GALOIS THEORY 2015/2016 EXERCISE SHEET 10

Questions marked with an asterisk are optional.

(1) Let $L_1|K$ and $L_2|K$ be Galois extensions with groups G_1 and G_2. Show that $L_1L_2|K$ is Galois, and show that there is an injective group homomorphism

$$\mathrm{Gal}(L_1L_2|K) \to G_1 \times G_2.$$

(2) Let K be a field of characteristic zero. Let $K(T_0, \ldots, T_{n-1})$ be the field of fractions of a polynomial algebra in n variables. Let $F(X)$ be the polynomial $X^n + \sum T_i X^i$, and let L be its splitting field. Let $B_1, \ldots B_n$ denote the roots of F. Let

$$\rho : S_n \xrightarrow{\simeq} \mathrm{Gal}(L|K)$$

be the isomorphism discussed in lectures sending σ to the automorphism sending B_i to $B_{\sigma(i)}$. Write down a generator of the subextension L' corresponding to the subgroup A_n.

(3) Let G be a finite group, and let $[G, G]$ be the set of elements of G of the form $ghg^{-1}h^{-1}$.
   (a) Show that $[G, G]$ is a normal subgroup of G.
   (b) Show that if $\alpha : G \to A$ is a group homomorphism and A is abelian, then $[G, G]$ is contained in the kernel of α.
   (c) Deduce that if $[G, G] = G$ and $|G| > 1$, then G is not solvable.

Comments, corrections, questions etc to netandogra@gmail.com.