbb{F}_\bigvee(P)$ *is join-generated by the image of* P.

However, in $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P)))$ *something else happens. The image of*

$$\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P))} \circ \beta_{\mathbb{F}_\bigvee(P)} \circ \alpha_P : P \longrightarrow \mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P)))$$

is the set containing finitary elements, the images $\downarrow\uparrow\downarrow\{n\}$ *of the elements* n *of* ω*, and one other element,* $\downarrow\uparrow\downarrow\{\top\}$*. We see that:*

$\downarrow\uparrow\downarrow\{n\}$ *is the image of* $n \in P$,

$I = \bigvee\{\downarrow\uparrow\downarrow\{m\} \mid m \in \omega\}$ *and*

$\downarrow\uparrow\downarrow\{\top\}$ *is the image of* $\top \in P$.

So the only element of which we are not sure if it can be obtained by taking joins of elements of the image of $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P))} \circ \beta_{\mathbb{F}_\bigvee(P)} \circ \alpha_P$ *is* $\downarrow\uparrow\{\omega\}$*. Note that for any downset* A *in* $\omega + 1$ *we have either*

$\bigvee\{\downarrow\uparrow\downarrow\{a\} \mid a \in A\} = \downarrow\uparrow\downarrow\{n\}$,

$\bigvee\{\downarrow\uparrow\downarrow\{a\} \mid a \in A\} = I$ *or*

$\bigvee\{\downarrow\uparrow\downarrow\{a\} \mid a \in A\} = \downarrow\uparrow\downarrow\{\top\}$.

From this fact we can conclude that there is no downset A *in* $\omega + 1$ *such that the join of the image of* A *under* $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P))} \circ \beta_{\mathbb{F}_\bigvee(P)} \circ \alpha_P$ *equals* $\downarrow\uparrow\{\omega\}$*. In other words,* $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P)))$ *is not generated by the image of* P *under* $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P))} \circ \beta_{\mathbb{F}_\bigvee(P)} \circ \alpha_P$.
In fact the same thing will happen every time you take a new join-completion. There will be a new element every time that is not join-generated by the image of P.

The example nicely illustrates the answer to our question: if we iterate these completions, as in $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(L))$ we can still embed L in this lattice, but it need not be the case that all elements of $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(L))$ can be obtained by taking $\bigvee$'s and $\bigwedge$'s of elements from L.

3.3 Canonical extensions of canonical extensions

It's quite a disappointment that when iterating the join- and meet-completion of a lattice, the acquired lattices are not always generated by the image of the original lattice. Let's just pose another question. What happens if we iterate constructing the canonical extension of a lattice?
To get a feeling for what is going on, I will start with an example.

Example 3.7 *We take a look at the Boolean algebra B consisting of all the finite and co-finite subsets of $\mathbb{N}$, $B := \{S \subseteq \mathbb{N} \mid S$ is finite or $\mathbb{N}\backslash S$ is finite$\}$. This set can be seen as a distributive lattice with union and intersection as operations, the empty set as bottom and $\mathbb{N}$ as top. The set of ultrafilters of $\mathbb{B} = (B, \cup, \cap, \emptyset, \mathbb{N})$ consists of the principal upsets of the singletons together with one non-principal ultrafilter containing all the cofinite sets, which I will represent with ∞.*

By theorem 2.10 we know that the canonical extension $\mathbb{B}^\delta$ is isomorphic to the power set of the ultrafilters of $\mathbb{B}$:

$$\mathbb{B}^\delta = \wp(Ult(\mathbb{B})) = \wp(\{\uparrow\{n\} \mid n \in \mathbb{N}\} \cup \{\infty\})$$

Because I don't want to get lost in all the set theoretic layers I will represent the set of ultrafilters of $\mathbb{B}$ from now on as $\mathbb{N} \cup \{\infty\}$; thus $\mathbb{B}^\delta \subseteq \wp(\mathbb{N} \cup \{\infty\})$.
For $i_1 : \mathbb{B} \hookrightarrow \mathbb{B}^\delta$ we have $T \longmapsto \{\mu \in Ult(\mathbb{B}) \mid T \in \mu\}$. Using the new representation we get

$$S \overset{i_1}{\longmapsto} S \qquad \text{if } S \text{ is finite ;}$$
$$S \overset{i_1}{\longmapsto} S \cup \{\infty\} \qquad \text{if } S \text{ is cofinite.}$$

In the same way we can take a look at the canonical extension of $\mathbb{B}^\delta$:

$$\mathbb{B}^{\delta^\delta} := (\mathbb{B}^\delta)^\delta = \wp(Ult(\wp(\mathbb{N} \cup \{\infty\})))$$

For $F \in Ult(\wp(\mathbb{N} \cup \{\infty\}))$ we have either $F = \uparrow S$ for some $S \subseteq \mathbb{N} \cup \{\infty\}$, in which case $S = \{x\}$ with $x \in \mathbb{N} \cup \{\infty\}$, or F is not principal. Let $\mathcal{P}$ be the set of non-principal filters in $Ult(\wp(\mathbb{N} \cup \{\infty\}))$.

The embedding $i_2 : \mathbb{B}^\delta \hookrightarrow \mathbb{B}^{\delta^\delta}$ is now $T \longmapsto \{\mu \in Ult(\mathbb{B}^\delta) \mid T \in \mu\}$. We should also try to understand how this embedding works, at least on the image of i_1 in $\mathbb{B}^\delta$.
Let $S = \{s_1, \ldots, s_n\}$ be a finite subset of $\mathbb{N}$ with $S \in \mu$ for some ultrafilter. Since $\{s_1, \ldots, s_n\} = \bigvee_{1 \leq i \leq n} \{s_i\} \in \mu$ and μ is an ultrafilter, there is an $s_i \in \mathbb{N} \cup \{\infty\}$ with $\{s_i\} \in \mu$. It follows that the filter μ is the principal upset $\uparrow\{s_i\}$.
If we define μ_n to be $\{T \subseteq \mathbb{N} \cup \{\infty\} \mid n \in T\} = \uparrow\{n\}$ we have $i_2(S) = \{\mu_n \mid n \in S\}$ if S is finite.

If S is a cofinite subset of $\mathbb{N} \cup \infty$ then it is clear that S is in the upset of $\{n\}$ for every $n \in S$. So $\{\mu_n \mid n \in S\}$ is a subset of $i_2(S)$. However, we are also interested in which part of $\mathcal{P}$ sits in $i_2(S)$.

It is known that for any ultrafilter $\mu \in \mathcal{P}$ and for any finite subset $S \subset \mathbb{N} \cup \{\infty\}$ we have $S \notin \mu$. But because of the ultrafilter properties of μ we then know that then $S^C \in \mu$. It follows that for any cofinite subset of $\mathbb{N} \cup \infty$ and for any ultrafilter $\mu \in \mathcal{P}$ we have $T \in \mu$. From this we can conclude that whole $\mathcal{P}$ is a subset *of $i_2(S)$ if S is a cofinite set.*

If we combine this with the statement above we get:

$$S \overset{i_2}{\longmapsto} \{\mu_n \mid n \in S\} \qquad \textit{if } S \textit{ is finite} ;$$

$$S \overset{i_2}{\longmapsto} \{\mu_n \mid n \in S\} \cup \mathcal{P} \qquad \textit{if } S \textit{ is cofinite.}$$

If we now take a look at the composition of i_1 and i_2 we notice that not so much happens. If $S \in \mathbb{B}$ is finite we have

$$(i_2 \circ i_1)(S) = i_2(S) = \{\mu_n \mid n \in S\},$$

and if $S \in B$ is cofinite we get

$$(i_2 \circ i_1)(S) = i_2(S \cup \{\infty\}) = \{\mu_n \mid n \in S\} \cup \{\mu_\infty\} \cup \mathcal{P}.$$

In other words, we showed that the image of $i_2 \circ i_1$ in $\mathbb{B}^{\delta^\delta}$ is isomorphic to $\mathbb{B}^\delta$.

Actually, something more general holds: the image of the composition of the two embeddings of a distributive lattice $\mathbb{L}$ into $\mathbb{L}^\delta$ and further into $\mathbb{L}^{\delta^\delta}$ generates a distributive lattice that is isomorphic to $\mathbb{L}^\delta$.

Lemma 3.8 *Let $\mathbb{L}$ be a distributive lattice, i_1 the embedding of $\mathbb{L}$ in $\mathbb{L}^\delta$ and i_2 the embedding of $\mathbb{L}^\delta$ in $\mathbb{L}^{\delta^\delta}$. The complete distributive lattice generated by $Im(i_2 \circ i_1)$ is isomorphic to $\mathbb{L}^\delta$.*

Proof: We know that $(i_1, \mathbb{L}^\delta)$ is the up to isomorphism unique completion of $\mathbb{L}$ such that i_1 is compact, and $Im(i_1)$ is dense in $\mathbb{L}^\delta$. So to prove this lemma above I only have to show that $i_2 \circ i_1$ is compact and $Im(i_2 \circ i_1)$ lies dense in $\mathbb{L}^* := \langle Im(i_2 \circ i_1) \rangle$.

Denseness
First note that $\mathbb{L}^*$ is a subalgebra of $\mathbb{L}^{\delta^\delta}$ and therefore $\mathbb{L}^*$ is completely distributive. Now, let b be an element of $\mathbb{L}^*$, in other words b is made of meets and joins of elements in $Im(i_2 \circ i_1)$. Because of complete distributivity we can write b in the join normal form or in the meet normal form,

$$b = \bigvee_{i \in I} \bigwedge_{j \in J_i} A_{ij} \quad \textit{and} \quad b = \bigwedge_{i \in I'} \bigvee_{j \in J'_{i'}} B_{ij},$$

with $A_{ij}, B_{ij} \in Im(i_2 \circ i_1)$. So clearly $Im(i_2 \circ i_1)$ also lies $\bigvee\bigwedge-$ and $\bigwedge\bigvee$-dense in $\mathbb{L}^*$.

Compactness
Let $S, T \subseteq L$ with $\bigwedge_{L^*} i_2(i_1(S)) \leq \bigvee_{L^*} i_2(i_1(T))$. Since $S_1 := i_1(S)$ and

$T_1 := i_1(T)$ are subsets of $\mathbb{L}^\delta$ and i_2 is compact, there are finite subsets $S'_1 \subseteq i_1(S)$ and $T'_1 \subseteq i_1(T)$ with:

$$\bigwedge i_2(S'_1) \leq \bigvee i_2(T'_1)$$

We now know that $S'_1, T'_1 \subseteq Im(i_1)$. Also i_1 is an injective function, therefore

$$S' = \{s \in S \mid i_1(s) \in S'_1\} \text{ and}$$

$$T' = \{t \in T \mid i_1(t) \in T'_1\}$$

are finite subsets of L with $i_1(S') = S'_1$ and $i_1(T') = T'_1$. It follows that

$$\bigwedge (i_2 \circ i_1)(S') \leq \bigvee (i_2 \circ i_1)(T'),$$

as required. We can conclude that the complete distributive lattice $\mathbb{L}^*$ generated by $Im(i_2 \circ i_1)$ is isomorphic to $\mathbb{L}^\delta$. □

The proof of above result can be easily extended to a proof for the next theorem.

Theorem 3.9 *Let $\mathbb{L}$ be a distributive lattice and $n \in \mathbb{N}, n \geq 2$. Also let*

i_1 be the embedding of $\mathbb{L}$ in $\mathbb{L}^\delta$

and for $j \in \{1, \cdots, n\}$: i_j the embedding of $\mathbb{L}^{\overbrace{\delta^{\cdots^\delta}}^{j}}$ in $\mathbb{L}^{\overbrace{\delta^{\cdots^\delta}}^{j+1}}$.

Then the complete distributive lattice generated by $Im(i_n \circ \ldots \circ i_1)$ is isomorphic to $\mathbb{L}^\delta$.

Proof: I will prove this theorem using induction.
In lemma 3.8 I already proved $\langle Im(i_2 \circ i_1)\rangle \cong \mathbb{L}^\delta$.
Let n be an arbitrary element of $\mathbb{N}$ and let us have the functions

$$\mathbb{L} \xhookrightarrow{i_1} \mathbb{L}^\delta \xhookrightarrow{i_2} \mathbb{L}^{\delta^\delta} \quad \cdots \quad \mathbb{L}^{\overbrace{\delta^{\cdots^\delta}}^{n}} \xhookrightarrow{i_{n+1}} \mathbb{L}^{\overbrace{\delta^{\cdots^\delta}}^{n+1}}.$$

and assume we know that $\langle Im(i_n \circ \ldots \circ i_1)\rangle \cong \mathbb{L}^\delta$. If we define j to be $i_n \circ \cdots \circ i_1$ we have $Im(i_{n+1} \circ i_n \circ \cdots \circ i_1) = Im(i_{n+1} \circ j)$. Since we are only interested in the images of the functions, we might as well consider the following diagram.

$$\mathbb{L} \xhookrightarrow{j} \langle Im(j)\rangle \cong \mathbb{L}^\delta \xhookrightarrow{i_{n+1}|\langle Im(j)\rangle} \langle Im(i_{n+1} \circ j)\rangle$$

To conclude that $\langle Im(i_{n+1} \circ j)\rangle \cong \mathbb{L}^\delta$ we have to proof that

$i_{n+1}|\langle Im(j)\rangle \circ j$ is compact and

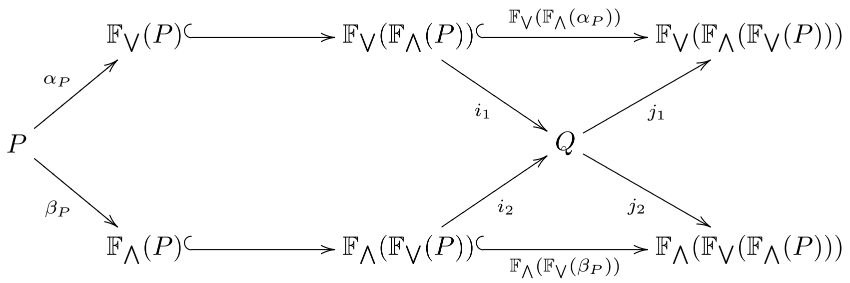
$Im(i_{n+1}|\langle Im(j)\rangle \circ j)$ lies both $\bigwedge\bigvee$- and $\bigvee\bigwedge$-dense in $\langle Im(i_{n+1} \circ j)\rangle$.

Using the arguments used in lemma 3.8 it is clear to see that this is in fact the case. □

3.4 The Δ_2 object

After finding the two *Delta*-structures, it is an interesting question whether it is possible to construct another two-sided completion, Δ_2, in the third layer. To investigate this question we will take a look at the example used in section 3.2.

Starting with an arbitrary poset P, we hope that there are intermediate structures Q that embed in $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P)))$ and $\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(P)))$ and in which $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(P))$ and $\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P))$ embed. These embeddings have to make the following diagram commute.



In other words, we would want

$j_1 \circ i_2 = \alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(P))}$ and

$j_2 \circ i_1 = \beta_{\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(P))}$.

Further we want

$j_1 \circ i_1 = \mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_P))$ and

$j_2 \circ i_2 = \mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(\beta_P))$.

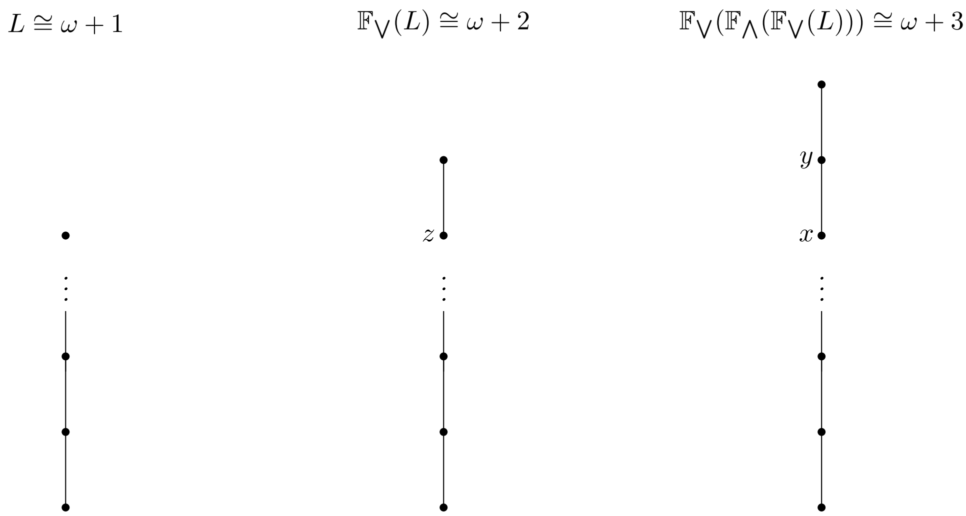
However, after examining the following example we will find out that these intermediate structures don't always exist.

Example 3.10 *Let L be the lattice $\omega + 1$. In section 3.2 we already saw that*

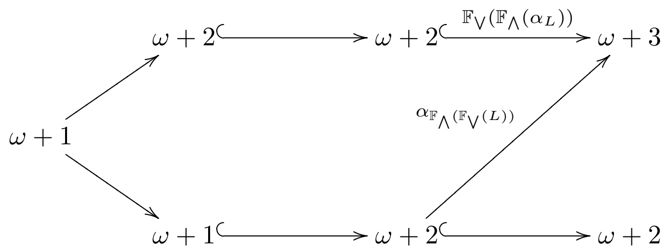
$\mathbb{F}_\bigwedge(L) \cong L$ *and*

$\mathbb{F}_\bigvee(L) \cong \mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L)) \cong \mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(L)) \cong \mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(L)))$.

In fact these last structures are all isomorph to $\omega + 2$. As discussed in example 3.6 we know that $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))) \cong \omega + 3$.



Let's take another look at our diagram, with the ordinals filled in for the structures.



The smallest interpolating structure that would possibly do the job is

$$\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L)) \cup \mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L)),$$

but since this isn't really a set we have to look at

$Im(\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(\alpha_L))) \cup Im(\alpha(\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))))$ *and*

$Im(\beta(\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L)))) \cup Im(\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(\beta_L)))$.

The embedding of $\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))$ and $\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L))$ into $\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L)))$ is obvious since these structures are isomorphic. More interesting is to see what the images of $\alpha_{\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))}$ and $\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(\alpha_L))$ are. These two functions are order-embeddings, so there is no discussing that the "finite" elements will be sent to "themselves" and the top will be sent to the top. In other words:

$$\uparrow\downarrow\{n\} \xmapsto{\alpha_{\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))}} \downarrow\uparrow\downarrow\{n\}, \quad \uparrow\downarrow\{\top\} = \uparrow\{L\} \xmapsto{\alpha_{\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))}} \downarrow\uparrow\downarrow\{\top\} = \mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))$$

and

$$\downarrow\uparrow\{n\} \xmapsto{\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))} \uparrow\downarrow\uparrow\{n\}, \quad \downarrow\uparrow\{\top\} = \mathbb{F}_\bigwedge(L) \xmapsto{\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))} \uparrow\downarrow\uparrow\{\top\} = \uparrow\mathbb{F}_\bigwedge(L).$$

The only interesting question is what happens with the element z.

Claim 3.11 *Let z be in $\mathbb{F}_\bigvee(P)$ and x, y in $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L)))$ as in the diagram above.*

1. $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = y$
2. $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))(z) = x$

Proof:

1. *Since $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}$ is an injection we know that it is not possible to have $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = \downarrow\uparrow\downarrow\{n\}$ for any $n \in \mathbb{N}$ or to have $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = \downarrow\uparrow\downarrow\{\top\}$. The only two remaining options are $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = x$ or $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = y$.*

   If we assume $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = x$ then clearly

   $$\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = x = \bigvee_{k\in\omega} \alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(k),$$

   so in particular

   $$\bigvee_{k\in\omega} \alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(k) \geq \alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z).$$

   By proposition 2.1 we know that $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}$ is compact, so from the assumption we have a $k \in \omega$ with $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(k) \geq \alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z)$. This contradicts the fact that $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}$ is an order-preserving map. We can conclude $\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}(z) = y$.

2. *By the definition of $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))$ we have*

   $$\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))(z) = \bigvee\{\mathbb{F}_\bigwedge(\alpha_L)(q) \mid q \leq z\}.$$

   Also $\bigvee\{\mathbb{F}_\bigwedge(\alpha_L)(q) \mid q \leq z\} = x$, so it follows immediately that $\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))(z) = x$.

□

Now that we proved this claim, let's compare these statements

$$z \xmapsto{\mathbb{F}_\bigvee(\mathbb{F}_\bigwedge(\alpha_L))} y$$

$$z \xmapsto{\alpha_{\mathbb{F}_\bigwedge(\mathbb{F}_\bigvee(L))}} x$$

and

$$z \xmapsto[\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(\beta_L))]{\beta_{\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L))}} z$$

Note that these embeddings don't match, the order on the images are not the same. Also we remark that

$$Im(\alpha_{\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))}) \cup Im(\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(\alpha_L))) = \mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))) \cong \omega + 3 \text{ and}$$

$$Im(\beta_{\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L))}) \cup Im(\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(\beta_L))) = \mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L))) \cong \omega + 2.$$

So, in contraposition to the Δ_1 object, there is no interpolating structure in which $\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L))$ and $\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L))$ embed which also embeds in $\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(L)))$ and $\mathbb{F}_{\bigwedge}(\mathbb{F}_{\bigvee}(\mathbb{F}_{\bigwedge}(L)))$ such that the order is preserved. With this conclusion all hope of finding a Δ_2 object in the hierarchy of this lattice is lost, since we chose our example to be the smallest that might do the job.

We are forced to conclude that there is no general way to define a Δ_2 object. Further we can use this example to see that this is also the case on any other level higher than two. In other words, Δ_0 and Δ_1 are the only largest interpolating objects, since there is no way to make a global definition for a Δ_n structure for any $n \geq 2$.

Epilogue Intuition is a tricky thing

The expedition has ended, so it is time to look back at the journey and see what all the discoveries really mean.
When I started this project my intuition and that of many others, was that this hierarchy has meaning as a hierarchy of completions of the original lattice. We expected that the lattices grow with every step and that one original lattice will generate each lattice in the hierarchy typically. Indeed, in the Boolean case, the hierarchy grows significantly with every step, as we saw in section 3.1. While every step is generated by the level before and the first step is generated by the original lattice, in section 3.2 it turned out that not every step is generated by the original lattice. This was counterintuitive to me.
In section 2.2 I described two two-sided completions of a lattice, the MacNeille completion and the canonical extension. These constructions are well known and much used for decades. Only in 2008 did Priestley and Gehrke put them in a new light, by pointing out their role in the hierarchy. Again, intuition would guide us to the idea that these kind of constructions arise in every step of the hierarchy and again we find our intuition wrong. The example in section 3.4 shows that on the third level, namely Δ_2, it does not even make sense to think about another intermediate structure, let alone a two-sided completion. In other words, the MacNeille completion and the canonical extension are the only natural two-sided completions arising in the hierarchy. This makes them even more canonical. In section 3.3 we saw that this is even more true, because taking the canonical extension of a canonical extension leads to an exact copy of the first one, at least in the distributive case.
Reading about all these idea's and seeing them being shut down can give the impression that this hierarchy of completions is not very interesting mathematically. Of course this is not true. We explored it from only one point of view, but there can be many other ways to fit the hierarchy into interesting mathematics. One idea for further research could be to find the link between the theorem of Solovay and our hierarchy. Solovay proved that there exists a countably generated complete Boolean algebra of every cardinality. Could this hierarchy be some kind of handle to build towards this massive construction, step by step? Also, our counterexample used in section 3.4 is a lattice. What if we look at Boolean algebras. Could there still happen something interesting?
There is no time to sit around and reading boring theses. Let's pack our pencils, pads and brains and start another expedition!

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