Universal Algebra

Syllabus

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Book:
A course in Universal Algebra
by Stanley Burris and H. P. Sankappanavar
Millenium Edition
Springer
Available from: `http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`

A rough outline of the course, which is subject to change as we go, is given on the next page. The exercises listed are automatically part of the assignment when we have covered the corresponding material. In addition I will be assigning some additional exercises that will be added to this document as we go (see page 3 and beyond).

Course outline:
Chapter II:
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- Sections 1, 2, (3), 5.

Section 3 is on lattice theory as it relates to the study of algebraic structures. It turns out that some properties of particular types of algebras (eg groups, rings, ...) actually hold because the lattices of subalgebras or of quotients of those algebras have certain lattice theoretical properties. We may skip 3.5.

Exercises: 1.5, 1.6, 5.2-5.6, 5.8, 5.9, (5.10-5.12)*.
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- Section 6: this is a big part of the course on the general isomorphism theorems.

Exercises in Section 6: 1-6.
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- Section 7.

Exercises in Section 7: 2-5.
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- Section 8: We could possibly skip this section completely. I suggest just skipping material on simple algebras.

Exercises: 1, 3*, 4, 5*, 7*, 9*.
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- Section 9.

Exercises: 1, 5.
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- Section 10: This and the next section is really the meat of the course. We will skip 10.13.

Exercises: 1, 4, 5, 7, and prove Theorem 10.16 in detail.
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- Section 11: Main theorem!

Exercises: 1-3, 4*, 5*.
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- Section 14: Deductive system for the logic of algebraic equations. This is not a necessity and can be left out.

Exercises: 10, 12.
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As time permits, we may look at material in Chapter IV, sections 5 and 6.

Additional exercises:

1. If $\beta : B \to A$ is an injective homomorphism, then $\Im(\beta)$ is a subalgebra of A isomorphic to B. Conversely, if B is a subalgebra of A then the inclusion map $i : B \to A$ is an injective homomorphism.

2. Prove each of the following:
(a) $X \subseteq E(X) \subseteq E^2(X) \subseteq \ldots \subseteq Sg(X)$;
(b) $Sg(X) = \bigcup_{n \geq 0} E^n(X)$;
(c) $\forall n \geq 0 \quad \forall a \in E^n(X) \quad \exists Y \subseteq X$ with Y finite and $a \in E^n(Y) \subseteq Sg(X)$.

3. Let $c : L \to L$ be a closure operator on a complete lattice. Show that the set of closed elements, $Cl_c(L)$, contains the top of L and is closed under infima taken in L. Conclude that $Cl_c(L)$ is a complete lattice with $\bigwedge_{Cl_c(L)} = \bigwedge_L$. Give an example to show that $\bigvee$ in L and in $Cl_c(L)$ may be different.

4. Find a kind of algebra (type plus properties) that 'occurs naturally' (preferably in something you have been or are studying) in which congruences are not determined by one of their equivalence classes.

5. Show that for a ring R, the principal congruences correspond to the principal ideals, and the finitely generated congruences correspond to the finitely generated ideals.

6. Derive the Principle of Noetherian Induction.

7. Let $(M, \cdot, 1)$ be a monoid. Show that there exists a type $\mathcal{F}$ and an algebra A of type $\mathcal{F}$ so that $(M, \cdot, 1) \cong (End(A), \circ, \mathsf{id}_A)$.

8. If $i : X \to A$ and $i' : X \to A'$ are two absolutely free algebras of type $\mathcal{F}$ over X, then they are isomorphic. That is, there is an $\mathcal{F}$-isomorphism between A and A' which commutes with the maps i and i'.

9. Prove the Formation Lemma.

10. Prove that if $f : X \to F$ is free over X for a class $\mathcal{K}$ of algebras of type $\mathcal{F}$, the algebra F is in $\mathcal{K}$, and $\mathcal{K} = S(\mathcal{K})$, then F is generated by X (strictly speaking by $Im(f)$) as an $\mathcal{F}$-algebra.

11.* (a) Give an example of a class $\mathcal{K}$ of algebras having an algebra that is free for $\mathcal{K}$ over each set X, but for which all these free algebras are isomorphic.
(b) Let $\mathcal{K}$ be a class of similar algebras. Assume that $\mathcal{K}$ contains at least one non-trivial finite algebra. Let X and Y be finite sets. Suppose that $i : X \to F$ is free for $\mathcal{K}$ over X and that $i : Y \to G$ is free for $\mathcal{K}$ over Y. Prove that, $F \cong G$ if and only if X and Y have the same cardinality.

12.* Let $\mathcal{F} = \{\cdot, f, g\}$ with $\cdot$ binary and f and g unary operations. Let $\mathcal{K}$ be the variety of $\mathcal{F}$-algebras defined by the identities $f(x \cdot y) \approx x$, $g(x \cdot y) \approx y$, and $f(x) \cdot g(x)) \approx x$.
(a) Assuming that $i : \{0\} \to F$ is free for $\mathcal{K}$ over $1 = \{0\}$ (and that F belongs to $\mathcal{K}$), show that for each $n \in \mathbb{N}$, there exists $i_n : n \to F_n$ which is free for $\mathcal{K}$ over n and that $F \cong F_n$ for each $n \in \mathbb{N}$.
(b) Show that there exists a non-trivial algebra in $\mathcal{K}$.

13. Let A and B be $\mathcal{F}$-algebras.

(a) Let $C \subseteq A$ and $D \subseteq B$. Show that $C \times D \subseteq A \times B$; show by example that, in general, not every subalgebra of $A \times B$ need be of this form.

(b) Assume that there is a binary term $t(x, y)$ such that $t^A(a, a') = a$ for all $a, a' \in A$ and $t^B(b, b') = b'$ for all $b, b' \in B$. Prove that in this case, every subalgebra of $A \times B$ is of the form $C \times D$ with $C \subseteq A$ and $D \subseteq B$.

(c) Let G and H be finite groups of order m and n, respectively. Prove that if $\gcd(m, n) = 1$, then there exists a binary term t satisfying the condition of part (a). Hence every subgroup of $G \times H$ is a product of subgroups of G and H.