Resumo

As álgebras semi-De Morgan constituem uma variedade definida por H. P. Sankappanavar que inclui como subvariedades a variedade $\mathcal{K}_{1,1}$ das álgebras de Ockham e uma generalização da variedade dos reticulados distributivos pseudocomplementados também estudada pelo mesmo autor, a variedade dos reticulados semi-pseudocomplementados. Mais tarde D. Hobby desenvolveu uma dualidade para estas álgebras e definiu a maior subvariedade das álgebras semi-De Morgan com a propriedade de extensão de congruências, a variedade $\mathcal{C}$.

Neste trabalho são apresentadas desigualdades, mais simples que as encontradas por D. Hobby, que caracterizam a variedade $\mathcal{C}$ e determinam-se equações que definem as congruências principais desta variedade. Como aplicação, caracterizam-se as álgebras subdirectamente irredutíveis de $\mathcal{C}$ provando que para além das álgebras subdirectamente irredutíveis de $\mathcal{K}_{1,1}$ e da variedade dos reticulados semi-pseudocomplementados só existem, a menos de isomorfismo, três outras álgebras subdirectamente irredutíveis.

Explica-se de maneira detalhada um processo para obtenção de dualidades para álgebras com redutos reticulados distributivos que se baseia na canonicidade e na teoria de Sahlqvist e que foi obtido por M. Gehrke, H. Nagahashi e Y. Venema num artigo recente. Como aplicação determina-se uma nova dualidade para a variedade das álgebras semi-De Morgan e caracterizam-se os espaços duais de algumas das suas subvariedades.

Estudam-se as álgebras semi-De Morgan que só têm congruências principais, caracterizando-se os espaços duais dos reticulados semi-pseudocomplementados com esta propriedade. Generalizam-se resultados que dizem respeito ao supremo ou ao ínfimo de congruências principais, obtidos por R. Beazer para reticulados pseudocomplementados, à variedade dos reticulados semi-pseudocomplementados.

Palavras chave: Semi-De Morgan algebras, variedade $\mathcal{C}$, congruências principais, dualidade, canonicidade, Sahlqvist.

Abstract

Semi-De Morgan algebras are a subvariety, defined by H. P. Sankappanavar, that includes as subvarieties the variety $\mathcal{K}_{1,1}$ of Ockham algebras and a generalization of pseudocomplemented distributive lattices also studied by the same author, the variety of demi-pseudocomplemented lattices. Later D. Hobby developed a duality for these algebras and defined the largest subvariety of semi-De Morgan algebras with the congruence extension property, the variety $\mathcal{C}$.

In this work simpler inequalities than those found by D. Hobby to characterize the variety $\mathcal{C}$ are presented and equations defining principal congruences in this variety are determined. As an application, the subdirectly irreducible algebras of $\mathcal{C}$ are characterized and it is proved that apart from the subdirectly irreducible algebras of $\mathcal{K}_{1,1}$ and of the variety of demi-pseudocomplemented algebras there are, up to isomorphism, just three other subdirectly irreducible algebras.

It is spelled out a process of obtaining dualities for distributive lattice-ordered algebras based on canonicity and on the Sahlqvist theory that was developed by M. Gehrke, H. Nagahashi and Y. Venema in a recent article. As an application it is determined a new duality for the variety of semi-De Morgan algebras and the dual spaces of some of its subvarieties are characterized.

Semi-De Morgan algebras having only principal congruences are also studied and the dual spaces of demi-pseudocomplemented lattices with this property are characterized. Results on the principal join property and the principal intersection property that were obtained by R. Beazer for pseudocomplemented distributive lattices are generalized to the variety of demi-pseudocomplemented lattices.

Keywords: Semi-De Morgan algebras, variety $\mathcal{C}$, principal congruences, duality, canonicity, Sahlqvist.

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Contents

Introduction

Many varieties of algebras consisting of bounded distributive lattices with additional unary operations have been studied as algebraic models for certain logics. They include well known examples, such as Ockham, MS, De Morgan, Stone, and Boolean algebras as well as pseudo-complemented lattices.

For some of these varieties there are several similar properties so it is natural to look for generalizations where these properties still hold.

This is the case of De Morgan algebras (DMAs) and distributive pseudo-complemented lattices (also called p-lattices) where the presence of common aspects made Sankappanavar consider in [38] the more general variety of semi-De Morgan algebras ($SDMA$s).

Sankappanavar continued the investigation of $SDMA$s in [39] and [40], where he concentrated on the subvariety of demi-pseudocomplemented lattices (also called demi-p-lattices, ($DMPL$s)) and on almost pseudocomplemented lattices (also called almost p-lattices, (APLs)), a subvariety of demi-p-lattices. These are generalizations of p-lattices and they don't include the variety of De Morgan algebras.

Using algebraic techniques, Sankappanavar characterized in [38] some important subvarieties of $SDMA$s and proved that certain elements of a semi-De Morgan algebra form a De Morgan algebra, extending the well known theorem for p-lattices due to Glivenko.

When he restricted his study to $DMPL$ in [39] and [40], Sankappanavar determined equations defining principal congruences and, as an application, characterized the subdirectly irreducible algebras. He also determined the lattice of subvarieties of $DMPL$.

Since we started this study, our main goal has been the investigation of the corresponding results for the variety $SDMA$. We studied in [26] the lattice of congruences of subdirectly irreducible algebras in $SDMA$. However, our attempt to go further using algebraic methods was not fruitful since $SDMA$

does not have the congruence extension property (CEP).

Using topological methods, D. Hobby developed in [25] a duality for $SDMA$ based on the Priestley duality for distributive lattices. As an application, he characterized the dual spaces of subdirectly irreducible semi-De Morgan algebras and he determined the largest subvariety of $SDMA$ with the congruence extension property which he called the variety $\mathcal{C}$.

Hobby's duality is described by himself as "tractable enough to be useful" however with conditions that "seem inelegant" so, he suggests, as Problem 1 of his article, a duality for $SDMA$s "more nicely stated".

Since the variety $SDMA$ does not have the congruence extension property, Hobby observed that perhaps it is too large to be useful as a common generalization of DMAs and p-lattices. Therefore $\mathcal{C}$ turns out to be very interesting because it has CEP and it contains all the previously studied subvarieties of $SDMA$, namely the subvariety $K_{1,1}$ of Ockham algebras (consequently DMA) and $DMPL$.

However the two inequalities (α and β) that characterize $\mathcal{C}$ as a subvariety of $SDMA$ in [25] are too complicated to make the study of $\mathcal{C}$ very tempting.

In fact Problem 2 in [25] is to find "nicer axioms for $\mathcal{C}$".

We solved this problem algebraically determining a new inequality (γ) such that $\mathcal{C}$ can be characterized by γ and β.

We determined the equations defining principal congruences in $\mathcal{C}$ and we characterized the subdirectly irreducible algebras of this variety.

Blyth and Varlet characterized in [9] the distributive lattices, the Stone, the De Morgan and the Heyting algebras that have only principal congruences. In [6], Beazer solved the same problem for p-lattices. It was natural to ask whether their results could be extended. Using algebraic techniques, analogous to those used by Beazer in [5] and [6], we proved, in [27], that the semi-De Morgan algebras with the referred property are finite so that, when applying Hobby's duality, we could drop the topology because, in this case, it is the discrete one. Using this method we generalized to demi-p-lattices results obtained by Beazer in [5] and [6].

We continued the study of principal congruences, concentrating on algebras that have the principal join property (PJP), i.e. those algebras such that the join of any two principal congruences is a principal congruence. Beazer characterizes p-lattices with this property in [5]. In [12], I. Chada calls these algebras congruence principal and, in [13] he studies algebras whose principal congruences form a sublattice of their congruence lattice. In

[29] we apply Hobby's duality to the study of this property in what concerns demi-p-lattices and we prove, generalizing a result by Beazer [5], that in this variety those algebras having the principal join property have the principal intersection property. This solves, for demi-p-lattices, problem 6 proposed by Hobby in [25].

The difficulties we found when dealing with Hobby's duality motivated us to try to find a simpler alternative for this duality.

While searching for a duality for $SDMA$ we decided to follow the suggestions of Professor M. Gehrke and to apply Canonicity and the Sahlqvist Theory to this study.

Canonical extensions provide a complete lattice theoretic view of topological duality for various types of lattice and even poset-ordered algebras [18, 20, 22]. They were first developed by Jónsson and Tarski for Boolean algebras with additional operations that preserve join in each coordinate. More recently, generalizations and stronger results about preservation of identities under canonical extensions have been obtained by M. Gehrke and B. Jónsson and others. The paper [21] is particularly useful for our purposes. In this paper M. Gehrke, H. Nagahashi and Y. Venema use the modern theory of canonical extensions to generalize powerful results from modal logic about preservation of identities and relational correspondents for these equations. The results in that paper are cast as a generalization of Sahlqvist theory for certain generalized modal logics based on distributive lattices. However, as the authors also comment, these results may also be seen as a general theory for manufacturing topological dualities for the distributive lattice-ordered algebras corresponding to the logics they treat. In this work we will spell out the parts of this process and apply it in the case of the variety of semi-De Morgan algebras and various subvarieties of this variety.

In this work, to avoid confusion with De Morgan algebras, we denote by unary quasi-operators algebras (UQAs) the distributive modal algebras considered in [21] by M.Gehrke, H. Nagahashi and Y. Venema, since the unary operations $\Diamond, \Box, \rhd$ and $\lhd$ are called unary quasi-operators in [22].

We follow [21] to determine a duality between the canonical extensions of these algebras and the corresponding ordered relational structures. We obtain the dual spaces of UQAs by defining a topology in these ordered spaces.

It is clear that these results also apply to algebras that are the reducts of UQAs so we consider distributive algebras with the unary operations $\Box$ and

$\triangleright$. In this setting we can characterize the variety $SDMA$ introducing additional inequalities satisfied by the unary operations. These inequalities are, according to the definition in [21], Sahlqvist inequalities and hence canonical.

M.Gehrke, H. Nagahashi and Y. Venema proved in [21] that every Sahlqvist modal sequent corresponds to a formula in the dual frame. It is clear that the same happens to Sahlqvist inequalities and conditions in the dual space. We applied this result to compute quite easily the formulas corresponding to Sahlqvist inequalities.

Since $SDMA$, as well as important subvarieties such as $\mathcal{C}$, $DMPL$, $K_{1,1}$, can be defined by Sahlqvist inequalities, we established a duality for $SDMA$ and we characterized the dual spaces of the referred subvarieties.

The first chapter of this thesis contains definitions and results that will be needed later.

We assume familiarity with basic concepts of universal algebra and lattice theory.

In section 1.4 we present the definition of semi-De Morgan algebras and important properties of these algebras. For a more detailed exposition see [38], [39] and [40].

In chapter 2, we solve Problem 2 from [25]. We determine algebraically an inequality (γ) such that inequalities γ and β characterize $\mathcal{C}$ as a subvariety of $SDMA$. In Sections 2 and 3 we include results that were obtained in [28]: We characterize the principal congruences on $\mathcal{C}$, extending the corresponding characterization for demi-p-lattices, due to Sankappanavar [39], and for the variety $\mathcal{K}_{1,1}$ due to J. Berman [8] and to M. Ramalho and M. Sequeira [33]. It is shown that $\mathcal{C}$ has equationally definable principal congruences, a result which strengthens Hobby's result that this variety has congruence extension property. We also determine the subdirectly irreducible algebras of the variety $\mathcal{C}$. The subdirectly irreducible demi-p-lattices were characterized by Sankappanavar in [39] and the subdirectly irreducible algebras of the variety $\mathcal{K}_{1,1}$ were identified in [37] and also in [5]. We use these results and the characterization of principal congruences to prove that apart from the subdirectly irreducible algebras of the varieties of demi-p-lattices and $\mathcal{K}_{1,1}$ there are, up to isomorphisms, three more subdirectly irreducible algebras in $\mathcal{C}$. We consider the set consisting of the isomorphism classes of finite subdirectly irreducible algebras of the variety $\mathcal{C}$ and we present the Hasse diagram of this poset. Using a theorem of B. Davey [16], we prove that the lattice of subvarieties of $\mathcal{C}$ is isomorphic to the lattice of order-ideals of this poset. In

Section 4 we give defining identities for some subvarieties of $\mathcal{C}$.

Chapters 3 and 4 concern the study of duality based on the theory of canonical extensions. This is a general program that will be illustrated here.

In chapter 3 we will give some results on canonical extensions for distributive lattices ($\mathcal{DL}$). As our results may be seen mainly as an application of the results in [21], we will refer to this paper and we will try to conform to the notation and nomenclature used there. For further references on canonical extensions we will refer to the most recent and comprehensive paper by Gehrke and Jónsson [20] even for results first proved earlier.

In section 3.2 we consider the class of Perfect Distributive Lattices ($\mathcal{DL}^+$). These are completely distributive lattices that are join generated by the set of completely join irreducible elements and they include the class of canonical extensions of distributive lattices.

Following [21], we establish a duality between $\mathcal{DL}^+$ and posets: Given a lattice in $\mathcal{DL}^+$ the completely join irreducible elements form a poset and, given a poset, the downsets form a lattice in $\mathcal{DL}^+$. In this way it is obtained a generalization of Birkhoff's duality for finite distributive lattices.

At the objects level this was already done by G. Raney [34], V. Balachandran [3] and P. Dwinger [4].

When we expand perfect distributive lattices with additional unary operations ($\diamond, \Box, \rhd$ and $\lhd$), we obtain the class of algebras that we denote by UQA^+ and we explain how in [21] the authors determine the corresponding duality endowing the dual posets with binary relations.

The fact that a lattice in $\mathcal{DL}$ is a lattice in $\mathcal{DL}^+$ has important consequences that we discuss in section 3.3.

The duality for the category $\mathcal{DL}^+$ is applied in section 3.4 to define a duality for bounded distributive lattices ($\mathcal{DL}$). This is done by introducing a topology in the dual poset of completely join irreducible elements of the canonical extensions of lattices in $\mathcal{DL}$. In the end what we obtain is a version of Priestley duality.

This kind of approach of Priestley duality for distributive lattices has, among others, the advantage of defining the dual space as a subset of the canonical extension of the distributive lattice. Besides, when dealing with finite distributive lattices, we fall directly in Birkhoff's duality.

To determine a duality for UQA, we have just to find how the topology in the Priestley space (dual of the underlying distributive lattice) interacts with the additional binary relations in the dual space of UQA^+. This is done in detail for the join preserving operation and then generalized to the other unary operations using the appropriate order duals of the distributive lattice.

Using Priestley duality, R. Goldblatt developed in [23], a representation for distributive lattices with operators that are meet or join preserving. Later, in [41], V. Sofronie-Stokermans generalized this duality to meet or join reversing operators. Though we work in a different setting their results were very useful.

As an application of this duality, we develop in chapter 4 a duality for $SDMA$. We start by considering, in section 4.1, a class of distributive lattices extended with the unary operations $\rhd$ and $\Box$. This is a reduct of UQA where we define the variety $SDMA$ by a set of inequalities that are satisfied by the unary operations.

These inequalities are Sahlqvist so, applying results established by M. Gehrke, H. Nagahashi and Y. Venema in [21], we conclude that they are canonical and hence $SDMA$ is a canonical variety. Therefore the canonical extension of an algebra in $SDMA$ is still in $SDMA$. In fact it is in a class of algebras that we call $SDMA^+$ which is the intersection of UQA^+ and $SDMA$. Since in the previous chapter we have already established a duality for UQA^+, we apply this duality and the Sahlqvist theory in [21] to compute the formulas corresponding to the Sahlqvist inequalities in the dual space. This way we characterize the binary relations that correspond to $\rhd$ and $\Box$ in the dual structure of $SDMA^+$. But these two relations are not independent so we can define morphisms between dual structures of $SDMA^+$ by less conditions than in UQA^+.

The minimal elements in the codomain of one of the binary relations are the maximal elements in the codomain of the other so that we can define a new binary relation having as codomain this set of elements. In section 4.2, we consider the case of algebras in $SDMA^+$ that are canonical extensions of algebras in $SDMA$. Then, this new binary relation is particularly interesting because we can obtain a much simpler duality for $SDMA^+$.

To capture a duality for $SDMA$ we have, as for UQA, to consider the topology in the dual space and to determine how the new binary relation behaves regarding this topology. This way we finish by establishing a full duality for $SDMA$.

As an application of this duality, we characterize, in section 4.3, the dual spaces of some important subvarieties of $SDMA$ that are defined by Sahlqvist inequalities and compare the duality we have established with the correspondent known dualities.

In chapter 5, applying the duality for $SDMA$s presented in the previous chapter, we study the properties of principal congruences in $SDMA$ considered in [27] and [29]. Generalizing results obtained by R. Beazer in [5] and [6], we show that $SDMA$s having only principal congruences are finite. Next, using duality, we show how demi-p-lattices and almost p-lattices having only principal congruences can be described.

We also characterize those demi-p-lattices having the principal join property extending the corresponding results obtained by Beazer in [5]. We also prove that those algebras in $DMPL$ having the principal join property have the principal intersection property.

We observe that some of the results presented in chapters 2 and 5 were joint work with Professor Raquel Santos.

Introdução

Muitas variedades de álgebras tais como as álgebras de Ockham, as álgebras MS, de De Morgan, de Stone e de Boole bem como os reticulados pseudocomplementados são exemplos bem conhecidos de álgebras tendo como reduto um reticulado distributivo limitado com uma operação unária adicional.

Nalgumas destas variedades, que têm sido estudadas como modelos algébricos de certas lógicas, há propriedades que são semelhantes pelo que é natural procurar generalizações em que estas propriedades se continuam a verificar.

É este o caso das álgebras de De Morgan (DMAs) e dos reticulados distributivos pseudocomplementados (também designados por reticulados-p) onde a presença de aspectos comuns levou Sankappanavar a considerar em [38] a variedade mais geral das álgebras semi-De Morgan ($SDMA$s).

Em [39] e em [40], Sankappanavar continuou a investigação sobre $SDMA$s concentrando o seu estudo nas subvariedades dos reticulados semi-pseudocomplementados (também designados por reticulados semi-p, ($DMPL$s)) e dos reticulados quase-pseudocomplementados (também designados por reticulados quase-p, (APLs)). Ambas são generalizações dos reticulados-p que não incluem a variedade das álgebras de De Morgan.

Usando técnicas algébricas, Sankappanavar caracterizou em [38] algumas subvariedades importantes de $SDMA$ e provou que determinados elementos de uma álgebra semi-De Morgan formam uma álgebra de De Morgan, extendendo assim o bem conhecido teorema de Glivenko para reticulados-p.

Quando restringiu o seu estudo a $DMPL$ em [39] e em [40], Sankappanavar determinou equações que definem as congruências principais e, como aplicação, caracterizou as álgebras subdirectamente irredutíveis desta variedade. Determinou, também, o reticulado das subvariedades de $DMPL$.

Quando iniciámos este estudo, o nosso principal objectivo era a investigação dos resultados correspondentes para a variedade $SDMA$. Estudámos em [26] o reticulado das congruências das álgebras subdirectamente irredutíveis de $SDMA$. Contudo, as tentativas para ir mais além usando métodos algébricos foram infrutíferas porque $SDMA$ não tem a propriedade de extensão de congruências (CEP).

Usando métodos topológicos, D. Hobby desenvolveu em [25] uma dualidade para $SDMA$ baseada na dualidade de Priestley para reticulados distributivos. Como aplicação desta dualidade, caracterizou os espaços duais das álgebras semi-De Morgan subdirectamente irredutíveis e determinou a maior subvariedade de $SDMA$ com a propriedade de extensão de congruências que designou por variedade $\mathcal{C}$.

A dualidade de Hobby é descrita por ele próprio como "suficientemente tratável para ser útil" contudo com condições que "parecem deselegantes". Em consequência, como Problema 1 do seu artigo, Hobby sugere a determinação de uma nova dualidade para $SDMA$s.

Como a variedade $SDMA$ não tem a propriedade de extensão de congruências, Hobby observou que talvez seja grande demais para ser útil como generalização comum das DMAs e dos reticulados-p. A variedade $\mathcal{C}$ torna-se portanto muito interessante porque tem CEP e contém todas as subvariedades de $SDMA$ previamente estudadas, nomeadamente a subvariedade $K_{1,1}$ das álgebras de Ockham (logo DMA) e $DMPL$. Contudo as desigualdades α e β que caracterizam $\mathcal{C}$ como subvariedade de $SDMA$ em [25] são excessivamente complicadas para tornar o estudo de $\mathcal{C}$ tentador.

De facto o Problema 2 de [25] é a determinação de axiomas mais elegantes para $\mathcal{C}$.

Resolvemos este problema algebricamente determinando uma nova desigualdade (γ) tal que $\mathcal{C}$ é caracterizável por γ e β.

Determinámos também equações que definem as congruências principais em $\mathcal{C}$ e caracterizámos as álgebras subdirectamente irredutíveis desta variedade.

Os reticulados distributivos, as álgebras de Stone, de De Morgan e de Heyting que só têm congruências principais foram caracterizados por Blyth e Varlet em [9]. Beazer resolveu o mesmo problema para reticulados-p em [6]. Pôs-se assim naturalmente a questão de saber se estes resultados são generalizáveis. Aplicando técnicas algébricas análogas às usadas por Beazer em [5] e em [6], provámos, em [27], que as álgebras semi-De Morgan que só têm congruências principais são finitas logo, ao aplicar a dualidade de Hobby, pudemos ignorar a topologia uma vez que neste caso é a topologia discreta. Usando este método generalizámos a reticulados semi-p resultados obtidos por Beazer em [5] e em [6].

Continuando o estudo das congruências principais, considerámos as álgebras tais que o supremo de quaisquer duas congruências principais é uma congruência principal (álgebras com a propriedade PJP). Beazer caracteriza os reticulados-p com esta propriedade em [5]. I. Chada [12], chama congruentes principais a estas álgebras e, em [13], estuda álgebras cujas congruências principais formam um subreticulado do reticulado das congruências. Em [29], aplicámos a dualidade de Hobby para estudar esta propriedade nos reticulados semi-p e provámos, generalizando um resultado de Beazer [5], que nesta variedade as álgebras que têm a propriedade PJP têm também a propriedade de a intersecção de quaisquer duas congruências principais ser uma congruência principal (propriedade PIP). Resolvemos assim para reticulados semi-p o Problema 6 proposto por Hobby em [25].

As dificuldades encontradas ao aplicar a dualidade de Hobby motivaram o estudo de uma alternativa mais simples a esta dualidade.

Ao procurar uma dualidade para $SDMA$ decidimos aceitar a sugestão da Professora M. Gehrke e aplicar a Canonicidade e a Teoria de Sahlqvist a este estudo.

As extensões canónicas foram desenvolvidas inicialmente por Jónsson e Tarski para álgebras Booleanas com operações adicionais que preservam o supremo em cada coordenada. Mais recentemente, generalizações e resultados mais fortes sobre a preservação de identidades em extensões canónicas foram obtidos por M. Gehrke e B. Jónsson e por outros. O artigo [21] é particularmente útil para os nossos objectivos. Neste artigo M. Gehrke, H. Nagahashi e Y. Venema usam a moderna teoria das extensões canónicas para generalizar resultados poderosos da lógica modal sobre a preservação de identidades e as condições relacionais que lhes correspondem. Os resultados desse artigo são apresentados como uma generalização da teoria de Sahlqvist para certas lógicas modais generalizadas baseadas em reticulados distributivos. Contudo, como os autores também comentam, estes resultados podem ser vistos como uma teoria geral para obter dualidades topológicas para as álgebras com redutos reticulados distributivos correspondentes às lógicas consideradas. Neste trabalho explicamos detalhadamente as partes deste processo e aplicamo-lo ao caso da variedade das álgebras semi-De Morgan e de algumas das suas subvariedades.

Para evitar confusão com as álgebras de De Morgan (DMAs), ao longo deste trabalho chamamos álgebras de quase-operadores unários (UQAs) às álgebras modais distributivas consideradas em [21] por M.Gehrke, H. Nagahashi e Y. Venema, uma vez que as operações unárias $\Diamond, \Box, \triangleright$ and $\triangleleft$ são designadas por quase-operadores unários em [22].

Seguimos [21] para determinar uma dualidade entre as extensões canónicas destas álgebras e certas estruturas relacionais ordenadas. Obtemos os espaços duais das UQAs definindo uma topologia nestas estruturas relacionais ordenadas.

Como estes resultados também se aplicam a álgebras que são redutos de UQAs, podemos considerar álgebras distributivas com as operações unárias $\Box$ e $\triangleright$. Neste contexto a variedade $SDMA$ pode ser caracterizada por meio de desigualdades adicionais nas operações unárias. Estas desigualdades são, de acordo com a definição dada em [21], desigualdades de Sahlqvist e portanto são canónicas.

Foi provado em [21] que toda a sequência modal de Sahlqvist corresponde a uma fórmula na estrutura dual. Como o mesmo acontece com desigualdades de Sahlqvist e condições no espaço dual, aplicámos este resultado para deduzir, com bastante facilidade, as fórmulas que correspondem às desigualdades de Sahlqvist.

Tanto $SDMA$ como subvariedades importantes tais como $\mathcal{C}$, $DMPL$ e $K_{1,1}$ podem ser definidas por desigualdades de Sahlqvist. Estabelecemos assim uma dualidade para $SDMA$ e caracterizamos o espaço dual das referidas subvariedades.

O primeiro capítulo desta tese contém definições e resultados que serão necessários posteriormente.

Pressupomos familiariedade com conceitos básicos de Álgebra Universal e da Teoria dos Reticulados.

Na secção 1.4 apresentamos a definição de álgebra semi-De Morgan e algumas propriedades importantes destas álgebras. Para informação mais detalhada sugerimos [38], [39] and [40].

No Capítulo 2, resolvemos o Problema 2 de [25]. Determinamos algebricamente uma desigualdade (γ) tal que as desigualdades γ and β caracterizam $\mathcal{C}$ como subvariedade de $SDMA$.

Nas Secções 2 e 3 incluimos resultados que foram obtidos em [28]: Caracterizamos as congruências principais em $\mathcal{C}$, generalizando a caracterização correspondente para reticulados semi-p obtida por Sankappanavar [39], e para a variedade $\mathcal{K}_{1,1}$ obtida por J. Berman [8] e por M. Ramalho e M. Sequeira [33]. Mostra-se que $\mathcal{C}$ tem congruências principais equacionalmente definíveis, um resultado que confirma que esta variedade tem a propriedade de extensão de congruências como Hobby provou. Determinamos também as álgebras subdirectamente irredutíveis da variedade $\mathcal{C}$. Os reticulados semi-p subdirectamente irredutíveis foram caracterizados por Sankappanavar em [39] e as álgebras subdirectamente irredutíveis da variedade $\mathcal{K}_{1,1}$ foram identificadas em [37] e também em [5]. Usamos estes resultados e a caracterização das congruências principais para provar que para além das álgebras subdirectamente irredutíveis das variedades dos reticulados semi-p e de $\mathcal{K}_{1,1}$ há, a menos de isomorfismo, mais três álgebras subdirectamente irredutíveis em $\mathcal{C}$. Apresentamos o diagrama de Hasse do conjunto parcialmente ordenado das classes de isomorfismo das álgebras finitas subdirectamente irredutíveis da variedade $\mathcal{C}$. Usando um teorema de B. Davey [16], provamos que o reticulado das subvariedades de $\mathcal{C}$ é isomorfo ao reticulado dos semi-ideais deste conjunto.

Na secção 2.4 apresentamos identidades que caracterizam algumas subvariedades de $\mathcal{C}$.

Os capítulos 3 e 4 incluem o estudo da dualidade baseada na teoria das extensões canónicas.

No capítulo 3 damos alguns resultados sobre extensões canónicas de reticulados distributivos ($\mathcal{DL}$). Como os nossos resultados podem ser considerados principalmente como uma aplicação dos resultados obtidos em [21], fazemos referência a este artigo e usamos notação e nomenclatura adaptada à que é aí usada. Para outras referências relativas a extensões canónicas recorremos ao artigo mais recente e mais abrangente de Gehrke e Jónsson [20] mesmo para resultados já provados anteriormente.

Na secção 3.2 consideramos a classe dos Reticulados Distributivos Perfeitos ($\mathcal{DL}^+$). São reticulados completamente distributivos que são sup-gerados pelo conjunto dos elementos completamente sup-irredutíveis e incluem a classe das extensões canónicas dos reticulados distributivos.

De acordo com [21], estabelecemos uma dualidade entre $\mathcal{DL}^+$ e conjuntos parcialmente ordenados: Dado um reticulado em $\mathcal{DL}^+$, os elementos completamente sup-irredutíveis constituem um conjunto parcialmente ordenado e, dado um conjunto parcialmente ordenado, os semi-ideais formam um reticulado em $\mathcal{DL}^+$. Deste modo obtem-se uma generalização da dualidade de Birkhoff para reticulados distributivos finitos.

Isto já havia sido feito, ao nível dos objectos, por G. Raney [34], V. Balachandran [3] e P. Dwinger [4].

Quando se expandem reticulados distributivos perfeitos com operações unárias adicionais ($\Diamond, \Box, \rhd$ and $\lhd$), obtem-se a classe de álgebras que designamos por UQA^+. Explicamos como em [21] M. Gehrke, H. Nagahashi e Y. Venema determinam a dualidade correspondente munindo os conjuntos parcialmente ordenados duais com relações binárias.

O facto de um reticulado em $\mathcal{DL}$ ser um reticulado em $\mathcal{DL}^+$ tem consequências importantes que discutimos na secção 3.3.

A dualidade para a categoria $\mathcal{DL}^+$ é aplicada na secção 3.4 para definir uma dualidade para reticulados distributivos limitados ($\mathcal{DL}$). Para o fazer introduzimos uma topologia no conjunto parcialmente ordenado dos elementos completamente sup-irredutíveis, dual da extensão canónica dum reticulado de $\mathcal{DL}$. No fim o que se obtém é uma versão da dualidade de Priestley.

Este tipo de abordagem da dualidade de Priestley para reticulados distributivos tem, entre outras, a vantagem de definir o espaço dual como um subconjunto da extensão canónica do reticulado distributivo e de, quando se consideram reticulados distributivos finitos, se ir cair directamente na dualidade de Birkhoff.

Para determinar uma dualidade para UQA, basta estudar o modo como a topologia no espaço de Priestley (dual do reticulado distributivo subjacente) interage com as relações binárias adicionais do espaço dual de UQA^+.

Este estudo é feito em detalhe para a operação que preserva o supremo e generalizado depois às outras operações unárias usando os duais de ordem apropriados.

Usando a dualidade de Priestley, R. Goldblatt desenvolveu em [23] uma representação para reticulados distributivos com operadores que preservam o supremo ou o ínfimo. Mais tarde, em [41], V. Sofronie-Stokermans generalizou esta dualidade a operadores que invertem o supremo ou o ínfimo. Apesar de trabalharmos num contexto diferente os resultados obtidos por estes autores foram-nos muito úteis.

Como aplicação da dualidade que obtivemos, desenvolvemos no capítulo 4 uma dualidade para $SDMA$. Começamos por considerar, na secção 4.1, uma classe de reticulados distributivos com as operações unárias $\rhd$ e $\Box$. Trata-se de um reduto de UQA onde definimos a variedade $SDMA$ por um conjunto de desigualdades que têm que ser satisfeitas pelas operações unárias. Estas são desigualdades de Sahlqvist e portanto, aplicando resultados estabelecidos por M. Gehrke, H. Nagahashi e Y. Venema em [21], concluímos que são canónicas logo $SDMA$ é uma variedade canónica. Consequentemente a extensão canónica de uma álgebra em $SDMA$ está ainda em $SDMA$. De facto está numa classe de álgebras a que chamamos $SDMA^+$ e que é a intersecção de UQA^+ e $SDMA$. Como no capítulo anterior já tinhamos estabelecido uma dualidade para UQA^+, aplicamos esta dualidade e a teoria de Sahlqvist de [21] para calcular as fórmulas que correspondem, no espaço dual, às desigualdades. Deste modo caracterizamos as relações binárias que correspondem a $\triangleright$ e a $\square$ na estrutura dual de $SDMA^+$. Como estas duas relações não são independentes podemos definir os morfismos entre estruturas duais de $SDMA^+$ por menos condições que em UQA^+.

Os elementos minimais do codomínio de uma das relações binárias são os elementos maximais do codomínio da outra de modo que é possível definir uma nova relação binária tendo como codomínio este conjunto de elementos. Na secção 4.2, consideramos o caso de álgebras de $SDMA^+$ que são extensões canónicas de álgebras de $SDMA$. Neste caso esta nova relação binária é particularmente interressante porque permite obter uma dualidade muito mais simples para $SDMA^+$.

Tal como em UQA, para obter uma dualidade para $SDMA$ consideramos a topologia do espaço dual e determinamos o modo como a nova relação binária se comporta relativamente a esta topologia.

Como aplicação desta dualidade, caracterizamos, na secção 4.3, os espaços duais de algumas subvariedades importantes de $SDMA$ que são definidas por desigualdades de Sahlqvist e comparamos a dualidade que estabelecemos com as correspondentes dualidades já conhecidas.

No capítulo 5, com a dualidade para $SDMA$s que apresentámos no capítulo anterior, estudamos as propriedades das congruências principais em $SDMA$ consideradas em [27] e [29]. Assim, generalizando resultados obtidos por Beazer em [5] e em [6], mostramos que $SDMA$s que só têm congruências principais são finitas e esclarecemos como podem ser descritos os reticulados semi-p e quase-p que só têm congruências principais.

Caracterizamos, também, os reticulados semi-p com a propriedade PJP fazendo uma extensão dos resultados correspondentes obtidos por Beazer em [5]. Provamos que as álgebras de $DMPL$ com a propriedade PJP têm ainda a propriedade PIP.

De notar que parte dos resultados referidos nos capítulos 2 e 5 foram um trabalho conjunto com a Professora Raquel Santos.

Chapter 1

Preliminaries

We assume that the basic notions of Universal Algebra and Lattice Theory are known. Anyway we list here some definitions and results that will be directly related with our study.

For more details concerning these subjects, we refer the reader to R. Balbes and P. Dwinger [4], S. Burris and H. P. Sankappanavar [11], B. A. Davey and H. A. Priestley [17].

1.1 Ordered Structures

Let $(X, \leq)$ be a partially ordered set *(poset)* and let $x, y \in X$. We say that *x is covered by y* and we write $x \ll y$ (or $y \gg x$) if $x < y$ and there is no element $z \in X$ for which $x < z < y$.

A subset S of a poset X is called *convex* if $x, y \in S$ and $z \in X$ and $x \leq z \leq y$ imply that $z \in S$.

If X is a poset then we say that X has *height* less than or equal to $k \in \mathbb{N}$ and we write $h(X) \leq k$ if every chain in X has at most $k+1$ elements.

Notice that a partially ordered set X is such that $h(X) \leq 1$ if and only if all its subsets are convex.

Let $(X, \leq)$ be a poset and let $S \subseteq X$. We say that S is a *downset* (or an *order ideal*) if, whenever $x \in S$, $y \in X$ and $y \leq x$, we have $y \in S$. Dually, S is an *upset* (or an *order filter*) if, whenever $x \in S$, $y \in X$ and $x \leq y$, we have $y \in S$.

Given an arbitrary subset $S \subseteq X$, we define

$$\downarrow S = \{y \in X : \exists x \in S \, y \leq x\} \text{ and } \uparrow S = \{y \in X : \exists x \in S \, y \geq x\}.$$

When $S = \{x\}$ with $x \in X$ we denote $\downarrow S$ and $\uparrow S$ by $\downarrow x$ and $\uparrow x$, respectively.

The family of all down-sets of X is denoted by $\mathcal{D}(X)$.

Let X be a poset. A non-empty subset $S \subseteq X$ is said to be *up-directed* if, for any $x, y \in S$, there exists an upper bound $z \in S$. Dually S is *down-directed* if, for any $x, y \in S$, there exists a lower bound $z \in S$.

A poset X is defined to be *up-complete* if every up-directed subset has a join in X. If every down-directed subset has a meet in X we say that X is *down-complete*.

Every lattice L can be regarded as an ordered set $(L, \leq)$ such that, for any $x, y \in L$, we have that $x \vee y$ and $x \wedge y$ exist.

A *distributive lattice* is a lattice L that satisfies the following equivalent *distributive laws*,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for every $a, b, c \in L$.

1.2 Universal Algebra

Generally, for an algebra $\mathbb{L}$ of type τ, we will denote by L the algebra and its universe and by $Con(L)$ the set of all congruences on the algebra L.

An algebra L is *congruence-distributive* if $Con(L)$ is a distributive lattice.

For an arbitrary set, $S \subseteq L$, the congruence *generated by* S, denoted by $\theta(S)$, is the intersection of all congruences containing $S \times S$. When $S = \{a, b\}$ with $a, b \in L$, the congruence generated by $\{a, b\}$ is called a *principal congruence* and is denoted by $\theta(a, b)$.

Let $\{L_i\}_{i \in I}$ be an indexed family of algebras of type τ, let $\Pi_{i \in I} L_i$ be their direct product and let π_i be the projection map from $\Pi_{i \in I} L_i$ onto L_i. Then an algebra L is a *subdirect product* of the family of algebras $\{L_i\}_{i \in I}$ if $L \leq \Pi_{i \in I} L_i$ and $\pi_i(L_i) = L_i$ for each $i \in I$.

An algebra L is *subdirectly irreducible* if $|L| > 1$ and, for every embedding $f : L \to \Pi_{i \in I} L_i$ such that $f(L)$ is a subdirect product of the family $\{L_i\}_{i \in I}$, there is an $i \in I$ such that $\pi_i \circ f : L \to L_i$ is an isomorphism,.

Proposition 1.2.1. *An algebra L is subdirectly irreducible if and only if $Con(L)\backslash\{\Delta\}$ has a minimum.*

An algebra L is *finitely subdirectly irreducible* if $|L| > 1$ and, for any $a, b, c, d \in L$ with $a \neq b$, $c \neq d$ we always have $\theta(a, b) \cap \theta(c, d) \neq \Delta$.

If an algebra is subdirectly irreducible, then it is finitely subdirectly irreducible.

An algebra L has the *congruence extension property* (CEP) if, for every subalgebra K of L, any congruence relation $\theta \in Con(K)$ is the restriction of a congruence relation in $Con(L)$.

1.2.1 Classes of algebras of the same type

Let K be a class of algebras of the same type τ. We will denote by $H(K)$, $I(K)$, $S(K)$ and $P(K)$, respectively, the classes of all the homomorphic images, isomorphic images, subalgebras and direct products of algebras of K.

A nonempty class of algebras of the same type τ is called a *variety* if it is closed under homomorphic images, subalgebras and direct products.

The smallest variety containing K is $HSP(K)$.

Let $p_i \approx q_i$ be identities for $i \in I$. The class of algebras of the same type τ satisfying all identities $p_i \approx q_i$, $i \in I$ is called an *equational class of algebras*.

From a known theorem by Birkhoff, we know that $\mathcal{V}$ is a variety if and only if $\mathcal{V}$ is an equational class of algebras.

1.3 Lattices

A *lattice* is an algebra $(L, \vee, \wedge)$ such that the following identities hold in L:

$$
\begin{aligned}
&a \vee a \approx a && a \wedge a \approx a \\
&a \vee b \approx b \vee a && a \wedge b \approx b \wedge a \\
&a \vee (b \vee c) \approx (a \vee b) \vee c && a \wedge (b \wedge c) \approx (a \wedge b) \wedge c \\
&a \vee (b \wedge a) \approx a && a \wedge (b \vee a) \approx a
\end{aligned}
$$

Given a lattice L, if $\bigwedge L$ and $\bigvee L$ exist, we denote these elements by 0 and 1 respectively and we say that L is *bounded*.

We will denote the *class of bounded distributive lattices* by $\mathcal{DL}$.

A lattice L is a *complete lattice* if, $\bigvee S$ and $\bigwedge S$ exist for any subset $S \subseteq L$.

Let L_1 and L_2 be lattices. A map $f : L_1 \to L_2$ is a *complete homomorphism* if, whenever $\bigvee S$ exists for a subset $S \subseteq L_1$, then $\bigvee f(S)$ exists and $f(\bigvee S) = \bigvee f(S)$ and, dually for $\bigwedge$.

A complete lattice L is *completely distributive* if, for every doubly indexed family of elements $\{a_{i,j}\}_{i\in I, j\in J}$ in L, the following equivalent conditions hold:

$$\bigwedge_{i\in I}\bigvee_{j\in J} a_{i,j} = \bigvee_{f\in J^I}\bigwedge_{i\in I} a_{i,f(i)} \text{ and } \bigvee_{i\in I}\bigwedge_{j\in J} a_{i,j} = \bigwedge_{f\in J^I}\bigvee_{i\in I} a_{i,f(i)}$$

where J^I denotes the set of all functions on I to J.

Let L be a complete lattice and let $k \in L$. Then k is said to be *compact* if, for every subset $S \subseteq L$, $k \le \bigvee S$ implies $k \le \bigvee T$ for some finite subset $T \subseteq S$.

A complete lattice L is *algebraic* if every element in L is the join of compact elements.

Let L be a lattice. An element $a \in L$ is *join-irreducible* if $a \neq 0$ and if, for any $a, b \in L$, $a = b \vee c$ implies $a = b$ or $a = c$. The set of join-irreducible elements of L will be denoted by $J(L)$.

We define *meet-irreducible* elements dually and we denote by $M(L)$ the set of meet-irreducible elements of L.

An element a of a complete lattice L is called *completely join-irreducible* if $a \neq 0$ and, for every subset S of L, $a = \bigvee S$ implies that $a \in S$. *completely meet-irreducible* elements are defined dually.

The sets of completely join-irreducible elements and completely meet-irreducible elements of L will be denoted by $J^\infty(L)$ and $M^\infty(L)$, respectively.

Let L be a distributive lattice, $a, b, x, y \in L$ and let $a \le b$. Then $(x, y) \in \theta(a, b)$ if and only if $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$.

From here it follows:

Lemma 1.3.1. *Let L be a distributive lattice, H a sublattice of L and $x, y \in L$. Then*

(i) $(x, y) \in \theta(H)$ if and only if there are $a, b \in H$ such that $a \le b$ and $(x, y) \in \theta(a, b)$.

(ii) $\theta(H) = \bigvee\{\theta(a,b) :\ a,b \in H \text{ and } a \leq b\}$.

Proof. (i) Let ρ be the relation defined by $(x,y) \in \rho$ iff there exist $a,b \in H$ such that $a \leq b$ and $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$. It is easy to verify that ρ is a lattice congruence and that ρ collapses the elements of H. So, we can conclude that $\theta\,(H) \leq \rho$.

Since $(x,y) \in \theta\,(a,b)$ iff $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$ it results that $\rho \leq \bigvee\{\theta\,(a,b) :\ a,b \in H \text{ and } a \leq b\} \leq \theta\,(H)$ and hence $\rho = \theta(H)$.

(ii) It follows from the proof of (i).

□

From Lemma 1.3.1 it follows

Corollary 1.3.2. *Let I and F be, respectively, an ideal and a filter in L. Then:*

(i) $\theta(I) = \bigvee_{i\in I} \theta(0,i)$.

(ii) $\theta(F) = \bigvee_{f\in F} \theta(f,1)$.

Proof. (i) Since $0, i \in I$ we have, by Lemma 1.3.1,

$$\theta(I) = \bigvee\{\theta(a,b) :\ a,b \in I \text{ and } a \leq b\} \geq \bigvee\{\theta(0,i) :\ i \in I\}.$$

On the other hand, if $a,b \in I$ are such that $a \leq b$, then $\theta(a,b) \leq \theta(0,b)$ so that

$$\bigvee\{\theta(a,b) :\ a,b \in I \text{ and } a \leq b\} \leq \bigvee\{\theta(0,i) :\ i \in I\}.$$

Dually we prove (ii). □

For any congruence $\varphi \in ConL$, we write $[a]_\varphi$ to denote the class of φ containing $a \in L$. The class $[a]_\varphi$ is a convex sublattice of the lattice L.

The restriction of $\theta(b,c)$ to $[a]_\varphi$ will be denoted by $\theta(b,c)|_{[a]_\varphi}$.

It is possible to prove, applying arguments similar to those used by Beazer in the proof of [6] Lemma 3.4, the following:

Lemma 1.3.3. *Let $L \in \mathcal{DL}$, $a \in L$ and let $\varphi \in ConL$. For any $i \in I$, let $(b_i, c_i)_{i\in I}$ be elements of $L \times L$ such that $b_i, c_i \in [a]_\theta$ and $b_i \leq c_i$. Then*

$$\left(\bigvee_{i\in I}\{\theta(b_i,c_i)\}\right)|_{[a]_\varphi} = \bigvee_{i\in I}\{\,\theta(b_i,c_i)|_{[a]_\varphi}\}.$$

1.3.1 Pseudocomplemented distributive lattices

A *pseudocomplemented distributive lattice*, which we will often denote by *p-lattice*, is a lattice $L \in \mathcal{DL}$ such that, for each $a \in L$ there exists an element a^* that is the maximum of the set $\{x \in L : \ a \wedge x = 0\}$.

According to what is stated in [4] we have the following

Proposition 1.3.4. *An algebra* $L = (L, \vee, \wedge, *, 0, 1)$ *of type* $(2, 2, 1, 0, 0)$ *is a p-lattice if the following conditions hold* $(a, b \in L)$*:*

(1) $(L, \vee, \wedge, 0, 1)$ *is a distributive lattice with* $0, 1$.

(2) $0^* \approx 1$ *and* $1^* \approx 0$.

(3) $a \wedge (a \wedge b)^* \approx a \wedge b^*$.

We will denote by B_ω this equational class of algebras.

For $L \in B_\omega$, we have $a \leq a^{**}$ for any $a \in L$.

An algebra $L \in B_\omega$ is subdirectly irreducible if and only if $L = L_1 \oplus 1$ where L_1 is a Boolean algebra.

The subvarieties of B_ω are in a chain:

$$B_0 \subset B_1 \subset ... \subset B_n \subset ... \subset B_\omega$$

where B_n with $n \in \mathbb{N}_0$ is the subvariety of B_ω generated by $\mathbf{2}^n \oplus 1$.

The subvarieties B_0 and B_1 are, respectively, the Boolean algebras and the Stone algebras.

1.3.2 Ockham algebras

Definition 1.3.5. An *Ockham algebra* is an algebra $(L, \vee, \wedge, ', 0, 1)$ for which $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice satisfying the identities

$$(a \vee b)' \approx a' \wedge b', \quad (a \wedge b)' \approx a' \vee b', \quad 0' \approx 1 \quad \text{and} \quad 1' \approx 0.$$

The subvariety $K_{1,1}$ of the variety of Ockham algebras, first considered by J. Berman in [8], is the class of Ockham algebras which satisfy $a' \approx a'''$.

An algebra of $K_{1,1}$ is a *De Morgan algebra* if and only if it satisfies $a'' \approx a$.

The subdirectly irreducible algebras of the variety $K_{1,1}$ (sometimes also denoted by $P_{3,1}$) were obtained by Sankappanavar in [37] and independently by Beazer in [5]. Their diagrams are presented in [10] pages 70 and 71 and the poset of these subdirectly irreducible algebras ordered according to a theorem of Davey [16] is presented in [10] page 91.

For an easier understanding of this work we recall that the subdirectly irreducible algebras of $K_{1,1}$ were denoted in [10] by: B, K, M, S, $\overline{S}$, S_1, K_1, $\overline{K_1}$, K_2, $\overline{K_2}$, K_3, $\overline{K_3}$, M_1, $\overline{M_1}$, L, $\overline{L}$, N, $\overline{N}$ and B_1.

It is well known that the subdirectly irreducible De Morgan algebras are B, K and M which diagrams we present in Figure 1.1

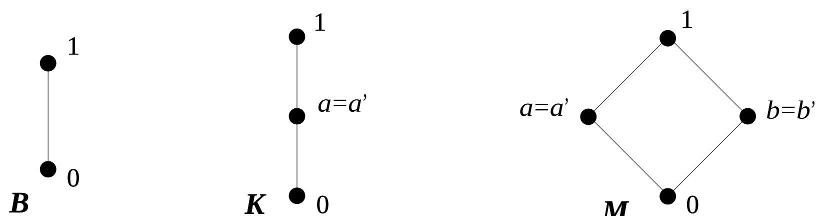


Figure 1.1: Subdirectly irreducible De Morgan algebras

1.4 Semi-De Morgan algebras

Definition 1.4.1. An algebra $L = (L, \vee, \wedge, ', 0, 1)$ is a *semi-De Morgan algebra* if the following five conditions hold $(a, b \in L)$:

(S1) $(L, \vee, \wedge, 0, 1)$ is a distributive lattice with $0, 1$.

(S2) $0' \approx 1$ and $1' \approx 0$.

(S3) $(a \vee b)' \approx a' \wedge b'$.

(S4) $(a \wedge b)'' \approx a'' \wedge b''$.

(S5) $a''' \approx a'$.

We will denote by $SDMA$ this equational class of algebras.

The following rules hold in $SDMA$:

(S6) $(a \wedge b)' \approx (a'' \wedge b'')' \approx (a \wedge b'')'$.

(S7) $(a \wedge b)' \approx (a' \vee b')''$.

(S8) $(a \wedge b)'' \approx (a' \vee b')'$.

(S9) $a \leq b$ implies $b' \leq a'$.

(S10) $a \wedge (a \wedge b)' \geq a \wedge b'$.

(S11) $(a \vee b)'' \approx (a' \wedge b')' \approx (a'' \vee b'')''$.

Remark 1.4.2. A semi-De Morgan algebra is a De Morgan algebra, or DMA, if and only if it satisfies the identity $a'' \approx a$.

In what follows DMA will denote the equational class of De Morgan algebras.

If $L \in SDMA$ then L_{lat} denotes the lattice reduct of L. The height of L_{lat} will be denoted by $h(L)$.

When studying congruences in $SDMA$, the congruence lattice of the semi-De Morgan algebra L will be denoted by $Con(L)$ and the corresponding congruence lattice on the lattice reduct of L will be denoted by $Con_{latL}(L)$. The principal congruence of $Con(L)$ collapsing the pair $a, b \in L$ is denoted by $\theta(a, b)$ and $\theta_{latL}(a, b)$ denotes the smallest congruence of $Con_{latL}(L)$ collapsing a, b.

Definition 1.4.3. If L is an $SDMA$, we write

$$DM(L) = \{a \in L : a = a''\}.$$

Then, by [38] Theorem 2.4, $\left(DM(L), \dot{\vee}, \wedge, ', 0, 1\right)$ is a DMA where $a \dot{\vee} b$ is defined to be $(a' \wedge b')'$.

The map $\beta : L \to L$ defined by $\beta(a) = a''$ is a homomorphism from L onto $DM(L)$ and its kernel is $\phi = \{(a, b) \in L \times L : a' = b'\}$. Therefore $L/\phi \cong DM(L)$ ([38] Lemma 3.1).

Definition 1.4.4. If L is an $SDMA$ satisfying the equation $a' \wedge a'' \approx 0$, then L is called a *demi-p-lattice* ($DMPL$). If L is an $SDMA$ and it satisfies $a \wedge a' \approx 0$, then L is called an *almost p-lattice* (APL).

Lemma 1.4.5. *An almost-p-lattice L is a distributive pseudocomplemented lattice (*p-lattice*) if and only if $a \leq a''$ holds in L.*

Note that $L \in SDMA$ is a demi-p-lattice if and only if $\left(DM(L), \dot{\vee}, \wedge, ', 0, 1\right)$ is a Boolean algebra ([38] Corollary 2.7).

For a demi-p-lattice, L, we let $B(L) = DM(L)$ and we write $D_0 = D_0(L) = \{a \in L : a' = 1\}$ and $D_1 = D_1(L) = \{a \in L : a' = 0\}$ as in [39]. It is clear that D_0 is an ideal and D_1 is a filter.

The intersection of the variety $SDMA$ with the variety of Ockham algebras is the variety $K_{1,1}$, so semi-De Morgan algebras are a generalization of $K_{1,1}$ algebras.

Remark 1.4.6. Observe that $K_{1,1}$ is characterized, as a subvariety of $SDMA$, by the identity $a' \vee b' \approx (a \wedge b)'$.

Most of these results were proved by H.P. Sankappanavar in [38] (see also [39]).

Lemma 1.4.7. *Let $L \in SDMA$ and let $DM(L)_{lat}$ be a chain. Then $L \in K_{1,1}$.*

Proof. Let $a, b \in L$. Then $a', b' \in DM(L)$ and, since $DM(L)_{lat}$ is a chain, we have $a' \wedge b' = a'$ or $a' \wedge b' = b'$.

Without loss of generality we may assume that $a' \wedge b' = a'$. Then, both in $DM(L)_{lat}$ and in L_{lat}, $a' \leq b'$ so in $DM(L)$ and L we have $a' \dot{\vee} b' = b'$ and $a' \vee b' = b'$ respectively. Therefore $a' \vee b' = a' \dot{\vee} b' = (a'' \wedge b'')' = (a \wedge b)'$ □

With Professor R. Santos we characterized in [26], by algebraic techniques, the congruence lattice of subdirectly irreducible semi-De Morgan algebras. We quote from there the following:

Proposition 1.4.8 ([26] Propositions 2.5, 2.6 and 2.7). *Let $L \in SDMA$ be a finitely subdirectly irreducible algebra. Then for each $a, b \in L$,*

(i) $|a/\phi| \leq 2$

(ii) $(a, b) \in \phi$ *implies* $a = b$ *or* $a = b''$ *or* $a'' = b$

(iii) $a = a''$ *or* $a \ll a''$ *or* $a'' \ll a$

(iv) Two distinct pairs of elements $a \neq a''$ *and* $b \neq b''$ *cannot be in the same chain.*

By [26], Theorem 2.10 we know that $L \in SDMA \setminus DMA$ is a finitely subdirectly irreducible algebra if and only if L is a subdirectly irreducible algebra. This equivalence is also true in $K_{1,1}$ (see [37] Theorem 2.8), so we have:

Proposition 1.4.9. *Let $L \in SDMA$. Then L is a finitely subdirectly irreducible algebra if and only if L is a subdirectly irreducible algebra.*

Proposition 1.4.10 ([26], Corollary 2.11). *Let $L \in SDMA$. L is a subdirectly irreducible algebra if and only if L is a subdirectly irreducible De Morgan algebra or ϕ is the minimum element of $Con(L) \setminus \{\Delta\}$.*

The finite subdirectly irreducible demi-p-lattices are described in [40] Corollary 5.3. Observe that the algebras of $DMPL$ denoted by $B_{0,0}$, $B_{0,1}$, $B_{1/2,0}$ and $B_{1,0}$ in [40] are the subdirectly irreducible algebras of $K_{1,1}$ referred in [10] as B, S, $\overline{S}$, and S_1 respectively. In fact the intersection of the set of subdirectly irreducible algebras of $DMPL$ with the set of subdirectly irreducible algebras in $K_{1,1}$ has exactly $B_{0,0}$, $B_{0,1}$, $B_{1/2,0}$ and $B_{1,0}$ as its elements.

The variety $SDMA$ does not have the Congruence Extension Property.

Chapter 2

The variety $\mathcal{C}$

In [25], Hobby characterized the largest subvariety of $SDMA$ with the congruence extension property. This variety, which he called $\mathcal{C}$, is also a common generalization of DMA and pseudocomplemented distributive lattices. In fact it contains $DMPL$ and the subvariety $\mathcal{K}_{1,1}$ of Ockham algebras, since it is well known that both of these varieties have the congruence extension property.

In this chapter we study some properties of the variety $\mathcal{C}$.

2.1 The variety $\mathcal{C}$ of semi-De Morgan algebras

Hobby characterized the variety $\mathcal{C}$ by the following inequalities:

$$(\alpha) \quad a' \vee b' \geq (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'$$
$$(\beta) \quad a' \vee (a' \wedge b \wedge b')' \geq (a \wedge b)'\,.$$

It is possible to obtain simpler inequalities characterizing $\mathcal{C}$. With this aim we will consider the following identities:

$$(\alpha_1) \quad a' \vee b' = a' \vee b' \vee \big((a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'\big)$$
$$(\beta_1) \quad a' \vee (a' \wedge b \wedge b')' = (a \wedge b)' \vee (a' \wedge b \wedge b')'.$$

These identities are equivalent to α and β, respectively, because $a \geq a \wedge b$ implies $a' \leq (a \wedge b)'$. We will use them to prove the following lemmas.

Lemma 2.1.1. *Let $L \in \mathcal{C}$ and $a, d \in L$. Then identity α_1 implies:*

$$(\alpha_2) \qquad (a' \wedge d'') \vee d' = ((a \wedge d)' \wedge d'') \vee d'$$
$$(\alpha_3) \qquad (a' \vee d') \wedge d'' = (a \wedge d)' \wedge d''$$

Proof. By (S3), $(a' \wedge d'') \vee d' = (a \vee d')' \vee d'$. Replacing b by $a \vee d'$, a by d and c by d' in identity α_1 and 1 commutativity, we obtain

$$\begin{aligned}
&(a \vee d')' \vee d' = \\
&= (a \vee d')' \vee d' \vee \big(((a \vee d') \wedge d)' \wedge (d \wedge d')' \wedge ((a \vee d') \wedge d')' \wedge ((a \vee d') \wedge d'')'\big) \\
&= (a \vee d')' \vee d' \vee \big(((a \vee d') \wedge d)' \wedge (d \wedge d')' \wedge d''\big)
\end{aligned}$$

because $((a \vee d') \wedge d'')' = ((a \vee d') \wedge d)'$ by (S6).

But $(d \wedge d')' \geq ((a \vee d') \wedge d)'$ since $d \wedge d' \leq (a \vee d') \wedge d$, hence it follows

$$\begin{aligned}
(a \vee d')' \vee d' &= (a \vee d')' \vee d' \vee \big(((a \vee d') \wedge d)' \wedge d''\big) \\
&= (a \vee d')' \vee d' \vee (((a \vee d') \wedge d) \vee d')' \qquad \text{by S3} \\
&= (a \vee d')' \vee d' \vee ((a \vee d') \wedge (d \vee d'))' \qquad \text{by distributivity} \\
&= d' \vee ((a \vee d') \wedge (d \vee d'))' \quad \text{because } (a \vee d')' \leq ((a \vee d') \wedge (d \vee d'))' \\
&= d' \vee ((a \wedge d) \vee d')' \qquad \text{by distributivity} \\
&= d' \vee \big((a \wedge d)' \wedge d''\big).
\end{aligned}$$

So, the identity α_2 holds.

Now, by distributivity, we obtain from α_2:

$$(a' \vee d') \wedge (d'' \vee d') = ((a \wedge d)' \vee d') \wedge (d'' \vee d').$$

Since $(a \wedge d)' \geq d'$, it follows

$$(a' \vee d') \wedge (d'' \vee d') = (a \wedge d)' \wedge (d'' \vee d')$$

and, meeting the two members with d'', we have α_3. □

Lemma 2.1.2. *Let $L \in \mathcal{C}$ and $a, b, e \in L$ be such that*

(i) $e' \geq e''$.

(ii) $(a \wedge e)' = (b \wedge e)'$.

(iii) $a' \wedge e'' = b' \wedge e''$.

Then $a' \vee e' = b' \vee e'$.

Proof. Using identity α_1, we have:

$$\begin{aligned} a' \vee e' &= a' \vee e' \vee \big((a \wedge e)' \wedge (a \wedge b)' \wedge (e \wedge b)' \wedge (e \wedge b')'\big) \\ &= a' \vee e' \vee \big((a \wedge e)' \wedge (a \wedge b)' \wedge (e \wedge a)' \wedge (e \wedge b')'\big) \quad \text{by (ii)} \\ &= a' \vee e' \vee \big((a \wedge e)' \wedge (a \wedge b)' \wedge (e'' \wedge b')'\big) \quad \text{by (S6)} \\ &= a' \vee e' \vee \big((a \wedge e)' \wedge (a \wedge b)' \wedge (e'' \wedge a')'\big) \quad \text{by (iii)} \\ &= \big(a' \vee e' \vee (a \wedge e)'\big) \wedge \big(a' \vee e' \vee (a \wedge b)'\big) \wedge \big(a' \vee e' \vee (e'' \wedge a')'\big) \\ &= (a \wedge e)' \wedge \big(e' \vee (a \wedge b)'\big) \wedge \big(a' \vee (e'' \wedge a')'\big) \end{aligned}$$

since $a', e' \leq (a \wedge e)'$, $a' \leq (a \wedge b)'$ and $e' \leq (e'' \wedge a')'$.

Note that,

$$\begin{aligned} a' \vee (e'' \wedge a')' &= a' \vee (a' \wedge e' \wedge e'')' \quad \text{by (i)} \\ &= a' \vee (a' \wedge e'' \wedge e''')' \quad \text{by (S5) and commutativity} \\ &= (a \wedge e'')' \vee (a' \wedge e'' \wedge e''')' \quad \text{by } \beta_1 \\ &= (a \wedge e)' \vee (a' \wedge e'' \wedge e')' \quad \text{by (S6) and (S5)} \\ &= (a \wedge e)' \vee (e'' \wedge a')' \quad \text{by (i).} \end{aligned}$$

Substituting the above into the previous equation, it follows that

$$a' \vee e' = (a \wedge e)' \wedge \big(e' \vee (a \wedge b)'\big).$$

Similarly

$$b' \vee e' = (b \wedge e)' \wedge \big(e' \vee (a \wedge b)'\big).$$

From (ii) we conclude $a' \vee e' = b' \vee e'$.

□

The search for simpler inequalities defining the variety $\mathcal{C}$ requires some rather nasty calculations so that we consider several lemmas before we can find our goal:

Lemma 2.1.3. *Let $L \in SDMA$ and let $a, b, c \in L$. Then the inequality α is equivalent to*

$$(\delta) \qquad (a' \wedge (b \wedge (c \vee c'))')\vee(b' \wedge (a \wedge c)') = (a\wedge b)'\wedge(a\wedge c)'\wedge((b \wedge (c \vee c'))'.$$

Proof. First note that α is equivalent to

$$(a' \vee b') \wedge (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')' = (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'$$

and, by distributivity, this identity is equivalent to

$$\begin{aligned}&(a' \wedge (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'\,) \vee \\ &\qquad \vee (b' \wedge (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'\,) = \\ &= (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'.\end{aligned}$$

By S9, it is known that a' is less than or equal to $(a \wedge b)'$ and to $(a \wedge c)'$ and that b' is also less than or equal to $(a \wedge b)', (b \wedge c)'$ and $(b \wedge c')'$. Therefore the previous identity is equivalent to

$$(a' \wedge (b \wedge c)' \wedge (b \wedge c')') \vee (b' \wedge (a \wedge c)') = (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'$$

and, by S3, also to

$$(a' \wedge ((b \wedge c) \vee (b \wedge c'))') \vee (b' \wedge (a \wedge c)') = (a \wedge b)' \wedge (a \wedge c)' \wedge ((b \wedge c) \vee (b \wedge c'))'.$$

Finally, by the distributivity of L, we conclude that α is equivalent to δ.

□

Lemma 2.1.4. *Let $L \in SDMA$ and let $a, b, c \in L$. Then*

$$(\alpha_3) \qquad (a' \vee b') \wedge b'' = (a \wedge b)' \wedge b'' \quad \textit{and}$$
$$(\beta_1) \qquad a' \vee (a' \wedge b \wedge b')' = (a \wedge b)' \vee (a' \wedge b \wedge b')'$$

imply

$$(\delta) \qquad (a' \wedge (b \wedge (c \vee c'))') \vee (b' \wedge (a \wedge c)') = (a \wedge b)' \wedge (a \wedge c)' \wedge ((b \wedge (c \vee c'))'.$$

Proof. Let us denote by A and B, respectively, the first and the second member of the identity δ. We are going to prove that β_1 and α_3 imply $A = B$ using the distributivity of L.

First we will verify that the joins of A and B with $(a' \wedge c' \wedge c'')'$ are equal:

$A \vee (a' \wedge c' \wedge c'')' =$
$= (a' \wedge (b \wedge (c \vee c'))') \vee (((b' \vee (a' \wedge c' \wedge c'')') \wedge ((a \wedge c)' \vee (a' \wedge c' \wedge c'')')))$
(by distributivity)
$= (a' \wedge (b \wedge (c \vee c'))') \vee (((b' \vee (a' \wedge c' \wedge c'')') \wedge (a' \vee (a' \wedge c' \wedge c'')')))$
(by β_1 and S6)
$= (a' \wedge (b \wedge (c \vee c'))') \vee (b' \wedge a') \vee (a' \wedge c' \wedge c'')'$ (by distributivity)
$= (a' \wedge (b \wedge (c \vee c'))') \vee (a' \wedge c' \wedge c'')'$
(because, by S9, $(b \wedge (c \vee c'))' \geq b'$ and thus $a' \wedge (b \wedge (c \vee c'))' \geq a' \wedge b'$).

$B \vee (a' \wedge c' \wedge c'')' =$
$= (((a \wedge b)' \wedge (b \wedge (c \vee c'))') \vee (a' \wedge c' \wedge c'')') \wedge ((a \wedge c)' \vee (a' \wedge c' \wedge c'')')$
(by distributivity)
$= (((a \wedge b)' \wedge (b \wedge (c \vee c'))') \vee (a' \wedge c' \wedge c'')') \wedge (a' \vee (a' \wedge c' \wedge c'')')$
(by β_1 and S6)
$= ((a \wedge b)' \wedge (b \wedge (c \vee c'))' \wedge a') \vee (a' \wedge c' \wedge c'')'$ (by distributivity)
$= (a' \wedge (b \wedge (c \vee c'))') \vee (a' \wedge c' \wedge c'')'$ because, by S9, $(a \wedge b)' \geq a'$.

Thus we proved that $A \vee (a' \wedge c' \wedge c'')' = B \vee (a' \wedge c' \wedge c'')'$.

Now we are going to see that the same is true with the meets. We will have to use identity α_3 so we must notice that, by S3, S11, S9 and S5,

$$(a' \wedge c' \wedge c'')' = (a \vee c \vee c')'' = (a \vee c'' \vee c')'' \geq (a' \wedge c' \wedge c'')'' = (a \vee c \vee c')'$$

and therefore, denoting by d the expression $a \vee c \vee c'$ we will have

$$(a' \wedge c' \wedge c'')' = d'' \geq d'$$

and thus:

$$\begin{aligned}
&A \wedge (a' \wedge c' \wedge c'')' = A \wedge d'' = \\
&= ((a' \wedge (b \wedge (c \vee c'))') \vee (b' \wedge (a \wedge c)')) \wedge d'' \\
&= \big((a \vee (b \wedge (c \vee c')))' \vee (b' \wedge (a \wedge c)')\big) \wedge d'' \quad \text{(by S3)} \\
&= \big(((a \vee b) \wedge (a \vee c \vee c'))' \wedge d''\big) \vee ((b' \wedge (a \wedge c)') \wedge d'') \\
&\qquad\qquad \text{(by distributivity)} \\
&= \big(((a \vee b) \wedge d)' \wedge d''\big) \vee ((b' \wedge (a \wedge c)') \wedge d'') \\
&\qquad\qquad \text{(by the definition of } d\,,\text{)} \\
&= (((a \vee b)' \vee d') \wedge d'') \vee ((b' \wedge (a \wedge c)') \wedge d'') \quad \text{(by } \alpha_3\text{)} \\
&= (((a' \wedge b') \vee d') \wedge d'') \vee ((b' \wedge (a \wedge c)') \wedge d'') \quad \text{(by S3)} \\
&= ((a' \wedge b') \vee d' \vee (b' \wedge (a \wedge c)') \wedge d'' \quad \text{(by distributivity)} \\
&= (d' \vee (b' \wedge (a \wedge c)') \wedge d'' \quad \text{(because } (a \wedge c)' \geq a'\text{)} \\
&= (b' \wedge (a \wedge c)' \wedge d'') \vee (d' \wedge d'') \quad \text{(by distributivity)} \\
&= (b' \wedge (a \wedge c)' \wedge d'') \vee d' \quad \text{(because } d'' \geq d'\text{).}
\end{aligned}$$

By a similar process:

$$\begin{aligned}
&B \wedge (a' \wedge c' \wedge c'')' = B \wedge d'' = \\
&= (a \wedge b)' \wedge (b \wedge (c \vee c'))' \wedge d'' \wedge (a \wedge c)' \quad \text{(by commutativity)} \\
&= ((a \wedge b) \vee (b \wedge (c \vee c')))' \wedge d'' \wedge (a \wedge c)' \quad \text{(by S3)} \\
&= (b \wedge (a \vee c \vee c'))' \wedge d'' \wedge (a \wedge c)' \quad \text{(by distributivity)} \\
&= (b \wedge d)' \wedge d'' \wedge (a \wedge c)' \quad \text{(by the definition of } d\text{)} \\
&= (b' \vee d') \wedge d'' \wedge (a \wedge c)' \qquad \text{(applying } \alpha_3\text{)} \\
&= ((b' \wedge d'') \vee (d' \wedge d'')) \wedge (a \wedge c)') \qquad \text{(by distributivity)} \\
&= ((b' \wedge d'') \vee d') \wedge (a \wedge c)') \qquad \text{(because } d'' \geq d' \text{)} \\
&= (b' \wedge d'' \wedge (a \wedge c)') \vee (d' \wedge (a \wedge c)') \qquad \text{(by distributivity)} \\
&= (b' \wedge d'' \wedge (a \wedge c)') \vee d' \\
&\qquad\qquad \text{(because, by S9, } d' = (a \vee c \vee c')' \leq (a \wedge c)'\text{).}
\end{aligned}$$

So, we have proved that

$$A \wedge (a' \wedge c' \wedge c'')' = B \wedge (a' \wedge c' \wedge c'')'.$$

By the characterization of $\theta_{latL}((a' \wedge c' \wedge c'')', (a' \wedge c' \wedge c'')')$ we conclude that $A = B$.

□

From the previous lemmas we obtain the following:

Lemma 2.1.5. *Let $L \in SDMA$, then $L \in \mathcal{C}$ if and only if the identities*

$$(\alpha_3) \qquad (a' \vee b') \wedge b'' = (a \wedge b)' \wedge b''$$

and

$$(\beta_1) \qquad a' \vee (a' \wedge b \wedge b')' = (a \wedge b)' \vee (a' \wedge b \wedge b')'$$

hold.

Proof. We proved in Lemma 2.1.1 that the identity α_3 is a consequence of α_1 which is equivalent to α.

Conversely, by Lemma 2.1.4 , (α_3 and β_1) imply (δ and β_1) and, by Lemma 2.1.3, these are equivalent to α and β_1. □

It is now possible to characterize $\mathcal{C}$ by simpler axioms solving Problem 2 in Hobby [25] :

Theorem 2.1.6. *The subvariety $\mathcal{C}$ of semi-De Morgan algebras can be characterized by inequalities γ and β:*

$$(\gamma) \qquad a' \vee b' \geq (a \wedge b)' \wedge b''$$

$$(\beta) \qquad a' \vee (a' \wedge b \wedge b')' \geq (a \wedge b)'$$

Proof. It is enough to prove that the identity α_3 of the previous lemma is equivalent to inequality γ.

By α_3 we have:

$$a' \vee b' \geq (a' \vee b') \wedge b'' = (a \wedge b)' \wedge b''.$$

Therefore α_3 implies γ.

On the other hand, from γ, we know that:

$$(a' \vee b') \wedge (a \wedge b)' \wedge b'' = (a \wedge b)' \wedge b''.$$

But $a' \leq (a \wedge b)'$ and $b' \leq (a \wedge b)'$ so that $a' \vee b' \leq (a \wedge b)'$ and therefore α_3 follows from γ. □

2.2 Principal congruences in the variety $\mathcal{C}$

In this section we shall give a characterization of principal congruences on $\mathcal{C}$ that extends the corresponding characterization for demi-p-lattices, due to Sankappanavar [39], and for the variety $\mathcal{K}_{1,1}$ due to J. Berman [8] and to M. Ramalho and M. Sequeira [33].

Theorem 2.2.1. *Let $L \in \mathcal{C}$, $a, b \in L$ with $a \leq b$, and let $t = a \vee b'$ and $s = a \wedge b'$. Then $(x, y) \in \theta(a, b)$ if and only if x, y satisfy:*

(1) $(x \wedge a \wedge t'') \vee t' = (y \wedge a \wedge t'') \vee t'$.

(2) $((x \vee b) \wedge t'') \vee t' = ((y \vee b) \wedge t'') \vee t'$.

(3) $x \wedge a \wedge s'' = y \wedge a \wedge s''$.

(4) $(x \vee b) \wedge s'' = (y \vee b) \wedge s''$.

(5) $(x \wedge a) \vee s' = (y \wedge a) \vee s'$.

(6) $x \vee b \vee s' = y \vee b \vee s'$.

(7) $(x \wedge t)' \wedge t'' = (y \wedge t)' \wedge t''$.

Proof. Let ψ denote the equivalence relation such that $(x, y) \in \psi$ if and only if conditions (1)-(7) are true.

Let $(x, y) \in \psi$ and $z \in L$.

Using distributivity, it is easy to check $(x \wedge z, y \wedge z)$ and $(x \vee z, y \vee z)$ satisfy conditions (1)-(6).

We will prove that they also satisfy (7).

Since (7) holds,

$$\left(\left((x \wedge t)' \wedge t''\right) \vee (z' \wedge t'')\right)'' = \left(\left((y \wedge t)' \wedge t''\right) \vee (z' \wedge t'')\right)''$$

and so, by distributivity,

$$\left(\left((x \wedge t)' \vee z'\right) \wedge t''\right)'' = \left(\left((y \wedge t)' \vee z'\right) \wedge t''\right)''.$$

By (S6),

$$\left(\left((x \wedge t)' \vee z'\right)'' \wedge t''\right)'' = \left(\left((y \wedge t)' \vee z'\right)'' \wedge t''\right)''$$

and by (S7)

$$\Big(\big((x\wedge t)''\wedge z''\big)'\wedge t''\Big)''=\Big(\big((y\wedge t)''\wedge z''\big)'\wedge t''\Big)''.$$

Then, by (S4),(S5) and (S6)

$$(x\wedge z\wedge t)'\wedge t''=(y\wedge z\wedge t)'\wedge t'',$$

so $(x\wedge z, y\wedge z)$ satisfies (7).

Also from (7), we have

$$(x\wedge t)'\wedge(z\wedge t)'\wedge t''=(y\wedge t)'\wedge(z\wedge t)'\wedge t''.$$

Hence, by (S3),

$$((x\wedge t)\vee(z\wedge t))'\wedge t''=((y\wedge t)\vee(z\wedge t))'\wedge t''$$

and, by distributivity,

$$((x\vee z)\wedge t)'\wedge t''=((y\vee z)\wedge t)'\wedge t'',$$

so $(x\vee z, y\vee z)$ satisfies (7).

Therefore $(x\wedge z, y\wedge z)\in\psi$ and $(x\vee z, y\vee z)\in\psi$.

To prove that ψ preserves $'$ consider $(x,y)\in\psi$. Observe that

$$\begin{aligned}(x'\wedge a\wedge t'')\vee t' &= ((x'\wedge t'')\vee t')\wedge(a\vee t') && \text{by distributivity}\\ &= \big(((x\wedge t)'\wedge t'')\vee t'\big)\wedge(a\vee t') && \text{by Lemma 2.1.1 } (\alpha_2).\end{aligned}$$

Analogously,

$$(y'\wedge a\wedge t'')\vee t'=\big(((y\wedge t)'\wedge t'')\vee t'\big)\wedge(a\vee t').$$

By (7) we conclude:

$$(x'\wedge a\wedge t'')\vee t'=(y'\wedge a\wedge t'')\vee t'.$$

In a similar way, we can show

$$((x'\vee b)\wedge t'')\vee t'=((y'\vee b)\wedge t'')\vee t'.$$

Thus we have proved that if $(x,y)\in\psi$ then (x',y') satisfies (1) and (2).

From (6) it follows that

$$(\ x \vee b \vee s')' = (\ y \vee b \vee s')'\,,$$

and so, by (S3),

$$x' \wedge b' \wedge s'' = y' \wedge b' \wedge s''.$$

Since $s'' = a'' \wedge b'$, we have

$$x' \wedge s'' = y' \wedge s''.$$

Then, it is clear that

$$x' \wedge a \wedge s'' = y' \wedge a \wedge s'' \quad \text{and} \qquad (x' \vee b) \wedge s'' = (y' \vee b) \wedge s''.$$

Thus we conclude that if $(x, y) \in \psi$ then (x', y') satisfies (3) and (4).
Since $s' = (a \wedge b')' = (a' \vee b)''$, $s'' = (a \wedge b')''$ and $b \geq a$ we have $s' \geq s''$.
By (3),

$$(\ x \wedge a \wedge s'')' = (\ y \wedge a \wedge s'')'\,.$$

By (S6) and (S5) we observe that

$$(a \wedge s'')' = (a'' \wedge s'')' = (a'' \wedge a'' \wedge b')' = s''' = s'.$$

So we conclude

$$(x \wedge s)' = (y \wedge s)'\,.$$

But we have already proved $x' \wedge s'' = y' \wedge s''$, so conditions (i), (ii) and (iii) of Lemma 2.1.2 hold. Hence

$$x' \vee s' = y' \vee s',$$

which implies

$$(x' \wedge a) \vee s' = (y' \wedge a) \vee s' \quad \text{and} \qquad x' \vee b \vee s' = y' \vee b \vee s'.$$

so (x', y') satisfies (5) and (6).
By Lemma 2.1.1, we know

$$((x' \wedge t'') \vee t')' = \big(((x \wedge t)' \wedge t'') \vee t'\big)'$$

and

$$((y' \wedge t'') \vee t')' = \big(((y \wedge t)' \wedge t'') \vee t'\big)'\,.$$

So, by (7),

$$((x' \wedge t'') \vee t')' = ((y' \wedge t'') \vee t')'$$

and then, by (S3) and (S6),

$$(x' \wedge t)' \wedge t'' = (y' \wedge t)' \wedge t''.$$

Thus, we have proved that if $(x, y) \in \psi$ then (x', y') also satisfies (7) and so ψ is a congruence of the semi-De Morgan algebra L.

Obviously (a, b) satisfies (1)-(6).
Since

$$\begin{aligned}
(b \wedge t)' \wedge t'' &= (b \wedge (a \vee b'))' \wedge (a \vee b')'' \\
&= ((b \wedge a) \vee (b \wedge b'))' \wedge (a' \wedge b'')' \text{ by distributivity and (S3)} \\
&= ((b \wedge a) \vee (b \wedge b') \vee (a' \wedge b))' \text{ by (S3) and (S6)} \\
&= (a \vee (a' \wedge b))' \text{ because } a' \geq b' \\
&= a' \wedge (a' \wedge b)' \text{ by (S3)} \\
&= (a \wedge (a \vee b'))' \wedge (a' \wedge b)' \\
&= (a \wedge (a \vee b'))' \wedge (a \vee b')'' \text{ by (S3) and (S6)} \\
&= (a \wedge t)' \wedge t'',
\end{aligned}$$

(a, b) satisfies (7).
Thus $(a, b) \in \psi$.

Finally, let ρ be any congruence relation of the semi-De Morgan algebra L such that $(a, b) \in \rho$ and let $(x, y) \in \psi$. Therefore $\left((a \vee b')'', (a' \vee b)''\right) \in \rho$, thus $(t'', s') \in \rho$ and $(t', s'') \in \rho$.

From $((y \wedge a \wedge t'') \vee t', (y \wedge a \wedge s') \vee s'') \in \rho$ and (1), we have

$$((x \wedge a \wedge t'') \vee t', (y \wedge a \wedge s') \vee s'') \in \rho.$$

Hence, taking the meet with $x \wedge a \wedge t''$,

$$(x \wedge a \wedge t'', (x \wedge a \wedge t'') \wedge ((y \wedge a \wedge s') \vee s'')) \in \rho.$$

Using distributivity and the fact that $s'' \leq t''$,

$$(x \wedge a \wedge t'', (x \wedge y \wedge a \wedge t'' \wedge s') \vee (x \wedge a \wedge s'')) \in \rho.$$

Similarly

$$(y \wedge a \wedge t'', (x \wedge y \wedge a \wedge t'' \wedge s') \vee (y \wedge a \wedge s'')) \in \rho.$$

By (3), we conclude

$$(x \wedge a \wedge t'', y \wedge a \wedge t'') \in \rho.$$

On the other hand, since

$$((x \wedge a) \vee t'', (x \wedge a) \vee s') \in \rho \quad \text{and} \quad ((y \wedge a) \vee t'', (y \wedge a) \vee s') \in \rho,$$

we have, by (5)

$$((x \wedge a) \vee t'', (y \wedge a) \vee t'') \in \rho.$$

Taking the meet with $x \wedge a$ and using distributivity,

$$(x \wedge a, (x \wedge y \wedge a) \vee (x \wedge a \wedge t'')) \in \rho.$$

Similarly

$$(y \wedge a, (x \wedge y \wedge a) \vee (y \wedge a \wedge t'')) \in \rho.$$

Thus $(x \wedge a, y \wedge a) \in \rho$.

In a similar way, it follows from conditions (2), (4) and (6) that $(x \vee b, y \vee b) \in \rho$.

Since $(a, b) \in \rho$, we have $(x \vee a, x \vee b)$, $(y \vee b, y \vee a) \in \rho$, therefore , by transitivity, $(x \vee a, y \vee a) \in \rho$.

Meeting with x (respectively y) and using distributivity,

$$(x, (x \wedge y) \vee (x \wedge a)) \in \rho \quad \text{and} \quad (y, (x \wedge y) \vee (y \wedge a)) \in \rho$$

thus $(x, y) \in \rho$.

So $\psi \leq \rho$ and the proof is complete. □

The equations defining the principal congruences become much simpler in some particular cases:

Corollary 2.2.2. *Let $L \in \mathcal{C}$ and $a \in L$. Then the following are equivalent:*

(i) $(x, y) \in \theta\,(a, 1)$.

(ii) $(x \wedge a \wedge a'') \vee a' = (y \wedge a \wedge a'') \vee a'$.

Proof. Let us use Theorem 2.2.1 and put 1 in for b. Then $t = a$ and $s = 0$.

This makes all but (1) and (7) vacuous.

For (1) we obtain

$$(x \wedge a \wedge a'') \vee a' = (y \wedge a \wedge a'') \vee a',$$

so it is clear that (i) implies (ii).

For the other direction observe that from (ii) it follows (1) and consequently,

$$((x \wedge a \wedge a'') \vee a')' = ((y \wedge a \wedge a'') \vee a')'.$$

By S3 we have,

$$(x \wedge a \wedge a'')' \wedge a'' = (y \wedge a \wedge a'')' \wedge a''$$

and, by S6,

$$(x \wedge a)' \wedge a'' = (y \wedge a)' \wedge a''$$

But this is none other than the equation that we obtained from (7). Hence (ii) implies (1) and (7).

□

Corollary 2.2.3. *Let $L \in \mathcal{C}$ and $b \in L$. Then the following are equivalent:*

(i) $(x, y) \in \theta(0, b)$.

(ii) $(x \vee b \vee b'') \wedge b' = (y \vee b \vee b'') \wedge b'$.

Proof. If $a = 0$ in Theorem 2.2.1, then $t = b'$ and $s = 0$ and all but (2) and (7) are vacuous.

For (2) we obtain

$$((x \vee b) \wedge b') \vee b'' = ((y \vee b) \wedge b') \vee b''$$

and from (2) and distributivity, it follows

$$(x \vee b \vee b'') \wedge (b' \vee b'') = (y \vee b \vee b'') \wedge (b' \vee b'').$$

Meeting with b' we obtain (ii). Therefore (i) implies (ii).

For the converse notice that, by distributivity, (ii) implies

$$((x \vee b) \wedge b') \vee (b'' \wedge b') = ((y \vee b) \wedge b') \vee (b'' \wedge b').$$

Making the join with b'' we have

$$((x \vee b) \wedge b') \vee b'' = ((y \vee b) \wedge b') \vee b''.$$

So (ii) implies (2) and also:

$$(((x \vee b) \wedge b') \vee b'')' = (((y \vee b) \wedge b') \vee b'')'.$$

From S3 and S5 it follows,

$$((x \vee b) \wedge b')' \wedge b' = ((y \vee b) \wedge b')' \wedge b'$$

and, from distributivity

$$((x \wedge b') \vee (b \wedge b'))' \wedge b' = ((y \wedge b') \vee (b \wedge b'))' \wedge b'.$$

Finally by S3 we have

$$(x \wedge b')' \wedge (b \wedge b')' \wedge b' = (y \wedge b')' \wedge (b \wedge b')' \wedge b'$$

and since by S9, $b' \leq (b \wedge b')'$,

$$(x \wedge b')' \wedge b' = (y \wedge b')' \wedge b'$$

which is the equation we obtained from (7).

Thus (ii) implies (2) and (7). □

Theorem 2.2.1 extends the corresponding results for demi p-lattices and for the variety $\mathcal{K}_{1,1}$ of Ockham algebras due to Sankappanavar[39] and J. Berman [8], respectively.

Corollary 2.2.4. *([39] Corollary 3.4)*

Let $L \in \mathcal{C}$ and $a, b \in L$ with $a \leq b$. If L is a demi p-lattice then $(x, y) \in \theta(a, b)$ if and only if x, y satisfy:

(1') $x \wedge a \wedge (a' \wedge b)' = y \wedge a \wedge (a' \wedge b)'$.

(2') $(x \vee b) \wedge (a' \wedge b)' = (y \vee b) \wedge (a' \wedge b)'$.

Proof. If L is a demi p-lattice then L satisfies $x' \wedge x'' = 0$.

Since $a \leq b$,

$$(a \wedge b')' = (a \wedge b \wedge b')' = (a \wedge b'' \wedge b')' = 0' = 1.$$

Thus, in Theorem 2.2.1, $s' = 1$ and $s'' = 0$ and so conditions (3), (4),(5) and (6) hold trivially. We have just to prove that in $DMPL$ (1), (2) and (7) are equivalent to (1') and (2').

We point out there that, $t = a \vee b'$, $t' = a' \wedge b''$ and $t'' = (a' \wedge b)'$.

Now,

$$x \wedge a \wedge t'' \wedge t' = y \wedge a \wedge t'' \wedge t'$$

by the demi p-lattice identity. And (1) states that

$$(x \wedge a \wedge t'') \vee t' = (y \wedge a \wedge t'') \vee t'.$$

Thus $x \wedge a \wedge t'' = y \wedge a \wedge t''$ by the characterization of $\theta_{lat}(t', t')$ and so, applying (S3) and (S6), we easily derive (1').

In a similar way we can prove that (2) implies (2').

It can be readily seen that (1') and (2') imply (1) and (2).

To show that (1') and (2') imply (7), note that (1') and (2') are equivalent to

$$x \wedge a \wedge t'' = y \wedge a \wedge t'' \quad \text{and} \quad (x \vee b) \wedge t'' = (y \vee b) \wedge t''$$

with $t = a \vee b'$.

Since $b \geq b \wedge t''$, we have by distributivity,

$$x \wedge t'' \wedge a = y \wedge t'' \wedge a \quad \text{and} \quad (x \wedge t'') \vee b = (y \wedge t'') \vee b.$$

Therefore $(x \wedge t'', y \wedge t'') \in \theta_{latL}(a, b)$.

As $L \in DMPL$ and $a \leq b$, $(x \wedge t'', y \wedge t'') \in \theta_{latL}(a, b)$ implies

$$\left((x \wedge t'')', (y \wedge t'')'\right) \in \theta_{latL}(t'', 1)$$

([39] Lemma 3.1). It follows that

$$(x \wedge t'')' \wedge t'' = (y \wedge t'')' \wedge t''$$

and, by (S6) we obtain (7). □

Corollary 2.2.5. *$L \in \mathcal{C}$ and $a, b \in L$ with $a \leq b$. If $L \in \mathcal{K}_{1,1}$ then $(x, y) \in \theta(a, b)$ if and only if x, y satisfy:*

(1") $(x \wedge a \wedge (a'' \vee b')) \vee (a' \wedge b'') = (y \wedge a \wedge (a'' \vee b')) \vee (a' \wedge b'')$.

(2") $((x \vee b) \wedge (a'' \vee b')) \vee (a' \wedge b'') = ((y \vee b) \wedge (a'' \vee b')) \vee (a' \wedge b'')$.

(3") $x \wedge a \wedge a'' \wedge b' = y \wedge a \wedge a'' \wedge b'$.

(4") $(x \vee b) \wedge a'' \wedge b' = (y \vee b) \wedge a'' \wedge b'$.

(5") $(x \wedge a) \vee a' \vee b'' = (y \wedge a) \vee a' \vee b''$.

(6") $x \vee b \vee a' \vee b'' = y \vee b \vee a' \vee b''$.

Proof. If $L \in \mathcal{K}_{1,1}$ then L satisfies $(x \wedge y)' = x' \vee y'$.

Replacing t by $a \vee b'$ and s by $a \wedge b'$ it is clear that conditions (1)-(6) of Theorem 2.2.1 are equivalent to conditions (1")-(6") when $L \in K_{1,1}$. So we only have to prove that, when $L \in \mathcal{K}_{1,1}$ and $a \leq b$, conditions (1")-(6") imply condition (7) of Theorem 2.2.1.

In fact, (1") implies

$$((x \wedge a \wedge (a'' \vee b')) \vee (a' \wedge b''))' = ((y \wedge a \wedge (a'' \vee b')) \vee (a' \wedge b''))' .$$

Applying axioms of $K_{1,1}$ we obtain

$$(x' \vee a') \wedge (a'' \vee b') = (y' \vee a') \wedge (a'' \vee b')$$

so meeting with a'' gives

$$(x' \vee a') \wedge a'' = (y' \vee a') \wedge a'' \qquad (*).$$

In a similar way, from (2") we can derive

$$(((x \vee b) \wedge (a'' \vee b')) \vee (a' \wedge b''))' = (((y \vee b) \wedge (a'' \vee b')) \vee (a' \wedge b''))' ,$$

and

$$(x' \vee b'') \wedge (a'' \vee b') \wedge a' = (y' \vee b'') \wedge (a'' \vee b') \wedge a'.$$

Then we obtain,

$$(x' \vee b'') \wedge b' = (y' \vee b'') \wedge b' \qquad (**).$$

From $(*)$ and $(**)$ it follows,

$$((x' \vee a') \wedge a'') \vee ((x' \vee b'') \wedge b') = ((y' \vee a') \wedge a'') \vee ((y' \vee b'') \wedge b')$$

and by distributivity and absorption

$$(x' \vee a' \vee b'') \wedge (x' \vee a' \vee b') \wedge (a'' \vee x' \vee b'') \wedge (a'' \vee b') =$$
$$= (y' \vee a' \vee b'') \wedge (y' \vee a' \vee b') \wedge (a'' \vee y' \vee b'') \wedge (a'' \vee b') .$$

Therefore,

$$(x' \vee a' \vee b'') \wedge (x' \vee a') \wedge (x' \vee b'') \wedge (a'' \vee b') =$$
$$= (y' \vee a' \vee b'') \wedge (y' \vee a') \wedge (y' \vee b'') \wedge (a'' \vee b') .$$

and so we conclude

$$(x' \vee a') \wedge (x' \vee b'') \wedge (a'' \vee b') = (y' \vee a') \wedge (y' \vee b'') \wedge (a'' \vee b') .$$

Applying axioms of $K_{1,1}$ it is easy to check that this condition is equivalent to condition (7) in Theorem 2.2.1. □

Observe that from this corollary we can conclude that conditions (1")-(6") are equivalent to the eight conditions characterizing principal congruences in $K_{1,1}$ presented in [10] Theorem 8.1.

Theorem 2.2.1 shows that, in the variety $\mathcal{C}$, $(x, y) \in \theta(a, b)$ is determined by a set of equations involving only x, y, a, b because equations (1)-(7) can be written down explicitly by replacing t by $a \vee b'$ and s by $a \wedge b'$. Thus the following theorem is immediate.

Theorem 2.2.6. *The subvariety $\mathcal{C}$ of SDMA has equationally definable principal congruences.*

If an equational class has equationally definable principal congruences then the congruence extension property (CEP) holds in that class.

In [25], Hobby proved, using duality, that variety $\mathcal{C}$ has the congruence extension property. Using Theorem 2.2.1 we have proved algebraically the same fact:

Corollary 2.2.7. *The subvariety $\mathcal{C}$ of SDMA has the congruence extension property.*

2.3 Subdirectly irreducibles in variety $\mathcal{C}$.

Lemma 2.3.1. *Let $L \in \mathcal{C}$ and $a \in L$ such that $a'' \leq a'$. Then $\theta(a'', a') = \theta_{latL}(a'', a')$.*

Proof. Clearly $\theta_{latL}(a'', a') \leq \theta(a'', a')$.

To prove the other inclusion let $(x, y) \in \theta(a'', a')$. From Theorem 2.2.1 (3), we obtain $x \wedge a'' = y \wedge a''$. Theorem 2.2.1 (6) implies $x \vee a' = y \vee a'$.

Thus $(x, y) \in \theta_{latL}(a'', a')$. □

Lemma 2.3.2. *Let $L \in \mathcal{C}$ and $a, c \in L$ such that $a'' < a' < c'$. Then $\theta(a'', a') \wedge \theta(a', c') = \Delta$.*

Proof. Let $(x, y) \in \theta(a'', a') \wedge \theta(a', c')$.

We will use Theorem 2.2.1 (1). Considering that $(x, y) \in \theta(a', c')$, we have $t = a' \vee c''$. Since $c'' \leq a'' < a'$ by (S9), we have $t = a'$. So (1) gives

$$(x \wedge a') \vee a'' = (y \wedge a') \vee a'' \quad (*).$$

By Lemma 2.3.1, $(x, y) \in \theta(a'', a')$ implies $x \wedge a'' = y \wedge a''$.Thus, since $a'' < a'$, we have,

$$x \wedge a' \wedge a'' = y \wedge a' \wedge a'' \qquad (**)$$

.

Then $(*)$ and $(**)$ imply $x \wedge a' = y \wedge a'$ by the characterization of $\theta_{latL}(a'', a'')$.

From Lemma 2.3.1 it follows $x \vee a' = y \vee a'$,therefore, by the characterization of $\theta_{latL}(a', a')$, we conclude $x = y$. □

Lemma 2.3.3. *Let $L \in \mathcal{C}$ be a subdirectly irreducible algebra. Then either $a' \wedge a'' = 0$ for every $a \in L$ or $a' = a''$ for every $a \in L$ such that $a'' \in L \backslash \{0, 1\}$.*

Proof. We prove first that, if $L \in \mathcal{C}$ is a subdirectly irreducible algebra, for every $a \in L$, $a' = a''$ or $a' \wedge a'' = 0$.

Suppose, by way of obtaining a contradiction, that there exists $a \in L$ such that $a' \neq a''$ and $a' \wedge a'' \neq 0$.

Since $a', a'' \in DM(L)$ and $a' \neq a''$, we have in $\left(DM(L), \wedge, \dot{\vee}, 0, 1\right)$, $a' \wedge a'' < a' \dot{\vee} a''$. But $a' \dot{\vee} a'' = (a \wedge a')'$, so $a' \wedge a'' < (a' \wedge a)'$.

Observe that $(a' \wedge a)' < 1$, for if $(a' \wedge a)' = 1$ then $(a' \wedge a'', 0) \in \phi$ and since $a' \wedge a'' \in DM(L)$ it would be 0, a contradiction.

Therefore we have a chain $(a' \wedge a)'' < (a' \wedge a)' < 0'$.

From Lemma 2.3.2 it follows that

$$\theta\left((a' \wedge a)'', (a' \wedge a)'\right) \wedge \theta\left((a' \wedge a)', 0'\right) = \Delta$$

, a contradiction since L is subdirectly irreducible.

Thus we have $a' = a''$ or $a' \wedge a'' = 0$, for every $a \in L$.

We claim that if there exists $a \in L$ such that $a' = a''$ then, for every $b'' \in L \setminus \{0,1\}$, $b' = b''$. To see this, assume there is $a \in L$ such that $a' = a''$ and consider $b'' \in L \setminus \{0,1\}$. As we have just proved, $b' = b''$ or $b' \wedge b'' = 0$.

Since $(a'' \wedge b'')'' = a'' \wedge b''$, we must have also,

$$(a'' \wedge b'')' = a'' \wedge b'' \text{ or } (a'' \wedge b'')' \wedge a'' \wedge b'' = 0.$$

First, let $(a'' \wedge b'')' = a'' \wedge b''$.

Since we have $(a'' \wedge b'')' = (a' \vee b')''$, by axiom (S3), it follows that $b' \leq (a' \vee b')'' = a'' \wedge b'' \leq b''$. Therefore $b' \wedge b'' = b' \neq 0$, so $b' = b''$.

Now let $(a'' \wedge b'')' \wedge a'' \wedge b'' = 0$.

Since $(a'' \wedge b'')' = (a' \vee b')'' = (a'' \vee b')'' \geq a'' \wedge b''$, we have $a'' \wedge b'' = 0$ and consequently $a' \dot{\vee} b' = (a'' \wedge b'')' = 1$.

Then we have in $\left(DM(L), \wedge, \dot{\vee}, 0, 1\right)$,

$$b'' = b'' \wedge \left(a' \dot{\vee} b'\right) = (b'' \wedge a') \dot{\vee} (b'' \wedge b').$$

If $b' \wedge b'' = 0$, then $b'' = b'' \wedge a'$. It follows $b'' \leq a'$ and, by (S9), $a'' \leq b'$. Since $a' = a''$, we conclude $b'' \leq b'$ and $b' \wedge b'' = b'' \neq 0$, a contradiction.

Therefore $b' = b''$. □

Lemma 2.3.4. *Let $L \in \mathcal{C}$ be a subdirectly irreducible algebra, then L is a subdirectly irreducible demi p-lattice or $DM(L)$ is a subdirectly irreducible De Morgan algebra.*

Proof. By Lemma 2.3.3, if $L \in \mathcal{C}$ is a subdirectly irreducible algebra then $a' \wedge a'' = 0$ for every $a \in L$ or $a' = a''$ for every $a'' \in L \setminus \{0,1\}$.

Suppose $a' \wedge a'' = 0$ for every $a \in L$. Then L is a subdirectly irreducible demi p-lattice.

Now suppose $a' = a''$ for every $a \in L$ such that $a'' \in L \setminus \{0,1\}$. Then every element of $DM(L) \setminus \{0,1\}$ is a fixed point. So, $DM(L)$ is of height < 3. Note also that the De Morgan algebra $DM(L)$ is such that $\{a' \in DM(L) : a' \wedge a'' = 0\} = \{0,1\}$. These last two facts imply, by [36] Theorem 3.4, that $DM(L)$ is a subdirectly irreducible De Morgan algebra. □

Lemma 2.3.5. *Let $L \in \mathcal{C}$ be a subdirectly irreducible algebra such that $DM(L)$ is a subdirectly irreducible De Morgan algebra. Then $h(L) < 4$.*

Proof. Suppose $h(L) \geq 4$. Then there exists a chain $0 < x_1 < x_2 < x_3 < 1$ in L. This leads to $0/\phi \leq x_1/\phi \leq x_2/\phi \leq x_3/\phi \leq 1/\phi$ in $L/\phi \simeq DM(L)$.

$DM(L)$ is a subdirectly irreducible De Morgan algebra, so $h(L) < 3$ and therefore we have at most three distinct ϕ classes in the chain above.

The argument below works in general, but for ease of presentation we will do the case where $(0, x_1) \in \phi$ and $(x_2, x_3) \in \phi$. By Proposition 1.4.8 (ii) we have $0 = x_1''$ and $x_2 = x_3''$ or $x_2'' = x_3$. This implies, by Proposition 1.4.8 (iii), $x_1'' \ll x_1$ and $x_2 = x_3'' \ll x_3$ or $x_1'' \ll x_1$ and $x_2 \ll x_3 = x_2''$. In both cases we have a contradiction with Proposition 1.4.8 (iv). □

In the next two lemmas our aim is to describe the subdirectly irreducible algebras $L \in \mathcal{C}$ such that $DM(L)$ is a subdirectly irreducible algebra of DMA.

Since the lattice reducts of the subdirectly irreducible De Morgan algebras B and K are chains it is clear, by Lemma 1.4.7 that if $L \in \mathcal{C}$ is a subdirectly irreducible algebra such that $DM(L) = B$ or $DM(L) = K$ then $L \in K_{1,1}$. So we conclude:

Lemma 2.3.6. *Let $L \in \mathcal{C}$ be a subdirectly irreducible algebra.*

(i) If $DM(L) = B$, then $L \in \{S_1, S, \overline{S}, B\}$

(ii) If $DM(L) = K$, then $L \in \{N, \overline{N}, L, \overline{L}, K_3, \overline{K}_3, K_2, \overline{K}_2, K_1, \overline{K}_1, K\}$.

To describe the subdirectly irreducible algebras $L \in \mathcal{C}$ such that $DM(L) = M$ we will use the following "strategy": We consider all the possible L_{lat} that are distributive lattices of height less than 4. For each of them we discuss the possible cardinalities of the ϕ classes in L applying Proposition 1.4.8 (i). When we obtain a semi-De Morgan algebra we test if it belongs to $\mathcal{C}$. This can be done algebraically. To decide the subdirect irreducibility of these algebras we can apply Proposition 1.4.10. We found that besides the subdirectly irreducible algebras of $K_{1,1}$ there exist three new subdirectly irreducible algebras in $\mathcal{C}$. They were named C_1, C_2 and C_3 and their diagrams appear in Figure 2.1.

In these diagrams, as well as in those of Figure 2.2, we use solid lines to show the ϕ classes with two elements. Solid dots denote the elements of $DM(L)$ and hollow dots mean the elements of $L \setminus DM(L)$.

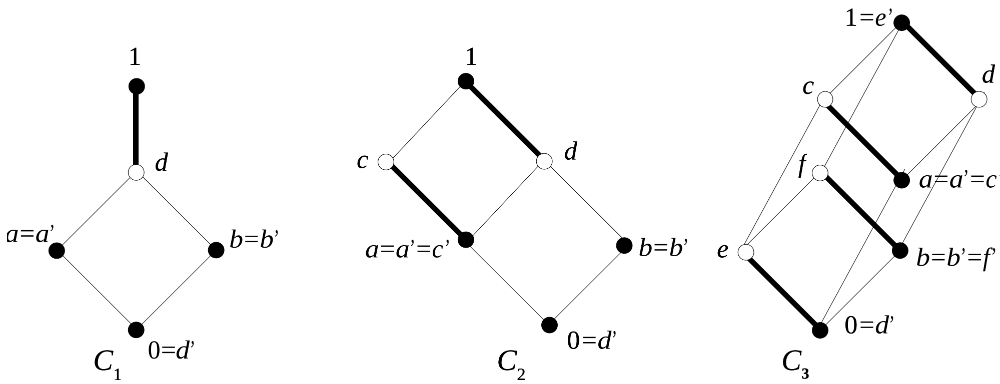


Figure 2.1:

The next lemma shows that C_1, C_2 and C_3 are, apart from the subdirectly irreducible algebras in $K_{1,1}$, the only subdirectly irreducible algebras in $\mathcal{C}$ such that $DM(L) = M$.

Observe that, for the sake of simplicity, we will often use "equal" instead of "isomorphic".

Lemma 2.3.7. [1]

Let $L \in \mathcal{C}$ be a subdirectly irreducible algebra such that $DM(L) = M$. Then, up to isomorphism,

$$L \in \left\{C_1, C_2, C_3, B_1, M_1, \overline{M}_1, M\right\}$$

.

Proof. Since L_{lat} is a finite distributive lattice and $h(L_{lat}) < 4$ then L_{lat} is a sublattice of $\mathbf{2}^3$ (see [24] Chapter II Theorem 4.1).

In what follows we will consider the different sublattices L_{lat} of $\mathbf{2}^3$, such that $L/\phi \simeq M$. Now $DM(L) = M$ and we know every ϕ class has at most two elements , by Proposition 1.4.8 (i). So we can discuss all the different cases.

[1]This lemma was joint work with Professor R. Santos

As we did in Figure 2.1, we will denote by a and b the fixed points of L.

Case 1:

L_{lat} is $\mathbf{2}^3$.

Every ϕ class has two elements, therefore we have the following possibilities:

(i) The fixed points are an atom and a co-atom.

Then L is equal to B_1.

(ii) The fixed points are two atoms.

Then we obtain C_3.

(iii) The fixed points are two co-atoms.

In this case, $L \notin SDMA$. In fact, supposing $0/\phi = \{0, e\}$, where $e = a \wedge b$, we have $a'' \wedge b'' = e \neq (a \wedge b)'' = 0$.

Case 2:

L_{lat} is the sublattice $\mathbf{2} \times \mathbf{3}$ of $\mathbf{2}^3$.

It is clear, since ϕ is a congruence relation in L_{lat}, that exactly one of the classes $0/\phi$ and $1/\phi$ has two elements and that exactly one of the classes a/ϕ and b/ϕ has two elements.

We only need to consider the following subcases.

(i) The fixed points are an atom and a co-atom.

Assuming, without loss of generality, that a is a co-atom and b is an atom:

If $|1/\phi| = 2$ then $|0/\phi| = 1$, $|a/\phi| = 2$ and $|b/\phi| = 1$. So we obtain M_1. If $|1/\phi| = 1$ then $|0/\phi| = 2$, $|a/\phi| = 1$ and $|b/\phi| = 2$. So we obtain $\overline{M}_1$.

(ii) The fixed points are two atoms.

In this case $|0/\phi| = 1$ and $|1/\phi| = 2$. Suppose without loss of generality that $|a/\phi| = 2$ then $|b/\phi| = 1$ and we obtain C_2.

(iii) The fixed points are two co-atoms.

In this case $|1/\phi| = 1$, $|0/\phi| = 2$. Using the same argument of Case 1 (iii), we conclude $L \notin SDMA$.

Case 3:

L_{lat} is the sublattice $\mathbf{2}^2 \oplus \mathbf{1}$ or the sublattice $\mathbf{1} \oplus \mathbf{2}^2$ of $\mathbf{2}^3$.

It is obvious that one and only one of the classes $0/\phi$ and $1/\phi$ has two elements.

If $L_{lat} = \mathbf{2}^2 \oplus \mathbf{1}$ then $|1/\phi| = 2$, and we obtain C_1

If $L_{lat} = \mathbf{1} \oplus \mathbf{2}^2$ then $|0/\phi| = 2$, and as in Case 1 (iii), $L \notin SDMA$.

Case 4:

L_{lat} is the sublattice $\mathbf{2}^2$ of $\mathbf{2}^3$.

Clearly in this case we have $L = M$.

□

Theorem 2.3.8. *Let $L \in \mathcal{C}$. Then the following are equivalent:*

(i) L is subdirectly irreducible.

(ii) L is a subdirectly irreducible demi p-lattice or L is a subdirectly irreducible algebra in the variety $K_{1,1}$ of Ockham algebras or L is up to isomorphism in $\{C_1, C_2, C_3\}$.

Proof. We first show (i) implies (ii). By Lemma 2.3.4 we know that if $L \in \mathcal{C}$ is a subdirectly irreducible algebra then L is a subdirectly irreducible algebra of $DMPL$ or $DM(L)$ is a subdirectly irreducible algebra of DMA. In this case it results from Lemmas 2.3.6 and 2.3.7 that L is a subdirectly irreducible algebra in the variety $K_{1,1}$ of Ockham algebras or L is up to isomorphism in $\{\mathrm{C}_1, C_2, C_3\}$.

It is trivial that (ii) implies (i). □

In Figure 2.2 we present the diagrams of the finite subdirectly irreducible algebras of $\mathcal{C}$: $B_{0,2}$, K_1, $\overline{K}_1$, $\overline{K}_2$ and $\overline{S}$ since they will be necessary in the next section.

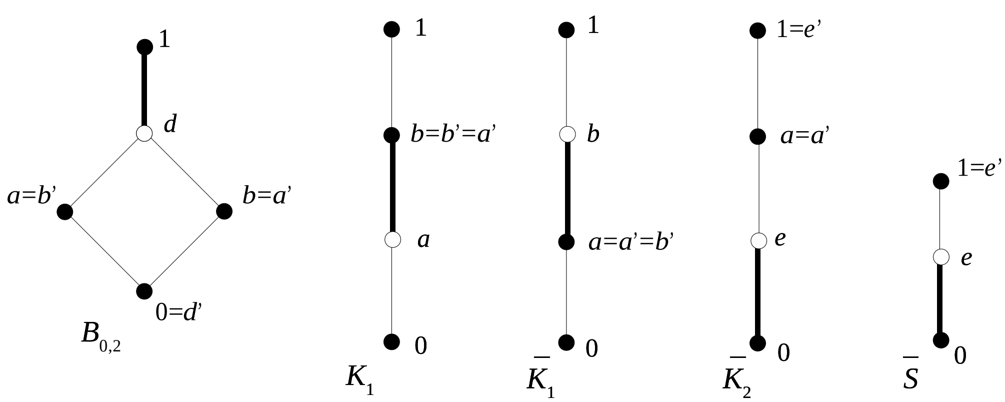


Figure 2.2:

Sankappanavar proved in [40] Corollary 3.2 that every subalgebra of a subdirectly irreducible demi-p-lattice is also subdirectly irreducible. The same is true in variety $\mathcal{C}$, since this is true for the subalgebras of the subdirectly irreducible algebras of $K_{1,1}$ and for the subalgebras of C_1, C_2 and C_3.

Corollary 2.3.9. *If $L \in \mathcal{C}$ is a subdirectly irreducible algebra, then so is every subalgebra of L.*

The variety $SDMA$ is known to be locally finite. A proof can be found in Theorem 5.1 of [25]. It is also well known that this variety is congruence-distributive. These results together with a fundamental theorem reached by Davey in [16] will help to describe a poset such that the lattice of its down-sets is isomorphic to the lattice of the subvarieties of variety $\mathcal{C}$.

Let $\Lambda(\mathcal{K})$ denote the lattice of subvarieties of a variety $\mathcal{K}$ and let $SI_F(\mathcal{K})$ be a set consisting of precisely one algebra from each of the isomorphism classes of the nontrivial finite subdirectly irreducible algebras in the variety $\mathcal{K}$.

Theorem 2.3.10. *(Davey [16], Theorem 3.3)*
Let $\mathcal{K}$ be a locally finite congruence-distributive variety and order the set $SI_F(\mathcal{K})$ by

$$L_1 \leq L_2 \quad \text{if and only if} \quad L_1 \in HS(L_2).$$

Then $\Lambda(\mathcal{K})$ is a completely distributive lattice and is isomorphic to $\mathcal{D}(SI_F(\mathcal{K}))$.

Sankappanavar described the poset $SI_F(DMPL)$ in [40] (see Definition 6.7 (iii) and (iv), Lemma 6.8 and Corollary 6.10). In [9] page 91, the poset $SI_F(K_{1,1})$ is described so, using these results and Theorem 2.3.8, it is easy to obtain the Hasse diagram of the poset $SI_F(\mathcal{C})$.

With this aim we note that both in $K_{1,1}$ and $DMPL$ every non-trivial homomorphic image of a finite subdirectly irreducible algebra L is isomorphic to a subalgebra of L (see [37] Corollary 2.9 and [40] Lemma 3.5.) so, for these algebras, $L_1 \in HS(L_2)$ is equivalent to $L_1 \in IS(L_2)$. This is not true for C_1, C_2 and C_3. In fact, if L is in $\{C_1, C_2, C_3\}$ then $L/\phi \simeq M$ and M is not isomorphic to a subalgebra of L. Since for such an L, ϕ is the minimum element of $Con(L) \setminus \{\Delta\}$, it follows that, for any $\theta \in Con(L) \setminus \{\Delta, \nabla\}$, L/θ is a subdirectly irreducible algebra of DMA.

Thus, for $L \in \{C_1, C_2, C_3\}$, we have $L_1 \leq L$ if and only if $L_1 \in S(L)$ or L_1 is a subdirectly irreducible De Morgan algebra.

We present the diagram of $SI_F(\mathcal{C})$ in Figure 2.3 where, as pointed in [40], $B \subset S \subset B_{0,2} \subset \cdots \subset B_{0,n} \subset \cdots$ is the chain of subdirectly irreducible pseudocomplemented lattices.

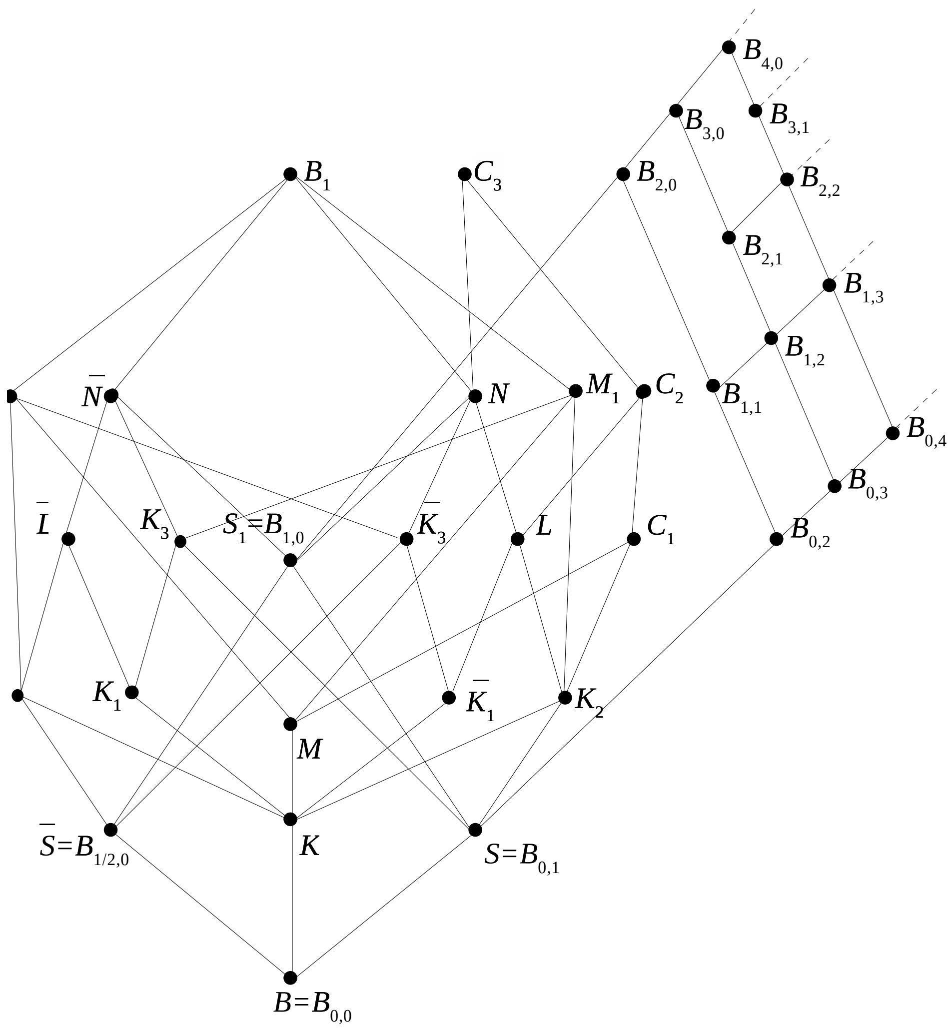


Figure 2.3: $SI_F(\mathcal{C})$

We can extend a result obtained by R. Beazer ([7] Corollary 7) for the variety $K_{1,1}$.

Corollary 2.3.11. *Let $L \in \mathcal{C} \setminus DMA$ be a finite subdirectly irreducible algebra. Then $Con(L)$=$\{\Delta,\ \phi,\ \nabla\}$ if and only if $L \in (K_{1,1} \setminus DMA) \cup \{C_1, C_2, C_3\}$.*

Proof. Let $L \in \mathcal{C} \setminus DMA$ be a finite subdirectly irreducible algebra.

Suppose $Con(L) = \{\Delta,\ \phi,\ \nabla\}$. Clearly $L \notin DMA$. Assume that $L \notin (K_{1,1} \setminus DMA) \cup \{C_1, C_2, C_3\}$. Then by the diagram of Figure 2.3 we have that L contains $B_{0,2}$ as a subalgebra. Since $Con(B_{0,2}) = \mathbf{1} \oplus \mathbf{2}^2$ and $DMPL$ satisfies CEP it is obvious $|Con(L)| > 3$, a contradiction.

If $L \in (K_{1,1} \setminus DMA) \cup \{C_1, C_2, C_3\}$ then by Theorem 2.3.8 L/ϕ is simple and $\Delta \neq \phi$ so $Con(L) = \{\Delta,\ \phi,\ \nabla\}$. □

2.4 Equational Bases for some Subvarieties.

In this section we describe equational bases for the subvarieties of $\mathcal{C}$ generated by C_1, by C_2 and by C_3.

Note that $K_{1,1}$ is characterized, as a subvariety of $\mathcal{C}$, by the identity $(a \wedge b)' = a' \vee b'$. So, when we consider each subvariety $\mathcal{V}$ of $K_{1,1}$, generated by a single subdirectly irreducible algebra, as a subvariety of $\mathcal{C}$ we only need to add to this identity the equational bases characterizing $\mathcal{V}$ as a subvariety of $K_{1,1}$. These equational bases are presented in [37] and [9].

If $L \in \mathcal{C}$ then $\mathcal{V}(L)$ denotes the variety generated by L. Please read "defined by" as "defined, modulo $\mathcal{C}$, by".

While reading the proofs of the following theorems it might be useful to refer to Figures 2.2 and 2.3

Theorem 2.4.1. *$\mathcal{V}(C_3)$ is defined by*

(λ) $a' \wedge a'' \leq a \wedge a'$.

(μ) $a \wedge a' \wedge b'' \wedge b' = b \wedge b' \wedge a'' \wedge a'$.

(ξ) $a' \vee a'' = (a' \wedge a)'$.

Proof. Observe $\mathcal{V}(C_3) \models (\lambda), (\mu)$, and (ξ), $B_{0,2} \not\models (\xi)$ because $a' \vee a'' = d$ and $(a' \wedge a)' = 1$, $K_1 \not\models (\lambda)$ because $a \wedge a' = a$ and $a'' \wedge a' = b$ and $\overline{K}_2 \not\models (\mu)$ because $e \wedge e' \wedge a'' \wedge a' = e$ and $a \wedge a' \wedge e'' \wedge e' = 0$.

Also observe that if L is a finite subdirectly irreducible algebra in $\mathcal{C}$ and $L \notin HS(C_3)$ then L contains, up to isomorphism, $B_{0,2}$ or K_1 or $\overline{K}_2$ as subalgebras. Thus, if $\mathcal{V}$ is the subvariety of $\mathcal{C}$ defined by (λ), (μ) and (ξ), then $SI_F(\mathcal{V}) = HS(C_3)$ so $\mathcal{V} = \mathcal{V}(C_3)$. □

Theorem 2.4.2. *$\mathcal{V}(C_2)$ is defined by*

(ξ) $a' \vee a'' = (a' \wedge a)'$.

(ρ) $a \wedge a' = a'' \wedge a'$.

Proof. Observe $\mathcal{V}(C_2) \models (\xi)$ and (ρ), $B_{0,2} \not\models (\xi)$ because $a' \vee a'' \neq (a' \wedge a)'$, $K_1 \not\models (\rho)$ because $a \wedge a' \neq a'' \wedge a'$ and $\overline{S} \not\models (\rho)$ because $e \wedge e' \neq e'' \wedge e'$.

If L is a finite subdirectly irreducible algebra in $\mathcal{C}$ and $L \notin HS(C_2)$ then L contains, up to isomorphism, $B_{0,2}$ or K_1 or $\overline{S}$ as subalgebras. Thus, if $\mathcal{V}$ is the subvariety of $\mathcal{C}$ defined by (ξ),and (ρ), then $SI_F(\mathcal{V}) = HS(C_2)$ so $\mathcal{V} = \mathcal{V}(C_2)$. □

Theorem 2.4.3. *$\mathcal{V}(C_1)$ is defined by*

(ξ) $a' \vee a'' = (a' \wedge a)'$.

(ρ) $a \wedge a' = a'' \wedge a'$.

(σ) $a \leq a''$.

Proof. Observe $\mathcal{V}(C_1) \models (\xi)$, (ρ) and (σ), $B_{0,2} \not\models (\xi)$, $K_1 \not\models (\rho)$ and $\overline{S} \not\models (\rho)$ as we saw in the previous theorem, and $\overline{K}_1 \not\models (\sigma)$ because $b'' < b$.

If L is a finite subdirectly irreducible algebra in $\mathcal{C}$ and $L \notin HS(C_1)$ then L contains, up to isomorphism, $B_{0,2}$ or K_1 or $\overline{K}_1$ or $\overline{S}$ as subalgebras. Thus, if $\mathcal{V}$ is the subvariety of $\mathcal{C}$ defined by (ξ), (ρ) and (σ), then $SI_F(\mathcal{V}) = HS(C_1)$ so $\mathcal{V} = \mathcal{V}(C_1)$. □

Chapter 3

Duality

We develop here a duality based on the theory of canonical extensions. The idea is that dualities are extracted by composing canonical extension and correspondence for complex algebras. This is a general program which this paper will illustrate.

3.1 Canonical extension and canonicity

3.1.1 Canonical extensions of Distributive Lattices

We will describe here some results on canonical extensions of bounded distributive lattices ($\mathcal{DL}$) obtained by M. Gehrke and B. Jónsson in [18], [19] and [20] and by M. Gehrke, H. Nagahashi and Y. Venema in [21].

In the interest of self containment we list here some important facts.

For a lattice L, we will denote by $J^{\infty}(L)$ the set of all completely join irreducible elements of L and by $M^{\infty}(L)$ the set of all completely meet irreducible elements of L. By $J^{\infty}_{\omega}(L)$ and $M^{\infty}_{\omega}(L)$ we denote the sets consisting, respectively, of all finite joins of elements of $J^{\infty}(L)$ and all finite meets of elements of $M^{\infty}(L)$. Observe that $0 \in J^{\infty}_{\omega}(L)$ but $0 \notin J^{\infty}(L)$ and that $1 \in M^{\infty}_{\omega}(L)$ but $1 \notin M^{\infty}(L)$. For convenience we set $J^{\infty}_{0}(L) = J^{\infty}(L) \cup \{0\}$ and $M^{\infty}_{1}(L) = M^{\infty}(L) \cup \{1\}$.

Definition 3.1.1. ([21] Definition 2.12, [20] Definition 2.4)

Suppose L is a bounded sublattice of a complete lattice L'. We say that

1. L is *dense* in L' if every element of L' can be expressed both as a join of meets and as a meet of joins of elements from L.

2. L is *separating* in L' if for all $p, q \in J^{\infty}(L')$ with $p \not\leq q$ there exists $a \in L$ such that $q \leq a$ and $p \not\leq a$.

3. L is *compact* in L' if, for all $S, T \subseteq L$ with $\bigwedge S \leq \bigvee T$ in L', there exist finite sets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

For a bounded distributive lattice L, the lattice of clopen down-sets of the dual Priestley space $(X, \leq, \tau)$ is dense and compact in the complete lattice of down-sets of $(X, \leq)$. On the other hand, one can show that between any two complete extensions of L that satisfy these two conditions there is a unique isomorphism so the following definition makes sense:

Definition 3.1.2. ([21] Definition 2.13)

The *canonical extension* of a $\mathcal{DL}$ L is a complete lattice L^{σ} containing L as a dense and compact sublattice.

And we have also the following,

Theorem 3.1.3 ([20]Theorem 2.5. (i)). *If L is a $\mathcal{DL}$, then L^{σ} contains L as a separating sublattice.*

Let $L \in \mathcal{DL}$. We will often use the lattice obtained from L by reversing the order which we call the *(order) dual lattice* of L.

Definition 3.1.4. ([21] Definition 2.10)

Given a bounded distributive lattice $L = (L, \vee, \wedge, 0, 1)$, let L^{∂} denote the *dual lattice*, that is, the structure $L^{\partial} = (L, \wedge, \vee, 1, 0)$. For technical convenience, let $L^1 = L$.

An element $\varepsilon \in \{1, \partial\}^n$ is called an *order type*; the i-th component of such an ε will be denoted by ε_i.

Given an order type $\varepsilon \in \{1, \partial\}^n$, let L^{ε} denote $L^{\varepsilon_1} \times ... \times L^{\varepsilon_n}$.

Obviously L and L^{∂} have the same universe. Since the characterization of canonical extensions is self-dual it is possible to identify $(L^{\sigma})^{\partial}$ with $(L^{\partial})^{\sigma}$ ([19] Corollary 3).

We always have $J^{\infty}(L^{\partial}) = M^{\infty}(L)$ and $M^{\infty}(L^{\partial}) = J^{\infty}(L)$.

The canonical extension of a distributive lattice is a *perfect* distributive lattice.

Definition 3.1.5. ([21] Definition 2.14)

A distributive lattice L is called *perfect* or a $\mathcal{DL}^+$ if it satisfies one of the following, equivalent, conditions:

1. L is doubly algebraic (that is, both L and L^∂ are algebraic).

2. L is complete, completely distributive and join generated by the set $J^\infty(L)$

3. L is isomorphic to a set-theoretic lattice based on the collection of down-sets of some partially ordered set.

Note that, since condition 1 is self-dual, the duals of conditions 2 and 3 also hold for perfect distributive lattices.

As is mentioned in [21], the density of L in L^σ implies that $J^\infty(L^\sigma)$ is contained in $K(L^\sigma)$, the meet closure of L in L^σ, and $M^\infty(L^\sigma)$ is contained in $O(L^\sigma)$, the join closure of L in L^σ.

Since the canonical extension of a bounded distributive lattice can be obtained by taking the poset of all order ideals or down-sets of its topological dual space, we refer to elements of L as *clopen* elements, to elements of $K(L^\sigma)$ as *closed* elements and to elements of $O(L^\sigma)$ as *open* elements of L^σ.

If L is a perfect distributive lattice, there is a natural isomorphism between the posets $J^\infty(L)$ and $M^\infty(L)$.

Definition 3.1.6. ([20] Definition 2.2)

Given a lattice $L \in \mathcal{DL}^+$ we define

(i) $\kappa(p) = \bigvee(L \setminus (\uparrow p))$ for all $p \in J^\infty(L)$.

(ii) $\kappa^{-1}(u) = \bigwedge(L \setminus (\downarrow u))$ for all $u \in M^\infty(L)$.

Theorem 3.1.7 ([20] Theorem 2.3). *For any $L \in \mathcal{DL}^+$,*

$$\kappa : (J^\infty(L), \leq) \simeq (M^\infty(L), \leq) : \kappa^{-1}$$

i.e., κ is an isomorphism from the poset $J^\infty(L)$ onto the poset $M^\infty(L)$ and κ^{-1} is the inverse of κ.

Remark 3.1.8. Observe that this isomorphism is such that, for any $a \in L$ and any $p \in J^\infty(L)$,

$$p \nleq a \Leftrightarrow \kappa(p) \geq a.$$

A map $f : L \to K$ between $L, K \in \mathcal{DL}$ can be extended in various ways to a map between L^σ and K^σ.

In order to define the extensions of a map f, six topologies will be introduced here.

Definition 3.1.9. ([20] Definition 2.6.)

Suppose $L \in \mathcal{DL}$. We denote by σ, $\sigma^\uparrow$ and $\sigma^\downarrow$ the topologies on L^σ having as bases, respectively, the open sets of the forms $\uparrow p \cap \downarrow u$, $\uparrow p$ and $\downarrow u$, with $p \in K(L^\sigma)$ and $u \in O(L^\sigma)$.

The other three topologies will be defined in the same manner, except that now the elements p and u are, respectively, compact and dually compact in the lattice theoretic sense and thus these topologies can be defined in an arbitrary doubly algebraic distributive lattice K.

Definition 3.1.10. ([20] Definition 2.7)

Suppose K is a doubly algebraic lattice. We denote by ι, $\iota^\uparrow$ and $\iota^\downarrow$ the topologies on K having as bases, respectively, the open sets of the forms $\uparrow p \cap \downarrow u$, $\uparrow p$ and $\downarrow u$, with p compact and u dually compact.

Observe that the elements of $J^\infty_\omega(K)$ are the compact elements in the sense of lattice theory and the members of $M^\infty_\omega(K)$ are the dually compact elements and that it follows that sets of the form $\uparrow p \cap \downarrow u$, $\uparrow p$ and $\downarrow u$, with $p \in J^\infty(K)$ and $u \in M^\infty(K)$ form subbases for the topologies ι, $\iota^\uparrow$ and $\iota^\downarrow$, respectively.

Obviously, the topology σ is the join of $\sigma^\uparrow$ and $\sigma^\downarrow$ and ι is the join of $\iota^\uparrow$ and $\iota^\downarrow$.

For any $L \in \mathcal{DL}$, the topologies ι, $\iota^\uparrow$ and $\iota^\downarrow$ on L^σ are weaker than σ, $\sigma^\uparrow$ and $\sigma^\downarrow$, respectively, because $J^\infty_\omega(L^\sigma) \subseteq K(L^\sigma)$ and $M^\infty_\omega(L^\sigma) \subseteq O(L^\sigma)$.

In general, if we have sets X and Y with topologies τ and μ and if the map $g : X \to Y$ is continuous relatively to τ on X and μ on Y we say that g is (τ, μ)-*continuous*.

Remark 3.1.11. Since continuity only needs to hold on a subbase we have:

The map $g : L^\sigma \to K^\sigma$ is $(\sigma, \iota^\uparrow)$-continuous means that, for all $u \in L^\sigma$ and for all $q \in J^\infty(K^\sigma)$, if $q \leq g(u)$ then there exist $K(L^\sigma) \ni x \leq u \leq y \in O(L^\sigma)$ so that $q \leq g(v)$ for all $v \in L^\sigma$ such that $x \leq v \leq y$.

And:

The map g is $(\sigma, \iota^{\downarrow})$-continuous means that, for all $u \in L^\sigma$ and for all $n \in M^\infty(K^\sigma)$, if $n \geq g(u)$ then there exist $K(L^\sigma) \ni x \leq u \leq y \in O(L^\sigma)$ so that $n \geq g(v)$ for all $v \in L^\sigma$ such that $x \leq v \leq y$.

Now we will describe the canonical extensions of arbitrary maps between distributive lattices and we will consider the continuity of these maps.

Definition 3.1.12. ([21] Definition 2.15)
Suppose $L, K \in \mathcal{DL}$. Given a map $f : L \to K$ we define two maps

$$f^\sigma, f^\pi : L^\sigma \longrightarrow K^\sigma$$

by

$$f^\sigma(u) = \bigvee \left\{ \bigwedge \{f(a) :\ a \in L \text{ and } x \leq a \leq y\} :\ K(L^\sigma) \ni x \leq u \leq y \in O(L^\sigma) \right\}$$

and

$$f^\pi(u) = \bigwedge \left\{ \bigvee \{f(a) :\ a \in L \text{ and } x \leq a \leq y\} :\ K(L^\sigma) \ni x \leq u \leq y \in O(L^\sigma) \right\}$$

for each $u \in L^\sigma$.

As proved in [20] Theorem 2.12 we have the following topological characterization of f^σ and f^π:

Theorem 3.1.13. *For any map $f : L \to K$ with $L, K \in \mathcal{DL}$, the maps f^σ and f^π are extensions of f. In fact, f^σ is the largest $(\sigma, \iota^{\uparrow})$-continuous extension of f to L^σ and f^π is the smallest $(\sigma, \iota^{\downarrow})$-continuous extension of f to L^σ.*

From [20] Theorem 2.11 (iii) it follows:

Corollary 3.1.14. *For any map $f : L \to K$ with $L, K \in \mathcal{DL}$, the maps f^σ and f^π are such that $f^\sigma \leq f^\pi$.*

Maps f for which $f^\sigma = f^\pi$ are particularly nice. These maps are called *smooth* and we have:

Corollary 3.1.15 ([20]Corollary 2.17). *If $f : L \to K$ is smooth, then the extension $f^\sigma = f^\pi$ of f is (σ, ι)-continuous. Conversely, if f has a (σ, ι)-continuous extension $g : L^\sigma \to K^\sigma$, then f is smooth and $f^\sigma = f^\pi = g$.*

Remark 3.1.16. If the map $f : L \to K$ is isotone then, from [20] Corollary 2.19, it follows that the descriptions of f^σ and f^π can be simplified . Then for all $u \in L^\sigma$

$$f^\sigma(u) = \bigvee\left\{\bigwedge\{f(a) :\ a \in L \text{ and } x \leq a\} :\ K(L^\sigma) \ni x \leq u)\right\}$$

and

$$f^\pi(u) = \bigwedge\left\{\bigvee\{f(a) :\ a \in L \text{ and } a \leq y\} :\ u \leq y \in O(L^\sigma)\right\}.$$

Also, for all $x \in K(L^\sigma)$ and $y \in O(L^\sigma)$,

$$f^\sigma(x) = f^\pi(x) = \bigwedge\{f(a) :\ a \in L \text{ and } x \leq a\},$$

$$f^\pi(y) = f^\sigma(y) = \bigvee\{f(a) :\ a \in L \text{ and } a \leq y\}.$$

So, f^σ and f^π agree on closed and open elements .

Remark 3.1.17. For isotone maps, the $(\sigma, \iota^\uparrow)$-continuity of f^σ yields the $(\sigma^\uparrow, \iota^\uparrow)$-continuity of f^σ.

This continuity may be stated as follows and was in fact already identified in this non-topological guise in [18]:

Lemma 3.1.18 ([18]Corollary 3.9). *Suppose $f : L \to K$, with $L, K \in \mathcal{DL}$, is isotone and $p \in J^\infty(K^\sigma)$. Below every element $u \in L^\sigma$ that satisfies the inclusion $p \leq f^\sigma(u)$ there is a minimal solution of this inclusion, and all the minimal solutions are closed elements.*

When composing isotone maps we have:

Theorem 3.1.19 ([19] Theorem 15). *Let $g : L_1 \to L_2$ and $f : L_2 \to L_3$ be isotone maps on $\mathcal{DL}$s. If g is meet preserving then $(f \circ g)^\sigma = f^\sigma \circ g^\sigma$. Dually, if g is join preserving, then $(f \circ g)^\pi = f^\pi \circ g^\pi$.*

Some special maps play an important part in our study so we will see how they behave in what concerns canonical extensions. To do so we have to give the following:

Definition 3.1.20. Let $L, K \in \mathcal{DL}^+$ and let $\{a_i :\ i \in I\}$ be any subset of L. We say that a map $f : L \to K$ is:

(i) *completely join preserving*, if $f(0) = 0$ and $f\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} f(a_i)$.

(ii) *completely meet preserving*, if $f(1) = 1$ and $f\left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} f(a_i)$.

(iii) *completely join reversing*, if $f(0) = 1$ and $f\left(\bigvee_{i \in I} a_i\right) = \bigwedge_{i \in I} f(a_i)$.

(iv) *completely meet reversing*, if $f(1) = 0$ and $f\left(\bigwedge_{i \in I} a_i\right) = \bigvee_{i \in I} f(a_i)$.

Properties of join preserving maps were studied first [18]. Then they were generalized to the other cases by using properties of the order dual of a distributive lattice.

Remark 3.1.21. ([21] Remark 2.11)

Given $L, K \in \mathcal{DL}$ and two order types (Definition 3.1.4) ε and ε', if we consider two maps $f : L^{\varepsilon} \to K$ and $g : L^{\varepsilon'} \to K$ that are set-theoretically identical (that is $f(a) = g(a)$ for all $a \in L^n$), such maps will be called *order variants*.

Observe that, from the definition of f^{σ} and g^{σ}, if f and g are order variants then so are f^{σ} and g^{σ} and these maps are also set-theoretically identical. Clearly, the same is also true for f^{π} and g^{π}.

Dualizing the order in the codomain however does make a difference. In fact, given a map $f : L \to K$, denoting by f^{∂} the same set-theoretic map as f but with the order dualized on both domain and codomain, $f^{\partial} : L^{\partial} \to K^{\partial}$, it is easy to see, from the definitions of f^{σ} and f^{π}, that $(f^{\partial})^{\sigma} = (f^{\pi})^{\partial}$ and $(f^{\partial})^{\sigma} = (f^{\pi})^{\partial}$.

Therefore, if a map f is smooth both its order variants and f^{∂} are smooth, and either extension of any order variant of f and f^{∂} all give the same set theoretic function.

Using these properties it is possible to prove:

Lemma 3.1.22 ([19]Theorem 10, [20]Corollary 2.28). *If the map $f : L \to K$, with $L, K \in \mathcal{DL}$, is a join preserving map then f^{π} is completely join preserving, if f is meet preserving then f^{σ} is completely meet preserving.*

In either case f is smooth.

and

Lemma 3.1.23 ([21] page 12, [20] Theorem 2.23). *Let $L, K \in \mathcal{DL}$.*

If the map $f : L \to K$ is a join reversing (or meet reversing) map then f^{σ} is completely join reversing (or meet reversing). In either case f is smooth.

3.1.2 Canonical extensions of Distributive Algebras

We will now consider algebras having as a reduct a bounded distributive lattice and unary operations that are meet or join preserving or meet or join reversing. These were called Distributive Modal Algebras (DMA) in [21] where they were considered as algebraic counterparts to Distributive Modal Logics. However, as our main object of study here will be semi-De Morgan Algebras which contain the De Morgan Algebras (DM) or (DMA), these latter algebras could easily be confused with the Distributive Modal Algebras. In [22], operations that turn joins or meets in any one coordinate to joins (meets) are called (dual) quasi-operators and thus we will use the name Unary quasi-operators algebras, UQA, instead of DMA for these algebras which consist of a distributive lattice with additional operations that are unary quasi-operators or unary dual quasi-operators.

Definition 3.1.24. Let UQA be the class of algebras

$$(L, \wedge, \vee, 0, 1, \Diamond, \Box, \rhd, \lhd)$$

such that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and $\Diamond, \Box, \rhd, \lhd$ are unary operations such that for any $a, b \in L$:

$$\begin{array}{rlcl} \text{(i)} & \Diamond 0 = 0 & \text{and} & \Diamond(a \vee b) = \Diamond a \vee \Diamond b. \\ \text{(ii)} & \Box 1 = 1 & \text{and} & \Box(a \wedge b) = \Box a \wedge \Box b. \\ \text{(iii)} & \rhd 0 = 1 & \text{and} & \rhd (a \vee b) = \rhd a \wedge \rhd b. \\ \text{(iv)} & \lhd 1 = 0 & \text{and} & \lhd (a \wedge b) = \lhd a \vee \lhd b. \end{array}$$

Let L^σ be the canonical extension of L. The operations $\Diamond^\sigma$ and $\Diamond^\pi$ are such that, for each $u \in L^\sigma$,

$$\Diamond^\sigma(u) = \bigvee\{\bigwedge\{\Diamond a : x \leq a \in L\} : u \geq x \in K(L^\sigma)\}$$

$$\Diamond^\pi(u) = \bigwedge\{\bigvee\{\Diamond a : y \geq a \in L\} : u \leq y \in O(L^\sigma)\}.$$

Since all maps that preserve joins are smooth (Lemma 3.1.22), we have $\Diamond^\sigma = \Diamond^\pi$.

Similarly, the other unary operations can be extended to L^σ and all these extensions are smooth so it makes sense the following definition:

Definition 3.1.25 ([21] Definition 2.19). Let $(L, , \wedge, \vee, 0, 1, \Diamond, \Box, \rhd, \lhd) \in UQA$. The *canonical or perfect extension* of L is the algebra

$$(L^\sigma, \wedge, \vee, 0, 1, \Diamond^\sigma, \Box^\sigma, \rhd^\sigma, \lhd^\sigma) = (L^\sigma, \wedge, \vee, 0, 1, \Diamond^\pi, \Box^\pi, \rhd^\pi, \lhd^\pi).$$

From Lemma 3.1.22 and subsequent remarks we can conclude that these canonical extensions satisfy the conditions of the following definition:

Definition 3.1.26. Let UQA^+ be the class of algebras

$$(L, \wedge, \vee, 0, 1, \Diamond, \Box, \rhd, \lhd)$$

such that $(L, \wedge, \vee, 0, 1) \in \mathcal{DL}^+$, and $\Diamond, \Box, \rhd, \lhd$ are unary operations such that for any subset $\{a_i : i \in I\} \subseteq L$:

(i) $\Diamond 0 = 0$ and $\Diamond \left(\bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} \Diamond a_i.$

(ii) $\Box 1 = 1$ and $\Box \left(\bigwedge_{i \in I} a_i \right) = \bigwedge \Box a_i.$

(iii) $\rhd 0 = 1$ and $\rhd \left(\bigvee_{i \in I} a_i \right) = \bigwedge_{i \in I} \rhd a_i.$

(iv) $\lhd 1 = 0$ and $\lhd \left(\bigwedge_{i \in I} a_i \right) = \bigvee_{i \in I} \lhd a_i.$

So,

Lemma 3.1.27 ([21] Lemma 2.21). *If L is a UQA then L^σ is a UQA^+.*

In what concerns UQA-homomorphisms observe that UQA-algebras are in the scope of the following definitions:

Definition 3.1.28. ([21] Definition 5.4)

A *Distributive Lattice Expansion* is any algebra $(L, (f_i)_{i \in I})$ consisting of a $\mathcal{DL}$, $L = (L, \vee, \wedge, 0, 1)$, and additional operations $f_i : L^{n_i} \to L$ for each $i \in I$. Such an algebra is said to be *monotone* provided each basic operation is *monotone*, that is, for each $i \in I$, there is an order type $\varepsilon_i \in \{1, \partial\}^{n_i}$ so that $f_i : L^{\varepsilon_i} \to L$ is order preserving. The sequence $(\varepsilon_i)_{i \in I}$ is called the *monotonicity type* of L.

For these algebras we have (from [20] Lemma 3.24) that

Every homomorphism $h : L \to K$ where L, K are monotone is preserved by canonical extensions. Even more, $h^\sigma : L^\sigma \to K^\sigma$ is a complete homomorphism.

So we can conclude that σ is a functor from the category of all UQAs and their homomorphisms into the category of all UQA^+s and their complete homomorphisms.

Remark 3.1.29. It is clear that these results apply to algebras that are reducts of UQAs.

Properties that are preserved by canonical extensions are of great interest. Such properties will be called *canonical*.

Definition 3.1.30. ([21] Definition 2.22)

A class of UQAs is *canonical* if it is closed under taking canonical extensions.

An equation, formula or set of formulas is called *canonical* if the class of UQAs defined by the equation, formula or set of formulas is canonical.

3.1.3 Sahlqvist inequalities and canonicity

Not all the properties in a UQA are canonical, but a broad class of canonical inequalities, called Sahlqvist inequalities, was identified for unary Boolean algebras with operators in [35] and for UQAs in [21]. The definitions in [21] were given for so-called sequents, but we will give them here for the corresponding inequalities.

With each term for UQAs associate two generation trees, the *positive* and the *negative generation tree*, depending on whether the sign of the root is $(+)$ and $(-)$, respectively. Each of these will be an expansion of the term's generation tree in which every node is signed with either $+$ or $-$. These signings are required to satisfy the following constraints:

- If a node is $\vee, \wedge, \Diamond$ or $\Box$, assign the same sign to its successor nodes.
- If a node is $\lhd$ or $\rhd$, assign the opposite sign to its successor node.

By these conditions, the sign of each node is determined by the sign of the root of the tree. Thus with each term we may associate two signed generation trees, the positive (with positive root) and the negative one (with negative root).

Definition 3.1.31. ([21] Definition 3.1)
A node in a signed generation tree of a UQA term is said to be

1. *positive* if it is signed "+" and *negative* if it is signed "−".
2. a *choice node* if it is either positive and labeled $\vee, \Diamond, \lhd$ or negative and labeled $\wedge, \Box, \rhd$.
3. a *universal node* if it is either positive and labeled $\Box$ or $\rhd$ or negative and labeled $\Diamond$ or $\lhd$.

Definition 3.1.32. ([21] Definition 3.4)
Let $\varepsilon \in \{1, \partial\}^n$ be an order type. A term $\alpha(x_1, ...x_n)$ is *ε-left Sahlqvist* (resp *ε-right Sahlqvist*) if it satisfies the following two conditions:

1. in the positive (resp. negative) generation tree, for all i with $\varepsilon_i = 1$, there are no paths from an occurrence of x_i with $+$ to the root along which one meets a choice node before a universal node.
2. in the positive (resp. negative) generation tree, for all i with $\varepsilon_i = \partial$, there are no paths from an occurrence of x_i with $-$ to the root along which one meets a choice node before a universal node.

Definition 3.1.33. ([21] Definition 3.4)
An *ε-Sahlqvist inequality* is an inequality $\alpha \leq \beta$ such that α is ε-left Sahlqvist and β is ε-right Sahlqvist.

An inequality is called simply a *Sahlqvist inequality* if it is an ε-Sahlqvist inequality for some order type ε.

As an example notice that the term $\Box \rhd x$ is both 1-left Sahlqvist and ∂-left Sahlqvist. The positive generation tree for this term is:

$\Box(+)$ **u.n**
$\rhd\,(+)$ **u.n**
$x(-)$

For 1-left Sahlqvist we have to worry only about positive occurrences of the variable x, but there aren't any. So this term is vacuously 1-left Sahlqvist.

For ∂-left Sahlqvist, we have to worry about negative occurrences of x. There is one, but since all nodes in the tree are universal, there is no problem anyway. Thus this term is also ∂-left Sahlqvist.

So the inequality $\Box \rhd x \leq x$ is Sahlqvist.

For Sahlqvist inequalities we have the following:

Theorem 3.1.34 ([21] Theorem 5.1). *Every Sahlqvist inequality is canonical in UQA.*

3.2 Discrete Duality

3.2.1 Basic technical ingredients

Correspondence between Perfect Lattices and Posets

A correspondence between the class of Perfect Distributive Lattices (Definition 3.1.5) and Posets was studied by G. Raney [34], V. Balachandran [3] and P. Dwinger [4]. More recently S.T.Thomason [42]and M.Gehrke, H.Nagashi and Y. Venema ([21]) have extended this correspondence in order to obtain a duality that generalizes Birkhoff's duality for finite distributive lattices.

Let L be a $\mathcal{DL}^+$ (Definition 3.1.5). The lattice L gives rise to a poset $(J^\infty(L), \leq)$ where $J^\infty(L)$ is the set of completely join irreducible elements of L and $\leq$ is the restriction to $J^\infty(L)$ of the order in the lattice L. In turn, for an arbitrary poset X, let $(\mathcal{D}(X), \cap, \cup, \emptyset, X)$ be the lattice of downsets of X. Then this lattice is in $\mathcal{DL}^+$. So, denoting by $\mathcal{P}$ posets $(X, \leq)$, we can define maps

$$()_+ : \mathcal{DL}^+ \to \mathcal{P} \quad \text{such that} \quad (L)_+ = J^\infty(L) \quad \text{for each} \quad L \in \mathcal{DL}^+$$

and

$$()^+ : \mathcal{P} \to \mathcal{DL}^+ \quad \text{such that} \quad (X)^+ = \mathcal{D}(X) \quad \text{for each} \quad X \in \mathcal{P}.$$

In Theorem 2 [34] G.Raney proved that for a complete distributive lattice L to be isomorphic to the lattice of downsets of $J^\infty(L)$ it is necessary and sufficient that every element of L is the join of a set of completely join irreducible elements. From the proof of this theorem it follows:

Theorem 3.2.1. *The maps $()_+ : \mathcal{DL}^+ \to \mathcal{P}$ and $()^+ : \mathcal{P} \to \mathcal{DL}^+$ are such that, for each $L \in \mathcal{DL}^+$, there is an isomorphism $\eta_L : L \to ((L)_+)^+$ defined by*

$$\eta_L(a) = \{x \in J^\infty(L) : x \leq a\} \qquad (= J^\infty(L) \cap \downarrow a)$$

for any $a \in L$.

On the other hand, for an arbitrary poset X, the completely join irreducible elements of the lattice $(\mathcal{D}(X), \cap, \cup, \emptyset, X)$ are exactly the principal downsets, $\downarrow x$ with $x \in X$, so we have:

Theorem 3.2.2. *The maps* $()_+ : \mathcal{DL}^+ \to \mathcal{P}$ *and* $()^+ : \mathcal{P} \to \mathcal{DL}^+$ *are such that for each* $X \in \mathcal{P}$ *there is an order isomorphism* $\varepsilon_X : X \to ((X)^+)_+$ *where*

$$\varepsilon_X(x) = \downarrow x$$

for any $x \in X$.

From the previous theorems it follows that the maps $()_+$ and $()^+$ determine, at the objects level, a duality between the categories $\mathcal{DL}^+$ and $\mathcal{P}$.

We want to extend these maps in order to obtain a duality so, we have to define the morphisms of these categories.

Correspondence between maps and order compatible relations

Naturally morphisms of $\mathcal{DL}^+$ are complete homomorphisms. Morphisms of $\mathcal{P}$ are order preserving maps. However, since we also have in mind extending the $\mathcal{DL}^+$s by additional operations and this will require that we produce duals for these, we will start by considering separately maps that preserve arbitrary joins and maps that preserve arbitrary meets.

We will obtain the duality between $\mathcal{DL}^+$- morphisms and $\mathcal{P}$-morphisms and we will determine the correspondents of unary operations that are join (meet) preserving as a corollary to this more general correspondence.

Let $L, K \in \mathcal{DL}^+$ and let $f : L \to K$ be a completely join preserving map (Definition 3.1.20).

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be posets such that $X = J^\infty(L)$ and $Y = J^\infty(K)$ with the order induced from L and K, respectively.

The map f is uniquely determined by its value on the completely join irreducible elements of L, $J^\infty(L)$. In fact, since L is join generated by $J^\infty(L)$, for any $a \in L$ we have $a = \bigvee_{p \leq_L a} p$ with $p \in J^\infty(L)$ so

$$f(a) = f\left(\bigvee_{p \leq_L a} p\right) = \bigvee_{p \leq_L a} f(p). \tag{3.2.1}$$

It would be convenient if we could define a binary relation S on the poset $Y \times X$ encoding f.

We cannot take S to be $\{(f(p), p) : p \in X\}$ since, in general, f won't map into Y but, for each $p \in X$, we have

$$f(p) = \bigvee_{\substack{q \in Y \\ q \leq_K f(p)}} q$$

since $f(p) \in K$ and K is join generated by Y.

Therefore we take $S \subseteq Y \times X$ such that,

$$S = \{(q, p) : q \leq_K f(p)\}.$$

The map f is order preserving, hence we have

$$\leq_Y \circ S \circ \leq_X \subseteq S.$$

On the other hand if we have posets $(X, \leq_X)$ and $(Y, \leq_Y)$ and a relation $S \subseteq Y \times X$ such that $\leq_Y \circ S \circ \leq_X \subseteq S$, it is easy to prove that the map

$$f_S : \mathcal{D}(X) \to \mathcal{D}(Y)$$

such that, for any $U \in \mathcal{D}(X)$,

$$f_S(U) = S^{-1}(U) = \{q \in Y : \exists s \in U \, qSs\}$$

is a completely join preserving map.

Since there is no danger of misunderstanding, in what follows we will denote the order relation by $\leq$ no matter the poset we are considering.

Thus we have established the following:

Lemma 3.2.3. *Let $L, K \in \mathcal{DL}^+$ and $(X, \leq)$ and $(Y, \leq)$ be the posets of completely join irreducible elements of L and K, respectively. Then, to any completely join preserving map $f : L \to K$ corresponds a binary relation*

$$S = \{(q, p) : q \leq f(p)\} \subseteq Y \times X$$

such that $\leq \circ S \circ \leq \subseteq S$.

Conversely, if $(X, \leq)$ and $(Y, \leq)$ are posets and $S \subseteq Y \times X$ is a binary relation such that $\leq \circ S \circ \leq \subseteq S$, then the map $f_S : \mathcal{D}(X) \to \mathcal{D}(Y)$ defined by

$$f_S(U) = S^{-1}(U),$$

for every $U \in \mathcal{D}(X)$, is a completely join preserving map.

In a way similar to the one we used for completely join preserving maps we can determine the relation corresponding to a completely meet preserving map.

To do this, given $L \in \mathcal{DL}^+$, we will often use the *order dual lattice* L^∂. It follows from the definition of this lattice that $M^\infty(L^\partial) = J^\infty(L)$ and $J^\infty(L^\partial) = M^\infty(L)$.

We will also use the order isomorphism $\kappa : J^\infty(L) \to M^\infty(L)$ (Definition 3.1.6).

Since $L \in \mathcal{DL}^+$ is meet generated by the completely meet preserving elements, we have L^∂ dually join generated by $\kappa(J^\infty(L))$. So, for any $a \in L^\partial$,

$$a = \bigwedge_{\substack{p \in J^\infty(L) \\ \kappa(p) \geq a}} \kappa(p) = \bigvee^{\partial}_{\substack{\kappa(p) \in J^\infty(L^\partial) \\ \kappa(p) \leq^\partial a}} \kappa(p).$$

Let $g : L \to K$ be a completely meet preserving map (Definition 3.1.20).

If we consider the map

$$g^\partial : L^\partial \to K^\partial.$$

This map is set theoretically identical to g and it is a completely join preserving map.

In fact for any $a \in L$,

$$g(a) = g^\partial \left(\bigvee^{\partial}_{\substack{\kappa(p) \in J^\infty(L^\partial) \\ \kappa(p) \leq^\partial a}} \kappa(p) \right) = \bigvee^{\partial}_{\substack{\kappa(p) \in J^\infty(L^\partial) \\ \kappa(p) \leq^\partial a}} g^\partial(\kappa(p)) = \bigwedge_{\substack{p \in J^\infty(L) \\ \kappa(p) \geq a}} g(\kappa(p)). \tag{3.2.2}$$

Consequently g is uniquely determined by its value on $\kappa(J^\infty(L))$.

Since, for any $p \in J^\infty(L)$, $g(\kappa(p))$ is in K and this lattice is meet generated by $M^\infty(L) = \kappa(J^\infty(L))$, we have:

$$g(\kappa(p)) = \bigwedge_{\substack{q \in J^\infty(K) \\ \kappa(q) \geq g(\kappa(p))}} \kappa(q).$$

Now we take $T \subseteq J^\infty(K) \times J^\infty(L)$ to be

$$T = \{(q, p) : \ \kappa(q) \geq g(\kappa(p)\}$$

and T is such that:

$$\leq^{\partial} \circ T \circ \leq^{\partial} \subseteq T$$

which is equivalent to

$$\geq \circ T \circ \geq \subseteq T.$$

If we have posets $(X, \leq)$ and $(Y, \leq)$ and a relation $T \subseteq Y \times X$ such that $\geq \circ T \circ \geq \subseteq T$, then the map

$$g_T : \mathcal{D}(X) \to \mathcal{D}(Y)$$

such that, for every $U \in \mathcal{D}(X)$,

$$g_T(U) = X \setminus T^{-1}(X \setminus U) = \{q \in Y : \forall s \in X \,(qTs \Rightarrow s \in U)\}$$

is a completely meet preserving map.

This way we obtain:

Lemma 3.2.4. *Let $L, K \in \mathcal{DL}^+$ and $(X, \leq)$ and $(Y, \leq)$ be the posets of completely join irreducible elements of L and K, respectively. Then, to any completely meet preserving map $g : L \to K$ corresponds a binary relation*

$$T = \{(q, p) : \kappa(q) \geq g(\kappa(p))\} \subseteq Y \times X$$

such that $\geq \circ T \circ \geq \subseteq T$.

If $(X, \leq)$ and $(Y, \leq)$ are posets and $T \subseteq Y \times X$ is a binary relation such that $\geq \circ T \circ \geq \subseteq T$, then the map $g_T : \mathcal{D}(X) \to \mathcal{D}(Y)$ defined by

$$g_T(U) = X \setminus T^{-1}(X \setminus U),$$

for every $U \in \mathcal{D}(X)$, is a completely meet preserving map.

Now it is time to see what happens to complete homomorphisms $h : L \to K$ with $L, K \in \mathcal{DL}^+$.

We will follow the process used for endomorphisms in [21] 6.2.

It is possible to consider h as a completely join preserving map f such that $f = g$ where g is completely meet preserving. Then f and g have to satisfy

$$\forall a \in L : f(a) \leq g(a) \text{ and } g(a) \leq f(a).$$

These inequalities are equivalent to conditions on the corresponding relations S and T.

To determine these conditions we are going to follow an algebraic method that will be often used in this chapter. This method consists in eliminating the universal quantifier on elements of L.

Condition $\forall a \in L : f(a) \leq g(a)$ is equivalent to:

$$\forall a \in L \forall q \in J^{\infty}(K) : \ (q \leq f(a) \Rightarrow q \leq g(a)) \qquad (*)$$

because K is join generated by $J^{\infty}(K)$.

Now, from equation (3.2.1) and the fact that q is completely join irreducible we conclude that $q \leq f(a)$ is equivalent to

$$\exists p \in J^{\infty}(L) : \ (p \leq a \text{ and } q \leq f(p)).$$

It follows, from the definition of S (Lemma 3.2.3), that this is equivalent to

$$\exists p \in J^{\infty}(L) : \ (p \leq a \text{ and } qSp)$$

From equation (3.2.2), $q \leq g(a)$ is equivalent to

$$\forall r \in J^{\infty}(L) : \ (\kappa(r) \geq a \Rightarrow q \leq g(\kappa(r)))$$

and to

$$\forall r \in J^{\infty}(L) : \ (q \not\leq g(\kappa(r)) \Rightarrow \kappa(r) \not\geq a.)$$

From Remark 3.1.8 and the definition of T (Lemma 3.2.4) we obtain the following equivalent expressions:

$$\forall r \in J^{\infty}(L) : \ (g(\kappa(r)) \leq \kappa(q) \Rightarrow r \leq a)$$

and

$$\forall r \in J^{\infty}(L) : \ (qTr \Rightarrow r \leq a.)$$

Therefore, condition (*) is equivalent to

$$\begin{aligned} &\forall a \in L \forall q \in J^{\infty}(K) : \\ &\quad ((\exists p \in J^{\infty}(L)(p \leq a \text{ and } qSp)) \quad \Rightarrow (\forall r \in J^{\infty}(L) : \ (qTr \Rightarrow r \leq a)). \end{aligned}$$

and to

$$\forall a \in L \forall q \in J^{\infty}(K) \forall p, r \in J^{\infty}(L) : ((p \leq a \text{ and } qSp) \Rightarrow (qTr \Rightarrow r \leq a)).$$

Now observe that, given $p \in J^\infty(L)$, the least a satisfying the antecedent is $a = p$, and if the consequent holds for some a then it also holds for any greater values of a. So the previous condition is equivalent to

$$\forall q \in J^\infty(K)\, \forall p, r \in J^\infty(L) : \, (qSp \Rightarrow (qTr \Rightarrow r \leq p)).$$

Notice that we have now eliminated the quantification over elements of L and are left with a condition on X and Y. We now simplify this further:

$$\forall q \in J^\infty(K)\, \forall p, r \in J^\infty(L) : \, ((qSp \text{ and } qTr) \Rightarrow r \leq p)).$$

We will follow a similar process for condition $\forall a \in L : \, g(a) \leq f(a)$:
This condition is equivalent to

$$\forall a \in L\, \forall q \in J^\infty(K)\, (q \leq g(a) \Rightarrow q \leq f(a)). \qquad (**)$$

As we have just proved the antecedent is equivalent to

$$\forall r \in J^\infty(L) : \, (qTr \Rightarrow r \leq a)$$

and, making $qT = \{r \in J^\infty(L) : qTr\}$, it follows that it is equivalent to $\bigvee qT \leq a$.

So (**) is equivalent to:

$$\forall a \in L\, \forall q \in J^\infty(K) : \, (\bigvee qT \leq a \Rightarrow q \leq f(a)).$$

Now, there is a least a that satisfies the antecedent, $a = \bigvee qT$, and, since f preserves the order, if the consequent holds for some a it also holds for any greater element so the previous implication is equivalent to

$$\forall q \in J^\infty(K) : \, q \leq f\left(\bigvee qT\right)$$

and to

$$\forall q \in J^\infty(K) : \, q \leq \bigvee f\,(qT)$$

because f is completely join preserving.

Since q is completely join irreducible, the previous condition is equivalent to

$$\forall q \in J^\infty(K)\, \exists s \in J^\infty(L) : \, qTs \text{ and } q \leq f(s).$$

According to the definition of S, this is equivalent to

$$\forall q \in J^\infty(K)\, \exists s \in J^\infty(L) : \, qTs \text{ and } qSs.$$

Thus we have proved that conditions

(i) $\forall q \in J^\infty(K)\,\forall p, r \in J^\infty(L) :\ ((qSp \text{ and } qTr) \Rightarrow r \leq p))$.

(ii) $\forall q \in J^\infty(K)\,\exists s \in J^\infty(L) :\ qTs \text{ and } qSs$.

are equivalent to $f(a) \leq g(a)$ and $g(a) \leq f(a)$, respectively.

If, as in [21] 6.2, we consider the relation

$$R = S \cap T \subseteq J^\infty(K) \times J^\infty(L),$$

it is easy to see that conditions (i) and (ii) imply

(i') $\forall q \in J^\infty(K)\,\forall p, r \in J^\infty(L) :\ ((qRp \text{ and } qRr) \Rightarrow r = p))$.

(ii') $\forall q \in J^\infty(K)\,\exists s \in J^\infty(L) :\ qRs$.

That is, R is the graph of a function from $J^\infty(K)$ to $J^\infty(L)$.

The relation R is such that

$$\leq \circ R \subseteq R \circ \leq .$$

In fact, for any $q \in J^\infty(K)$ and $p \in J^\infty(L)$, $q \leq \circ Rp$ implies the existence of $r \in J^\infty(K)$ such that

$$q \leq r \text{ and } rSp \text{ and } rTp.$$

From Lemma 3.2.3 it follows that qSp and, from (ii) there is $s \in J^\infty(L)$ such that $q(S \cap T)s$ so, since we have simultaneously qSp and qTs, we know by (i) that $s \leq p$.

This means that, R is the graph of an order preserving map.

Observe that from (i) it follows that, for any $q \in J^\infty(K)$, the element $s \in J^\infty(L)$ such that qRs is simultaneously the minimum of qS and the maximum of qT. So we have $min(qS) = max(qT)$.

Thus we have the following

Lemma 3.2.5. *Let $L, K \in \mathcal{DL}^+$, let $h : L \to K$ be a complete homomorphism and let $S, T \subseteq J^\infty(K) \times J^\infty(L)$ be relations such that $S = \{(q, p) : q \leq h(p)\}$ and $T = \{(q, p) : \kappa(q) \geq h(\kappa(p))\}$. Then, the relation $S \cap T$ is the graph of an order preserving map,*

$$\varphi_h : J^\infty(K) \to J^\infty(L)$$

where, for any $q \in J^\infty(K)$*,* $\varphi_h(q) \in J^\infty(L)$ *is the only element such that* $q(S \cap T)\varphi_h(q)$*.*

Further, for any $q \in J^\infty(K)$*, the set* $qS = \{p \in J^\infty(L) : qSp\}$ *has a minimum and the set* $qT = \{r \in J^\infty(L) : qTr\}$ *has a maximum and*

$$min(qS) = max(qT) = \varphi_h(q).$$

By the previous lemma, for any $q \in K$,

$$\varphi_h(q) = min\{p \in J^\infty(L) : q \leq h(p)\} = max\{p \in J^\infty(L) : \kappa(q) \geq h(\kappa(p))\}.$$

But

$$min\{p \in J^\infty(L) : q \leq h(p)\} = min\{a \in L : q \leq h(a)\}.$$

In fact from $\{p \in J^\infty(L) : q \leq h(p)\} \subseteq \{a \in L : q \leq h(a)\}$ it follows that

$$\bigwedge\{p \in J^\infty(L) : q \leq h(p)\} \geq \bigwedge\{a \in L : q \leq h(a)\}.$$

On the other hand, for any $a \in L$, $h(a) = \bigvee_{p \leq a} h(p)$ with $p \in J^\infty(L)$, so $q \leq h(a)$ implies that there is some $p \in J^\infty(L)$ such that $p \leq a$ and $q \leq h(p)$. Thus we have also

$$\bigwedge\{p \in J^\infty(L) : q \leq h(p)\} \leq \bigwedge\{a \in L : q \leq h(a)\}.$$

Since both meets are elements of these sets the equality is proved.

Therefore we have, for any $q \in J^\infty(K^\sigma)$,

$$\varphi_h(q) = min\{a \in L : q \leq h(a)\}$$

and it is not difficult to prove that as a consequence of

$$\varphi_h(q) = max\{p \in J^\infty(L) : \kappa(q) \geq h(\kappa(p))\}$$

we have also

$$\varphi_h(q) = max\{a \in L : \kappa(q) \geq h(a)\}.$$

Conversely, if $(X, \leq)$ and $(Y, \leq)$ are posets and $\varphi : Y \to X$ is an order preserving map, then

$$R_\varphi = \{(q, \varphi(q)) : q \in Y\} \subseteq Y \times X$$

satisfies (i') , (ii') and $\leq \circ R_\varphi \subseteq R_\varphi \circ \leq$.

Then we can define relations

$$S = R_\varphi \circ \leq \text{ and } T = R_\varphi \circ \geq .$$

Clearly (i) and (ii) hold so that the corresponding functions ,

$$f_S : \mathcal{D}(X) \to \mathcal{D}(Y) \, \text{such that} \, f_S(U) = S^{-1}(U)$$

and

$$g_T : \mathcal{D}(X) \to \mathcal{D}(Y) \, \text{such that} \, g_T(U) = X \setminus T^{-1}(X \setminus U)$$

are respectively completely join preserving and completely meet preserving and we have $f_S = g_T$.

Thus it makes sense to define a complete homomorphism $h_\varphi : \mathcal{D}(X) \to \mathcal{D}(Y)$ such that, for every $U \in \mathcal{D}(X)$, $h_\varphi(U) = f_S(U) = g_T(U)$.

For this homomorphism

$$h_\varphi(U) = S^{-1}(U) = \{q \in Y : \exists s \in U \, qSs\}$$

Since $S = R_\varphi \circ \leq$, we have $q \in h_\varphi(U)$ equivalent to

$$\exists s \in U \exists r \in X : \, qR_\varphi r \text{ and } r \leq s$$

which is equivalent to

$$\exists r \in U : \, qR_\varphi r$$

because U is a downset.

From the definition of R_φ it follows that $h_\varphi(U) = \varphi^{-1}(U)$ for any $U \in \mathcal{D}(X)$.

Thus we have established a correspondence between order-preserving maps and complete homomorphisms.

It is now possible to define the functors $()_+$ and $()^+$ on morphisms. We will denote by $\mathcal{DL}^+(L, K)$ the set of complete homomorphisms from L to K and by $\mathcal{P}(Y, X)$ the set of order preserving maps from Y to X. We want $()_+$ and $()^+$ to be contravariant functors such that:

For any $L, K \in \mathcal{DL}^+$ there exists, for each $h \in \mathcal{DL}^+(L, K)$, a map $(h)_+ \in \mathcal{P}((K)_+, (L)_+)$. For each $X, Y \in \mathcal{P}$ there exists, for any $\varphi \in \mathcal{P}(Y, X)$, a map $(\varphi)^+ \in \mathcal{DL}^+((X)^+, (Y)^+)$.

For this correspondence of morphisms we have the following generalization of [17] Theorem 8.24,

Theorem 3.2.6. *Let $L, K \in \mathcal{DL}^+$ and let $X, Y \in \mathcal{P}$. Given a complete homomorphism $h \in \mathcal{DL}^+(L, K)$, there is an associated order preserving map $\varphi_h \in \mathcal{P}(J^\infty(K), J^\infty(L))$ defined by*

$$\begin{aligned}\varphi_h(p) &= min\{a \in L : p \leq h(a)\}\\ &= max\{a \in L : \kappa(p) \geq h(a)\}\\ &= p(S_h \cap T_h)\end{aligned}$$

for all $p \in J^\infty(K)$.

Given an order-preserving map $\varphi \in \mathcal{P}(Y, X)$, there is an associated complete homomorphism $h_\varphi \in \mathcal{DL}^+(\mathcal{D}(X), \mathcal{D}(Y))$ defined by

$$h_\varphi = \varphi^{-1}(U)$$

for each $U \in \mathcal{D}(X)$.

The maps $()_+ : \mathcal{DL}^+(L, K) \longrightarrow \mathcal{P}(J^\infty(K), J^\infty(L))$ such that $(h)_+ = \varphi_h$ and $()^+ : \mathcal{P}(Y, X) \longrightarrow \mathcal{DL}^+(\mathcal{D}(X), \mathcal{D}(Y))$ such that $(\varphi)^+ = h_\varphi$ are bijections and, for each $a \in L$ and each $p \in J^\infty(K)$,

$$p \leq h(a) \Longleftrightarrow \varphi_h(p) \leq a \qquad (\star)$$

The diagrams

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ {\scriptstyle \eta_L}\downarrow & & \downarrow{\scriptstyle \eta_K} \\ \mathcal{D}(J^\infty(L)) & \xrightarrow[h_{\varphi_h}]{} & \mathcal{D}(J^\infty(K)) \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \varepsilon_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ J^\infty(\mathcal{D}(Y)) & \xrightarrow[\varphi_{h_\varphi}]{} & J^\infty(\mathcal{D}(X)) \end{array}$$

commute.

Further:

(i) h is one to one if and only if φ_h is onto.

(ii) h is onto if and only if φ_h is an order embedding.

Proof. We have already seen that the maps $()_+$ and $()^+$ are well defined and $(\star)$ is a consequence of the definition of $(h)_+ = \varphi_h$.

To prove that the diagrams commute observe that:

For each $a \in L$ we have, by Theorem 3.2.1, $\eta_L(a) = \{p \in J^\infty(L) : p \leq a\}$ so

$$\begin{aligned} h_{\varphi_h}(\eta_L(a)) &= \varphi_h^{-1}(\eta_L(a)) \\ &= \{r \in J^\infty(K) : \varphi_h(r) \leq a\} \\ &= \{r \in J^\infty(K) : r \leq h(a)\} \qquad \text{(by } (\star)\text{)} \\ &= \eta_K(h(a)) \end{aligned}$$

therefore $\eta_K \circ h = h \circ \eta_L$.

For any $q \in Y$ we know that $\varepsilon_Y(q) = \downarrow q$ hence

$$\begin{aligned} \varphi_{h_\varphi}(\varepsilon_Y(q)) &= \varphi_{h_\varphi}(\downarrow q) \\ &= min\{U \in \mathcal{D}(X) : \downarrow q \subseteq h_\varphi(U)\} \\ &= min\{U \in \mathcal{D}(X) : \downarrow q \subseteq \varphi^{-1}(U)\} \\ &= min\{U \in \mathcal{D}(X) : \varphi(q) \in U\} \\ &= \downarrow \varphi(q) \\ &= \varepsilon(\varphi(q)) \end{aligned}$$

thus $\varphi_{h_\varphi} \circ \varepsilon_Y = \varepsilon_X \circ \varphi$.

Now, to prove that $()_+$ is a bijection, let $\varphi \in \mathcal{P}(J^\infty(K), J^\infty(L))$ and let $h = \eta_K^{-1} \circ h_\varphi \circ \eta_L$.

To prove that $\varphi = \varphi_h = (h)_+$ observe that from the diagram it follows that

$$h_{\varphi_h} = \eta_K \circ h \circ \eta_L^{-1} = \eta_K \circ \eta_K^{-1} \circ h_\varphi \circ \eta_L \circ \eta_L^{-1} = h_\varphi$$

so, for any $U \in \mathcal{D}(J^\infty(L))$, we have $\varphi_h^{-1}(U) = \varphi^{-1}(U)$.

If there was some $p \in J^\infty(K)$ such that $\varphi_h(p) \neq \varphi(p)$ we could suppose that $\varphi_h(p) \not\leq \varphi(p)$ and then we would obtain $p \in \varphi^{-1}(\downarrow \varphi(p))$ and $p \notin \varphi_h^{-1}(\downarrow \varphi(p))$, which is a contradiction.

So $()_+$ is onto.

For any $h_1, h_2 \in \mathcal{DL}^+(L, K)$ such that $\varphi_{h_1} = \varphi_{h_2}$ we have

$$h_1 = \eta_K^{-1} \circ h_{\varphi_{h_1}} \circ \eta_L = \eta_K^{-1} \circ h_{\varphi_{h_2}} \circ \eta_L = h_2$$

so $()_+$ is one to one.

In a similar way we prove that $()^+$ is a bijection.

In order to prove (i) let $h : L \to K$ be a complete homomorphism and let $\varphi_h : J^\infty(K) \to J^\infty(L)$ be the associated order preserving map. Suppose h is one to one and let $p \in J^\infty(L)$.

We have $p \in L$ so, $h(p) \in K$ thus $h(p) = \bigvee\{q \in J^\infty(K) : \, q \le h(p)\}$.

For each q with $q \le h(p)$ we have, by condition $\star$, $\varphi_h(q) \le p$.

Let $c = \bigvee\{\varphi_h(q) : \, q \le h(p)\}$. Then $c \le p$.

Suppose $c < p$. Then $h(c) < h(p)$ because h is one to one. Consequentely, there is $q \in J^\infty(K)$ such that $q \le h(p)$ and $q \not\le h(c)$. Hence $\varphi_h(q) \not\le c$ contradicting the definition of c.

Therefore $c = p$. So $p = \bigvee\{\varphi_h(q) : \, q \le h(p)\}$ and, since p is completely join irreducible, there is some q such that $p = \varphi_h(q)$. Thus φ_h is onto.

Conversely suppose φ_h is onto and let $a, b \in L$ be such that $h(a) = h(b)$.

Since $L \in \mathcal{DL}^+$, $a = \bigvee\{p \in J^\infty(L) : \, p \le a\}$ and, for each element $p \le a$, there is $r \in J^\infty(K)$ such that $p = \varphi_h(r)$ because φ_h is onto. Then $\varphi_h(r) \le a$ and, from $(\star)$ it follows that $r \le h(a)$.

But $h(a) = h(b)$ so we have also $r \le h(b)$. Therefore $\varphi_h(r) = p \le b$ and consequently $a \le b$.

Analogously, $b \le a$ so h is one to one.

To prove (ii):

Let $h : L \to K$ be onto and let $\varphi_h : J^\infty(K) \to J^\infty(L)$. Let $q, r \in J^\infty(K)$ be such that $\varphi_h(q) \ge \varphi_h(r)$.

By the definition of φ_h we have

$$\{a \in L : \, q \le h(a)\} \subseteq \{a \in L : \, r \le h(a)\}$$

because these two sets are upsets.

Since h is onto and $q \in K$ there is $c \in L$ such that $q = h(c)$. Then c is an element of the first set and consequently it is also in the second set. Therefore $r \le h(c) = q$ so φ_h is an order embedding.

Conversely, let φ_h be an order embedding and let $b \in K$.

Then $b = \bigvee\{q \in J^\infty(K) : \, q \le b\}$.

For each $q \in J^\infty(K)$ such that $q \le b$ let $p \in J^\infty(L)$ be such that $p = \varphi_h(q)$ and consequently $((\star))$ $q \le h(p)$. So,

$$b \le \bigvee\{h(p) : \, p = \varphi_h(q) \text{ and } q \le b\} = h\left(\bigvee\{p : \, p = \varphi_h(q) \text{ and } q \le b\}\right)$$

because h is completely join preserving.

Let $e \in K$ be such that

$$e = \bigvee\{h(p) : p = \varphi_h(q) \text{ and } q \leq b\} = h\left(\bigvee\{p : p = \varphi_h(q) \text{ and } q \leq b\}\right).$$

But $e = \bigvee\{r \in J^\infty(K) : r \leq e\}$.

By the definition of e, for each $r \in J^\infty(K)$ such that $r \leq e$, there is an element $p_r \in J^\infty(L)$ and an element $q_r \in J^\infty(K)$ such that $r \leq h(p_r)$ and $p_r = \varphi_h(q_r)$ with $q_r \leq b$.

As a consequence of $(\star)$, $\varphi_h(r) \leq p_r$ and, since φ_h is an order embedding, $r \leq q_r$.

Thus we have got:

$$b \leq e \leq \bigvee\{q_r \in J^\infty(K) : r \leq e\} \leq b.$$

So

$$b = e = h\left(\bigvee\{p : p = \varphi(q) \text{ and } q \leq b\}\right).$$

Therefore h is onto.

□

So we have proved that functors $()_+$ and $()^+$ establish a dual equivalence between categories $\mathcal{DL}^+$ and $\mathcal{P}$.

This duality extends the Basic Birkhoff duality from finite to perfect $\mathcal{DL}^+$ as a categorical duality.

3.2.2 A duality for UQA^+s

The categorical duality we have been studying can be extended to objects with additional unary operations that are completely $\bigvee$ or $\bigwedge$ preserving or reversing i.e. to UQA^+s (Definition 3.1.26). The paper [21] by M. Gehrke, H. Nagashi and Y. Venema, particularly Section 2.3, was very useful for this purpose.

Let $(L, \wedge, \vee, 0, 1, \Diamond, \Box, \rhd, \lhd)$ be a UQA^+.

To define the dual structures of UQA^+ we will have to obtain binary relations on $J^\infty(L)$ encoding the unary operations considered on L.

The unary operations $\Diamond : L \to L$ and $\Box : L \to L$ are, respectively, completely join preserving and completely meet preserving maps. We have seen in section 3.2.1 how to define the corresponding binary relations, thus we can use here the same correspondence.

For the operation $\Diamond$ we denote the binary relation by $R_\Diamond$ and we take, as in Lemma 3.2.3, $R_\Diamond \subseteq J^\infty(L) \times J^\infty(L)$ to be

$$R_\Diamond = \{(p, q) : p \leq \Diamond q\}$$

Since the operation $\Diamond$ is order preserving, we have as in Lemma 3.2.3

$$\leq \circ R_\Diamond \circ \leq \subseteq R_\Diamond.$$

For the operation $\Box$, we take $R_\Box \subseteq J^\infty(L) \times J^\infty(L)$ to be as in Lemma 3.2.4,

$$R_\Box = \{(p, q) :\ \kappa(p) \geq \Box(\kappa(q)\}$$

and $R_\Box$ is such that:

$$\geq \circ R_\Box \circ \geq \subseteq R_\Box.$$

In what concerns $\triangleright$ we will follow a similar way:

Since $\triangleright$: $L \longrightarrow L$ sends arbitrary joins into meets, it is set-theoretically identical to a completely join preserving map $\triangleright^\partial$: $L \longrightarrow L^\partial$ such that, for each $a \in L$, $\triangleright^\partial (a) = \triangleright (a)$.

For any $a \in L$, $a = \bigvee_{p \leq a} p$ with $p \in J^\infty(L)$ and therefore

$$\triangleright a = \triangleright^\partial \left(\bigvee_{p \leq a} p \right) = \bigvee_{p \leq a}^{\partial} \triangleright^\partial p = \bigwedge_{p \leq a} \triangleright p.$$

For each $q \in J^\infty(L)$ we have $\triangleright q = \triangleright^\partial q \in L^\partial$ and L^∂ is dually join generated by $J^\infty(L^\partial) = M^\infty(L) = \{\kappa(p) :\ p \in J^\infty(L)\}$ so

$$\triangleright q = \triangleright^\partial q = \bigvee^{\partial}_{\substack{\kappa(p) \in J^\infty(L^\partial) \\ \kappa(p) \leq^\partial \triangleright^\partial q}} \kappa(p) = \bigwedge_{\substack{p \in J^\infty(L) \\ \kappa(p) \geq \triangleright q}} \kappa(p).$$

Therefore we take $R_\triangleright \subseteq J^\infty(L) \times J^\infty(L)$ to be

$$R_\triangleright = \{(p, q) \in J^\infty(L) \times J^\infty(L) :\ \kappa(p) \geq \triangleright q\}.$$

It is easy to prove that $R_{\triangleright}$ is such that:

$$\geq \circ R_{\triangleright} \circ \leq = \leq^{\partial} \circ R_{\triangleright} \circ \leq \subseteq R_{\triangleright}.$$

For the operation $\triangleleft$ that sends arbitrary meets to joins, we will consider the map $\triangleleft\colon L^{\partial} \longrightarrow L$ that is set theoretically identical to $\triangleleft$ and completely join preserving.

Now, for any $a \in L^{\partial}$,

$$a = \bigvee^{\partial}_{\substack{\kappa(p) \in J^{\infty}(L^{\partial}) \\ \kappa(p) \leq^{\partial} a}} \kappa(p) = \bigwedge_{\substack{p \in J^{\infty}(L) \\ \kappa(p) \geq a}} \kappa(p)$$

so

$$\triangleleft\, a = \triangleleft \left(\bigvee^{\partial}_{\substack{\kappa(p) \in J^{\infty}(L^{\partial}) \\ \kappa(p) \leq^{\partial} a}} \kappa(p) \right) = \bigvee_{\substack{\kappa(p) \in J^{\infty}(L^{\partial}) \\ \kappa(p) \leq^{\partial} a}} \triangleleft\, (\kappa(p)) = \bigvee_{\substack{p \in J^{\infty}(L) \\ \kappa(p) \geq a}} \triangleleft\, (\kappa(p))$$

Since $\kappa(p)$ is in L, we have:

$$\triangleleft\, (\kappa(q)) = \bigvee_{\substack{p \in J^{\infty}(L) \\ p \leq \triangleleft(\kappa(q))}} p$$

Therefore we take $R_{\triangleleft} \subseteq J^{\infty}(L) \times J^{\infty}(L)$ to be

$$R_{\triangleleft} = \{(p, q) :\ p \leq \triangleleft\, (\kappa(q)\}$$

and, clearly, $R_{\triangleleft}$ is such that:

$$\leq \circ R_{\triangleleft} \circ \geq = \leq \circ R_{\triangleleft} \circ \leq^{\partial} \subseteq R_{\triangleleft}.$$

It is now possible to define the class of structures corresponding to UQA^{+}:

Definition 3.2.7. Let PR be the class of relational structures

$$(X, \leq, R_{\diamond}, R_{\Box}, R_{\triangleright}, R_{\triangleleft})$$

where $(X, \leq)$ is a partially ordered set and $R_{\diamond}, R_{\Box}, R_{\triangleright}, R_{\triangleleft}$ are binary relations on $\mathcal{P}$ such that:

(i) $\leq \circ R_\Diamond \circ \leq \subseteq R_\Diamond$

(ii) $\geq \circ R_\Box \circ \geq \subseteq R_\Box$

(iii) $\geq \circ R_\rhd \circ \leq \subseteq R_\rhd$

(iv) $\leq \circ R_\lhd \circ \geq \subseteq R_\lhd$

Note that in [23], R. Goldblatt considers ordered structures with relations of types (i) and (ii). The class PR is contained in the class of *ordered relational structures* studied by V. Sofronie-Stokermans in [41]. Both $R_\Diamond$ and $R_\rhd$ are what they call *increasing relations* and $R_\Box$ and $R_\lhd$ are *decreasing relations*. To those structures they associated the distributive lattice of upsets of X with the corresponding operators.

We have just proved that there is a map

$$()_+ : UQA^+ \longrightarrow PR$$

where

$$(L, \Diamond, \Box, \rhd, \lhd)_+ = (J^\infty(L), \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd).$$

On the other hand if we have a relational structure,

$$(X, \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd) \in PR,$$

we saw in the previous section (Lemmas 3.2.3 and 3.2.4) how to define the maps $\Diamond_R$, and $\Box_R$ from $\mathcal{D}(X)$ to $\mathcal{D}(X)$. Therefore we will have unary operations $\Diamond_R, \Box_R$, in $\mathcal{D}(X)$ such that, for every $U \in \mathcal{D}(X)$,

$$\begin{aligned} \Diamond_R(U) &= R_\Diamond^{-1}(U) \\ \Box_R(U) &= X \setminus R_\Box^{-1}(X \setminus U). \end{aligned}$$

For $\rhd$ and $\lhd$ we will define the operations:

$$\begin{aligned} \rhd_R (U) &= X \setminus R_\rhd^{-1}(U) \\ \lhd_R (U) &= R_\lhd^{-1}(X \setminus U). \end{aligned}$$

(Really it should be $\Diamond_{R_\Diamond}, \Box_{R_\Box}, \rhd_{R_\rhd}, \lhd_{R_\lhd}$ but, since there is no danger of misunderstanding, we simplify the notation.)

It is easy to prove that $(\mathcal{D}(X), \cap, \cup, \emptyset, X, \Diamond_R, \Box_R, \rhd_R, \lhd_R)$ is in UQA^+.

So there is a map

$$()^+ : PR \rightarrow UQA^+.$$

We denote these maps by $()_+$ and $()^+$ as in the previous section since they are restrictions of those defined for $\mathcal{DL}^+$ and $\mathcal{P}$.

As in Theorems 3.2.1 and 3.2.2 we have:

Theorem 3.2.8. *There exist maps*

$$()_+ : UQA^+ \longrightarrow PR$$

such that, for each $L \in UQA^+$,

$$(L)_+ = (J^\infty(L), \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$$

with:

$$pR_\Diamond q \text{ if and only if } p \leq \Diamond q, \qquad pR_\Box q \text{ if and only if } \kappa(p) \geq \Box(\kappa(q))$$
$$pR_\rhd q \text{ if and only if } \kappa(p) \geq \rhd q, \qquad pR_\lhd q \text{ if and only if } p \leq \lhd (\kappa(q))$$

for any $p, q \in J^\infty(L)$ *and,*

$$()^+ : PR \longrightarrow UQA^+$$

such that, for each $X \in PR$,

$$(X)^+ = (\mathcal{D}(X), \cap, \cup, \emptyset, X, \Diamond_R, \Box_R, \rhd_R, \lhd_R)$$

where, for any $U \in \mathcal{D}(X)$,

$$\Diamond_R(U) = R_\Diamond^{-1}(U), \qquad \Box_R(U) = X \setminus R_\Box^{-1}(X \setminus U)$$
$$\rhd_R (U) = X \setminus R_\rhd^{-1}(U), \qquad \lhd_R (U) = R_\lhd^{-1}(X \setminus U).$$

These maps satisfy the following conditions:

(i) For each $L \in UQA^+$ *the function* $\eta_L : L \longrightarrow ((L)_+)^+$ *is an isomorphism.*

(ii) For each $X \in PR$ *the function* $\varepsilon_X : X \rightarrow ((X)^+)_+$ *is an isomorphism for the relations* $\leq$ *and* $R_\Diamond, R_\Box, R_\rhd$ *and* $R_\lhd$.

Proof. (i) We will only prove that for any $a \in L$, $\eta_L(\Diamond a) = \Diamond_R(\eta(a))$ because the proofs for the other operations are similar.

Let $p \in J^\infty(L)$ be such that $p \in \eta_L(\Diamond a)$. By Theorem 3.2.1, this is equivalent to $p \leq \Diamond a$.

Since $\Diamond$ is join preserving we have

$$p \leq \Diamond a \Leftrightarrow p \leq \bigvee_{\substack{q \in J^\infty(L) \\ q \leq a}} \Diamond q$$

and this is equivalent to the existence of some $r \in J^\infty(L)$ and $r \leq a$ such that $p \leq \Diamond r$.

But this means that there is some $r \in J^\infty(L)$ such that $r \in \eta_L(a)$ and $pR_\Diamond r$ or equivalently

$$p \in R_\Diamond^{-1}(\eta_L(a)) = \Diamond_R(\eta_L(a)).$$

Thus

$$p \in \eta_L(\Diamond a) \Leftrightarrow p \in \Diamond_R(\eta_L(a)).$$

(ii) By Theorem 3.2.2 we know that ε_X is an isomorphism for the relation $\leq$. We will prove that the same happens for the relation $R_\Diamond$:

Let $p, q \in X$ such that $pR_\Diamond q$. Thus $p \leq \Diamond q$.

Now let $r \in \varepsilon_X(p) = \downarrow p$. Then we have also $r \leq \Diamond q$ so $rR_\Diamond q$ and consequently $r \in R_\Diamond^{-1}(\downarrow q)$. Therefore $r \in R_\Diamond^{-1}(\varepsilon(q))$. So we have

$$\varepsilon(p) \subseteq \Diamond_R(\varepsilon(q))$$

and thus

$$\varepsilon(p) R_{\Diamond_R}(\varepsilon(q)).$$

The proofs for the other relations are similar. □

The unary operations $\triangleright$ and $\Box$ will be very useful later in our study so, we will prove here the following:

Lemma 3.2.9. *Let $L \in UQA^+$ and $X = J^\infty(L)$. Then, for any $a \in L$ and any $p \in X$:*

(i) $\kappa(p) \geq \triangleright a$ is equivalent to $p \not\leq \triangleright a$ and to $\exists q \in X\,(q \leq a \text{ and } pR_\triangleright q)$.

(ii) $p \leq \triangleright a$ is equivalent to $\forall q \in X\,(pR_\triangleright q \Rightarrow q \not\leq a)$ and to $a \leq \bigwedge \kappa(pR_\triangleright)$.

(iii) $\kappa(p) \geq \Box a$ *is equivalent to* $p \not\leq \Box a$ *and to* $\exists q \in X\ (q \not\leq a$ *and* $pR_\Box q)$*.*

(iv) $p \leq \Box a$ *is equivalent to* $\bigvee pR_\Box \leq a$ *where* $pR_\Box = \{q \in X : pR_\Box q\}$*.*

Proof. Observe that the first equivalences in (i) and (iii) follow from Remark 3.1.8.

To prove the second equivalence in (i) notice that, since $\rhd$ is join reversing, $\rhd a = \bigwedge_{q \leq a} \rhd q$, so that $p \not\leq \rhd a$ is equivalent to $\exists q \in X\ (q \leq a$ and $p \not\leq \rhd q)$ and, by Remark 3.1.8, this holds if and only if, $\exists q \in X\,(q \leq a$ and $k(p) \geq \rhd q)$ which is equivalent to $\exists q \in X\ (q \leq a$ and $pR_\rhd q)$ by the definition of $R_\rhd$.

It is obvious that the first equivalence in (ii) follows from (i). The second equivalence follows from Remark 3.1.8 and the fact that L is complete.

For the second equivalence in (iii), remember that $\Box$ is completely meet preserving so, $\Box a = \bigwedge_{\kappa(q) \geq a} \Box(\kappa(q))$ and, consequently, $p \not\leq \Box a$ is equivalent to $\exists q \in X\ (k(q) \geq a$ and $p \not\leq \Box(k(q)))$. By Remark 3.1.8, this is true if and only if $\exists q \in X\ (q \not\leq a$ and $k(p) \geq \Box(k(q)))$ and, by the definition of $R_\Box$ we obtain $\exists q \in X\ (q \not\leq a$ and $pR_\Box q)$.

From here it follows (iv) because $p \leq \Box a$ is equivalent to $\forall q \in X\,(pR_\Box q \Rightarrow q \leq a\}$ and, since L is complete, to $\bigvee pR_\Box \leq a$. $\Box$

Now we are going to study the morphisms of the categories UQA^+ and PR.

Definition 3.2.10. Let $L, K \in UQA^+$. A map $h : L \to K$ is a morphism $h \in UQA^+(L, K)$ if h is a complete homomorphism such that, for each $a \in L$:

(i) $h(\Diamond a) = \Diamond(h(a))$.

(ii) $h(\Box a) = \Box(h(a))$.

(iii) $h(\rhd a) = \rhd (h(a))$.

(iv) $h(\lhd a) = \lhd (h(a))$.

To define the set of morphisms $PR(Y, X)$ from Y to X with $Y, X \in PR$ we must find the conditions that correspond to (i),...,(iv).

The key to these proofs is the elimination of the universal quantifier on elements of L. This is a process that we have already used for homomorphisms in $\mathcal{DL}^+$.

Suppose $h \in UQA^+(L, K)$. Then by Theorem 3.2.6 there is an order preserving map $\varphi : J^\infty(K) \longrightarrow J^\infty(L)$ corresponding to h.

Since K is join generated by $J^\infty(K)$ it follows that the equation (i) holds if and only if we have simultaneously

$$\forall a \in L\, \forall q \in J^\infty(K) : \quad (q \leq h(\Diamond a) \Rightarrow q \leq \Diamond(h(a))) \qquad (*)$$

and

$$\forall a \in L\, \forall q \in J^\infty(K) : \quad (q \leq \Diamond(h(a)) \Rightarrow (q \leq h(\Diamond a)). \qquad (**)$$

Now, on one hand, $q \leq h(\Diamond a) \iff \varphi(q) \leq \Diamond a$ by Theorem 3.2.6 and, since $\Diamond$ is completely join preserving we get $\varphi(q) \leq \Diamond a$ if and only if $\varphi(q) \leq \bigvee_{\substack{p \in J^\infty(L) \\ p \leq a}} \Diamond p$.

But $\varphi(q) \in J^\infty(L)$ thus, there exists $p \in J^\infty(L)$ such that $p \leq a$ and $\varphi(q) \leq \Diamond p$. Thus we obtain

$$q \leq h(\Diamond a) \Leftrightarrow \exists p \in J^\infty(L) : \quad (p \leq a \text{ and } \varphi(q) R_\Diamond p).$$

On the other hand, since $h(a) = \bigvee_{\substack{r \in J^\infty(K) \\ r \leq h(a)}} r$ we have that

$$q \leq \Diamond(h(a)) \iff q \leq \Diamond \left(\bigvee_{\substack{r \in J^\infty(K) \\ r \leq h(a)}} r \right) \iff q \leq \bigvee_{\substack{r \in J^\infty(K) \\ r \leq h(a)}} \Diamond r.$$

Since $q \in J^\infty(K)$ there is $r \in J^\infty(K)$ such that $r \leq h(a)$ and $q \leq \Diamond r$ and hence

$$q \leq \Diamond(h(a)) \Leftrightarrow \exists r \in J^\infty(K) : \quad (\varphi(r) \leq a \text{ and } q R_\Diamond r).$$

Consequently $(*)$ is true if and only if

$$\forall a \in L\, \forall q \in J^\infty(K) :$$
$$((\exists p \in J^\infty(L) : (p \leq a \text{ and } \varphi(q) R_\Diamond p) \Rightarrow (\exists r \in J^\infty(K) : (\varphi(r) \leq a \text{ and } q R_\Diamond r))$$

if and only if

$$\forall a \in L\, \forall q \in J^\infty(K)\, \forall p \in J^\infty(L) :$$
$$((p \leq a \text{ and } \varphi(q) R_\Diamond p) \Rightarrow (\exists r \in J^\infty(K) : (\varphi(r) \leq a \text{ and } q R_\Diamond r))$$

if and only if

$$\forall q \in J^\infty(K)\, \forall p \in J^\infty(L) : \\ (\varphi(q) R_\diamond p \Rightarrow (\exists r \in J^\infty(K) :\ (\varphi(r) \leq p \text{ and } q R_\diamond r))$$

Notice that the last equivalence holds because, given $p \in J^\infty(L)$, the least a satisfying the antecedent is $a = p$ and, if the consequent holds for some a, then it also holds for any greater values of a.

In what concerns condition $(**)$, the implication is equivalent to
$\forall a \in L\, \forall q \in J^\infty(K)$:

$$((\exists r \in J^\infty(K) :\ \varphi(r) \leq a \text{ and } q R_\diamond r) \Rightarrow (\exists p \in J^\infty(L) :\ (p \leq a \text{ and } \varphi(q) R_\diamond p))$$

which holds if and only if
$\forall a \in L\, \forall q, r \in J^\infty(K)$:

$$((\varphi(r) \leq a \text{ and } q R_\diamond r) \Rightarrow (\exists p \in J^\infty(L) :\ (p \leq a \text{ and } \varphi(q) R_\diamond p))\,.$$

The least a for which the antecedent holds is $a = \varphi(r)$ and, if some a verifies the consequent then any greater values of a do the same. Therefore the previous condition is equivalent to
$\forall q, r \in J^\infty(K)$:

$$(q R_\diamond r \Rightarrow (\exists p \in J^\infty(L) :\ (p \leq \varphi(r) \text{ and } \varphi(q) R_\diamond p))$$

which holds if and only if

$$\forall q, r \in J^\infty(K) :\ (q R_\diamond r \Rightarrow \varphi(q) R_\diamond \circ \leq \varphi(r))$$

which is equivalent to

$$\forall q, r \in J^\infty(K) :\ (q R_\diamond r \Rightarrow \varphi(q) R_\diamond \varphi(r))$$

by Definition 3.2.7 (i).

So we conclude that

$$\forall q, r \in J^\infty(K) :\ (q R_\diamond r \Rightarrow \varphi(q) R_\diamond \varphi(r))$$

and

$$\forall q \in J^\infty(K) \forall p \in J^\infty(L) : (\varphi(q) R_\diamond p \Rightarrow (\exists r \in J^\infty(K) : (\varphi(r) \leq p \,\text{and}\, q R_\diamond r))$$

are equivalent to (i).

Using a similar process we can determine the conditions in PR that correspond to equations (ii) ,(iii) and (iv) and see that the set of order preserving maps satisfying these conditions is closed for the composition of maps.

We obtain the following

Definition 3.2.11. Let $X, Y \in PR$. An order preserving map $\varphi : X \longrightarrow Y$ is a morphism in PR if it satisfies the following conditions:

(i) (a) $\forall q, r \in X : \ (qR_\diamond r \Rightarrow \varphi(q)R_\diamond\varphi(r))$.

(b) $\forall q \in X\, \forall p \in Y : \ (\varphi(q)R_\diamond p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \text{ and } qR_\diamond r))$.

(ii) (a) $\forall q, r \in X : \ (qR_\Box r \Rightarrow \varphi(q)R_\Box\varphi(r))$.

(b) $\forall q \in X\, \forall p \in Y : \ (\varphi(q)R_\Box p \Rightarrow (\exists r \in X : \ (\varphi(r) \geq p \text{ and } qR_\Box r))$.

(iii) (a) $\forall q, r \in X : \ (qR_\triangleright r \Rightarrow \varphi(q)R_\triangleright\varphi(r))$.

(b) $\forall q \in X\, \forall p \in Y : \ (\varphi(q)R_\triangleright p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \text{ and } qR_\triangleright r))$.

(iv) (a) $\forall q, r \in X : \ (qR_\triangleleft r \Rightarrow \varphi(q)R_\triangleleft\varphi(r))$.

(b) $\forall q \in X\, \forall p \in Y : \ (\varphi(q)R_\triangleleft p \Rightarrow (\exists r \in X : \ (\varphi(r) \geq p \text{ and } qR_\triangleleft r))$.

In [23] Goldblatt, maps between ordered relational structures satisfying these conditions are called *bounded morphisms*.

As a consequence of Theorem 3.2.6 we have the following:

Theorem 3.2.12. *Let $L, K \in UQA^+$ and let $X, Y \in PR$. Given a morphism $h \in UQA^+(L, K)$, there is an associated morphism $\varphi_h \in PR(J^\infty(K), J^\infty(L))$ defined by*

$$\varphi_h(p) = min\{a \in L : p \leq h(a)\} = max\{a \in L : \kappa(p) \geq h(a)\}$$

for all $p \in J^\infty(K)$.

Given a morphism $\varphi \in PR(Y, X)$, there is an associated morphism $h_\varphi \in UQA^+(\mathcal{D}(X), \mathcal{D}(Y))$ defined by

$$h_\varphi = \varphi^{-1}(U)$$

for each $U \in \mathcal{D}(X)$.

The maps $()_+ : UQA^+(L,K) \longrightarrow PR(J^\infty(K), J^\infty(L))$ *such that* $(h)_+ = \varphi_h$ *and* $()^+ : PR(Y,X) \longrightarrow UQA^+(\mathcal{D}(X), \mathcal{D}(Y))$ *such that* $(\varphi)^+ = h_\varphi$ *are bijections.*

The diagrams

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ {\scriptstyle \eta_L}\downarrow & & \downarrow{\scriptstyle \eta_K} \\ \mathcal{D}(J^\infty(L)) & \xrightarrow[h_{\varphi_h}]{} & \mathcal{D}(J^\infty(K)) \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \varepsilon_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ J^\infty(\mathcal{D}(Y)) & \xrightarrow[\varphi_{h_\varphi}]{} & J^\infty(\mathcal{D}(X)) \end{array}$$

commute.

Further:

(i) h *is one to one if and only if* φ_h *is onto.*

(ii) h *is onto if and only if* φ_h *is an order embedding.*

Now it is clear that the functors $()_+$ and $()^+$ establish a dual equivalence between the categories UQA^+ and PR.

3.3 Discrete duality for Canonical Extensions

Given a Distributive lattice L, the canonical extension L^σ falls in the category $\mathcal{DL}^+$ that we have described in Section 3.1.

In fact $\sigma : \mathcal{DL} \to \mathcal{DL}^+$ is a functor ([19] Theorem 17). However as it is noticed in [19] Remark 3 it is not clear exactly what the image objects are.

The distributive lattices in the codomain of the functor σ are those lattices $\mathcal{D}(X)$ for which X is a *representable poset* or a *spectral poset*, that is a poset that is order isomorphic to the set of prime ideals (or filters) of a distributive lattice, ordered by inclusion.

As is referred by H. Priestley in [32] page 112 "In order that a poset X be spectral it is necessary (but not sufficient) that it satisfies the conditions

(K1) X is up-complete and down-complete.

(K2) X is weakly atomic (that is, given $x < y$ in X there exist $u, v \in X$ such that $x \leq u < v \leq y$ and $u \leq z < v$ implies $z = u$).

These properties hold in the underlying ordered set of a Priestley space".

By *up-complete* we mean a poset X where every up-directed subset has a join in X. By *down-complete*, a poset such that every down-directed subset has a meet in X.

The fact that a $\mathcal{DL}^+$ is the canonical extension of a distributive lattice implies among other things that Lemma 3.1.18 ([18] Corollary 3.9) holds and as consequence we have:

Lemma 3.3.1. *Let $L, K \in \mathcal{DL}$.*

If $f : L \to K$ is a join preserving map and $f^\sigma : L^\sigma \to K^\sigma$ is the canonical extension of f, then for each $q \in J^\infty(K^\sigma)$ and $p \in J^\infty(L^\sigma)$ with $q \leq f^\sigma(p)$ there is $r \in J^\infty(L^\sigma)$ with $r \leq p$ such that r is minimal in $\{s \in J^\infty(L^\sigma) : q \leq f^\sigma(s)\}$.

If $g : L \to K$ is a meet preserving map and $g^\sigma : L^\sigma \to K^\sigma$ is the canonical extension of g, then, for each $q \in J^\infty(K^\sigma)$ and $p \in J^\infty(L^\sigma)$ with $\kappa(q) \geq g^\sigma(\kappa(p))$ there is $r \in J^\infty(L^\sigma)$ with $r \geq p$ such that r is maximal in $\{s \in J^\infty(L^\sigma) : \kappa(q) \geq g^\sigma(\kappa(s))\}$.

Proof. Since f is join preserving, f is isotone so, it satisfies the conditions of Lemma 3.1.18 and therefore, for each $q \in J^\infty(K^\sigma)$ and $p \in J^\infty(L^\sigma)$ with $q \leq f^\sigma(p)$, there is $x \in K(L^\sigma)$ such that $x \leq p$ and x is minimal in $\{v \in Ł^\sigma : q \leq f^\sigma(v)\}$.

But $x = \bigvee\{r \in J^\infty(L^\sigma) : r \leq x\}$ because L^σ is join generated by $J^\infty(L^\sigma)$. Thus, from $q \leq f^\sigma(x)$, it follows $q \leq f^\sigma\left(\bigvee_{r \leq x} r\right)$ and consequently $q \leq$

$\bigvee_{r \leq x} f^\sigma(r)$ since f^σ is completely join preserving by Lemma 3.1.22. Then, there is $r \in J^\infty(L^\sigma)$ such that $r \leq x$ and $q \leq f^\sigma(r)$ because q is completely join irreducible. Since x is a minimal solution of $q \leq f^\sigma(x)$, we have $x = r$. Finally, if r is minimal in $\{v \in Ł^\sigma : q \leq f^\sigma(v)\}$, it is also minimal in the subset $\{s \in J^\infty(L^\sigma) : q \leq f^\sigma(s)\}$

In what concerns the meet preserving map g, notice that it is possible to consider the join preserving map

$$g^\partial : L^\partial \to K^\partial$$

which is set theoretically identical to g.

For this map g^∂ we have just proved that for each $\kappa(q) \in J^\infty((K^\partial)^\sigma) = M^\infty(K^\sigma)$ and $\kappa(p) \in J^\infty((L^\partial)^\sigma) = M^\infty(L^\sigma)$ with $\kappa(q) \leq^\partial (g^\partial)^\sigma(\kappa(p))$ there is a $\kappa(r) \in J^\infty((L^\partial)^\sigma) = M^\infty(L^\sigma)$ with $\kappa(r) \leq^\partial \kappa(p)$ and $\kappa(r)$ is (order dually) minimal in $\{v \in Ł^\sigma : \kappa(q) \leq^\partial (g^\partial)^\sigma(v)\}$.

From the characterization of L^∂ and K^∂ it follows that for each $q \in J^\infty(K^\sigma)$ and $p \in J^\infty(L^\sigma)$ with $\kappa(q) \geq g^\sigma(\kappa(p))$ there is a $r \in J^\infty(L^\sigma)$ with $r \geq p$ and $\kappa(r)$ is maximal in $\{v \in Ł^\sigma : \kappa(q) \geq g^\sigma(v)\}$. But this implies that r is maximal in $\{s \in J^\infty(L^\sigma) : \kappa(q) \geq g^\sigma(\kappa(s))\}$.

In fact, $r \in \{s \in J^\infty(L^\sigma) : \kappa(q) \geq g^\sigma(\kappa(s))\}$ and, for any $s \geq r$, we have $\kappa(s) \geq \kappa(r)$. Since $\kappa(r)$ is maximal in $\{v \in Ł^\sigma : \kappa(q) \geq g^\sigma(v)\}$, we conclude that $\kappa(r) = \kappa(s)$ and consequently $r = s$. □

The previous Lemma can be generalized to join reversing or meet reversing maps, hence we can apply it to the canonical extensions of the unary operations $\Diamond, \Box, \rhd$ and $\lhd$.

Though these canonical extensions are denoted by $\Diamond^\sigma, \Box^\sigma, \rhd^\sigma$ and $\lhd^\sigma$, since there is no danger of misunderstanding we will denote the corresponding relations in $J^\infty(L^\sigma)$ by $R_\Diamond, R_\Box, R_\rhd$ and $R_\lhd$ as usual and, for each $q \in J^\infty(L^\sigma)$, we will consider the subsets:

$$qR_\Diamond = \{r \in J^\infty(L^\sigma) : qR_\Diamond r\}, \quad qR_\rhd = \{r \in J^\infty(L^\sigma) : qR_\rhd r\},$$

$$qR_\Box = \{r \in J^\infty(L^\sigma) : qR_\Box r\}, \quad qR_\lhd = \{r \in J^\infty(L^\sigma) : qR_\lhd r\}.$$

From these definitions and from Lemma 3.3.1 it follows:

Theorem 3.3.2. *Let* $(L, \Diamond, \Box, \rhd, \lhd) \in UQA$ *and let* $(L^\sigma, \Diamond^\sigma, \Box^\sigma, \rhd^\sigma, \lhd^\sigma) \in UQA^+$ *be its canonical extension. Then*

(i) *For every* $p \in J^\infty(L^\sigma)$ *such that* $qR_\diamond p$ *there is a minimal element* $r \in qR_\diamond$ *such that* $r \leq p$.

(ii) *For every* $p \in J^\infty(L^\sigma)$ *such that* $qR_\triangleright p$ *there is a minimal element* $r \in qR_\triangleright$ *such that* $r \leq p$.

(iii) *For every* $p \in J^\infty(L^\sigma)$ *such that* $qR_\Box p$ *there is a maximal element* $r \in qR_\Box$ *such that* $r \geq p$.

(iv) *For every* $p \in J^\infty(L^\sigma)$ *such that* $qR_\triangleleft p$ *there is a maximal element* $r \in qR_\triangleleft$ *such that* $r \geq p$

Proof. It is clear that (i) and (iii) are a direct consequence of Lemma 3.3.1.

For (ii) and (iv) observe that, considering the order dual of L as in the proof of Lemma 3.3.1, we have the completely join preserving maps

$$\triangleright^\partial \colon L \to L^\partial \text{ and } \triangleleft\colon L^\partial \to L.$$

Therefore we can apply Lemma 3.3.1 to these maps. □

Now we can define the relation $R_\diamond^{min}$ such that $qR_\diamond^{min}p$ if and only if $qR_\diamond p$ and p is minimal in $qR_\diamond$.

In a similar way we define $R_\triangleright^{min}, R_\Box^{max}$, and $R_\triangleleft^{max}$.

Observe that for a canonical extension $L^\sigma \in UQA^+$ we have, for each $q \in J^\infty(L^\sigma)$,

$$qR_\diamond = \uparrow qR_\diamond^{min},\ qR_\triangleright = \uparrow qR_\triangleright^{min},\ qR_\Box = \downarrow qR_\Box^{max} \text{ and } qR_\triangleleft = \downarrow qR_\triangleleft^{max}.$$

We can also prove the following corollary that shows how these relations behave in what concerns morphisms:

Corollary 3.3.3. *Let* $X = J^\infty(L^\sigma)$ *and* $Y = J^\infty(K^\sigma)$ *where* L^σ *and* K^σ *are the canonical extensions of* $L, K \in \mathcal{DL}$. *Let* $\varphi : X \to Y$ *be a morphism in* PR.

Then $\varphi(X)$ *is closed under the relations* $R_\diamond^{min}, R_\triangleright^{min}, R_\Box^{max}$ *and* $R_\triangleleft^{max}$.

Proof. Let $x \in X$ and let $p \in Y$ be such that $\varphi(x)R_\diamond^{min}p$. By Definition 3.2.11 there is $r \in X$ such that $\varphi(r) \leq p$ and $xR_\diamond r$.

But, again by Definition 3.2.11, φ preserves the relation $R_\diamond$ so we have also $\varphi(x)R_\diamond\varphi(r)$ and, since p is minimal in $\varphi(x)R_\diamond$ we conclude that $p = \varphi(r)$.

The proofs for the other relations are similar. □

3.4 Basic Topological duality

We can apply the duality established for the category $\mathcal{DL}^+$ to define a duality for bounded distributive lattices.

3.4.1 Bounded Distributive Lattices

Suppose that L is a bounded distributive lattice, ($L \in \mathcal{DL}$). The canonical extension of L, L^σ is, according to [20] Theorem 2.2, a $\mathcal{DL}^+$ containing L as a separating compact sublattice (Definition 3.1.2 and Theorem 3.1.3)

We have seen in section 3.2.1 (Theorems 3.2.1 and 3.2.6) that there is a dual equivalence between the category $\mathcal{DL}^+$ of perfect bounded distributive lattices and the category $\mathcal{P}$ of posets.

Since $L^\sigma \in \mathcal{DL}^+$, we know that $(J^\infty(L^\sigma), \leq) \in \mathcal{P}$.

If we want to recapture L from L^σ however, we need to remember how L sits inside L^σ.

Of course, if we want Priestley duality, the way to capture L from the dual of L^σ, $X = J^\infty(L^\sigma)$, is by considering the sets

$$v(a) = \{p \in J^\infty(L^\sigma) : \, p \leq a\} \text{ and } v(a)^c = J^\infty(L^\sigma) \setminus v(a)$$

for $a \in L$, and by generating a topology τ on X having the set

$$\{v(a), v(a)^c : \, a \in L\}$$

as a subbasis.

By Definition 3.1.2 and Theorem 3.1.3 ([20] Theorem 2.5), $(J^\infty(L^\sigma), \leq, \tau)$ is a compact totally order disconnected space and, denoting by $Clop\mathcal{D}(J^\infty(L^\sigma))$ the set of clopen downsets of $J^\infty(L^\sigma)$, it is obvious that

$$(Clop\mathcal{D}(J^\infty(L^\sigma)), \cap, \cup, \emptyset, J^\infty(L^\sigma)),$$

being a sublattice of $(\mathcal{D}(J^\infty(L^\sigma)), \cap, \cup, \emptyset, J^\infty(L^\sigma))$ is a bounded distributive lattice.

On the other hand, if we consider a compact totally order disconnected space $(X, \leq, \tau)$, then $(X, \leq)$ is a poset and $\mathcal{D}(X)$ is in $\mathcal{DL}^+$. Clearly $\mathcal{D}(X)$ is the canonical extension of the bounded distributive lattice $Clop\mathcal{D}(X)$.

Let us denote by $\mathcal{DL}$ the category of bounded distributive lattices having as morphisms {0,1}-homomorphisms and by $\mathcal{P}_\tau$ the category of Priestley spaces having as morphisms continuous order preserving maps.

We have just seen how to define, at the objects level, functors from $\mathcal{DL}$ to $\mathcal{P}_\tau$ and from $\mathcal{P}_\tau$ to $\mathcal{DL}$.

We will use D and E to denote these functors so that we have:

Lemma 3.4.1. *There exist maps $D : \mathcal{DL} \longrightarrow \mathcal{P}_\tau$ and $E : \mathcal{P}_\tau \longrightarrow \mathcal{DL}$ such that for each $L \in \mathcal{DL}$ and each $X \in \mathcal{P}_\tau$,*

$$D(L) = (J^\infty(L^\sigma), \leq, \tau) \text{ and } E(X) = (Clop(\mathcal{D}(X)), \cup, \cap, \emptyset, X).$$

Further, the following is also true:

Lemma 3.4.2. *Let L be a bounded distributive lattice and L^σ its canonical extension. Then the map*

$$v : a \longmapsto \{p \in J^\infty(L^\sigma) :\ p \leq a\}$$

is an isomorphism of L onto $Clop\mathcal{D}(J^\infty(L^\sigma))$.

Proof. By the definition of v we know that $v = \eta \restriction_L$ where $\eta : L^\sigma \longrightarrow \mathcal{D}(J^\infty(L^\sigma))$ is, by Theorem 3.2.1, an isomorphism. Consequently v is a homomorphism and v is injective. We only have to prove that v is onto.

Let $U \in Clop\mathcal{D}(J^\infty(L^\sigma))$.

For each $p \in U$ and $q \notin U$, since U is a downset, we have $q \not\leq p$, and thus, by Theorem 3.1.3, there is $a_{p,q} \in L$ such that $p \leq a_{p,q}$ but $q \not\leq a_{p,q}$.

We have, for any given $p \in U$ and $q \in U^c$, $q \notin v(a_{p,q})$ and thus

$$U^c \subseteq \bigcup \{v(a_{p,q})^c :\ q \in U^c\}.$$

Since U^c is closed and therefore compact, we get $q_1, ..., q_n$ such that:

$$U^c \subseteq \bigcup_{i=1}^{n} v(a_{p,q_i})^c,$$

that is

$$U \supseteq \bigcap_{i=1}^{n} v(a_{p,q_i}) = v\left(\bigwedge_{i=1}^{n} a_{p,q_i}\right) = v(a_p)$$

where $a_p = \bigwedge_{i=1}^{n} a_{p,q_i}$.

That is, for each $p \in U$, there is $a_p \in L$ with $p \leq a_p$ and $v(a_p) \subseteq U$. But then $U = \bigcup_{p \in U} v(a_p)$ and, by compactness of U, we get $p_1, ...p_m \in U$ with

$$U \subseteq \bigcup_{i=1}^{m} v(a_{p_i}) \subseteq \bigcup_{p \in U} v(a_p) = U$$

so

$$U = \bigcup_{i=1}^{m} v(a_{p_i}) = v\left(\bigvee_{i=1}^{m} a_{p_i}\right)$$

and thus U is in the image of v.

□

If we consider a compact totally order disconnected space $(X, \leq, \tau)$, there is, by Lemma 3.4.1, a bounded distributive lattice $E(X) = Clop\mathcal{D}(X)$. To this lattice corresponds a Priestley space $(J^\infty(\mathcal{D}(X)), \leq, \tau)$ where τ has as a subbasis

$$\{v(U), v(U)^c : \ U \in Clop\mathcal{D}(X)\}.$$

Notice that the completely join irreducible elements of $\mathcal{D}(X)$ are the principal ideals $\downarrow x$ with $x \in X$ so, for any $U \in Clop\mathcal{D}(X)$, we have

$$v(U) = \{\downarrow x \in J^\infty(\mathcal{D}(X)) : \downarrow x \subseteq U\} = \{\downarrow x \in J^\infty(\mathcal{D}(X)) : \ x \in U\}$$

and we can prove:

Lemma 3.4.3. *Let $(X, \leq, \tau)$ be a Priestley space. Then the map $\varepsilon : X \longrightarrow J^\infty(\mathcal{D}(X))$ such that for each $x \in X$ we have $\varepsilon(x) = \downarrow x$ is an order homeomorphism.*

Proof. By Theorem 3.2.2, we know that ε is an order isomorphism. To prove that it is an order homeomorphism it is only necessary to prove that, for any $U \in Clop(\mathcal{D}(X))$, the subsets $\varepsilon^{-1}(v(U))$ and $\varepsilon^{-1}(v(U)^c)$ are clopen.

Let $U \in Clop\mathcal{D}(X)$, then

$$\begin{aligned} \varepsilon^{-1}(v(U)) &= \{x \in X : \ \varepsilon(x) \in v(U)\} \\ &= \{x \in X : \downarrow x \in v(U)\} \\ &= \{x \in X : \ x \in U\} \\ &= U. \end{aligned}$$

Therefore $\varepsilon^{-1}(v(U))$ is clopen and $\varepsilon^{-1}(v(U)^c) = X \setminus \varepsilon^{-1}(v(U))$ is also clopen.

□

At this point we have defined, at the objects level, a duality between the categories $\mathcal{DL}$ and $\mathcal{P}_\tau$.

Now we are going to see how to define maps D and E between morphisms of the two categories.

Let $L, K \in \mathcal{DL}$ and let h be a $\{0,\ 1\}$-homomorphism from L to K (i.e. $h \in \mathcal{DL}(L, K)$).

It is possible to define extensions of h to L^σ. Since h is isotone we have, by Remark 3.1.16, extensions

$$h^\sigma : L^\sigma \longrightarrow K^\sigma \text{ and } h^\pi : L^\sigma \longrightarrow K^\sigma$$

such that for each $u \in L^\sigma$:

$$h^\sigma(u) = \bigvee \left\{ \bigwedge \{h(a) :\ x \leq a \in L\} :\ u \geq x \in K(L^\sigma) \right\}$$

where $K(L^\sigma)$ is the meet closure of L in L^σ and

$$h^\pi(u) = \bigwedge \left\{ \bigvee \{h(a) :\ y \geq a \in L\} :\ u \leq y \in O(L^\sigma) \right\}.$$

where $O(L^\sigma)$ is the join closure of L in L^σ.

By Lemmas 3.1.22 and 3.1.23 ([20] Theorem 2.23), h^π is completely join preserving because h is join preserving and, since h is meet preserving, h^σ is completely meet preserving and $h^\pi = h^\sigma$.

So we conclude that h^σ is a complete homomorphism i.e. $h^\sigma \in \mathcal{DL}^+(L^\sigma, K^\sigma)$.

Further, we have:

Lemma 3.4.4. *Every complete homomorphism $H : L^\sigma \longrightarrow K^\sigma$ that extends h is equal to h^σ.*

Proof. This consequence was not made clear in [20] but essentially follows from the fact that h^σ is the greatest $(\sigma, \iota^\uparrow)$-continuous extension of h and h^π is the least $(\sigma, \iota^\downarrow)$-continuous extension of h (see Theorem 3.1.13 or [20] Theorem 2.15).

To see this, notice that, by Remarks 3.1.11 and 3.1.17, for an order preserving map $H : L^\sigma \to K^\sigma$ we have, for any $u \in L^\sigma$,

$$H(u) = \bigvee \{H(x) : K(L^\sigma) \ni x \leq u\}$$

if and only if H is $(\sigma, \iota^{\uparrow})$-continuous.

It is clear that $H(u) = \bigvee\{H(x) : K(L^\sigma) \ni x \leq u\}$ implies the $(\sigma^{\uparrow}, \iota^{\uparrow})$-continuity of H and consequently its $(\sigma, \iota^{\uparrow})$-continuity.

Conversely, if H is $(\sigma, \iota^{\uparrow})$-continuous then, by Remark 3.1.17, H is $(\sigma^{\uparrow}, \iota^{\uparrow})$-continuous so that $q \in J^\infty(L^\sigma)$ and $q \leq H(u)$ implies the existence of an $x \in K(L^\sigma)$ such that, for any $v \in L^\sigma$ such that $x \leq v$ we have $q \leq H(v)$ and hence $q \leq H(x)$. Thus $q \leq \bigvee\{H(x) : K(L^\sigma) \ni x \leq u\}$.

On the other hand, if $q \in J^\infty(L^\sigma)$ is $q \leq \bigvee\{H(x) : K(L^\sigma) \ni x \leq u\}$, then, since q is completely join irreducible there is $x \leq u$ such that $q \leq H(x)$. But H is order preserving so that $q \leq H(u)$.

Thus the equality is proved.

Dually

$$H(u) = \bigwedge\{H(y) : O(L^\sigma) \ni y \geq u\}$$

if and only if H is $(\sigma, \iota^{\downarrow})$-continuous.

But these properties certainly hold for any complete homomorphism. So, by Theorem 3.1.13 and Corollary 3.1.14, we have

$$H \leq h^\sigma \leq h^\pi \leq H$$

and thus $H = h^\sigma$.

□

Having, for each $h \in \mathcal{DL}(L, K)$, a complete homomorphism $h^\sigma \in \mathcal{DL}^+(L^\sigma, K^\sigma)$ we have also, by Theorem 3.2.6, an order preserving map $\varphi_{h^\sigma} \in \mathcal{P}(J^\infty(K^\sigma), J^\infty(L^\sigma))$ such that

$$\varphi_{h^\sigma}(q) = min\{u \in L^\sigma : q \leq h^\sigma(u)\} = \bigwedge\{u \in L^\sigma : q \leq h^\sigma(u)\}$$

for all $q \in J^\infty(K^\sigma)$.

It is possible to define φ_{h^σ} in a different way.

Lemma 3.4.5. *Let L^σ and K^σ be the canonical extensions of the distributive lattices L and K, respectively. Let h be a homomorphism from L to K. Then, for every $q \in J^\infty(K^\sigma)$,*

$$\varphi_{h^\sigma}(q) = \bigwedge\{u \in L^\sigma :\ q \leq h^\sigma(u)\} = \bigwedge\{a \in L :\ q \leq h(a)\} \qquad (\bigwedge \text{ in } L^\sigma).$$

Proof. We know that L is a sublattice of L^σ and, for every $a \in L$, $h^\sigma(a) = h(a)$ so,

$$\{u \in L^\sigma :\ q \leq h^\sigma(u)\} \supseteq \{a \in L :\ q \leq h(a)\}.$$

and, consequently

$$\bigwedge\{u \in L^\sigma :\ q \leq h^\sigma(u)\} \leq \bigwedge\{a \in L :\ q \leq h(a)\}.$$

On the other hand, since h^σ is $(\sigma, \iota^\uparrow)$-continuous and isotone, we know by Remarks 3.1.11 and 3.1.17 that, for every $u \in L^\sigma$ and every $q \in J^\infty(K^\sigma)$ such that $q \leq h^\sigma(u)$, there is $x \in K(L^\sigma)$ such that $x \leq u$ and for any $v \in L^\sigma$,

$$x \leq v \Rightarrow q \leq h^\sigma(v).$$

So

$$\{a \in L :\ x \leq a\} \subseteq \{a \in L :\ q \leq h(a)\}.$$

But $x = \bigwedge\{a \in L :\ x \leq a\}$ because $x \in K(L^\sigma)$. Thus $x \geq \bigwedge\{a \in L : q \leq h(a)\}$ and, from $u \geq x$ it follows $u \geq \bigwedge\{a \in L :\ q \leq h(a)\}$.

Therefore

$$\bigwedge\{u \in L^\sigma :\ q \leq h^\sigma(u)\} \geq \bigwedge\{a \in L :\ q \leq h(a)\}.$$

□

For $Y, X \in \mathcal{P}_\tau$ let us denote by $\mathcal{P}_\tau(Y, X)$ the class of continuous order preserving maps from Y to X. It is now possible to prove

Lemma 3.4.6. *Let $L, K \in \mathcal{DL}$. Given a $\{0, 1\}$-homomorphism $h \in \mathcal{DL}(L, K)$, there is a continuous order preserving map*

$$\varphi_h \in \mathcal{P}_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$$

defined by

$$\varphi_h(q) = \bigwedge\{a \in L : q \leq h(a)\}$$

for all $q \in J^\infty(K^\sigma)$.

These maps are such that, for each $a \in L$ and each $q \in J^\infty(K^\sigma)$,

$$q \leq h(a) \text{ if and only if } \varphi_h(q) \leq a. \qquad (\star)$$

Proof. The equivalence

$$q \leq h(a) \text{ if and only if } \varphi_h(q) \leq a$$

is an immediate consequence of Theorem 3.2.6 because we have, for each $a \in L$, $h(a) = h^\sigma(a)$.

By Theorem 3.2.6 and Lemma 3.4.5, we know that φ_h is an order preserving map so, we have only to prove that φ_h is continuous. To do so it is enough to prove that for any $a \in L$, $\varphi_h^{-1}(v(a))$ is a clopen subset of $J^\infty(K^\sigma)$.

$$\begin{aligned} \varphi_h^{-1}(v(a)) &= \varphi_h^{-1}(\{p \in J^\infty(L^\sigma) : \, p \leq a\}) \\ &= \{q \in J^\infty(K^\sigma) : \, \varphi_h(q) \leq a\} \\ &= \{q \in J^\infty(K^\sigma) : \, q \leq h(a)\} \\ &= v(h(a)) \end{aligned}$$

So $\varphi_h^{-1}(v(a))$ is clopen. □

On the other hand we have:

Lemma 3.4.7. *Let $X, Y \in \mathcal{P}_\tau$ and let $\varphi \in \mathcal{P}_\tau(Y, X)$ be a continuous order-preserving map. Then, there is an associated $\{0,\ 1\}$-homomorphism $h_\varphi \in \mathcal{DL}(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ defined by*

$$h_\varphi(U) = \varphi^{-1}(U)$$

for each $U \in Clop\mathcal{D}(X)$.

Proof. Since φ is an order preserving map we know from Theorem 3.2.6 that there is a complete homomorphism H_φ from $\mathcal{D}(X)$ to $\mathcal{D}(Y)$ such that $H_\varphi(U) = \varphi^{-1}(U)$.

But φ is continuous so H_φ maps $Clop\mathcal{D}(X)$ into $Clop\mathcal{D}(Y)$ and consequently $h_\varphi = H_\varphi \restriction Clop\mathcal{D}(X)$. □

So we can consider contravariant functors between the categories $\mathcal{DL}$ and $\mathcal{P}_\tau$ arriving this way at a duality that is none other than Priestley duality ([17] **10.25**).

Theorem 3.4.8. *Let $L, K \in \mathcal{DL}$ and let $X, Y \in \mathcal{P}_\tau$.*

Given a $\{0,\ 1\}$-homomorphism $h \in \mathcal{DL}(L, K)$, there is an associated continuous order preserving map $D(h) \in \mathcal{P}_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$.

Given a continuous order-preserving map $\varphi \in \mathcal{P}_\tau(Y, X)$, there is an associated $\{0, 1\}$-homomorphism $E(\varphi) \in \mathcal{DL}(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$.

The maps $D : \mathcal{DL}(L, K) \longrightarrow \mathcal{P}_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ and $E : \mathcal{P}_\tau(Y, X) \longrightarrow \mathcal{DL}(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ are bijections and the diagrams

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ {\scriptstyle v_L}\downarrow & & \downarrow{\scriptstyle v_K} \\ Clop\mathcal{D}(J^\infty(L^\sigma)) & \xrightarrow[ED(h)]{} & Clop\mathcal{D}(J^\infty(K^\sigma)) \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \varepsilon_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ J^\infty(\mathcal{D}(Y)) & \xrightarrow[DE(\varphi)]{} & J^\infty(\mathcal{D}(X)) \end{array}$$

commute.

Further:

(i) h is one to one if and only if $D(h)$ is onto.

(ii) h is onto if and only if $D(h)$ is an order embedding.

Proof. The maps D and E are defined in Lemmas 3.4.6 and 3.4.7.

To prove that D is one to one let us consider $h_1, h_2 \in \mathcal{DL}(L, K)$ such that $h_1 \neq h_2$.

Clearly, for the extensions h_1^σ, h_2^σ, we have $h_1^\sigma \neq h_2^\sigma$ and, by Theorem 3.2.6, the corresponding order preserving maps $\varphi_{h_1^\sigma}$ and $\varphi_{h_2^\sigma}$ are also different. But, as we proved in Lemma 3.4.5, $\varphi_{h_1^\sigma} = \varphi_{h_1}$ and $\varphi_{h_2^\sigma} = \varphi_{h_2}$ with $\varphi_{h_1}, \varphi_{h_2} \in \mathcal{P}_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ so $D(h_1) \neq D(h_2)$.

To prove that D is onto let us consider $\psi \in \mathcal{P}_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$. Since we have also $\psi \in \mathcal{P}(J^\infty(K^\sigma), J^\infty(L^\sigma))$ we know by Theorem 3.2.6 that there is $H \in \mathcal{DL}^+(L^\sigma, K^\sigma)$ such that $\psi = \varphi_H$.

Denoting by H_{φ_H} the complete homomorphism

$$H_{\varphi_H} \in \mathcal{DL}^+(\mathcal{D}(J^\infty(L^\sigma)), \mathcal{D}(J^\infty(K^\sigma)))$$

such that $H_{\varphi_H} = \varphi_H^{-1}$. It follows that H_{φ_H} maps $Clop\mathcal{D}(J^\infty(L^\sigma))$ into $Clop\mathcal{D}(J^\infty(K^\sigma))$ because φ_H is continuous.

By Theorem 3.2.6 we know that $H = \eta_K^{-1} \circ H_{\varphi_H} \circ \eta_L$.

For any $a \in L$ we have $\eta_L(a) = \upsilon_L(a) \in Clop\mathcal{D}(J^\infty(L^\sigma))$ so,

$$H_{\varphi_H} \circ \eta_L(a) = \varphi_H^{-1}(\eta_L(a)) \in Clop\mathcal{D}(J^\infty(K^\sigma)).$$

Therefore, by Lemma 3.4.2, $\eta_K^{-1} \circ H_{\varphi_H} \circ \eta_L(a) = \upsilon_K^{-1}(H_{\varphi_H} \circ \eta_L(a)) \in K$.

So $H_{\restriction L} \in \mathcal{DL}(L, K)$ and H extends $H_{\restriction L}$. By Lemmas 3.4.4 and 3.4.5, $\psi = \varphi_H = D(H_{\restriction L})$.

In what concerns E, we are going to consider $\varphi_1, \varphi_2 \in \mathcal{P}_\tau(Y, X)$ such that $E(\varphi_1) = E(\varphi_2)$, Then, for every $U \in Clop\mathcal{D}(X), \varphi_1^{-1}(U) = \varphi_2^{-1}(U)$.

Now suppose, in order to obtain a contradiction, that $\varphi_1 \neq \varphi_2$.

Then there is $p \in Y$ such that $\varphi_1(p) \neq \varphi_2(p)$. Without loss of generality we may suppose that $\varphi_1(p) \not\leq \varphi_2(p)$. Since X is a Priestley space, there is $V \in Clop\mathcal{D}(X)$ such that $\varphi_2(p) \in V$ and $\varphi_1(p) \notin V$ and consequently $p \in \varphi_2^{-1}(V)$ and $p \notin \varphi_1^{-1}(V)$. A contradiction.

So E is one to one.

To see that E is onto let us consider $h \in \mathcal{DL}(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$. Then $h^\sigma \in \mathcal{DL}^+(\mathcal{D}(X), \mathcal{D}(Y))$ and, by Theorem 3.2.6 there is an order preserving map $\varphi \in \mathcal{P}(Y, X)$ such that $h^\sigma(V) = \varphi^{-1}(V)$ for every $V \in \mathcal{D}(X)$.

Since h^σ maps $Clop\mathcal{D}(X)$ in $Clop\mathcal{D}(Y)$ because it extends h it follows that φ is continuous.

Then we have $\varphi \in \mathcal{P}_\tau(Y, X)$ and, for every $U \in Clop\mathcal{D}(X)$, $h(U) = \varphi^{-1}(U)$ so $h = E(\varphi)$.

To prove that the diagrams commute notice that, for every $a \in L$,

$$\upsilon_K(h(a)) = \{q \in J^\infty(K^\sigma) : \ q \leq h(a)\} = \{q \in J^\infty(K^\sigma) : \ \varphi_h(q) \leq a\}$$

and that

$$h_{\varphi_h}(\upsilon_L(a)) = \varphi_h^{-1}(\{p \in J^\infty(L^\sigma) : \ p \leq a\}) = \{q \in J^\infty(K^\sigma) : \ \varphi_h(q) \leq a\}$$

so $\upsilon_K \circ h = h_{\varphi_h} \circ \upsilon_L$.

On the other hand, for every $q \in Y$,

$$\varepsilon_X(\varphi(q)) = \downarrow \varphi(q)$$

and, by Lemma 3.4.5,

$$\begin{aligned}\varphi_{h\varphi}(\varepsilon_Y(q)) &= \varphi_{h\varphi}(\downarrow q)\\ &= \bigwedge\{U \in Clop\mathcal{D}(Y) : \downarrow q \subseteq h_\varphi(U)\}\\ &= min\{V \in \mathcal{D}(Y) : \downarrow q \subseteq \varphi^{-1}(V)\}\\ &= min\{V \in \mathcal{D}(Y) :\ \varphi(q) \in V\}\\ &= \downarrow \varphi(q)\end{aligned}$$

so $\varepsilon_X \circ \varphi = \varphi_{h\varphi} \circ \varepsilon_Y$.

As for (i) and (ii), notice that these conditions are a direct consequence of Theorem 3.2.6 and Theorem 17 in [19].

□

In [17], Priestley studies the correspondents of congruences in the dual spaces. Since congruences play an important role in Lattice Theory we will also refer here the representation of congruences of a distributive lattice $L \in \mathcal{DL}$. We will denote by $Con(L)$ the lattice of congruences of L.

To a congruence $\theta \in Con(L)$ corresponds a homomorphism h of L onto L/θ. Therefore, by the previous theorem, the map $D(h) = \varphi_h : J^\infty((L/\theta)^\sigma) \to J^\infty(L^\sigma)$ is a continuous order embedding and consequently its range is a compact and hence a closed subset of $J^\infty(L^\sigma)$ that is homeomorphic to the dual space of L/θ.

Denoting by Z this range we have, for any $a, b \in L$:

$$a\theta b \Leftrightarrow h(a) = h(b) \Leftrightarrow ED(h)(v(a)) = ED(h)(v(b)) \Leftrightarrow \varphi_h^{-1}(v(a)) = \varphi_h^{-1}(v(b)).$$

Since Z is the range of the order embedding φ_h, it is easy to prove that

$$\varphi_h^{-1}(v(a)) = \varphi_h^{-1}(v(b)) \Leftrightarrow v(a) \cap Z = v(b) \cap Z$$

and consequently

$$a\theta b \quad \text{if and only if} \quad v(a) \cap Z = v(b) \cap Z.$$

It is possible to show, following [17], that this correspondence between $Con(L)$ and the order dual of the lattice of closed subsets of $J^\infty(L^\sigma)$ is an isomorphism so we have:

Corollary 3.4.9. *Let* $L \in \mathcal{DL}$ *and* $X = J^\infty(L^\sigma)$. *To any congruence* $\theta \in Con(L)$ *corresponds a closed subset* $Z \subseteq X$ *that is homeomorphic to* $J^\infty((L/\theta)^\sigma)$ *and that is such that, for any* $a, b \in L$,

$$a\theta b \Leftrightarrow v(a) \cap Z = v(b) \cap Z.$$

The lattice of congruences of L, $Con(L)$, *is isomorphic to the order dual of the lattice of closed subsets of* X.

The duality for distributive lattices that we have been developing can also be applied to distributive lattices with operations that are meet or join preserving and meet or join reversing.

This was done in a different setting by R. Goldblatt in [23]. There he used Priestley duality to develop a representation for distributive lattices with operators that are meet or join preserving. Later, in [41], V. Sofronie-Stokermans extended this duality to meet or join reversing operators.

In the present work we will restrict our study to unary operations as this is all we need.

3.4.2 Distributive Algebras

Following [21] we can define relational structures that are the duals of UQAs (Definition 3.1.24). We will start by considering algebras with a single unary operation that is join-preserving. This way we will start by studying in detail algebras

$$(L, \vee, \wedge, \Diamond, 0, 1)$$

that are the reducts of algebras in UQA. Then it will be easy, using the order dual of L, the generalization to algebras

$$(L, \vee, \wedge, \Diamond, \Box, \rhd, \lhd, 0, 1) \in UQA.$$

Definition 3.4.10. A $UQ\Diamond$ is an algebra $(L, \vee, \wedge, 0, 1, \Diamond)$ where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\Diamond$ is an unary operation satisfying

$$\Diamond 0 = 0 \text{ and } \Diamond(a \vee b) = \Diamond a \vee \Diamond b$$

The unary operation $\Diamond$ is an order preserving map $\Diamond : L \longrightarrow L$ so we can extend $\Diamond$ to L^σ and we can define unambiguously as in Definition 3.1.25:

Definition 3.4.11. Let $L = (L, \vee, \wedge, 0, 1, \Diamond) \in UQ\Diamond$. The canonical extension of L is the algebra

$$L^\sigma = (L^\sigma, \vee, \wedge, 0, 1, \Diamond^\sigma) = (L^\sigma, \vee, \wedge, 0, 1, \Diamond^\pi).$$

We will denote by $UQ\Diamond^+$ those algebras in $UQ\Diamond$ such that $L \in \mathcal{DL}^+$ and $\Diamond$ is completely join preserving.

From Lemma 3.1.27 it follows that

If L is a $UQ\Diamond$, then L^σ is a $UQ\Diamond^+$

so, for canonical properties (Definition 3.1.30), we can move to the canonical extension of an algebra $L \in UQ\Diamond$.

In Section 3.2, we studied the correspondents of completely join preserving maps. From there we can conclude that the dual of $(L^\sigma, \vee, \wedge, 0, 1, \Diamond^\sigma) \in UQ\Diamond^+$ is a relational structure, $(J^\infty(L^\sigma), \leq, R_\Diamond) \in PR\Diamond$, where $PR\Diamond$ is the following class of structures:

Definition 3.4.12. A $PR\Diamond$ is a relational structure $(X, \leq, R_\Diamond)$ such that $(X, \leq)$ is a partially ordered set and $R_\Diamond \subseteq X \times X$ is a binary relation satisfying

$$\leq \circ R_\Diamond \leq \subseteq R_\Diamond.$$

To get a relational structure that is the dual of $(L, \vee, \wedge, 0, 1, \Diamond)$ we have to find out additional conditions involving the relation $R_\Diamond$ and the topology in the dual space of $(L, \vee, \wedge, 0, 1)$, (i.e. the Priestley space $(J^\infty(L^\sigma), \leq, \tau)$).

The conditions that we will obtain correspond to the restrictions of those obtained by R. Goldblatt in [23] (**2.2**).

Lemma 3.4.13. *Let $L \in UQ\Diamond$ and let $R_\Diamond$ be the binary relation on $J^\infty(L^\sigma)$ such that, for $p, q \in J^\infty(L^\sigma)$, $pR_\Diamond q$ if and only if $p \leq \Diamond^\sigma q$. Then $R_\Diamond$ satisfies:*

(i) For any $U \in Clop\mathcal{D}(J^\infty(L^\sigma))$, we have $R_\Diamond^{-1}(U) \in Clop\mathcal{D}(J^\infty(L^\sigma))$.

(ii) For each $p \in J^\infty(L^\sigma)$ the set $pR_\Diamond = \{q \in J^\infty(L^\sigma) : pR_\Diamond q\}$ is a closed subset of $(J^\infty(L^\sigma), \leq, \tau)$.

Proof. i) For every $U \in Clop\mathcal{D}(J^\infty(L^\sigma))$ there exists $a \in L$ such that $U = v(a)$ (Lemma 3.4.2), so

$$\begin{aligned} R_\diamond^{-1}(U) &= R_\diamond^{-1}(v(a)) \\ &= \{p \in J^\infty(L^\sigma) :\ \exists s \in J^\infty(L^\sigma)\ (s \in v(a) \text{ and } pR_\diamond s)\} \\ &= \{p \in J^\infty(L^\sigma) :\ \exists s \in J^\infty(L^\sigma)(s \le a \text{ and } p \le \diamond^\sigma s)\}. \end{aligned}$$

Since $\diamond^\sigma$ is order preserving, ($s \le a$ and $p \le \diamond^\sigma s$) implies $p \le \diamond^\sigma a$.

Conversely if $p \le \diamond^\sigma a$ we get, since $\diamond^\sigma$ is completely join preserving,

$$p \le \bigvee\{\diamond^\sigma r :\ r \in J^\infty(L^\sigma) \text{ and } r \le a\}.$$

Consequently there is some $r \le a$ such that $p \le \diamond^\sigma r$ because $p \in J^\infty(L^\sigma)$.

So $\exists s \in J^\infty(L^\sigma) : (s \le a$ and $p \le \diamond^\sigma s)$ is equivalent to $p \le \diamond^\sigma a$ and, since for each $a \in L$ we have $\diamond^\sigma a = \diamond a$, we get:

$$R_\diamond^{-1}(v(a)) = \{p \in J^\infty(L^\sigma) :\ p \le \diamond a\} = v(\diamond a) \in Clop\mathcal{D}(J^\infty(L^\sigma)).$$

ii) Let us consider $r \in J^\infty(L^\sigma)$ such that $r \in (pR_\diamond)^c$. Then $r \notin \{q \in J^\infty(L^\sigma) :\ p \le \diamond^\sigma q\}$ so $p \nleq \diamond^\sigma r$.

Since $J^\infty(L^\sigma) \subseteq K(L^\sigma)$ we have, by Remark 3.1.16, that for each $r \in J^\infty(L^\sigma)$

$$\diamond^\sigma r = \bigwedge\{\diamond a :\ a \in L \text{ and } r \le a\}.$$

So $p \nleq \bigwedge\{\diamond a :\ r \le a\}$ and hence there is $a \in L$ such that $r \le a$ and $p \nleq \diamond a$.

Then $v(a)$ is an open neighborhood of r and we may prove that $v(a) \subseteq (pR_\diamond)^c$.

In fact, if there was some $s \in v(a)$ such that $s \in pR_\diamond$, we would have

$$p \le \diamond^\sigma s \le \diamond^\sigma a = \diamond a$$

since $s \le a$ and $\diamond^\sigma$ is order preserving, a contradiction.

Thus, for each $r \in (pR_\diamond)^c$ we have $a \in L$ with $r \in v(a)$ and $v(a) \subseteq (pR_\diamond)^c$ hence

$$(pR_\diamond)^c = \bigcup\{v(a) :\ v(a) \subseteq (pR_\diamond)^c\}$$

and consequently

$$pR_\diamond = \bigcap\{v(a)^c :\ v(a)^c \supseteq pR_\diamond\}. \tag{3.4.1}$$

□

Now we can define the relational structures that are the duals of the algebras in $UQ\diamondsuit$.

Definition 3.4.14. Let $(X, \leq, \tau, R_\diamond)$ be a relational structure such that $(X, \leq, \tau)$ is in $\mathcal{P}_\tau$ and $R_\diamond$ satisfies:

(i) $\leq \circ R_\diamond \circ \leq = R_\diamond$.

(ii) For each $U \in Clop\mathcal{D}(X)$, $R_\diamond^{-1}(U) \in Clop\mathcal{D}(X)$.

(iii) For each $p \in X$, $pR_\diamond = \{q \in X : pR_\diamond q\}$ is closed.

We will call $(X, \leq, \tau, R_\diamond)$ a $PR\diamondsuit_\tau$.

From Lemma (3.4.1) and Lemma (3.4.13) we conclude

Lemma 3.4.15. *There is a map* $D : UQ\diamondsuit \longrightarrow PR\diamondsuit_\tau$ *such that*

$$D(L, \wedge, \vee, 0, 1, \diamondsuit) = (J^\infty(L^\sigma), \leq, \tau, R_\diamond)$$

where $R_\diamond$ *is such that, for any* $p, q \in J^\infty(L^\sigma)$, $pR_\diamond q$ *if and only if* $p \leq \diamondsuit^\sigma q$.

On the other hand we have also

Lemma 3.4.16. *There is a map* $E : PR\diamondsuit_\tau \longrightarrow UQ\diamondsuit$ *such that*

$$E(X, \leq, \tau, R_\diamond) = (Clop\mathcal{D}(X), \cup, \cap, \emptyset, X, \diamondsuit_R)$$

where $\diamondsuit_R(U) = R_\diamond^{-1}(U)$ *for any* $U \in Clop\mathcal{D}(X)$.
Further $R_\diamond^{-1} = \diamondsuit_R^\sigma$.

Proof. We have already seen that $(Clop\mathcal{D}(X), \cup, \cap, \emptyset, X)$ is a distributive lattice (Lemma 3.4.1) and that $\mathcal{D}(X) = (Clop\mathcal{D}(X))^\sigma$.

By Definition 3.4.14, for each $U \in Clop\mathcal{D}(X)$ we have $R_\diamond^{-1}(U) \in Clop\mathcal{D}(X)$ so $R_\diamond^{-1} \restriction Clop\mathcal{D}(X) = \diamondsuit_R$ is a unary operation in $Clop\mathcal{D}(X)$.

It is clear that $R_\diamond^{-1}$ is an extension of $\diamondsuit_R$. We will prove that in fact

$$R_\diamond^{-1} = \diamondsuit_R^\sigma.$$

These are both completely completely join preserving so we just need to show that they agree on $J^\infty(\mathcal{D}(X)) = \{\downarrow q : q \in X\}$.

Since $J^\infty(\mathcal{D}(X)) \subseteq K(\mathcal{D}(X))$ we have by Remark 3.1.16:

$$\Diamond_R^\sigma(\downarrow q) = \bigcap \{\Diamond_R(U) : U \in Clop\mathcal{D}(X) \text{ and } q \in U\}$$

which is equivalent to

$$\Diamond_R^\sigma(\downarrow q) = \bigcap \left\{R_\Diamond^{-1}(U) : U \in Clop\mathcal{D}(X) \text{ and } q \in U\right\}.$$

From here it follows that:

$$p \in \Diamond_R^\sigma(\downarrow q) \Leftrightarrow \forall U : ((U \in Clop\mathcal{D}(X) \text{ and } q \in U) \Rightarrow \exists r \in U : pR_\Diamond r).$$

But this means that q is in the closure of $pR_\Diamond$ and $(X, \leq, \tau, R_\Diamond) \in PR\Diamond_\tau$ implies $pR_\Diamond$ is closed so:

$$p \in \Diamond_R^\sigma(\downarrow q) \Leftrightarrow q \in \overline{(pR_\Diamond)} \Leftrightarrow q \in pR_\Diamond \Leftrightarrow p \in R_\Diamond^{-1}(\downarrow q).$$

□

Now it is possible to prove:

Theorem 3.4.17. *There exist maps*

$$D : UQ\Diamond \longrightarrow PR\Diamond_\tau \text{ and } E : PR\Diamond_\tau \longrightarrow UQ\Diamond$$

such that

(i) For each $L \in UQ\Diamond$ there is an isomorphism $v_L : L \longrightarrow E(D(L))$.

(ii) For each $X \in P\Diamond_\tau$ there is a map $\varepsilon_X : X \longrightarrow D(E(X))$ that is an order homeomorphism and an isomorphism for the relation $R_\Diamond$.

Proof. The maps D and E were defined in Lemmas 3.4.15 and 3.4.16.

(i) We have already proved in Lemma 3.4.2 that v_L is an isomorphism of $L \in \mathcal{DL}$ onto $Clop\mathcal{D}(J^\infty(L^\sigma)) \in \mathcal{DL}$ so we have only to prove that, for each $a \in L$, $v_L(\Diamond a) = \Diamond_R(v_L(a))$. But this is already done because, by the previous lemma $\Diamond_R(v_L(a)) = R_\Diamond^{-1}(v_L(a))$ and, by the proof of Lemma 3.4.13 $R_\Diamond^{-1}(v_L(a)) = v_L(\Diamond a)$.

(ii) In Lemma 3.4.3 we proved that ε_X is an order homeomorphism so we have to see if for every $p, q \in X$,

$$pR_\Diamond q \text{ if and only if } \varepsilon_X(p) R_{\Diamond_R} \varepsilon_X(q).$$

Notice that by the definition of ε_X and Lemma 3.4.16

$$\varepsilon(p)R_{\Diamond_R}\varepsilon(q) \Leftrightarrow \downarrow p \subseteq \Diamond_R^\sigma \downarrow q \Leftrightarrow \downarrow p \subseteq R_\Diamond^{-1}(\downarrow q).$$

Now, if $pR_\Diamond q$ then $p \in R_\Diamond^{-1}(\downarrow q)$ and since $R_\Diamond^{-1}(\downarrow q)$ is a downset, $\downarrow p \subseteq R_\Diamond^{-1}(\downarrow q)$.

For the converse, if $\downarrow p \subseteq R_\Diamond^{-1}(\downarrow q)$ then $p \in R_\Diamond^{-1}(\downarrow q)$ so there is $r \in X$ such that $r \leq q$ and $pR_\Diamond r$. Hence $pR_\Diamond \circ \leq q$ and, by Definition 3.4.14, $pR_\Diamond q$. □

Having established a duality between objects in $UQ\Diamond$ and in $PR\Diamond_\tau$ let us look at morphisms.

Definition 3.4.18. Let $L, K \in UQ\Diamond$. A morphisms $h \in UQ\Diamond(L, K)$ is a $\{0, 1\}$-homomorphism from the algebra L to K.

We saw in Remark 3.1.16 how to extend a map $h : L \to K$ to L^σ by defining

$$h^\sigma : L^\sigma \longrightarrow K^\sigma \text{ and } h^\pi : L^\sigma \longrightarrow K^\sigma$$

and we mentioned (Lemma 3.1.22) how $h^\pi = h^\sigma$ and $h^\sigma \in \mathcal{DL}^+(L^\sigma, K^\sigma)$.

To prove that h^σ is a complete homomorphism we have to find out whether, for every $u \in L^\sigma$,

$$h^\sigma(\Diamond^\sigma(u)) = \Diamond^\sigma(h^\sigma(u)).$$

By Theorem 3.1.19 and the fact that h is smooth and it preserves $\Diamond$ we have,

$$h^\sigma(\Diamond^\sigma(u)) = (h \circ \Diamond)^\sigma(u) = (\Diamond \circ h)^\sigma(u) = \Diamond^\sigma(h^\sigma(u))$$

so h^σ is a complete homomorphism from L^σ to K^σ.

We will denote by $UQ\Diamond^+(L^\sigma, K^\sigma)$, complete homomorphisms from L^σ to K^σ with $L^\sigma, K^\sigma \in UQ\Diamond^+$. Since $UQ\Diamond^+$ is a reduct of UQA^+ and $h^\sigma \in UQ\Diamond^+(L^\sigma, K^\sigma)$, to this map corresponds, by Theorem 3.2.12, a morphism $\varphi_{h^\sigma} \in PR\Diamond(J^\infty(K^\sigma), J^\infty(L^\sigma))$ satisfying the conditions of Definition 3.2.11 (i).

On the other hand we know from Lemmas 3.4.5 and 3.4.6 that $\varphi_{h^\sigma} = \varphi_h$ is a continuous order preserving map. So, we define morphisms in $PR\Diamond_\tau$ by:

Definition 3.4.19. Let $X, Y \in PR\Diamond_\tau$. A continuous order preserving map $\varphi : X \longrightarrow Y$ is a morphism in $PR\Diamond_\tau(X, Y)$ if it satisfies the following conditions:

(i)

$$\forall q, r \in X : \ (qR_\diamond r \Rightarrow \varphi(q)R_\diamond\varphi(r))\,.$$

(ii)

$$\forall q \in X\, \forall p \in Y : \ (\varphi(q)R_\diamond p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \text{ and } qR_\diamond r)).$$

Thus, we conclude that to each $h \in UQ\diamond(L, K)$ corresponds a map $\varphi_h \in PR\diamond_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$.

Conversely, if $X, Y \in PR\diamond_\tau$, for every map $\varphi \in PR\diamond_\tau(Y, X)$ we know, since $\varphi \in PR\diamond(Y, X)$ that there exists a complete homomorphism $H_\varphi \in UQ\diamond^+(\mathcal{D}(X), \mathcal{D}(Y))$ such that $H_\varphi(U) = \varphi^{-1}(U)$ for every $U \in \mathcal{D}(X)$ (Theorem 3.2.12).

But φ is continuous thus φ^{-1} maps $Clop\mathcal{D}(X)$ in $Clop\mathcal{D}(Y)$ therefore $H_\varphi \restriction Clop\mathcal{D}(X) \in UQ\diamond(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$.

So, applying Theorem 3.4.8, we can define the functors between morphisms of these two categories:

Theorem 3.4.20. *Let $L, K \in UQ\diamond$ and let $X, Y \in PR\diamond_\tau$. Given a morphism $h \in UQ\diamond(L, K)$, there is an associated morphism $D(h) = \varphi_h \in PR\diamond_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ defined by*

$$\varphi_h(p) = \bigwedge\{a \in L : p \leq h(a)\}$$

for all $p \in J^\infty(K^\sigma)$.

Given a morphism $\varphi \in PR\diamond_\tau(Y, X)$, there is an associated morphism $E(\varphi) = h_\varphi \in UQ\diamond(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ defined by

$$h_\varphi = \varphi^{-1}(U)$$

for each $U \in Clop\mathcal{D}(X)$.

Again as a consequence of Theorems 3.4.8, we can state that there is a dual equivalence between $UQ\diamond$ and $PR\diamond_\tau$.

This duality can be extended to the class of UQAs with unary operations $\diamond, \Box, \lhd$ and $\rhd$ (Definition 3.1.24).

This is after all a different way of presenting for unary operations the dualities established by R. Goldblatt [23] for distributive lattices with meet or join preserving operators and by V. Sofronie-Stokermans [41] for meet or join reversing operators.

We will consider

$$(L, \wedge, \vee, 0, 1, \Diamond, \Box, \rhd, \lhd) \in UQA.$$

The canonical or perfect extension of L is, by Lemma 3.1.27, the algebra

$$(L^\sigma, \wedge, \vee, 0, 1, \Diamond^\sigma, \Box^\sigma, \rhd^\sigma, \lhd^\sigma) \in UQA^+.$$

As we have already done in Section 3.2 when we established a duality for UQA^+ we can use alternatively the algebra L and its order dual L^∂ in order to consider the unary operations $\Box, \rhd$ and $\lhd$ as join preserving maps. Following a similar process to that we used for the operation $\Diamond$ and the relation $R_\Diamond$, we can consider the binary relations $R_\Box, R_\rhd$ and $R_\lhd$ in $J^\infty(L^\sigma)$ and define the dual structures of UQA:

Definition 3.4.21. Let $(X, \leq, \tau, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ be a relational structure such that $(X, \leq, \tau)$ is in $\mathcal{P}_\tau$ and the binary relations $R_\Diamond, R_\Box, R_\rhd$ and $R_\lhd$ satisfy:

(i) $\leq \circ R_\Diamond \circ \leq = R_\Diamond$.

(ii) For each $U \in Clop\mathcal{D}(X)$, $R_\Diamond^{-1}(U)$ is clopen.

(iii) For each $p \in X$, $pR_\Diamond = \{q \in X : pR_\Diamond q\}$ is closed.

(iv) $\geq \circ R_\Box \circ \geq = R_\Box$.

(v) For each $U \in Clop\mathcal{D}(X)$, $R_\Box^{-1}(X \setminus U)$ is clopen.

(vi) For each $p \in X$, $pR_\Box = \{q \in X : pR_\Box q\}$ is closed.

(vii) $\geq \circ R_\rhd \circ \leq = R_\rhd$.

(viii) For each $U \in Clop\mathcal{D}(X)$, $R_\rhd^{-1}(U)$ is clopen.

(ix) For each $p \in X$, $pR_\rhd = \{q \in X : pR_\rhd q\}$ is closed.

(x) $\leq \circ R_\lhd \circ \geq = R_\lhd$.

(xi) For each $U \in Clop\mathcal{D}(X)$, $R_{\triangleleft}^{-1}(X \setminus U)$ is clopen.

(xii) For each $p \in X$, $pR_{\triangleleft} = \{q \in X : pR_{\triangleleft}q\}$ is closed.

Then we will call $(X, \leq, \tau, R_{\diamond}, R_{\Box}, R_{\triangleright}, R_{\triangleleft})$ a PR_{τ}.

To this structures correspond algebras $(Clop\mathcal{D}(X), \diamond_R, \Box_R, \triangleright_R, \triangleleft_R)$ such that, for each $U \in Clop\mathcal{D}(X)$, we have as in Theorem 3.2.8:

$$\begin{aligned} \diamond_R(U) &= R_{\diamond}^{-1}(U) \\ \Box_R(U) &= X \setminus R_{\Box}^{-1}(X \setminus U) \\ \triangleright_R (U) &= X \setminus R_{\triangleright}^{-1}(U) \\ \triangleleft_R (U) &= R_{\triangleleft}^{-1}(X \setminus U). \end{aligned}$$

We can prove, as in Lemma 3.4.16 that

$$(Clop\mathcal{D}(X), \cap, \cup, \emptyset, X, \diamond_R, \Box_R, \triangleright_R, \triangleleft_R) \in UQA.$$

So we have, generalizing Theorem 3.4.17,

Theorem 3.4.22. *There exist maps*

$$D : UQA \longrightarrow PR_{\tau} \qquad \text{and} \qquad E : PR_{\tau} \longrightarrow UQA.$$

such that for each $L \in UQA$ and each $X \in PR_{\tau}$,

$$D(L) = (J^{\infty}(L^{\sigma}), \leq, \tau, R_{\diamond}, R_{\Box}, R_{\triangleright}, R_{\triangleleft})$$

and

$$E(X) = (Clop\mathcal{D}(X), \cap, \cup, \emptyset, X, \diamond_R, \Box_R, \triangleright_R, \triangleleft_R).$$

These maps satisfy the following conditions:

(i) For each $L \in UQA$ the function $\eta_L : L \to E(D(L))$ is an isomorphism.

(ii) For each $X \in PR_{\tau}$ the function $\varepsilon_X : X \to D(E(X))$ is an order homeomorphism and an isomorphism for the relations $R_{\diamond}, R_{\Box}, R_{\triangleright}$ and $R_{\triangleleft}$.

In what concerns morphisms we have in UQA, $\{0, 1\}$-homomorphisms. In PR_{τ}, morphisms are defined in a similar way to that used for $PR\diamond_{\tau}$ (see also the corresponding definitions in [23] 2.3 and [41] Definition 4):

Definition 3.4.23. Let $X, Y \in PR_\tau$. A continuous order preserving map $\varphi : X \longrightarrow Y$ is a morphism, $\varphi \in PR_\tau(X, Y)$, if it satisfies the following conditions:

(i)
$$\forall q, r \in X : \ (qR_\diamond r \Rightarrow \varphi(q) R_\diamond \varphi(r)) \, .$$
$$\forall q \in X \, \forall p \in Y : \ (\varphi(q) R_\diamond p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \text{ and } qR_\diamond r)).$$

(ii)
$$\forall q, r \in X : \ (qR_\Box r \Rightarrow \varphi(q) R_\Box \varphi(r)) \, .$$
$$\forall q \in X \, \forall p \in Y : \ (\varphi(q) R_\Box p \Rightarrow (\exists r \in X : \ (\varphi(r) \geq p \text{ and } qR_\Box r)).$$

(iii)
$$\forall q, r \in X : \ (qR_\triangleright r \Rightarrow \varphi(q) R_\triangleright \varphi(r)) \, .$$
$$\forall q \in X \, \forall p \in Y : \ (\varphi(q) R_\triangleright p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \text{ and } qR_\triangleright r)).$$

(iv)
$$\forall q, r \in X : \ (qR_\triangleleft r \Rightarrow \varphi(q) R_\triangleleft \varphi(r)) \, .$$
$$\forall q \in X \, \forall p \in Y : \ (\varphi(q) R_\triangleleft p \Rightarrow (\exists r \in X : \ (\varphi(r) \geq p \text{ and } qR_\triangleleft r)).$$

From the study done for morphisms in $UQ\Diamond$ and in $PR\Diamond_\tau$ (Theorem 3.4.20) together with Theorem 3.2.12 it follows that for the categories UQA and PR_τ we obtain:

Theorem 3.4.24. *Let $L, K \in UQA$ and let $X, Y \in PR_\tau$. Given a morphism $h \in UQA(L, K)$, there is an associated morphism $\varphi_h \in PR_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ defined by*
$$\varphi_h(p) = \bigwedge \{a \in L : p \leq h(a)\}$$
for all $p \in J^\infty(K^\sigma)$.

Given a morphism $\varphi \in PR_\tau(Y, X)$, there is an associated morphism $h_\varphi \in UQA(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ defined by
$$h_\varphi = \varphi^{-1}(U)$$
for each $U \in Clop\mathcal{D}(X)$.

The maps $D : UQA(L, K) \to PR_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ such that $D(h) = \varphi_h$ and $E : PR_\tau(Y, X) \to UQA(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ such that $E(\varphi) = h_\varphi$ are bijections.

The diagrams

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ {\scriptstyle v_L}\downarrow & & \downarrow{\scriptstyle v_K} \\ Clop\mathcal{D}(J^{\infty}(L^{\sigma})) & \xrightarrow[h_{\varphi_h}]{} & Clop\mathcal{D}(J^{\infty}(K^{\sigma})) \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \varepsilon_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ J^{\infty}(\mathcal{D}(Y)) & \xrightarrow[\varphi_{h_{\varphi}}]{} & J^{\infty}(\mathcal{D}(X)) \end{array}$$

commute.
Further:

(i) h is one to one if and only if φ_h is onto.

(ii) h is onto if and only if φ_h is an order embedding.

Thus we have a dual equivalence between UQA and PR_{τ}.

Chapter 4

A duality for $SDMA$s

4.1 Canonicity

We can apply the Sahlqvist Theory and canonicity of UQAs to the study of semi-De Morgan algebras and some of its subvarieties.

In fact, it is possible to identify $SDMA$s with algebras $(L, \wedge, \vee, \rhd, 0, 1)$ where the unary operation is $\rhd ='$ subjected to the additional restrictions

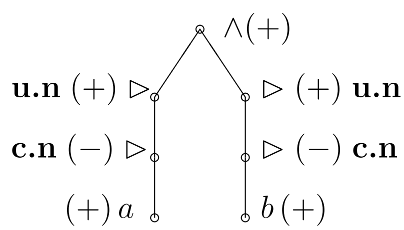
1. $\rhd 1 \leq 0$
2. $\rhd\rhd a \wedge \rhd\rhd b \leq \rhd\rhd (a \wedge b)$.
3. $\rhd\rhd\rhd a \leq \rhd a$ and $\rhd a \leq \rhd\rhd\rhd a$.

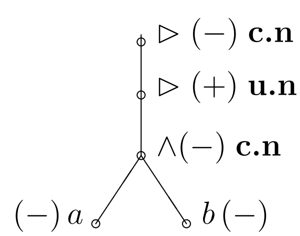
It is clear that, by the definition of $\rhd$ (Definition 3.1.24 (iii)), this set of inequalities characterizes $SDMA$ because the dual of condition 2 is a consequence of $\rhd\rhd$ being order preserving.

However not all these inequalities are Sahlqvist (Definition 3.1.33).

We are going to show that $\rhd\rhd a \wedge \rhd\rhd b \leq \rhd\rhd (a \wedge b)$ is neither 1-Sahlqvist nor ∂-Sahlqvist.

The generation trees for these terms are respectively:





The term $\rhd\rhd\, a\wedge \rhd\rhd\, b$ is not 1-left Sahlqvist and it is ∂-left Sahlqvist.

For 1-left Sahlqvist we have to worry about the positive occurrences of the variables a and b. On the paths from these occurrences to the root we have the prohibited configurations: we meet a choice node and then afterwards a universal node as we travel from the variable up to the root. This is why this term is not 1-left Sahlqvist.

For ∂-left Sahlqvist, we have only to worry about the negative occurrences of a and b. There aren't any and thus the term is ∂-left Sahlqvist.

The term $\rhd\rhd\, (a\wedge b)$ is 1-right Sahlqvist because in the negative generation tree there are no occurrences of a or b with (+) but it is not ∂-right Sahlqvist because there is a prohibited configuration in the paths of occurrences of a and b with (-).

Therefore the inequality is not Sahlqvist.

In [21] Theorem 3.7, M. Gehrke, H. Nagashi and Y. Venema prove that the first order formulas corresponding to Sahlqvist sequents can be effectively computed. The theorem that is referred to yields the following:

Theorem 4.1.1. *Every Sahlqvist inequality in a UQA^+ corresponds to a formula in the first order language of the dual relational structure. This first order formula can be effectively computed from the inequality.*

From the previous theorem it follows that Sahlqvist inequalities are particularly useful. This is not the case of 2 so, it is advantageous for nicer correspondence results (and thus duality) to consider a term-equivalent description of $SDMA$s which casts them as UQAs with basic operations $\rhd ='$ and $\Box =''$. Accordingly we define:

Definition 4.1.2. Let $SDMA$ be the class of algebras $(L, \wedge, \vee, \rhd, \Box, 0, 1)$ satisfying

(M1) $(L, \wedge, \vee, 0, 1) \in \mathcal{DL}$

(M2) $\rhd\, 1 \leq 0$.

(M3) $\Box a \leq \rhd\rhd\, a$.

(M4) $\rhd\rhd\, a \leq \Box a$.

(M5) $\rhd\, a \leq \Box \rhd\, a$.

(M6) $\Box \rhd a \leq \rhd a$.

All the inequalities M2,...,M6 are Sahlqvist (see Appendix). This is rather interesting because comparing this set of conditions with 1, 2 and 3 we conclude that Sahlqvistness is not preserved under term equivalence.

For any $(L, \wedge, \vee, \rhd, \Box, 0, 1) \in SDMA$ there is, as in UQA, a natural embedding from this algebra in $(L^\sigma, \wedge, \vee, \rhd^\sigma, \Box^\sigma, 0, 1)$.

From Theorem 3.1.34 it follows that the inequalities M2,...,M6 are canonical so, if they are true in L, they are also true in L^σ. This means that the variety $SDMA$ is canonical and consequently $(L, \wedge, \vee, \rhd, \Box, 0, 1) \in SDMA$ implies $(L^\sigma, \wedge, \vee, \rhd^\sigma, \Box^\sigma, 0, 1)$ is what we might call a $SDMA^+$, i.e. the intersection of $SDMA$ with UQA^+.

So we can define the following class of algebras.

Definition 4.1.3. Let $SDMA^+$ be the class of algebras

$$(L, \wedge, \vee, 0, 1, \rhd, \Box) \in SDMA$$

such that $L \in \mathcal{DL}^+$ and, for any subset $\{a_i : i \in I\}$

1. $\rhd 0 = 1$ and $\rhd \left(\bigvee_{i \in I} a_i\right) = \bigwedge_{i \in I} \rhd a_i$.
2. $\Box 1 = 1$ and $\Box \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} \Box a_i$.

Since $SDMA$ is canonical, we have:

Theorem 4.1.4. *If $L \in SDMA$ then $L^\sigma \in SDMA^+$.*

In a similar way we can state that any subvariety of $SDMA$ defined by Sahlqvist inequalities is canonical. It is easy to characterize some important subvarieties of $SDMA$ using conditions on $\Box$ and $\rhd$. In fact from Section 1.4 (Remark 1.4.6, Remark 1.4.2, Definition 1.4.4 and Lemma 1.4.5), respectively, it follows:

Lemma 4.1.5. *Let $(L, \wedge, \vee, \rhd, \Box, 0, 1) \in SDMA$. Then:*

(i) L is in $K_{1,1}$ if and only if $\rhd (a \wedge b) \leq \rhd a \vee \rhd b$ for any $a, b \in L$.

(ii) L is a De Morgan algebra if and only if $\Box a \leq a$ and $a \leq \Box a$ for each $a \in L$.

(iii) *L is a demi p-lattice if and only if* $\rhd a \wedge \Box a \leq 0$ *for each* $a \in L$.

(iv) *L is an almost p-lattice if and only if* $\rhd a \wedge a \leq 0$ *for each* $a \in L$.

(v) *L is a p-lattice if and only if L is a demi p-lattice and* $a \leq \Box a$ *for each* $a \in L$.

(vi) *L is in the variety* $\mathcal{C}$ *if and only if for any* $a, b \in L$ *inequalities* γ *and* β *hold:*

$$(\gamma) \quad \rhd (a \wedge b) \wedge \Box b \leq \rhd a \vee \rhd b$$

$$(\beta) \quad \rhd (a \wedge b) \leq \rhd a \vee \rhd (\rhd a \wedge b \wedge \rhd b)$$

Remark 4.1.6. It is now a simple exercise to verify that all these inequalities are Sahlqvist (see Appendix) and therefore they are all canonical.

4.1.1 Discrete Correspondence

Based on the duality established in Section 3.2 between UQA^+ and PR we are going to determine the structures corresponding to $SDMA^+$s.

For an algebra $L \in SDMA^+$ (Definition 4.1.3), the dual structure is a relational structure $(J^\infty(L), \leq, R_\rhd, R_\Box)$ where the relations $R_\rhd$ and $R_\Box$ have to satisfy conditions corresponding to (M2),...(M6).

We have already proved that all these inequations are Sahlqvist so as a consequence of Theorem 4.1.1 we know that we can effectively compute the corresponding conditions:

For M2 we have:

$$\begin{aligned} \rhd 1 \leq 0 &\Leftrightarrow \forall p \in J^\infty(L) : p \not\leq \rhd 1 \\ &\Leftrightarrow \forall p \in J^\infty(L)\, \exists q \in J^\infty(L) :\ (q \leq 1 \text{ and } pR_\rhd q) \quad \text{by Lemma 3.2.9(i)} \\ &\Leftrightarrow \forall p \in J^\infty(L)\, \exists q \in J^\infty(L) :\ pR_\rhd q \end{aligned}$$

or equivalently,

$$R_\rhd^{-1}(J^\infty(L)) = J^\infty(L)$$

For M3 we have:

$$\forall a \in L :\ \Box a \leq \rhd\rhd a \quad \text{if and only if}$$

$$\forall a \in L\, \forall p \in J^\infty(L) :\ (p \leq \Box a \Rightarrow p \leq \rhd\rhd a)$$

because L is join generated by $J^\infty(L)$.

Our objective now is to eliminate the universal quantifier in L, but first observe that, by Lemma 3.2.9(iv), the previous condition is equivalent to:

$$\forall a \in L \forall p \in J^\infty(L) : \left(\bigvee(pR_\Box) \leq a \Rightarrow p \leq \rhd\rhd a\right).$$

Notice that, given p, there is a least a for which the antecedent holds, namely $a = \bigvee(pR_\Box)$.

Also notice that if the conclusion holds for some $a \in L$ then it holds for all greater a. Thus we get:

$$\forall p \in J^\infty(L) \left(p \leq \rhd\rhd \left(\bigvee(pR_\Box)\right)\right). \tag{4.1.1}$$

Now, by Lemma 3.2.9(ii):

$$p \leq \rhd\rhd \left(\bigvee(pR_\Box)\right) \Leftrightarrow \forall q \in J^\infty(L) \left(pR_\rhd q \Rightarrow q \not\leq \rhd \left(\bigvee(pR_\Box)\right)\right)$$

and, by Lemma 3.2.9(i):

$$q \not\leq \rhd \left(\bigvee(pR_\Box)\right) \Leftrightarrow \exists r \in J^\infty(L)(r \leq \bigvee(pR_\Box) \text{ and } qR_\rhd r).$$

Since $r \in J^\infty(L)$ and $pR_\Box$ is a downset, we have $r \leq \bigvee(pR_\Box)$ if and only if $pR_\Box r$.

Consequently, condition 4.1.1 is equivalent to

$$\forall p, q \in J^\infty(L)\ (pR_\rhd q \Rightarrow \exists r \in J^\infty(L)(pR_\Box r \text{ and } qR_\rhd r))$$

so that M3 holds in L if and only if

$$R_\rhd \subseteq R_\Box \circ R_\rhd^\partial$$

For M4, observe that, since L is meet generated by $M^\infty(L) = \kappa(J^\infty(L))$ (Theorem 3.1.7), $\rhd\rhd a \leq \Box a$ holds in L if and only if

$$\forall a \in L\, \forall p \in J^\infty(L)(\kappa(p) \geq \Box a \Rightarrow \kappa(p) \geq \rhd\rhd a).$$

So we get, from Lemma 3.2.9 (iii):

$$\forall a \in L\, \forall p \in J^\infty(L)\ ((\exists q \in J^\infty(L)(pR_\Box q \text{ and } q \not\leq a)) \Rightarrow \kappa(p) \geq \rhd\rhd a)$$

or equivalently

$$\forall a \in L\, \forall p, q \in J^{\infty}(L)((pR_{\Box}q \text{ and } \kappa(q) \geq a) \Rightarrow \kappa(p) \geq \rhd\rhd\, a).$$

Given $p, q \in J^{\infty}(L)$, notice that the greatest $a \in L$ for which $\kappa(q) \geq a$ holds is $a = \kappa(q)$ and if the conclusion holds for some a it also holds for all smaller elements because $\rhd\rhd$ is order preserving. Thus we get

$$\forall p, q \in J^{\infty}(L)((pR_{\Box}q \Rightarrow \kappa(p) \geq \rhd\rhd\, \kappa(q)).$$

Now, by Lemma 3.2.9 (i) and (ii) and by Remark 3.1.8, respectively, we obtain:

$$\begin{aligned}\kappa(p) \geq \rhd\rhd\, \kappa(q) &\Leftrightarrow \exists t \in J^{\infty}(L)\, (pR_{\rhd}t \text{ and } t \leq \rhd\, \kappa(q))) \\ &\Leftrightarrow \exists t \in J^{\infty}(L)\, (pR_{\rhd}t \text{ and } (\forall s \in J^{\infty}(L)\, (tR_{\rhd}s \Rightarrow s \not\leq \kappa(q))) \\ &\Leftrightarrow \exists t \in J^{\infty}(L)\, (pR_{\rhd}t \text{ and } (\forall s \in J^{\infty}(L)\, (tR_{\rhd}s \Rightarrow s \geq q))\,.\end{aligned}$$

So M4 holds in L if and only if

$$\forall p, q \in J^{\infty}(L)\, (pR_{\Box}q \Rightarrow (\exists t \in J^{\infty}(L)\, (pR_{\rhd}t \text{ and } \forall s \in J^{\infty}(L)(tR_{\rhd}s \Rightarrow s \geq q)))\,.$$

For M5, $\rhd\, a \leq \Box \rhd a$ holds in L if and only if

$$\forall a \in L\, \forall p \in J^{\infty}(L)(\kappa(p) \geq \Box \rhd a \Rightarrow \kappa(p) \geq \rhd\, a)$$

because L is meet generated by $M^{\infty}(L)$.

But, by Lemma 3.2.9 (iii) and (i):

$$\begin{aligned}&\kappa(p) \geq \Box \rhd a \\ &\Leftrightarrow \exists q \in J^{\infty}(L)\, (pR_{\Box}q \text{ and } q \not\leq \rhd\, a) \\ &\Leftrightarrow \exists q \in J^{\infty}(L)\, (pR_{\Box}q \text{ and } (\exists r \in J^{\infty}(L)\, (qR_{\rhd}r \text{ and } r \leq a)) \\ &\Leftrightarrow \exists q, r \in J^{\infty}(L)\, (pR_{\Box}q \text{ and } qR_{\rhd}r \text{ and } r \leq a) \\ &\Leftrightarrow \exists r \in J^{\infty}(L)\, (pR_{\Box} \circ R_{\rhd}r \text{ and } r \leq a)\end{aligned}$$

So we get,

$$\forall a \in L \forall p \in J^{\infty}(L)\, ((\exists r \in J^{\infty}(L)\, (pR_{\Box} \circ R_{\rhd}r \text{ and } r \leq a)) \Rightarrow \kappa(p) \geq \rhd\, a)$$

which is equivalent to

$$\forall a \in L \forall p, r \in J^{\infty}(L)\, ((pR_{\Box} \circ R_{\rhd}r \text{ and } r \leq a) \Rightarrow \kappa(p) \geq \rhd\, a)$$

Given $p, r \in J^\infty(L)$, the least a satisfying the antecedent is $a = r$, and if the conclusion holds for some a it holds for all greater values because $\rhd$ is order reversing. Thus we get

$$\forall p, r \in J^\infty(L)\ ((pR_\Box \circ R_\rhd r) \Rightarrow \kappa(p) \geq \rhd r)$$

That is, by the definition of $R_\rhd$(equation (3.2.3)),

$$\forall p, r \in J^\infty(L)\ (p(R_\Box \circ R_\rhd)r \Rightarrow pR_\rhd r)$$

which is equivalent to

$$R_\Box \circ R_\rhd \subseteq R_\rhd$$

For M6, $\Box \rhd a \leq \rhd a$ holds in L is equivalent to

$$\forall a \in L\, \forall p \in J^\infty(L)\ (\kappa(p) \geq \rhd a \Rightarrow \kappa(p) \geq \Box \rhd a).$$

By Lemma 3.2.9 (i), this is equivalent to

$$\forall a \in L \forall p \in J^\infty(L)((\exists q \in J^\infty(L)(pR_\rhd q \text{ and } q \leq a)) \Rightarrow \kappa(p) \geq \Box \rhd a)$$

and to

$$\forall a \in L \forall p, q \in J^\infty(L)((pR_\rhd q \text{ and } q \leq a) \Rightarrow \kappa(p) \geq \Box \rhd a)$$

Given $p, q \in J^\infty(L)$, the least a satisfying the antecedent is $a = q$ and if the conclusion holds for some a it also holds for all greater elements, so we get

$$\forall p, q \in J^\infty(L)\ (pR_\rhd q \Rightarrow \kappa(p) \geq \Box \rhd q)$$

Now

$$\begin{aligned} \kappa(p) \geq \Box \rhd q &\Leftrightarrow \exists r \in J^\infty(L)\ (pR_\Box r \text{ and } r \not\leq \rhd q) && \text{by Lemma 3.2.9(iii)} \\ &\Leftrightarrow \exists r \in J^\infty(L)\ (pR_\Box r \text{ and } rR_\rhd q) && \text{by the definition of } R_\rhd \\ &\Leftrightarrow pR_\Box \circ R_\rhd q \end{aligned}$$

That is, M6 holds in L if and only if

$$\forall p, q \in J^\infty(L)(pR_\rhd q \Rightarrow pR_\Box \circ R_\rhd q)$$

or equivalently

$$R_{\triangleright} \subseteq R_{\Box} \circ R_{\triangleright}.$$

Notice that in all the previous proofs the correspondence between Sahlqvist equations and conditions in $R_{\triangleright}$ and $R_{\Box}$ was always obtained using the strategy of eliminating the universal quantifier on elements of L.

It is now possible to define the dual structures of $SDMA^+$s. We will denote by PRM these structures:

Definition 4.1.7. Let PRM be the class of relational structures $(X, \leq, R_{\Box}, R_{\triangleright})$ where $(X, \leq)$ is a partially ordered set and $R_{\Box}, R_{\triangleright}$ are binary relations on X such that:

(F1) $\geq \circ R_{\Box} \circ \geq = R_{\Box}$

(F2) $\geq \circ R_{\triangleright} \circ \leq = R_{\triangleright}$

(F3) $R_{\triangleright}^{-1}(X) = X$

(F4) $R_{\triangleright} \subseteq R_{\Box} \circ R_{\triangleright}^{\partial}$

(F5) $\forall p, q \in X \ (pR_{\Box}q \Rightarrow \exists t \in X \ (pR_{\triangleright}t \text{ and } \forall s \in X (tR_{\triangleright}s \Rightarrow s \geq q)) \, .$

(F6) $R_{\triangleright} = R_{\Box} \circ R_{\triangleright}$

Since we have proved the equivalence of conditions M2,...,M6 and F3,...,F6 it is possible to define a map $()_+ : SDMA^+ \rightarrow PRM$. On the other hand, when we have a relational structure, $(X, \leq, R_{\Box}, R_{\triangleright}) \in PRM$ it is easy to prove that $(\mathcal{D}(X), \cap, \cup, \emptyset, X, \Box_R, \triangleright_R)$ is in $SDMA^+$.

So we conclude, applying Theorem 3.2.8,

Theorem 4.1.8. *There exist maps*

$$()_+ : SDMA^+ \longrightarrow PRM \qquad \textit{and} \qquad ()^+ : PRM \longrightarrow SDMA^+$$

such that for each $L \in SDMA^+$ *and each* $X \in PRM$,

$$(L)_+ = (J^{\infty}(L), \leq, R_{\Box}, R_{\triangleright}) \textit{ and } (X)^+ = (\mathcal{D}(X), \cup, \cap, \emptyset, X, \Box_R, \triangleright_R).$$

The function $\eta_L : L \longrightarrow ((L)_+)^+$ *is a complete isomorphism and the function* $\varepsilon_X : X \longrightarrow ((X)^+)_+$ *is an isomorphism for the relations* $\leq$, $R_{\Box}$ *and* $R_{\triangleright}$.

Naturally, morphisms in $SDMA^+$ are complete {0,1}-homomorphisms and morphisms in PRM are order preserving maps satisfying conditions (ii) and (iii) from definition 3.2.11 so that we can establish as in Theorem 3.2.12 a correspondence between these morphisms.

This way we obtain a dual equivalence between $SDMA^+$ and PRM.

We are going to see that the conditions defining morphisms in PRM are not independent:

Theorem 4.1.9. *Let X, Y be relational structures in PRM and let $\varphi : X \to Y$ be an order preserving map. Then $\varphi \in PRM(X, Y)$ if and only if the following conditions hold:*

(a) $\forall q, r \in X : \ (qR_\triangleright r \Rightarrow \varphi(q)R_\triangleright\varphi(r))$.

(b) $\forall q \in X\, \forall p \in Y : \ (\varphi(q)R_\triangleright p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \textit{ and } qR_\triangleright r))$.

Proof. We have only to prove that (a) and (b) imply conditions (ii) from definition 3.2.11.

For the first implication of (ii) we have, for any $q, r \in X$,

$qR_\square r \Rightarrow$
$\Rightarrow \exists t \in X \ (qR_\triangleright t \text{ and } (\forall s \in X \ (tR_\triangleright s \Rightarrow s \geq r)))$ by F5
$\Rightarrow \exists t \in X \ (\varphi(q)R_\triangleright\varphi(t) \text{ and } (\forall s \in X \ (tR_\triangleright s \Rightarrow s \geq r)))$ by (a)
$\Rightarrow \exists t \in X \exists u \in Y \ (\varphi(q)R_\square u \text{ and } \varphi(t)R_\triangleright u \text{ and } (\forall s \in X \ (tR_\triangleright s \Rightarrow s \geq r)))$ by F4
$\Rightarrow \exists t \in X \exists u \in Y \ (\varphi(q)R_\square u \text{ and } \exists v \in X \ (\varphi(v) \leq u \text{ and } tR_\triangleright v)$ and
and $(\forall s \in X \ (tR_\triangleright s \Rightarrow s \geq r))$ by (b) .

From here it follows,

$$\exists t, v \in X \ (\varphi(q)R_\square \circ \geq \varphi(v) \text{ and } v \geq r)) \, .$$

By F1 and since φ is order preserving we have

$$\exists v \in X \ (\varphi(q)R_\square\varphi(v) \text{ and } \varphi(v) \geq \varphi(r))$$

and, again by F1,

$$\varphi(q)R_\square\varphi(r).$$

In what concerns the second implication of (ii):

For every $q \in X$ and $p \in Y$,

$\varphi(q) R_\Box p \Rightarrow$
$\Rightarrow \exists r \in Y\ (\varphi(q) R_\triangleright r \text{ and } (\forall s \in Y\ r R_\triangleright s \Rightarrow s \geq p))$ by F5
$\Rightarrow \exists r \in Y \exists t \in X\ (\varphi(t) \leq r \text{ and } q R_\triangleright t \text{ and } (\forall s \in Y\ (r R_\triangleright s \Rightarrow s \geq p)))$ by (b)
$\Rightarrow \exists r \in Y \exists t, u \in X\ (\varphi(t) \leq r \text{ and } q R_\Box u \text{ and } t R_\triangleright u \text{ and } (\forall s \in Y\ (r R_\triangleright s \Rightarrow s \geq p)))$ by F4
$\Rightarrow \exists r \in Y \exists t, u \in X\ (r \geq \varphi(t) \text{ and } \varphi(t) R_\triangleright \varphi(u) \text{ and } q R_\Box u \text{ and }$
$(\forall s \in Y\ (r R_\triangleright s \Rightarrow s \geq p)))$ because φ is order preserving
$\Rightarrow \exists r \in Y \exists u \in X\ (r \geq \circ R_\triangleright \varphi(u) \text{ and } q R_\Box u \text{ and } (\forall s \in Y\ (r R_\triangleright s \Rightarrow s \geq p)))$

As a consequence of F1 we have

$$\exists r \in Y \exists u \in X\ (r R_\triangleright \varphi(u) \text{ and } (\forall s \in Y :\ r R_\triangleright s \Rightarrow s \geq p) \text{ and } q R_\Box u)$$

and finally we obtain

$$\exists u \in X\ (\varphi(u) \geq p \text{ and } q R_\Box u).$$

□

This way we can define morphisms in PRM as order preserving maps satisfying conditions (a) and (b) of the previous theorem.

In order to simplify the duality we have obtained for $SDMA^+$, we are going to prove some other properties that are true in PRM.
First observe that, from F4, it follows that

$$R_\triangleright^\partial \subseteq R_\triangleright \circ R_\Box^\partial$$

so, again by F4,

$$R_\triangleright \subseteq R_\Box \circ R_\triangleright \circ R_\Box^\partial.$$

Finally, by F6 ,

(F7) $R_\triangleright \subseteq R_\triangleright \circ R_\Box^\partial$.

Now we are going to study those elements of $X \in PRM$ that are maximal in $pR_\Box$ or minimal in $pR_\triangleright$ for some $p \in X$. For these elements we have:

Lemma 4.1.10. *Let $X \in PRM$ and let $p, q \in X$ be such that q is a maximal element of $pR_\Box$. Then there is $t \in X$ such that $pR_\triangleright t$ and q is the minimum of $tR_\triangleright$.*

Proof. From F5, we know that $pR_\Box q$ implies:

$$\exists t \in X\ (pR_\triangleright t \text{ and } (\forall u \in X\ (tR_\triangleright u \Rightarrow u \geq q))).$$

But, by F4, it follows from $pR_\triangleright t$ that there is $v \in X$ such that $pR_\Box v$ and $tR_\triangleright v$. Therefore $v \geq q$.

Since q is maximal in $pR_\Box$ we must have $v = q$.

Then we conclude that q is the minimum of $tR_\triangleright$ because we have $tR_\triangleright q$ and $\forall u \in X,\ (tR_\triangleright u \Rightarrow u \geq q)$.

□

Lemma 4.1.11. *Let $X \in PRM$ and let $p, q \in X$ be such that q is a minimal element of $pR_\triangleright$. Then there is $t \in X$ such that $pR_\triangleright t$ and q is the maximum of $tR_\Box$*

Proof. Since we have $pR_\triangleright q$ we know, by F6, that

$$\exists r \in X(pR_\Box r \text{ and } rR_\triangleright q)$$

and, from F5 and $pR_\Box r$,

$$\exists t \in X\ (pR_\triangleright t \text{ and } \forall u \in X : (tR_\triangleright u \Rightarrow u \geq r))\,. \qquad (*)$$

Now, for any $s \in X$ such that $tR_\Box s$, we have, by F5, that t is such that

$$\exists y \in X(tR_\triangleright y \text{ and } \forall w \in X(yR_\triangleright w \Rightarrow w \geq s)). \qquad (**)$$

By condition $(*)$, $tR_\triangleright y$ implies $y \geq r$ and, since $rR_\triangleright q$, we have, by F2, $yR_\triangleright q$. Therefore, by $(**)$, $q \geq s$.

So we proved that t and q are such that

$$\forall s \in X(tR_\Box s \Rightarrow q \geq s). \qquad (***)$$

From $pR_\triangleright t$ and F7 it follows that there is $z \in X$ such that $pR_\triangleright z$ and $tR_\Box z$.

So we have $q \geq z$ by condition $(***)$ and, since q is minimal in $pR_\triangleright$, we have $z = q$ and hence $tR_\Box q$.

By $(***)$, we conclude that q is the maximum of $tR_\Box$.

□

Now we can define

Definition 4.1.12. Let $X \in PRM$. We will denote by W the subset

$$W = \{q \in X : \exists p \in X \, q \, \text{is minimal in} \, pR_{\triangleright}\}$$

It is clear from Lemmas 4.1.10 and 4.1.11 that the following holds:

$$W = \{q \in X : \exists p \in X \, q \, \text{is maximal in} \, pR_{\Box}\}. \tag{4.1.2}$$

It is possible to obtain another characterization of W:

Lemma 4.1.13. *Let $X \in PRM$ and let $q \in X$. Then $q \in W$ if and only if*

$$qR_{\Box}q \textit{ and } \forall r \in X(qR_{\triangleright}r \Rightarrow rR_{\triangleright}q).$$

Proof. ($\Leftarrow$) Let $q \in X$ be such that $qR_{\Box}q$ and $\forall r \in X(qR_{\triangleright}r \Rightarrow rR_{\triangleright}q)$.

Since we have $qR_{\Box}q$ then, by F5, there is $t \in X$ such that

$$qR_{\triangleright}t \text{ and } \forall s \in X \quad (tR_{\triangleright}s \Rightarrow s \geq q).$$

Then we have also $tR_{\triangleright}q$, so we conclude that q is the minimum of $tR_{\triangleright}$ and thus, $q \in W$.

($\Rightarrow$)

Let $q \in W$. Then, there is $p \in X$ such that $pR_{\Box}q$ and q is maximal in $pR_{\Box}$ by condition (4.1.2).

By Lemma 4.1.10 there is $s \in X$ such that q is the minimum of $sR_{\triangleright}$.

From F7 and $sR_{\triangleright}q$ it follows that there is $u \in X$ such that $sR_{\triangleright}u$ and $qR_{\Box}u$ so, since q is the minimum of $sR_{\triangleright}$, we have $u \geq q$ and, by F2, $qR_{\Box}q$.

Since q is minimal in $sR_{\triangleright}$ there is $t \in X$ such that q is the maximum of $tR_{\Box}$.

Now, for any $r \in X$, $qR_{\triangleright}r$ implies $tR_{\triangleright}r$ by F6 and, from F4 it follows that there is $v \in X$ such that $tR_{\Box}v$ and $rR_{\triangleright}v$ so $v \leq q$ because q is the maximum of $tR_{\Box}$ and, by F1, $rR_{\triangleright}q$.

$\Box$

In order to obtain a simpler duality for $SDMA$s we will use the binary relation $R_{\triangleright}^{min}$ that we considered in Section 3.3. For practical reasons we will denote this relation by S.

Definition 4.1.14. Let $X \in PRM$ and let $p, q \in X$. We call S the binary relation such that

$$pSq \quad \text{if and only if} \quad pR_{\triangleright}q \text{ and } q \text{ is minimal in } pR_{\triangleright}$$

From Definition 4.1.12 it follows that

$$W = Im(S).$$

It is obvious, for those familiar with Hobby's duality for $SDMA$s [25] that this relation S is the correspondent of the binary relation $\leftarrow$ defined by D. Hobby in the poset of prime ideals of a $SDMA$.

4.2 Correspondence for canonical extensions

The relation $S \subseteq X \times X$ becomes particularly interesting when there is a semi-De Morgan algebra L such that $X = J^{\infty}(L^{\sigma})$ where L^{σ} is the canonical extension of L. In fact, when this happens we can apply Theorem 3.3.2 (ii) and, from Definition 4.1.7 (F3) and Definition 4.1.12, we conclude:

Lemma 4.2.1. *Let $L^{\sigma} \in SDMA^{+}$ be the canonical extension of $L \in SDMA$ and let $X = J^{\infty}(L^{\sigma})$. Then*

$$S^{-1}(X) = S^{-1}(W) = X.$$

When we consider the restriction of S to the subset W we have:

Lemma 4.2.2. *Let $L \in SDMA$ and $X = J^{\infty}(L^{\sigma})$. Then, $S \cap (W \times W)$ is an order reversing involution.*

Proof. We have to prove that S is such that:

(i) For any $q \in W$ there is one and only one element $r \in W$ such that qSr and, for this element r we have also rSq.

(ii) For any $p, q, r, u \in W$ such that qSr and pSu, if $q \geq p$ then $u \geq r$.

To prove (i) note that for every $q \in W$ we have $qR_{\square}q$ so, according to F5,

$$\exists t \in X\ (qR_{\triangleright}t \text{ and } \forall s \in X(tR_{\triangleright}s \Rightarrow s \geq q)). \qquad (*)$$

From Theorem 3.3.2, we know that there is $r \in W$ such that r is a minimal element of $qR_\rhd$ and $r \leq t$. Therefore r is such that qSr.

By Lemma 4.1.13, we have also $rR_\rhd q$.

On the other hand, by F2, we know that for any $s \in X$, $rR_\rhd s$ implies $tR_\rhd s$ because $t \geq r$.

It follows from $(*)$ that

$$\forall s \in X \; (rR_\rhd s \Rightarrow s \geq q)$$

so q is the minimum of $rR_\rhd$ and hence rSq.

In an analogous way, from $r \in W$ and $rR_\Box r$ it follows that there is $x \in W$ such that rSx and xSr and r is the minimum of $xR_\rhd$. But we know that q is the minimum of $rR_\rhd$ so we have $q = x$ and, since both sets $qR_\rhd$ and $rR_\rhd$ have minimums, these elements are unique.

To prove (ii) notice that, by F2, $q \geq pSu$ implies $qR_\rhd u$. Since r is the minimum of $qR_\rhd$ we obtain $u \geq r$. □

From the previous Lemma it follows that, for any $p \in W$, the set $pR_\rhd$ has a minimum and that

$$\forall p, q \in W, \; pS^2q \Leftrightarrow p = q.$$

It is clear how Lemma 4.2.2 generalizes the dual description of De Morgan Algebras obtained by Cornish and Fowler in [14]. In fact, when $X = W$, the space X is the dual of a De Morgan Algebra and the relation S corresponds to the involution ξ of Cornish and Fowler's duality.

We have already studied the relation S^2 in $W \times W$. When we consider the relation $S^2 \subseteq X \times X$ we have:

Lemma 4.2.3. *Let $X = J^\infty(L^\sigma)$ with $L \in SDMA$. Then, for any $p, q \in X$, we have*

$$pS^2q \quad \textit{if and only if} \quad q \textit{ is maximal in } pR_\Box.$$

Proof. $(\Leftarrow)$

Suppose that q is maximal in $pR_\Box q$. Then $q \in W$ so, by Lemma 4.2.2, there is one and only one $r \in W$ such that qSr and, by F6, $pR_\rhd r$.

By Theorem 3.3.2 there is $t \in W$ such that $t \leq r$ and pSt and, by Lemma 4.2.2, there is $u \in W$ such that tSu and uSt.

Since S is an involution in W, $t \leq r$ implies $q \leq u$.

On the other hand, from pSt it follows, by F4, that there is $v \in X$ such that $pR_{\Box}v$ and $tR_{\triangleright}v$. But $t \in W$ so u is the minimum of $tR_{\triangleright}$ and hence $u \leq v$.

So we have $q \leq u \leq v$ and $pR_{\Box}v$. Since q is maximal in $pR_{\Box}$, $q = u = v$ and consequently $pStSq$.

($\Rightarrow$)

For the converse suppose that p, q are such that pS^2q. Then $q \in W$ and there is $u \in W$ such that pSu and uSq.

From F4 and pSu, there is $x \in X$ such that $pR_{\Box}x$ and $uR_{\triangleright}x$. But $u, q \in W$ so, $x \geq q$ because q is the minimum of $uR_{\triangleright}$. From F1 it follows that $pR_{\Box}q$.

By Theorem 3.3.2 there is $w \in W$ such that $w \geq q$ and w is maximal in $pR_{\Box}$.

As a consequence of the if part, pS^2w holds so that there is $z \in W$ such that pSz and zSw.

But then we get zSw, uSq and $w \geq q$ with all these elements in W. It follows from Lemma 4.2.2 that $z \leq u$. Since we have simultaneously pSu and pSz we conclude that $z = u$ and consequently $q = w$.

So, q is maximal in $pR_{\Box}$.

□

From the previous Lemmas we conclude the following theorem that characterizes the relation S:

Theorem 4.2.4. *Let $L \in SDMA$ and let $X = J^{\infty}(L^{\sigma}) \in PRM$. Then the binary relation $S \subseteq X \times X$ has the following properties:*

(H1) $\geq \circ S \subseteq S \circ \leq$.

(H2) $Dom(S) = \{p \in X : \exists q \in X\ pSq\} = X$.

(H3) For each $p \in X$, pS is an antichain.

(H4) On $W = \{q \in X : \exists p \in X\ pSq\}$, S is an involution.

Proof. It is clear that (H1) follows from F1 and Theorem 3.3.2 and that (H2) is a direct consequence of Lemma 4.2.1.

The definition of S implies (H3) and condition (H4) follows from Lemma 4.2.2. □

Now we can think of a relational structure $(X, \leq, S)$ where $X = J^\infty(L^\sigma)$ for some $L \in SDMA$ and S satisfies conditions H1,...,H4. We will denote these structures by PRS.

We can prove the following:

Theorem 4.2.5. *Let $(X, \leq, S) \in PRS$ and let*

$$R_\triangleright = S\circ \leq \quad \text{and} \quad R_\Box = S^2\circ \geq .$$

Then $(X, \leq, R_\Box, R_\triangleright) \in PRM$ and, for any $U \in \mathcal{D}(X)$,

$$R_\triangleright^{-1}(U) = S^{-1}(U) \quad \text{and} \quad R_\Box^{-1}(U) = S^{-1}(X \setminus S^{-1}(U)).$$

Proof. From the definition of $R_\triangleright$ it is obvious that F2 follows from H1 while F3 is a direct consequence of H2.

In what concerns F1 notice that, for any $p, q \in X$,

$$p \geq \circ R_\Box \circ \geq q \Leftrightarrow p \geq \circ S^2 \circ \geq q \Leftrightarrow \exists y, z \in W,\ p \geq \circ SySz \geq q.$$

By H1 this implies

$$\exists y, z \in W,\ pS\circ \leq ySz \geq q \Leftrightarrow \exists x, y, z \in W,\ pSx \leq ySz \geq q.$$

Since $x, y, z \in W$, it follows from H4 that there is $t \in W$ such that $t \geq z$ and xSt and hence we obtain:

$$pSxSt \geq q \Leftrightarrow pS^2\circ \geq q \Leftrightarrow pR_\Box q.$$

For F4 observe that from the definition of $R_\triangleright$ and H4,

$$pR_\triangleright q \Leftrightarrow \exists x \in W,\ pSx \leq q \Leftrightarrow \exists x, y \in W,\ pSxSySx \leq q.$$

But this implies pS^2y and $q \geq xSy$ or, by H1, $qS\circ \leq y$ and consequently $pR_\Box y$ and $qR_\triangleright y$.

To prove F5:

$$pR_\Box q \Leftrightarrow pS^2\circ \geq q \Leftrightarrow \exists x, y \in W,\ pSxSy \geq q.$$

This implies $pR_\triangleright x$ and since $x \in W$,

$$\forall s \in X,\ xR_\triangleright s \Rightarrow xSy \leq s \Rightarrow s \geq y \geq q.$$

For F6, observe that on one side we have, by the definition of $R_\triangleright$ and H4:

$$pR_\triangleright q \Leftrightarrow \exists x \in W,\ pSx \leq q \Leftrightarrow \exists x, y \in W,\ pSxSySx \leq q$$

from here it follows that pS^2y and $yS\circ \leq q$. Consequently $pR_\Box y$ and $yR_\triangleright q$.

For the converse we have, by H1 and H4 respectively, that for any $y \in X$:

$$pR_\Box y \text{ and } yR_\triangleright q \Leftrightarrow pS^2\circ \geq yS\circ \leq q \Rightarrow pS^3\circ \leq q \Rightarrow pS\circ \leq q \Leftrightarrow pR_\triangleright q.$$

So we proved that $(X, \leq, R_\Box, R_\triangleright) \in PRM$.

Now, to prove that $R_\triangleright^{-1}(U) = S^{-1}(U)$ for any $U \in \mathcal{D}(X)$, observe that the inclusion $R_\triangleright^{-1}(U) \supseteq S^{-1}(U)$ is evident.

For the other inclusion notice that, for any $p \in R_\triangleright^{-1}(U)$ there is $q \in U$ such that $pR_\triangleright q$ and, by the definition of $R_\triangleright$, $pS\circ \leq q$ so, there is $t \in X$ such that $t \leq q$ and pSt. But U is a downset so $t \in U$ and thus $p \in S^{-1}(U)$.

For the proof of the second equation suppose that $p \in X$ is such that $p \in S^{-1}(X \setminus S^{-1}(U))$.

Then there is $q \notin S^{-1}(U)$ such that pSq. By H4, there is one and only one element $r \in W$ such that qSr. It is clear now that $r \in X \setminus U$ and, since pS^2r we have $pR_\Box r$ so that $p \in R_\Box^{-1}(X \setminus U)$. Thus $R_\Box^{-1}(X \setminus U) \supseteq S^{-1}(X \setminus S^{-1}(U))$.

Now let $p \in R_\Box^{-1}(X \setminus U)$. Then there is $q \notin U$ such that $pR_\Box q$ or, by the definition of $R_\Box$, $pS^2\circ \geq q$.

Hence there are $y, t \in W$ such that $pSySt \geq q$. Since U is a downset and $q \notin U$ we have also $t \notin U$ and, since t is the only element such that ySt, $y \in X \setminus S^{-1}(U)$. Therefore $p \in S^{-1}(X \setminus S^{-1}(U))$.

Consequently $R_\Box^{-1}(X \setminus U) \subseteq S^{-1}(X \setminus S^{-1}(U))$. □

From the previous theorems we conclude that there is an alternative duality between the class of structures $(X, \leq, S) \in PRS$ and $SDMA^+$.

From Theorem 4.2.5, it is obvious that we can define the unary operations in $\mathcal{D}(X)$ by

$$\triangleright(U) = X \setminus S^{-1}(U) \quad \text{and} \quad \Box(U) = X \setminus (S^{-1}(X \setminus S^{-1}(U)) \tag{4.2.1}$$

for any $U \in \mathcal{D}(X)$.

Now we are going to study morphisms between structures $(X, \leq, S)$ and $(Y, \leq, S)$ in PRS:

Theorem 4.2.6. *Let $L, K \in SDMA$ and let $X = J^\infty(L^\sigma)$ and $Y = J^\infty(K^\sigma)$ and $\varphi : X \to Y$ be an order preserving map. Then $\varphi \in PRM(X, Y)$ if and only if*

(c) $\forall q \in X \forall p \in Y,\ (\varphi(q)S\circ \leq p \Leftrightarrow (\exists r \in X,\ \varphi(r) \leq p$ *and* $qSr))$.

Proof. First suppose that $\varphi \in PRM(X, Y)$ so that the conditions (a) and (b) in Theorem 4.1.9 hold.

($\Rightarrow$)

Let $q \in X$ and $p \in Y$ be such that $\varphi(q)S\circ \leq p$. Then $\varphi(q)R_\triangleright p$ and from (b),

$$\exists t \in X\ (\varphi(t) \leq p \text{ and } qR_\triangleright t).$$

So, by the definition of $R_\triangleright$,

$$\exists r, t \in X\ (\varphi(t) \leq p \text{ and } qSr \text{ and } r \leq t)$$

and, since φ is order preserving,

$$\exists r \in X\ (\varphi(r) \leq p \text{ and } qSr).$$

($\Leftarrow$)

Now suppose that $q \in X$ and $p \in Y$ and that there is $r \in X$ such that $\varphi(r) \leq p$ and qSr.

By the definition of $R_\triangleright$, $\varphi(r) \leq p$ and $qR_\triangleright r$ so, by condition (a),

$$\varphi(r) \leq p \text{ and } \varphi(q)R_\triangleright\varphi(r).$$

Thus $\varphi(q)R_\triangleright\circ \leq p$ and consequently $\varphi(q) \geq \circ S\circ \leq p$. By H1 we conclude that $\varphi(q)S\circ \leq p$.

For the converse observe that condition (b) is a direct consequence of the definition of S and condition (c).

For condition (a), notice that if $qR_\triangleright r$ then there is $x \leq r$ such that qSx and, since φ is an order preserving map $\varphi(x) \leq \varphi(r)$. So, by condition (c), $\varphi(q)S\circ \leq \varphi(r)$ and, consequently, $\varphi(q)R_\triangleright\varphi(r)$. □

Thus we have the following

Definition 4.2.7. Let $X, Y \in PRS$ and let $\varphi : X \to Y$ be an order preserving map. We say that $\varphi \in PRS(X, Y)$ if and only if

(c) $\forall q \in X \forall p \in Y,\ (\varphi(q)S\circ \leq p \Leftrightarrow (\exists r \in X,\ \varphi(r) \leq p$ and $qSr))$.

Observe that if, for any $p \in X$, we consider the subset $pS = \{q \in X : pSq\}$ then condition (c) is equivalent to the simpler equality

$$\uparrow (\varphi(q)S) = \uparrow \varphi(qS)$$

for any $q \in X$. This shows how $\varphi(q)S$ and $\varphi(qS)$ generate the same upset.

Since the ranges of morphisms play an important role in the study of congruences, notice that, as in Corollary 3.3.3, we have also:

Corollary 4.2.8. *Let $X = J^\infty(L^\sigma)$ and $Y = J^\infty(K^\sigma)$ where $J^\infty(L^\sigma)$ and $J^\infty(K^\sigma)$ are the canonical extensions of $L, K \in SDMA$. Let $\varphi : X \to Y$ be a morphism in PRS.*

Then $\varphi(X)$ is closed for the relation S.

4.2.1 Topological duality for $SDMA$s

As an application of the duality we have established for UQAs we are going to present a duality for semi-De Morgan Algebras.

We will identify the variety of $SDMA$s with the class of algebras $(L, \wedge, \vee, \rhd, \Box, 0, 1)$ from Definition 4.1.2 as we have been doing in the previous sections.

Since the variety $SDMA$ is canonical we will consider the natural embedding of an $SDMA$ $(L, \wedge, \vee, 0, 1, \rhd, \Box)$ in $(L^\sigma, \wedge, \vee, 0, 1, \rhd^\sigma, \Box^\sigma) \in SDMA^+$ (Theorem 4.1.4).

We have already established, in section 4.1.1, Theorems 4.1.8 and 4.1.9, a duality between perfect $SDMA^+$ and the relational structures PRM.

To find the dual spaces of $SDMA$s we have to consider, as in UQA, a topology in the relational space. Therefore, we will define the following relational structures:

Definition 4.2.9. Let PRM_τ be the class of relational structures $(X, \leq, \tau, R_\Box, R_\rhd)$ where $(X, \leq, \tau)$ is a Priestley space and $R_\Box, R_\rhd$ are binary relations on X such that:

(F1) (i) $\geq \circ R_\Box \circ \geq = R_\Box$.

(ii) For each $U \in Clop\mathcal{D}(X)$, $X \setminus R_\Box^{-1}(X \setminus U) \in Clop\mathcal{D}(X)$.

(iii) For each $p \in X$, $pR_\Box = \{q \in X : pR_\Box q\}$ is closed.

(F2) (i) $\geq \circ R_\rhd \circ \leq = R_\rhd$

(ii) For each $U \in Clop\mathcal{D}(X)$, $X \setminus R_\rhd^{-1}(U) \in Clop\mathcal{D}(X)$.

(iii) For each $p \in X$, $pR_\rhd = \{q \in X : pR_\rhd q\}$ is closed.

(F3) $R_\rhd^{-1}(X) = X$

(F4) $R_\rhd \subseteq R_\Box \circ R_\rhd^\partial$

(F5) $\forall p, q \in X \ (pR_\Box q \Rightarrow \exists t \in X \ (pR_\rhd t \text{ and } \forall s \in X (tR_\rhd s \Rightarrow s \geq q)))$.

(F6) $R_\rhd = R_\Box \circ R_\rhd$

We are going to prove:

Theorem 4.2.10. *There exist maps*

$$D : SDMA \longrightarrow PRM_\tau \qquad \text{and} \qquad E : PRM_\tau \longrightarrow SDMA.$$

such that the function $\eta_L : L \longrightarrow E(D(L))$ is an isomorphism and the function $\varepsilon_X : X \longrightarrow D(E(X))$ is an order homeomorphism and an isomorphism for the relations $R_\Box$ and $R_\rhd$.

Proof. To define D notice that if $L \in SDMA$ then, by Theorem 4.1.4, we have $L^\sigma \in SDMA^+$.

But for this class of algebras we have already a duality so, by Theorem 4.1.8, $(J^\infty(L^\sigma), \leq, R_\Box, R_\rhd) \in PRM$ and consequently conditions F1 (i), F2(i) ,F3, F4, F5 and F6 hold.

On the other hand we know by Theorem 3.4.22 that $(J^\infty(L^\sigma), \leq, \tau, R_\Box, R_\rhd)$ is a Priestley space where conditions F1 (i), (ii),(iii) and F2 (i), (ii), (iii) are satisfied.

So we can take $D(L)$ to be $(J^\infty(L^\sigma), \leq, \tau, R_\Box, R_\rhd) \in PRM_\tau$

To define E, let us consider $X \in PRM_\tau$. Then we have also $X \in PRM$ and by Theorem 4.1.8, $(\mathcal{D}(X), \cap, \cup, \emptyset, X, \Box_R, \rhd_R)$ where

$$\Box_R(U) = X \setminus R_\Box^{-1}(X \setminus U) \text{ and } \rhd_R(U) = X \setminus R_\rhd^{-1}(U)$$

is in $SDMA^+$.

By Theorem 3.4.22 we can conclude that $(Clop\mathcal{D}(X), \cap, \cup, \emptyset, X, \Box_R, \rhd_R)$ is in $SDMA$. In fact if conditions (M2),...,(M6) from definition 4.1.2 hold in $\mathcal{D}(X)$ then they are also true in $Clop\mathcal{D}(X)$.

Thus we will define $E(X) = (Clop\mathcal{D}(X), \cap, \cup, \emptyset, X, \Box_R, \rhd_R) \in SDMA$

The properties of η_L and ε_X follow from Theorem 3.4.22.

□

Thus we have defined at the objects level a duality between $SDMA$ and PRM_τ.

To define functors D and E between morphisms of these two categories observe that morphisms in $SDMA$ are {0,1}-homomorphisms while morphisms in PRM_τ have to respect the topology so they are continuous order preserving maps. On the other hand, since for every $(X, \leq, \tau, R_\Box, R_\rhd) \in PRM_\tau$ we have also $(X, \leq, R_\Box, R_\rhd) \in PRM$, we will have by Theorem 4.1.9 the following

Definition 4.2.11. Let X, Y be relational structures in PRM_τ and let $\varphi : X \to Y$ be a continuous order preserving map. Then we say $\varphi \in PRM_\tau(X, Y)$ if and only if the following conditions hold:

(a) $\forall q, r \in X : \ (qR_\triangleright r \Rightarrow \varphi(q)R_\triangleright\varphi(r))$.

(b) $\forall q \in X \,\forall p \in Y : \ (\varphi(q)R_\triangleright p \Rightarrow (\exists r \in X : \ (\varphi(r) \leq p \text{ and } qR_\triangleright r))$.

We can prove:

Theorem 4.2.12. *Let $L, K \in SDMA$ and let $X, Y \in PRM_\tau$. Given a morphism $h \in SDMA(L, K)$, there is an associated morphism $\varphi_h \in PRM_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ defined by*

$$\varphi_h(p) = \bigwedge\{a \in L : p \leq h(a)\}$$

with all $p \in J^\infty(K^\sigma)$.

Given a morphism $\varphi \in PRM_\tau(Y, X)$, there is an associated morphism $h_\varphi \in SDMA(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ defined by

$$h_\varphi = \varphi^{-1}(U)$$

for each $U \in Clop\mathcal{D}(X)$.

The maps $D : SDMA(L, K) \longrightarrow PRM_\tau(J^\infty(K^\sigma), J^\infty(L^\sigma))$ such that $D(h) = \varphi_h$ and $E : PRM_\tau(Y, X) \longrightarrow SDMA(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$ such that $E(\varphi) = h_\varphi$ are bijections.

The diagrams

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ {\scriptstyle v_L}\downarrow & & \downarrow{\scriptstyle v_K} \\ Clop\mathcal{D}(J^\infty(L^\sigma)) & \xrightarrow[h_{\varphi_h}]{} & Clop\mathcal{D}(J^\infty(K^\sigma)) \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \varepsilon_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ J^\infty(\mathcal{D}(Y)) & \xrightarrow[\varphi_{h_\varphi}]{} & J^\infty(\mathcal{D}(X)) \end{array}$$

commute.

Further:

(i) h is one to one if and only if φ_h is onto.

(ii) h is onto if and only if φ_h is an order embedding.

Proof. Let $h \in SDMA(L, K)$ Then we have, since h is a {0,1}-homomorphism that $h^\sigma \in SDMA^+(L^\sigma, K^\sigma)$ and, as we saw in Theorem 4.2.12, $\varphi_h \in PRM$. By Lemma 3.4.6, φ_h is continuous so $\varphi_h \in PRM_\tau$.

Let $\varphi \in PRM_\tau(Y, X)$. Then $\varphi \in \mathcal{P}_\tau(Y, X)$ and, from Lemma 3.4.7, it follows that h_φ such that $h_\varphi = \varphi^{-1}(U)$ for every $U \in Clop\mathcal{D}(X)$ is a morphism in $\mathcal{DL}(Clop\mathcal{D}(X), Clop\mathcal{D}(Y))$.

On the other hand we have also $\varphi \in PRM(Y, X)$ so there is a homomorphism $H_\varphi \in SDMA^+(\mathcal{D}(X), \mathcal{D}(Y))$ such that for each $V \in \mathcal{D}(X)$ we have $H_\varphi(V) = \varphi^{-1}(V)$. Then $h_\varphi = (H_\varphi)_{\restriction Clop\mathcal{D}(X)}$ so $h_\varphi \in SDMA$.

The other statements in the theorem are a consequence of Theorem 3.4.8 □

So we have established that the categories $SDMA$ and PRM_τ are dually equivalent.

Naturally we will obtain a much simpler duality if we consider, as in Section 4.2, ordered structures with the relation S (Definition 4.1.14). But now we have also to bear in mind that there is a topology in the ordered space. Let us find out how S interacts with the topology τ in the new structures.

Definition 4.2.13. Let PRS_τ be the class of relational structures $(X, \leq, \tau, S)$ where $(X, \leq, \tau)$ is a Priestley space and S is a binary relation on X such that:

(H1) (i) $\geq \circ S \subseteq S \circ \leq$.

(ii) For each $U \in Clop\mathcal{D}(X)$, $X \setminus S^{-1}(U) \in Clop\mathcal{D}(X)$.

(iii) For each $p \in X$, $\uparrow(pS) = \{q \in X : pS\circ \leq q\}$ is closed.

(H2) $Dom(S) = \{p \in X : \exists q \in X\, pSq\} = X$.

(H3) For each $p \in X$, pS is an antichain.

(H4) On $W = \{q \in X : \exists p \in X\, pSq\}$, S is an involution.

Now we can prove:

Lemma 4.2.14. *Let* $(X, \leq, \tau, R_\Box, R_\rhd) \in PRM_\tau$ *and let* $S = R_\rhd^{min}$. *Then* $(X, \leq, \tau, S) \in PRS_\tau$.

Proof. We have already proved in Theorem 4.2.5 that $R_\rhd^{-1}(U) = S^{-1}(U)$ for every $U \in \mathcal{D}(X)$ and it is clear that $\uparrow (pS) = pR_\rhd$ by the definition of S so H1 (ii) and (iii) are verified.

The other conditions follow from Theorem 4.2.4. □

Conversely

Lemma 4.2.15. *Let* $(X, \leq, \tau, S) \in PRS_\tau$ *and let* $R_\rhd = S\circ \leq$ *and* $R_\Box = S^2\circ \geq$. *Then* $(X, \leq, \tau, R_\Box, R_\rhd) \in PRM_\tau$.

Proof. The equivalence between H2,..., H4 and F3,..., F6 was already verified in Theorems 4.2.4 and 4.2.5 and conditions F1 (i) and F2 (i) are a direct consequence of the way $R_\rhd$ and $R_\Box$ were defined there.

From Theorem 4.2.5 and H1 (ii), it is clear that conditions F1 (ii) and F2 (ii) hold in X so we have just to prove that $pR_\Box$ is closed.

To do so observe that $pR_\Box = \downarrow pS^2$ by the definition of $R_\Box$.

Now, let $q \notin \downarrow pS^2$. Then for every $r \in pS^2$, $q \not\leq r$. Since X is a Priestley space there is a clopen downset U_r such that $r \in U_r$ and $q \in X \setminus U_r$. Clearly $\downarrow pS^2 \subseteq \bigcup_{r\in pS^2} U_r$.

By H1(ii), for each $r \in pS^2$, $S^{-1}(U_r)$ is a clopen upset so, to prove that $\uparrow pS \subseteq \bigcup_{r\in pS^2} S^{-1}(U_r)$, it is enough to see that $pS \subseteq \bigcup_{r\in pS^2} S^{-1}(U_r)$.

Let $r \in pS$. Then $r \in W$ and pSr so there is one and only one $t \in W$ such that $pSrSt$. Then $t \in pS^2$ so that $t \in U_t$ and consequently $r \in S^{-1}(U_t)$ so $r \in \bigcup_{r\in pS^2} S^{-1}(U_r)$.

Now remember that $\uparrow pS$ is closed so that there is a finite number of upsets $S^{-1}(U_{r_1}), ..., S^{-1}(U_{r_n})$ such that $\uparrow pS \subseteq S^{-1}(U_{r_1}) \cup ... \cup S^{-1}(U_{r_n})$. We are going to prove that $\downarrow pS^2 \subseteq U_{r_1} \cup ...U_{r_n}$.

Let $x \in pS^2$ then $x \in W$ so, there is $y \in W$ such that $pSySx$. Then $y \in \downarrow pS$ so, there is U_{r_k} with $1 \leq k \leq n$ such that $y \in S^{-1}(U_{r_k})$. Since $y \in W$, x is the only element such that ySx so $x \in U_{r_k}$.

Therefore $pS^2 \subseteq U_{r_1} \cup ...U_{r_n}$ and since $\downarrow pS^2$ is a downset, $\downarrow pS^2 \subseteq U_{r_1} \cup ...U_{r_n}$.

From here we conclude that $X \setminus U_{r_1} \cap ... \cap X \setminus U_{r_n}$, is an open neighborhood of q that is disjoint from $\downarrow pS^2$. So, $\downarrow pS^2$ is closed.

□

It is now obvious that at the objects level we can establish a duality between $SDMA$ and PRS_τ.

Regarding morphisms, Theorem 4.2.6 shows how to define such maps in order that both categories PRM and PRS_τ are equivalent. So we have:

Definition 4.2.16. Let $L, K \in SDMA$ and let $X = J^\infty(L^\sigma)$ and $Y = J^\infty(K^\sigma)$ and $\varphi : X \to Y$ be a continuous order preserving map. Then $\varphi \in PRS_\tau$ if and only if

(c) $\forall q \in X \forall p \in Y,\ (\varphi(q)S\circ \leq p \Leftrightarrow (\exists r \in X,\ \varphi(r) \leq p \text{ and } qSr))$.

Applying Theorem 4.2.10, Lemma 4.2.14 and Lemma 4.2.15 we can define directly the functors D and E between $SDMA$ and PRS_τ obtaining the following:

Theorem 4.2.17. *Let $L \in SDMA$ and let $D(L) = (J^\infty(L^\sigma), \leq, \tau, S)$ where S is such that pSq if and only if q is a minimal element in $\{r \in J^\infty(L^\sigma) : \kappa(p) \geq\rhd q\}$.*

Let $(X, \leq, \tau, S) \in PRS_\tau$ and let $E(X) = (Clop\mathcal{D}(X), \cap, \cup, \emptyset, X, \rhd, \Box)$ where, for any $U \in Clop\mathcal{D}(X)$, $\rhd (U) = X \setminus S^{-1}(U)$ and $\Box(U) = X \setminus S^{-1}(X \setminus S^{-1}(U))$.

Then D maps $SDMA$ in PRS_τ and E maps PRS_τ in $SDMA$.

Further, D and E define a dual equivalence between $SDMA$ and PRS_τ.

Clearly we obtained this way a category PRS_τ that is dually equivalent to $SDMA$.

In this duality notice that the correspondents of congruences in $SDMA$s are closed subsets of the dual space that are closed for S. In fact, from Corollary 3.4.9 it follows that, since any congruence θ of a semi-De Morgan algebra L is a congruence of the distributive lattice, there is a a closed subset $Z \subseteq J^\infty(L^\sigma)$ corresponding to θ and, for any $a, b \in L$:

$$a\theta b \Leftrightarrow v(a) \cap Z = v(b) \cap Z$$

where $v(a) = \{p \in J^\infty(L^\sigma) :\ p \leq a\}$ and $v(b) = \{p \in J^\infty(L^\sigma) :\ p \leq b\}$.

Since the subset Z is the range of the morphism in PRS_τ that corresponds to θ, we know, by Corollary 4.2.8, that Z is such that for any $p \in Z$, $pS \subseteq Z$.

So we have:

Theorem 4.2.18. *Let $L \in SDMA$ and $X = J^\infty(L^\sigma)$. The lattice of $SDMA$-congruences of L, $Con(L)$, is isomorphic to the order dual of the lattice of closed subsets of X that are closed for S.*

Observe that in this isomorphism the subset W corresponds to the congruence relation ϕ defined in Section 1.4.

For $(L, \wedge, \vee, \rhd, \Box, 0, 1) \in SDMA$ the congruence relation ϕ is such that

$$\forall a, b \in L\ (a\phi b \text{ if anf only if } \rhd a = \rhd b)$$

and we can prove:

Corollary 4.2.19. *Let $L \in SDMA$ and let $J^\infty(L^\sigma) \in PRS_\tau$ be the dual space of L. Then $W = \{q \in J^\infty(L^\sigma) : \exists p \in J^\infty(L^\sigma)\, pSq\}$ is such that for any $a, b \in L$,*

$$a\phi b \text{ if and only if } v(a) \cap W = v(b) \cap W$$

where $v(a) = \{p \in J^\infty(L^\sigma) : p \leq a\}$ and $v(b) = \{p \in J^\infty(L^\sigma) : p \leq b\}$.

Consequently W is a closed subset of $J^\infty(L^\sigma)$ that is closed for the relation S and is homeomorphic to $J^\infty((L/\phi)^\sigma)$.

Proof. Suppose first that $\rhd a \leq \rhd b$.

Since we have also $a, b \in L^\sigma$ this is equivalent to:

$$\forall p \in J^\infty(L^\sigma)\ (\kappa(p) \geq \rhd b \Rightarrow \kappa(p) \geq \rhd a).$$

By Lemma 3.2.9 (i), the antecedent holds if and only if there is $q \in J^\infty(L^\sigma)$ such that $q \leq b$ and $pR_\rhd q$. This means that $v(b) \cap pR_\rhd \neq \emptyset$.

With the same arguments we can prove that the consequent is equivalent to $v(a) \cap pR_\rhd \neq \emptyset$ so the implication holds if and only if

$$\forall p \in J^\infty(L^\sigma)\ (v(b) \cap pR_\rhd \neq \emptyset \Rightarrow v(a) \cap pR_\rhd \neq \emptyset).$$

Since $v(a)$ and $v(b)$ are downsets, the previous condition is equivalent to

$$\forall p \in J^\infty(L^\sigma)\ (v(b) \cap pS \neq \emptyset \Rightarrow v(a) \cap pS \neq \emptyset). \qquad (*)$$

Now, this is equivalent to

$$\forall x \in W\ (v(b) \cap xS \neq \emptyset \Rightarrow v(a) \cap xS \neq \emptyset). \qquad (**)$$

It is obvious that $*$ implies $**$. For the converse observe that if $p \in J^\infty(L^\sigma)$ and $q \in v(b) \cap pS$ then $q \in W$ and, by Lemma 4.2.2, there is one and only one

$x \in W$ such that qSx and xSq. Since q is the only element in xS it follows that $v(b) \cap xS = \{q\}$ and, by $**$, $\{q\} = v(a) \cap xS$ so that $q \in v(a) \cap pS$.

Since, for any $x \in W$, xS is a singleton we conclude that $**$ is equivalent to

$$v(b) \cap W \subseteq v(a) \cap W.$$

Analogously $\rhd b \leq \rhd a$ holds if and only if $v(a) \cap W \subseteq v(b) \cap W$.

From Theorem 4.2.18 it follows that W, corresponding to the congruence relation ϕ, is a closed subset of $J^\infty(L^\sigma)$ that is closed for the relation S and, from Corollary 3.4.9, W is isomorphic to $J^\infty((L/\phi)^\sigma)$. □

4.3 The dual spaces of some subvarieties of $SDMA$

As an application of the duality between $SDMA$ and PRS_τ we are going to characterize the dual spaces of some important subvarieties of $SDMA$.

We will show how this duality generalizes the well known dualities of De Morgan algebras, $K_{1,1}$ and p-lattices.

Finally we will discuss the dual characterization of the subvariety $\mathcal{C}$.

For De Morgan algebras we have the following:

Theorem 4.3.1. *Let $L \in SDMA$ and $X = J^\infty(L^\sigma)$. Then L is a De Morgan algebra if and only if $X = W$.*

Proof. By Lemma 4.1.5, $L \in SDMA$ is a De Morgan algebra if and only if $a = \Box a$ for any $a \in L$. But this condition is equivalent to

$$\forall a, b \in L\, (\rhd a = \rhd b \Rightarrow a = b).$$

Therefore an algebra $L \in SDMA$ is a De Morgan algebra if and only if the congruence relation ϕ is such that $\phi = \triangle$.

Naturally the subset of X corresponding to $\triangle$ in the isomorphism referred in Theorem 4.2.18 is X so, by Corollary 4.2.19, $\phi = \triangle$ is equivalent to $X = W$. □

This Theorem shows that, for De Morgan algebras, the relation S is the correspondent of the involution ξ defined by Cornish and Fowler in the space of prime ideals of a De Morgan algebra [14].

For the other subvarieties that we are considering, the characterization of their dual spaces will appear as a consequence of the fact that they can be defined as subvarieties of $SDMA$, by Sahlqvist inequalities (Remark 4.1.6). Therefore, we know, by Theorem 4.1.1, that we can effectively compute the corresponding relations in PRM.

Observe that in all the following proofs the key for this correspondence is the elimination of the universal quantifier on elements of L^σ and that this process is generally quite simple.

When we consider the subvariety $K_{1,1}$ we obtain:

Theorem 4.3.2. *Let $L \in SDMA$ and $X = J^\infty(L^\sigma)$. Then L is in $K_{1,1}$ if and only if the relation $S \subseteq X \times X$ is functional.*

Proof. We have already referred in Lemma 4.1.5 that $K_{1,1}$ is characterized, as a subvariety of $SDMA$ by the Sahlqvist inequality,

$$\forall a, b \in L \;\; \rhd (a \wedge b) \leq \rhd a \vee \rhd b$$

thus we conclude that this inequality is canonical and hence it is also true in L^σ:

$$\forall a, b \in L^\sigma \;\; \rhd^\sigma (a \wedge b) \leq \rhd^\sigma a \vee \rhd^\sigma b.$$

Now we can compute the corresponding condition in $X = J^\infty(L^\sigma)$ because the previous inequality is equivalent to

$$\forall a, b \in L^\sigma \; \forall p \in X \; (p \leq \rhd^\sigma (a \wedge b) \Rightarrow (p \leq \rhd^\sigma a \text{ or } p \leq \rhd^\sigma b))$$

that holds if and only if

$$\forall a, b \in L^\sigma \; \forall p \in X \; ((p \not\leq \rhd^\sigma a \text{ and } p \not\leq \rhd^\sigma b) \Rightarrow p \not\leq \rhd^\sigma (a \wedge b)).$$

By Lemma 3.2.9, this condition is equivalent to

$$\forall a, b \forall p \; ((\exists q \; (q \leq a \;\text{ and } pR_\rhd q)) \text{ and } (\exists r \; r \leq b \text{ and } pR_\rhd r)) \Rightarrow$$
$$\Rightarrow (\exists t \; t \leq a \wedge b \text{ and } pR_\rhd t))$$

and to

$$\forall a, b \forall p, q, r \; (((q \leq a \;\text{ and } pR_\rhd q) \text{ and } (r \leq b \text{ and } pR_\rhd r)) \Rightarrow$$
$$\Rightarrow (\exists t \; t \leq a \wedge b \text{ and } pR_\rhd t))$$

and this holds if and only if

$$\forall p, q, r\ ((pR_{\triangleright}q \text{ and } pR_{\triangleright}r) \Rightarrow (\exists t\ (t \leq q \wedge r \text{ and } pR_{\triangleright}t))). \qquad (*)$$

because q and r are, respectively, the least a and b for which the antecedent holds and, if the conclusion is verified by some a and b it holds for all greater elements.

From the previous implication it follows that, when we consider the relation S, for any p, q, r such that pSq and pSr the antecedent holds so, there is $t \leq q \wedge r$ and $x \leq t$ such that $pSx \leq t$. Since, by H3, pS is an antichain we must have $x = q = r$.

Therefore, condition $(*)$ implies

$$\forall p, q, r\ ((pSq \text{ and } pSr) \Rightarrow q = r).$$

Conversely, from this implication we can obtain condition $(*)$ directly from the definition of S and Theorem 3.3.2. □

Observe that the function defined by S when $L \in K_{1,1}$ is none other than the correspondent of the function defined by Urquhart in the dual spaces of algebras in $K_{1,1}$ in [43].

The varieties of demi-p-lattices and almost p-lattices were considered by Sankappanavar in [38], [39] and [40]. They are generalizations of p-lattices so, when we study the dual spaces of algebras of these varieties, it is natural to obtain characteristics that generalize properties of the dual spaces of p-lattices.

Theorem 4.3.3. *Let $L \in SDMA$ and $X = J^{\infty}(L^{\sigma})$. Then L is a demi p-lattice if and only if for any $q \in W$, qSq.*

Proof. We know from Lemma 4.1.5 that L is a demi-p-lattice if and only if, for any $a \in L$,

$$\triangleright\, a \wedge \Box a \leq 0$$

and, since this is a Sahlqvist inequality, it is canonical so,

$$\forall a \in L^{\sigma}\ \ \triangleright^{\sigma} a \wedge \Box^{\sigma} a \leq 0.$$

To compute the correspondent condition in $X = J^{\infty}(L^{\sigma})$ observe that, since L^{σ} is join generated by $J^{\infty}(L^{\sigma})$, the previous inequality is equivalent to

$$\forall a \in L^{\sigma}\ \forall p \in X\ (p \leq \Box^{\sigma} a \Rightarrow p \not\leq \triangleright^{\sigma} a).$$

Now, by Lemma 3.2.9, this condition is equivalent to

$$\forall a\, \forall p\, (\bigvee pR_{\Box} \leq a \Rightarrow (\exists r : (r \leq a \text{ and } pR_{\triangleright} r))$$

and, since the least a for which the antecedent holds is $a = \bigvee pR_{\Box}$ and, since the consequent holds for all greater elements than a if it holds for some a, the previous condition is also equivalent to

$$\forall p\, (\exists r : (r \leq \bigvee pR_{\Box} \text{ and } pR_{\triangleright} r)$$

Finally we obtain

$$\forall p\, (\exists r : (pR_{\Box} r \text{ and } pR_{\triangleright} r)). \qquad (*)$$

because $r \in J^{\infty}(L^{\sigma})$ and $pR_{\Box}$ is a downset.

For any $q \in W$, it follows from this condition that:

$$\exists r \in X\, (qS^2 \circ \geq r \text{ and } qS \circ \leq r).$$

From H4 we obtain:

$$\exists r \in X \exists s \in W\, (q \geq r \text{ and } qSs \leq r)$$

and hence $q \geq r \geq s$.

Now, since we have $s \in W$ and q is the only element such that sSq, it also follows from condition $*$ and the previous arguments that

$$\exists y \in X\, (s \geq y \text{ and } sSq \leq y)$$

so that $s \geq y \geq q$.

Thus $s = q$ and consequently qSq.

The converse is obvious so, $(*)$ is equivalent to $\forall q \in W\; qSq$. □

From the previous Lemma it follows that, when L is a demi-p-lattice, pSp and qSq and $p \leq q$ implies $p = q$ because S is an order reversing involution in W so that

Corollary 4.3.4. *When L is a demi-p-lattice the set $W \subseteq J^{\infty}(L^{\sigma})$ is an antichain.*

This property of the dual space of demi-p-lattices is also a consequence of the fact that a demi-p-lattice L is a $SDMA$ such that L/ϕ is a Boolean algebra (Section 1.4) and consequently W is the dual space of a Boolean algebra.

Theorem 4.3.5. *Let $L \in SDMA$ and $X = J^{\infty}(L^{\sigma})$. Then L is an almost p-lattice if and only if for any $p, q \in X$,*

$$q \text{ is minimal in } \downarrow p \Rightarrow pSq$$

Proof. By Lemma 4.1.5, L is an almost p-lattice if and only if

$$\forall a \in L\ a \wedge \rhd a \leq 0.$$

This is a Sahlqvist inequality so it is canonical and therefore,

$$\forall a \in L^{\sigma}\ a \wedge \rhd^{\sigma} a \leq 0$$

which holds if and only if

$$\forall a \in L^{\sigma}\ \forall p \in X\ (p \leq a \Rightarrow p \nleq \rhd^{\sigma} a).$$

By Lemma 3.2.9 this is equivalent to

$$\forall a \in L^{\sigma}\ \forall p \in X\ (p \leq a \Rightarrow (\exists q \in X\ (q \leq a \text{ and } pR_{\rhd}q))).$$

Now, since the least a for which the antecedent holds is p and since the consequent is such that if it is verified by some a it is also verified by all greater elements, we obtain the equivalent condition

$$\forall p\ \exists q\ (q \leq p \text{ and } pR_{\rhd}q)$$

and, by F2,

$$\forall p\ pR_{\rhd}p$$

which is equivalent to:

$$\forall p\ pS\circ \leq p.$$

So, if the previous condition holds and if q is minimal in $\downarrow p$, then q being a minimal element of X implies $qS\circ \leq q$ and hence qSq. Thus $q \in W$.

Consequentely we have $p \geq qSq$ and, by H1 (i), $pS\circ \leq q$ which implies pSq, because q is minimal.

Conversely, suppose that q minimal in $\downarrow p$ implies pSq. Then, since X is a Priestley space, for any $p \in X$ there is $r \in X$ that is minimal in $\downarrow p$ and consequently $\forall p\ pS\circ \leq p$.

□

We have proved that if L is an almost p-lattice and if an element q is minimal in $J^\infty(L^\sigma)$, then we have qSq so $q \in W$.

Conversely, if $q \in W$ then, since L is a demi-p-lattice, qSq. If $r \in J^\infty(L^\sigma)$ is a minimal element of $\downarrow q$ then, by the previous theorem qSr, and consequently $q = r$ because $q \in W$.

Thus we have proved:

Corollary 4.3.6. *If L is an almost p-lattice, the set $W \subseteq J^\infty(L^\sigma)$ is the set of minimal elements of $J^\infty(L^\sigma)$.*

Observe that in an almost p-lattice we can have $pS \not\subseteq \downarrow p$. This cannot happen in p-lattices.

Theorem 4.3.7. *Let $L \in SDMA$ and $X = J^\infty(L^\sigma)$. Then L is a p-lattice if and only if, for any $p, q \in X$,*

$$q \textit{ is minimal in } \downarrow p \Leftrightarrow pSq$$

Proof. From Lemma 4.1.5 we know that L is a p-lattice if and only if L is a demi p-lattice such that for any $a \in L$, $a \leq \Box a$ and it is clear that this is equivalent to L being an almost p-lattice such that $a \leq \Box a$.

Since this inequality is Sahlqvist, it is also true in L^σ so we have,

$$\forall a \in L^\sigma \; a \leq \Box^\sigma a.$$

This is equivalent to

$$\forall a \in L^\sigma \; \forall p \in X \; (p \leq a \Rightarrow p \leq \Box^\sigma a)$$

because L^σ is join generated by X.

By Lemma 3.2.9(iv), it is also equivalent

$$\forall a \in L^\sigma \; \forall p \in X \; (p \leq a \Rightarrow \bigvee pR_\Box \leq a).$$

But p is the least a satisfying the antecedent and since the consequent, being valid for some a, holds for any greater elements we obtain the equivalent condition

$$\forall p \in X \; (\bigvee pR_\Box \leq p).$$

which holds if and only if

$$\forall p, q \in X \; (pR_\Box q \Rightarrow q \leq p). \qquad (*)$$

We are going to prove that, in an almost p-lattice L, $(*)$ is equivalent to

$$\forall p, q \in X \ (q \text{ is minimal in } \downarrow p \Leftrightarrow pSq). \qquad (**)$$

The if part in condition $**$ was proved in Theorem 4.3.5.

For the other implication observe that, from pSq it follows that qSq since L is a demi-p-lattice. Thus pS^2q and, by condition $(*)$, $p \geq q$.

Now suppose that $r \in X$ is minimal in $\downarrow q$. Then, by Theorem 4.3.5, qSr and, since $q \in W$, $q = r$.

Consequently q is minimal in $\downarrow p$.

To prove that condition $**$ implies condition $*$, observe that, in an almost p-lattice, $pR_\Box q$ is equivalent to pS^2q because the elements in W are minimal in X and $pR_\Box$ is a downset. Thus we obtain $pSqSq$ because L is a demi p-lattice. From condition $**$ we conclude that $p \geq q$. $\Box$

This Theorem corresponds to the characterization of the dual spaces of p-lattices obtained by Priestley in [31]. Notice that, from the previous theorem it follows that, when X is the dual space of a p-lattice, we have for any $U \in Clop\mathcal{D}(X)$, $S^{-1}(U) = \uparrow U$. Therefore, the unary operation $\rhd_R$ is such that $\rhd_R (U) = X \setminus \uparrow U$.

Finally we are going to study the dual spaces of algebras in the variety $\mathcal{C}$. We will see how the conditions corresponding to the inequalities that characterize this variety restrict the graphs of the relation S.

The conditions that we present here could have be obtained as a consequence of the study, done by D. Hobby in [25], of the partial diagrams that are omited in the dual space of an algebra in the variety $\mathcal{C}$.

In the dual space of an algebra $L \in \mathcal{C}$ if we have, for any $p, q, r \in J^\infty(L^\sigma)$, pSq and pSr with $q \neq r$ as in Figure 4.1

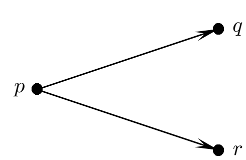


Figure 4.1:

then we must have one of the following situations:

Figure 4.2:

Theorem 4.3.8. *Let $L \in SDMA$ and $X = J^{\infty}(L^{\sigma})$. Then L is in the variety $\mathcal{C}$ if and only if for any $p, q, r \in X$ such that $q \neq r$,*

$$pSq \textit{ and } pSr \quad \textit{implies} \quad (qSr \textit{ and } rSq) \textit{ or } (qSq \textit{ and } rSr)$$

Proof. As we have already referred, the inequalities γ and β from Lemma 4.1.5 are Sahlqvist so, by Theorem 4.1.1 we can compute the correspondent conditions.

The inequality β holds in L^{σ} if and only if

$$\forall a, b \in L^{\sigma}\ \forall p \in X\ (p \leq \rhd^{\sigma} (a \wedge b) \Rightarrow p \leq \rhd^{\sigma} a \vee \rhd^{\sigma} (\rhd^{\sigma} a \wedge b \wedge \rhd^{\sigma} b))$$

and this implication is equivalent to

$$\forall a, b \in L^{\sigma}\ \forall p \in X\ (p \not\leq \rhd^{\sigma} a \text{ and } p \not\leq \rhd^{\sigma} (\rhd^{\sigma} a \wedge b \wedge \rhd^{\sigma} b) \Rightarrow p \not\leq \rhd^{\sigma} (a \wedge b)).$$

Now, by Lemma 3.2.9, this holds if and only if

$$\forall a, b \forall p((\exists q(q \leq a \text{ and } pR_{\rhd}q) \ \text{ and } \exists r(r \leq \rhd^{\sigma} a \wedge b \wedge \rhd^{\sigma} b \text{ and } pR^{\sigma}_{\rhd} r)) \Rightarrow$$
$$\Rightarrow \exists t(t \leq a \wedge b \text{ and } pR^{\sigma}_{\rhd} t))$$

Thus the inequality β is equivalent to

$$\forall a, b \forall p((\exists q(q \leq a \text{ and } pR_{\rhd}q) \ \text{ and } \exists r(r \leq \rhd^{\sigma} a \text{ and } r \leq b \text{ and } r \leq \rhd^{\sigma} b \text{ and } pR_{\rhd} r)) \Rightarrow$$
$$\Rightarrow \exists t(t \leq a \wedge b \text{ and } pR_{\rhd} t)).$$

and consequently, also to

$$\forall a, b \forall p, q, r((q \leq a \text{ and } pR_{\rhd}q \ \text{ and } r \leq \rhd^{\sigma} a \text{ and } r \leq b \text{ and } r \leq \rhd^{\sigma} b \text{ and } pR_{\rhd} r) \Rightarrow$$
$$\Rightarrow \exists t(t \leq a \wedge b \text{ and } pR_{\rhd} t))$$

and to

$$\forall a, b \forall p, q, r((q \leq a \text{ and } pR_{\triangleright}q \text{ and } r \leq b \text{ and } pR_{\triangleright}r) \Rightarrow$$
$$\Rightarrow (r \not\leq \triangleright^{\sigma} a \text{ or } r \not\leq \triangleright^{\sigma} b \text{ or } (\exists t(t \leq a \wedge b \text{ and } pR_{\triangleright}t))).$$

Observe that we have moved to the consequent the negation of conditions that were in the antecedent in order to obtain a simpler equivalent condition.

This is something that is not possible when we are dealing with the inequations in the algebra.

The previous condition is also equivalent to

$$\forall p, q, r \;\; ((pR_{\triangleright}q \text{ and } pR_{\triangleright}r) \Rightarrow$$
$$\Rightarrow (r \not\leq \triangleright^{\sigma} q \text{ or } r \not\leq \triangleright^{\sigma} r \text{ or } (\exists t \; (t \leq q \wedge r \text{ and } pR_{\triangleright}t))).$$

because q and r are, respectively, the least a and b for which the antecedent holds and, if the conclusion holds for some a and b then it also holds for all greater elements because $\triangleright^{\sigma}$ is order reversing.

Now we can move to the antecedent the negation of a condition in the consequent and conclude that β is equivalent to

$$\forall p, q, r \; ((pR_{\triangleright}q \text{ and } pR_{\triangleright}r \text{ and } (\forall t \; (t \leq q \wedge r \Rightarrow p \not R_{\triangleright}t))) \Rightarrow (rR_{\triangleright}q \text{ or } rR_{\triangleright}r)). \quad (*)$$

In what concerns inequality γ we know that it holds in L^{σ} so that we obtain

$$\forall a, b \in L^{\sigma} \forall p \in X \; (p \leq \triangleright^{\sigma} (a \wedge b) \wedge \Box b \Rightarrow p \leq \triangleright^{\sigma} a \vee \triangleright^{\sigma} b).$$

This is equivalent to

$$\forall a, b \in L^{\sigma} \forall p \in X \; ((p \not\leq \triangleright^{\sigma} a \text{ and } p \not\leq \triangleright^{\sigma} b) \Rightarrow (p \not\leq \triangleright^{\sigma} (a \wedge b) \text{ or } p \not\leq \Box^{\sigma} b)).$$

and to

$$\forall a, b \in L^{\sigma} \forall p \in X \; ((p \not\leq \triangleright^{\sigma} a \text{ and } p \not\leq \triangleright^{\sigma} b \text{ and } p \leq \Box^{\sigma} b) \Rightarrow (p \not\leq \triangleright^{\sigma} (a \wedge b))).$$

Now, by Remark 3.2.9, this is true if and only if

$$\forall p, q, r \; ((q \leq a \text{ and } pR_{\triangleright}q \text{ and } r \leq b \text{ and } pR_{\triangleright}r \text{ and } \bigvee pR_{\Box} \leq b \Rightarrow$$
$$\Rightarrow (\exists t \; (t \leq a \wedge b \text{ and } pR_{\triangleright}t)).$$

Now, the least a and b for which the antecedent holds are q and $r \vee (\bigvee pR_\Box)$. Notice also that if the consequent holds for these a and b it also holds for all greater elements so we conclude for the equivalence to,

$$\forall p, q, r\ ((pR_\triangleright q \text{ and } pR_\triangleright r \Rightarrow (\exists t\ (t \leq q \wedge (r \vee (\bigvee pR_\Box)) \text{ and } pR_\triangleright t)).$$

and, by distributivity (and associativity), to

$$\begin{aligned}&\forall p, q, r\ ((\ pR_\triangleright q \text{ and } pR_\triangleright r \Rightarrow \\ &\qquad \Rightarrow (\exists t\ (t \leq q \wedge r \text{ and } pR_\triangleright t) \text{ or } (\exists v\ (v \leq q \text{ and } v \leq \bigvee pR_\Box) \text{ and } pR_\triangleright v)).\end{aligned}$$

This holds if and only if

$$\begin{aligned}&\forall p, q, r\ ((\ pR_\triangleright q \text{ and } pR_\triangleright r \text{ and } (\forall t\ (t \leq q \wedge r \Rightarrow p \not R_\triangleright t)) \Rightarrow \\ &\qquad \Rightarrow (\exists v\ (v \leq q \text{ and } v \leq \bigvee pR_\Box) \text{ and } pR_\triangleright v))).\end{aligned}$$

Finally we conclude that the inequality γ is equivalent to

$$\begin{aligned}&\forall p, q, r\ (\ pR_\triangleright q \text{ and } pR_\triangleright r \text{ and } (\forall t\ (t \leq q \wedge r \Rightarrow p \not R_\triangleright t)) \Rightarrow \qquad (**) \\ &\qquad \Rightarrow (\exists v\ (v \leq q \text{ and } pR_\Box v \text{ and } pR_\triangleright v)).\end{aligned}$$

Observe that the conditions obtained by interchanging q and r both in condition $(**)$ and $(*)$ are also true in $\mathcal{C}$.

Now we are going to see how these conditions become much simpler when we use the relation S.

Suppose pSq and pSr and $q \neq r$. Then notice that the antecedent of conditions $(*)$ and $(**)$ holds and so we have simultaneously:

(i) $rS\circ \leq q$ or $rS\circ \leq r$ by $(*)$

(ii) $qS\circ \leq r$ or $qS\circ \leq q$ by $(*)$

(iii) $\exists v \in X\ (v \leq q$ and $pS^2\circ \geq v$ and $pS\circ \leq v)$ by $(**)$

(iv) $\exists u \in X\ (u \leq r$ and $pS^2\circ \geq u$ and $pS\circ \leq u)$ by $(**)$

Since we have pSq it follows from (iii) and the fact that pS is an antichain that $q = v$ so,

$$pS^2\circ \geq q.$$

By the same argument and (iv) we have also

$$pS^2\circ \geq r.$$

Now suppose that in (i) we have $rS\circ \leq q$ and that $t \in W$ is such that $rSt \leq q$. From $pS^2\circ \geq r$ we have $pS^2\circ \geq rSt \leq q$ which implies, by H1 in Definition 4.2.13, $pS^3\circ \leq t \leq q$ and, since S is an involution in W, we obtain $pSt \leq q$ and consequently $t = q$ because pS is an antichain.

So we have rSq and since these two elements are in W, qSr.

Now suppose that we have $rS\circ \leq r$ in (i) and that $t \in W$ is such that $rSt \leq r$. Then, from $pS^2\circ \geq r$ it follows $pS^2\circ \geq rSt \leq r$ and, by H1 in Definition 4.2.13, $pS^3\circ \leq t \leq r$ so that $pSt \leq r$ and thus $t = r$ so that rSr

In a similar way, from $qS\circ \leq q$ in (ii) and $pS^2\circ \geq q$ it follows that qSq.

Observe now that when rSr it is not possible to have $qS\circ \leq r$ in (ii) because then we would obtain $q \geq r$ because S is an involution so we obtain qSq. Similarly when qSq it is not possible to have $rS\circ \leq q$ so we must have rSr.

Thus we proved that conditions $(*)$ and $(**)$ imply

$$(pSq \text{ and } pSr \text{ and } q \neq r) \Rightarrow ((qSr \text{ and } rSq) \text{ or } (qSq \text{ and } rSr)) \qquad (***)$$

For the converse observe that for any $p, q, r \in X$ such that

$$pR_{\triangleright}q \text{ and } pR_{\triangleright}r \text{ and } (\forall t\ (t \leq q \wedge r \Rightarrow p \not R_{\triangleright} t)$$

there is $x, y \in X$ such that $x \leq q$ and $y \leq r$ and pSx and pSy and $x \neq y$ so, the antecedent of $(***)$ holds.

Therefore ySx or ySy and consequently $rR_{\triangleright}q$ or $rR_{\triangleright}r$ so that $(*)$ holds.

Since we have also pSy and ySx and pSx this implies pS^2x and pSx so that x is such that $x \leq q$ and $pR_{\Box}x$ and $pR_{\triangleright}x$. Therefore $(**)$ holds.

$\square$

Chapter 5

Some properties of congruences

In [9] Blyth and Varlet characterized the distributive lattices and the Stone, the De Morgan and the Heyting algebras that have only principal congruences. Later on, Beazer in Corollary 3. 9 in [6] identified the p-lattices that have the same property.

The p-lattices having the principal join property, i.e. those p-lattices such that the join of any two principal congruences is a principal congruence were characterized by Beazer in [5]. In a variety the algebras such that the join of any two principal congruences is a principal congruence are called congruence principal by I. Chajda in [12] and they are defined there as being such that every compact congruence is principal.

In this chapter we will study these properties in $SDMA$s generalizing to demi-p-lattices results already known for p-lattices. The results we obtain here were previously determined in [27] and [29] using Hobby's duality. Here we will apply the duality that we developed in the previous chapters.

5.1 Congruences and duality in $\mathcal{DL}$

We start by proving somme lemmas concerning principal congruences in distributive lattices that will be important to our study.

As we referred in Corollary 3.4.9, if L is a bounded distributive lattice and $X = J^{\infty}(L^{\sigma}) \in \mathcal{P}_{\tau}$ is its dual space, then the lattice of congruences of L, $Con(L)$, is isomorphic to the order dual of the lattice of closed subsets of X. It is clear that the order dual of this lattice is isomorphic to the lattice of the (complementary) open subsets of X. Since it is more convenient in the present context, we will prefer this order-isomorphism. This corresponds to what Priestley does in [31]

Thus, from Corollary 3.4.9, it follows:

Proposition 5.1.1. *Let $L \in \mathcal{DL}$ and $X = J^{\infty}(L^{\sigma})$. To any congruence $\theta \in Con(L)$ corresponds an open subset $A \subseteq X$ such that for any $a, b \in L$,*

$$a\theta b \Leftrightarrow v(a) \cap (X \setminus A) = v(b) \cap (X \setminus A)$$

where $v(a) = \{p \in X : p \leq a\}$ and $v(b) = \{p \in X : p \leq b\}$.

This correspondence is an order isomorphism.

Now we can study the open sets corresponding to some special congruences.

Lemma 5.1.2. *Let $L \in \mathcal{DL}$ and let $X \in \mathcal{P}_{\tau}$ be its dual space. Let $\theta(e, 1)$ and $\theta(0, f)$, with $e, f \in L$, be principal congruences of L. Then, for any $a, b \in L$:*

(i) $(a, b) \in \theta(e, 1)$ if and only if $v(a) \cap (X \setminus v(e)^c) = v(b) \cap (X \setminus v(e)^c)$.

(ii) $(a, b) \in \theta(0, f)$ if and only if $v(a) \cap (X \setminus v(f)) = v(b) \cap (X \setminus v(f))$.

Proof. (i) Since, for every $a, b \in L$, we know that $(a, b) \in \theta(e, 1)$ is equivalent to $a \wedge e = b \wedge e$ it follows from the isomorphism between L and $Clop\mathcal{D}(X)$ that this condition holds if and only if

$$v(a) \cap v(e) = v(b) \cap v(e)$$

and it is obvious that this proves (i).

(ii) For any $a, b \in L$, we have $(a, b) \in \theta(0, f)$ if and only if $a \vee f = b \vee f$. This is equivalent to $v(a) \cup v(f) = v(b) \cup v(f)$ in $Clop\mathcal{D}(X)$.

It is not difficult to prove that this equality is equivalent to

$$v(a) \cap (X \setminus v(f)) = v(b) \cap (X \setminus v(f)).$$

□

As a consequence of the previous lemma we have:

Theorem 5.1.3. *Let $L \in \mathcal{DL}$ and let $X \in \mathcal{P}_{\tau}$ be its dual space. A congruence $\theta \in Con(L)$ is principal if and only if the subset of X that corresponds to θ in the order isomorphism between $Con(L)$ and the lattice of open subsets of X is clopen and convex.*

Proof. Since in a distributive lattice L a principal congruence $\theta(e,f)$ with $e \leq f$ is such that $\theta(e,f) = \theta(e,1) \cap \theta(0,f)$, we conclude, from Lemma 5.1.2, that the open subset of the dual space corresponding to this principal congruence is the clopen convex subset of X, $v(e)^c \cap v(f) = v(f) \setminus v(e)$.

The converse is also true. In fact, if $Q \subseteq X$ is a clopen convex subset of X, then there exist $v(e)$ and $v(f)$ clopen decreasing subsets of X such that $Q = v(f) \setminus v(e)$ and the corresponding congruence is the principal congruence $\theta(e,f)$.

To prove this claim, we will follow the proof of Lemma 3 in [2].

Since X is a Priestley space and Q is clopen, $\downarrow Q, \uparrow Q, \downarrow Q \setminus Q$ and $\uparrow Q \setminus Q$ are closed subsets of X.

Since Q is convex, no element in $\uparrow Q \setminus Q$ is less than or equal to any element in $\downarrow Q$ so, there is $v(g) \in Clop\mathcal{D}(X)$, with $g \in L$, such that $\downarrow Q \subseteq v(g)$ and $\uparrow Q \setminus Q \subseteq v(g)^c$.

Analogously, there is no element in $\uparrow Q$ that is less than or equal to any element in $\downarrow Q \setminus Q$ and hence there is $v(e) \in Clop\mathcal{D}(X)$, with $e \in L$, such that $\downarrow Q \setminus Q \subseteq v(e)$ and $\uparrow Q \subseteq v(e)^c$.

Therefore, $\downarrow Q \subseteq v(e) \cup Q$ and, since Q is convex, $v(e) \cup Q$ is decreasing and clopen. Thus there is $h \in L$ such that $v(e) \cup Q = v(h)$. From here it follows that $\downarrow Q \subseteq v(g) \cap v(h) = v(g \wedge h)$.

Now $Q = \downarrow Q \cap \uparrow Q$ because Q is convex and $\downarrow Q \cap \uparrow Q \subseteq v(g \wedge h) \cap v(e)^c$.

To prove that $Q = v(g \wedge h) \cap v(e)^c$ let $p \in X$ be such that $p \in v(g \wedge h) \cap v(e)^c$. Then we have $p \in v(e) \cup Q$ because $p \in v(h)$ and $p \in v(e)^c$. Consequently $p \in Q$.

So $Q = v(g \wedge h) \cap v(e)^c$.

□

The characterization of the open subsets of the Priestley space of prime ideals, dual of a distributive lattice L, corresponding to a congruence $\theta(F)$, where F is a filter of L, was obtained in [31] Proposition 12.

We intend to determine the open subsets of $X = J^\infty(L^\sigma)$ that correspond to a congruence $\theta(F)$, as well as those corresponding to a congruence $\theta(I)$ where I is an ideal of L. For this purpose, we prove the following:

Lemma 5.1.4. *Let L be a distributive lattice and let I and F be an ideal and a filter in L, respectively. If A_I and A_F are the open subsets of $X = J^\infty(L^\sigma)$ such that, for any $a, b \in L$:*

$$(a,b) \in \theta(I) \qquad \textit{iff} \qquad v(a) \cap (X \setminus A_I) = v(b) \cap (X \setminus A_I),$$

$$(a,b) \in \theta(F) \qquad \textit{iff} \qquad v(a) \cap (X \setminus A_F) = v(b) \cap (X \setminus A_F),$$

then

(i) *A_I is the decreasing subset $\{p \in J^\infty(L^\sigma) : \exists f \in I \; p \leq f\}$.*

(ii) *A_F is the increasing subset $\{p \in J^\infty(L^\sigma) : \exists e \in F \; \kappa(p) \geq e\}$.*

Proof. (i) From Corollary 1.3.2 it follows that $\theta(I) = \bigvee_{f \in I} \theta(0,f)$ and consequently

$$A_I = \bigcup_{f \in I} v(f) = \{p \in J^\infty(L^\sigma) : \exists f \in I \; p \leq f\}.$$

(ii) In a similar way $\theta(F) = \bigvee_{e \in F} \theta(e,1)$. Therefore

$$A_F = \bigcup_{e \in F} v(e)^c = \{p \in J^\infty(L^\sigma) : \exists e \in F \; p \not\leq e\}.$$

But we know by Remark 3.1.8 that $p \not\leq e$ is equivalent to $\kappa(p) \geq e$ so,

$$A_F = \{p \in J^\infty(L^\sigma) : \exists e \in F \; \kappa(p) \geq e\}.$$

□

An ideal $I \subseteq L$ is a sublattice of L but, when $I \neq L$, $1 \notin I$. Anyway we can take the element $\bigvee I \in O(L^\sigma)$ to be the element 1_I so that the lattice $I \cup \{1_I\} \in \mathcal{DL}$. Then $J^\infty((I \cup \{1_I\})^\sigma)$ is the dual space of this lattice.

Dually, taking $\bigwedge F \in K(L^\sigma)$ to be the element 0_F, we have $F \cup \{0_F\} \in \mathcal{DL}$ and its dual space is $J^\infty((F \cup \{0_F\})^\sigma)$.

For the sake of simplicity we will use the following:

Definition 5.1.5. Let L be a distributive lattice, and let I and F be an ideal and a filter in L, respectively. We will denote by $D(I)$ the subset $J^\infty((I \cup \{1_I\})^\sigma)$ and by $D(F)$ the subset $J^\infty((F \cup \{0_F\})^\sigma)$.

It is not difficult to prove that the sets A_I and $D(I)$ are identical.

Dually

$$M^\infty((F \cup \{0_F\})^\sigma) = \{\kappa(p) \in M^\infty(L^\sigma) : \exists e \in F \; \kappa(p) \geq e\}$$

because $J^\infty((F \cup \{0_F\})^\sigma)$ is order isomorphic to $\kappa(J^\infty((F \cup \{0_F\})^\sigma)) = M^\infty((F \cup \{0_F\})^\sigma)$ by Theorem 3.1.7.

Therefore A_F is order isomorphic to $D(F)$.

Thus we have:

Lemma 5.1.6. *If I and F are, respectively, an ideal and a filter in a distributive lattice L, then A_I is order isomorphic to $D(I)$ and A_F is order isomorphic to $D(F)$.*

5.2 Congruences and duality in $SDMA$

We are going to study the lattice of congruences of a semi-De Morgan algebra L, $Con(L)$. To avoid any misunderstanding, we will always denote the lattice of congruences of the distributive lattice L by $Con_{latL}(L)$ and by $\theta_{latL}(a,b)$ with $a,b \in L$, the principal congruences of the distributive lattice.

We start by proving algebraically a lemma that was inspired by [6] Lemma 3.1 where the corresponding result was stated for p-algebras and for quasi-modular p-algebras.

Lemma 5.2.1. *Let L be an $SDMA$, $a,b,c \in L$, $b \leq c$ and $b' = c'$. Then*

(i) $\theta\,(b,c) = \theta_{latL}\,(b,c)$,

(ii) *(a)* $\theta\,(b,c) = \theta\,(b \vee b',\, c \vee b') \vee \theta\,(b \wedge b', c \wedge b')$,

(b) $\theta_{latL}\,(d,e)_{|[a]_\phi} = \theta_{lat[a]_\phi}\,(d,e)$ *for any* $d,e \in [a]_\phi$.

Proof. (i) Since $b' = c'$, we have $(b\,,\,c) \in \phi$, thus $\theta_{latL}\,(b,c) \leq \phi$ and consequently $\theta_{latL}\,(b,c) \in Con(L)$. Thus $\theta_{latL}\,(b,c) = \theta\,(b,c)$.

(ii) (a) Let $\delta = \theta\,(b \vee b',\, c \vee b') \vee \theta\,(b \wedge b', c \wedge b')$. It is clear that $\delta \leq \theta\,(b,c)$.

But $(b, (b \wedge c) \vee (b\, \wedge b')) \in \delta$, $((b \wedge c) \vee (b' \wedge c)\,, c) \in \delta$ and $((b \wedge c) \vee (b \wedge b')\,, (b \wedge c) \vee (c \wedge b')) \in \delta$. Thus $(b,c) \in \delta$ and $\theta\,(b,c) \leq \delta$.

(ii) (b) Since $d,e \in [a]_\phi$ we have $\theta_{lat[a]_\phi}\,(d,e) \leq \theta_{latL}\,(d,e)_{|[a]_\phi}$.

If $(f,g) \in \theta_{latL}\,(d,e)_{|[a]_\phi}$ then $f \wedge d = g \wedge d$ and $f \vee e = g \vee e$. Thus $(f,g) \in \theta_{lat[a]_\phi}\,(d,e)$. So $\theta_{latL}\,(d,e)_{|[a]_\phi} \leq \theta_{lat[a]_\phi}\,(d,e)$. □

As we proved in Theorem 4.2.18, $Con(L)$, where $L \in SDMA$, is order isomorphic to the order dual of the lattice of closed subsets of the dual space $D(L) = (J^\infty(L^\sigma), \leq, \tau, S)$ that are closed under S. It is obvious that the complements of these sets are open subsets that are closed under S^∂ so, as in the previous section, we will use the order-isomorphism between $Con(L)$ and the open subsets $A \subseteq J^\infty(L^\sigma)$ that are closed under S^∂.

Thus we have, as in Proposition 5.1.1,

Proposition 5.2.2. *Let $L \in SDMA$ and $X = J^\infty(L^\sigma)$. To any congruence $\theta \in Con(L)$ corresponds an open subset $A \subseteq X$ such that for any $p, q \in J^\infty(L^\sigma)$,*

$$(pSq \text{ and } q \in A) \Rightarrow p \in A$$

and, for any $a, b \in L$,

$$a\theta b \Leftrightarrow v(a) \cap (X \setminus A) = v(b) \cap (X \setminus A).$$

This correspondence is an order isomorphism.

As a consequence of Corollary 4.2.19, the open subset of $J^\infty(L^\sigma)$ corresponding to the congruence ϕ in the referred order-isomorphism is

$$X \setminus W = \{q \in J^\infty(L^\sigma) : \forall p \in J^\infty(L^\sigma) \ p \not S q\}.$$

where $p \not S q$, means that pSq is not true.

Furthermore, since every lattice congruence, $\theta \in Con_{latL}(L)$, such that $\theta \leq \phi$ is a congruence of the semi-De Morgan algebra L, the intervals $[\Delta, \phi] \subseteq Con(L)$ and $[\Delta, \phi] \subseteq Con_{latL}(L)$ coincide. So, to the elements $\theta \in Con(L)$ such that $\theta \leq \phi$, correspond the open subsets of $X \setminus W$.

It is now possible to characterize the open subsets of the dual space of a semi-De Morgan algebra that correspond, under duality, to principal congruences below ϕ.

From Theorem 5.1.3 and Lemma 5.2.1(i) it follows:

Lemma 5.2.3. *Let $L \in SDMA$ and θ a congruence less than or equal to ϕ. Then θ is a principal congruence of the semi-De Morgan algebra L if and only if the corresponding open subset of the dual space $D(L)$ is a clopen convex subset of $D(L)$ contained in $X \setminus W$.*

When L is a demi-p-lattice we denote by D_0 the congruence class $[0]_\phi = \{a \in L : a' = 1\}$ and by D_1 the congruence class $[1]_\phi = \{a \in L : a' = 0\}$. These classes are respectively an ideal and a filter.

It is clear that congruences $\theta(D_0)$ and $\theta(D_1)$ are below congruence ϕ. Therefore $\theta(D_0) = \theta_{latL}(D_0)$ and $\theta(D_1) = \theta_{latL}(D_1)$. According to Lemma 5.1.4 the open subsets of $J^\infty(L^\sigma)$ corresponding to these congruences are respectively a decreasing subset and an increasing subset that will be denote by A_0 and A_1.

Now, as in Definition 5.1.5, we can consider the lattices $D_0 \cup 1_{D_0}, D_1 \cup 0_{D_1} \in \mathcal{DL}$ and their dual spaces. By Lemma 5.1.6, we have:

Lemma 5.2.4. *Let L be a demi p-lattice, let A_0 be the decreasing subset*

$$\{p \in J^\infty(L^\sigma) : \exists f \in D_0 \; p \leq f\}$$

and let A_1 be the increasing subset

$$\{p \in J^\infty(L^\sigma) : \exists e \in D_1 \; \kappa(p) \geq e\}.$$

Then A_0 is order isomorphic to $D(D_0)$ and A_1 is order isomorphic to $D(D_1)$.

In [39] Theorem 2.5, Sankappanavar shows that, if L is a demi-p-lattice, $\phi = \theta_{latL}(D_0) \vee \theta_{latL}(D_1)$. Therefore, since the intervals $[\Delta, \phi] \subseteq Con(L)$ and $[\Delta, \phi] \subseteq Con_{latL}(L)$ coincide, we have by Lemma 5.1.4:

Lemma 5.2.5. *Let L be a demi-p-lattice. Then $X \setminus W = A_0 \cup A_1$.*

In an almost p-lattice, $D_0 = \{0\}$ (see [39], Theorem 2.2) so, as a consequence of the previous lemmas, we have:

Lemma 5.2.6. *Let L be an almost p-lattice. Then $X \setminus W = A_1$ and $X \setminus W$ is order isomorphic to $D(D_1)$.*

Now we will prove algebraically the extension to demi-p-lattices of a lemma proved for p-lattices by Beazer ([5] Lemma 3.2).

Lemma 5.2.7. *A congruence relation of a demi-p-lattice L is principal if and only if it is of the form $\theta(0, a) \vee \theta(d, e)$ for some $a \in B(L)$ and some $d, e \in L$ with $d \leq e$ and $d' = e'$.*

Proof. First we will show that, for any $b, c \in L$ with $b \leq c$,

$$\theta(b, c) = \theta(0, b' \wedge c'') \vee \theta(b \wedge (c \wedge b')', c \wedge (c \wedge b')').$$

Let ρ denote $\theta(0, b' \wedge c'') \vee \theta(b \wedge (c \wedge b')', c \wedge (c \wedge b')')$.

Note that $(b'', c'') \in \theta(b, c)$ so $(b' \wedge b'', b' \wedge c'') \in \theta(b, c)$ and, since L is a demi-p-lattice, $(0, b' \wedge c'') \in \theta(b, c)$. Thus $\theta(0, b' \wedge c'') \leq \theta(b, c)$.

It is obvious that $\theta(b \wedge (c \wedge b')', c \wedge (c \wedge b')') \leq \theta(b, c)$. Therefore $\rho \leq \theta(b, c)$.

Since $(0, b' \wedge c'') \in \rho$, we have by S6 $(1, (b' \wedge c'')') = (1, (b' \wedge c)') \in \rho$ and consequently $(b, b \wedge (c \wedge b')') \in \rho$ and $(c, c \wedge (c \wedge b')') \in \rho$. It follows from $(b \wedge (b' \wedge c)', c \wedge (b' \wedge c)') \in \rho$ that $(b, c) \in \rho$ and so $\theta(b, c) \leq \rho$.

Thus we have $\theta(b, c) = \rho$.

It is clear that $b' \wedge c'' \in B(L)$ and $b \wedge (c \wedge b')' \leq c \wedge (c \wedge b')'$.

To show that $(b \wedge (c \wedge b')')' = (c \wedge (c \wedge b')')'$, observe that by S6, S5 and S3 respectively,

$$(b \wedge (c \wedge b')')' = (b'' \wedge (c'' \wedge b')')' = (b' \vee (c'' \wedge b'))'' = b''' = b'$$

and that

$$\begin{aligned}
(c \wedge (c \wedge b')')' &= (c'' \wedge (c'' \wedge b')')' \qquad \text{by S6} \\
&= (c' \vee (c'' \wedge b'))'' \qquad \text{by S3} \\
&= ((c' \vee c'') \wedge (c' \vee b'))'' \qquad \text{by distributivity} \\
&= (c' \vee c'')'' \wedge (c' \vee b')'' \qquad \text{by S4} \\
&= (c'' \wedge b'')' \quad \text{because } L \text{ is a demi p-lattice and by S3} \\
&= (c \wedge b)' \qquad \text{by S6} \\
&= b'.
\end{aligned}$$

For the converse, suppose that $a \in B(L), d, e \in L$, $d \leq e$ and $d' = e'$. We will show that $\theta(0, a) \vee \theta(d, e) = \theta(a' \wedge d, a \vee e)$.

Let θ denote $\theta(0, a) \vee \theta(d, e)$. Since $(d, e) \in \theta$, we have $(a' \wedge d, a' \wedge e) \in \theta$. From $(0, a) \in \theta$ it follows that $(a', 1) \in \theta$ and $(a' \wedge e, e) \in \theta$. From $(0, a) \in \theta$ we have also $(e, a \vee e) \in \theta$.

Thus, by transitivity, $(a' \wedge d, a \vee e) \in \theta$, so $\theta(a' \wedge d, a \vee e) \leq \theta$.

For the reverse inclusion, observe that $(d \vee (a' \wedge d), d \vee a \vee e) = (d, a \vee e) \in \theta(a' \wedge d, a \vee e)$ and therefore $(d \wedge e, (a \vee e) \wedge e) = (d, e) \in \theta(a' \wedge d, a \vee e)$.

Thus $\theta(d, e) \leq \theta(a' \wedge d, a \vee e)$.

It follows from $a \in B(L)$ that $a \wedge a' = 0$ and, therefore, $(0, a) \in \theta(a' \wedge d, a \vee e)$.

Thus $\theta(0, a) \leq \theta(a' \wedge d, a \vee e)$.

It is now clear that $\theta \leq \theta(a' \wedge d, a \vee e)$. □

5.3 $SDMA$s having only principal congruences

Blyth and Varlet characterized in [9], via the poset of their join-irreducible elements, the distributive lattices and the De Morgan algebras having only principal congruences. They proved there that these structures are finite.

We know that for a finite distributive lattice L we have $L = L^\sigma$ so that the dual space of L is $D(L) = J^\infty(L^\sigma) = J(L)$.

The following two theorems were proved in [9]. We will include them here because they are fundamental for our study.

Denoting by h the height of a poset, we have:

Theorem 5.3.1 ([9], Theorem 1). *A distributive lattice L has only principal congruences if and only if L is finite and $h(J(L)) \leq 1$.*

Theorem 5.3.2 ([9] Theorem 3). *A De Morgan algebra L has only principal congruences if and only if L is finite and $h(J(L)) \leq 3$.*

To study algebras in $SDMA$ having only principal congruences, we start by proving the following:

Lemma 5.3.3. *If L is an $SDMA$ that has only principal congruences then so does $[a]_\phi$, for any $a \in L$.*

Proof. Let α be a congruence of $[a]_\phi$ and let us define a congruence $\bar{\alpha}$ of L by $\bar{\alpha} = \bigvee \{\theta(b,c) : (b,c) \in \alpha\}$. By hypothesis, $\bar{\alpha}$ is principal and so there exist $d, e \in L$ with $d \leq e$ such that $\bar{\alpha} = \theta(d,e)$. Clearly $\bar{\alpha} \leq \phi$ so we have $d' = e'$ and, by Lemma 5.2.1 (i), $\theta(d,e) = \theta_{latL}(d,e)$.

We will prove that there exist $f, g \in [a]_\phi$, with $f \leq g$ such that $\theta(d,e) = \theta(f,g)$.

Since $\bar{\alpha} = \theta(d,e) \leq \theta_{latL}([a]_\phi)$, by Lemma 1.3.1, there exist $m, n \in [a]_\phi$ with $m \leq n$ such that $d \wedge m = e \wedge m$ and $d \vee n = e \vee n$.

It is clear that $\theta_{latL}((d \wedge n) \vee m, (e \wedge n) \vee m) \leq \theta_{latL}(d,e)$.

Since

$$d \wedge ((d \wedge n) \vee m) = d \wedge n = e \wedge ((d \wedge n) \vee m)$$

and

$$d \vee ((e \wedge n) \vee m) = e \vee m = e \vee ((e \wedge n) \vee m)$$

we can conclude that

$$\theta_{latL}(d,e) \leq \theta_{latL}((d \wedge n) \vee m, (e \wedge n) \vee m).$$

Thus

$$\theta_{latL}(d,e) = \theta_{latL}((d \wedge n) \vee m, (e \wedge n) \vee m) = \theta_{latL}(f,g)$$

where $f = (d \wedge n) \vee m$ and $g = (e \wedge n) \vee m$.

But

$$((d \wedge n) \vee m)' = (d \wedge n)' \wedge m' = (d'' \wedge n'')' \wedge m' =$$
$$= (d'' \wedge a'')' \wedge a' = ((d \wedge a) \vee a)' = a'.$$

Analogously $((e \wedge n) \vee m)' = a'$ and therefore $f, g \in [a]_\phi$.

It follows

$$\bar{\alpha}_{|[a]_\phi} = \theta_{latL}(f,g)_{|[a]_\phi} = \theta_{lat[a]_\phi}(f,g)$$

by Lemma 5.2.1 (ii) (b).

Therefore,

$$\begin{aligned} \theta_{lat[a]_\phi}(f,g) &= \left(\bigvee \{\theta(b,c) : (b,c) \in \alpha\}\right)_{|[a]_\phi} \\ &= \left(\bigvee \{\theta_{latL}(b,c) : (b,c) \in \alpha\}\right)_{|[a]_\phi} \quad \text{by Lemma 5.2.1 (i)} \\ &= \bigvee \left\{\theta_{lat[a]_\phi}(b,c) : (b,c) \in \alpha\right\} \quad \text{by Lemma 1.3.3} \\ &= \alpha \end{aligned}$$

□

As a consequence of the Theorem 5.3.1 and Lemma 5.3.3 we have the following

Corollary 5.3.4. *Let L be an $SDMA$ which has only principal congruences. Then $[a]_\phi$ is a finite lattice and $h\left(J\left([a]_\phi\right)\right) \leq 1$, for any $a \in L$.*

For an algebra $L \in SDMA$, we know that the algebra $\left(DM(L), \dot{\vee}, \wedge, ', 0, 1\right)$, where $DM(L) = \{a \in L : a = a''\}$ and $a \dot{\vee} b$ is defined to be $(a' \wedge b')'$, is a DMA. Thus, from Theorem 5.3.2, it follows:

Corollary 5.3.5. *Let L be an $SDMA$ which has only principal congruences. Then $DM(L)$ has only principal congruences, it is finite and $h(J(DM(L))) \leq 3$.*

Proof. Since $DM(L)$ is an homomorphic image of L, $DM(L)$ has only principal congruences (see the proof of Lemma 3.3 in [6]). But $DM(L)$ is a DMA so, by Theorem 5.3.2, we have that $DM(L)$ is finite and $h(J(DM(L))) \leq 3$. □

As an immediate consequence of Corollary 5.3.4 and Corollary 5.3.5 we have:

Corollary 5.3.6. *Let L be an $SDMA$ which has only principal congruences. Then L is finite, $h\left(J\left([a]_\phi\right)\right) \leq 1$, for any $a \in L$ and $h\left(J\left(DM\left(L\right)\right)\right) \leq 3$.*

From now on we will use the previous results to determine the dual spaces of algebras $L \in SDMA$ having only principal congruences. For these algebras we have $D(L) = J(L)$ because we have just proved that they are finite.

Since we know, by Corollary 4.2.19, that for any finite $L \in SDMA$ the set $W \subseteq J(L)$ is the dual of the De Morgan algebra L/ϕ which is isomorphic to $DM(L)$, we have

Theorem 5.3.7. *Let L be an $SDMA$ which has only principal congruences. Then L is finite, $J(L) \setminus W$ is a convex subset of $J(L)$ and $h\left(J(L) \setminus W\right) \leq 1$. Furthermore $h\left(W\right) \leq 3$ and $h\left(J\left(L\right)\right) \leq 5$.*

Proof. Since every congruence less than or equal to ϕ is principal it follows from Lemma 5.2.3 that $J(L) \setminus W$ is a convex subset of $J(L)$ and $h\left(J(L) \setminus W\right) \leq 1$ because every subset of $J(L) \setminus W$ has to be convex (see Preliminaries, 1.1) .

From Corollary 4.2.19 it follows that $W = D(L/\phi)$. But, by Corollary 5.3.5, the DMA L/ϕ is a finite one and $h\left(J\left(L/\phi\right)\right) = h\left(D\left(L/\phi\right)\right) \leq 3$.

It is now immediate that $h\left(J\left(L\right)\right) \leq 5$. □

When we consider demi-p-lattices and almost p-lattices we can apply the characterization of their dual space obtained in Chapter 4. There we proved that a semi-De Morgan algebra L is a demi p-lattice if and only if for any $p \in W$, pSp (Theorem 4.3.3) and, as we noticed in Corollary 4.3.4, the subset $W \subseteq J^\infty(L^\sigma)$ is an antichain.

As a consequence we have:

Lemma 5.3.8. *Let L be a demi-p-lattice such that $X = D(L)$ has finite height. Then $h\left(X \setminus W\right) \leq h\left(X\right) \leq h\left(X \setminus W\right) + 1$.*

Now we can characterize demi p-lattices having only principal congruences:

Theorem 5.3.9. [1]

Let L be a demi-p-lattice. Then L has only principal congruences if and only if L is finite, $J(L) \setminus W$ is a convex subset of $J(L)$ and $h(J(L) \setminus W) \leq 1$.

[1] This theorem was joint work with Professor R. Santos

Proof. If a demi-p-lattice L has only principal congruences then, by Theorem 5.3.7, L is finite and $J(L) \setminus W$ is a convex set and $h\,(J(L) \setminus W) \leq 1$.

Conversely, assume that L is a demi-p-lattice such that L is finite and $h\,(J(L) \setminus W) \leq 1$ and $J(L) \setminus W$ is a convex set. Let B be a subset of $J(L)$ corresponding to a congruence. Then B is closed under S^{∂}, by Proposition 5.2.2, and we will show that B is a convex set.

Let $p, q, r \in J(L)$ be such that $p < q < r$, with $p, r \in B$. Since $h(W) = 0$, the set $J(L) \setminus W$ is convex and $h(J(L) \setminus W) \leq 1$, we have to consider the following two cases:

Case 1:
$p \in W$ and $q, r \in J(L) \setminus W$.

Since L is a demi-p-lattice we have pSp. Thus $q > pSp$ and, by H1 (i) in Definition 4.2.13, there is $u \in W$ such that $qSu \leq p$. Then $u = p$ because W is an antichain. Therefore, qSp and consequently $q \in B$ because B is closed for S^{∂}.

Case 2:
$r \in W$ and $p, q \in J(L) \setminus W$.

Then rSr because L is a demi p-lattice and there is $v \in W$ such that qSv.

Consequently $r > qSv$ and, by Definition 4.2.13 H1 (i), we must have $rSr \leq v$. Since W is an antichain, $r = v$ and hence qSr.

Therefore $q \in B$ because B is closed for S^{∂}. □

Corollary 5.3.10. *Let L be a demi-p-lattice. Then L has only principal congruences if and only if L is finite and all congruences less than or equal to congruence ϕ are principal congruences.*

Proof. The necessity of the finiteness follows from Theorem 5.3.9

Suppose that L is finite and all congruences less than or equal to the congruence ϕ are principal.

From Proposition 5.2.2, it follows that all subsets B of $J(L) \setminus W$ are closed under S^{∂} and, by Lemma 5.2.3, they are convex subsets of $J(L)$. Then $J(L) \setminus W$ is a convex subset of $J(L)$ and we must have $h(J(L) \setminus W) \leq 1$ so that all the subsets of $J(L) \setminus W$ are convex. Applying Theorem 5.3.9 it results that L has only principal congruences. □

When the algebra L is an almost p-lattice we know from Corollary 4.3.6 that W is the set of minimal elements of the dual space $X = D(L)$, so it follows:

Lemma 5.3.11. *Let L be an almost p-lattice such that $X = D(L)$ has finite height. Then $h(X) = h(X \setminus W) + 1$.*

As a consequence, we can extend to almost p-lattices [6], Corollary 3.9 proved by Beazer for p-lattices:

Theorem 5.3.12. *Let L be an almost p-lattice. Then the following are equivalent:*

(i) L has only principal congruences.

(ii) L is finite and $h(J(D_1)) \leq 1$.

(iii) L is finite and $h(J(L)) \leq 2$.

Proof. Assume (i) holds.

By Theorem 5.3.9, L is finite and $h(J(L) \setminus W) \leq 1$.

From Lemma 5.3.11, it follows that $h(J(L)) \leq 2$. So (i) implies (iii).

Now assume (iii). Since, by Corollary 4.3.6, W is the set of minimal elements of $J(L)$ we know that $h(J(L) \setminus W) \leq 1$ and $J(L) \setminus W$ is convex. Applying Theorem 5.3.9 we conclude that L has only principal congruences. Thus (iii) implies (i).

The equivalence between (ii) and (iii) is a direct consequence of Lemma 5.2.6 and the fact that W is the set of minimal elements of $J(L)$. □

5.4 The principal join property

Here we characterize those demi-p-lattices having the principal join property extending the corresponding results obtained for p-lattices by Beazer in [5]. There Beazer characterized the Priestley spaces of prime ideals that are the duals of such lattices. Since for any $L \in \mathcal{DL}$ this space is isomorphic to $J^\infty(L^\sigma)$, we will use, as in the previous sections, the dual spaces $D(L) = J^\infty(L^\sigma)$.

I. Chajda [13] studies algebras whose principal congruences form a sublattice of their congruence lattice. We also provide an answer to this problem in what concerns demi-p-lattices since we prove, generalizing a result of Beazer [5], that, in this equational class, those algebras having the principal join property have the principal intersection property.

Definition 5.4.1. We will say that an algebra L has the *principal join property*, abbreviated PJP, if the join of any two principal congruences on L is again a principal congruence.

An algebra L has the *principal intersection property*, PIP, if the intersection of any two principal congruences is a principal congruence.

In [1], Adams and Beazer characterized the distributive lattices with the PJP. They proved the following lemma that we include here because it is very important in what follows.

Lemma 5.4.2. *Let L be a distributive lattice. Then the following are equivalent:*

(i) L has the PJP.

(ii) There is no 3-element chain in the poset of prime ideals of L.

(iii) For any $d, e, f, g \in L$, with $d \leq e \leq f \leq g$ there exists $m, n \in L$ such that $d = e \wedge m, e \vee m = f \wedge n$ and $f \vee n = g$.

Since the poset of prime ideals of a distributive lattice L is order isomorphic to $J^{\infty}(L^{\sigma})$, (ii) is equivalent to:
(ii') There is no 3-element chain in $J^{\infty}(L^{\sigma})$.

Generalizing to $SDMA$s a proposition from [5] Theorem 3.4 we have:

Lemma 5.4.3. *Let L be a $SDMA$ having the PJP. If $a \in L$, then the sublattice $[a]_{\phi}$ of the lattice reduct of L has the PJP.*

Proof. Let L be a $SDMA$ having the PJP. Let d, e, f, g be elements of $[a]_{\phi}$ such that $d \leq e$ and $f \leq g$.

We want to show that $\theta_{lat[a]_{\phi}}(d, e) \vee \theta_{lat[a]_{\phi}}(f, g)$ is a principal congruence of the sublattice $[a]_{\phi}$.

Clearly $\theta(d, e) \leq \phi$ and $\theta(f, g) \leq \phi$. Thus $\theta(d, e) = \theta_{latL}(d, e)$ and $\theta(f, g) = \theta_{latL}(f, g)$.

Since L has the PJP there exist $k, l \in L$ with $k \leq l$ such that $\theta(d, e) \vee \theta(f, g) = \theta(k, l)$. But $\theta(k, l) \leq \phi$ so $k' = l'$ and $\theta(k, l) = \theta_{latL}(k, l)$.

Observe that $\theta(d, e) \vee \theta(f, g) \leq \theta_{latL}([a]_{\phi})$. So $\theta(k, l) \leq \theta_{latL}([a]_{\phi})$.

By Lemma 1.3.1, we know that there exist $m, n \in [a]_{\phi}$ with $m \leq n$ such that $k \wedge m = l \wedge m$ and $k \vee n = l \vee n$.

It is clear that $\theta_{latL}((k \wedge n) \vee m, (l \wedge n) \vee m) \leq \theta_{latL}(k, l)$.
Since

$$k \wedge ((k \wedge n) \vee m) = k \wedge n = l \wedge ((k \wedge n) \vee m)$$

and

$$k \vee ((l \wedge n) \vee m) = (l \wedge (k \vee n)) \vee m = l \vee m = l \vee ((l \wedge n) \vee m)$$

we conclude that $\theta_{latL}(k, l) \leq \theta_{latL}((k \wedge n) \vee m, (l \wedge n) \vee m)$.

Thus $\theta_{latL}(k, l) = \theta_{latL}((k \wedge n) \vee m, (l \wedge n) \vee m) = \theta_{latL}(h, j)$ where $h = (k \wedge n) \vee m$ and $j = (l \wedge n) \vee m$. Since $m' = n' = a'$, we have, by S3, S7 and S5, respectively,

$$((k \wedge n) \vee m)' = ((k \wedge n'') \vee m'')' = ((k \wedge a'') \vee a'')' = a''' = a'.$$

Analogously, $((l \wedge n) \vee m)' = a'$. Therefore $h, j \in [a]_\phi$.

Using Lemma 5.2.1 (ii)(b) and Corollary 1.3.3 we have:

$$\begin{aligned}
\theta_{lat[a]_\phi}(d, e) \vee \theta_{lat[a]_\phi}(f, g) &= \theta_{latL}(d, e)|_{[a]_\phi} \vee \theta_{latL}(f, g)|_{[a]_\phi} = \\
&= (\theta_{latL}(d, e) \vee \theta_{latL}(f, g))|_{[a]_\phi} \\
&= \theta_{latL}(h, j)|_{[a]_\phi} \\
&= \theta_{lat[a]_\phi}(h, j).
\end{aligned}$$

Thus $[a]_\phi$ has the PJP. □

From Lemma 5.4.3 and Lemmas 5.4.2, 5.2.5 and 5.2.4 we infer the following.

Lemma 5.4.4. *Let L be a demi-p-lattice having the PJP. Then there is no 3-element chain neither in $D(D_0)$ nor in $D(D_1)$. Equivalently, there is no 3-element chain in any of the subsets A_0 and A_1 of $D(L)$.*

and we can prove:

Lemma 5.4.5. *Let L be a demi-p-lattice. If L has the PJP then there exists no 3-element chain $x < y < z$ in $D(L)$ such that $x \in A_0$ and $z \in A_1$.*

Proof. Suppose that $x < y < z$ is a chain of elements of $J^\infty(L^\sigma)$ such that $x \in A_0$ and $z \in A_1$. Since $D(L) = J^\infty(L^\sigma)$ is a Priestley space and $x \not\geq y$ there is a clopen decreasing set $v(a)$ with $a \in L$ such that $x \in v(a)$ and $y \notin v(a)$.

We know that $\downarrow x$ and $X \setminus A_0$ are closed subsets of $D(L)$ and they are disjoint since $\downarrow x$ is decreasing, $X \setminus A_0$ is increasing and $x \in A_0$ so, by compactness and total disconnectedness, there is a clopen decreasing subset of $D(L), v(b)$ with $b \in L$ such that $\downarrow x \subseteq v(b)$ and $X \setminus A_0 \subseteq v(b)^c$. Therefore $v(b) \subseteq A_0$ and $x \in v(b)$. Consequently $x \in v(a) \cap v(b) = v(a \wedge b) \subseteq A_0$ because A_0 is decreasing and $y \notin v(a \wedge b)$ because $y \notin v(a)$.

Now, $v(a \wedge b)$ is the clopen decreasing (convex) subset corresponding to the principal congruence $\theta_{latL}(0, a \wedge b)$. Since $v(a \wedge b) \subseteq A_0$, we know that $\theta_{latL}(0, a \wedge b) = \theta(0, a \wedge b)$.

In a similar way, since $y \not\geq z$ there is a $v(c)$ with $c \in L$ such that $y \in v(c)$ and $z \in v(c)^c$ and, from the fact that $X \setminus A_1$ is a closed decreasing subset such that $z \in A_1$ there is a $v(d)$ with $d \in L$ such that $X \setminus A_1 \subseteq v(d)$ and $z \in v(d)^c$. Then $z \in v(c)^c \cap v(d)^c = v(c \vee d)^c \subseteq A_1$ because A_1 is increasing and $y \in v(c \vee d)$ because $y \in v(c)$.

Thus we obtained $v(c \vee d)^c$, the clopen increasing subset corresponding to the principal congruence $\theta_{latL}(c \vee d, 1) = \theta(c \vee d, 1)$.

It is clear that $v(a \wedge b) \cup v(c \vee d)^c$ is an open subset of $X \setminus W$ so it corresponds to a congruence of the demi-p-lattice below ϕ. It is not a convex subset because $x \in v(a \wedge b), z \in v(c \vee d)^c$ and $y \notin v(a \wedge b) \cup v(c \vee d)^c$. Therefore the corresponding congruence $\theta(0, a \wedge d) \vee \theta(c \vee d, 1)$ is not a principal congruence. A contradiction. □

In what follows we shall see that it is possible to characterize those algebras with the PJP in the variety of demi-p-lattices by the principal congruences that are less than or equal to ϕ.

Theorem 5.4.6. *Let L be a demi-p-lattice. Then L has the PJP if and only if the join of any two principal congruences less than or equal to ϕ is a principal congruence.*

Proof. $(\Rightarrow)$ is obvious.

$(\Leftarrow)$ Let $\theta(h, j)$ and $\theta(k, l)$ be two principal congruences of L such that $h \leq j$ and $k \leq l$. We wish to prove that $\theta(h, j) \vee \theta(k, l)$ is a principal congruence of L.

By Lemma 5.2.7 there exist $a, b \in B(L)$ and $d, e, f, g \in L$ such that $d \leq e, f \leq g, d' = e'$ and $f' = g'$ satisfying $\theta(h, j) = \theta(0, a) \vee \theta(d, e)$ and $\theta(k, l) = \theta(0, b) \vee \theta(f, g)$.

Then $\theta(h, j) \vee \theta(k, l) = \theta(0, a) \vee \theta(0, b) \vee \theta(d, e) \vee \theta(f, g)$.

First we will show that $\theta(0,a) \vee \theta(0,b) = \theta(0,(a \vee b)'')$. In fact, since $a, b \in B(L)$, $a = a'' \leq (a \vee b)''$ and $b = b'' \leq (a \vee b)''$, it is clear that

$$\theta(0,a) \vee \theta(0,b) \leq \theta(0,(a \vee b)'') \tag{5.4.1}$$

By [39] Corollary 3.6, we know that $\theta(0,b'') = \theta_{latL}(b',1)$.

Since $b = b''$, $(x,y) \in \theta(0,b)$ if and only if $x \wedge b' = y \wedge b'$. Therefore $(a,(a \vee b)'') \in \theta(0,b)$ because, by S4 and S5 and by distributivity, respectively,

$$(a \vee b)'' \wedge b' = ((a \vee b) \wedge b')'' = ((a \wedge b') \vee (b \wedge b'))'' = a'' \wedge b' = a \wedge b'.$$

Since $(0,a) \in \theta(0,a)$ and $(a,(a \vee b)'') \in \theta(0,b)$ we have $(0,(a \vee b)'') \in \theta(0,a) \vee \theta(0,b)$. Thus

$$\theta\left(0,(a \vee b)''\right) \leq \theta(0,a) \vee \theta(0,b). \tag{5.4.2}$$

From (5.4.1) and (5.4.2) it follows that $\theta(0,a) \vee \theta(0,b) = \theta\left(0,(a \vee b)''\right)$.

Now observe that $\theta(d,e) \leq \phi$ and $\theta(f,g) \leq \phi$ so, by the hypothesis, there exist $m, n \in L$ such that $\theta(d,e) \vee \theta(f,g) = \theta(m,n)$ with $m \leq n$. Since $\theta(m,n) \leq \phi$ we have $m' = n'$.

Then $\theta(h,j) \vee \theta(k,l) = \theta\left(0,(a \vee b)''\right) \vee \theta(m,n)$ where $(a \vee b)'' \in B(L)$, $m' = n'$ and $m \leq n$.

Applying Lemma 5.2.7 we conclude that $\theta(h,j) \vee \theta(k,l)$ is a principal congruence of L. □

Now we can characterize demi-p-lattices with the PJP.

Theorem 5.4.7. *Let L be a demi-p-lattice. Then the following are equivalent:*

(i) L has the PJP.

(ii) There is no 3-element chain in $X \setminus W$ and $X \setminus W$ is a convex subset of $D(L)$.

(iii) There exists no 3-element chain neither in $D(D_0)$ nor in $D(D_1)$, and there exists no 3-element chain $x < y < z$ in $D(L)$ such that $x \in A_0$ and $\kappa(z) \in A_1$.

Proof. (i)$\Longrightarrow$ (ii):

By Corollary 5.2.5 we know that $X \setminus W = A_0 \cup A_1$.

Now observe that if there is a chain $x < y < z$ with $x \in X \setminus W$, such that $x \in A_1$, then $y, z \in A_1$ because, by Lemma 5.2.5, A_1 is an increasing subset. This contradicts Lemma 5.4.4. Therefore, if $x \in X \setminus W$, $x \in A_0$ and $x \notin A_1$.

By Lemma 5.2.5 A_0 is a decreasing subset therefore, if $z \in A_0$, the 3-element chain would be in A_0 which is absurd by Lemma 5.4.4.

But, by Lemma 5.4.5, $z \notin A_1$, thus it is impossible to have a 3-element chain $x < y < z$ with $x, z \in X \setminus W$.

So we conclude that there is no 3-element chain in $X \setminus W$ and therefore $X \setminus W$ is a convex subset of $D(L)$.

(ii)$\Longrightarrow$(i) Now suppose that (ii) holds. In order to show that L has the PJP it is enough, by Theorem 5.4.6, to show that for any two principal congruences $\theta_1 \leq \phi$ and $\theta_2 \leq \phi$, $\theta_1 \vee \theta_2$ is principal.

Let A_{θ_1} and A_{θ_2} be the open subsets of $D(L)$ corresponding to θ_1 and θ_2 respectively. By Lemma 5.2.3 we know that A_{θ_1} and A_{θ_2} are clopen convex subsets of $D(L)$ contained in $X \setminus W$ so it is clear that $A_{\theta_1} \cup A_{\theta_2}$ is also a clopen subset of $D(L)$ and that it is a convex subset since it is contained in the convex subset $X \setminus W$ and there is no 3-element chain in $X \setminus W$. Therefore, by Lemma 5.2.3, the congruence $\theta_1 \vee \theta_2$ corresponding to $A_{\theta_1} \cup A_{\theta_2}$ is a principal congruence of the demi-p-lattice L.

(ii)$\Longleftrightarrow$ (iii) by Lemma 5.1.6 and Corollary 5.2.5. □

By Corollary 4.3.4, we know that, if L is a demi-p-lattice, the subset $W \subseteq D(L)$ is an antichain so we conclude

Corollary 5.4.8. *If L is a demi-p-lattice with the PJP then there is no 4-element chain in $D(L)$.*

Now we can extend to almost p-lattices the corresponding theorem stated for p-lattices with the PJP by Beazer ([5] Theorem 3.8).

Theorem 5.4.9. *Let L be an almost-p-lattice. Then the following are equivalent:*

(i) L has the PJP.

(ii) There is no 4-element chain in $D(L)$.

(iii) D_1 *has the PJP.*

(iv) *There is no 3-element chain in* $D(D_1)$.

(v) *For any* $d, e, f \in D_1$, *with* $d \leq e \leq f$ *there exists* $m, n \in D_1$ *such that* $d = e \wedge m, e \vee m = f \wedge n$ *and* $f \vee n = 1$.

Proof. (i) $\Longrightarrow$ (ii) by Corollary 5.4.8

(ii) $\Longrightarrow$ (i): When L is an almost p-lattice W is the set of minimal elements of $D(L)$, by Corollary 4.3.6. So, for almost p-lattices, the existence of a 4-element chain in $D(L)$ is equivalent to the existence of a 3-element chain in $X \setminus W$.

But, from [39] Theorem 2.5, $\phi = \theta(D_1)$. Then, in an almost-p-lattice, $X \setminus W$ is always a convex subset of $D(L)$ since $X \setminus W = A_1$.

Thus by Theorem 5.4.7 L has the PJP.

(i) $\Longleftrightarrow$ (iv) by Theorem 5.4.7, since $\phi = \theta(D_1)$ and consequently $X \setminus W = A_1$.

The equivalence of statements (iii), (iv) and (v) is given by Beazer [5] Lemma 3.5. □

5.5 The Principal Intersection Property

For p-lattices it is known ([5] Theorem 3.10) that any algebra having P.J.P has the PIP. The same is true for demi-p-lattices.

Theorem 5.5.1. *Let* L *be a demi-p-lattice. If* L *has the PJP, then* L *has the PIP.*

Proof. Let L be a demi-p-lattice with the PJP. Let $\theta(a, b)$ and $\theta(c, d)$ be elements of $Con(L)$. By Lemma 5.2.7, there exist $e, f \in B(L)$ and $g, h, i, j \in L$, with $g \leq h, i \leq j, g' = h'$ and $i' = j'$ such that, $\theta(a, b) = \theta(0, e) \vee \theta(g, h)$ and $\theta(c, d) = \theta(0, f) \vee \theta(i, j)$.

Then

$$\begin{aligned} \theta(a, b) \wedge \theta(c, d) = &(\theta(0, e) \wedge \theta(0, f)) \vee \\ &\vee (\theta(0, e) \wedge \theta(i, j)) \vee \\ &\vee (\theta(0, f) \wedge \theta(g, h)) \vee \\ &\vee (\theta(g, h) \wedge \theta(i, j)) . \end{aligned}$$

We have, by [39] Corollary 3.6, $\theta(0,e)\wedge\theta(0,f)=\theta_{latL}(e',1)\wedge\theta_{latL}(f',1)$. It is known that $\theta_{latL}(e',1)\wedge\theta_{latL}(f',1)=\theta_{latL}(e'\vee f',1)$.

We claim that $\theta_{latL}(e'\vee f',1)=\theta(e'\vee f',1)$.

In fact, again by [39] Corollary 3.6, $(x,y)\in\theta(e'\vee f',1)$ if and only if $x\wedge(e'\vee f')\wedge(e'\vee f')''=y\wedge(e'\vee f')\wedge(e'\vee f')''$.

By distributivity S5 and S9 we have

$$(e'\vee f')\wedge(e'\vee f')''=(e'''\wedge(e'\vee f')'')\vee(f'''\wedge(e'\vee f')'')=e'\vee f'$$

Thus $(x,y)\in\theta(e'\vee f',1)$ if and only if $x\wedge(e'\vee f')=y\wedge(e'\vee f')$ which is equivalent to $(x,y)\in\theta_{latL}(e'\vee f',1)$. Hence $\theta_{latL}(e'\vee f',1)=\theta(e'\vee f',1)$ and consequently $\theta(0,e)\wedge\theta(0,f)=\theta(e'\vee f',1)$.

From [39] Corollary 3.6 and from $i'=j'$ it is clear that $\theta(0,e)\wedge\theta(i,j)=\theta_{latL}(e',1)\wedge\theta_{latL}(i,j)$.

Since in a distributive lattice the PIP holds there is $k,l\in L$ such that $\theta_{latL}(e',1)\wedge\theta_{latL}(i,j)=\theta_{latL}(k,l)$ and $\theta_{latL}(k,l)=\theta(k,l)$ because $\theta_{latL}(k,l)\leq\theta_{latL}(i,j)\leq\phi$.

Therefore $\theta(0,e)\wedge\theta(i,j)=\theta(k,l)$.

Using the same arguments we can show that there exists $m,n\in L$ such that $\theta(0,f)\wedge\theta(g,h)=\theta(m,n)$.

In a similar way, since $g'=h'$ and $i'=j'$, we have $\theta(g,h)\wedge\theta(i,j)=\theta_{latL}(g,h)\wedge\theta_{latL}(i,j)$. But distributive lattices have the PIP therefore, there exist $r,s\in L$ such that $\theta_{latL}(g,h)\wedge\theta_{latL}(i,j)=\theta_{latL}(r,s)$ and $\theta_{latL}(r,s)=\theta(r,s)$ since $\theta_{latL}(r,s)\leq\theta_{latL}(i,j)\leq\phi$.

Hence all these meets are principal congruences of the demi-p-lattice L. It is now clear, since the PJP holds in L, that $\theta(a,b)\wedge\theta(c,d)$ is a principal congruence of L. So L has the PIP $\square$

Appendix A

Appendix

Generation trees of some Sahlqvist inequalities

$$\text{M3:}\quad \Box a \leq \rhd\rhd\, a$$

u.n $(+)\,\Box$
$(+)\,a$

$\rhd\,(-)$ **c.n**
$\rhd\,(+)$ **u.n**
$a\,(-)$

. . . .

$$\text{M4:}\quad \rhd\rhd\, a \leq \Box a$$

u.n $(+)\,\rhd$
c.n $(-)\,\rhd$
$(+)\,a$

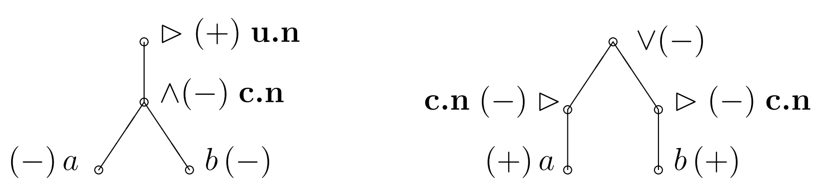
$\Box(-)$ **c.n**
$a\,(-)$

Observe that this inequality is not 1-Sahlqvist but, since the term $\rhd\rhd\, a$ is ∂-left Sahlqvist and the term $\Box a$ is ∂-right Sahlqvist the inequality is ∂ Sahlqvist.

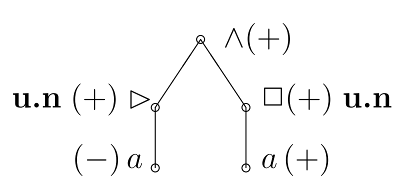
M5: $\rhd a \leq \Box \rhd a$

M6: $\Box \rhd a \leq \rhd a$

$\rhd (a \wedge b) \leq \rhd a \vee \rhd b$



$\rhd a \wedge \Box a \leq 0$



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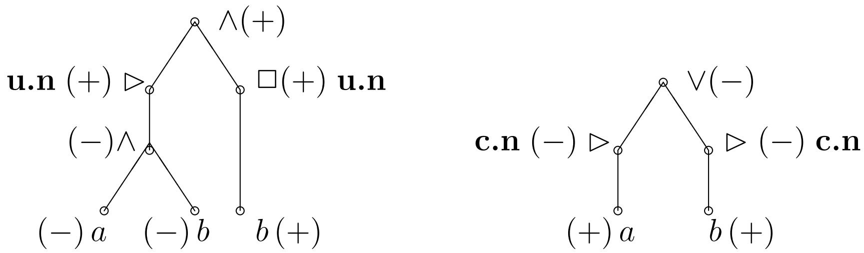
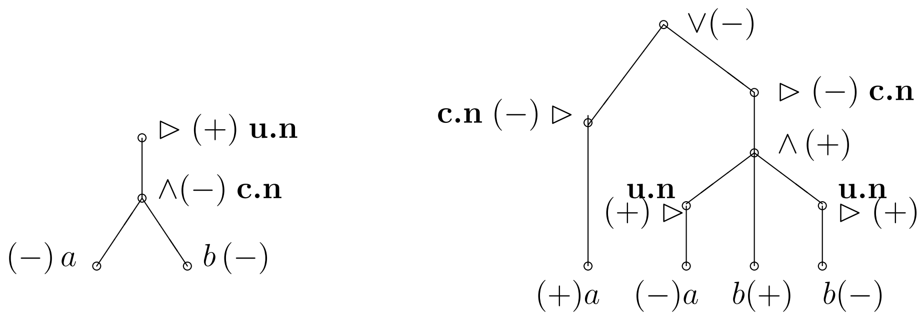
$$\rhd (a \wedge b) \wedge \Box b \leq \rhd a \vee \rhd b$$



$$\rhd (a \wedge b) \leq \rhd a \vee \rhd (\rhd a \wedge b \wedge \rhd b)$$



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