SECTION 11: KILLING HOMOTOPY GROUPS: POSTNIKOV AND WHITEHEAD TOWERS

In the previous section we used the technique of adjoining cells in order to construct CW approximations for arbitrary spaces. Here we will see that the same technique allows us to modify spaces by killing all homotopy groups above a certain dimension. This will be used to 'approximate' a connected space by a tower of spaces which have only non-trivial homotopy groups *below* or *above* a fixed dimension where they are isomorphic to the ones of the given space. The first case gives rise to the *Postnikov tower* and the second one to the *Whitehead tower*. Moreover, the homotopy groups of two subsequent levels in these towers only differ in one dimension. In fact, the maps belonging to the towers are fibrations and the fibers have precisely one non-trivial homotopy group.

We know that if $\alpha\colon \partial e^{n+1} \to X$ represents an element $[\alpha] \in \pi_n(X, x_0)$, then $[\alpha] = 0$ if and only if α extends to a map $e^{n+1} \to X$. Thus if we enlarge X to a space $X' = X \cup_\alpha e^{n+1}$ by adjoining an $(n+1)$-cell with α as attaching map, then the inclusion $i\colon X \to X'$ induces a map $i_*\colon \pi_n(X, x_0) \to \pi_n(X', x_0)$ with $i_*[\alpha] = 0$. We say that $[\alpha]$ 'has been killed'. (Naively, we think of X' as a smallest extension of X that does that. Some justification for this thinking will be provided in the exercises.) The following lemma expresses what happens to the homotopy groups in lower dimensions. The proof is similar to the one that the inclusion of the n-skeleton of a CW complex is an n-equivalence and will hence not be given.

Lemma 1. *Let (X, x_0) be a pointed space, and let $X' = X \cup_\alpha e^{n+1}$ be obtained from X by adjoining an $(n+1)$-cell. Then the inclusion $i\colon X \to X'$ induces a map $\pi_k(X, x_0) \to \pi_k(X', x_0)$ which is an isomorphism for $k < n$ and surjective for $k = n$.*

It is difficult to control what happens to the higher homotopy groups. For example, since $\pi_3(S^2)$ is non-trivial, adding a 2-cell to an element in π_1 may well add elements in π_3. However, we can 'kill' all of π_n without changing π_k for $k < n$, by iterating the procedure of Lemma 1.

Lemma 2. *Let (X, x_0) be a pointed space. Then there exists a relative CW complex $i\colon X \to Y$, constructed by adjoining $(n+1)$-cells only, such that $i_*\colon \pi_k(X, x_0) \to \pi_k(Y, y_0)$ is bijective for $k < n$ and such that $\pi_n(Y, y_0) = 0$.*

Proof. Let A be a set of representatives α of generators $[\alpha]$ of the group $\pi_n(X, x_0)$. Let Y be obtained from X by attaching an $(n+1)$-cell e^{n+1}_α along $\alpha\colon \partial e^{n+1}_\alpha \to X$ for each $\alpha \in A$:

$$\begin{array}{ccc} A \times \partial e^{n+1} & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle i} \\ A \times e^{n+1} & \longrightarrow & Y. \end{array}$$

Then by an iterated application of Lemma 1, the map $i\colon X \to Y$ induces isomorphisms in π_k for $k < n$, and induces the zero-map on π_n. Since this map is also surjective, we conclude that $\pi_n(Y)$ has to vanish. □

For the proof of the next theorem, recall that any map $f\colon U \to V$ can be factored as $f = p \circ \phi$,

$$f\colon U \xrightarrow{\phi} P(f) \xrightarrow{p} V,$$

where p is a Serre fibration and ϕ is a homotopy equivalence ('mapping fibration', see Section 5). We say that (up to homotopy), any map 'can be turned into a fibration'.

Theorem 3. ***(Postnikov tower)*** *For any connected space X, there is a 'tower' of fibrations*

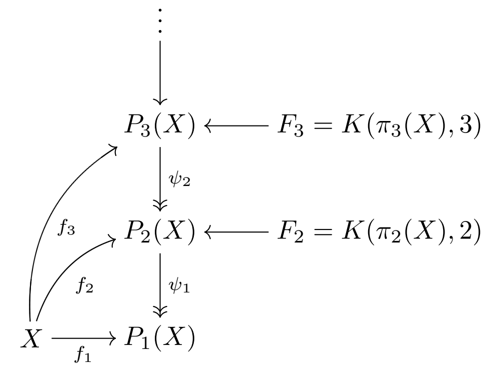
$$P_1(X) \xleftarrow{\psi_1} P_2(X) \xleftarrow{\psi_2} P_3(X) \longleftarrow \cdots$$

and compatible maps $f_i\colon X \to P_i(X)$ (compatible in the sense that $\psi_n \circ f_{n+1} = f_n\colon X \to P_n(X)$), with the following properties:

(1) *$\pi_k(P_n(X)) = 0$ for $k > n$*
(2) *$\pi_k(X) \to \pi_k(P_n(X))$ is an isomorphism for $k \leq n$ (and hence so is $\pi_k P_n(X) \to \pi_k P_{n-1}(X)$ for $k < n$)*
(3) *The fiber F_n of ψ_{n-1} has the property that $\pi_n(F_n) \cong \pi_n(X)$ and $\pi_k(F_n) = 0$ for all $k \neq n$.*

Remark 4. A space like this fiber F_n with only one non-trivial homotopy group is called an **Eilenberg-MacLane space**. If Z is such a space with $\pi_k(Z) = 0$ for all $k \neq n$ and $\pi_n(Z) \cong A$, one says that Z is a $K(A, n)$-space (strictly speaking one always means the space Z together with a chosen isomorphism $\pi_n(Z) \cong A$). We will discuss these spaces in more detail in a later lecture.

With this terminology the situation of the theorem can be depicted as follows



where we used $\longrightarrow\!\!\!\!\!\rightarrow$ ($\twoheadrightarrow$) to denote a fibration.

Proof. (of Theorem 3) Let $i_n\colon X \to Y_n$ be a space obtained from X by killing $\pi_k(X)$ for all $k > n$, i.e., such that

(1) $(i_n)_*\colon \pi_k(X) \to \pi_k(Y_n)$ is an isomorphism for all $k \leq n$.
(2) $\pi_k(Y_n) = 0$ for all $k > n$.

Such a space Y_n can be obtained by repeated application of the procedure of Lemma 2,

$$X \subseteq Y_n^{(n+1)} \subseteq Y_n^{(n+2)} \subseteq \ldots$$

where $Y_n^{(n+1)}$ kills $\pi_{n+1}(X)$ by adjoining $(n+2)$-cells, $Y_n^{(n+2)}$ kills $\pi_{n+2}(Y_n^{(n+1)})$ by adjoining $(n+3)$-cells to $Y_n^{(n+1)}$, and so on. The resulting space $Y_n = \bigcup_{m>n} Y_n^{(m)}$, the union endowed with the weak topology, has the desired property, as is immediate from the fact that any map $K \to Y_n$ with K compact (e.g., $K = S^k$ or $K = S^k \times [0,1]$) must factor through some $Y_n^{(m)}$. If you see what this construction does, then it is clear that there is a canonical inclusion $\phi_n \colon Y_{n+1} \to Y_n$ making the following diagram commute (we need to adjoin 'more cells' for Y_n than for Y_{n+1}):

$$\begin{array}{ccc} & \overset{i_{n+1}}{\nearrow} & Y_{n+1} \\ & & \downarrow{\scriptstyle \phi_n} \\ X & \xrightarrow[i_n]{} & Y_n \end{array}$$

Thus, X is 'approximated' by smaller and smaller relative CW complexes

$$X \subseteq \ldots \subseteq Y_{n+1} \subseteq Y_n \subseteq \ldots \subseteq Y_2 \subseteq Y_1.$$

Now let $P_1(X) = Y_1$, and let $f_1 \colon X \to P_1(X)$ be $i_1 \colon X \to P_1(X)$. Let $P_2(X)$ be the space fitting into a factorization of

$$Y_2 \xrightarrow{\phi_1} Y_1 \xrightarrow{\text{id}} P_1(X)$$

into a homotopy equivalence j_2 followed by a fibration ψ_1. Next factor $j_2\phi_2$ in a similar way as $\psi_2 j_3$, and so on, all fitting into a diagram

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i_3} & Y_3 & \xrightarrow[\simeq]{j_3} & P_3(X) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \psi_2} \\ X & \xrightarrow{i_2} & Y_2 & \xrightarrow[\simeq]{j_2} & P_2(X) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \psi_1} \\ X & \xrightarrow[i_1]{} & Y_1 & \xrightarrow[=]{} & P_1(X). \end{array}$$

Write $f_n \colon X \to P_n(X)$ for the composition $j_n i_n$, and denote the fiber of $\psi_{n-1} \colon P_n(X) \to P_{n-1}(X)$ by $F_n \subseteq P_n(X)$.

Now let us look at the homotopy groups. By construction we have (1) and (2) above, and hence the same is true for $P_n(X)$ instead of Y_n:

(1) $(f_n)_* \colon \pi_k(X) \to \pi_k(P_n(X))$ is an isomorphism for all $k \leq n$.
(2) $\pi_k(P_n(X)) = 0$ for all $k > n$.

We can feed this information in the long exact sequence of the fibration $F_n \subseteq P_n(X) \overset{\psi_{n-1}}{\to} P_{n-1}(X)$, a part of which looks like

$$\ldots \to \pi_{k+1}(P_n) \to \pi_{k+1}(P_{n-1}) \to \pi_k(F_n) \to \pi_k(P_n) \to \pi_k(P_{n-1}) \to \ldots$$

where for simplicity we write P_n for $P_n(X)$, and omit all base points from the notation. So, we clearly have:

(1) For $k > n$, the group $\pi_k(F_n)$ lies between two zero groups, hence is itself the zero-group.
(2) For $k < n$, the group $\pi_k(F_n)$ lies between a surjection and an isomorphism,

$$\bullet \twoheadrightarrow \bullet \longrightarrow \pi_k(F_n) \longrightarrow \bullet \xrightarrow{\cong} \bullet\ ,$$

hence is zero again.

(3) For $k = n$, the relevant part of the sequence looks like

$$0 \to 0 \to \pi_n(F_n) \to \pi_n(P_n) \to 0$$

whence $\pi_n(F_n)$ is isomorphic to $\pi_n(P_n) \cong \pi_n(X)$.

This tells us that F_n is a $K(\pi_n(X), n)$-space and hence proves the theorem. □

Remark 5. Much more can be said about these Postnikov towers: under some conditions, the fibration $P_n \to P_{n-1}$ is even a fiber bundle.

The Postnikov tower builds up the homotopy groups of X (together with all relations between them, such as the action of π_1 on π_n) 'from below', by constructing for each n a space with homotopy groups $\pi_1, \ldots, \pi_n$ only. There is also a construction 'from above', called the **Whitehead tower** of X, as described in the following theorem.

Theorem 6. *(**Whitehead tower**) Let X be a connected space. There exists a tower*

$$X \longleftarrow W_1(X) \longleftarrow W_2(X) \longleftarrow W_3(X) \longleftarrow \cdots$$

with the following properties:

(1) $\pi_k(W_n(X)) = 0$ *for* $k \leq n$
(2) *The map* $\pi_k(W_n(X)) \to \pi_k(X)$ *is an isomorphism for all* $k > n$.
(3) *The map* $W_n(X) \to W_{n-1}(X)$ *is a fibration whose fiber is a* $K(\pi_n(X), n-1)$*-space.*

Proof. As in the proof of the Postnikov tower, X can be approximated by extensions

$$X \subseteq \ldots \subseteq Y_{n+1} \subseteq Y_n \subseteq \ldots \subseteq Y_2 \subseteq Y_1.$$

where $\pi_k(Y_n) = 0$ for $k > n$ and $\pi_k(X) \to \pi_k(Y_n)$ is an isomorphism for $k \leq n$. For $X \subseteq Y$, let $\bar{W}_n(X)$ be the space of paths in Y_n from the base point to X, as in the pullback

$$\begin{array}{ccc} \bar{W}_n(X) & \longrightarrow & Y_n^{[0,1]} \\ \downarrow & & \downarrow \\ X \cong 1 \times X & \xrightarrow[x_0 \times i_n]{} & Y_n \times Y_n \end{array}$$

So $\bar{W}_n(X) \to X$ is a fibration. (Remember we used this fibration to describe relative homotopy groups of the pair (Y_n, X) in the exercises to Section 4.) These spaces fit naturally into a sequence

$$X \longleftarrow \bar{W}_1(X) \xleftarrow{\supseteq} \bar{W}_2(X) \xleftarrow{\supseteq} \bar{W}_3(X) \xleftarrow{\supseteq} \cdots$$

Now turn these inclusions into fibrations (by factoring into a homotopy equivalence followed by a fibration as before) to obtain a diagram

$$\begin{array}{ccccccccc} X & \longleftarrow & \bar{W}_1(X) & \xleftarrow{\supseteq} & \bar{W}_2(X) & \xleftarrow{\supseteq} & \bar{W}_3(X) & \xleftarrow{\supseteq} & \cdots \\ = \downarrow & & = \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\ X & \twoheadleftarrow & W_1(X) & \twoheadleftarrow & W_2(X) & \twoheadleftarrow & W_3(X) & \twoheadleftarrow & \cdots \end{array}$$

where the lower horizontal maps are all fibrations and the vertical ones are homotopy equivalences.

Now let us look at the homotopy groups: We know $\pi_k(\bar{W}_n X) \cong \pi_k(W_n X)$, and there are two fibrations to play with, viz $\bar{W}_n(X) \to X$ and $W_n(X) \to W_{n-1}(X)$. The fiber of the first one is the loop space ΩY_n of Y_n, and the fiber of the second one will be denoted G_n. Then the long exact sequence of $\bar{W}_n(X) \to X$ looks like

$$\cdots \longrightarrow \pi_k(\Omega Y_n) \longrightarrow \pi_k(\bar{W}_n X) \longrightarrow \pi_k(X) \longrightarrow \pi_{k-1}(\Omega Y_n) \longrightarrow \cdots$$

or equivalently

$$\cdots \longrightarrow \pi_{k+1}(Y_n) \longrightarrow \pi_k(\bar{W}_n X) \longrightarrow \pi_k(X) \longrightarrow \pi_k(Y_n) \longrightarrow \cdots$$

But $\pi_k(Y_n) = 0$ for $k > n$ and $\pi_k(X) \to \pi_k(Y_n)$ is an isomorphism for $k \leq n$, so

$$\pi_k(\bar{W}_n(X)) \cong \pi_k(X), \quad k > n, \qquad \text{and} \qquad \pi_k(\bar{W}_n) = 0, \quad k \leq n,$$

and hence the same is true for W_n instead of $\bar{W}_n$. Next, the long exact sequence associated to $W_n(X) \to W_{n-1}(X)$ looks like

$$\cdots \longrightarrow \pi_{k+1}(W_n) \longrightarrow \pi_{k+1}(W_{n-1}) \longrightarrow \pi_k(G_n) \longrightarrow \pi_k(W_n) \longrightarrow \pi_k(W_{n-1}) \longrightarrow \cdots$$

(where we write W_n for $W_n(X)$, etc), and we notice:

(1) if $k > n$ then $\pi_k(G_n)$ is squeezed in between two isomorphisms, so $\pi_k(G_n) = 0$.
(2) if $k \leq n-2$ then $\pi_k(G_n)$ sits between two zero groups hence is zero itself.
(3) for $k = n$ we obtain $\pi_{n+1}(W_n) \longrightarrow \pi_{n+1}(W_{n-1}) \longrightarrow \pi_n(G_n) \longrightarrow 0$ and the first map is an isomorphism so that $\pi_n(G_n) = 0$.
(4) in the remaining case $k = n-1$ the sequence looks like $0 \to \pi_n(W_{n-1}) \to \pi_{n-1}(G_n) \to 0$, so that we have an isomorphism $\pi_n(X) \cong \pi_n(W_{n-1}) \cong \pi_{n-1}(G_n)$.

Thus, this tells us that G_n is a $K(\pi_n(X), n-1)$-space. □

Note that the spaces $\bar{W}_n(X)$ used in the proof of the Whitehead tower are precisely the *homotopy fibers* of the maps $i_n\colon X \to Y_n$ constructed in the proof of the Postnikov tower. The remaining work in the proof of Theorem 6 then consists of turning a certain sequence of maps between the homotopy fibers in a sequence of fibrations and analyzing what happens at the level of homotopy groups. This observation is sometimes referred to by saying that the Whitehead tower is obtained from the Postnikov tower 'by passing to homotopy fibers'.