GALOIS THEORY 2015/2016 EXERCISE SHEET 5

(1) Do the following extensions of fields exist? In each case, either give an example or show non-existence.
   (a) A finite extension $L|K$ which is normal but not separable.
   (b) A finite extension $L|K$ which is separable but not normal.
   (c) A finite extension $L|K$, and a subextension F, such that $L|K$ is separable but $L|F$ is not separable.
   (d) A finite extension $L|K$, and a subextension F, such that $L|K$ is separable but $F|K$ is not separable.
   (e) A finite extension $L|K$, and a subextension F, such that $L|K$ is normal but $L|F$ is not normal.
   (f) A finite extension $L|K$, and a subextension F, such that $L|K$ is normal but $F|K$ is not normal.
   (g) A finite extension $L|K$, and a subextension F, such that $F|K$ and $L|F$ are normal but $L|K$ is not normal.
   (h) A finite extension $L|K$ which is neither normal nor simple.

(2) Let $L|K$ be a finite extension. Let F_1 and F_2 be subextensions.
   (a) Suppose F_1 and F_2 are separable. Show that F_1F_2 is separable.
   (b) Suppose F_1 and F_2 are normal. Show that F_1F_2 is normal.

(3) Let $p > 2$ be an odd prime number.
   (a) Show that $f(X) = X^4 + 1$ is not irreducible in $\mathbb{F}_p[X]$. Find the irreducible factors of $f(X)$ when $p = 3, 5$ and 17.
   (b) For which p is $X^2 + 1$ irreducible in $\mathbb{F}_p[X]$?

Comments, corrections, questions etc to netandogra@gmail.com.