GALOIS THEORY 2015/2016 EXERCISE SHEET 2

(1) Define the polynomials $\Phi_n(X) \in \mathbb{Q}(X)$ inductively as follows. $\Phi_1(X)$ is defined to be $X-1$. For $n > 1$, define $\Phi_n(X)$ by
$$\Phi_n(X) := \frac{X^n - 1}{\prod \Phi_d(X)}$$
where the product is over all integers $1 \leq d < n$ such that d divides n.
  (a) Show that $\Phi_n(X)$ is in $\mathbb{Q}[X]$ and that its roots in $\mathbb{C}$ are exactly the primitive nth roots of unity.
  (b) Let ζ be a root of $\Phi_n(X)$. Calculate the size of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\zeta), \mathbb{C})$.
  (c) Deduce that the polynomial $\Phi_n(X)$ is irreducible over $\mathbb{Q}$.

(2) Let ζ be a primitive 7th root of unity in $\mathbb{C}$.
  (a) Show that
$$\alpha := \zeta + \zeta^2 + \zeta^4$$
  is a root of the polynomial $X^2 + X + 2$.
  (b) Suppose that, over $\mathbb{Q}(\alpha)$, $\Phi_7(X)$ factorises as
$$\Phi_7(X) = \prod_{i=1}^{n} F_i(X),$$
  where $F_i(X)$ is a monic irreducible polynomial in $\mathbb{Q}(\alpha)[X]$ of degree $d_i > 0$. What is n, and what are $d_1, \ldots, d_n$?

(3) For each of the following fields K and polynomials $f(X)$ in $K[X]$, find the degree $[L : K]$ of a splitting field L, and the size of the group $\mathrm{Hom}_K(L, L)$.
  (a) $K = \mathbb{C}(T)$, $f(X) = X^n - T$.
  (b) $K = \mathbb{Q}(\zeta_3)$, $f(X) = X^3 - 2$.
  (c) $K = \mathbb{Q}$, $f(X) = X^3 - 2$.
  (d) $K = \mathbb{F}_p(T)$, $f(X) = X^p - T$.

Comments, corrections, questions etc to netandogra@gmail.com.