# Lattice Theory — Fall 2009

# Worksheet 1

The exercises marked with * should be turned in.

1. Let $P$ be a set on which a binary relation $<$ is defined such that, for all $x, y, z \in P$,
   (a) $x < x$ is false,
   (b) $x < y$ and $y < z$ implies $x < z$.
   Prove that if $\leq$ is defined by
   $$x \leq y \iff (x < y \text{ or } x = y),$$
   then $\leq$ is a partial order on $P$, and moreover every partial order on $P$ arises from a relation $<$ satisfying (a) and (b). (A binary relation satisfying (a) and (b) is called a strict order.)

2. * There are 16 Hasse diagrams that represent all the possible 4 element partially ordered sets. Draw them.

3. Go through the examples from social and computer science and for each of 1.7-1.13 write down what the basic partial order is.

4. * Read the section on down-sets and up-sets (pages 20-23). Let $P$ be the power set of $\{1, 2, 3\}$ ordered by inclusion.
   (a) Draw the Hasse diagram of $P$. Show in particular that it has 8 elements. Starting from the bottom and moving from left to right on each layer, label them a,b,c,d,e,f,g,h.
   (b) Find $\mathcal{O}(P)$ and draw the Hasse diagram of $\mathcal{O}(P)$.

5. Draw the Hasse diagram for the poset
   $$Q = \{0\} \cup \{2^n 3^m \mid 0 \leq n, m \leq 5\}$$
   with the order of divisibility.

6. * Let $(Q, \preccurlyeq)$ be a *quasi-ordered* set, that is, $\preccurlyeq$ is a reflexive and transitive relation on $Q$.

   (a) Show that the relation $\cong = (\preccurlyeq \cap (\preccurlyeq)^\partial)$ is an equivalence relation;

(b) On the quotient $P = Q/\cong$, define $\leq$ by

$$[q_1] \leq [q_2] \quad \iff \quad q_1 \preccurlyeq q_2.$$

Show that $\leq$ is well-defined and a partial order.

7. * Given a partial order $\leq$ on a set $X$, the *dual order*, $\leq^\partial$, is defined by

$$x \leq^\partial y \quad \iff \quad x \geq y.$$

Given a property $\Phi$ about a partial order $\leq$, the *dual property*, denoted $\Phi^\partial$ is the property defined by

$$\Phi^\partial \text{ holds about } \leq \quad \iff \quad \Phi \text{ holds about } \leq^\partial.$$

(a) Show that if $\leq$ is a partial order on $X$, then so is the relation $\leq^\partial$;

(b) Identify the order dual properties for each of the following: maximum element of $S \subseteq X$, and least upper bound or $\bigvee$ of $S \subseteq X$;

(c) Prove the Order Duality Principle: If a property is true of all partial orders, then so is the dual property;

(d) Prove that both maxima and minima of subsets of posets, when they exist, are unique;