Partial Functions in Data Manipulation

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Abstract

In this paper we take a look at partial functions and their applications in computer science. The axiomatization from [1] has been introduced to formally derive equality between partial functions. We work with algebras satisfying this axiomatization and a finite generation assumption which is satisfied by all finite algebras of the axiomatic class specified in [1].

First we introduce the algebra of partial functions and give some useful properties. We distinguish most basic functions, which are the building blocks of partial function algebras. These functions cannot be decomposed in smaller elements, so are in fact atoms.

After that, we take a look at a more general family of algebras, namely the subalgebras of partial function algebras. In particular, we will show that these subalgebras can easily be described up to isomorphism, by placing a restriction on the images of partial functions.

We will also note that the most basic functions can be used to construct a coordinate system, as every partial function can be composed of most basic functions.

Finally we consider algebras that satisfy the axioms proposed in [1]. These algebras also contain basic elements, similar to the most basic functions of partial function algebras. In the formal system, elements also can be decomposed into most basic elements. A special ordering will give a basis for structural induction. We will use this scheme of structural induction to show that elements with a similar decomposition are equal to each other.

The main result of this thesis (Theorem 4.5.2) is a theorem which states that every algebra which satisfies the axioms given in [1] and is in a way finite[1] may be embedded as a subalgebra of a full partial function algebra.

[1]We will define more clearly what we mean by this in Chapter 4.

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Chapter 1

Partial Functions in Practice

Partial functions play an important role in Computing Science applications since they can handle tagged information sets in a very natural way. For example, when creating a database, partial functions can assign values to attributes. Or in a web-application, partial functions can be used to efficiently send data between the client and the server. We will take a closer look at the latter.

1.1 Partial Functions and XML

Today, many services on the internet want to offer their clients some form of interactivity. Famous examples include many services offered by Google. To realise this, the client has to communicate with the server non-stop, so to reduce traffic, it is essential to keep communication efficient and clean. Many people choose to use the XML-format for this.

XML, which is an abbreviation for Extended Markup Language, is a language that has been designed to be readable by humans, but especially by computers.

We can describe XML by using partial functions. An XML-file is a collection of attributes with values assigned to them. We can also assign these values to attributes by using partial functions. Take a look at the following piece of XML:

```
<book>
    <ISBN>978-0-387-25790-7</ISBN>
    <author>Volker Runde</author>
    <title>A Taste of Topology</title>
</book>
```

```
<book>
    <ISBN>0-14-012499-3</ISBN>
    <author>G. Polya</author>
    <title>How To Solve It</title>
</book>

<book>
    <ISBN>0-13-198199-4</ISBN>
    <author>Wade Trappe & Lawrence Washington</author>
    <title>Introduction to Cryptography
           with Coding Theory</title>
</book>
```

This block of code actually describes a set of partial functions (of type "book"), $\{t_1, t_2, t_3\}$, such that for example

$$t_1 : \text{“author”} \mapsto \text{“Volker Runde”}$$

and

$$t_2 : \text{“ISBN”} \mapsto \text{“0-14-012499-3”}.$$

A server receives an XML-file to be processed, manipulates this file and delivers the resulting XML-file back to the client. For example, after receiving the XML-file above, the server will produce the following XML-file:

```
<book>
    <ISBN>978-0-387-25790-7</ISBN>
    <author>Volker Runde</author>
    <title>A Taste of Topology</title>
    <price>3995</price>
</book>

<book>
    <ISBN>0-14-012499-3</ISBN>
    <author>G. Polya</author>
    <title>How To Solve It</title>
</book>

<book>
    <ISBN>0-13-198199-4</ISBN>
    <author>Wade Trappe & Lawrence Washington</author>
    <title>Introduction to Cryptograph
           with Coding Theory</title>
    <price>6495</price>
</book>
```

The server has added a price for each book. Do note however, that in this particular example, the server did not find a price for the book “How To Solve It”. This result can be described as a set of partial functions, $\{t'_1, t'_2, t'_3\}$. So in a way, the server has transformed the incoming partial functions to the outgoing partial functions.

The server did not find the book “How To Solve It”, which was of course because the author was not “G. Polya”, but “George Polya”, so the client needs to fix his error. Instead of creating a new file, and sending this, the client just has to send the updated information:

```
<book>
    <ISBN>0-14-012499-3</ISBN>
    <author>George Polya</author>
</book>
```

The client only includes the information that needs to be updated, and the ISBN to be able to identify the right book. The information that was sent by the client can also be seen as a partial function t^*, such that

$$t^* : \text{“ISBN”} \mapsto \text{“0-14-012499-3”}$$

and

$$t^* : \text{“author”} \mapsto \text{“George Pólya”}.$$

The server will now search for the partial function t_i, such that t_i : “ISBN” $\mapsto$ “0-14-012499-3”, and will then update t_i with t^*.

Imagine we have an online book-store. When someone orders a book, they send information to the server, the ISBN of the book they want to order, shipping data, billing data, etcetera. For example, they might send the following XML-file to the server:

```
<order>
    <book>
        <ISBN>...</ISBN>
    </book>
    <name>...</name>
    <address>...</address>
    ...
</order>
```

The server will receive this file, has to identify the correct book, and then send the order to Shipping.

1.2 The Axioms

So we have seen how we are able to describe the communication between client and server by using partial functions and a few operators. This requires a powerful underlying partial function algebra. This algebra can also be described by a finite set of axioms, enabling to reason about the equality of partial functions, based on these axioms. In this paper we want to find out if the following set of axioms describes the algebra we need.

$$\begin{aligned} x \rhd x &= x \\ x - x &= \emptyset \\ x \rhd y &= (y - x) \rhd x \\ x - (y - z) &= (x - y) \rhd \big(x - (x - z)\big) \\ (x \rhd y) - z &= (x - z) \rhd (y - z) \end{aligned}$$

The layout of this paper is as follows: In chapter 2 we formally introduce the partial function algebra (for flat partial functions). Then in chapter 3 we are going to look at what the subalgebras of partial function algebras are; this is needed to be able to embed the algebra induced by our axioms in the partial function algebras. In chapter 4 we are going to try to embed the algebra induced by our axioms in the partial function algebras.

Chapter 2

Partial Function Algebras

So far we have seen a couple of examples of operators on partial functions. Let f and g be partial functions. One of these examples was the "update"-operator. $f\langle g\rangle$ is the partial function that updates f with g. The next useful function we encountered was "override". $f \rhd g$ is the partial function that is obtained by overriding g with f. Finally, there is a "minus"-operator (which we have not encountered yet). $f - g$ is the partial function that is obtained by restricting f to the domain $\mathrm{dom}(f) \setminus \mathrm{dom}(g)$. Finally, we have a special function, $\emptyset$, which has empty domain.

The binary operations override and minus, together with the nullary operation $\emptyset$ behave very much like addition, subtraction, and the neutral element under addition. These operations would provide a natural structure on partial functions. We will give a more precise definition of these operations.

In this chapter, we will first introduce some basic definitions, which are needed to be able to talk about partial functions. After we introduced these definitions, we can talk about override, update, and minus, and we will be able to give concise definitions of these operators. After this has been done, we can define partial function algebras. Once we know what partial function algebras are, we will look at a special type of partial functions, called "most basic functions".

2.1 Definitions

Definition 2.1.1. Let X and Y be sets. Let $f \subseteq X \times Y$. If, for every $x \in X$, there is at most one $y \in Y$ such that $(x, y) \in f$, then we call f a partial function. Notation: Let $X \to Y$ be the set of all partial functions with domain X and codomain Y.

Let $x \in X$. We say f is defined on x iff there exists a $y \in Y$ such that $(x, y) \in f$.

Notation: $f \downarrow x$. If no such y exists, we say f is not defined on x. Notation: $f \not\downarrow x$.

We say $f(x) = y$ iff $(x, y) \in f$. If $f \not\downarrow x$, sometimes we will write $f(x) = \lambda$.

Let $\mathrm{dom}(f) = \{x \in X \mid f \downarrow x\}$. We call $\mathrm{dom}(f)$ the domain of f.

Let D be a set.

$$f \upharpoonright D \stackrel{\texttt{def}}{=} \{(x, y) \in f \mid x \in D\}$$

We call $f \upharpoonright D$ "f restricted to D".

Now, we are going to formally define our operators, "minus", "override", and $\emptyset$. We will also define "update". In the following definitions, let X and Y be sets, and let $f, g \in X \to Y$.

Definition 2.1.2. The "minus"-operator $(-)$ is defined as follows:

$$f - g \stackrel{\texttt{def}}{=} f \setminus \{(x, y) \in f \mid g \downarrow x\}$$

Another way to write this is:

$$f - g = \{(x, y) \in f \mid g \not\downarrow x\}$$

This definition is equivalent with:

$$f - g = f \upharpoonright (\mathrm{dom}(f) \setminus \mathrm{dom}(g))$$

Definition 2.1.3. The "override"-operator $(\rhd)$ is defined as follows:

$$f \rhd g \stackrel{\texttt{def}}{=} f \cup \{(x, y) \in g \mid f \not\downarrow x\}$$

This definition is equivalent with:

$$f \rhd g = f \cup g \upharpoonright (\mathrm{dom}(g) \setminus \mathrm{dom}(f))$$

Definition 2.1.4. We will call the set $\emptyset$ "the empty function".

Definition 2.1.5. The "update"-operator $(\cdot\langle\cdot\rangle)$ is defined as follows:

$$f\langle g\rangle \stackrel{\texttt{def}}{=} \{(x, y) \in f \mid g \not\downarrow x\} \cup \{(x, y) \in g \mid f \downarrow x\}$$

Definition 2.1.6. The "intersection"-operator $(@)$ is defined as follows:

$$f@g \stackrel{\texttt{def}}{=} f - (f - g)$$

Definition 2.1.7. The algebra $\mathcal{F} = (X \to Y, \rhd, -, \emptyset)$ of type $(2, 2, 0)$ is called a "full partial function algebra".

2.2 Some Properties

In this section we will look at some properties of partial functions. The first property we will take a look at is "idempotency".

Proposition 2.2.1. $f \rhd f = f$

Proof. $f \rhd f = f \cup \{(x, y) \in f \mid f \not\downarrow x\} = f \cup \emptyset = f$ □

Now, let us take a look at $f - f$.

Proposition 2.2.2. $f - f = \emptyset$

Proof. $f - f = f \setminus \{(x, y) \in f \mid f\downarrow x\} = f \setminus f = \emptyset$ □

The $\rhd$-operator has a few interesting properties.

Lemma 2.2.3. $f \rhd g = f \cup (g - f)\big(= f \rhd (g - f)\big)$

Proof. $f \rhd g = f \cup g \restriction (\mathrm{dom}(g) \setminus \mathrm{dom}(f)) = f \cup (g - f)$. □

Lemma 2.2.4. If the domains of f and g are disjoint, then $f \rhd g = f \cup g$.

Proof. Assume $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$. Then

$$f \rhd g = f \cup g \restriction (\mathrm{dom}(g) \setminus \mathrm{dom}(f)) = f \cup g.$$

□

One property we might be interested in, is if the $\rhd$-operator is commutative. Unfortunately though, it is not. Take a look at the following example. Let $X = \{0, 1\}$, $Y = \{0, 1\}$, $f = \{(0, 0), (1, 1)\}$, and $g = \{(0, 1), (1, 0)\}$. Then $f \rhd g = f$, but $g \rhd f = g$. So $f \rhd g \neq g \rhd f$. We can however prove a weaker property, namely that $f \rhd g = (g - f) \rhd f$.

Proposition 2.2.5. $f \rhd g = (g - f) \rhd f$

Proof.

$$\begin{aligned}(g - f) \rhd f &= (g - f) \cup f \\ &= f \cup (g - f) \\ &\overset{2.2.3}{=} f \rhd g\end{aligned}$$

□

Let us take a look at $f - (g - h)$ now. We claim that $f - (g - h) = (f - g) \rhd (f - (f - h))$.

Proposition 2.2.6. $f - (g - h) = (f - g) \rhd \big(f - (f - h)\big)$

Proof. For the sake of brevity, let $F = \mathrm{dom}(f)$, $G = \mathrm{dom}(g)$ and $H = \mathrm{dom}(h)$.

$$\begin{aligned}(f - g) \rhd \big(f - (f - h)\big) &= f \restriction (F - G) \rhd f \restriction (F - (F - H)) \\ &= f \restriction (F - G) \rhd f \restriction (F \cap H) \\ &= f \restriction (F - G) \cup f \restriction ((F \cap H) - (F - G)) \\ &= f \restriction ((F - G) \cup ((F \cap H) - (F - G)))\end{aligned}$$

Note that for sets A and B, $A \cup (B - A) = A \cup B$.

$$= f \restriction ((F - G) \cup (F \cap H))$$

Claim: for sets A, B and C, $(A - B) \cup (A \cap C) = A - (B - C)$. Proof will follow later.

$$\begin{aligned}&= f \restriction (F - (G - H)) \\ &= f \restriction (F - \mathrm{dom}(g - h)) \\ &= f - (g - h)\end{aligned}$$

□

Lemma 2.2.7. Let A, B and C be sets. Then $(A - B) \cup (A \cap C) = A - (B - C)$.

Proof.

$$\begin{aligned}x \in (A - B) \cup (A \cap C) &\Leftrightarrow x \in (A - B) \lor x \in (A \cap C) \\ &\Leftrightarrow (x \in A \land x \notin B) \lor (x \in A \land x \in C) \\ &\Leftrightarrow x \in A \land (x \notin B \lor x \in C) \\ &\Leftrightarrow x \in A \land \neg(x \in B \land x \notin C) \\ &\Leftrightarrow x \in A \land \neg x \in (B - C) \\ &\Leftrightarrow x \in A - (B - C)\end{aligned}$$

□

Lastly, we will prove that the minus-operator is right-distributive over the $\rhd$-operator, or $(f \rhd g) - h = (f - h) \rhd (g - h)$.

Proposition 2.2.8. $(f \rhd g) - h = (f - h) \rhd (g - h)$

Proof.

$$\begin{aligned}(f \rhd g) - h &= \big(f \cup (g - f)\big) - h \\ &= (f - h) \cup \big((g - f) - h\big) \\ &= (f - h) \cup \big((g - h) - (f - h)\big) \\ &= (f - h) \rhd (g - h)\end{aligned}$$

□

2.3 Most Basic Functions

When working with certain groups, we can consider the basic elements of that group. For example, when we consider $(\mathbb{Z}^*, \times, 1)$ (the multiplicative group of the integers), we have a notion of *prime numbers*. Not considering the element 1, we can decompose every element from $\mathbb{Z}^*$ into prime numbers.

Or, if one would look at vector spaces, one would consider the basis of the vector space. Then every vector in the vector space can be produced as a linear combination of elements from the basis.

In our partial function algebra, we also have a notion of "most basic functions". These are the functions that are defined on exactly *one* element. Let f and g be $\mathfrak{mbf}$'s. Then $f - g$ either is f or $\emptyset$.

Definition 2.3.1. Let $f \in X \to Y$, such that $f \neq \emptyset$. We define f to be a "most basic function", denoted as $P_{\mathfrak{mbf}}(f)$, iff for each $g \in X \to Y$, $f - g = \emptyset$ or $f - g = f$. With other words:

$$P_{\mathfrak{mbf}}(f) \stackrel{\text{def}}{=} \forall_{g \in X \to Y} \left[(f - g = \emptyset) \vee (f - g = f) \right]$$

To show that this definition matches our idea of a $\mathfrak{mbf}$, we will prove two lemmas. First we will show that partial functions that are defined on only one point are in fact $\mathfrak{mbf}$'s, then we will show that if a function is a $\mathfrak{mbf}$, then its domain can contain only one point.

Lemma 2.3.2. Let $f \in X \to Y$, such that $f \neq \emptyset$, and

$$\forall_{x_1, x_2 \in X} \left[(f \downarrow x_1 \wedge f \downarrow x_2) \Rightarrow (x_1 = x_2) \right],$$

then f is a $\mathfrak{mbf}$.

This lemma states that if the domain of a partial function contains exactly one element, it must be a $\mathfrak{mbf}$.

Proof. We need to prove that for arbitrary $g \in X \to Y$, $f - g$ either is f or it is $\emptyset$. Let $x \in X$ be such that $\text{dom}(f) = x$. Let $g \in X \to Y$ be arbitrary. Then either $g \downarrow x$, in which case $f - g = f \upharpoonright (\text{dom}(f) \setminus \text{dom}(g)) = f \upharpoonright \emptyset = \emptyset$, or $g \not\downarrow x$, in which case $\text{dom}(f) \cap \text{dom}(g) = \emptyset$, so $f - g = f \upharpoonright (\text{dom}(f) \setminus \text{dom}(g)) = f \upharpoonright \text{dom}(f) = f$. □

Lemma 2.3.3. Let $f \in X \to Y$, such that f is a $\mathfrak{mbf}$. Then the following statement holds:

$$\forall_{x_1, x_2 \in X} \left[(f \downarrow x_1 \wedge f \downarrow x_2) \Rightarrow (x_1 = x_2) \right]$$

This lemma states that if a partial function is a $\mathfrak{mbf}$, then its domain contains at most one element. Since, by definition, the domain of a $\mathfrak{mbf}$ contains at least one element, the domain of a $\mathfrak{mbf}$ has exactly one element.

Proof. Let $x_1, x_2 \in X$ be arbitrary. Assume $f\downarrow x_1$ and $f\downarrow x_2$. Let $g \in X \to Y$, such that $g\downarrow x_1$. Then, by definition of "minus", $(f-g)\not\downarrow x_1$. So $f-g \neq f$. By definition of $\mathfrak{mbf}$, $f-g$ must be equal to $\emptyset$. So $g\downarrow x_2$. However, g was arbitrary, therefore $x_1 = x_2$. $\square$

For future reference, we will summarize the results of these two lemmas in one corollary.

Corollary 2.3.4. The $\mathfrak{mbf}$'s of a full partial function algebra $\mathcal{F}$ are exactly the partial functions whose domains consist of one single element.

We now have a fair idea of what a most basic function is. We will use these $\mathfrak{mbf}$'s to describe other partial functions. Any partial function can be written as a combination of $\mathfrak{mbf}$'s. Let $f \in X \to Y$. Then for every $x \in X$ such that $f\downarrow x$, we can create a partial function $f_x = \{(x, f(x))\}$, which is a $\mathfrak{mbf}$. We can now write f as the join of all these $\mathfrak{mbf}$'s.

Before we take a closer look at this, we will first introduce one more definition.

Definition 2.3.5. Let f and g be partial functions such that $f \neq g$. We define f and g to be "disjoint" iff $f \neq g$ and $f \rhd g = g \rhd f$.

As a corollary of 2.3.4 we can show that $\mathfrak{mbf}$'s are disjoint exactly when their domains are disjoint.

Corollary 2.3.6. Let f and g be $\mathfrak{mbf}$'s such that $f \neq g$. Then f and g are disjoint exactly when $f - g \neq \emptyset \neq g - f$.

Proof. By 2.3.4 it will suffice to prove $f - g \neq \emptyset$. Also by 2.3.4 we know that $\text{dom}(f) = \{x_1\}$ and $\text{dom}(g) = \{x_2\}$ for certain x_1, x_2. Also, let $y_1, y_2 \in Y$ be so that $f(x_1) = y_1$ and $g(x_2) = y_2$.

If $f - g = \emptyset$, then necessarily $x_1 = x_2$. By assumption $f \neq g$, so necessarily $y_1 \neq y_2$. Now by definition of override, $\big(f \rhd g\big)(x_1) = y_1$ and $\big(g \rhd f\big)(x_2) = y_2$. So $f \rhd g \neq g \rhd f$. So f and g are not disjoint.

By contraposition, if f and g are disjoint, then $f - g \neq \emptyset$.

If $f - g \neq \emptyset$, then necessarily $x_1 \neq x_2$. Then, by definition of override:

$$\begin{aligned} f \rhd g &= f \cup g \restriction (\text{dom}(g) \setminus \text{dom}(f)) \\ &= f \cup g \restriction \text{dom}(g) \\ &= f \cup g = g \cup f \\ &= g \cup f \restriction \text{dom}(f) \\ &= g \cup f \restriction (\text{dom}(f) \setminus \text{dom}(g)) \\ &= g \rhd f \end{aligned}$$

So f and g are disjoint iff $f - g \neq \emptyset$. $\square$

Theorem 2.3.7. For any partial function f with a finite domain, there exist pairwise disjoint $\mathfrak{mbf}$'s $f_1, f_2, \ldots, f_n$ such that $f = f_1 \rhd \ldots \rhd f_n$. These $\mathfrak{mbf}$'s are also unique up to rearrangement of the list.

Proof. Let f be any partial function. For every x in $\text{dom}(f)$ define a partial function $\mathtt{1}_x$

$$\mathtt{1}_x\,(y) \stackrel{\texttt{def}}{=} \begin{cases} f\,(x) & y = x \\ \lambda & \text{otherwise} \end{cases}.$$

Since these partial functions are defined on exactly one place, they are $\mathfrak{mbf}$'s. Furthermore, they are pairwise disjoint. Therefore,

$$f = \rhd_{x \in (\text{dom}(f))} \mathtt{1}_x.$$

Also, note that these $\mathtt{1}_x$ are uniquely determined (up to rearrangement). □

2.4 Relations on Partial Functions

We will now define some relations on partial functions. The first relation we introduce is an equivalence relation on $\mathfrak{mbf}$'s. We will call two $\mathfrak{mbf}$'s related exactly when they have a common domain. This relation will prove useful later on in this thesis. Formally, we define this relation as follows.

Definition 2.4.1. Let $f, g \in X \to Y$ be $\mathfrak{mbf}$'s. Then we say $f \sim g$ iff $f \rhd g = f$.

To show that this definition is exactly what we want, we have to show two things.

Lemma 2.4.2. Let $f, g \in X \to Y$ be $\mathfrak{mbf}$'s such that $f \sim g$. Then $\text{dom}(f) = \text{dom}(g)$.

Proof. Let $\text{dom}(f) = F$ and $\text{dom}(g) = G$. Then

$$\begin{aligned} \text{dom}(f \rhd g) &= \text{dom}\left(f \cup g \restriction (G \setminus F)\right) \\ &= \text{dom}(f) \cup \text{dom}(g \restriction (G \setminus F)) \\ &= F \cup (G \setminus F) \end{aligned}$$

By definition, $f \rhd g = f$, so $F \cup (G \setminus F) = F$. Therefore $G \setminus F$ has to be a subset of F. So G has to be a subset of F. Since f is a $\mathfrak{mbe}$, this means that either $F = G$, or $G = \emptyset$. Since g also is a $\mathfrak{mbe}$, $F = G$. □

Lemma 2.4.3. Let $f, g \in X \to Y$ be $\mathfrak{mbf}$'s such that $\text{dom}(f) = \text{dom}(g)$. Then $f \sim g$.

Proof. Let $\mathrm{dom}(f) = F$ and $\mathrm{dom}(g) = G$. Then

$$\begin{aligned} f \rhd g &= f \cup g \restriction (G \setminus F) \\ &= f \cup g \restriction \emptyset \\ &= f \end{aligned}$$

□

So the definition matches our intuition. Now we want to show that this relation is in fact an equivalence relation.

Proposition 2.4.4. The relation $\sim$ between $\mathfrak{mbf}$'s is an equivalence relation.

Proof. For this, we have to show three properties.

- $f \sim f$

  This is already proven in proposition 2.2.1.

- *If $f \sim g$, then $g \sim f$.*

  If $f \sim g$, then they have common domains. Which makes $g \sim f$ trivial.

- *If $f \sim g$ and $g \sim h$, then $f \sim h$.*

  If $f \sim g$, then f and g have common domains. That is to say, $\mathrm{dom}(f) = \mathrm{dom}(g)$. Since $g \sim h$, we have $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathrm{dom}(h)$. So $f \sim h$.

□

The second relation we introduce on partial functions is an ordering. If we have partial functions f and g, then we say f is "smaller" than g exactly when $f \subseteq g$. More formally, we can say:

Definition 2.4.5. Let f and g be partial functions. We say $f \leq g$ iff $f \rhd g = g$.

This definition again matches our intiution – the proof of this is similar to the prove given before.

Lemma 2.4.6. Let f and g be partial functions such that $f \rhd g = g$. Then $f \subseteq g$.

Proof. Let f and g be partial functions such that $f \rhd g = g$. By definition of override, $f \rhd g = f \cup g \restriction \mathrm{dom}(g) \setminus \mathrm{dom}(f)$. Therefore, $f \subseteq g$. □

Lemma 2.4.7. Let f and g be partial functions such that $f \subseteq g$. Then $f \rhd g = g$.

Proof. Let f and g be partial functions such that $f \subseteq g$. Then $f \rhd g = f \cup g \restriction \mathrm{dom}(g) \setminus \mathrm{dom}(f) = g$. □

Proposition 2.4.8. The relation $\leq$ between partial functions is an ordering.

Proof. For this, we have to show three properties.

- $f \leq f$
  This is already proven in proposition 2.2.1.
- *If* $f \leq g$ *and* $g \leq f$*, then* $f = g$*.*
  Since $f \leq g$, $f \subseteq g$. Since $g \leq f$, $g \subseteq f$. Therefore, $f = g$.
- *If* $f \leq g$ *and* $g \leq h$*, then* $f \leq h$*.*
  Since $f \leq g$, $f \subseteq g$. Since $g \leq h$, $g \subseteq h$. Therefore $f \subseteq h$ and thus $f \leq h$.

□

Chapter 3

Subalgebras of Partial Function Algebras

In this chapter we will look at a more general family of algebras, namely the subalgebras of partial function algebras. We show that each algebra satisfying the axioms in 1.2 is in fact isomorphic to a subalgebra of a partial function algebra.

We have a hunch that a subalgebra of a partial function algebra, is actually an algebra of restricted partial functions, or a "restricted partial function algebra". Before we go in more detail about these restricted partial function algebras, let us first define general partial function algebras. In the section after that, we will give a characterisation of restricted partial function algebras.

3.1 General Partial Function Algebras

Definition 3.1.1. Let $\mathcal{F}$ be a full partial function algebra. Let G be a subset of $X \to Y$, such that G is closed under $\triangleright$ (override), $-$ (minus), and $\emptyset \in G$. We then call $\mathcal{G} = (G, \triangleright, -, \emptyset)$ a subalgebra of $\mathcal{F}$. We will call these subalgebras "general partial function algebras" or GPFAs.

Note that definition 2.3.1 is still valid for partial function algebras, therefore we can speak of $\mathfrak{mbf}$'s of a GPFA. In our normal PFAs, the $\mathfrak{mbf}$'s were exactly partial functions with singleton domains. However, for GPFAs this is not necessarily true.

Example 3.1.2. Let $X = \{0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$. Let G be the set containing the following elements:

$$\begin{aligned}\emptyset &= \{\}\\ f_1 &= \{(0,0)\}\\ f_2 &= \{(1,1),(2,2)\}\\ f_3 &= \{(1,3),(2,4)\}\\ f_4 &= \{(0,0),(1,1),(2,2)\}\\ f_5 &= \{(0,0),(1,3),(2,4)\}\end{aligned}$$

Clearly G is a subset of $X \to Y$. It is also not hard to verify that G is closed under the operations $-$ and $\triangleright$. So G is a GPFA. However, if we try to find out what the $\mathfrak{mbf}$'s of G are, we find that these are f_1, f_2 and f_3, while the domains of f_2 and f_3 clearly have more than one element.

So we have seen that in a GPFA, $\mathfrak{mbf}$'s can have more than one point in their domain. This makes us wonder if $\mathfrak{mbf}$'s can have overlapping domains. Luckily, this is not the case. From this point on, let G be a GPFA.

Lemma 3.1.3. Let $f, g \in G$ be $\mathfrak{mbf}$'s of G. Then either $f@g = \emptyset$ or $f@g = f$.

Proof. $f@g = f-(f-g)$. Since f is a $\mathfrak{mbf}$ it is trivial to show that $f-(f-g) \in \{\emptyset, f\}$. □

If we have two $\mathfrak{mbf}$'s, then either their domains are identical, or their domains are disjoint. This means that definition 2.3.5 also is still valid. We will now extend theorem 2.3.7 for GPFAs.

Theorem 3.1.4. Let G be a GPFA. For any partial function $f \in G$ with a finite domain, there exist pairwise disjoint $\mathfrak{mbf}$'s $f_1, f_2, \ldots, f_n$ such that $f = f_1 \triangleright \ldots \triangleright f_n$. These $\mathfrak{mbf}$'s are also unique up to rearrangement of the list.

Proof. Let G be a GPFA and let $f \in G$ such that $\mathrm{dom}(f)$ is finite. If f is $\mathfrak{mbf}$, then the theorem holds for f. Now assume the theorem holds for all $h \in G$ such that $h \lneq f$. Since f is not $\mathfrak{mbf}$, there exists a $g \in G$ such that $f-g \neq \emptyset$ and $f-g \neq f$. Let $f_1 = f-g$ and $f_2 = f@g$. Then $f = f_1 \triangleright f_2$ and $f_1 \lneq f$ and $f_2 \lneq f$ while $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2) = \emptyset$. □

Now we can make the following construction.

Algorithm 3.1.5. From the $\mathfrak{mbf}$'s of G we are going to construct a new set of $\mathfrak{mbf}$'s, such that the domain of each $\mathfrak{mbf}$ has exactly one point.

1. Let $H = \{f \in G \mid \forall_g [f-g=\emptyset \vee f-g=f]\}$ (the set of all $\mathfrak{mbf}$'s in G).
2. We start with $H' = \emptyset$, $X' = X$, and $Y' = Y$.

3. For each element f of H:
   (a) Let f_d be the domain of f and $f_i = f(f_d)$ the image of f.
   (b) Append $f' = \{(f_d, f_i)\}$ to H'.
   (c) If f_d isn't already an element of X', then
      - $X' := (X' \setminus f_d) \cup \{f_d\}$; and
      - $Y' := Y' \cup \{f_i\}$.
4. Let $G' = \{f : X' \to Y' \mid \exists_{e_1,\ldots,e_n \in H'} [f = e_1 \rhd \ldots \rhd e_n]\}$. The elements in H' are exactly the $\mathfrak{mbf}$'s of G'.

If we apply this construction on example 3.1.2, we end up with the following values for X', Y' and H'. For the sake of brevity, let $a_i = \{i\}$ for $0 \le i \le 4$, $a_{12} = \{1, 2\}$ and $a_{34} = \{3, 4\}$.

- $X' = \big\{a_0, a_{12}\big\}$
- $Y' = \big\{a_0, a_1, a_2, a_3, a_4, a_{12}, a_{34}\big\}$
- $H' = \Big\{\big\{(a_0, a_0)\big\}, \big\{(a_{12}, a_{12})\big\}, \big\{(a_{12}, a_{34})\big\}\Big\}$

So G' contains the following elements:

$$\begin{aligned}
\emptyset &= \big\{\big\} \\
f'_1 &= \big\{(a_0, a_0)\big\} \\
f'_2 &= \big\{(a_{12}, a_{12})\big\} \\
f'_3 &= \big\{(a_{12}, a_{34})\big\} \\
f'_4 &= \big\{(a_0, a_0), (a_{12}, a_{12})\big\} \\
f'_5 &= \big\{(a_0, a_0), (a_{12}, a_{34})\big\}
\end{aligned}$$

Proposition 3.1.6. The algebra G' as constructed in algorithm 3.1.5 is isomorphic to G.

Proof. Let H' be the set of $\mathfrak{mbf}$'s of G' and H the set of $\mathfrak{mbf}$'s of G. Our algorithm gives a natural one-to-one mapping $h : H \to H'$. We want to extend this mapping h to include every element of G. By theorem 2.3.7 we know that for each element a of G, there exist $\mathfrak{mbf}$'s $a_1, \ldots, a_n$ such that $a = a_1 \rhd \ldots \rhd a_n$. We now define $h(a) \stackrel{\text{def}}{=} h(a_1) \rhd \ldots \rhd h(a_n)$, and $h(\emptyset) \stackrel{\text{def}}{=} \emptyset$. Now h is a mapping $G \to G'$.

First we will show that $h(a - b) = h(a) - h(b)$.

$$\begin{aligned}
h(a - b) &= h(a_1 \rhd \ldots \rhd a_n - b) \\
&\stackrel{2.2.8}{=} h\big((a_1 - b) \rhd \ldots \rhd (a_n - b)\big) \\
&= h(a_1 - b) \rhd \ldots \rhd h(a_n - b)
\end{aligned}$$

Since for each i, a_i is a $\mathfrak{mbf}$, $a_i - b$ either is $\emptyset$ or a_i. Assume $a_i - b = \emptyset$. Also assume $b = b_1 \rhd \ldots \rhd b_k$. Then there exists a j such that a_i and b_j have the same domain. By construction, $h(a_i)$ and $h(b_j)$ then have the same domain, so $h(a_i - b) = \emptyset = h(a_i) - h(b_j) = h(a_i) - h(b)$.

Now assume $a_i - b = a_i$. Again, $b = b_1 \rhd \ldots \rhd b_k$. Then for each j, b_j does not share its domain with a_i. By construction $h(b_j)$ also does not share its domain with $h(a_i)$. So $h(a_i - b) = h(a_i) = h(a_i) - h(b)$.

$$\begin{aligned}
&= \big(h(a_1) - h(b)\big) \rhd \ldots \rhd \big(h(a_n) - h(b)\big) \\
&\overset{2.2.8}{=} h(a_1) \rhd \ldots \rhd h(a_n) - h(b) \\
&= h(a) - h(b)
\end{aligned}$$

Next, we will show that $h(a \rhd b) = h(a) \rhd h(b)$.

$$h(a \rhd b) \overset{2.2.3}{=} h\big(a \rhd (b - a)\big)$$

Note that a and $b - a$ are disjoint, therefore by definition of h:

$$\begin{aligned}
&= h(a) \rhd h(b - a) \\
&= h(a) \rhd \big(h(b) - h(a)\big) \\
&\overset{2.2.3}{=} h(a) \rhd h(b)
\end{aligned}$$

So h is an isomorphism between G and G'. Therefore, G is isomorphic to G'. □

Corollary 3.1.7. Every GPFA which doesn't have the property that its $\mathfrak{mbf}$'s have a one-point domain is isomorphic to a GPFA which *does* have this property. Therefore, *every* GPFA is isomorphic to a GPFA with the property that its $\mathfrak{mbf}$'s have a one-point domain.

3.2 Restricted Partial Function Algebras

If a GPFA has the property that its $\mathfrak{mbf}$'s all have one-point domains, then its elements are actually partial functions with restrictions on their image. Whereas in full partial function algebras the image of an element $x \in X$ under a partial function could be any $y \in Y$, now we only want to consider $y \in Y_x$ where $Y_x \subset Y$. To do this, we shall first give a more precise definition of what we mean by a "restriction".

Definition 3.2.1. Let $r : X \to \mathcal{P}(Y)$. We call r a "restriction". Now consider the set $X \to Y|_r = \{f : X \to Y \mid \forall_{x \in X}[f(x) \in r(x)]\}$. Functions from this set are called "partial functions restricted under r".

Note that every partial function in particular is a restricted partial function (with the restriction $r(x) = Y$).

Definition 3.2.2. Let r be a restriction. Let G_r be the set all of partial functions restricted under r. Let override ($\triangleright$) and minus ($-$) be the same as for regular partial functions. Then the algebra $\mathcal{F}_r = (G_r, \triangleright, -, \emptyset)$ is called the "restricted partial function algebra".

Note that we can use the same $\mathfrak{mbf}$'s in our restricted partial function algebras as we have used in our partial function algebras. There are just less of them (obviously, if we still had all of them, then we would again get the full partial function algebra).

Theorem 3.2.3. Let $\mathcal{F}$ and $\mathcal{G}$ be restricted partial function algebras. Let $F_{\mathfrak{mbf}}$ be the set of $\mathfrak{mbf}$'s of $\mathcal{F}$ and $G_{\mathfrak{mbf}}$ the set of $\mathfrak{mbf}$'s of $\mathcal{G}$. Then $\mathcal{G}$ is a subalgebra of $\mathcal{F}$ iff $G_{\mathfrak{mbf}}$ is a subset of $F_{\mathfrak{mbf}}$.

Proof. Note that for a restricted partial function algebra, the set of $\mathfrak{mbf}$'s is exactly the set of the partial functions with a singleton domain.

If $\mathcal{G}$ is a subalgebra of $\mathcal{F}$, then the underlying set G of $\mathcal{G}$ is a subset of the underlying set F of $\mathcal{F}$. Necessarily, $G_{\mathfrak{mbf}}$ is a subset of $F_{\mathfrak{mbf}}$.

Let $G_{\mathfrak{mbf}}$ be a subset of $F_{\mathfrak{mbf}}$. By theorem 2.3.7 we know that any partial function in $\mathcal{G}$ can be constructed from $\mathfrak{mbf}$'s in $G_{\mathfrak{mbf}}$, which is a subset of $F_{\mathfrak{mbf}}$. So any function in $\mathcal{G}$ can be constructed from $\mathfrak{mbf}$'s in $F_{\mathfrak{mbf}}$, so $\mathcal{G}$ is a subalgebra of $\mathcal{F}$. $\square$

Corollary 3.2.4. Let $\mathcal{F}$ be a partial function algebra with domain $X_{\mathcal{F}}$ and codomain $Y_{\mathcal{F}}$. Let $\mathcal{G}$ be an algebra of the same type as $\mathcal{F}$. Then $\mathcal{G}$ is a subalgabra of $\mathcal{F}$ iff there exists a restricted partial function algebra $\mathcal{H}$ with domain $X_{\mathcal{H}}$ and codomain $Y_{\mathcal{F}}$, such that $X_{\mathcal{H}} \subset X_{\mathcal{F}}$ and $Y_{\mathcal{H}} \subset Y_{\mathcal{F}}$, and $\mathcal{H}$ is isomorphic to $\mathcal{G}$.

Chapter 4

Embedding $\mathbb{A}$ in $\mathcal{F}$

The goal of this chapter is to show that in any algebra satisfying the axioms in 1.2, we can decompose the elements into most basic elements. In order to prove this, we introduce a predicate, stating that an element is decomposable. In order to stress the finiteness of the elements of the algebra, we introduce a scheme of structural induction. Using structural induction we show that each element, except for $\emptyset$, has an unique decomposition into coordinates. The coordinate-values will correspond to the restriction of arguments, as described in the previous chapter.

In [1], several properties of the algebras $\mathbb{A}$ have been proven using a proving system. In this chapter, we will use some of the results proven in the paper. For convenience, these have been relisted in Appendix A.

4.1 The Formal System

Consider the algebra $\mathbb{A} = (A, \rhd, -, \emptyset)$ satisfying the identities Σ:

$$x \rhd x = x \tag{4.1}$$

$$x - x = \emptyset \tag{4.2}$$

$$x \rhd y = (y - x) \rhd x \tag{4.3}$$

$$x - (y - z) = (x - y) \rhd \big(x - (x - z)\big) \tag{4.4}$$

$$(x \rhd y) - z = (x - z) \rhd (y - z). \tag{4.5}$$

Also consider the algebra $\mathcal{F} = (X \to Y, \rhd, -, \emptyset)$.

Lastly, we introduce the following notation:

Definition 4.1.1. We define the operation "intersection" as follows

$$x@y = x - (x - y)$$

4.2 Most Basic Elements in $\mathbb{A}$

Now we want to see if we can find $\mathfrak{mbf}$'s in our algebra $\mathbb{A}$. Remember that in a partial function algebra, we called a partial function f a $\mathfrak{mbf}$, iff for each partial function g either $f - g = \emptyset$ or $f - g = f$.

Definition 4.2.1. Let a in $\mathbb{A}$ such that $a \neq \emptyset$. We define a to be a "most basic element" ($\mathfrak{mbe}$) iff for any b in $\mathbb{A}$, $a - b = \emptyset$ or $a - b = a$. With other words:

$$P_{\mathfrak{mbe}}(a) \stackrel{\texttt{def}}{=} \forall_{b \in \mathbb{A}} [(a - b = \emptyset) \vee (a - b = a)]$$

In previous chapters we also defined some properties on $\mathfrak{mbf}$'s. We want to define these properties on our $\mathfrak{mbe}$'s. In chapter 2 we defined the notion of being disjoint. We can use the same definition for our $\mathfrak{mbe}$'s.

Definition 4.2.2. Let $a, b \in \mathbb{A}$ be $\mathfrak{mbe}$'s. We define a and b to be disjoint iff $a \neq b$ and $a \rhd b = b \rhd a$.

In chapter 3 we defined a relation on $\mathfrak{mbf}$'s. The same relation can be defined for $\mathfrak{mbe}$'s. We will also show that this relation still is an equivalence relation.

Definition 4.2.3. Let a_1 and a_2 be $\mathfrak{mbe}$'s, then:

$$a_1 \sim a_2 \stackrel{\texttt{def}}{=} a_1 \rhd a_2 = a_1$$

Lemma 4.2.4. $\sim$ is an equivalence relation over $\mathfrak{mbe}$'s.

Proof. We have to prove three properties.

- $a_1 \sim a_1$, since $a_1 \rhd a_1 = a_1$ (by 4.1).
- If $a_1 \sim a_2$, then $a_2 \sim a_1$

  First we will prove that for $\mathfrak{mbe}$'s a_1, a_2, we have $a_1 - a_2 = \emptyset$ iff $a_2 - a_1 = \emptyset$. Assume $a_1 - a_2 = \emptyset$ and $a_2 - a_1 = a_2$. Then

  $$\emptyset = a_1 - a_2 = a_1 - (a_2 - a_1) \stackrel{A.22}{=} a_1.$$

  However, since a_1 was a $\mathfrak{mbe}$, $a_1 \neq \emptyset$. Thus we have a contradiction. So if $a_1 - a_2 = \emptyset$, then also $a_2 - a_1 = \emptyset$. Assume $a_2 - a_1 = \emptyset$ and $a_1 - a_2 = a_1$. Then

  $$\emptyset = a_2 - a_1 = a_2 - (a_1 - a_2) \stackrel{A.22}{=} a_2,$$

  which again yields a contradiction since a_2 is a $\mathfrak{mbe}$. Therefore, $a_1 - a_2 = \emptyset$ iff $a_2 - a_1 = \emptyset$.

Next we will prove that, if $a_1 \sim a_2$—which is to say $a_1 \rhd a_2 = a_1$—then $a_2 - a_1 = \emptyset$, and therefore also $a_1 - a_2 = \emptyset$.

$$\begin{aligned} a_2 - a_1 &= \emptyset \rhd (a_2 - a_1) = (a_1 - a_1) \rhd (a_2 - a_1) \\ &= (a_1 \rhd a_2) - a_1 = a_1 - a_1 = \emptyset \end{aligned}$$

Now we can easily prove that, if $a_1 \sim a_2$, then $a_2 \rhd a_1 = a_2$.

$$a_2 \rhd a_1 = (a_1 - a_2) \rhd a_2 = \emptyset \rhd a_2 = a_2$$

So if $a_1 \sim a_2$, then also $a_2 \sim a_1$.

- If $a_1 \sim a_2$ and $a_2 \sim a_3$, then $a_1 \sim a_3$.

  We know that $a_1 \rhd a_2 = a_1$, and by our previous item we know that $a_3 \sim a_2$, so $a_3 \rhd a_2 = a_3$. Therefore, we may conclude:

$$\begin{aligned} a_1 \rhd a_3 &= (a_1 \rhd a_2) \rhd (a_3 \rhd a_2) \\ &= (a_1 \rhd (a_2 \rhd a_3)) \rhd a_2 \\ &= (a_1 \rhd a_2) \rhd a_2 \\ &= a_1 \rhd a_2 \\ &= a_1 \end{aligned}$$

  So $a_1 \rhd a_3 = a_1$.

□

4.3 Structural Induction

In this section we will introduce a predicate dec (a), which is intended to express the decomposability of elements from $\mathbb{A}$ into basic elements.

Definition 4.3.1. If an element $a \in \mathbb{A}$ is decomposable, we will write

$$\operatorname{dec}(a) \Leftrightarrow \exists_{a_1, a_2 \in A} [a_1, a_2 \notin \{\emptyset, a\} \wedge a = a_1 \rhd a_2]$$

Finally, we want to restrict ourselves to objects that have a finite construction. For this purpose we will add one more axiom, this scheme of structural induction.

Axiom 4.6. Let $F(a)$ be a property on objects from $\mathbb{A}$, such that if

1. $F(\emptyset)$ (the property holds for $\emptyset$),
2. $\forall_{a \in \mathbb{A}} [P_{\mathfrak{mbe}}(a) \implies F(a)]$ (the property holds for all $\mathfrak{mbe}$'s), and
3. $F(a_1) \wedge F(a_2) \implies F(a_1 \rhd a_2)$ (if the property holds for two elements a_1, a_2 from $\mathbb{A}$, then the property also holds for $a_1 \rhd a_2$),

then we may conclude that $\forall_{a \in \mathbb{A}} [F(a)]$ (the property holds for every single element of $\mathbb{A}$).

Note that if an algebra $\mathbb{A}$ is finite, this axiom always holds! We will prove this later in the thesis, as a consequence of lemma 4.4.2.

It is interesting to note that, in stead of adding the axioms of structural induction, we could also have added the Descending Chain Condition. In Definition 4.4.8 we will define an order on $\mathbb{A}$, which we will use in section 4.4.1 to elaborate on this idea.

4.4 Decomposing Elements

In our previous chapters we have been working with partial functions. Partial functions can easily be decomposed into $\mathfrak{mbf}$'s, because we understand what these functions look like and how they are constructed. However, in our algebra $\mathbb{A}$, we don't know what our underlying domain is. To also be able to decompose elements from $\mathbb{A}$, we have to introduce some lemma's. Our goal in this section is to decide whether or not an element can be decomposed into smaller elements. We will also give a formal definition of what we mean by "smaller".

Lemma 4.4.1. Let $a \in \mathbb{A}$ such that $a \neq \emptyset$ and a is not a $\mathfrak{mbe}$. Let $b \in \mathbb{A}$ such that $a - b \neq a$, then it follows that $a@b \neq \emptyset$. Also, if $a - b \neq \emptyset$, then $a@b \neq a$.

Proof. By contraposition, we need to prove $a@b = \emptyset$ implies $a - b = a$.

$$\begin{aligned} a - b &= (a - b) \triangleright \emptyset = (a - b) \triangleright (a@b) \\ &= (a - b) \triangleright \big(a - (a - b)\big) \\ &\overset{4.4}{=} a - (b - b) \\ &\overset{4.2}{=} a - \emptyset \overset{A.4}{=} a \end{aligned}$$

Secondly, we prove (again by contraposition) $a@b = a$ implies $a - b = \emptyset$.

$$a - b = (a@b) - b \overset{A.28}{=} a@(b - b) = a@\emptyset \overset{A.30}{=} \emptyset$$

□

Lemma 4.4.2. Let $a \in \mathbb{A}$ with $a \neq \emptyset$. If a is not a $\mathfrak{mbe}$, then there exist $a_1, a_2 \in \mathbb{A}$ such that $a = a_1 \triangleright a_2$ and $a_1, a_2 \neq \emptyset, a$, and also $a_1 \triangleright a_2 = a_2 \triangleright a_1$.

Proof. Let $a \in \mathbb{A}$ not be a $\mathfrak{mbe}$, with other words there exist $b \in \mathbb{A}$ so that $b \neq \emptyset, a$ and $a - b \neq \emptyset, a$.

We can take $a_1 = a - b$ and $a_2 = a@b$. After all,

$$(a - b) \rhd (a@b) \overset{\texttt{def}}{=} (a - b) \rhd \big(a - (a - b)\big) \overset{4.4, 4.2, A.4}{=} a$$

By assumption $a - b \neq \emptyset, a$. By 4.4.1 $a@b \neq \emptyset, a$.

Also note that $a_1 \rhd a_2 = a_2 \rhd a_1$.

$$\begin{aligned} a_1 \rhd a_2 = (a - b) \rhd (a@b) &\overset{A.20}{=} a \\ &\overset{A.19}{=} (a@b) \rhd (a - b) = a_2 \rhd a_1 \end{aligned}$$

□

Corollary 4.4.3. Assume axioms 4.1 through 4.5 hold and assume A is finite. Then axiom 4.6 also holds.

Proof. Assume $\mathbb{A}$ is finite. Let $a \in \mathbb{A}$ such that a isn't a $\mathfrak{mbe}$ and $a \neq \emptyset$. Then, by lemma 4.4.2, there exist $a_1, a_2 \in \mathbb{A}$ such that $a = a_1 \rhd a_2$ and $a_1, a_2 \neq \emptyset, a$. Since $\mathbb{A}$ only has a finite number of elements, we can repeat this proces until we've decomposed a into $\mathfrak{mbe}$'s. Therefore, every element in $\mathbb{A}$ is finitely decomposable into $\mathfrak{mbe}$'s. □

Lemma 4.4.4. Let $a, e \in \mathbb{A}$ such that $P_{\mathfrak{mbe}}(e)$ and $a@e \neq \emptyset$. Then $P_{\mathfrak{mbe}}(a@e)$.

Proof. We need to show that for each $b \in \mathbb{A}$ either $(a@e) - b = \emptyset$ or $(a@e) - b = a@e$.

Assume $e - b = \emptyset$.

$$\begin{aligned} (a@e) - b &= (a - b)@(e - b) \\ &= (a - b)@\emptyset \\ &= \emptyset \end{aligned}$$

Assume $e - b = e$.

$$\begin{aligned} (a@e) - b &= a@(e - b) \\ &= a@e \end{aligned}$$

So we may conclude that $a@e$ is a $\mathfrak{mbe}$, if e is a $\mathfrak{mbe}$. □

Corollary 4.4.5. Let $a \in \mathbb{A}$, $a \neq \emptyset$. If a is not a $\mathfrak{mbe}$, then there exist $a_1, a_2 \in \mathbb{A}$ such that a_2 is a $\mathfrak{mbe}$, $a = a_1 \rhd a_2$, $a_1, a_2 \neq \emptyset, a$, and $a_1 \rhd a_2 = a_2 \rhd a_1$.

Proof. We already proved we can find a'_1 and a'_2 such that $a = a'_1 \rhd a'_2$ and $a'_1, a'_2 \neq \emptyset, a$. We did that by finding a b, such that $b \neq \emptyset, a$ and $a - b \neq \emptyset, a$. Since $b \neq \emptyset$, we either have b is a $\mathfrak{mbe}$, or b isn't. If b is a $\mathfrak{mbe}$, then $a@b$ also is a $\mathfrak{mbe}$.

If b isn't a $\mathfrak{mbe}$, then there exists a $\mathfrak{mbe}$ e, such that $b@e \neq \emptyset$. Since e is a $\mathfrak{mbe}$, $b@e$ also is a $\mathfrak{mbe}$. Choose $b' = b@e$. Note that, since b' is a $\mathfrak{mbe}$, $b' \neq \emptyset, a$ and $a@b'$ is a $\mathfrak{mbe}$. So $a@b' \neq \emptyset, a$. By 4.4.1, $a - b' \neq \emptyset, a$. So instead of b, we might as well have choosen b', which is a $\mathfrak{mbe}$.

Since we've usen the same construction as in Lemma 4.4.2, $a_1 \rhd a_2 = a_2 \rhd a_1$ follows in the same way. □

So we have proven that, given an element $a \in \mathbb{A}$ which isn't a $\mathfrak{mbe}$, we can find a b, so that we can decompose a into $a_1 = a - b$ and $a_2 = a@b$. Now we can wonder if these elements have an empty intersection. By "empty intersection", we mean $a_1@a_2 = \emptyset$ and $a_2@a_1 = \emptyset$.

Lemma 4.4.6. Let a_1 and a_2 be as in Corollary 4.4.5. Then it follows that $a_1@a_2 = \emptyset$

Proof. First note that $a_1@a_2 = a_1 - (a_1 - a_2) = (a - b) - \big((a - b) - (a@b)\big)$, so if we can prove that $a - b = (a - b) - (a@b)$, we may conclude that $a_1@a_2 = (a - b) - (a - b) = \emptyset$.

$$\begin{aligned}
(a - b) - (a@b) &= (a - b) - \big(a - (a - b)\big) \\
&\overset{4.4}{=} \big((a - b) - a\big) \rhd \Big((a - b) - \big((a - b) - (a - b)\big)\Big) \\
&\overset{4.2, A.4}{=} \big((a - b) - a\big) \rhd (a - b) \\
&\overset{A.25}{=} a - b
\end{aligned}$$

□

Lemma 4.4.7. Let a_1 and a_2 be as in Corollary 4.4.5. Then it follows that $a_2@a_1 = \emptyset$

Proof.

$$\begin{aligned}
a_2@a_1 &= (a@b)@(a-b) \\
&\overset{A.17}{=} \big(a@(a-b)\big)@b \\
&\overset{A.28}{=} \big((a@a)-b\big)@b \\
&\overset{A.8}{=} (a-b)@b \\
&\overset{A.27}{=} (a@b)-b \\
&\overset{A.28}{=} a@(b-b) \\
&\overset{4.2}{=} a@\emptyset \\
&\overset{A.4}{=} (a@\emptyset)-\emptyset \\
&\overset{A.1}{=} \emptyset
\end{aligned}$$

□

Now we know how to decompose elements from $\mathbb{A}$, it makes sense to define an ordering. If $a \in \mathbb{A}$, we can decompose a into a_1, a_2 (such that $a = a_1 \rhd a_2$). We want to be able to say that a_1 and a_2 are, in a way, "smaller" than a.

Definition 4.4.8. We will now define a relation on elements from $\mathbb{A}$.

$$a \leq b \overset{\texttt{def}}{\iff} a \rhd b = b$$

We also define:

$$a \lneq b \overset{\texttt{def}}{\iff} a \leq b \wedge a \neq b$$

Lemma 4.4.9. $\leq$ is an ordering.

Proof. To show $\leq$ is an ordering, we have to show three things.

- $a \leq a$

  $a \rhd a = a$

- $a \leq b \wedge b \leq a \implies a = b$

  Assume $a \leq b$ and $b \leq a$. So $a \rhd b = b$ and $b \rhd a = a$. Thus we have $a = b \rhd a = (a \rhd b) \rhd a$ and therefore $((a \rhd b) \rhd a) - a = a - a = \emptyset$. By derived law A.13, we may conclude $\emptyset = ((a \rhd b) \rhd a) - a = (b-a) \rhd (a-a) = (b-a) \rhd \emptyset = b-a$ $(b-a = \emptyset)$. Therefore, $b = a \rhd b \overset{A.24}{=} a \rhd (b-a) = a \rhd \emptyset = a$.

- $a \leq b \wedge b \leq c \implies a \leq c$

  $a \leq b$, so $a \rhd b = b$. $b \leq c$, so $b \rhd c = c$.

$$\begin{aligned}
a \rhd c &= a \rhd (b \rhd c) = (a \rhd b) \rhd c \\
&= b \rhd c = c
\end{aligned}$$

□

Corollary 4.4.10. Let $a \in \mathbb{A}$, such that $a \neq \emptyset$ and a isn't a $\mathfrak{mbe}$. So there exists a $\mathfrak{mbe}$ e such that $(a' = a - e)$ $a = a' \rhd e$. Then $a' \lneq a$.

Proof. First of all, $a' \neq a$, so we just have to prove that $a' \leq a$. Since $a = a' \rhd e$, this is trivial. □

4.4.1 Descending Chain Condition

The Descending Chain Condition is defined as follows:

Definition 4.4.11. A partially ordered set is said to satisfy the Descending Chain Condition (or DCC) iff for any descending sequence

$$a_1 \geq a_2 \geq \dots$$

there exists a $N \in \mathbb{N}$ such that $\forall_{n,m \geq N} [a_n = a_m]$.

This leads us to the following conjecture:

Conjecture 4.4.12. We suspect that the DCC holds for our order $\leq$ iff the axiom of Structural Induction (axiom 4.6) hold.

4.5 The Main Theorem

This section revolves around the main theorem of this thesis. The main theorem states that if we decompose two elements of $\mathbb{A}$, and their components are the same, then they are in fact the same element. Before we make this formal, we first introduce a relation on $\mathbb{A}$. We will call two elements in $\mathbb{A}$ "similar", if their decomposition is the same.

Definition 4.5.1. Let $a, b \in \mathbb{A}$. Then we define the relation $\approx$ as follows.

$$a \approx b \overset{\text{def}}{\iff} \forall_{e \in \mathbb{A}} [P_{\mathfrak{mbe}}(e) \implies a@e = b@e]$$

The main theorem now states that elements which are similar, are in fact the same.

Theorem 4.5.2.

$$a \approx b \implies a = b$$

Proof. Assume $a \approx b$. We want to show that $a = b$. We are going to do so using structural induction.

- Assume $a = \emptyset$. Then

$$\forall_e \left[P_{\mathfrak{mbe}}(e) \implies a@e = \emptyset\right]$$

So

$$\forall_e \left[P_{\mathfrak{mbe}}(e) \implies b@e = \emptyset\right]$$

  - Assume b is $\mathfrak{mbe}$. Then $b@b = b \neq \emptyset$. So $\exists_e \left[P_{\mathfrak{mbe}}(e) \wedge b@e \neq \emptyset\right]$. $\lightning$
  - Assume b is neither $\emptyset$ or a $\mathfrak{mbe}$. Then, by corollary 4.4.5, there exists a $\mathfrak{mbe}$ e such that (if $b' = b - e$) $b = b' \rhd e$. Then:

$$\begin{aligned} b@e &= (b \rhd e)@e \\ &= (b@e) \rhd (e@e) \\ &= (b@e) \rhd e \overset{A.31}{\neq} \emptyset \end{aligned}$$

    So $\exists_e \left[P_{\mathfrak{mbe}}(e) \wedge b@e \neq \emptyset\right]$. $\lightning$

  So by process of elimination, we can conclude that $b = \emptyset$.

- Assume a is $\mathfrak{mbe}$. Then for each $\mathfrak{mbe}$ e we either have $a - e = \emptyset$ or $a - e = a$. Assume $a@e \neq \emptyset$, then $a - e = \emptyset$ (if $a - e = a$, then $a@e = a - (a - e) = a - a = \emptyset$). So $e \rhd a = (a - e) \rhd e = \emptyset \rhd e = e$. So $e \sim a$.
  - Assume $b = \emptyset$. Then, by the proof in the previous bullet, $a = \emptyset$. $\lightning$
  - Assume b is neither $\emptyset$ or a $\mathfrak{mbe}$. Then, by corollary 4.4.5, there exists a $\mathfrak{mbe}$ g such that (if $b' = b - g$) $b = b' \rhd g$. Note that:

$$\begin{aligned} b@g &= (b' \rhd g)@g \\ &= (b'@g) \rhd (g@g) \\ &= \emptyset \rhd g \\ &= g \neq \emptyset \end{aligned}$$

    So $a@g = b@g \neq \emptyset$. So $g \sim a$.
    Now assume b' isn't a $\mathfrak{mbe}$. Then there exists g' such that (if $b'' = b' - g'$) $b' = b'' \rhd g'$. Then:

$$\begin{aligned} b@g' &= (b' \rhd g)@g' \\ &= \big((b'' \rhd g') \rhd g\big)@g' \\ &= \big((b'' \rhd g')@g'\big) \rhd (g@g') \\ &= \big((b''@g') \rhd (g'@g')\big) \rhd (g@g') \\ &= (\emptyset \rhd g') \rhd (g@g') \\ &= g' \rhd (g@g') = g' \neq \emptyset \end{aligned}$$

    So $a@g' = b@g' \neq \emptyset$. So $g' \sim a$. So $g' \sim g$. So $g' \rhd g = g'$.

$$\begin{aligned} b &= b' \rhd g = (b'' \rhd g') \rhd g \\ &= b'' \rhd (g' \rhd g) = b'' \rhd g' = b' \end{aligned}$$

But this contradicts our assumption that we decomposed b into g and b', as $b' = b$. So apparently our assumption that b' isn't a $\mathfrak{mbe}$ was wrong. $\lightning$

Assume b' is a $\mathfrak{mbe}$. We still have $b = b' \triangleright g$ and $a \sim e$. However,

$$\begin{aligned}
b@b' &= (b' \triangleright e)@b' \\
&= (b' \triangleright e) - \big((b' \triangleright e) - b'\big) \\
&= (b' \triangleright e) - \big((b' - b') \triangleright (e - b')\big) \\
&= (b' \triangleright e) - (e - b') \\
&= \big((e - b') \triangleright b'\big) - (e - b') \\
&= \big((e - b') - (e - b')\big) \triangleright \big(b' - (e - b')\big) \\
&= b' - (e - b') = b' \neq \emptyset
\end{aligned}$$

So $a@b' = b@b' \neq \emptyset$, so $a \sim b'$, so $b' \sim e$. So $b = b' \triangleright e = b'$, which contradicts our assumption that $b \neq b'$. So apparently our assumption that b' isn't a $\mathfrak{mbe}$ also is wrong. $\lightning$

But if both our assumption that b' is a $\mathfrak{mbe}$ is wrong, and our assumption that b' isn't a $\mathfrak{mbe}$ also is wrong, then we get a contradiction with our assumption that b' exists. So our original assumption, that b wasn't a $\mathfrak{mbe}$ must have been wrong. $\lightning$

So by process of elimination, if a is a $\mathfrak{mbe}$, then b also is a $\mathfrak{mbe}$.

- Assume a neither is $\emptyset$ or a $\mathfrak{mbe}$. So there exists a $\mathfrak{mbe}$ f such that ($a' = a - f$) $a = a' \triangleright f$. Also assume that for each c such that $c \lneq a$ we already have proven that if for certain d we have $c \approx d$, then $c = d$.

  Let $b' = b - f$. Let e be an arbitrary $\mathfrak{mbe}$.

  Assume $e \not\sim f$. If $f - e = \emptyset$, then $e \triangleright f = (f - e) \triangleright e = \emptyset \triangleright e = e$, which contradicts our assumption that $e \not\sim f$. So $f - e = f$. So $f@e = f - (f - e) = f - f = \emptyset$.

$$\begin{aligned}
a@e &= b@e \\
(a' \triangleright f)@e &= (b' \triangleright f)@e \\
(a'@e) \triangleright (f@e) &= (b'@e) \triangleright (f@e) \\
(a'@e) \triangleright \emptyset &= (b'@e) \triangleright \emptyset \\
a'@e &= b'@e
\end{aligned}$$

  Assume $e \sim f$. Note that, since $e \sim f$ (so also $f \sim e$), for arbitrary c we have:

$$\begin{aligned}
c@e &= c@(e \triangleright f) = (c@e) \triangleright (c@f) \\
&= (c@f) \triangleright (c@e) = c@(f \triangleright e) \\
&= c@f
\end{aligned}$$

So $a'@e = a'@f = \emptyset = b'@f = b'@e$.

Since $a' \approx b'$, we now know that $a' = b'$. So $a = a' \triangleright f = b' \triangleright f = b$.

$\square$

Appendix A

Derived Laws

We will use the following derived laws, which were proven in [1].

Empty domain.

$$x@y - y = \emptyset \tag{A.1}$$
$$x \rhd \emptyset = x \tag{A.2}$$
$$\emptyset \rhd x = x \tag{A.3}$$
$$x - \emptyset = x \tag{A.4}$$
$$\emptyset - x = \emptyset \tag{A.5}$$

Associativity.

$$x \rhd (y \rhd z) = (x \rhd y) \rhd z \tag{A.6}$$
$$x@(y@z) = (x@y)@z \tag{A.7}$$

Idempotence.

$$x@x = x \tag{A.8}$$

Distributivity.

$$(x \rhd y)@z = (x@z) \rhd (y@z) \tag{A.9}$$
$$x@(y \rhd z) = (x@y) \rhd (x@z) \tag{A.10}$$
$$(x@y) - z = (x - z)@(y - z) \tag{A.11}$$
$$x \rhd (y@z) = (x \rhd y)@(x \rhd z) \tag{A.12}$$
$$\big((x \rhd y) \rhd z\big) - x = (y - x) \rhd (z - x) \tag{A.13}$$

Weak commutativity.

$$(x - y) - z = (x - z) - y \tag{A.14}$$
$$x - (y \triangleright z) = x - (z \triangleright y) \tag{A.15}$$
$$(x - y) \triangleright (x - z) = (x - z) \triangleright (x - y) \tag{A.16}$$
$$(x@y)@z = (x@z)@y \tag{A.17}$$
$$x - (y@z) = x - (z@y) \tag{A.18}$$

Partitioning.

$$(x@y) \triangleright (x - y) = x \tag{A.19}$$
$$(x - y) \triangleright (x@y) = x \tag{A.20}$$

Combine minus.

$$(x - y) - z = x - (y \triangleright z) \tag{A.21}$$

Double minus.

$$x - (y - x) = x \tag{A.22}$$

Overlapping.

$$x \triangleright (y@x) = x \tag{A.23}$$

Agree.

$$x \triangleright (x - y) = x \tag{A.24}$$
$$(x - y) \triangleright x = x \tag{A.25}$$

Compatible.

$$x \triangleright (y - x) = x \triangleright y \tag{A.26}$$

Reordering.

$$(x - y)@z = (x@z) - y \tag{A.27}$$
$$x@(y - z) = (x@y) - z \tag{A.28}$$

We will also use the following derived laws, which we shall prove ourselves.

Law A.29.

$$\begin{aligned} (x \triangleright y)@x &\overset{A.9}{=} (x@x) \triangleright (y@x) \\ &\overset{A.8}{=} x \triangleright (y@x) \\ &\overset{A.23}{=} x \end{aligned}$$

Law A.30.

$$\begin{aligned} x@\emptyset &\overset{A.4}{=} x@\emptyset - \emptyset \\ &\overset{A.1}{=} \emptyset \end{aligned}$$

Law A.31. If $x \rhd y = \emptyset$, then $x = \emptyset = y$.

Proof.

$$\begin{aligned} y &\overset{A.20}{=} (y - x) \rhd (y@x) \\ &\overset{A.12}{=} \big((y - x) \rhd y\big)@\big((y - x) \rhd x\big) \\ &\overset{4.3}{=} \big((y - x) \rhd y\big)@(x \rhd y) \\ &= \big((y - x) \rhd y\big)@\emptyset \\ &\overset{A.30}{=} \emptyset \end{aligned}$$

Since $\emptyset = x \rhd y = x \rhd \emptyset = x$ we may conclude $x = \emptyset = y$. □

Law A.32. If $(a \rhd b)@e \neq \emptyset$, then $a@e \neq \emptyset$ or $b@e \neq \emptyset$.

Proof. Assume $a@e = \emptyset = b@e$. Then

$$\begin{aligned} \emptyset &\overset{A.4}{=} (a@e) \rhd (b@e) \\ &\overset{A.9}{=} (a \rhd b)@e \end{aligned}$$

So by contradiction, $(a \rhd b)@e \neq \emptyset \implies a@e \neq \emptyset \lor b@e \neq \emptyset$. □

Bibliography

[1] Jasper Berendsen, David N. Jansen, Julien Schmaltz, and Frits W. Vaandrager. The axiomatization of overriding and update. *Journal of Applied Logic*, 8(1):141–150, 2010.