THE BRANCHING RULES FOR $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ REVISITED: A SPHERICAL APPROACH AND APPLICATIONS TO ORTHOGONAL POLYNOMIALS

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Abstract. We study the branching rules for the pair $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ by means of the extended weight semigroup. We obtain an alternative proof for the classical branching rules in this case as well as an approximation of the corresponding equivariant embeddings of the representation spaces. As an application we obtain approximations of the corresponding spherical functions. In particular, we obtain new examples of matrix weights for matrix-valued orthogonal polynomials. For one class of examples we determine the number of generators of the commutative algebra that has the spherical functions as simultaneous eigenfunctions.

1. Introduction

Let G be a connected reductive group defined over $\mathbb{C}$ and let $H\subset G$ be a connected reductive subgroup. Let $\pi:G\to\mathrm{GL}(V)$ be an irreducible holomorphic representation. Since H is reductive, the restriction $\pi|_H:H\to\mathrm{GL}(V)$ decomposes as a finite sum of irreducible H-representations $V=\bigoplus V'$. Branching rules are concerned with describing the multiplicity $[\pi|_H:\pi']:=\dim\mathrm{Hom}_H(V',V)$ of an irreducible H-representations in this decomposition. Suppose that the multiplicity of π' in π is one. A more refined problem is to find an explicit H-equivariant embedding $V'\to V$. Such embeddings are important for the study of the spherical functions, which, in our multiplicity free setting, give rise to vector- and matrix-valued orthogonal polynomials.

1.1. **Statements of results.** In this note we revisit both problems for the pair $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$. The branching rules for this pair are classically given by the interlacing conditions after H. Weyl and the H-equivariant embeddings can be retrieved from the Gelfand-Tsetlin basis of the representation spaces. We propose an alternative approach to both problems and apply it to calculate certain matrix coefficients.

We provide a proof of the branching rules for the pair $(\mathrm{SL}(n+1,\mathbb{C}),\mathrm{GL}(n,\mathbb{C}))$ using the theory of spherical varieties. It amounts to calculate the generators of the extended weight semigroup for a particular spherical pair, which we explain below. It turns out that the extended weight semigroup also encodes approximations of the embeddings that we are looking for. This description of the embeddings is of a less combinatorial nature than the one using the Gelfand-Tsetlin basis.

We apply our results to obtain explicit expressions of approximations of spherical functions which are used to provide examples of families of matrix-valued orthogonal polynomials. These polynomials have nice properties, for example, they are uniquely determined (up to scaling) as simultaneous eigenfunctions of a commutative algebra of differential operators. Although we only find approximations of these polynomials, the results can still be used to investigate some of their properties. Moreover, our method allows for an easy implementation in a computer algebra package.

2010 *Mathematics Subject Classification.* 22E46, 33C47.

Key words and phrases. branching rules, spherical varieties, matrix-valued orthogonal polynomials.

Similar approaches of the branching rules can be found in [6], but the weights of the colors are not calculated in that paper. The generators of Lemma 2.2 have also been obtained in [1, Thm. 7]. However, we go one step further and show the equivalence with the classical branching rules. This interpretation is needed to obtain the approximation of the spherical functions.

We proceed to discuss the content of this note in more detail, thereby introducing the necessary concepts and notations.

1.2. **Spherical varieties and representations.** A pair (G,P) with G a connected reductive group (always over $\mathbb{C}$ from now on) and $P\subset G$ a connected subgroup is called a spherical pair if a Borel subgroup $B\subset G$ has an open orbit in the quotient G/P.

The group of characters $P\to\mathbb{C}^\times$ is denoted by $X(P)$. Let $\mu\in X(P)$ and consider the associated G-line bundle $G\times^P\mathbb{C}_\mu\to G/P$. Its space of global sections $\Gamma(G\times^P\mathbb{C}_\mu)$ is isomorphic to $\{f\in\mathbb{C}[G]|\forall(p,g)\in P\times G:\ f(gp)=\mu(p)^{-1}f(g)\}$. The group G acts on $\Gamma(G\times^P\mathbb{C}_\mu)$ by $g\cdot f(g')=f(g^{-1}g')$. The pair (G,P) is spherical if and only if for all $\mu\in X(P)$ the representation space $\Gamma(G\times^P\mathbb{C}_\mu)$ decomposes multiplicity free into irreducible G-representations, see e.g. [24, Thm.25.1].

Fix a Borel subgroup $B\subset G$ with Levi decomposition $B=TB^u$, where $T\subset B$ is a maximal torus and B^u the maximal unipotent subgroup of B. Let $X^+(T)$ be the group of characters of T that are positive with respect to B. The irreducible G-representations are determined by their highest weight, i.e. if $\pi:G\to\mathrm{GL}(V)$ is an irreducible representation, then there is a unique line $\mathbb{C}v$ that is stable under B. The torus T acts on $\mathbb{C}v$ with a character called the highest weight of π. The vector $v\in V$ is called a highest weight vector and it is unique up to scaling. An irreducible representation of highest weight $\lambda\in X^+(T)$ is denoted by $\pi_\lambda:G\to\mathrm{GL}(V_\lambda)$ or simply by π_λ. We denote the highest weight of the dual representation V_λ^* by λ^*. Let $P\subset G$ be a connected subgroup. We are interested in the set of pairs $(\lambda,\mu)\in X^+(T)\times X(P)$ such that

$$\mathbb{C}[G]^{(B\times P)}_{(\lambda,\mu)}:=\{f\in\mathbb{C}[G]|\,\forall(b,g,p)\in B\times G\times P:\ f(b^{-1}gp)=\lambda(b)\mu(p)f(g)\}$$

is non-trivial, where the extension $\lambda:B\to\mathbb{C}^\times$ is defined by $\lambda(B^u)=1$. The collection of all these pairs is called the extended weight semigroup,

$$\widehat{\Lambda}_+(G,P):=\{(\lambda,\mu)\in X^+(T)\times X(P)|\,\mathbb{C}[G]^{(B\times P)}_{(\lambda,\mu)}\neq\{0\}\}.$$

The extended weight semigroup has been studied for example in [2, 3]. Note that $(\lambda^*,\mu)\in\widehat{\Lambda}_+(G,P)$ if and only if $(V_\lambda)^{(P)}_\mu:=\{v\in V_\lambda|\,\forall p\in P:\pi_\lambda(p)v=\mu(p)v\}$ is non-trivial. Moreover, if G is simply connected, then $\widehat{\Lambda}_+(G,P)$ is freely generated by indecomposable elements, i.e. the generators are not multiples of other elements in $X^+(T)\times X(P)$. The reason is that these generators are the weights of the B-stable prime divisors on G/P. See e.g. [3, §1.2-3] for a discussion of these facts.

Definition 1.1. *Let G be a connected reductive group, $H\subset G$ a reductive connected spherical subgroup and $P\subset H$ a parabolic subgroup. The triple (G,H,P) is called a multiplicity free system if (G,P) is a spherical pair.*

The multiplicity free systems (G,H,P) have been classified in [11, Thm.6.2] and [26] for (G,H) symmetric and non-symmetric spherical respectively. Let $T_H\subset B_H\subset H$ be a maximal torus of H contained in B_H where $B_H\subset P$. If $\mu\in X^+(P)$, i.e. μ is the extension to P of a positive character $\mu\in X^+(T_H)$ that is trivial on the unipotent radical of P, then

$$\mathrm{ind}_P^H(-\mu):=\{f\in\mathbb{C}[H]|\,\forall p\in P:f(hp)=\mu(p)f(h)\}$$

is an irreducible H-representation of highest weight μ^*, i.e. isomorphic to π_{μ^*}. Induction in stages shows that $\mathrm{ind}_P^G(-\mu)=\mathrm{ind}_H^G\pi_{\mu^*}$ and Frobenius reciprocity implies that $[\mathrm{ind}_P^G(-\mu):\pi_\lambda]=1$ if and only if $[\pi_\lambda|_H:$

$\pi_{\mu^*}] = 1$ which is equivalent to $(\lambda^*, \mu) \in \widehat{\Lambda}_+(G,P)$. Hence a multiplicity free triple (G,H,P) with $P \subset H$ proper, provides an abundance of irreducible H-representations that induce multiplicity free to G.

Definition 1.2. *Let $P_G^+(\mu)$ denote the set of all $\lambda \in X^+(T)$ such that $[\pi_\lambda|_H : \pi_\mu] \geq 1$. The set $P_G^+(\mu)$ is called the μ-well.*

The set $P_G^+(\mu)$ describes the irreducible subrepresentations of $\mathrm{ind}_H^G \pi_\mu$. If (G,H,P) is a multiplicity free system and $\mu \in X^+(P)$, then it is obtained from $\widehat{\Lambda}_+(G,P)$ by fixing the second coordinate to be μ and replacing the first coordinate by its dual.

The description of $P_G^+(\mu)$ is a generalization of the Cartan-Helgason theorem which describes the set $P_G^+(0)$ where $H \subset G$ is symmetric, see [13, Thm.8.49]. The sets $P_G^+(\mu)$ have been calculated for many examples of multiplicity free systems, see e.g. [7, 12, 25]. The calculations in these references are all based on the inversion of classical branching rules. Moreover, all these examples are of rank one, i.e. the multiplicity free systems (G,H,P) are such that (G,H) is a spherical pair of rank one. By the rank of a spherical pair (G,H), denoted by $\mathrm{rank}(G/H)$, we mean rank of the abelian subgroup of $X(T)$ that consists of weights of non-trivial semi-invariant rational functions on G/H. See e.g. [24, §5] for more details.

1.3. **Relation to matrix-valued orthogonal polynomials.** The μ-well $P_G^+(\mu)$ has also been calculated for some multiplicity free systems of higher rank: in [23] for G/H symmetric and $\mu : H \to \mathbb{C}^\times$ a character and in [17, 26] for higher dimensional irreducible H-representations. In all these cases it turns out that the μ-well is of a particular shape

$$P_G^+(\mu) \cong B(\mu) \times \mathbb{N}_0^r, \tag{1}$$

where $B(\mu) \subset P_G^+(\mu)$ is a finite set and $r = \mathrm{rank}(G,H)$. This shape and additional properties of $B(\mu)$ allow us to describe certain sets of matrix coefficients, namely the spherical functions Φ_λ^μ of type μ associated to $\lambda \in P_G^+(\mu)$, see Definition 5.1, by means of matrix-valued orthogonal polynomials. For example, the zonal spherical functions on a symmetric space G/H, i.e. the spherical functions of type $0 \in P_H^+$, have the structure of a polynomial algebra with $\mathrm{rank}(G/H)$ generators. Behind this polynomial structure are the recurrence relations that encode the branching of tensor product representations, see [29]. Another way to see the polynomial nature is as follows. Being eigenfunctions to the Casimir operator, the zonal spherical functions can be written as hypergeometric functions. Since the parameters are integral, the hypergeometric series are only finite, hence polynomial. We refer to [12, 17, 25, 26] for this connection with matrix-valued orthogonal polynomials. The families of matrix-valued orthogonal polynomials that are associated to the representation theory of spherical pairs can sometimes be made completely explicit, see e.g. [15, 16, 21].

The first examples of matrix-valued orthogonal polynomials related to the representation theory of spherical pairs can be found in [18]. In [10] the spherical functions for $(\mathrm{SL}(3,\mathbb{C}), \mathrm{GL}(2,\mathbb{C}))$ are studied by bringing invariant differential operators into the game. As a result of further studies the authors obtain families of matrix-valued orthogonal polynomials in subsequent papers. One of the ingredients to see that the solutions of their differential equations are of polynomial nature, is the observation that the invariant differential operators can be brought into hypergeometric form. This implies that the solutions have a power series of hypergeometric nature on the one hand, while on the other hand the series is finite because the spectral parameter satisfies an integrality condition. The way to hypergeometrize the involved operators was first observed by Román and Tirao in [22] for the case $(\mathrm{SL}(3,\mathbb{C}), \mathrm{GL}(2,\mathbb{C}))$ and has later been modified for the one-step representations of $(\mathrm{SL}(n+1,\mathbb{C}), \mathrm{GL}(n,\mathbb{C}))$ in [19]. By a one-step representation we mean that the highest weight is a multiple of one fundamental weight plus a multiple of the determinant representation.

In the general set-up [12, 17, 26] the link between the representation theory and the orthogonal polynomials is given by the set of spherical functions Φ^{μ}_{λ}, where $\lambda \in B(\mu) \subset P^+_G(\mu)$. It is clear from the theory that these functions give the necessary hypergeometrizations, although this is not used for an *a priori* construction of the polynomials. However, it is important to have control over the indicated set of spherical functions because they determine the matrix weight that describes the orthogonality of the polynomials. Using the approximations of the embeddings $V_\mu \to V_\lambda$ in this paper we provide approximations of the spherical functions. This means roughly that up to lower order and up to an invertible upper triangular matrix, we can describe packages of spherical functions by functions that we can calculate more easily.

This upper triangular matrix encodes a branching problem that is in general not multiplicity free. We give an explicit example of families where this matrix can be calculated. This provides a new family of matrix weights for which there exists a family of orthogonal polynomials whose members are completely determined (up to scaling) as simultaneous eigenfunctions of a commutative algebra of differential operators.

In fact, we show that this algebra is generated by differential operators of order two when ever the H-representation is zero-, one- or two-step. Here, the number of steps is the number of fundamental weights whose coefficient in the highest weight is non-zero.

1.4. **Notation and conventions.** From this point on we fix $n \geq 2$ in $\mathbb{N}$ and throughout the rest of this paper the symbols G and H denote

$$G = \mathrm{SL}(n+1, \mathbb{C}) \quad \text{and} \quad H = \mathrm{GL}(n, \mathbb{C}).$$

We view H as a subgroup of G via the embedding $h \mapsto \mathrm{diag}(h, \det(h)^{-1})$. Note that (G, H) is a spherical pair. Moreover, if $B_H \subset H$ is a Borel subgroup, then (G, B_H) is a spherical pair. For a reference see e.g. [11] or [25, note 2.2.14].

Let $B \subset G$ be the standard Borel subgroup consisting of upper triangular matrices. Let $T \subset B$ be the maximal torus consisting of diagonal elements. Let $\epsilon_i : T \to \mathbb{C}^\times : t = \mathrm{diag}(t_1, \ldots, t_{n+1}) \mapsto t_i$. We denote the characters additively. The set of roots of the pair (G, T) is denoted by $\Delta(G, T) = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq n+1\}$. The set of positive roots is $\Delta^+(G, T) = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n+1\}$. The set of simple roots is $\Pi(G, T) = \{\epsilon_i - \epsilon_{i+1} | \, i = 1, \ldots, n\}$. The Killing form is identified with the pairing $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. The fundamental weights are given by $\varpi_i = \sum_{j=1}^{i} \epsilon_j - j(\epsilon_1 + \cdots + \epsilon_{n+1})/(n+1)$ for $i = 1, \ldots, n$.

Let $B_H \subset H$ denote the standard Borel subgroup consisting of upper triangular matrices. The maximal torus $T \subset G$ is also maximal in H. Therefore we can describe the roots and weights of H in terms of $\epsilon_1, \ldots, \epsilon_{n+1}$. We have $\Delta(H, T) = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq n\}$. The set of positive roots is $\Delta^+(G, T) = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}$. The set of simple roots is $\Pi(H, T) = \{\epsilon_i - \epsilon_{i+1} | \, i = 1, \ldots, n-1\}$. The fundamental weights are $\varpi_1, \ldots, \varpi_{n-1}$. The character ϖ_n is the highest weight of the representation $H \to \mathbb{C}^\times : H \mapsto \det(h)$. Note that the positive Weyl chamber of H contains the positive Weyl chamber of G. We have visualized this in Figure 1.

The pair (G, H) is a symmetric pair, because $H \subset G$ is the set of fixed points for the involutive automorphism $\theta : G \to G : g \mapsto I_{n,1} g I_{n,1}$, where $I_{n,1} = \mathrm{diag}(1, \ldots, 1, -1)$. Let $A \subset G$ denote the one-dimensional torus with elements

$$a_w := \begin{pmatrix} \frac{1}{2}(w + w^{-1}) & 0 & -\frac{1}{2}(w - w^{-1}) \\ 0 & I_n & 0 \\ \frac{1}{2}(w - w^{-1}) & 0 & \frac{1}{2}(w + w^{-1}) \end{pmatrix}, \quad w \in \mathbb{C}^\times.$$

Note that $\theta(a_w) = a_w^{-1}$. Let $H_* = Z_A(H)$, the centralizer of A in H. The elements of H_* are given by $\mathrm{diag}(z, y, z)$, where $z \in \mathbb{C}^\times, y \in \mathrm{GL}(n-1, \mathbb{C})$ and $z^2 \det(y) = 1$. Let $T_{H_*} \subset H_*$ denote the maximal torus consisting of diagonal elements. The roots, positive roots and fundamental weights of H_* are given by

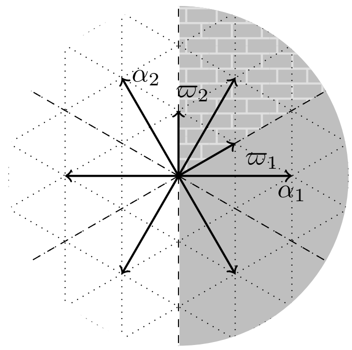


FIGURE 1. Roots and fundamental weights for $\mathrm{SL}(3,\mathbb{C})$. The positive Weyl chamber of $\mathrm{GL}(2,\mathbb{C})$ is light gray and it contains the positive Weyl chamber of $\mathrm{SL}(3,\mathbb{C})$ which is filled with bricks.

$\Delta(H_*,T_{H_*}) = \{\pm(\epsilon_i-\epsilon_j) \,|\, 2\le i<j\le n\}, \Delta^+(H_*,T_{H_*}) = \{\epsilon_i-\epsilon_j \,|\, 2\le i<j\le n\}$ and $\{\widetilde{\varpi}_2,\ldots,\widetilde{\varpi}_{n-1}\}$, where $\widetilde{\varpi}_i = \varpi_i - \frac{1}{2}(\epsilon_1-\epsilon_{n+1})$. The representation $H_*\to\mathbb{C}^\times : \mathrm{diag}(z,A,z)\mapsto z$ is of highest weight $\widetilde{\varpi}_1$.

We describe the irreducible representations of G, H and H_* by their highest weights. Let P_G^+, P_H^+ and $P_{H_*}^+$ denote the semigroups of dominant integral weights. We have

$$\begin{aligned} P_G^+ &= \mathbb{N}_0\varpi_1\oplus\cdots\oplus\mathbb{N}_0\varpi_{n-1}\oplus\mathbb{N}_0\varpi_n,\\ P_H^+ &= \mathbb{N}_0\varpi_1\oplus\cdots\oplus\mathbb{N}_0\varpi_{n-1}\oplus\mathbb{Z}\varpi_n,\\ P_{H_*}^+ &= \mathbb{Z}\widetilde{\varpi}_1\oplus\mathbb{N}_0\widetilde{\varpi}_2\oplus\cdots\oplus\mathbb{N}_0\widetilde{\varpi}_{n-1}. \end{aligned}$$

Let $U=\mathrm{SU}(n+1)\subset G$ be the set of unitary matrices, i.e. the elements $g\in G$ for which $\langle gv,gw\rangle=\langle v,w\rangle$ for all $v,w\in\mathbb{C}^{n+1}$, where $\mathbb{C}^n$ is endowed the standard Hermitian inner product $\langle\cdot,\cdot\rangle$. Then $U\subset G, U\cap H\subset H$ and $U\cap H_*\subset H_*$ are maximal compact subgroups. We endow each representation space of G, H and H_* with a Hermitian inner product for which the actions of the indicated maximal compact subgroups are unitary. Whenever we say a map between representation spaces is isometric, it will be with respect to these Hermitian structures.

Finally we discuss the symmetric powers of representations. Let $\pi: G\to\mathrm{GL}(V)$ be a a representation of G. Let $v_1,\ldots,v_d$ be an orthonormal basis of weight vectors. Fix $k\in\mathbb{N}$. Let $S^k(V)$ denote the symmetric power of V. Viewed as a subspace of the k-th tensor product it inherits the canonical Hermitian structure. Let $\rho\in\mathbb{N}_0^d$ be a partition of k, i.e. $|\rho| := \sum_{i=1}^d \rho_i = k$. Denote by v_ρ the element $v_1^{\rho_1}\cdots v_d^{\rho_d}\in S^k(V)$. Denote by $\binom{k}{\rho} = k!/(\rho_1!\cdots\rho_d!)$ the multinomial of k and ρ. The elements $\binom{k}{\rho}^{-1/2} v_\rho$ with $\rho\in\mathbb{N}_0^d$ such that $|\rho|=k$ constitute an orthonormal basis of $S^k(V)$.

2. Inverting the branching rule for $\mathrm{SL}(n+1,\mathbb{C})$ to $\mathrm{GL}(n,\mathbb{C})$

The classical branching rules for the general linear groups $\mathrm{GL}(n,\mathbb{C})\subset\mathrm{GL}(n+1,\mathbb{C})$ can be described by interlacing properties of the weights, see e.g. [13, Thm.9.14]. These rules have been proved by Weyl for the maximal compact subgroups $U(n)\subset U(n+1)$. The branching rules for $H\subset G$ can be deduced from these rules and they can be formulated as follows.

Theorem 2.1. *Let $\lambda\in P_G^+$ and $\mu\in P_H^+$ and write $\lambda=\sum_{i=1}^{n+1} a_i\epsilon_i$ and $\mu=\sum_{i=1}^{n+1} b_i\epsilon_i$. Then $[\pi_\lambda|_H:\pi_\mu]=1$ if and only if* (i) *$a_i-b_i\in\mathbb{Z}$ and* (ii) *$a_i\ge b_i$ and $b_i\ge a_{i+1}$ for $1\le i\le n$.*

The following lemma and its corollary are needed to prepare for the proof of Theorem 2.1. A proof of Lemma 2.2 can also be reconstructed from the results in [2], but to keep this paper self contained we provide an alternative argument. We use the convention $\varpi_0 = \varpi_{n+1} = 0$.

Lemma 2.2. *The extended weight semigroup $\widehat{\Lambda}^+(G, B_H)$ is generated by*

$$(\varpi_{n+1-i}, \varpi_i), \quad (\varpi_{n+1-i}, \varpi_{i-1} - \varpi_n), \quad i = 1, \dots, n. \tag{2}$$

Proof. Consider the map $G/P \to G/H$ given by the inclusion $P \subset H$. The spherical variety G/H has two B-stable divisors, see [5]. A B-stable prime divisor on G/P either maps onto one of the two B-stable prime divisors on G/H or it intersects the fiber H/B_H in a B_{H_*}-stable divisor. The Bruhat decomposition of H/B_H shows that there are $n-1$ prime divisors that are B_H-stable and thus B_{H_*}-stable. The open cell of H/B_H admits $n-1$ prime divisors that are B_{H_*}-stable. Indeed, these B_{H_*}-stable prime divisors correspond to the $T_{H_*} \cong (\mathbb{C}^\times)^{n-1}$-stable prime divisors on $\mathrm{Lie}(U_H)/\mathrm{Lie}(U_{H_*}) \cong \mathbb{C}^{n-1}$, the quotient of the Lie algebras of the maximal unipotent subgroups of B_H and B_{H_*} respectively. It follows that the rank of $\widehat{\Lambda}^+(G, B_H)$ is at most $2n$.

The elements (2) are linearly independent and indivisible. Furthermore it is clear that $(\varpi_{n+1-i}, \varpi_i) \in \widehat{\Lambda}^+(G, B_H)$, because $V^*_{\varpi_{n+1-i}} = V_{\varpi_i}$ has a B_H-stable line on which T_H acts with character ω_i. To see that the H-module $V_{\varpi_{i-1}-\varpi_n}$ is contained in V_{ϖ_i}, note the decomposition

$$\bigwedge^i \mathbb{C}^{n+1} = \bigwedge^i (\mathbb{C}^n \oplus \mathbb{C}_{-1}) = \left(\bigwedge^{i-1} \mathbb{C}^n \otimes \mathbb{C}_{-1} \right) \oplus \bigwedge^i \mathbb{C}^n \tag{3}$$

of H-representation, where $h \in H$ acts on $\mathbb{C}_{-1}$ via multiplication with $\det(h)^{-1}$. $\square$

Corollary 2.3. *Let $\mu = \sum_{i=1}^n \mu_i \varpi_i$. An element $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in P_G^+(\mu)$ is uniquely determined by the pair of n-tuples $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ such that*

- *$r_i + s_i = \lambda_i$ for $i = 1, \dots, n$,*
- *$r_i + s_{i+1} = \mu_i$ for $i = 1, \dots, n-1$,*
- *$r_n - (s_1 + \dots + s_n) = \mu_n$.*

Proof. For $\lambda \in P_G^+(\mu)$ we have $(\lambda^*, \mu) \in \widehat{\Lambda}^+(G, B_H)$, so by Lemma 2.2 there exists a unique pair of n-tuples $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ such that $(\lambda, \mu) = \sum_{i=1}^n (r_i(\varpi_i, \varpi_i) + s_i(\varpi_i, \varpi_{i-1} - \varpi_n))$. The result follows from comparison of the coefficients. $\square$

Proof of Theorem 2.1. Let $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in P_G^+$ and $\mu = \sum_{i=1}^n \mu_i \varpi_i \in P_H^+$. Then (λ, μ) satisfies the interlacing conditions of Theorem 2.1 if and only if there exists $s \in \mathbb{N}_0^n$ with (1) $s_i \leq \lambda_i$ for $i = 1, \dots, n$ and (2) $\lambda - \mu = \sum_{i=1}^n s_i(\epsilon_i - \epsilon_{n+1})$.

Suppose that we are in this situation. Write $r_i = \lambda_i - s_i$ for $i = 1, \dots, n$. Then $r \in \mathbb{N}_0^n$. We have to check that $\mu_i = r_i + s_{i+1}$ for $i = 1, \dots, n-1$ and $\mu_n = r_n - (s_1 + \dots + s_n)$. To this end write $\lambda = \sum_{i=1}^{n+1} a_i \epsilon_i$ and $\mu = \sum_{i=1}^{n+1} b_i \epsilon_i$. Observe that $s_i = a_i - b_i$ for $i = 1, \dots, n$. This implies that $r_i = b_i - a_{i+1}$ for $i = 1, \dots, n-1$. Hence $r_i + s_{i+1} = b_i - b_{i+1} = \mu_i$ for $i = 1, \dots, n-1$. Finally we use $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i = 0$ to deduce $r_n - (s_1 + \ldots + s_n) = \mu_n$.

Given a pair $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, we define (λ, μ) by the conditions of Corollary 2.3. We have to show that the interlacing conditions of Theorem 2.1 are satisfied. This follows from the observations that (1) $s_i \leq \lambda_i$ and (2) $\lambda - \mu = \sum_{i=1}^n s_i(\epsilon_i - \epsilon_{n+1})$, in view of the remark at the beginning of the proof. $\square$

Note that $P_G^+(\mu)$ is stable under addition of $\mathbb{N}_0$-multiples of $\varpi_1 + \varpi_n = \alpha_1 + \dots + \alpha_n = \epsilon_1 - \epsilon_{n+1}$. Moreover, given $\lambda \in P_G^+(\mu)$ we can subtract $r(\varpi_1 + \varpi_n)$ for $r = 0, \dots, \min(r_n, s_1)$ without leaving the

r_1	r_2	s_1	s_2	λ
4	0	1	0	$5\varpi_1$
3	0	0	1	$3\varpi_1 + \varpi_2$
2	1	0	2	$2\varpi_1 + 3\varpi_2$
1	2	0	3	$\varpi_1 + 5\varpi_2$
0	3	0	4	$7\varpi_2$
$\mu = 4\varpi_1 - \varpi_2$				

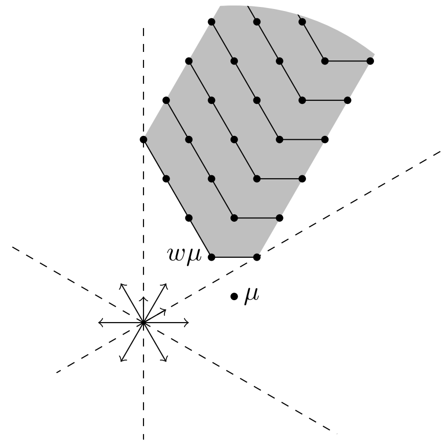


FIGURE 2. The μ-well for $(\mathrm{SL}(3,\mathbb{C}), \mathrm{GL}(2,\mathbb{C}))$ and $\mu = 4\varpi_1 - \varpi_2$.

μ-well, i.e. $\lambda - r(\varpi_1 + \varpi_n) \in P_G^+(\mu)$ for all these r. Define

$$B(\mu) := \{\lambda \in P_G^+(\mu) : \lambda - (\varpi_1 + \varpi_n) \notin P_G^+(\mu)\},$$

the *bottom* of the μ-well $P_G^+(\mu)$. We have

$$P_G^+(\mu) = B(\mu) + \mathbb{N}_0(\varpi_1 + \varpi_n). \tag{4}$$

Remark 2.4. The structure (4) of the μ-well is also available for other multiplicity free systems, see e.g. [12, 26, 17].

Example 2.5. Let $n = 2$ and take $\mu = 4\varpi_1 - \varpi_2$. In the table in Figure 2 we have listed the five quadruples (r_1, r_2, s_1, s_2) with $\min(r_2, s_1) = 0$. The corresponding λ are the elements of the bottom $B(\mu)$, i.e. those elements in $P_G^+(\mu)$ of μ-degree zero. In Figure 2 we have drawn part of the μ-well $P_G^+(\mu)$ for this example.

For a fixed $\mu \in P_H^+$ we define the μ-degree of $\lambda \in P_G^+(\mu)$ by $d_\mu(\lambda) := \min(r_n, s_1)$.

Lemma 2.6. *Let $\lambda \in P_G^+(\mu)$, $\alpha \in \Delta^+(G, T_G)$ and suppose that $\lambda - \alpha \in P_G^+(\mu)$. Then $d_\mu(\lambda - \alpha) \leq d_\mu(\lambda)$.*

Proof. The elements $(\alpha, 0) \in \widehat{\Lambda}^+(G, B_H) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $\alpha \in \Pi(G, T)$ can be written as

$$(\alpha_{n+1-j}, 0) = -(\varpi_{n+2-j}, \varpi_{j-1}) + (\varpi_{n+1-j}, \varpi_j) + (\varpi_{n+1-j}, \varpi_{j-1} - \varpi_n) - (\varpi_{n-j}, \varpi_j - \varpi_n).$$

Let $\alpha = \sum_{i=1}^n c_i \alpha_i \in \Delta^+(G, T)$ with $c_i \in \mathbb{N}_0$. Then $(\lambda', \mu) := (\lambda, \mu) - (\alpha, 0)$ has $r_n' \leq r_n$ and $s_1' \leq s_1$, whence the claim. $\square$

Remark 2.7. In the same way we can recover the classical branching laws from $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$. Indeed, in this case the induction of any irreducible $\mathrm{SO}(n)$-module to $\mathrm{SO}(n+1)$ decomposes multiplicity free as an $\mathrm{SO}(n+1)$-module, which is equivalent to saying that the a Borel subgroup of $\mathrm{SO}(n)$ remains spherical in $\mathrm{SO}(n+1)$. Since we have control over the spectra of all such induced representations via the extended weight semigroup, we can also understand the branching problems in this case through this method.

3. The decomposition of the H_*-module V_μ

The branching rules from H to H_* are described in [4, Thm.4.4]. We give an alternative proof that relates to branching from G to H.

Theorem 3.1. *Let $\mu \in P_H^+$ and $\nu \in P_{H_*}^+$ and write $\mu = \sum_{i=1}^{n+1} b_i\epsilon_i$ and $\nu = \sum_{i=1}^{n+1} c_i\epsilon_i$. Then $[\pi_\mu|_{H_*} : \pi_\nu] = 1$ if and only if* (i) *$b_i - c_i \in \mathbb{Z}$ for $2 \le i \le n$ and* (ii) *$b_i \ge c_{i+1}$ and $c_{i+1} \ge b_{i+1}$ for $1 \le i \le n-1$.*

The subgroup H_* acts spherically on H/B_H. This is a general feature for multiplicity free systems, see e.g. [26, L.2.4]. As a result the restriction $\pi_\mu|_{H_*}$ decomposes multiplicity free into irreducible H_*-representations. Note that $H_* \subset G$ is contained in the Levi subgroup with simple roots $\{\alpha_2, \dots, \alpha_{n-1}\}$. Let $Q \subset G$ denote the parabolic subgroup with this Levi subgroup that contains B. Let $Q = LU_Q$ be a Levi decomposition with $H_* \subset L_Q$. The representation of L on $V_\lambda^{U_Q}$ is irreducible of highest weight λ. Since the commutator subgroup of L_Q is contained in H_*, the group group H_* also acts irreducibly on $V_\lambda^{U_Q}$, with highest weight $\lambda_* := \lambda|_{T_{H_*}}$.

Given $\mu \in P_H^+$ we collect the irreducible H_*-representations that occur in the decomposition of $\pi_\mu|_{H_*}$,

$$P_{H_*}^+(\mu) := \{\nu \in P_{H_*}^+ \,|\, [\pi_\mu|_{H_*} : \pi_\nu] \ge 1\}.$$

Since H/B_H is H_*-spherical, we actually have $[\pi_\mu|_{H_*} : \pi_\nu] = 1$ for all $\nu \in P_{H_*}^+(\mu)$.

Lemma 3.2. *The map $B(\mu) \to P_{H_*}^+(\mu) : \lambda \mapsto \lambda_*$ is a bijection.*

Proof. The map is surjective, which is a general feature of multiplicity free systems, see [26, Thm.3.1]. To show it is injective, let $\lambda, \lambda' \in B(\mu)$ with $\lambda_* = \lambda'_*$. The set $B(\mu)$ consists of pairs (λ, μ) associated to pairs of n-tuples $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ with $\min(r_n, s_1) = 0$. Since $(\lambda - \lambda_*, 0) \in \widehat{\Lambda}^+(G, B_H)$ is a multiple of $(\varpi_1 + \varpi_n, 0)$, we conclude that $(\lambda - \lambda', 0) = t(\varpi_1 + \varpi_n, 0)$ and in fact $t \in \mathbb{N}_0$. If $t \ne 0$, then either $\lambda \notin B(\mu)$ or $\lambda' \notin B(\mu)$. This is absurd and hence injectivity is proved. $\square$

Proof of Theorem 3.1. Suppose that μ, ν satisfy the interlacing conditions of Theorem 3.1. Then, for suitable $t \in \frac{1}{2}\mathbb{N}_0$, we have $\lambda := \nu + t(\epsilon_1 - \epsilon_{n+1}) \in P_G^+(\mu)$. It follows that $\lambda_* = \nu \in P_{H_*}^+(\mu)$ by Lemma 3.2.

Conversely, if $\nu \in P_{H_*}^+(\mu)$ then there is $\lambda \in B(\mu)$ with $\lambda_* = \nu$. But $\nu = \lambda - t(\epsilon_1 - \epsilon_{n+1})$ for suitable $t \in \frac{1}{2}\mathbb{N}_0$ so that the coefficients of ϵ_i of λ and ν are equal for $i = 2, \dots, n-1$. The interlacing conditions follow from Theorem 2.1. $\square$

4. Equivariant embeddings

Given $\lambda \in P_G^+(\mu)$, we will study the H-equivariant embeddings $V_\mu \to V_\lambda$ by passing to bigger representation spaces, so called ambient spaces. Recall that the fundamental representations of $\mathrm{SL}(n+1, \mathbb{C})$, i.e. those whose highest weight is a fundamental weight ϖ_i, are realized as the natural representation on $\bigwedge^i \mathbb{C}^{n+1}$. The space $\bigwedge^i \mathbb{C}^{n+1}$ carries an inner product for which an orthonormal basis of the weight vectors is given by $e_{(j_1,\dots,j_i)} := e_{j_1} \wedge \dots \wedge e_{j_i}$, where $1 \le j_1 < \dots < j_i \le n+1$.

Definition 4.1. *For $\lambda = \sum_{i=1}^n \lambda_i\varpi_i \in P_G^+$ we define $S(G, \lambda) = \bigotimes_{i=1}^n S^{\lambda_i}(\bigwedge^i(\mathbb{C}^{n+1}))$, the ambient G-module for V_λ.*

For $\mu = \sum_{i=1}^n \mu_i\varpi_i \in P_G^+$ we define $S(H, \mu) = \bigotimes_{i=1}^{n-1} S^{\mu_i}(\bigwedge^i(\mathbb{C}^n)) \otimes \mathbb{C}_{\mu_n}$, the ambient H-module for V_μ. Here $\mathbb{C}_{\mu_n}$ is the representation space for the representation $h \mapsto \det(h)^{\mu_n}$.

Note that $\mathbb{C}_{\mu_n}$ can be identified with $S^{\mu_n}(\bigwedge^n \mathbb{C}^n)$ if $\mu_n \ge 0$ and with $S^{|\mu_n|}(\mathbb{C}e_{n+1})$ if $\mu_n \le 0$. Here $h \in H$ acts on e_{n+1} by multiplication with $\det(h)^{-1}$. Recall that the symmetric powers of fundamental representations are in general reducible. For later reference we record the following result.

Lemma 4.2. *(a) The weight space of $S(G, \lambda)$ is weight λ is one-dimensional. (b) The weight space of $S(H, \mu)$ is weight μ is one-dimensional.*

Moreover, if $\lambda, \lambda' \in P_G^+(\mu)$ and $V_{\lambda'} \subset S(G, \lambda)$, then $d_\mu(\lambda') \le d_\mu(\lambda)$ and $\lambda' \le \lambda$ in the usual partial order.

Proof. Part (a) and (b) are clear, the final statement follows from Lemma 2.6. □

Definition 4.3. *We fix a G-equivariant isometric embedding $S(\lambda) : V_\lambda \to S(G,\lambda)$ by fixing a highest weight vector $v_\lambda \in V_\lambda$ of length one, which we send to $e_1^{\lambda_1} \otimes \cdots \otimes e_{(1,\ldots,n)}^{\lambda_n}$. Similarly we fix an H-equivariant isometric embedding $S(\mu) : V_\mu \to S(H,\mu)$ by fixing a highest weight vector $v_\mu \in V_\mu$ of length one, which we send to $e_1^{\mu_1} \otimes \ldots \otimes e_{(1,\ldots,n-1)}^{\mu_{n-1}} \otimes e_{(1,\ldots,n)}^{\mu_n}$ if $\mu_n \geq 0$ and to $e_1^{\mu_1} \otimes \ldots \otimes e_{(1,\ldots,n-1)}^{\mu_{n-1}} \otimes e_{n+1}^{-\mu_n}$ otherwise.*

Let $S^*(\lambda) : S(G,\lambda) \to V_\lambda$ and $S^*(\mu) : S(H,\mu) \to V_\mu$ denote the equivariant orthogonal projections that satisfy $S^*(\lambda) \circ S(\lambda) = \mathrm{Id}$ and $S^*(\mu) \circ S(\mu) = \mathrm{Id}$.

We explain how to embed $S(H,\mu)$ into $S(G,\lambda)$. To this end we consider the the restriction of $S(G,\lambda)$ to H and the embedding of $S(H,\mu)$ into an even bigger tensor product. The match between these maps is given by the pair $(r,s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ that determines the pair $(\lambda^*,\mu) \in \widehat{\Lambda}_+(G,B_H)$. We start with the decomposition of the H-module $S(G,\lambda)$ into specific H-submodules.

Lemma 4.4. $S^{\lambda_i}(\bigwedge^i(\mathbb{C}^{n+1})) = \bigoplus_{u_i+v_i=\lambda_i} S^{v_i}(\bigwedge^{i-1}\mathbb{C}^n) \otimes \mathbb{C}_{-v_i} \otimes S^{u_i}(\bigwedge^i \mathbb{C}^n)$ *as H-modules.*

Proof. This follows from the decomposition (3) as H-modules together with properties of the symmetric products. □

Corollary 4.5. *As H-module we have*

$$S(G,\lambda) = \bigoplus_{(u,v):\, u_i+v_i=\lambda_i} \left(S^{u_1}(\bigwedge^1 \mathbb{C}^n) \otimes S^{v_2}(\bigwedge^1 \mathbb{C}^n) \right) \otimes \cdots \tag{5}$$
$$\cdots \otimes \left(S^{u_{n-1}}(\bigwedge^{n-1} \mathbb{C}^n) \otimes S^{v_n}(\bigwedge^{n-1} \mathbb{C}^n) \right) \otimes \left(S^{u_n}(\bigwedge^{n} \mathbb{C}^n) \otimes \mathbb{C}_{-(v_1+\cdots+v_n)} \right).$$

Proof. The isomorphism is obtained by applying Lemma 4.4 to each factor of $S(G,\lambda)$. The trivial factor $S^{v_1}(\bigwedge^0 \mathbb{C}^n)$ is left out and the factors $\mathbb{C}_{-v_i}$ are taken together. □

Definition 4.6. *Each pair $(u,v) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ with $u_i + v_i = \lambda_i$ for all $i = 1,\ldots n$ induces a canonical G-equivariant orthogonal projection $p_{(u,v)}$ from the right hand side of* (5) *onto the summand of the left hand side of* (5) *indexed by (u,v). With $j_{(u,v)}$ we denote its G-equivariant isometric section with $p_{(u,v)} \circ j_{(u,v)} = \mathrm{Id}$.*

To describe the embedding of $S(H,\mu)$ into $S(G,\lambda)$ we have to introduce some notation. Given $a,b \in \mathbb{N}_0$ and $\rho \in \mathbb{N}_0^n$ with $|\rho| = a+b$ define

$$\mathrm{Mat}((a,b),\rho) = \{(\tau^1,\tau^2) \in \mathbb{N}_0^n \times \mathbb{N}_0^n |\ (1)\, \tau_i^1 + \tau_i^2 = \rho_i,\ (2)\, |\tau^1| = a,\ (3)\, |\tau^2| = b\},$$

which can be viewed as the set of matrices with coefficients in $\mathbb{N}_0$ whose i-th column adds up to ρ_i and whose first and second row add up to a and b respectively. The following result is a generalization of a special case of the Clebsch-Gordan embedding theorems for $\mathrm{SL}(2,\mathbb{C})$.

Lemma 4.7. *Let $a,b \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$i_{(a,b)} : S^{a+b}(\mathbb{C}^p) \to S^a(\mathbb{C}^p) \otimes S^b(\mathbb{C}^p) : e_\rho \mapsto \sum_{\tau \in \mathrm{Mat}((a,b),\rho)} \binom{a+b}{\rho}^{-1} \left(\binom{a}{\tau^1} e^{\tau^1} \otimes \binom{b}{\tau^2} e^{\tau^2} \right)$$

is an isometric $\mathrm{GL}(p,\mathbb{C})$-equivariant map.

Proof. This is a special case of [17, L.6.2]. □

Let $\lambda \in P_G^+(\mu)$ and let $(r,s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ be the pair determined by $(\lambda^*, \mu) \in \widehat{\Lambda}_+(G, B_H)$. Then (r,s) gives rise to an H-equivariant isometric embedding

$$S(H,\mu) \to \left(\bigotimes_{i=1}^{n-1} S^{r_i}(\bigwedge^i \mathbb{C}^n) \otimes S^{s_{i+1}}(\bigwedge^i \mathbb{C}^n) \right) \otimes S^{r_n}(\bigwedge^n \mathbb{C}^n) \otimes \mathbb{C}_{-(s_1+\ldots+s_n)}. \tag{6}$$

Indeed, on the first $n-1$ factors of $S(H,\mu)$ we apply Lemma 4.7. The last factor of $S(H,\mu)$ is $\mathbb{C}_{\mu_n}$ which is equal to $\mathbb{C}_{r_n} \otimes \mathbb{C}_{-(s_1+\ldots+s_n)}$ by the conditions we imposed on (r,s). The image of (6) is the summand with $(u,v) = (r,s)$ in the decomposition of $S(G,\lambda)$ into H-submodules by Corollary 4.5. We denote the H-equivariant isometry that we obtain in this way by $\iota_{(\mu,\lambda)} : S(H,\mu) \to S(G,\lambda)$.

Theorem 4.8. *The composition $S^*(\lambda) \circ \iota_{(\mu,\lambda)} \circ S(\mu) : V_\mu \to V_\lambda$ is injective.*

Proof. We calculate the image of the highest weight vector of V_μ under $\iota_{(\lambda,\mu)} \circ S(\mu)$. Then we apply $\prod_{j=2}^n E_{j,1}^{s_j}$ to obtain a vector of weight $\mu + \sum_{j=1}^{s_j}(\epsilon_j - \epsilon_1)$. To this vector, $\prod_{j=2}^n E_{j,1}^{s_j}((\iota_{(\lambda,\mu)} \circ S(\mu))(v_\mu))$, we apply $E_{1,n+1}^{s_1+\cdots+s_{n+1}}$ to obtain

$$E_{1,n+1}^{s_1+\cdots+s_{n+1}} \prod_{j=2}^n E_{j,1}^{s_j}(\iota_{(\lambda,\mu)} \circ S(\mu)(v_\mu)), \tag{7}$$

a vector of weight λ because $\mu + \sum_{j=1}^{s_j}(\epsilon_j - \epsilon_1) = \lambda - \sum_{j=1}^n s_j(\epsilon_1 - \epsilon_{n+1})$. The weight space of $S(G,\lambda)$ of weight λ is one-dimensional by Lemma 4.2. If the vector (7) is non-zero, then it is not perpendicular to $S(\lambda)(V_\lambda)$. Hence $\iota_{(\lambda,\mu)}(S(\mu))(V_\mu))$ is not perpendicular to $S(\lambda)(V_\lambda)$. This shows that $S^*(\lambda) \circ \iota_{(\lambda,\mu)} \circ S(\mu)$ is injective, provided (7) is non-zero.

To see that (7) is non-zero, note that

$$(\iota_{(\lambda,\mu)} \circ S(\mu))(v_\mu) = e_{n+1}^{s_1} e_1^{r_1} \otimes \ldots \otimes e_{(1,\ldots,i-1,n+1)}^{s_i} e_{(1,\ldots,i)}^{r_i} \otimes \ldots \otimes e_{(1,\ldots,n-1,n+1)}^{s_n} e_{(1,\ldots,n)}^{r_n}. \tag{8}$$

Application of $E_{1,n+1}^{s_1+\cdots+s_n} \prod_{j=2}^n E_{j,1}^{s_j} = (e_1 \partial_{e_{n+1}})^{s_1+\cdots+s_n} \prod_{j=2}^n (e_j \partial_{e_1})^{s_j}$ to this vector yields

$$\begin{aligned}\sum_{\sigma,\sigma^2,\ldots,\sigma^n} &\left(E_{1,n+1}^{\sigma_1} \prod_{j=2}^n E_{j,1}^{\sigma_1^j} \right) e_{n+1}^{s_1} e_1^{r_1} \otimes \cdots \\ &\cdots \otimes \left(E_{1,n+1}^{\sigma_i} \prod_{j=2}^n E_{j,1}^{\sigma_i^j} \right) e_{(1,\ldots,i-1,n+1)}^{s_i} e_{(1,\ldots,i)}^{r_i} \otimes \cdots \otimes \left(E_{1,n+1}^{\sigma_n} \prod_{j=2}^n E_{j,1}^{\sigma_n^j} \right) e_{(1,\ldots,n-1,n+1)}^{s_i} e_{(1,\ldots,n)}^{r_1},\end{aligned} \tag{9}$$

where the sum is taken over all tuples $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ and $\sigma^j = (\sigma_1^j, \ldots, \sigma_n^j) \in \mathbb{N}_0^n, j = 2, \ldots, n$ such that $|\sigma| = |s|$ and $|\sigma^j| = s_j, j = 2, \ldots, n$. Many terms will be zero. In fact, the only non-zero terms are those with $\sigma = s$ and $\sigma^j = (0, \ldots, 0, s_j, 0, \ldots, 0)$. Indeed, any term of (9) is of the form $v_1 \otimes \ldots \otimes v_n$ of weight λ, with the vectors v_i of weight $\lambda_i \varpi_i - \tau_i$, where τ_i is an integer linear combination of positive roots with non-negative coefficients. It follows that all the τ_i are zero. The weight $\lambda_i \varpi_i$ of v_i is obtained by application of the root vectors of the roots $\epsilon_1 - \epsilon_{n+1}$ and $\epsilon_j - \epsilon_1$ with $j = 2, \ldots, n$, which are linearly independent. Hence there is only one possibility, and this is the one indicated above. It follows that (9) equals

$$\binom{|s|}{s} \prod_{j=1}^n s_j! \prod_{j=2}^n s_j! e_1^{s_1} e_1^{r_1} \otimes \ldots \otimes e_{(2,\ldots,i,1)}^{s_i} e_{(1,\ldots,i)}^{r_i} \otimes \ldots \otimes e_{(2,\ldots,n,1)}^{s_n} e_{(1,\ldots,n)}^{r_n},$$

which is a non-zero multiple of $S(\lambda)(v_\lambda)$. $\square$

Remark 4.9. We say that the embedding $\iota_{(\lambda,\mu)} \circ S(\mu) : V_\mu \to S(G, \lambda)$ is an approximation of the embedding $V_\mu \to V_\lambda$. It means that the submodule $V_\mu \subset S(G, \lambda)$ is seen by the submodule V_λ, i.e. the composition

$$S^*(\lambda) \circ \iota_{(\lambda,\mu)} \circ S(\mu) : V_\mu \to S(H, \mu) \to S(G, \lambda) \to V_\lambda$$

is injective. There may also be other irreducible G-submodules $V_{\lambda'} \subset S(G, \lambda)$ that see V_μ, but these λ' either have lower μ-degree or the same μ-degree, but then $\lambda' \leq \lambda$ in the usual partial ordering.

5. Application to spherical functions

Definition 5.1. *Let $\mu \in P_H^+$, $\lambda \in P_G^+(\mu)$ and let $j : V_\mu \to V_\lambda$ and $p : V_\lambda \to V_\mu$ be H-equivariant maps with j isometric and with $p \circ j = \mathrm{Id}$. The function $\Phi_\lambda^\mu : G \to \mathrm{End}(V_\mu) : g \mapsto p \circ \pi_\lambda(g) \circ j$ is called the spherical function of type μ associated to λ.*

The spherical functions satisfy $\Phi_\lambda^\mu(h_1 g h_2) = \pi_\mu(h_1)\Phi_\lambda^\mu(g)\pi_\mu(h_2)$ for all $h_1, h_2 \in H, g \in G$. We want to describe the spherical functions for a fixed element $\mu \in P_H^+$. To this end we make a number of reductions.

First of all we restrict Φ_μ^λ to the maximal compact subgroup $U = \mathrm{SU}(n+1)$ of G. Denote $K = U(n) \subset H$. The pair (U, K) is a compact symmetric pair. There exists a one dimensional torus $A_c \subset U$ such that $U = KA_cK$. In view of this decomposition and the transformation behavior of Φ_λ^μ it is enough to understand $\Phi_\lambda^\mu|_{A_c}$ and the values $\pi_\mu(k), k \in K$ to know the values of Φ_λ^μ on U. In this particular example the torus A_c consists of the elements

$$a(t) := \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & I_{n-1} & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix}, \quad t \in [0, 2\pi].$$

Let $M = Z_K(A_c)$. Then the complexification of M is equal to H_*. Note that $\Phi_\lambda^\mu(a) \in \mathrm{End}_{H_*}(V_\mu)$. As we have indicated in Section 3 the representation $\pi_\mu|_{H_*}$ is multiplicity free. As a basis of V_μ we take the union of the orthonormal bases that consist of T_{H_*}-weight vectors of the H_*-isotypical constituents. With respect to this basis the matrix $\Phi_\lambda^\mu(a)$ is block-diagonal, the blocks being multiples of the identity. The multiple is given by $\langle \pi_\lambda(a) v_\nu, v_\nu \rangle$, where $v_\nu \in V_\nu \subset V_\mu \subset V_\lambda$ is the highest weight vector of weight $\nu \in P_{H_*}^+(\mu)$.

Definition 5.2. *Let $N = \dim(\mathrm{End}_{H_*}(V_\mu))$ be the number of irreducible H_*-subrepresentations of V_μ and write $V_\mu = \oplus_{i=1}^N V_{\nu_i}$. Let $v_{\nu_i} \in V_{\nu_i} \subset V_\mu$ be a highest weight vector of H_* of weight ν_i.*

Let $\{V_1, \ldots, V_N\}$ be a collection of G-modules with a fixed compatible unitary structure. Suppose that we are given a collection of $\Gamma := \{\gamma_i : V_\mu \to V_i | i = 1, \ldots, N\}$ of H-equivariant embeddings. Define the matrix-valued function

$$\Psi_\Gamma^\mu : A_c \to \mathrm{Mat}(N \times N, \mathbb{C}), \quad (\Psi_\Gamma(a))_{i,j} := \langle \pi_j(a)\gamma_j(v_{\nu_i}), \gamma_j(v_{\nu_i}) \rangle.$$

The two main examples of such collections, with $d \in \mathbb{N}_0$, are given as follows.

- Let $\Upsilon_d := \{V_\mu \to V_\lambda | \lambda \in B(\mu) + d(\varpi_1 + \varpi_n)\}$, where $V_\mu \to V_\lambda$ is an isometric H-equivariant embedding. Then $\Psi_{\Upsilon_d}^\mu(a)$ is the matrix whose columns are essentially the restricted spherical functions evaluated in $a \in A_c$.
- By $\widetilde{\Upsilon}_d$ we denote the approximated version of Υ_d, i.e. $\widetilde{\Upsilon}_d := \{S(\mu) \circ \iota_{(\lambda,\mu)} : V_\mu \to S(G, \lambda) | \lambda \in B(\mu) + d(\varpi_1 + \varpi_n)\}$.

For convenience we introduce the notation $\Psi_d^\mu = \Psi_{\Upsilon_d}^\mu$ and $\widetilde{\Psi}_d^\mu = \Psi_{\widetilde{\Upsilon}_d}^\mu$. We close this section by relating Ψ_d^μ and $\widetilde{\Psi}_d^\mu$.

Proposition 5.3. *Fix $\mu \in P_H^+$ and a total ordering on $P_{H_*}^+(\mu)$ as in Section 3. Then*

$$\widetilde{\Psi}_d^\mu(a) = \sum_{k=0}^{d} \Psi_k^\mu(a) \cdot C^\mu(d,k)$$

for $a \in A_c$, where the matrices $C^\mu(d,k) \in \mathrm{Mat}(N \times N, \mathbb{C})$ are uniquely determined and where $C^\mu(d,d)$ is upper triangular and invertible.

Proof. Decompose $S(G,\lambda)$ into the direct sum of irreducible G-modules $V_{\lambda'}$ with $\lambda' \in P_G^+(\mu)$ and other G-modules. Then $d_\mu(\lambda') \leq d_\mu(\lambda)$. If the degrees are the same then $\lambda' \leq \lambda$ in the usual partial ordering. This implies the decomposition and the nature of $C^\mu(d,k)$ after some bookkeeping. To show that $C^\mu(d,d)$ is invertible we have to see that the diagonal entries are non-zero, which follows from Theorem 4.8. The matrices are unique because the spherical functions are linearly independent. □

Remark 5.4. In case $n = 2$ we observe that $C^\mu(d,d) = \mathrm{Id}$ because symmetric powers of fundamental representations are irreducible in this case.

Proposition 5.3 shows that $\widetilde{\Psi}_d^\mu$ can be seen as an approximation of Ψ_d^μ. The function Ψ_0^μ is of particular interest in the link of spherical functions with matrix-valued orthogonal polynomials. Indeed, the matrix weight of interest is given by

$$a \mapsto (\Psi_0^\mu(a))^* D^\mu \Psi_0^\mu(a), \tag{10}$$

where $D^\mu = \mathrm{diag}(\dim V_{\nu_1}, \ldots, \dim V_{\nu_N})$, see e.g. [26, §6].

The weight $a \mapsto (\widetilde{\Psi}_0^\mu(a))^* D^\mu \widetilde{\Psi}_0^\mu(a)$ differs from (10) by conjugation with $C^\mu(0,0)$, which is upper triangular and invertible. For certain properties of the weight this is immaterial, for example for its reducibility properties or for the existence of shift operators, which are invariant for conjugation with an invertible matrix.

Once the vectors $v_{\nu_1}, \ldots, v_{\nu_N} \in V_\mu$ are determined, the functions $\widetilde{\Psi}_d^\mu$ can be implemented in a computer algebra package without too much pain.

The functions Ψ_d^μ in can also be implemented, but then we have to calculate kernels of root vectors acting on high-dimensional representation spaces. We also have to implement an invariant Hermitian inner product, for which we have to invert very big matrices. Such calculations soon take too much memory.

6. Example

As an example we calculate $\widetilde{\Psi}_0^\mu$ for the case $(G,H) = (\mathrm{SL}(4,\mathbb{C}), \mathrm{GL}(3))$ and $\mu = \varpi_1 + \varpi_2 + m\varpi_3$, with $m \geq 0$. This is a two-step representation. It turns out that we can calculate Ψ_0^μ in this case. The bottom is given by $B(\mu) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where the λ_i are displayed in Table 1.

i	λ_i	s_1	s_2	s_3	r_1	r_2	r_3	$\dim V_{\lambda_i}$	$\dim V_{\nu_i}$
1	$\varpi_1 + \varpi_2 + m\varpi_3$	0	0	0	1	1	m	$\frac{4(m+1)(m+3)(m+5)}{3}$	2
2	$\varpi_1 + (m+2)\varpi_3$	0	0	1	1	0	$m+1$	$\frac{(m+3)(m+4)(m+6)}{2}$	1
3	$2\varpi_2 + (m+1)\varpi_3$	0	1	0	0	1	$m+1$	$(m+2)(m+5)(m+6)$	3
4	$\varpi_2 + (m+3)\varpi_3$	0	1	1	0	0	$m+2$	$\frac{(m+4)(m+6)(m+7)}{2}$	2
$\mu = \varpi_1 + \varpi_2 + m\varpi_3$, $m \geq 0$, $\dim V_\mu = 8$.									

TABLE 1. Some data to calculate the spherical functions

Note that $\lambda_2 = \lambda_1 + \alpha_3$, $\lambda_2 = \lambda_1 + \alpha_2 + \alpha_3$ and $\lambda_4 = \lambda_1 + \alpha_2 + 2\alpha_3$. This shows that the total ordering on $B(\mu)$ given by $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ is compatible with the usual ordering on the weight lattice. The dimensions of the representation spaces of highest weight $\lambda_i = \lambda_{i,1}\varpi_1 + \lambda_{i,2}\varpi_2 + \lambda_{i,3}\varpi_3$ are calculated using Weyl's dimension formula,

$$\dim V_{\lambda_i} = (\lambda_{i,1}+1)(\lambda_{i,2}+1)(\lambda_{i,3}+1)\frac{\lambda_{i,1}+\lambda_{i,2}+2}{2}\frac{\lambda_{i,2}+\lambda_{i,3}+2}{2}\frac{\lambda_{i,1}+\lambda_{i,2}+\lambda_{i,3}+3}{3}.$$

The H-representation V_μ decomposes into four irreducible H_*-representations of highest weights $\nu_i := \lambda_{i,*}$. The dimensions of these representation spaces are collected in Table 1 and they add up to eight, the dimension of V_μ. The highest weight vectors v_{ν_i} for H_* are displayed in Table 2. Their lengths need not be one, but all these vectors are non-zero and all are killed by $E_{3,2}$, the root vector of the only positive root of H_*. The root vectors $E_{i,j}$ act by $e_i\partial_{e_j}$, and since $\iota_{(\lambda_i,\mu)}$ is H-equivariant we can calculate the vectors $\iota_{(\lambda_i,\mu)}(v_{\nu_j})$ explicitly, they are displayed in Table 2.

i	v_{ν_i}	$\iota_{(\lambda_i,\mu)}(v_\mu)$
1	v_μ	$e_1 \otimes e_{(1,2)} \otimes e^m_{(1,2,3)}$
2	$(2E_{3,1} - E_{3,2}E_{2,1})v_\mu$	$e_1 \otimes e^{m+1}_{(1,2,3)}e_{(1,2,4)}$
3	$E_{2,1}v_\mu$	$e_{(1,2)}e_{(1,4)} \otimes e^{m+1}_{(1,2,3)}$
4	$E_{3,1}E_{2,1}v_\mu$	$e_{(1,4)} \otimes e^{m+2}_{(1,2,3)}e_{(1,2,4)}$

TABLE 2. Higest weight vectors

The element $a(t)$ acts on these vectors and to obtain $\langle a(t)\iota_{(\lambda_i,\mu)}(v_{\nu_j}), \iota_{(\lambda_i,\mu)}(v_{\nu_j})\rangle$ we only have to compare coefficients. This calculation is a matter of careful bookkeeping. After normalizing all weight vectors to have length one we obtain

$$\widetilde{\Psi}^\mu_0(a(t)) = c^m \begin{pmatrix} c^2 & c^2 & c^2 & c^2 \\ c & c(2(m+2)c^2 - 2(m+1)+1)/3 & c & c((2m+7)c^2-(2m+4))/3 \\ c & c & c(2c^2-1) & c^3 \\ 1 & (m+2)c^2-(m+1) & c^2 & c^2((m+3)c^2-(m+2)) \end{pmatrix},$$

where $c = \cos(t)$. Now we explain how to obtain Ψ^μ_0. To this end we collect the subrepresentations of the ambient representation spaces $S(G, \lambda_i)$ whose highest weights are in $P^+_G(\mu)$:

- $S(G,\lambda_1) = V_{\lambda_1}$+other summands,
- $S(G,\lambda_2) = V_{\lambda_2}$+other summands,
- $S(G,\lambda_3) = V_{\lambda_3} + V_{\lambda_1}$+other summands,
- $S(G,\lambda_4) = V_{\lambda_4} + V_{\lambda_2}$.

Note that $\Psi^\mu_0(e) = \widetilde{\Psi}^\mu_0(e)$ are both matrices with only ones. It follows that $C^\mu(0,0)$ is of the form

$$C^\mu(0,0) = \begin{pmatrix} 1 & 0 & 1-c_1 & 0 \\ 0 & 1 & 0 & 1-c_2 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_2 \end{pmatrix}$$

where $0 < c_1, c_2 \leq 1$.. The function $a \mapsto (\Psi^\mu_0(a))^* D^\mu \Psi^\mu_0(a)$ that we discussed below (10) is a matrix-valued polynomial W^μ_{pol} in the variable $\cos^2 t$. It can be tracked back to a function on U where it can be integrated component wise against the normalized Haar measure du. By Schur orthogonality the outcome is a diagonal matrix with entries $(\dim V_\mu)^2/\dim V_{\lambda_i}$. Another way to calculate this integral is by

$$\frac{1}{3}\int_0^1 W^\mu_{\mathrm{pol}}(x)(1-x)^2 dx,$$

see e.g. [25, L.3.5.7]. Putting all this information together we find $c_1 = 3/(m+4)$ and $c_2 = 3/(m+5)$. This gives the formula for Ψ_0^μ restricted to A_c,

$$\Psi_0^\mu(a(t)) = \\ c^m \begin{pmatrix} c^2 & c^2 & c^2 & c^2 \\ c & \frac{c(2(m+2)c^2-2(m+1)+1)}{3} & c & c((m+3)c^2-(m+2)) \\ c & c & \frac{c(2(m+4)c^2-(2m+5))}{3} & \frac{c((m+5)c^2-(m+2))}{3} \\ 1 & (m+2)c^2-(m+1) & \frac{(m+4)c^2-(m+1)}{3} & \frac{(m+5)(m+3)c^4-(m+2)(2m+7)c^2+(m+2)(m+1)}{3} \end{pmatrix},$$

where $c = \cos(t)$. Note that $\det \Psi_0^\mu(a(t)) = \frac{4(m+2)(m+3)(m+4)(m+5)}{27} \cos^{8+8m}(t)\left(1-\cos^2(t)\right)^4$, as expected by [27, Cor. 3.4]. Moreover, the matrix weight that we obtain is indecomposable in the sense that it cannot be conjugated with a constant matrix into a weight with blocks on the diagonal.

6.1. **Algebra of differential operators for step two representations.** Let $\mu = a\varpi_i + b\varpi_j + m\varpi_n$ with $1 \le i < j < n$ be the highest weight of an irreducible $\mathrm{GL}(n)$-representation. We are interested in the commutative algebra

$$\mathbb{D}(\mu) = U(\mathfrak{g})^{\mathfrak{k}} / \left(U(\mathfrak{g})^{\mathfrak{k}} \cap U(\mathfrak{g}) I_\mu\right),$$

where $I_\mu \subset U(\mathfrak{gl}(n))$ is the annihilator of $\mathrm{End}(V_\mu)$, which plays an important role in the theory of vector- and matrix-valued orthogonal polynomials. Indeed, the spherical functions of type μ are determined uniquely (up to scaling) as simultaneous eigenfunctions of this algebra of differential operators. For example, in [20] the spherical functions of type $a\omega_i + m\omega_n$ are calculated by solving a system of differential equations obtained from this algebra. The authors expect that the algebra $\mathbb{D}(a\omega_i + m\omega_n)$ is generated by two elements [20, §5.2]. We show that this is correct and moreover, that the algebra $\mathbb{D}(\mu)$ is generated the elements whose radial parts are of order 2.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to the involution that defines $\mathfrak{h}$. Then $\mathfrak{p}$ is isomorphic to $\mathbb{C}^n \oplus \mathbb{C}^{n*}$ as an $H' = \mathrm{SL}_n$-module. To calculate the number of generators of $\mathbb{D}(\mu)$ we use the isomorphism $\mathbb{D}(\mu) \cong S(\mathfrak{p}) \otimes_H \mathrm{End}(V_\mu^H)$ of vector spaces from [8, p. 100], which also respects the filtration by degree. Our aim is to determine the number of H-invariants for each degree, i.e. to determine $\dim \left(S^d(\mathfrak{p}) \otimes \mathrm{End}(V_\mu^H)\right)^H$ for each $d \in \mathbb{N}_0$. To this end we start with decomposing $S^d(\mathfrak{p})$ into irreducible H-modules.

Consider the embedding $\mathrm{GL}_n \to \mathbb{C}^\times \times \mathrm{GL}_n : A \mapsto (1, \det(A)A)$. The group $\mathbb{C}^\times \times \mathrm{GL}_n$ acts on $(\mathbb{C}_1 \otimes \mathbb{C}^n) \oplus \mathbb{C}^{n*}$ via $(c, A)(v, f) = (cAv, f \circ A^{-1})$. This induces an action on $S((\mathbb{C}_1 \otimes \mathbb{C}^n) \oplus \mathbb{C}^{n*})$ and the weight semigroup of this representation is generated by $\omega_1 + \omega_1', \omega_{n-1}', \omega_1$, where $\deg(\omega_1 + \omega_1') = \deg(\omega_{n-1}')$ and $\deg(\omega_1) = 2$, see [14].

The representation of GL_n on $\mathfrak{p} = (\mathbb{C}_1 \otimes \mathbb{C}^n) \oplus (\mathbb{C}_{-1} \otimes \mathbb{C}^{n*})$ is obtained by restricting the representation of $\mathbb{C}^\times \times \mathrm{GL}_n$ that acts on $(\mathbb{C}_1 \otimes \mathbb{C}^n) \oplus \mathbb{C}^{n*}$ to GL_n. Note that the weights $\omega_1 + \omega_1', \omega_{n-1}', \omega_1$ restrict to the weights $\omega_1 + \omega_n, \omega_{n-1} - \omega_n, 0$ of degree $1, 1, 2$ respectively. It follows that $S^k(\mathfrak{p})$ decomposes (without multiplicities) into irreducible representations of highest weight $x(\omega_1 + \omega_n) + y(\omega_{n-1} - \omega_n) + z(0) = x\omega_1 + y\omega_{n-1} + (x-y)\omega_n$ with $x + y + 2z = k$.

The center of H acts trivially on each constituent of $\mathrm{End}(V_\mu^H)$. Hence we are only interested in constituents of $S^d(\mathfrak{p})$ of highest weight $k(\omega_1 + \omega_{n-1})$, the other constituents are irrelevant for this problem. Note that the relevant representations occur only in even degree.

$\mu = 5\omega_3 + 3\omega_5$ $\qquad\qquad$ $\mu^* = 3\omega_2 + 5\omega_4$

FIGURE 3. In this example, with $n = 7$, μ corresponds to the partition $[8, 8, 8, 3, 3]$ and μ^* corresponds to the partition $[8, 8, 5, 5]$.

Lemma 6.1. *Consider the representation V_μ of $H = \mathrm{GL}_n$. We have*

$$\mathrm{End}(V_\mu) = \sum_{\ell=0}^{a+b} a_k(k\omega_1 + k\omega_{n-1}) + \ldots$$

where the dots represent irrelevant modules. The number a_k counts the number of lattice points (n_1, n_2) in the rectangle $[0, a] \times [0, b]$ intersected with the line $n_1 + n_2 = k$.

Proof. Without loss of generality we assume that $a \geq b$ (if not, take the dual; the tensor products $\lambda^* \otimes \lambda$ and $\lambda \otimes \lambda^*$ are equivalent via the flip.). There are three cases:

- $0 \leq k \leq b$, multiplicity is $k + 1$,
- $b \leq k \leq a$, multiplicity is $b + 1$,
- $a \leq k \leq a + b$, multiplicity is $a + b + 1 - k$.

Following the Littlewood-Richardson rule [9] we have to count the number Littlewood-Richardson skew tableaux of shape λ'_k/μ of weight μ^*. The rules are the following:

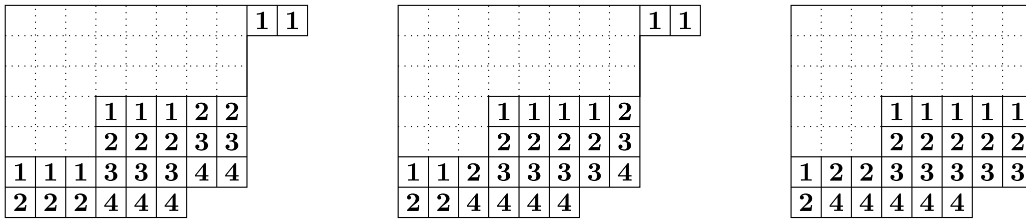
- Rule $\downarrow$: entries in a column must strictly increase from top to bottom.
- Rule $\rightarrow$: entries in a row must weakly increase from left to right.
- Rule $\leftrightarrow$: each first prefix of the reversed lattice word must contain at least as many $\mathbf{r}$'s as $\mathbf{r+1}$'s, for $r = 1, 2, \ldots$.

Here, λ'_k is the tableau of shape λ_k with a block $[a + b - k, \ldots, a + b - k]$ (n entries) attached to it. The dual of μ is $\mu^* = b\omega_{n-j} + a\omega_{n-i}$ which corresponds to the partition $[a + b, \ldots, a + b, a, \ldots, a]$, so the weight amounts to $a + b$ $\mathbf{1}$'s up to $a + b$ $\mathbf{n-i}$'s, a $\mathbf{n-i+1}$'s up to a $\mathbf{n-j}$'s, see Figure 3. If a $\mathbf{1}$ appears in a column, then it has to be on the top row (by $\downarrow$). Moreover, the first row is filled with $\mathbf{1}$'s (by $\leftrightarrow$). The $\mathbf{1}$'s in a row must be placed next to each other, starting in the left box (by $\rightarrow$). One verifies that the number of possibilities of putting the ones in the diagram are given by the indicated multiplicities.

The next step is to show that each of the above possibilities yields a unique Littlewood-Richardson skew tableau of weight μ^*.

Suppose $0 \leq k \leq b$. There are $k + 1$ configurations A_ℓ with $\ell = 0, \ldots, k$ of $\mathbf{1}$'s: row 1 is filled with $\mathbf{1}$'s, row $i + 1$ has $a - \ell$ $\mathbf{1}$'s and row $j + 1$ has $b - k + \ell$ $\mathbf{1}$'s. With induction to $n - j$ one shows that the $\mathbf{2}$'s up to the $\mathbf{n-j}$'s can be placed in the tableau in precisely one way. Indeed, in each case there are $a + b$ columns left and the claim follows from $\downarrow$.

There are $b + k - \ell$ columns left to place the $\mathbf{n-j+1}$'s. The only possibility is to start in the empty top box of column $a+b$ by $\leftrightarrow$ and this dictates where to place the rest of the $\mathbf{n-j+1}$'s, namely in the top empty boxes of columns $a + b - 1, \ldots, a + 1$. With induction one shows that the numbers $\mathbf{n-j+2}, \ldots, \mathbf{n-i-1}$ can be placed in the tableau in precisely one way. The final a numbers $\mathbf{n-i}$ are placed in the empty boxes and the result is the unique Littlewood-Richardson skew tableau of shape λ'_k/μ of weight μ^* with the given



A_2 A_1 A_0

FIGURE 4. In this example we have $n = 7$, μ, μ^* as in Figure 3 and $k = 2$. There are three configurations of $\mathbf{1}$'s leading to three Littlewood-Richardson skew tableaux.

configuration of $\mathbf{1}$'s. An example of this case is depicted in Figure 4. The other two cases are proved in a similar way. $\square$

Theorem 6.2. *The dimension of* $\left(S^{2k}(\mathfrak{p}) \otimes \mathrm{End}(V_\mu)\right)^H$ *is* $\sum_{\ell=0}^{\min(a+b,k)} a_\ell$. *The algebra* $(S(\mathfrak{p}) \otimes \mathrm{End}(V_\mu))^H$ *is generated by the elements in degree* 2 *as a* $\mathbb{C}$*-algebra.*

Proof. Assume $a \geq b$ and write $V = V_\mu$. The relevant modules are of highest weight $\ell(\omega_1 + \omega_{n-1})$. The ones that occur in $\mathrm{End}(V)$ are $\ell(\omega_1 + \omega_{n-1})$ with $\ell = 0, \ldots, a + b$, the multiplicities being a_k (depending on (a, b)). On the other hand, $S^{2k}(\mathfrak{p})$ contains $\ell(\omega_1 + \omega_{n-1})$ for $\ell = 0, \ldots, k$. It follows that $\dim \mathrm{Hom}_K(S^{2k}(\mathfrak{p}), \mathrm{End}(V)) = a_0 + \ldots + a_{\min(a+b,k)}$.

Let us consider the three cases (i) $a = b = 0$, (ii) $a \neq 0, b = 0$, (iii) $a, b \neq 0$. In the first case $V = \mathbb{C}$ is the trivial representation and $a_0 = 1$. This implies that there is a surjective homomorphism $S(\mathbb{C}) \to (S(\mathfrak{p}) \otimes \mathrm{End}(V))^H$, which implies that $S(\mathfrak{p})^H$ is generated as an algebra by an element of degree two.

In the second case there are $a_0 + a_1$ generators of degree two. With $b = 0$ we have $a_\ell = 1$ for $\ell = 0, \ldots, a$, so there are 2 generators of degree 2. In fact, the space $S^{2k}(\mathfrak{p}) \otimes_K \mathrm{End}(V)$ has dimension $a_0 + a_1 + \ldots + a_{\min(a,k)} \leq 2k + 1$. The dimension of $S^d(\mathbb{C}^2)$ equals $\binom{d+1}{d} = d + 1$. This implies that there is a surjective homomorphism

$$S(\mathbb{C}^2) \to (S(\mathfrak{p}) \otimes \mathrm{End}(V))^H$$

and we deduce that $\mathbb{D}(\mu)$ is the quotient of an algebra generated by two elements. Note that the kernel contains elements of positive degree.

In the third case $a_0 + a_1 = 3$ and $\dim S^{2k}(\mathfrak{p}) \otimes_K \mathrm{End}(V) = a_0 + \cdots + a_{\min(a+b,k)} \leq 1 + 2 + \cdots + k + (k+1) = \binom{k+2}{k}$. Since $\dim S^k(\mathbb{C}^3) = \binom{k+2}{k}$ we have a surjective homomorphism $S(\mathbb{C}^3) \to (S(\mathfrak{p}) \otimes \mathrm{End}(V))^H$ which finishes the proof. $\square$

As a result, the algebra $\mathbb{D}(\mu)$ of the example is generated by 3 differential operators of order two. Moreover, the algebra of differential operators in [20], which are related to one-step representations, is always generated by two elements of order two. In the latter case it is known how the two generators depend on the root multiplicities and in fact that these examples give rise to families of matrix-valued classical pairs with a shift operator [28]. It would be instructive to calculate the radial parts of the generators for a two-step representation for a couple of low-dimensional examples and to identify the root multiplicities in these expressions.

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