AN INTRODUCTION TO C*-ALGEBRAS FROM SYMBOLIC DYNAMICS

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ABSTRACT. This is a crash course into the realm of operator algebraic tecniques in the study of symbolic dynamics, focusing on a pedagogical example, that of Cuntz and Cuntz-Krieger algebras.

Contents

1. Introduction

The paradigm/philosophy of noncommutative geometry is the idea of using the language of operator algebras to study objects coming from different fields in mathematics hoping to get some insight from invariants coming from their associated C*-algebras.

One starts from an *input*, typically a dynamical object, like a dynamical system or a graph, or a geometric object, like a foliation or a singular space, and associates to this object a C*-algebra that encodes *all the relevant information*. By looking at the properties of this algebra and its invariants, e.g. its K-theory, we want to get information that we can relate back to the input [1].

Very often, the first passage in this receipe involves passing through an intermediate topological object, a groupoid.

This talk will hopefully give you a hint of what happens in the first passage in the case of dynamical systems. We will consider a particular class of dynamical systems, namely shifts and subshifts of finite type, and we will construct, using a groupoid model, their corresponding C*-algebras, that go under the name of Cuntz [1] and Cuntz-Krieger [2] algebras.

2. Shifts and subshifts of finite type

2.1. A class of Markov chains.

Roughly, a Markov chain is a model describing a sequence of possible events in which the probability of each event only depends on the state obtained in the previous event.

Start from an $N \times N$ matrix $\mathrm{A} := \{A(i,j)\}_{i,j=1}^{N}$ with entries zero and one and with no row nor column equal to zero.

The matrix is undestood as to define paths in a discrete Markov chain. A jump from i to j is admissible if and only if $A(i,j) \neq 0$. In that case, the probability is given by 1 over the number of entries different from zero in the i-th row.

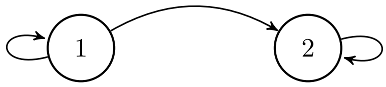
The assumption that no row nor column is equal to zero ensures that there is always at least one jumb out of any letter $i \in \{1, \ldots, N\}$.

[1]As I learnt from R. Deeley (Hawaii) during the Matrix School *Refining C*-algebraic invariants from dynamics using KK-theory*, this algorithm has the nice name of *C*-game*. I assume this name is his invention.

2.1.1. *Some examples.* The first case we consider is the 2×2 matrix

$$\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1}$$

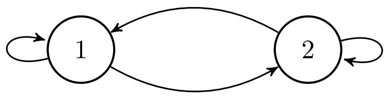
which corresponds to the following Markov chain:



Another interesting example is given by the 2×2 matrix

$$\mathrm{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2}$$

which corresponds to



Finally, to the 3×3 matrix given by

$$\mathrm{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{3}$$

one associates the Markov chain

2.2. **The space of infinite words and the shift operator.** We will construct a dynamical system out of these ingredients. The first space we can look at is the space of infinite admissible words:

$$\Omega_{\mathrm{A}} := \{(x_k)_{k\in\mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid \forall k \in \mathbb{N}, A(x_k, x_{k+1}) = 1\}.$$

We give Ω_{A} the topology induced by the compact product topology on $\{1, \dots, N\}^{\mathbb{N}}$. With respect to this topology Ω_{A} is totally disconnected.

Next, we define a shift operator on Ω_{A}:

$$\sigma : \Omega_{\mathrm{A}} \to \Omega_{\mathrm{A}} \qquad (x_k)_{k\in\mathbb{N}} \mapsto (x_{k+1})_{k\in\mathbb{N}}. \tag{4}$$

The resulting dynamical system is named a *subshift of finite type.* If the matrix $\mathrm{A} \equiv 1$, then we talk of *shifts of finite type.*

The space Ω_{A} admits a realisation as a boundary of a tree made of finite admissible words. A finite word of length M is a collection

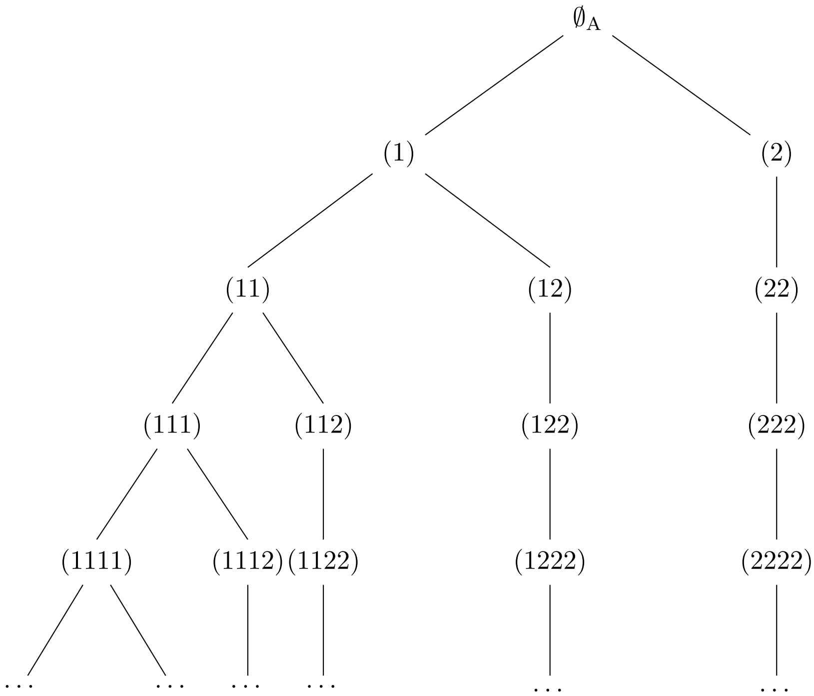
$$\mu := (\mu_i)i = 1^M.$$

We denote the length of the word by $|\mu| = M$. A word μ is admissible if and only if $A(\mu_i, \mu_{i+1}) = 1$.

Note that the empty word is an admissible word of length zero. We denote it with $\emptyset_{\mathrm{A}}$.

The set of all admissible finite words is denoted by $\mathcal{V}_{\mathrm{A}}$. It becomes a three by allowing an edge between μ and ν whenever $|\mu| < |\nu|$ and there exists η with $|\eta| = |\mu| - |\nu|$ such that $\nu = \mu\eta$. Moreover, if we choose the empty word $\emptyset_{\mathrm{A}}$ as base point, then $\mathcal{V}_A$ becomes a rooted three.

As an example we work out the rooted three for the matrix in (1)



In this example Ω_{A} contains the infinite words of the form $(2^\infty), (12^\infty), (1^2 2^\infty), (1^3 2^\infty), \ldots (1^n 2^\infty), \ldots, (1^\infty)$.

On the space of finite words we can define, for all $k \in \mathbb{N}$ the function ϕ given by

$$\phi(k) = \#\{\mu \in \mathcal{V}_{\mathrm{A}} | \ |\mu| = k\}.$$

For the matrix (1), $\phi(k) = k + 1$.

Exercise 2.1. *Show that for the matrix in* (3) *the function* ϕ *is given by*

$$\phi(k) = 2\phi(k-1) + \phi(k-2),$$

with $\phi(0) = 1$ *and* $\phi(1) = 3$.

The space Ω_A splits into cylinder sets given by the collection of infinite words that start with a certain finite word:

$$C_\mu := \{(x_k)_{k\in\mathbb{N}} \mid x_1 \cdot x_{|\mu|} = \mu\}, \qquad C_{\emptyset_{\mathrm{A}}} = \Omega_{\mathrm{A}}.$$

A finite word is admissible if and only if $C_\mu \neq 0$.

Finally, one can define a distance function on Ω_{A} by setting

$$d(x, y) = e^{-\min\{n\in N | x_n \neq y_n\}}, \quad \forall x, y \in \Omega_{\mathrm{A}}.$$

The metric topology on Ω_{A} agrees with the one coming from the compact product topology.

3. Groupoids for (sub)shifts of finite type and their C*-algebras

3.1. **Étale groupoids.** The shortest definition of a groupoid is also the most mysterious one: a small category in which every arrow is invertible. Some of you will be perfectly fine with such a definition. I prefer, however, to use the following equivalent definition:

Definition 3.1 ([4])**.** *A groupoid is a set* G *with a product map*

$$G \times G \supseteq G^{(2)} \to G, \quad (x, y) \mapsto xy$$

and an involution

$$^{-1} : G \to G \quad x \mapsto x^{-1}$$

satisfying the following conditions:

- *for all* $(x,y),(y,z) \in G^{(2)}$, $(xy,z),(x,yz) \in G^{(2)}$ *and* $(xy)z = x(yz)$ *in* G;

- $(x,x^{-1}) \in G(2)$ *and we have that for* $(x,y) \in G^{(2)}$, $x^{-1}(xy) = y$;
- $(x^{-1},x) \in G(2)$ *and we have that for* $(z,x) \in G^{(2)}$, $(zx)x^{-1} = z$;

If $x \in G$, *we define source and range of* x *as* $s(x) = x^{-1}x$ *and* $r(x) = xx^{-1}$. *A pair* $(x,y) \in G \times G$ *is composable, i.e. belongs to* $G^{(2)}$ *if and only if* $s(x) = r(y)$.

$G^{(0)} := r(G) = s(G)$ *is the unit space of* G. *Its objects are units in the sense that for all* xG *we have* $xs(x) = x$ *and* $r(x)x = x$.

Groups are groupoid with one object.

On the other hand of the spectrum, sets are groupoids with $G^{(0)} = X$ and morphisms the identity at every element.

An interesting example of groupoids that will be of use later is that of equivalence relations: let $\sim$ be an equivalence relation on X and

$$R = \{(x,y) \in X \times X \ | x \sim y\}.$$

Then R has a natural structure of groupoid by setting

$$G^{(2)} := \{((x,y),(y,z)) \in R \times R\}.$$

The product

$$(x,y)(y,z) = (x,z)$$

is a well defined element of R because of transitivity. The inverse is given by

$$(x,y)^{-1} = (y,x)$$

by symmetry of the equivalence relation.

The interesting fact is that given any groupoid we can naturally construct an equivalence relation on the unit space $G^{(0)}$ by setting

$$R := \{(r(g),s(g)) \ \forall g \in G\}.$$

Moreover, whenever the map

$$\alpha : G \to R \quad g \mapsto (r(g),s(g))$$

is injective, the groupoid G is isomorphic to the groupoid constructed from the equivalence relation R.

We described topological spaces arising from dynamics, so we need to work in the following settin:

Definition 3.2. *A* topological groupoid *is a groupoid* G *such that* G, $G^{(0)}$ *are topological spaces,* $G^{(2)}$ *is closed in* $G \times G$ *with respect to the induce topology, and the multiplication and inversion maps are continuous.*

As a consequence, the map $x \mapsto x^{-1}$ is a homeomorphism and the source and range maps are continuous.

In the following we will also assume the groupoid G to be Hausdorff.

Definition 3.3. *A groupoid* G *is* r*-discrete if* $r^{-1}(G^{(0)})$ *is open.*

As the name suggest, if G is r-discrete, then the space $r^{-1}(g)$ is discrete for all $g \in G$.

Definition 3.4. *A groupoid* G *is* étale *if* r *and* s *are local homeomorphisms.*

Every étale groupoid is r-discrerte, but the converse is not true in general.

Given an étale groupoid, one can define a notion of integration on G, thus allowing one to construct a C*-algebra out of the space $C_c(G)$ of compactly supported functions thereon. This requires the presence of a Haar system, a generalisation of the notion of Haar measure on groups.

An étale groupoid carries a canonical Haar system given by the counting measure in each fiber of r. Then the convolution product on $C_c(G)$ is given by

$$f * g(\eta) = \sum_{\xi : r(\xi) = r(\eta)} f(\xi) g(\xi^{-1}\eta) = \sum_{\xi \in r^{-1}(r(\eta))} f(\xi) g(\xi^{-1}\eta),$$

which is a finite sum because f is compactly supported and $r^{-1}(r(\eta))$ is discrete, thus turning the space $C_c(G)$ into an algebra.

With respect to the involution

$$f^*(\eta) = \overline{f(\eta^{-1})},$$

$C_c(G)$ becomes a *-algebra.

There is a canonical C*-norm on $C_c(G)$ coming from the condtional expectation $\rho : C_c(G) \to C_0(G^{(0)}$ given by restriction.

Then one defines a $C_0(G^{(0)})$ valued inner product on $C_c(G)$ given by

$$\langle f, g\rangle(x) = \sum_{y \in r^{-1}(x)} f(y^{-1})g(y^{-1}) = \rho(f^* * g)(x)$$

for $f, g \in C_c(G)$ and $x \in C_0(G^{(0)})$.

We define E_G to be the completion of $C_c(G)$ as a Hilbert C*-module, and we let $C_c(G)$ act on $C_0(G^{(0)})$ by convolution, getting the left regular representation $\lambda : C_c(G) \to \mathcal{L}(E_G)$:

$$\lambda(f)\phi = f * \phi.$$

The completion of $C_c(G)$ in the norm coming from the regular representation, i.e the closure of $\lambda(C_c(G)) \subseteq \mathcal{L}(E_G)$ is the *reduced*[2]

4. The Renault groupoid and Cuntz-Krieger algebras

There exists a locally compact Hausdorff groupoid encoding the dynamics of the pair $(\Omega_{\mathrm{A}}, \sigma)$.

$$G := \{(x, n, y) \in \Omega_{\mathrm{A}} \times \mathbb{Z} \times \Omega_{\mathrm{A}} \mid \exists k \in N : \sigma^{n+k}(x) = \sigma^k(y)\},$$

with $r(x, n, y) = x$ and $s(x, n, y) = y$ and composition given by

$$(x, n, y)(y, m, z) = (x, n + m, z).$$

Clearly the object space agrees with Ω_{A}.

4.1. Cuntz-Krieger algebras.

Cuntz-Krieger algebras are defined as universal C*-algebras associated to partial isometries.

Definition 4.1. *Let* A *be an* $N \times N$ *matrix with entries zero and one and no row or column equal to zero. The C*-algebra* $\mathcal{O}_{\mathrm{A}}$ *is the universal C*-algebra generated by* N *elements* S_i *satisfying*

$$S_i^* S_i = \sum_{j=1}^{N} \mathrm{A}(i, j) S_j S_j^* \tag{5}$$

$$\sum_{i=1}^{n} S_i S_i^* = 1 \tag{6}$$

$$S_i S_i^* S_j S_j^* = S_j S_j^* S_i S_i^* = \delta_{ij} S_i S_i^*. \tag{7}$$

One can define the range and source projections for S_i as $Q_i = S_i^* S_i$ and $P_i = S_i S_i^*$. Then the above relations become

$$P_i P_j = \delta_{ij} P_i, \tag{8}$$

$$Q_i = \sum_{i=1}^{n} \mathrm{A}(i, j) P_j. \tag{9}$$

[2]The name *reduced* comes from the fact that there is also a *full* groupoid C*-algebra, where the C*-norm is given by taking the supremum over all possible representations:

$$\|f\| = \sup_{\{L \text{ bounded representation of } C_c(G)\}} \|L(f)\|.$$

Let μ be a finite word $\mu = \mu_1 \cdots \mu_M$ and let

$$S_\mu := S_{\mu_1} \cdots S_{\mu_M}.$$

Then the above relations tell us that S_μ is non-zero if and only if μ is an admissible word.

We can now state the main result:

Theorem 4.1 ([4, 5])**.** *There is a canonical isomorphism between the groupoid C^*-algebra $C^*_r(G_{\mathrm{A}})$ and $\mathcal{O}_{\mathrm{A}}$.*

References

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