GALOIS THEORY 2015/2016 EXERCISE SHEET 1

Exercises marked with a star are entirely optional, and more designed for general pondering.

(1) (a) Let K be a subfield of $\mathbb{C}$ such that $[K : \mathbb{Q}] = 2$. Show that there exists a squarefree $d \in \mathbb{Z}$ with $d \neq 0, 1$ such that $K = \mathbb{Q}(\sqrt{d})$ (here *squarefree* means that for all n in $\mathbb{Z}_{>0}$, $n^2 | d$ implies $n = 1$).
   (b) Let d_1 and d_2 be squarefree integers $\neq 0, 1$. Suppose $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$. Show that $d_1 = d_2$.

(2) Let $L|K$. Let α and β be elements of L.
   (a) Show that if α and β are algebraic over K, then $\alpha + \beta$ and $\alpha\beta$ are algebraic over K.
   (b) Show that if $L|K$ is an extension of fields, then the subset of elements α in L which are algebraic over K is a field.
   (c) Now suppose the characteristic of K is not 2. Show that if $\alpha\beta$ and $\alpha + \beta$ are algebraic over K, then α and β are algebraic over K.
   (d) * What if the characteristic of K is 2?

(3) (a) Write down all elements of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\zeta_5), \mathbb{C})$.
   (b) Write down an element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{C}, \mathbb{C})$ that doesn't lie in $\mathrm{Hom}_{\mathbb{Q}(i)}(\mathbb{C}, \mathbb{C})$.
   (c) Let K denote the subextension of $\mathbb{C}|\mathbb{Q}$ consisting of elements α algebraic over $\mathbb{Q}$, as in question 2. Can you write down an element of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ that doesn't lie in $\mathrm{Hom}_{\mathbb{Q}(\sqrt{3})}(K, \mathbb{C})$?
   (d) * Can you write down an element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{C}, \mathbb{C})$ that doesn't lie in $\mathrm{Hom}_{\mathbb{Q}(\sqrt{3})}(\mathbb{C}, \mathbb{C})$?

(4) * Let $G : K[T] \to K((T))$ be the ring homomorphism sending a polynomial $f(T) = \sum_{i=0}^{n} a_i T^i$ to the power series $\sum_{i=0}^{\infty} a_i T^i$ (where $a_i := 0$ for $i > n$). Show that G extends to a homorphism of fields $K(T) \to K((T))$. Show that $[K((T)) : K(T)]$ is infinite (hint: when does a power series come from a fractional polynomial?).

Comments, corrections, questions etc to netandogra@gmail.com.