Algebraic stacks

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Algebraic spaces & algebraic stacks

Goal: arrive at/motivate a definition of algebraic stack
The def^n of alg stack fits into a pattern of defining "spaces" by gluing local models.
Part 1: explain how this works for sheaves, rather than stacks.
Part 2: explain how the sheaf constructions can be mimicked within stacks.

§1: Algebraic spaces

Let me start with a simpler question:
$\mathrm{Aff} = \mathrm{Ring}^{op} \subseteq \mathrm{Sch}$ with τ = étale topology.

Then $\mathrm{Sh}(\mathrm{Sch}) \xrightarrow[\simeq]{\text{restrict}} \mathrm{Sh}(\mathrm{Aff})$, so $\mathrm{Sch} \subseteq \mathrm{Sh}(\mathrm{Aff})$ fully faithful.
Q: when is a sheaf $X : \mathrm{Aff}^{op} \to \mathrm{Set}$ repr. by a scheme?
express this purely in terms of affines & sheaves.

Let P := class of Zariski open inclusions $U = \mathrm{Spec}(A) \to \mathrm{Spec}(B) = V$.

Def: • A sheaf X is (-2)-representable if $X = \mathrm{Spec}(A)$.
• A map $f: X \to Y$ is (-2)-representable if for all $U = \mathrm{Spec}(A) \to Y$, the pullback $X \times_Y U$ is (-2)-representable.
• f is of class (-2)-P if furthermore, the map of affines $X \times_Y U \to U$ is P.

Proceed inductively:
Def 1 For $n > -2$:
• X is n-representable if there exists a map $p: \coprod_{i \in I} U_i \to X$ (atlas) such that:
(a) p a surjection of sheaves.
(b) each $U_i = \mathrm{Spec}(A_i)$ is affine and $U_i \to X$ is of class $(n-1)$-P.
• $f: X \to Y$ is n-representable if for all $U = \mathrm{Spec}(A) \to Y$, $U \times_Y X$ is n-representable.
• f is of class n-P if furthermore
(a) for an atlas $\coprod \mathrm{Spec}(B_i) \to U \times_Y X$, each composite $\mathrm{Spec}(B_i) \to U \times_Y X \to U = \mathrm{Spec}(A)$ is P.
(b) f is mono.

Formal properties: • n-representable and n-P maps stable under base change
• any map between n-representables is n-representable.

Formal properties:
- n-representable and n-P maps stable under base change
- any map between n-representables is n-representable ($\Rightarrow$ n-representables stable under pb).
- n-reps stable under $\coprod$.
- for $m \geq n$: f n-rep + m-P $\Rightarrow f$ is n-P.

Def: Say that $f: X \to Y$ is of class P if it is n-P for some n.

- Let $f: X \to Y$ and $\coprod Y_i \to Y$ surjection, each $Y_i \to Y$ is P. Then f is n-rep (n-P) iff each $X \times_Y Y_i \to Y_i$ is (i.e.: these properties are "Zariski local" on the target).
- Similarly "Zariski local" on domain.

Lem: If X is n-representable, then X is 0-representable ($\Rightarrow$ for $n \geq 0$: all notions coincide).

Proof: Let $\coprod U_i \to X$ an atlas, $U_i = \mathrm{Spec}(A_i) \to X$ class $(n-1)$-P.

To show: $U_i \to X$ is (-1)-rep.

Take $V = \mathrm{Spec}(B) \to X$, then $U_i \times_X V$ is $(n-1)$-rep.

Let $\coprod_j V_{i,j} = \coprod_j \mathrm{Spec}(B_j) \to U_i \times_X V$ an atlas.

To show: $V_{i,j} \to U_i \times_X V$ is (-2)-rep.

Pick $W = \mathrm{Spec}(C) \to U_i \times_X V$. Then
$$V_{i,j} \times_{(U_i \times_X V)} W \cong V_{i,j} \times_{U_i \times V} W$$
is a pullback of affines, hence affine! $\square$.

Example / Proposition:
- X (-2)-rep $\Leftrightarrow$ X represent by affine
- X (-1)-rep $\Leftrightarrow$ representable by a scheme with affine diagonal

Reason: "$\Leftarrow$" take $\{U_i \to X\}$ an affine open cover of the scheme X. Then $\coprod U_i \to X$ is surj and for all affine W:

$$\begin{tikzcd} U_i \times_X W \arrow[r, hook] \arrow[d] & U_i \times W \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

is affine, so $U_i \times_X W \to W$ affine open subscheme.

"$\Rightarrow$" Take $\coprod U_i \to X$ atlas, consider
$$\coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \to X$$
coequalizer of schemes.

Each $U_i \times_X U_j \hookrightarrow U_i$, $U_i \times_X U_j \hookrightarrow U_j$ affine open subscheme

The scheme obtained by gluing the U_i along these open subschemes represents X.

- X 0-rep $\Leftrightarrow$ X representable by a scheme.

- A map between schemes $X \to Y$ is of class P $\Leftrightarrow$ open imm. in usual sense.

Observe: The above definitions also make sense when
$P =$ étale maps
(or $P =$ smooth maps)
and *without* the condition that maps in $(n-P)$ are mono!
(otherwise: mono + étale = open immersion).

$\rightsquigarrow$ get notion of 0-representable sheaf *more general* than a scheme.
All of the above results remain true, with one (important) extra:

Lem: Let $f : X \to Y$, suppose $\coprod Y_i \to Y$ any surjection of sheaves.
If each $X \times_Y Y_i \to Y_i$ is n-representable $(n-P)$, then f is too.
In other words, it is *étale* local on the target.

Proof: Representability only, by induction on n.
$n = (-2)$: Let $U = \mathrm{Spec}(A)$ affine, $U \to Y$.
There is an étale cover $U_i = \mathrm{Spec}(A_i) \to U$ s.t.
$$\begin{tikzcd} U_i \arrow[r, dashed, "\exists"] \arrow[dr] & Y_i \arrow[d] \\ & Y \end{tikzcd}.$$
We get
$$\begin{tikzcd} \coprod_{i,j} (U_i \times_U U_j) \times_Y X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_i U_i \times_{Y_i} (Y_i \times_Y X) \arrow[r] \arrow[d] & U \times_Y X \arrow[d] \\ \coprod U_i \times_U U_j \arrow[r, shift left] \arrow[r, shift right] & \coprod_i U_i \arrow[r] & U \end{tikzcd}$$

By assumption, the left two maps are affine maps of schemes.
Since affine maps satisfy étale descent, the colimit $U \times_Y X \to U$ is also affine.

For $n > -2$: Exactly the same: $\coprod_i U_i \times_{Y_i} (Y_i \times_Y X) \longrightarrow U \times_Y X$ is (-2)-étale with n-represent. domain (by ass).
Then an atlas for $\nearrow$ is also an atlas for $\nearrow$.

Cor: Let $R \hookrightarrow X \times X$ be an equivalence relation on X (internal to sheaves)
and let $R \rightrightarrows X \longrightarrow X/R$ be the quotient sheaf.
If R, X are 0-representable and the maps $R \rightrightarrows X$ are étale, then X/R is 0-representable.

Pf: It suffices to show that X/R is 1-representable ($= 0$-rep!).
Let $U = \coprod U_i \twoheadrightarrow X$ be an atlas and $\tilde{R} = R \times_{X \times X} U \times U$ the induced equivalence relation. Then the map $U/\tilde{R} \to X/R$ is an iso of sheaves
and $\tilde{R} \rightrightarrows U$ satisfies the conditions as well $\rightsquigarrow$ may assume X a coprod of affines.
In that case, $X \twoheadrightarrow X/R$ is 0-étale ($\Rightarrow X/R$ 1-rep).
Indeed, pulling back along $X \twoheadrightarrow X/R$, we get
$$\begin{tikzcd} X \times_{X/R} X \arrow[r] & X \\ R \arrow[u, phantom, "\cong"] \arrow[ur, "\pi_1"'] & \end{tikzcd}$$
, which is étale & 1-rep.

X/R
$\cong$
$R \xrightarrow{\pi_1}$

Rmk: In other words, the 0-repr sheaves form the smallest class of sheaves containing affines & closed under $\amalg$ & étale equivalence relations.

Thm: For a sheaf X, TFAE:

(1) X is 0-representable for P = étale (or P = smooth)

(2) There is a map $p: U \to X$ such that:

(a) p surj of sheaves

(b) U is a scheme and for any $V = \operatorname{Spec}(B) \to X$, $U \times_X V \to V$ is an étale map of schemes.

(c) The diagonal $\Delta: X \to X \times X$ is representable by schemes.

Def: An algebraic space is a sheaf $X \in Sh(Aff) \simeq Sh(Sch)$ satisfying (2) in the above Thm.

Rmk: Variants: [Knutson, L-MB]: also require $U \times_X U \to U \times U$ quasicompact.
[Stacks project,]: X is an fppf-sheaf
(this is equivalent by [Stacks 076L])

"Idea of proof:" (2)⇒(1) obvious: schemes are definitely 0-rep, 0-reps closed under étale equiv rel.
(1)⇒(2) Suffices to show that algebraic spaces are also closed under quotients of étale (smooth) equiv relations. $R \rightrightarrows X \to X/R$.
This is quite complicated [St.Pr 04S5, L-MB Prop. 1.6].
Step 1: (formal) can reduce to X a scheme.
Step 2: $R \underset{\text{mono}}{\hookrightarrow} X \times X$ and $R \to X$ $\underset{\text{smooth}}{\text{étale}}$ $\Rightarrow$ R is also a scheme.
([L-MB A.2], uses that quotients of schemes by finite flat equiv rel^s are schemes [SGA 3, V])

Step 3 Quotients of schemes by étale (smooth) equiv rel^s are algebraic spaces [Knutson, II 1.3], uses étale descent for separated, locally quasi-finite schemes.
[Stacks Proj 04S5]

Examples: • S a scheme, $G \curvearrowright S$ free action by a finite group
($s \in S$ a pt, if $g \cdot s = s$, then $g \curvearrowright k(s)$ nontrivially).
Then the quotient S/G is an algebraic space.
E.g: $\operatorname{char}(k) = 0$, $\mathbb{Z} \curvearrowright \mathbb{A}^1_k$ by $n \cdot x = x + n$. Then $\mathbb{A}^1_k/\mathbb{Z}$ is an A.S.

• $\operatorname{char}(k) \neq 2$: $U = \mathbb{A}^1_k$, let $R = \Delta(U) \amalg \Gamma^-(U) \hookrightarrow U \times U$ where $\Delta^-(U) = \{(x, -x) : x \neq 0 \in \mathbb{A}^1_k\}$.
Then $\mathbb{A}^1_k/R$ is an A.S.

• $Gal(k'/k) = \{1, \sigma\}$, let $S = \mathbb{A}^1_k$, $U = \mathbb{A}^1_{k'}$.
Then $U \times_S U = \operatorname{Spec}(k' \otimes_k k'[x]) = \Delta(U) \amalg \Delta^\sigma(U)$
where $\Delta^\sigma = (1, \sigma): U \to U \times U$

Then $U \times_S U = \mathrm{Spec}(k \otimes_k k[x]) = \Delta(U) \amalg \Delta^\sigma(U)$

where $\Delta^\sigma = (1, \sigma): U \to U \times_S U$.

Let $R = \Delta(U) \amalg \Delta^\sigma(U \setminus \{o_U\})$

Then $R \hookrightarrow U \times U$ étale equiv relation $\rightsquigarrow X = U/R$ an algebraic space over S.

Properties:

- $X \to S$ iso over $S - \{o_S\}$
- $X \to S$ étale, since $U \to S$ is.
- fiber over $\{o_S\}$ is $\overset{\{o_X\} =}{\{o_U\}} = \mathrm{Spec}(k')$.
- X not a scheme: otherwise $\mathcal{O}_{X, o_X}$ would be a local dom with fraction field $k(x)$, residue field k'.
- $X \times_S U \cong \mathbb{A}^1_{k'} \amalg_{\mathbb{G}_{m,k'}} \mathbb{A}^1_{k'}$ is a scheme (line w./ double origin).

What can you do with them: · any sort of map which is étale local on source & target induces a kind of map between A.S.

· X has $\begin{cases} \text{underlying top space} \\ \text{étale topos} \\ \text{quasi-coh sheaves.} \end{cases}$

§2: Algebraic stacks.

Now: interpret definition 1 not in sheaves over Aff, but stacks over Aff.

Def: An algebraic stack is a étale stack X over Aff (or Sch) which is 1-representable in the sense of Definition 1, where P = smooth maps.

Equivalently: X is an algebraic stack if

(1): $X \xrightarrow{\Delta} X \times X$ representable by algebraic spaces

(2) there is a scheme U, a surjection of stacks $p: U \to X$ which is representable by algebraic spaces and smooth.

Category theory of stacks

Preliminary remark: $\{\text{fibered categories } p: \mathbb{D} \to \mathbb{C} \text{ in groupoids}\} \supseteq PSh(\mathbb{C})$.

$\mathbb{C}/F \to \mathbb{C} \quad \longleftarrow \quad F$

There is an enhancement of this construction:

Grothendieck construction: Suppose $F: \mathbb{C}^{op} \to \mathrm{Gpd}$ a (strict) diagram of groupoids: $F(d) \xrightarrow{\alpha^*} F(c)$ for $\alpha: c \to d$.

Define a cat $\int_{\mathbb{C}} F$: objects (c, x) where $c \in \mathbb{C}$, $x \in F(c)$

maps: $(c,x) \xrightarrow{(\alpha, f)} (d,y)$: $\alpha: c \to d \in \mathbb{C}$

$f: c \to \alpha^*(d) \in F(c)$

comp: $(\alpha, f)(\beta, g) = (\alpha\beta, \beta^*(f) g)$.

Then the obv. projection $\int_{\mathbb{C}} F \to \mathbb{C}$ is fibered in gpds, fibers are just $F(c)$'s.

Inverse: $p : \mathbb{D} \to \mathbb{C}$ fibered in groupoids.

$St(p) : \mathbb{C}^{op} \to Gpd : c \mapsto$ groupoid of functors
$$\begin{tikzcd} \mathbb{C}/c \arrow[rr, "g"] \arrow[dr] & & \mathbb{D} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$
preserving cartesian maps.

For $\alpha : c \to d$, $\alpha^* : St(p)(d) \to St(p)(c)$ restricts along $\mathbb{C}/c \to \mathbb{C}/d$.
$(x \to c) \mapsto (x \to c \xrightarrow{\alpha} d)$.

Then $St(p)(c) \xrightarrow{ev(c=c)} p^{-1}(c)$ an equiv of cats.

Thm: These two constructions are mutually inverse, up to (natural) equivalence.
So a stack is really just a certain presheaf of groupoids. In particular, it determines a sheaf $\tilde{\pi}_0(X) : \mathbb{C}^{op} \to Set$, the assoc sheaf of the presheaf $c \mapsto Ob(X(c))/iso$.

Def: A map of stacks $X \to Y$ is a surjection iff the map $\tilde{\pi}_0(X) \to \tilde{\pi}_0(Y)$ is a surjection.

Weak limits of stacks

let $\mathbb{C} : I \to Cat$ a strict diagram of categories: $\mathbb{C}_i \xrightarrow{\alpha_!} \mathbb{C}_j$ for $\alpha \in I$.
The pseudo-limit $\underleftarrow{holim}\, \mathbb{C}$ of $\mathbb{C}$ is the following category:

- objects: tuples of $\cdot\, c_i \in \mathbb{C}_i$ for each $i \in I$
  $\cdot$ iso $\alpha_!(c_i) \underset{\cong}{\xrightarrow{\delta_\alpha}} c_j$ for each $\alpha \in I$, respect composition
- maps: $(c_i, \delta_\alpha) \to (d_i, g_\alpha)$ are tuples $h_i : c_i \to d_i$ in $\mathbb{C}_i$ s.t
  $h_j \delta_\alpha = g_\alpha h_i$.

This is universal (in some sense). among cones of the form
$$\begin{tikzcd} & \mathbb{D} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbb{C}_i \arrow[r, "\alpha"'] & \mathbb{C}_j \arrow[r, "\beta"'] & \mathbb{C}_k \end{tikzcd}$$

Ex: G a group, BG = 1 obj cat w $Aut(*) = G$. Then $* \times^h_B * \cong G$.

For : $X : \mathbb{C}^{op} \to Gpd$, $\{U_i \to U\}$ a cover in $\mathbb{C}$.

$\rightsquigarrow$ diagram $X(U) \dashrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j) \substack{\to\\\to\\\to} \prod_{i,j,k} X(U_i \times_U U_j \times_U U_k)$

Then $Desc(\mathcal{U}, X) \simeq$ holim of the solid diagram, i.e
X is a stack if $\uparrow$ is a holim diagram.

Homotopy limits of stacks: $\cdot$ In terms of functors $\mathbb{C}^{op} \to Gpd$: pointwise holim of gpds.

$\cdot$ In terms of fibered categories: compute the holim over $\mathbb{C}$.
(for pullbacks, it does not matter).

Ex: $Vect^n : \mathbb{C}^{op} \to Gpd$ is an algebraic stack.

Ex: $\mathrm{Vect}^n : \mathbb{C}^{op} \longrightarrow \mathrm{Gpd}$ is an algebraic stack.

An atlas is given by $* \longrightarrow \mathrm{Vect}^n$ taking the trivial vector bundle.