Generalised Kripke semantics for the Lambek–Grishin calculus *

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1 Introduction

Kripke semantics plays a fundamental role for modal logics. (?) suggests a natural generalisation of Kripke semantics for modal logics to an analogous kind of semantics (generalised Kripke semantics) for a broader setting including that of substructural logics. The algebraic structure underlying a traditional Kripke model is the *powerset* of a set of worlds. The generalisation used here allows us to build models on a more general algebraic structure, viz. on a kind of *lattice* generalising powersets. This allows the modular treatment of canonicity and correspondence as in the modal logic setting.

In this paper we will consider a kind of substructural logic (the so-called Lambek-Grishin calculus) that was first mentioned in (?) and discussed more in detail in (?). This calculus extends the implication-fusion fragment (the non-associative Lambek calculus) in the following way: together with the product residuated family of connectives, there is a second residuated family. In addition to this, some postulates of interaction between these two families of connectives are introduced. The Kripke semantics of the Lambek-Grishin calculus was discussed in (?) and (?).

This paper reflects our work in progress on generalised Kripke semantics for the Lambek-Grishin calculus (it is an abstract of work we have just started). This is a particularly interesting example to apply these new methods to: The residuated families of the Lambek-Grishin calculus have order dual properties and the symmetric set-up required by the general lattice setting lends itself particularly well to this. In addition, the interaction postulates provide an interesting case study for exploring the expressive power of these semantics compared to the traditional ones. In this preliminary note we consider Sahlqvist-style correspondence for one of the interaction postulates identified as pertinent to linguistic phenomena in (?).

The paper is structured as follows: in section 2 we briefly repeat the theory of generalised Kripke models. In section 3 we apply it to capture the basis of the Lambek-Grishin calculus. Finally, in section 4 we discuss the influence of interaction postulates.

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2 Complete semantics for some substructural logics

2.1 Generalising the set of worlds

In traditional Kripke semantics, a frame is built on a non-empty set of worlds $\mathcal{W}$ which does not, generally speaking, have any particular structure. An interpretant of a logical formula in a frame is an arbitrary set of worlds. The idea of the generalised Kripke frames is that an interpretant is not only described by its worlds, but also by 'information quanta' contained in it (so-called **co-worlds**), moreover, either of these completely determines the interpretant.

A **polarity** is a triple $F = (X, Y, \leqslant)$, where X and Y are non-empty sets and $\leqslant \subseteq X \times Y$ is a binary relation from X to Y. Here X is a **set of worlds**, Y a **set of co-worlds**, and a co-world, or information quantum, $y \in Y$ is a **part of the world** $x \in X$ provided $x \leqslant y$ holds.

Such a polarity yields a Galois connection:

$$\begin{array}{rcrcl} (\_)^{\leqslant} & : & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & & A & \mapsto & \{y \in Y \mid \forall x \in X\ (x \in A \Rightarrow x \leqslant y)\} \end{array}$$

$$\begin{array}{rcrcl} (\_)^{\geqslant} & : & \mathcal{P}(Y) & \to & \mathcal{P}(X) \\ & & B & \mapsto & \{x \in X \mid \forall y \in Y\ (y \in B \Rightarrow x \leqslant y)\} \end{array}$$

Interpretants in generalised Kripke semantics are Galois-stable subsets of X:

$$\mathcal{G}(F) = \{A \subseteq X \mid A = (A^{\leqslant})^{\geqslant}\}$$

Define $\uparrow x = \{y \in Y \mid x \leqslant y\}$ and $\downarrow y = \{x \in X \mid x \leqslant y\}$. A polarity F is called an **S-frame (a separating frame)** provided that

$$\forall x_1, x_2 \in X \quad (\uparrow x_1 = \uparrow x_2 \ \Rightarrow \ x_1 = x_2) \quad \text{and} \quad \forall y_1, y_2 \in Y \quad (\downarrow y_1 = \downarrow y_2 \ \Rightarrow \ y_1 = y_2).$$

Informally, these conditions mean that $\uparrow x$ ($\downarrow y$, respectively) completely describes the world x (the co-world y).

An S-frame F is called a **reduced** (or an **RS-frame**) if the following two properties hold:

$$\forall x \exists y \left(x \nleqslant y \wedge \forall x' \ (\uparrow x \subset \uparrow x' \Rightarrow x' \leqslant y)\right) \quad \text{and} \quad \forall y \exists x \left(x \nleqslant y \wedge \forall y' \ (\downarrow y \subset \downarrow y' \Rightarrow x \leqslant y')\right).$$

These two conditions basically say that all worlds are completely join-irreducible (i.e., it is not possible that $x = \bigvee\{x' \mid \uparrow x \subset \uparrow x'\}$) and all co-worlds are completely meet-irreducible. This notion of an RS-frame gives a generalisation of a Kripke set of worlds $\mathcal{W}$.

Finally, continuing a generalisation of a Kripke structure, let us introduce the notion of a valuation function. Given an RS-frame F and a set of variables Var, a **valuation** is a mapping $V \colon Var \to \mathcal{G}(F)$.

2.2 Models for substructural logics

The paper (**?**) offers complete semantics for the implication-fusion fragment of various substructural logics. The basic logic under consideration is the non-associative Lambek calculus. Thus, the set of formulae $\mathcal{F}$ is built from the set of variables Var with a product connective (which will be denoted as $\otimes$ here) and its two residuals denoted as $/$ and $\backslash$. The rules of the system are as follows ($A, B, C \in \mathcal{F}$):

- an axiom scheme: $A \vdash A$;
- a transitivity rule: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
- residuation rules: $A \vdash C/B \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A \backslash C$.

In order to capture this logic, we need to add a ternary relation $R_\otimes \subseteq X \times X \times Y$ to model the behaviour of the connectives. Since interpretants are Galois-stable sets, the following compatibility conditions on $R_\otimes$ are imposed:

$$\begin{array}{lcl} \forall x_1, x_2 \in X & (R_\otimes(x_1, x_2, \_)^{\geqslant})^{\leqslant} & = & R_\otimes(x_1, x_2, \_); \\ \forall x_1 \in X, y \in Y & (R_\otimes(x_1, \_, y)^{\leqslant})^{\geqslant} & = & R_\otimes(x_1, \_, y); \\ \forall x_2 \in X, y \in Y & (R_\otimes(\_, x_2, y)^{\leqslant})^{\geqslant} & = & R_\otimes(\_, x_2, y). \end{array}$$

A **model** is a triple $\mathcal{M} = (F, R_\otimes, V)$ where $F = (X, Y, \leqslant)$ is an RS-frame, $R_\otimes$ is a compatible relation, and $V\colon Var \to \mathcal{G}(F)$ is a valuation of variables. Given such a model, relations $\Vdash \subseteq X \times \mathcal{F}$ and $\succ \subseteq Y \times \mathcal{F}$ are defined inductively:

- for $p \in Var$: $x \Vdash p \;\Leftrightarrow\; x \leqslant V(p)$ and $y \succ p \;\Leftrightarrow\; V(p) \leqslant y$;
- if $x \Vdash A$, $x \Vdash B$, $y \succ A$, $y \succ B$ are defined, the relations for complex formulae are defined as follows:

$$\begin{array}{rcl} y \succ A \otimes B & \Leftrightarrow & \forall x_1, x_2 \in X \; (x_1 \Vdash A \;\wedge\; x_2 \Vdash B \Rightarrow R_\otimes(x_1, x_2, y)) \\ x \Vdash A \otimes B & \Leftrightarrow & \forall y \in Y \; (y \succ A \otimes B \Rightarrow x \leqslant y) \\ x \Vdash A \backslash B & \Leftrightarrow & \forall x' \in X, \; \forall y \in Y \; (x' \Vdash A \;\wedge\; y \succ B \Rightarrow R_\otimes(x', x, y)) \\ y \succ A \backslash B & \Leftrightarrow & \forall x \in X \; (x \Vdash A \backslash B \Rightarrow x \leqslant y) \\ x \Vdash B / A & \Leftrightarrow & \forall x' \in X, \; \forall y \in Y \; (x' \Vdash A \;\wedge\; y \succ B \Rightarrow R_\otimes(x, x', y)) \\ y \succ B / A & \Leftrightarrow & \forall x \in X \; (x \Vdash B / A \Rightarrow x \leqslant y) \end{array} \tag{1}$$

The sequent $A \vdash B$ is valid over the class of frames if and only if it holds in every model, i.e. for every RS-frame equipped with a compatible relation and for every valuation. The following theorem (proven in (**?**)) states soundness and completeness of the non-associative Lambek calculus with respect to the class of models defined above:

Theorem: For any formulae of the Lambek calculus A and B the sequent $A \vdash B$ is derivable if and only if it is valid over the class of all models.

In that paper, a construction of a canonical model is provided. This construction can be carried out for any kind of substructural logic.

3 Generalised Kripke semantics for $LG_\emptyset$

In this section we illustrate the theory described above on another kind of substructural logic, viz., the Lambek-Grishin calculus. We start with its minimal version $LG_\emptyset$, in which no interaction axioms are added. The formulae of $LG_\emptyset$ are built from variables taken from the set Var with six connectives: product $\otimes$ and its residuals $/$ and $\backslash$, and plus $\oplus$ and its residuals $\oslash$ and $\obslash$. The rules are as follows:

- an axiom scheme: $A \vdash A$;
- a transitivity rule: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
- residuation rules for the product family: $A \vdash C/B \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A \backslash C$;
- residuation rules for the plus family: $B \obslash C \vdash A \;\Leftrightarrow\; C \vdash B \oplus A \;\Leftrightarrow\; C \oslash A \vdash B$.

In order to model the behaviour of the plus family of connectives, we add to the frame for the non-associative Lambek calculus another ternary relation $R_\oplus \subseteq X \otimes Y \otimes Y$ satisfying the appropriate compatibility conditions:

$$\begin{array}{lll}
\forall y_1, y_2 \in Y & (R_\oplus(_, y_1, y_2)^{\leqslant})^{\geqslant} = & R_\oplus(_, y_1, y_2); \\
\forall y_1 \in Y, x \in X & (R_\oplus(x, y_1, _)^{\geqslant})^{\leqslant} = & R_\oplus(x, y_1, _); \\
\forall y_2 \in Y, x \in X & (R_\oplus(x, _, y_2)^{\geqslant})^{\leqslant} = & R_\oplus(x, _, y_2).
\end{array}$$

The truth conditions for the new connectives are defined as follows:

$$\begin{array}{lll}
x \Vdash A \oplus B & \Leftrightarrow & \forall y_1, y_2 \in Y\ (y_1 \succ A \ \wedge\ y_2 \succ B \Rightarrow R_\oplus(x, y_1, y_2)) \\
y \succ A \oplus B & \Leftrightarrow & \forall x \in X\ (x \Vdash A \oplus B \Rightarrow x \leqslant y) \\
 & & \\
y \succ A \obslash B & \Leftrightarrow & \forall y' \in Y,\ \forall x \in X\ (y' \succ A \ \wedge\ x \Vdash B \Rightarrow R_\oplus(x, y', y)) \\
x \Vdash A \obslash B & \Leftrightarrow & \forall y \in Y\ (y \succ A \obslash B \Rightarrow x \leqslant y) \\
 & & \\
y \succ B \oslash A & \Leftrightarrow & \forall y' \in Y,\ \forall x \in X\ (y' \succ A \ \wedge\ x \Vdash B \Rightarrow R_\oplus(x, y, y')) \\
x \Vdash B \oslash A & \Leftrightarrow & \forall y \in Y\ (y \succ B \oslash A \Rightarrow x \leqslant y)
\end{array} \tag{2}$$

We provide a formulation of the soundness and completeness theorem for the basic Lambek-Grishin calculus $LG_\emptyset$ with respect to the class of models described above:

Theorem: For any formulae of $LG_\emptyset$ A and B the sequent $A \vdash B$ is derivable in $LG_\emptyset$ if and only if it is valid over the class of all models.

This theorem is directly obtainable from the completeness theorem for the non-associative Lambek calculus given in the previous section by order dualisation. While this basic completeness theorem for $LG_\emptyset$ is a direct consequence of the work in (**?**), it is non-trivial to develop the modular correspondence theory for the interaction axioms. As in Kripke semantics for modal logics, two components are needed: canonicity and correspondence. In the next section we describe our preliminary results for correspondence.

4 Frame conditions for interaction postulates

In this section we extend $LG_\emptyset$ with one of the families of interderivable interaction postulates and show how to adapt the class of Kripke frames accordingly.

4.1 Interaction postulates extending $LG_\emptyset$

Interactions involving only connectives from one family are well-known from the Lambek calculus. For example, adding to the non-associative Lambek calculus a postulate $A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$ yields the associative Lambek calculus.

In the setting of $LG_\emptyset$ it is most interesting to consider mixed axioms, i.e. those involving connectives from both residuated families. In (?) Grishin has studied and classified all of these. (?) identified the postulates in Grishin's fourth group as most relevant for linguistic applications. However, from the logical perspective, Grishin's first group of postulates seems equally interesting. Because of the rules of the basic calculus, $LG_\emptyset$, the additional postulates come in groups of six that are mutually interderivable on the basis of $LG_\emptyset$. We will consider here one such group of postulates:

$$
\begin{array}{llll}
(a) & (B \backslash C) \oslash A \vdash B \backslash (C \oslash A) & (d) & (A \backslash C) \obslash B \vdash C \obslash (A \otimes B) \\
(b) & B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A & (e) & (A \oplus B) / C \vdash A / (C \oslash B) \\
(c) & A \otimes (C \oslash B) \vdash (A \otimes C) \oslash B & (f) & A \oslash (B \obslash C) \vdash (C / A) \backslash B
\end{array} \qquad (3)
$$

4.2 Correspondence for interaction postulates

In the paper (?) completeness theorems for the implication-fusion fragment of various substructural logics are proven algebraically. For each postulate that is added to the pure implication-fusion fragment, a first-order frame condition is derived by Sahlqvist correspondence methods and it is proven that these conditions are canonical (i.e. hold in the canonical frame for each substructural logic considered). We stress here that this approach is parallel to the one for modal logics and that this was the first modular approach to these completeness results.

For example, the frame condition imposed by postulating associativity for the product connective looks as follows:

$$
\begin{array}{l}
\forall x_1, x_2, x_3 \in X \ \forall y \in Y \\
\Big[\forall x \in X \ \Big(R^{\downarrow}_{\otimes}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_2, x, y)\Big) \ \Leftrightarrow \ \forall x \in X \ \Big(R^{\downarrow}_{\otimes}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)\Big)\Big].
\end{array}
$$

Here $R^{\downarrow}_{\otimes}$ is a ternary relation in $X \times X \times X$ which is defined as follows:

$$
(x_1, x_2, x) \in R^{\downarrow}_{\otimes} \ \Leftrightarrow \ \forall y \in Y \ \Big((x_1, x_2, y) \in R_{\otimes} \Rightarrow x \leqslant y\Big)
$$

Informally, $R_{\otimes}(x_1, x_2, y)$ reflects the fact that the information y is contained in the 'fusion of the worlds' x_1 and x_2, whereas $R^{\downarrow}_{\otimes}(x_1, x_2, x)$ means that the world x contains more information than the 'fusion of the worlds' x_1 and x_2.

The current paper is work in progress, so let us illustrate how adding to the basic Lambek-Grishin calculus only one postulate of interaction between the $\otimes$ and $\oplus$ families influences the relations $R_\otimes$ and $R_\oplus$. As an example, we chose the family of postulates given above in (3) and work on the form given as the postulate (b): $B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A$. The condition (in prenex normal form) looks as follows (all variables named x (y), with or without indices, are presupposed to be elements of X (Y):

$$\forall x, y_1, y_2\ \exists x_1\ \forall x_2\ \exists y_3\ \forall x_3$$
$$\Big[\Big(\Big[\Big(R^{\downarrow}_{\otimes}(x_2, x, x_3) \Rightarrow R_\oplus(x_3, y_3, y_1)\Big) \Rightarrow R_\otimes(x_2, x_1, y_1)\Big] \Rightarrow x_1 \leqslant y_2\Big) \Rightarrow R_\oplus(x, y_2, y_1)\Big].$$

In order to be able to sketch the proof of this correspondence result, we work in the 'complex algebra' for these non-distributive structures. That is, we work in $\mathcal{G}(X, Y, \leqslant)$. Now the x's are completely join irreducible elements of $\mathcal{G}(X, Y, \leqslant)$ (atoms in the Boolean setting), the y's are completely meet irreducible elements of $\mathcal{G}(X, Y, \leqslant)$ (co-atoms), and A, B, C are general elements of $\mathcal{G}(X, Y, \leqslant)$. Also $\otimes, /, \backslash$ and $\oplus, \oslash, \obslash$ are operations on $\mathcal{G}(X, Y, \leqslant)$ specified by (1) and (2), where $A \vdash B$ means $A \leqslant B$, $x \Vdash A$ means $x \leqslant A$ and $y \succ A$ means $y \geqslant A$. The content of the completeness theorem for the basic Lambek-Grishin calculus is, in algebraic terms, that on $\mathcal{G}(X, Y, \leqslant)$ the connectives $\otimes, /, \backslash$ and $\oplus, \oslash, \obslash$ satisfy the rules of $LG_\emptyset$.

Take the axiom: $\forall A, B, C\ \Big(B \backslash (C \oplus A) \leqslant (B \backslash C) \oplus A\Big)$. A completeness result for this axiom would follow from the following combination: canonicity of this axiom, that is, if the axiom holds in an $LG_\emptyset$-algebra then it holds in its canonical extension *and* a correspondence result for this axiom, that is, a proof that the axiom holds in an algebra $\mathcal{G}(X, Y, \leqslant)$ if and only if the above first-order condition holds for the underlying frame $(X, Y, \leqslant)$. The second part of this is what we sketch here. The axiom holds in $\mathcal{G}(X, Y, \leqslant)$ if and only if

$$\forall x\ \forall A, B, C\ \Big(x \leqslant B \backslash (C \oplus A)\ \Rightarrow\ x \leqslant (B \backslash C) \oplus A\Big)$$

because every element of $\mathcal{G}(X, Y, \leqslant)$ is a join of completely join irreducibles. By residuation rules,

$$\forall x\ \forall A, B, C\ \Big(B \leqslant (C \oplus A) \,/\, x\ \Rightarrow\ x \leqslant (B \backslash C) \oplus A\Big) \tag{4}$$

This certainly implies

$$\forall x\ \forall A, C\ \Big(x \leqslant \Big([(C \oplus A) \,/\, x] \backslash C\Big) \oplus A\Big) \tag{5}$$

simply by instantiating $B = (C \oplus A) \,/\, x$. On the other hand, if (4) holds for some B_1, then for any $B_2 \leqslant B_1$ we have $x \leqslant (B_1 \backslash C) \oplus A \leqslant (B_2 \backslash C) \oplus A$, so (4) also holds for B_2. Thus, (4) is equivalent to (5). Analogously to the first step, (5) is equivalent to

$$\forall x, y\ \forall A, C\ \Big(y \geqslant A\ \Rightarrow\ x \oslash y \leqslant [(C \oplus A) \,/\, x] \backslash C\Big)$$

The particular case $y = A$ does the job for A_1 less or equal to A:

$$\forall x, y\ \forall C\ \Big(x \oslash y \leqslant [(C \oplus y) \,/\, x] \backslash C\Big) \tag{6}$$

By residuation, this is if and only if

$$\forall x, y\ \forall C\ \Big([(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant C\Big)$$

Equivalently,

$$\forall x, y, y' \; \forall C \; \Big(y' \geqslant C \;\Rightarrow\; y' \geqslant [(C \oplus y) \,/\, x] \otimes [x \oslash y] \Big) \tag{7}$$

Consider the conclusion of the implication: $y' \geqslant [(C \oplus y) \,/\, x] \otimes [x \oslash y] = \cup_{x' \leqslant C}[(x' \oplus y) \,/\, x] \otimes [x \oslash y]$. Then the conclusion holds if and only if $\forall x, x', y, y' \; \forall C \; \Big(x' \leqslant C \;\Rightarrow\; [(x' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y' \Big)$. Thus, (7) is equivalent to

$$\forall x, x', y, y' \; \forall C \; \Big(x' \leqslant C \leqslant y' \;\Rightarrow\; [(x' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y' \Big)$$

Removing C:

$$\forall x, x', y, y' \; \Big(x' \leqslant y' \;\Rightarrow\; [(x' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y' \Big)$$

Now we can remove y':

$$\forall x, x', y \; \Big([(x' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant x' \Big)$$

which, by residuation, is equivalent to

$$\forall x, x', y \; \Big((x' \oplus y) \,/\, x \leqslant x' \,/ (x \oslash y) \Big) \tag{8}$$

Notice that this is precisely the shape of the interaction postulate (e) listed in (3). What has happened however is that some of the second-order variables A, B, C have been replaced by variables from X, others by variables from Y. Is such a replacement always possible (at least in some of Grishin's groups)? If so, is this process understandable in some direct way? We observe that, in this case, the variable that occurred negatively in the original axiom (that is B) is the one that leads to the Y variable if we use Grishin's variable encoding for the class of equivalent axioms. To understand this better we will need to understand why Grishin encoded his variables as he did.

We now proceed with the purpose of obtaining a first-order condition in terms of the relations $R_\otimes$ and $R_\oplus$. We start from (6) as this seems to lead to a condition with less variables than translating in the most straightforward way from (8). We replaced y with y_1.

By residuation, (6) is equivalent to

$$\forall x, y_1 \; \forall C \; \Big[x \leqslant \Big[([C \oplus y_1] \,/\, x) \setminus C \Big] \oplus y_1 \Big]$$

I.e.

$$\forall x, y_1, y_2 \; \Big[[(C \oplus y_1) \,/\, x] \setminus C \leqslant y_2 \;\Rightarrow\; x \leqslant y_2 \oplus y_1 \Big]$$

Replacing the premiss of the implication gives us

$$\forall x, y_1, y_2 \; \Big[\forall x_1 \; \Big(x_1 \leqslant ([C \oplus y_1] \,/\, x) \setminus C \;\Rightarrow\; x_1 \leqslant y_2 \Big) \;\Rightarrow\; x \leqslant y_2 \oplus y_1 \Big] \tag{9}$$

Let us consider $x_1 \leqslant ([C \oplus y_1] \,/\, x) \setminus C$ separately. It is equivalent to

$$\forall x_2 \; \Big[x_2 \leqslant (C \oplus y_1) \,/\, x \;\Rightarrow\; x_1 \leqslant x_2 \setminus C \Big]$$

Applying residuation twice,

$$\forall x_2 \left[x_2 \otimes x \leqslant C \oplus y_1 \ \Rightarrow \ x_2 \otimes x_1 \leqslant C \right]$$

$$\forall x_2 \left[(x_2 \otimes x) \oslash y_1 \leqslant C \ \Rightarrow \ x_2 \otimes x_1 \leqslant C \right]$$

Now we can remove C:

$$\forall x_2 \left[x_2 \otimes x_1 \leqslant (x_2 \otimes x) \oslash y_1 \right]$$

Now let us insert it into (9):

$$\forall x, y_1, y_2 \left[\forall x_1 \left(\forall x_2 \left[x_2 \otimes x_1 \leqslant (x_2 \otimes x) \oslash y_1 \right] \ \Rightarrow \ x_1 \leqslant y_2 \right) \ \Rightarrow \ x \leqslant y_2 \oplus y_1 \right]$$

The following transformations are to reformulate in terms of relations:

$$\forall x, y_1, y_2 \left[\forall x_1 \left(\forall x_2 \left[\forall y_3 \left((x_2 \otimes x) \oslash y_1 \leqslant y_3 \ \Rightarrow \ x_2 \otimes x_1 \leqslant y_3 \right) \right] \ \Rightarrow \ x_1 \leqslant y_2 \right) \ \Rightarrow \ R_{\oplus}(x, y_2, y_1) \right];$$

$$\forall x, y_1, y_2 \left[\forall x_1 \left(\forall x_2 \left[\forall y_3 \left(\forall x_3 \left[x_3 \leqslant x_2 \otimes x \ \Rightarrow \ x_3 \oslash y_1 \leqslant y_3 \right] \ \Rightarrow \ R_{\otimes}(x_2, x_1, y_3) \right) \right] \ \Rightarrow \right. \right.$$
$$\left. \left. x_1 \leqslant y_2 \right) \ \Rightarrow \ R_{\oplus}(x, y_2, y_1) \right]$$

Rewriting the same condition in relation terms gives us

$$\forall x, y_1, y_2 \left[\forall x_1 \left(\forall x_2 \left[\forall y_3 \left(\forall x_3 \left[R^{\downarrow}_{\otimes}(x_2, x, x_3) \ \Rightarrow \ R_{\oplus}(x_3, y_3, y_1) \right] \ \Rightarrow \ R_{\otimes}(x_2, x_1, y_3) \right) \right] \ \Rightarrow \right. \right.$$
$$\left. \left. x_1 \leqslant y_2 \right) \ \Rightarrow \ R_{\oplus}(x, y_2, y_1) \right]$$

5 Conclusion

In this paper we presented our work in progress on generalised Kripke semantics for the Lambek-Grishin calculus. At this stage, we know that the soundness and completeness theorem for the basic calculus $LG_{\emptyset}$ is easily obtainable from the analogous result for the non-associative Lambek calculus. A more interesting piece of work concerns interaction postulates: we have obtained a correspondence result for one of them. Questions for future exploration include:

- Correspondence results for other interaction axioms. In particular, we would like to understand when and how one can just replace the second-order variables by first-order variables as is the case in the example treated above.
- The canonicity of this and various other interaction axioms. We conjecture that all of Grishin's well-behaved classes are canonical but that other axioms might not be. This might be interesting for understanding the relative expressivity of traditional Kripke semantics versus these generalised Kripke semantics.
- The relative strength and content of various *groups* of Grishin postulates.