GRADED HECKE ALGEBRAS FOR DISCONNECTED REDUCTIVE GROUPS

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Abstract. We introduce graded Hecke algebras $\mathbb{H}$ based on a (possibly disconnected) complex reductive group G and a cuspidal local system $\mathcal{L}$ on a unipotent orbit of a Levi subgroup M of G. These generalize the graded Hecke algebras defined and investigated by Lusztig for connected G.

We develop the representation theory of the algebras $\mathbb{H}$, obtaining complete and canonical parametrizations of the irreducible, the irreducible tempered and the discrete series representations. All the modules are constructed in terms of perverse sheaves and equivariant homology, relying on work of Lusztig. The parameters come directly from the data $(G, M, \mathcal{L})$ and they are closely related to Langlands parameters.

Our main motivation for considering these graded Hecke algebras is that the space of irreducible $\mathbb{H}$-representations is canonically in bijection with a certain set of "logarithms" of enhanced L-parameters. Therefore we expect these algebras to play a role in the local Langlands program. We will make their relation with the local Langlands correspondence, which goes via affine Hecke algebras, precise in a sequel to this paper.

Erratum. Theorem 3.4 and Proposition 3.22 were not entirely correct as stated. This is repaired in a new appendix.

Contents

Date: August 9, 2024.
2010 *Mathematics Subject Classification.* 20C08,14F43,20G20.

Introduction

The study of Hecke algebras and more specifically their simple modules is a powerful tool in representation theory. They can be used to build bridges between different objects. Indeed they can arise arithmetically (as endomorphism algebras of a parabolically induced representation) or geometrically (using K-theory or equivariant homology). For example, this strategy was successfully used by Lusztig in his Langlands parametrization of unipotent representations of a connected, adjoint simple unramified group over a nonarchimedean local field [Lus6, Lus8]. This paper is part of a series, whose final goal is to generalize these methods to arbitrary irreducible representations of arbitrary reductive p-adic groups. In the introduction we discuss the results proven in the paper, and in Section 1 we shed some light on the envisaged relation with the Langlands parameters.

After [AMS], where the authors extended the generalized Springer correspondence in the context of a reductive disconnected complex group, this article is devoted to generalize in this context several results of the series of papers of Lusztig [Lus3, Lus5, Lus7]. Let G be an complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Although we do not assume that G is connected, it has only finitely components because it is algebraic. Let L be a Levi factor of a parabolic subgroup P of G°, $T = Z(L)^\circ$ the connected center of L, $\mathfrak{t}$ its Lie algebra and $v \in \mathfrak{l} = \mathrm{Lie}(L)$ be nilpotent. Let $\mathcal{C}_v^L$ be the adjoint orbit of v and let $\mathcal{L}$ be an irreducible L-equivariant cuspidal local system on $\mathcal{C}_v^L$. The triples $(L, \mathcal{C}_v^L, \mathcal{L})$ (or more precisely their G-conjugacy classes) defined by data of the above kind will be called *cuspidal supports* for G. We associate to $\tau = (L, \mathcal{C}_v^L, \mathcal{L})_G$ a twisted version $\mathbb{H}(G, L, \mathcal{L}) = \mathbb{H}(G, \tau)$ of a graded Hecke algebra and study its simple modules. More precisely, let $W_\tau = N_G(\tau)/L$, $W_\tau^\circ = N_{G^\circ}(\tau)/L$ and $\mathfrak{R}_\tau = N_G(P, \mathcal{L})/L$. Then $W_\tau = W_\tau^\circ \rtimes \mathfrak{R}_\tau$. Let $\mathbf{r}$ be an indeterminate and $\natural_\tau : \mathfrak{R}_\tau^2 \to \mathbb{C}^\times$ be a (suitable) 2-cocycle. The twisted graded Hecke algebra associated to τ is the vector space

$$\mathbb{H}(G, \tau) = \mathbb{C}[W_\tau, \natural_\tau] \otimes S(\mathfrak{t}^*) \otimes \mathbb{C}[\mathbf{r}],$$

with multiplication as in Proposition 2.2. As $W_\tau = W_\tau^\circ \rtimes \mathfrak{R}_\tau$ and W_τ plays the role of W_τ° in the generalized Springer correspondence for disconnected groups, the algebra $\mathbb{H}(G, \tau)$ contains the graded Hecke algebra $\mathbb{H}(G^\circ, \tau)$ defined by Lusztig in [Lus3] and plays the role of the latter in the disconnected context. More precisely, let $y \in \mathfrak{g}$ be nilpotent and let $(\sigma, r) \in \mathfrak{g} \oplus \mathbb{C}$ be semisimple such that $[\sigma, y] = 2ry$. Let $\sigma_0 = \sigma - rh \in \mathfrak{t}$ with $h \in \mathfrak{g}$ a semisimple element which commutes with σ and which arises in a $\mathfrak{sl}_2$-triple containing y. Then we have $\pi_0(Z_G(\sigma, y)) = \pi_0(Z_G(\sigma_0, y))$, where $Z_G(\sigma, y)$ denotes the simultaneous centralizer of σ and y in G, and respectively for σ_0. We also denote by Ψ_G the cuspidal support map defined in [Lus1, AMS], which associates to every pair (x, ρ) with $x \in \mathfrak{g}$ nilpotent and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(x))\big)$ (with $Z_G(x)$ the centralizer of x in G) a cuspidal support $(L', \mathcal{C}_{v'}^{L'}, \mathcal{L}')$.

Using equivariant homology methods, we define standard modules in the same way as in [Lus3] and denote by $E_{y,\sigma,r}$ (resp. $E_{y,\sigma,r,\rho}$) the one which is associated to y, σ, r (resp. y, σ, r and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$). They are modules over $\mathbb{H}(G, \tau)$ and we have the following theorem:

Theorem 1. *Fix* $r \in \mathbb{C}$.

(a) *Let $y, \sigma \in \mathfrak{g}$ with y nilpotent, σ semisimple and $[\sigma, y] = 2ry$. Let $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma_0, y))\big)$ such that $\Psi_{Z_G(\sigma_0)}(y, \rho) = \tau = (L, \mathcal{C}_v^L, \mathcal{L})_G$. With these data we associate a $\mathbb{H}(G, \tau)$-module $E_{y,\sigma,r,\rho}$. The $\mathbb{H}(G, \tau)$-module $E_{y,\sigma,r,\rho}$ has a distinguished irreducible quotient $M_{y,\sigma,r,\rho}$, which appears with multiplicity one in $E_{y,\sigma,r,\rho}$.*

(b) *The map $M_{y,\sigma,r,\rho} \longleftrightarrow (y, \sigma, \rho)$ gives a bijection between $\mathrm{Irr}_r(\mathbb{H}(G, \tau))$ and G-conjugacy classes of triples as in part (a).*

(c) *The set $\mathrm{Irr}_r(\mathbb{H}(G, \tau))$ is also canonically in bijection with the following two sets:*
   - *G-orbits of pairs (x, ρ) with $x \in \mathfrak{g}$ and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(x))\big)$ such that $\Psi_{Z_G(x_S)}(x_N, \rho) = \tau$, where $x = x_S + x_N$ is the Jordan decomposition of x.*
   - *$N_G(L)/L$-orbits of triples $(\sigma_0, \mathcal{C}, \mathcal{F})$, with $\sigma_0 \in \mathfrak{t}$, $\mathcal{C}$ a nilpotent $Z_G(\sigma_0)$-orbit in $Z_{\mathfrak{g}}(\sigma_0)$ and $\mathcal{F}$ a $Z_G(\sigma_0)$-equivariant cuspidal local system on $\mathcal{C}$ such that $\Psi_{Z_G(\sigma_0)}(\mathcal{C}, \mathcal{F}) = \tau$.*

Next we investigate the questions of temperedness and discrete series of $\mathbb{H}(G, \tau)$-modules. Recall that the vector space $\mathfrak{t} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ has a decomposition $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \oplus i\mathfrak{t}_{\mathbb{R}}$ with $\mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Hence any $x \in \mathfrak{t}$ can be written uniquely as $x = \Re(x) + i\Im(x)$. We obtain the following:

Theorem 2. *(see Theorem 3.25)*
Let y, σ, ρ be as above with $\sigma, \sigma_0 \in \mathfrak{t}$.

(a) *Suppose that $\Re(r) \leq 0$. The following are equivalent:*
   - *$E_{y,\sigma,r,\rho}$ is tempered;*
   - *$M_{y,\sigma,r,\rho}$ is tempered;*
   - *$\sigma_0 \in i\mathfrak{t}_{\mathbb{R}}$.*

(b) *Suppose that $\Re(r) \geq 0$. Then part (a) remains valid if we replace tempered by anti-tempered.*

Assume further that G° is semisimple.

(c) *Suppose that $\Re(r) < 0$. The following are equivalent:*
   - *$M_{y,\sigma,r,\rho}$ is discrete series;*
   - *y is distinguished in $\mathfrak{g}$, that is, it is not contained in any proper Levi subalgebra of $\mathfrak{g}$.*

   Moreover, if these conditions are fulfilled, then $\sigma_0 = 0$ and $E_{y,\sigma,r,\rho} = M_{y,\sigma,r,\rho}$.

(d) *Suppose that $\Re(r) > 0$. Then part (c) remains valid if we replace (i) by: $M_{y,\sigma,r,\rho}$ is anti-discrete series.*

(e) *For $\Re(r) = 0$ there are no (anti-)discrete series representations on which $\mathbf{r}$ acts as r.*

Moreover, using the Iwahori–Matsumoto involution we give another description of tempered modules when $\Re(r)$ is positive, and this is more suitable in the context of the Langlands correspondence.

The last section consists of the formulation of the previous results in terms of cuspidal quasi-supports, which is more adapted than cuspidal supports in the context of Langlands correspondence, as it can be seen in [AMS, §5–6].

Recall that a quasi-Levi subgroup of G is a group of the form $M = Z_G(Z(L)^\circ)$, where L is a Levi subgroup of G°. Thus $Z(M)^\circ = Z(L)^\circ$ and $M \longleftrightarrow L = M^\circ$ is a bijection between quasi-Levi subgroups of G and the Levi subgroups of G°.

A *cuspidal quasi-support* for G is the G-conjugacy class of $q\tau$ of a triple $(M, \mathcal{C}_v^M, q\mathcal{L})$, where M is a quasi-Levi subgroup of G, $\mathcal{C}_v^M$ is a nilpotent $\mathrm{Ad}(M)$-orbit in $\mathfrak{m} =$

Lie(M) and $q\mathcal{L}$ is a M-equivariant cuspidal local system on $\mathcal{C}_v^M$, i.e. as M°-equivariant local system it is a direct sum of cuspidal local systems. We denote by $q\Psi_G$ the cuspidal quasi-support map defined in [AMS, §5]. With the cuspidal quasi-support $q\tau = (M, \mathcal{C}_v^M, q\mathcal{L})_G$, we associate a twisted graded Hecke algebra denoted $\mathbb{H}(G, q\tau)$.

Theorem 3. *The analog of Theorem 2with cuspidal quasi-supports instead of cuspidal ones holds true.*

The article is organized as follows. The first section is introductory, it explains why and how the study of enhanced Langlands parameters motivated this paper. The second section contains the definition of the twisted graded Hecke algebra associated to a cuspidal support. After that we study the representations of these Hecke algebras in the third section. To do that we define the standard modules and we relate them to the standard modules defined in the connected case by Lusztig. As preparation we study precisely the modules annihilated by $\mathbf{r}$. By Clifford theory, as explained in [AMS, §1], we show then that the simple modules over $\mathbb{H}(G, \tau)$ can be parametrized in a compatible way by the objects in part (c) and (d) of the first theorem in this introduction. We deduce then the first theorem. After that we study temperedness and discrete series, resulting in the second theorem of the introduction. Note that we show a version of the ABPS conjecture for the involved Hecke algebras. To conclude, the last section is devoted to the adaption of the previous results for a cuspidal quasi-support as described above.

Acknowledgements.
The third author is supported by a NWO Vidi grant "A Hecke algebra approach to the local Langlands correspondence" (nr. 639.032.528).

We thank Dan Ciubotaru and George Lusztig for helpful discussions.

1. The relation with Langlands parameters

This article is part of a series the main purpose of which is to construct a bijection between enhanced Langlands parameters for $\mathcal{G}(F)$ and a certain collection of irreducible representations of twisted affine Hecke algebras, with possibly unequal parameters. The parameters appearing in Theorems 1 and 3 are quite close to those in the local Langlands correspondence, and with the exponential map one can make that precise. To make optimal use of Theorem 3, we will show that the parameters over there constitute a specific part of one Bernstein component in the space of enhanced L-parameters for one group. Let us explain this in more detail.

Let F be a local non-archimedean field, let $\mathbf{W}_F$ be the Weil group of F, $\mathbf{I}_F$ the inertia subgroup of $\mathbf{W}_F$, and $\mathrm{Frob}_F \in \mathbf{W}_F$ a geometric Frobenius element. Let $\mathcal{G}$ be a connected reductive algebraic group defined over F, and $\mathcal{G}^\vee$ be the complex dual group of $\mathcal{G}$. The latter is endowed with an action of $\mathbf{W}_F$, which preserves a pinning of $\mathcal{G}^\vee$. The Langlands dual group of the group $\mathcal{G}(F)$ of the F-rational points of $\mathcal{G}$ is ${}^L\mathcal{G} := \mathcal{G}^\vee \rtimes \mathbf{W}_F$.

A Langlands parameter (L-parameter for short) for ${}^L\mathcal{G}$ is a continuous group homomorphism

$$\phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{G}^\vee \rtimes \mathbf{W}_F$$

such that $\phi(w) \in \mathcal{G}^\vee w$ for all $w \in \mathbf{W}_F$, the image of $\mathbf{W}_F$ under ϕ consists of semisimple elements, and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

We call a L-parameter *discrete*, if $Z_{\mathcal{G}^\vee}(\phi)^\circ = Z(\mathcal{G}^\vee)^{\mathbf{W}_F,\circ}$. With [Bor, §3] it is easily seen that this definition of discreteness is equivalent to the usual one with proper Levi subgroups.

Let $\mathcal{G}^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group $\mathcal{G}^\vee_{\mathrm{der}}$. Let $Z_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi)$ be the image of $Z_{\mathcal{G}^\vee}(\phi)$ in the adjoint group $\mathcal{G}^\vee_{\mathrm{ad}}$. We define

$$Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi) = \text{ inverse image of } Z_{\mathcal{G}^\vee_{\mathrm{ad}}}(\phi) \text{ under } \mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee.$$

To ϕ we associate the finite group $\mathcal{S}_\phi := \pi_0(Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi))$. An *enhancement* of ϕ is an irreducible representation of $\mathcal{S}_\phi$. The group $\mathcal{S}_\phi$ coincides with the group considered by both Arthur in [Art] and Kaletha in [Kal, §4.6].

The group $\mathcal{G}^\vee$ acts on the collection of enhanced L-parameters for ${}^L\mathcal{G}$ by

$$g \cdot (\phi, \rho) = (g\phi g^{-1}, g \cdot \rho).$$

Let $\Phi_e({}^L\mathcal{G})$ denote the collection of $\mathcal{G}^\vee$-orbits of enhanced L-parameters.

Let us consider $\mathcal{G}(F)$ as an inner twist of a quasi-split group. Via the Kottwitz isomorphism it is parametrized by a character of $Z(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$, say $\zeta_{\mathcal{G}}$. We say that $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$ is relevant for $\mathcal{G}(F)$ if $Z(\mathcal{G}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$ acts on ρ as $\zeta_{\mathcal{G}}$. The subset of $\Phi_e({}^L\mathcal{G})$ which is relevant for $\mathcal{G}(F)$ is denoted $\Phi_e(\mathcal{G}(F))$.

As well-known, $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$ is already determined by $\phi|_{\mathbf{W}_F}, u_\phi := \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$ and ρ. Sometimes we will also consider $\mathcal{G}^\vee$-conjugacy classes of such triples $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$ as enhanced L-parameters. An enhanced L-parameter $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ will often be abbreviated to $(\phi_v, q\epsilon)$.

For $(\phi, \rho) \in \Phi_e({}^L\mathcal{G})$ we write

$$G_\phi := Z^1_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}), \tag{1}$$

a complex (possibly disconnected) reductive group. We say that (ϕ, ρ) is *cuspidal* if ϕ is discrete and $(u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big), \rho)$ is a cuspidal pair for G_ϕ: this means that ρ corresponds to a G_ϕ-equivariant cuspidal local system $\mathcal{F}$ on $\mathcal{C}^{G_\phi}_{u_\phi}$. We denote the collection of cuspidal L-parameters for ${}^L\mathcal{G}$ by $\Phi_{\mathrm{cusp}}({}^L\mathcal{G})$, and the subset which is relevant for $\mathcal{G}(F)$ by $\Phi_{\mathrm{cusp}}(\mathcal{G}(F))$.

Let G be a complex (possibly disconnected) reductive group. We define the *enhancement of the unipotent variety* of G as the set:

$$\mathcal{U}_e(G) := \{(\mathcal{C}^G_u, \rho) \,:\, \text{with } u \in G \text{ unipotent and } \rho \in \mathrm{Irr}(\pi_0(Z_G(u))\},$$

and call a pair $(\mathcal{C}^G_u, \rho)$ an *enhanced unipotent class*. Let $\mathfrak{B}(\mathcal{U}_e(G))$ be the set of G-conjugacy classes of triples $(M, \mathcal{C}^M_v, q\epsilon)$, where M is a quasi-Levi subgroup of G, and $(\mathcal{C}^M_v, q\epsilon)$ is a cuspidal enhanced unipotent class in M.

In [AMS, Theorem 5.5], we have attached to every element $q\tau \in \mathfrak{B}(\mathcal{U}_e(G))$ a 2-cocycle

$$\kappa_{q\tau} \colon W_{q\tau}/W^\circ_{q\tau} \times W_{q\tau}/W^\circ_{q\tau} \to \mathbb{C}^\times$$

where $W_{q\tau} := N_G(q\tau)/M$ and $W^\circ_{q\tau} := N_{G^\circ}(M^\circ)/M^\circ$, and constructed a *cuspidal support map*

$$q\Psi_G \colon \mathcal{U}_e(G) \to \mathfrak{B}(\mathcal{U}_e(G))$$

such that

$$\mathcal{U}_e(G) = \bigsqcup_{q\tau \in \mathfrak{B}(\mathcal{U}_e(G))} q\Psi_G^{-1}(q\tau), \tag{2}$$

where $q\Psi_G^{-1}(q\tau)$ is in bijection with the set of isomorphism classes of irreducible representations of twisted algebra $\mathbb{C}[W_{q\tau}, \kappa_{q\tau}]$. Our construction is an extension of, and is based on, the Lusztig's construction of the generalized Springer correspondence for G° in [Lus1].

Let $(\phi, \rho) \in \Phi_e(\mathcal{G}(F))$. We will first apply the construction above to the group $G = G_\phi$ in order to obtain a partition of $\Phi_e(\mathcal{G}(F))$ in the spirit of (2). We write $q\Psi_{G_\phi} = [M, v, q\epsilon]_{G_\phi}$. We showed in [AMS, Proposition 7.3] that, upon replacing (ϕ, ρ) by $\mathcal{G}^\vee$-conjugate, there exists a Levi subgroup $\mathcal{L}(F) \subset \mathcal{G}(F)$ such that $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ is a cuspidal L-parameter for $\mathcal{L}(F)$. Moreover,

$$\mathcal{L}^\vee \rtimes \mathbf{W}_F = Z_{\mathcal{G}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ).$$

We set

$${}^L\Psi(\phi, \rho) := (\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi|_{\mathbf{W}_F}, v, q\epsilon).$$

The right hand side consists of a Langlands dual group and a cuspidal L-parameter for that. Every enhanced L-parameter for ${}^L\mathcal{G}$ is conjugate to one as above, so the map ${}^L\Psi$ is well-defined on the whole of $\Phi_e({}^L\mathcal{G})$.

In [AMS], we defined Bernstein components of enhanced L-parameters. Recall from [Hai, §3.3.1] that the group of unramified characters of $\mathcal{L}(F)$ is naturally isomorphic to $Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F}$. We consider this as an object on the Galois side of the local Langlands correspondence and we write

$$X_{\mathrm{nr}}({}^L\mathcal{L}) = Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ_{\mathbf{W}_F}.$$

Given $(\phi', \rho') \in \Phi_e(\mathcal{L}(F))$ and $z \in X_{\mathrm{nr}}({}^L\mathcal{L})$, we define $(z\phi', \rho') \in \Phi_e(\mathcal{L}(F))$ by

$$z\phi' = \phi' \text{ on } \mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C}) \text{ and } (z\phi')(\mathrm{Frob}_F) = \tilde{z}\phi'(\mathrm{Frob}_F),$$

$\tilde{z} \in Z(\mathcal{L}^\vee \rtimes \mathbf{I}_F)^\circ$ represents z. By definition, an *inertial equivalence class* for $\Phi_e(\mathcal{G}(F))$ consists of a Levi subgroup $\mathcal{L}(F) \subset \mathcal{G}(F)$ and a $X_{\mathrm{nr}}({}^L\mathcal{L})$-orbit $\mathfrak{s}^\vee_{\mathcal{L}}$ in $\Phi_{\mathrm{cusp}}(\mathcal{L}(F))$. Another such object is regarded as equivalent if the two are conjugate by an element of $\mathcal{G}^\vee$. The equivalence class is denoted $\mathfrak{s}^\vee$.

The Bernstein component of $\Phi_e(\mathcal{G}(F))$ associated to $\mathfrak{s}^\vee$ is defined as

$$\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee} := {}^L\Psi^{-1}(\mathcal{L} \rtimes \mathbf{W}_F, \mathfrak{s}^\vee_{\mathcal{L}}). \tag{3}$$

In particular $\Phi_e(\mathcal{L}(F))^{\mathfrak{s}^\vee_{\mathcal{L}}}$ is diffeomorphic to a quotient of the complex torus $X_{\mathrm{nr}}({}^L\mathcal{L})$ by a finite subgroup, albeit not in a canonical way.

With an inertial equivalence class $\mathfrak{s}^\vee$ for $\Phi_e(\mathcal{G}(F))$ we associate the finite group

$$W_{\mathfrak{s}^\vee} := \text{ stabilizer of } \mathfrak{s}^\vee_{\mathcal{L}} \text{ in } N_{\mathcal{G}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee.$$

It plays a role analogous to that of the finite groups appearing in the description of the Bernstein centre of $\mathcal{G}(F)$. We expect that the local Langlands correspondence for $\mathcal{G}(F)$ matches every Bernstein component $\mathrm{Irr}^{\mathfrak{s}}(\mathcal{G}(F))$ for $\mathcal{G}(F)$, where $\mathfrak{s} = [\mathcal{L}(F), \sigma]_{\mathcal{G}(F)}$, with $\mathcal{L}$ an F-Levi subgroup of an F-parabolic subgroup of $\mathcal{G}$ and σ an irreducible supercupidal smooth representation of $\mathcal{L}(F)$, with a Bernstein component $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$, where $\mathfrak{s}^\vee = [\mathcal{L}(F), \mathfrak{s}^\vee_{\mathcal{L}}]_{\mathcal{G}^\vee}$, and that the (twisted) affine Hecke algebras on both sides will correspond.

Let $W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}$ be the isotropy group of $(\phi_v, q\epsilon) \in \mathfrak{s}^\vee_{\mathcal{L}}$. Let $\mathcal{L}^\vee_c \subset \mathcal{G}^\vee_{\mathrm{sc}}$ denote the preimage of $\mathcal{L}^\vee$ under $\mathcal{G}^\vee_{\mathrm{sc}} \to \mathcal{G}^\vee$. With the generalized Springer correspondence, applied to the group $G_\phi \cap \mathcal{L}_c^\vee$, we can attach to any element of ${}^L\Psi^{-1}(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi_v, q\epsilon)$ an irreducible projective representation of $W_{\mathfrak{s}^\vee,\phi_v,q\epsilon}$. More precisely, set

$$q\tau := [G_\phi \cap \mathcal{L}_c^\vee, v, q\epsilon]_{G_\phi}.$$

By [AMS, Lemma 8.2] $W_{q\tau}$ is canonically isomorphic to $W_{\mathfrak{s}^\vee,\phi_v,q\epsilon}$. To the data $q\tau$ we will attach (in Section 4) twisted graded Hecke algebras, whose irreducible representations are parametrized by triples (y, σ_0, ρ) related to $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. Explicitly, using the exponential map for the complex reductive group $Z_{\mathcal{G}^\vee}(\phi(\mathbf{W}_F))$, we can construct $(\phi', \rho') \in \Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ with $u_{\phi'} = \exp(y)$ and $\phi'(\mathrm{Frob}_F) = \phi_v(\mathrm{Frob}_F)\exp(\sigma_0)$.

In the sequel [AMS2] to this paper, we associate to every Bernstein component $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$ a twisted affine Hecke algebra $\mathcal{H}(\mathcal{G}(F), \mathfrak{s}^\vee, \vec{z})$ whose irreducible representations are naturally parametrized by $\Phi_e(\mathcal{G}(F))^{\mathfrak{s}^\vee}$. Here $\vec{z}$ is an abbreviation for an array of complex parameters.

For general linear groups (and their inner forms) and classical groups, it is proved in [AMS2] that there are specializations $\vec{z}$ such that the algebras $\mathcal{H}(\mathcal{G}(F), \mathfrak{s}^\vee, \vec{z})$ are those computed for representations. In general, we expect that the simple modules of $\mathcal{H}(\mathcal{G}(F), \mathfrak{s}^\vee, \vec{z})$ should be in bijection with that of the Hecke algebras for types in reductive p-adic groups (which is the case for special linear groups and their inner forms), and in this way they should contribute to the local Langlands correspondence.

2. The twisted graded Hecke algebra of a cuspidal support

Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Let L be a Levi subgroup of G° and let $v \in \mathfrak{l} = \mathrm{Lie}(L)$ be nilpotent. Let $\mathcal{C}_v^L$ be the adjoint orbit of v and let $\mathcal{L}$ be an irreducible L-equivariant cuspidal local system on $\mathcal{C}_v^L$. Following [Lus1, AMS] we call $(L, \mathcal{C}_v^L, \mathcal{L})$ a cuspidal support for G.

Our aim is to associate to these data a graded Hecke algebra, possibly extended by a twisted group algebra of a finite group, generalizing [Lus3]. Since most of [Lus3] goes through without any problems if G is disconnected, we focus on the parts that do need additional arguments.

Let $P = LU$ be a parabolic subgroup of G° with Levi factor L and unipotent radical U. Write $T = Z(L)^\circ$ and $\mathfrak{t} = \mathrm{Lie}(T)$. By [AMS, Theorem 3.1.a] the group $N_G(L)$ stabilizes $\mathcal{C}_v^L$. Let $N_G(\mathcal{L})$ be the stabilizer in $N_G(L)$ of the local system $\mathcal{L}$ on $\mathcal{C}_v^L$. It contains $N_{G^\circ}(L)$ and it is the same as $N_G(\mathcal{L}^*)$, where $\mathcal{L}^*$ is the dual local system of $\mathcal{L}$. Similarly, let $N_G(P, \mathcal{L})$ be the stabilizer of $(P, L, \mathcal{L})$ in G. We write

$$\begin{aligned}
&W_{\mathcal{L}} = N_G(\mathcal{L})/L,\\
&W_{\mathcal{L}}^\circ = N_{G^\circ}(L)/L,\\
&\mathfrak{R}_{\mathcal{L}} = N_G(P, \mathcal{L})/L,\\
&R(G^\circ, T) = \{\alpha \in X^*(T) \setminus \{0\} : \alpha \text{ appears in the adjoint action of } T \text{ on } \mathfrak{g}\}.
\end{aligned}$$

Lemma 2.1. (*a*) *The set $R(G^\circ, T)$ is (not necessarily reduced) root system with Weyl group $W_{\mathcal{L}}^\circ$.*

(*b*) *The group $W_{\mathcal{L}}^\circ$ is normal in $W_{\mathcal{L}}$ and $W_{\mathcal{L}} = W_{\mathcal{L}}^\circ \rtimes \mathfrak{R}_{\mathcal{L}}$.*

Proof. (a) By [Lus3, Proposition 2.2] $R(G^\circ, T)$ is a root system, and by [Lus1, Theorem 9.2] $N_{G^\circ}(L)/L$ is its Weyl group.

(b) Also by [Lus1, Theorem 9.2], $W_{\mathcal{L}}^\circ$ stabilizes $\mathcal{L}$, so it is contained in $W_{\mathcal{L}}$. Since

G° is normal in G, $W_\mathcal{L}^\circ$ is normal in $W_\mathcal{L}$. The group $\mathfrak{R}_\mathcal{L}$ is the stabilizer in $W_\mathcal{L}$ of the positive system $R(P,T)$ of $R(G^\circ,T)$. Since $W_\mathcal{L}^\circ$ acts simply transitively on the collection of positive systems, $\mathfrak{R}_\mathcal{L}$ is a complement for $W_\mathcal{L}^\circ$. $\square$

Now we give a presentation of the algebra that we want to study. Let $\{\alpha_i : i \in I\}$ be the set of roots in $R(G^\circ,T)$ which are simple with respect to P. Let $\{s_i : i \in I\}$ be the associated set of simple reflections in the Weyl group $W_\mathcal{L}^\circ = N_{G^\circ}(L)/L$. Choose $c_i \in \mathbb{C}$ $(i \in I)$ such that $c_i = c_j$ if s_i and s_j are conjugate in $W_\mathcal{L}$. We can regard $\{c_i : i \in I\}$ as a $W_\mathcal{L}$-invariant function $c : R(G^\circ,T)_{\mathrm{red}} \to \mathbb{C}$, where the subscript "red" indicates the set of indivisible roots.

Let $\natural : (W_\mathcal{L}/W_\mathcal{L}^\circ)^2 \to \mathbb{C}^\times$ be a 2-cocycle. Recall that the twisted group algebra $\mathbb{C}[W_\mathcal{L},\natural]$ has a $\mathbb{C}$-basis $\{N_w : w \in W_\mathcal{L}\}$ and multiplication rules

$$N_w \cdot N_{w'} = \natural(w,w') N_{ww'}.$$

In particular it contains the group algebra of $W_\mathcal{L}^\circ$.

Proposition 2.2. *Let* $\mathbf{r}$ *be an indeterminate, identified with the coordinate function on* $\mathbb{C}$. *There exists a unique associative algebra structure on* $\mathbb{C}[W_\mathcal{L},\natural] \otimes S(\mathfrak{t}^*) \otimes \mathbb{C}[\mathbf{r}]$ *such that:*

- *the twisted group algebra* $\mathbb{C}[W_\mathcal{L},\natural]$ *is embedded as subalgebra;*
- *the algebra* $S(\mathfrak{t}^*) \otimes \mathbb{C}[\mathbf{r}]$ *of polynomial functions on* $\mathfrak{t} \oplus \mathbb{C}$ *is embedded as a subalgebra;*
- $\mathbb{C}[\mathbf{r}]$ *is central;*
- *the braid relation* $N_{s_i}\xi - {}^{s_i}\xi N_{s_i} = c_i \mathbf{r}(\xi - {}^{s_i}\xi)/\alpha_i$ *holds for all* $\xi \in S(\mathfrak{t}^*)$ *and all simple roots* α_i*;*
- $N_w \xi N_w^{-1} = {}^w\xi$ *for all* $\xi \in S(\mathfrak{t}^*)$ *and* $w \in \mathfrak{R}_\mathcal{L}$.

Proof. It is well-known that there exists such an algebra with $W_\mathcal{L}^\circ$ instead of $W_\mathcal{L}$, see for instance [Lus4, §4]. It is called the graded Hecke algebra, over $\mathbb{C}[\mathbf{r}]$ with parameters c_i, and we denote it by $\mathbb{H}(\mathfrak{t}, W_\mathcal{L}^\circ, c\mathbf{r})$.

Let $\mathfrak{R}_\mathcal{L}^+$ be a finite central extension of $\mathfrak{R}_\mathcal{L}$ such that the 2-cocycle $\natural$ lifts to the trivial 2-cocycle of $\mathfrak{R}_\mathcal{L}^+$. For $w^+ \in W_\mathcal{L}^\circ \rtimes \mathfrak{R}_\mathcal{L}^+$ with image $w \in W_\mathcal{L}$ we put

$$\phi_{w^+}(N_{w'}\xi) = N_{ww'w^{-1}} {}^w\xi \qquad w' \in W_\mathcal{L}^\circ, \xi \in S(\mathfrak{t}^*) \otimes \mathbb{C}[\mathbf{r}].$$

Because of the condition on the c_i, $w^+ \mapsto \phi_{w^+}$ defines an action of $\mathfrak{R}_\mathcal{L}^+$ on $\mathbb{H}(\mathfrak{t}, W_\mathcal{L}^\circ, c\mathbf{r})$ by algebra automorphisms. Thus the crossed product algebra

$$\mathfrak{R}_\mathcal{L}^+ \ltimes \mathbb{H}(\mathfrak{t}, W_\mathcal{L}^\circ, c\mathbf{r}) = \mathbb{C}[\mathfrak{R}_\mathcal{L}^+] \ltimes \mathbb{H}(\mathfrak{t}, W_\mathcal{L}^\circ, c\mathbf{r})$$

is well-defined. Let $p_\natural \in \mathbb{C}[\ker(\mathfrak{R}_\mathcal{L}^+ \to \mathfrak{R}_\mathcal{L})]$ be the central idempotent such that

$$p_\natural \mathbb{C}[\mathfrak{R}_\mathcal{L}^+] \cong \mathbb{C}[\mathfrak{R}_\mathcal{L},\natural].$$

The isomorphism is given by $p_\natural w^+ \mapsto \lambda(w^+) N_w$ for a suitable $\lambda(w^+) \in \mathbb{C}^\times$. Then

$$p_\natural \mathbb{C}[\mathfrak{R}_\mathcal{L}^+] \ltimes \mathbb{H}(\mathfrak{t}, W_\mathcal{L}^\circ, c\mathbf{r}) \subset \mathbb{C}[\mathfrak{R}_\mathcal{L}^+] \ltimes \mathbb{H}(\mathfrak{t}, W_\mathcal{L}^\circ, c\mathbf{r}) \tag{4}$$

is an algebra with the required relations. $\square$

We denote the algebra of Proposition 2.2 by $\mathbb{H}(\mathfrak{t}, W_\mathcal{L}, c\mathbf{r}, \natural)$. It is a special case of the algebras considered in [Wit], namely the case where the 2-cocycle $\natural_\mathcal{L}$ and the braid relations live only on the two different factors of the semidirect product $W_\mathcal{L} = W_\mathcal{L}^\circ \rtimes \mathfrak{R}_\mathcal{L}$. Let us mention here some of its elementary properties.

Lemma 2.3. *$S(\mathfrak{t}^*)^{W_{\mathcal{L}}} \otimes \mathbb{C}[\mathbf{r}]$ is a central subalgebra of $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural)$. If $W_{\mathcal{L}}$ acts faithfully on $\mathfrak{t}$, then it equals the centre $Z(\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural))$.*

Proof. The case $W_{\mathcal{L}} = W_{\mathcal{L}}^\circ$ is [Lus3, Theorem 6.5]. For $W_{\mathcal{L}} \neq W_{\mathcal{L}}^\circ$ and $\natural = 1$ see [Sol2, Proposition 5.1.a]. The latter argument also works if $\natural$ is nontrivial. □

If V is a $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural)$-module on which $S(\mathfrak{t}^*)^{W_{\mathcal{L}}} \otimes \mathbb{C}[\mathbf{r}]$ acts by a character $(W_{\mathcal{L}}x, r)$, then we will say that the module admits the central character $(W_{\mathcal{L}}x, r)$.

A look at the defining relations reveals that there is a unique anti-isomorphism

$$* : \mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural) \to \mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural^{-1}) \tag{5}$$

such that $*$ is the identity on $S(\mathfrak{t}^*) \otimes \mathbb{C}[\mathbf{r}]$ and $N_w^* = (N_w)^{-1}$, the inverse of the basis element $N_w \in \mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural^{-1})$. Hence $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural^{-1})$ is the opposite algebra of $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural)$, and $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}^\circ, c\mathbf{r})$ is isomorphic to its opposite.

Suppose that $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{z}$ is a decomposition of $W_{\mathcal{L}}$-representations such that $\mathrm{Lie}(Z(L) \cap G_{\mathrm{der}}) \subset \mathfrak{t}'$ and $\mathfrak{z} \subset \mathfrak{t}^{W_{\mathcal{L}}}$. Then

$$\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural) = \mathbb{H}(\mathfrak{t}', W_{\mathcal{L}}, c\mathbf{r}, \natural) \otimes_{\mathbb{C}} S(\mathfrak{z}^*). \tag{6}$$

For example, if $W_{\mathcal{L}} = W_{\mathcal{L}}^\circ$ we can take $\mathfrak{t}' = \mathrm{Lie}(Z(L) \cap G_{\mathrm{der}})$ and $\mathfrak{z} = \mathrm{Lie}(Z(G))$.

Now we set out to construct $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural)$ geometrically. In the process we will specify the parameters c_i and the 2-cocycle $\natural$.

If X is a complex variety equiped with a continuous action of G and stratified by some algebraic stratification, we denote by $\mathcal{D}_c^b(X)$ the bounded derived category of constructible sheaves on X and by $\mathcal{D}_{G,c}^b(X)$ the G-equivariant bounded derived category as defined in [BeLu]. We denote by $\mathcal{P}(X)$ (resp. $\mathcal{P}_G(X)$) the category of perverse sheaves (resp. G-equivariant perverse sheaves) on X. Let us recall briefly how $\mathcal{D}_{G,c}^b(X)$ is defined. First, if $p\mathcal{P} \to X$ is a G-map where P is a free G-space and $q \colon P \to G\backslash P$ is the quotient map, then the category $\mathcal{D}_G^b(X, P)$ consists in triples $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \beta)$ with $\mathcal{F}_X \in \mathcal{D}^b(X)$, $\overline{\mathcal{F}} \in \mathcal{D}^b(G\backslash P)$, and an isomorphism $\beta : p^*\mathcal{F}_X \simeq q^*\overline{\mathcal{F}}$. Let $I \subset \mathbb{Z}$ be a segment. If $p \colon P \to X$ is an n-acyclic resolution of X with $n \geqslant |I|$, then $\mathcal{D}_G^I(X)$ is defined to be $\mathcal{D}_G^b(X, P)$ and this does not depend on the choice of P. Finally, the G-equivariant derived category $\mathcal{D}_G^b(X)$ is defined as the limit of the categories $\mathcal{D}_G^I(X)$. Moreover, $\mathcal{P}_G(X)$ is the subcategory of $\mathcal{D}_G^b(X)$ consisting of objects $\mathcal{F}$ such that $\mathcal{F}_X \in \mathcal{P}(X)$. All the usual functors, Verdier duality, intermediate extension, etc., exist and are well-defined in this category. We will denote by For: $\mathcal{D}_G^b(X) \to \mathcal{D}^b(X)$ the functor which associates to every $\mathcal{F} \in \mathcal{D}_G^b(X)$ the complex $\mathcal{F}_X$.

Consider the varieties

$$\begin{aligned}
\dot{\mathfrak{g}} &= \{(x, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t} + \mathfrak{u}\}, \\
\dot{\mathfrak{g}}^\circ &= \{(x, gP) \in \mathfrak{g} \times G^\circ/P : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t} + \mathfrak{u}\}, \\
\dot{\mathfrak{g}}_{RS} &= \{(x, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t}_{\mathrm{reg}} + \mathfrak{u}\}, \\
\dot{\mathfrak{g}}_{RS}^\circ &= \{(x, gP) \in \mathfrak{g} \times G^\circ/P : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t}_{\mathrm{reg}} + \mathfrak{u}\}
\end{aligned}$$

where $\mathfrak{t}_{\mathrm{reg}} = \{x \in \mathfrak{t} : Z_{\mathfrak{g}}(x) = \mathfrak{l}\}$. We let $G \times \mathbb{C}^\times$ act on these varieties by

$$(g_1, \lambda) \cdot (x, gP) = (\lambda^{-2}\mathrm{Ad}(g_1)x, g_1 gP).$$

Assume first that $\dot{\mathfrak{g}} = \dot{\mathfrak{g}}^\circ$ and so $\dot{\mathfrak{g}}_{RS} = \dot{\mathfrak{g}}^\circ_{RS}$. Consider the maps

$$\begin{array}{ll} \mathcal{C}_v^L \xleftarrow{f_1} \{(x,g) \in \mathfrak{g} \times G : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t} + \mathfrak{u}\} \xrightarrow{f_2} \dot{\mathfrak{g}}, & \\ f_1(x,g) = \mathrm{pr}_{\mathcal{C}_v^L}(\mathrm{Ad}(g^{-1})x), & f_2(x,g) = (x, gP). \end{array}$$

The group $G \times \mathbb{C}^\times \times P$ acts on $\{(x,g) \in \mathfrak{g} \times G : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t} + \mathfrak{u}\}$ by

$$(g_1, \lambda, p) \cdot (x,g) = (\lambda^{-2}\mathrm{Ad}(g_1)x, g_1 g p).$$

Let $\dot{\mathcal{L}}$ be the unique G-equivariant local system on $\dot{\mathfrak{g}}$ such that $f_2^* \dot{\mathcal{L}} = f_1^* \mathcal{L}$. The map

$$\mathrm{pr}_1 : \dot{\mathfrak{g}}_{RS} \to \mathfrak{g}_{RS} := \mathrm{Ad}(G)(\mathcal{C}_v^L + \mathfrak{t}_{\mathrm{reg}} + \mathfrak{u})$$

is a fibration with fibre $N_G(L)/L$, so $(\mathrm{pr}_1)_! \dot{\mathcal{L}}$ is a local system on $\mathfrak{g}_{RS}$. Let $\mathcal{V} := \mathrm{Ad}(G)(\overline{\mathcal{C}_v^L} + \mathfrak{t} + \mathfrak{u})$, $j : \mathcal{C}_v^L \hookrightarrow \overline{\mathcal{C}_v^L}$ and $\widehat{j} : \dot{\mathfrak{g}}_{RS} \hookrightarrow \mathcal{V}$. Since $\mathcal{L}$ is a cuspidal local system, by [Lus3, 2.2.b)] it is clean, so $j_! \mathcal{L} = j_* \mathcal{L} \in \mathcal{D}^b_L(\overline{\mathcal{C}_v^L})$. It follows (by unicity and base changes) that $\widehat{j}_! \dot{\mathcal{L}} = \widehat{j}_* \dot{\mathcal{L}} \in \mathcal{D}^b_G(\widehat{\mathfrak{g}}_{RS})$. Let $K_1 = \mathrm{IC}_G(\mathfrak{g}_{RS}, (\mathrm{pr}_1)_! \dot{\mathcal{L}})$ be the equivariant intersection cohomology complex defined by $(\mathrm{pr}_1)_! \dot{\mathcal{L}}$.

Considering pr_1 as a map $\dot{\mathfrak{g}} \to \mathfrak{g}$, we get (up to a shift) a G-equivariant perverse sheaf $K = (\mathrm{pr}_1)_! \dot{\mathcal{L}} = i_! K_1$ on $\mathfrak{g}$, where $i : \mathcal{V} \hookrightarrow \mathfrak{g}$. Indeed, by definition it is enough to show that $\mathrm{For}(K_1) \in \mathcal{D}^b(\mathfrak{g})$ is perverse. But the same arguments of [Lus3, 3.4] apply here (smallness of $\mathrm{pr}_1 : \dot{\mathfrak{g}} \to \mathcal{V}$, equivariant Verdier duality, etc) and the forgetful functor commutes with $(\mathrm{pr}_1)_!$ by [BeLu, 3.4.1].

Now, if $\dot{\mathfrak{g}} \neq \dot{\mathfrak{g}}^\circ$, then $\dot{\mathfrak{g}} = G \times_{G^S} \dot{\mathfrak{g}}^\circ$ where G^S is the largest subgroup of G which preserves $\dot{\mathfrak{g}}^\circ$. Using [BeLu, 5.1. Proposition (ii)] it follows that K is a perverse sheaf. Notice that $(\mathrm{pr}_1)_! \dot{\mathcal{L}}^*$ is another local system on $\mathfrak{g}_{RS}$. In the same way we construct K_1^* (on $\mathfrak{g}_{RS}$) and $K^* = (\mathrm{pr}_1)_! \dot{\mathcal{L}}^* = i_! K_1^*$ (on $\mathfrak{g}$).

Remark 2.4. In [AMS, §4] the authors consider a local system $\pi_* \tilde{\mathcal{E}}$ on a subvariety Y of G°. The local systems $(\mathrm{pr}_1)_! \dot{\mathcal{L}}$ and $(\mathrm{pr}_1)_! \dot{\mathcal{L}}^*$ on $\mathfrak{g}_{RS}$ are the direct analogues of $\pi_* \tilde{\mathcal{E}}$, when we apply the exponential map to replace G° by its Lie algebra $\mathfrak{g}$. As Lusztig notes in [Lus3, 2.2] (for connected G), this allows us transfer all the results of [AMS] to the current setting. In this paper we will freely make use of [AMS] in the Lie algebra setting as well.

In [AMS, Proposition 4.5] we showed that the G-endomorphism algebras of $(\mathrm{pr}_1)_! \dot{\mathcal{L}}$ and $(\mathrm{pr}_1)_! \dot{\mathcal{L}}^*$, in the category of equivariant local systems, are isomorphic to twisted group algebras:

$$\begin{array}{rcl} \mathrm{End}_G\big((\mathrm{pr}_1)_! \dot{\mathcal{L}}\big) & \cong & \mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}], \\ \mathrm{End}_G\big((\mathrm{pr}_1)_! \dot{\mathcal{L}}^*\big) & \cong & \mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}^{-1}], \end{array} \tag{7}$$

where $\natural_{\mathcal{L}} : (W_{\mathcal{L}} / W_{\mathcal{L}}^\circ)^2 \to \mathbb{C}^\times$ is a 2-cocycle. The cocycle $\natural_{\mathcal{L}}^{-1}$ in (7) is the inverse of $\natural_{\mathcal{L}}$, necessary because we use the dual $\mathcal{L}^*$.

Remark 2.5. In fact there are two good ways to let (7) act on $(\mathrm{pr}_1)_! \dot{\mathcal{L}}$. For the moment we subscribe to the normalization of Lusztig from [Lus1, §9], which is based on identifying a suitable cohomology space with the trivial representation of $W_{\mathcal{L}}^\circ$. However, later we will switch to a different normalization, which identifies the same space with the sign representation of the Weyl group $W_{\mathcal{L}}^\circ$.

According to [Lus3, 3.4] this gives rise to an action of $\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}^{-1}]$ on K_1^* and then on K^*. (And similarly without duals, of course.) Applying the above with the group $G \times \mathbb{C}^\times$ and the cuspidal local system $\mathcal{L}$ on $\mathcal{C}_v^L \times \{0\} \subset \mathfrak{l} \oplus \mathbb{C}$, we see that all these endomorphisms are even $G \times \mathbb{C}^\times$-equivariant.

Define $\mathrm{End}_G^+((\mathrm{pr}_1)_! \dot{\mathcal{L}})$ as the subalgebra of $\mathrm{End}_G((\mathrm{pr}_1)_! \dot{\mathcal{L}})$ which also preserves $\mathrm{Lie}(P)$. Then

$$\begin{array}{lll} \mathrm{End}_G^+\big((\mathrm{pr}_1)_! \dot{\mathcal{L}}\big) & \cong & \mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural_\mathcal{L}], \\ \mathrm{End}_G^+\big((\mathrm{pr}_1)_! \dot{\mathcal{L}}^*\big) & \cong & \mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural_\mathcal{L}^{-1}], \end{array} \tag{8}$$

The action of the subalgebra $\mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural_\mathcal{L}^{-1}]$ on K^* admits a simpler interpretation. For any representative $\bar{w} \in N_G(P, L)$ of $w \in \mathfrak{R}_\mathcal{L}$, the map $\mathrm{Ad}(\bar{w}) \in \mathrm{Aut}_\mathbb{C}(\mathfrak{g})$ stabilizes $\mathfrak{t} = \mathrm{Lie}(Z(L))$ and $\mathfrak{u} = \mathrm{Lie}(U) \lhd \mathrm{Lie}(P)$. Furthermore $\mathcal{C}_v^L$ supports a cuspidal local system, so by [AMS, Theorem 3.1.a] it also stable under the automorphism $\mathrm{Ad}(\bar{w})$. Hence $\mathfrak{R}_\mathcal{L}$ acts on $\dot{\mathfrak{g}}$ by

$$w \cdot (x, gP) = (x, g w^{-1} P). \tag{9}$$

The action of $w \in \mathfrak{R}_L$ on $(\dot{\mathfrak{g}}, \dot{\mathcal{L}}^*)$ lifts (9), extending the automorphisms of $(\dot{\mathfrak{g}}_{RS}, \dot{\mathcal{L}}^*)$ constructed in [AMS, (44) and Proposition 4.5]. By functoriality this induces an action of w on K^*.

For $\mathrm{Ad}(G)$-stable subvarieties $\mathcal{V}$ of $\mathfrak{g}$, we define, as in [Lus3, §3],

$$\begin{aligned} \dot{\mathcal{V}} &= \{(x, gP) \in \dot{\mathfrak{g}} : x \in \mathcal{V}\}, \\ \ddot{\mathcal{V}} &= \{(x, gP, g'P) : (x, gP) \in \dot{\mathcal{V}}, (x, g'P) \in \dot{\mathcal{V}}\}. \end{aligned}$$

The two projections $\pi_{12}, \pi_{13} : \ddot{\mathcal{V}} \to \dot{\mathcal{V}}$ give rise to a $G \times \mathbb{C}^\times$-equivariant local system $\ddot{\mathcal{L}} = \dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}^*$ on $\ddot{\mathcal{V}}$. As in [Lus3], the action of $\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}^{-1}]$ on K^* leads to

$$\text{actions of } \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}] \otimes \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}^{-1}] \text{ on } \ddot{\mathcal{L}} \text{ and on } H_j^{G^\circ \times \mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}}), \tag{10}$$

denoted $(w, w') \mapsto \Delta(w) \otimes \Delta(w')$. By [Lus3, Proposition 4.2] there is an isomorphism of graded algebras

$$H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}) \cong S(\mathfrak{t}^* \oplus \mathbb{C}) = S(\mathfrak{t}^*) \otimes \mathbb{C}[\mathbf{r}],$$

where $\mathfrak{t}^* \oplus \mathbb{C}$ lives in degree 2. This algebra acts naturally on $H_*^{G \times \mathbb{C}^\times}(\dot{\mathcal{V}}, \dot{\mathcal{L}})$ and that yields two actions $\Delta(\xi)$ (from π_{12}) and $\Delta'(\xi)$ (from π_{13}) of $\xi \in S(\mathfrak{t}^* \oplus \mathbb{C})$ on $H_*^{G \times \mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}})$.

Let $\Omega \subset G$ be a $P - P$ double coset and write

$$\ddot{\mathcal{V}}^\Omega = \{(x, gP, g'P) \in \ddot{\mathcal{V}} : g^{-1} g' \in \Omega\}.$$

Given any sheaf $\mathcal{F}$ on a variety $\mathcal{V}$, we denote its stalk at $v \in \mathcal{V}$ by $\mathcal{F}_v$ or $\mathcal{F}|_v$.

Proposition 2.6. *Let $\mathcal{V} = \ddot{\mathfrak{g}}$ or $\mathcal{V} = \ddot{\mathfrak{g}_N}$, where $\mathfrak{g}_N$ is the variety of nilpotent elements in $\mathfrak{g}$.*

(*a*) *The $S(\mathfrak{t}^* \oplus \mathbb{C})$-module structures Δ and Δ' define isomorphisms*

$$S(\mathfrak{t}^* \oplus \mathbb{C}) \otimes H_0^{G \times \mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}}) \rightrightarrows H_*^{G \times \mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}}).$$

(*b*) *As $\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}]$-modules*

$$H_0^{G \times \mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}}) = \bigoplus\nolimits_{w \in W_\mathcal{L}} \Delta(w) H_0^{G \times \mathbb{C}^\times}(\ddot{\mathcal{V}}^P, \ddot{\mathcal{L}}) \cong \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}].$$

Proof. We have to generalize [Lus3, Proposition 4.7] to the case where G is disconnected. We say that a $P-P$ double coset $\Omega \subset G$ is good if it contains an element of $N_G(L,\mathcal{L})$, and bad otherwise. Recall from [Lus1, Theorem 9.2] that $N_{G^\circ}(L,\mathcal{L}) = N_{G^\circ}(T)$. Let us consider $H_0^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}^\Omega, \ddot{\mathcal{L}})$.

- If Ω is good, then Lusztig's argument proves that $H_0^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}^\Omega, \ddot{\mathcal{L}}) \cong S(\mathfrak{t}^*\oplus\mathbb{C})$.
- If Ω does not meet $PN_G(L)P$, then Lusztig's argument goes through and shows that $H_0^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}^\Omega, \ddot{\mathcal{L}}) = 0$.
- Finally, if $\Omega \subset PN_G(L)P \setminus PN_G(L,\mathcal{L})P$, we pick any $g_0 \in \Omega$. Then [Lus3, p. 177] entails that

$$
\begin{aligned}
(11)\quad H_0^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}^\Omega, \ddot{\mathcal{L}}) &\cong H_0^{L\times\mathbb{C}^\times}(\mathcal{C}_v^L, \mathcal{L}\boxtimes \mathrm{Ad}(g_0)^*\mathcal{L}^*) \cong \\
& H_0^{Z_{L\times\mathbb{C}^\times}(v)}\big(\{v\}, (\mathcal{L}\boxtimes \mathrm{Ad}(g_0)^*\mathcal{L}^*)_v\big) \cong \\
& H_0^{Z_{L\times\mathbb{C}^\times}(v)}(\{v\}) \otimes (\mathcal{L}\boxtimes \mathrm{Ad}(g_0)^*\mathcal{L}^*)_v^{Z_{L\times\mathbb{C}^\times}(v)} = 0,
\end{aligned}
$$

  because $\mathrm{Ad}(g_0)^*\mathcal{L}^* \not\cong \mathcal{L}^*$.

We also note that (11) with P instead of Ω gives

$$
\begin{aligned}
& H_*^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}^P, \ddot{\mathcal{L}}) \cong H_*^{L\times\mathbb{C}^\times}(\mathcal{C}_v^L, \mathcal{L}\boxtimes\mathcal{L}^*) \cong \\
& \quad H_*^{Z_{L\times\mathbb{C}^\times}(v)}(\{v\}) \otimes (\mathcal{L}\boxtimes\mathcal{L}^*)_v^{Z_{L\times\mathbb{C}^\times}(v)} = H_*^{Z_{L\times\mathbb{C}^\times}(v)}(\{v\}) \otimes \mathrm{End}_{Z_{L\times\mathbb{C}^\times}(v)}(\mathcal{L}_v).
\end{aligned}
$$

By the irreducibility of $\mathcal{L}$ the right hand side is isomorphic to $H_*^{Z_{L\times\mathbb{C}^\times}(v)}(\{v\})$, which by [Lus3, p. 177] is

$$
S\big(\mathrm{Lie}(Z_{L\times\mathbb{C}^\times}(v))^*\big) = S(\mathfrak{t}^*\oplus\mathbb{C}).
$$

In particular $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}^P, \ddot{\mathcal{L}})$ is an algebra contained in $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}})$.

These calculations suffice to carry the entire proof of [Lus3, Proposition 4.7] out. It establishes (a) and

$$
\dim H_0^{G\times\mathbb{C}^\times}(\ddot{\mathcal{V}}, \ddot{\mathcal{L}}) = |W_\mathcal{L}|.
$$

Then (b) follows in the same way as [Lus3, 4.11.a]. □

The $W_\mathcal{L}$-action on T induces an action of $W_\mathcal{L}$ on $S(\mathfrak{t}^*)\otimes\mathbb{C}[\mathbf{r}]$, which fixes $\mathbf{r}$. For α in the root system $R(G^\circ,T)$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the associated eigenspace for the T-action. Let $\alpha_i \in R(G^\circ,T)$ be a simple root (with respect to P) and let $s_i \in W_\mathcal{L}^\circ$ be the corresponding simple reflection. We define $c_i \in \mathbb{Z}_{\geq 2}$ by

$$
(12)\qquad \begin{array}{lll} \mathrm{ad}(v)^{c_i-2}: & \mathfrak{g}_{\alpha_i}\oplus\mathfrak{g}_{2\alpha_i} \to \mathfrak{g}_{\alpha_i}\oplus\mathfrak{g}_{2\alpha_i} & \text{is nonzero,} \\ \mathrm{ad}(v)^{c_i-1}: & \mathfrak{g}_{\alpha_i}\oplus\mathfrak{g}_{2\alpha_i} \to \mathfrak{g}_{\alpha_i}\oplus\mathfrak{g}_{2\alpha_i} & \text{is zero.} \end{array}
$$

By [Lus3, Proposition 2.12] $c_i = c_j$ if s_i and s_j are conjugate in $N_G(L)/L$. According to [Lus3, Theorem 5.1], for all $\xi \in S(\mathfrak{t}^*\oplus\mathbb{C}) = S(\mathfrak{t}^*)\otimes\mathbb{C}[\mathbf{r}]$:

$$
(13)\qquad \begin{array}{lll} \Delta(s_i)\Delta(\xi) - \Delta({}^{s_i}\xi)\Delta(s_i) & = & c_i\Delta(\mathbf{r}(\xi - {}^{s_i}\xi)/\alpha_i), \\ \Delta'(s_i)\Delta'(\xi) - \Delta'({}^{s_i}\xi)\Delta'(s_i) & = & c_i\Delta'(\mathbf{r}(\xi - {}^{s_i}\xi)/\alpha_i). \end{array}
$$

Lemma 2.7. *For all $w \in \mathfrak{R}_\mathcal{L}$ and $\xi \in S(\mathfrak{t}^*\oplus\mathbb{C})$:*

$$
\begin{array}{lll} \Delta(w)\Delta(\xi) & = & \Delta({}^w\xi)\Delta(w), \\ \Delta'(w)\Delta'(\xi) & = & \Delta'({}^w\xi)\Delta'(w). \end{array}
$$

Proof. Recall that $\Delta(\xi)$ is given by $S(\mathfrak{t}^* \oplus \mathbb{C}) \cong H^*_{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}})$ and the product in equivariant (co)homology

$$H^*_{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}}) \otimes H_*^{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}},\dot{\mathcal{L}}) \to H_*^{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}},\dot{\mathcal{L}}).$$

As explained after (9), the action of $w \in \mathfrak{R}_\mathcal{L}$ on $(\dot{\mathfrak{g}},\dot{\mathcal{L}})$ is a straightforward lift of the action (9) on $\dot{\mathfrak{g}}$. It follows that

$$\Delta(w)\Delta(\xi)\Delta(w)^{-1} = \Delta({}^{\bar{w}}\xi),$$

where $\xi \mapsto {}^{\bar{w}}\xi$ is the action induced by (9). Working through all the steps of the proof of [Lus3, Proposition 4.2], we see that this corresponds to the natural action $\xi \mapsto {}^{w}\xi$ of $\mathfrak{R}_\mathcal{L}$ on $S(\mathfrak{t}^* \oplus \mathbb{C})$. □

Let $\mathbb{H}(G,L,\mathcal{L})$ be the algebra $\mathbb{H}(\mathfrak{t},W_\mathcal{L},c\mathbf{r},\natural_\mathcal{L})$, with the 2-cocycle $\natural_\mathcal{L}$ and the parameters c_i from (12). By (5) its opposite algebra is

$$\mathbb{H}(G,L,\mathcal{L})^{op} \cong \mathbb{H}(G,L,\mathcal{L}^*) = \mathbb{H}(\mathfrak{t},W_\mathcal{L},c\mathbf{r},\natural_\mathcal{L}^{-1}). \tag{14}$$

Using (8) we can interpret

$$\mathbb{H}(G,L,\mathcal{L}) = \mathbb{H}(\mathfrak{t},W_\mathcal{L},c\mathbf{r}) \rtimes \mathrm{End}^+_G\big((pr_1)_!\dot{\mathcal{L}}\big). \tag{15}$$

Lemma 2.8. *With the above interpretation $\mathbb{H}(G,L,\mathcal{L})$ is determined uniquely by $(G,L,\mathcal{L})$, up to canonical isomorphisms.*

Proof. The only arbitrary choices are P and $\natural_\mathcal{L} : \mathfrak{R}_\mathcal{L}^2 \to \mathbb{C}^\times$.

A different choice of a parabolic subgroup $P' \subset G$ with Levi factor L would give rise to a different algebra $\mathbb{H}(G,L,\mathcal{L})'$. However, Lemma 2.1.a guarantees that there is a unique (up to P) element $g \in G^\circ$ with $gPg^{-1} = P'$. Conjugation with g provides a canonical isomorphism between the two algebras under consideration.

The 2-cocycle $\natural_\mathcal{L}$ depends on the choice of elements $N_\gamma \in \mathrm{End}^+_G\big((pr_1)_!\dot{\mathcal{L}}\big)$. This choice is not canonical, only the cohomology class of $\natural_\mathcal{L}$ is uniquely determined. Fortunately, this indefiniteness drops out when we replace $\mathbb{C}[\mathfrak{R}_\mathcal{L},\natural_\mathcal{L}]$ by $\mathrm{End}^+_G\big((pr_1)_!\dot{\mathcal{L}}\big)$. Every element of $\mathbb{C}^\times N_\gamma \subset \mathrm{End}^+_G\big((pr_1)_!\dot{\mathcal{L}}\big)$ has a well-defined conjugation action on $\mathbb{H}(\mathfrak{t},W_\mathcal{L},c\mathbf{r})$, depending only on $\gamma \in \mathfrak{R}_\mathcal{L}$. This suffices to define the crossed product $\mathbb{H}(\mathfrak{t},W_\mathcal{L},c\mathbf{r}) \rtimes \mathrm{End}^+_G\big((pr_1)_!\dot{\mathcal{L}}\big)$ in a canonical way. □

The group $W_\mathcal{L}$ and its 2-cocycle $\natural_\mathcal{L}$ from [AMS, §4] can be constructed using only the finite index subgroup $G^\circ N_G(P,\mathcal{L}) \subset G$. Hence

$$\mathbb{H}(G,L,\mathcal{L}) = \mathbb{H}(G^\circ N_G(P,\mathcal{L}),L,\mathcal{L}). \tag{16}$$

With (10), (13) and Lemma 2.7 we can define endomorphisms $\Delta(h)$ and $\Delta'(h')$ of $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N,\ddot{\mathcal{L}})$ for every $h \in \mathbb{H}(G,L,\mathcal{L})$ and every $h' \in \mathbb{H}(G,L,\mathcal{L}^*)$.

Let $1 \in H_0^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N^P,\ddot{\mathcal{L}}) \cong S(\mathfrak{t}^* \oplus \mathbb{C})$ be the unit element.

Corollary 2.9. (*a*) *The map $\mathbb{H}(G,L,\mathcal{L}) \to H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N,\ddot{\mathcal{L}}) : h \mapsto \Delta(h)1$ is bijective.*

(*b*) *The map $\mathbb{H}(G,L,\mathcal{L}^*) \to H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N,\ddot{\mathcal{L}}) : h' \mapsto \Delta'(h')1$ is bijective.*

(*c*) *The operators $\Delta(h)$ and $\Delta'(h')$ commute, and $(h,h') \mapsto \Delta(h)\Delta'(h')$ identifies $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N,\ddot{\mathcal{L}})$ with the biregular representation of $\mathbb{H}(G,L,\mathcal{L})$.*

Proof. This follows in the same way as [Lus3, Corollary 6.4], when we take Proposition 2.6 and (14) into account. □

3. Representations of twisted graded Hecke algebras

We will extend the construction and parametrization of $\mathbb{H}(G, L, \mathcal{L})$-modules from [Lus3, Lus5] to the case where G is disconnected. In this section we work under the following assumption:

Condition 3.1. The group G equals $N_G(P, \mathcal{L})G^\circ$.

In view of (16) this does not pose any restriction on the collection of algebras that we consider.

3.1. Standard modules.

Let $y \in \mathfrak{g}$ be nilpotent and define

$$\mathcal{P}_y = \{gP \in G/P : \mathrm{Ad}(g^{-1})y \in \mathcal{C}_v^L + \mathfrak{u}\}.$$

The group

$$M(y) = \{(g_1, \lambda) \in G \times \mathbb{C}^\times : \mathrm{Ad}(g_1)y = \lambda^2 y\}$$

acts on $\mathcal{P}_y$ by $(g_1, \lambda) \cdot gP = g_1 gP$. Clearly $\mathcal{P}_y$ contains an analogous variety for G°:

$$\mathcal{P}_y^\circ := \{gP \in G^\circ/P : \mathrm{Ad}(g^{-1})y \in \mathcal{C}_v^L + \mathfrak{u}\}.$$

Since $\mathcal{C}_v^L$ is stable under $\mathrm{Ad}(N_G(L))$, $\mathcal{C}_v^L + \mathfrak{u}$ is stable under $\mathrm{Ad}(N_G(P))$. As $N_{G^\circ}(P) = P$ and $N_G(P, \mathcal{L})P/P \cong \mathfrak{R}_\mathcal{L}$, there is an isomorphism of $M(y)$-varieties

$$\mathcal{P}_y^\circ \times \mathfrak{R}_\mathcal{L} \to \mathcal{P}_y : (gP, w) \mapsto gw^{-1}P. \tag{17}$$

The local system $\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}$ restricts to a local system on $\mathcal{P}_y \cong \{y\} \times \mathcal{P}_y \subset \dot{\mathfrak{g}}$. We will endow the space

$$H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) \tag{18}$$

with the structure of an $\mathbb{H}(G, L, \mathcal{L})$-module. With the method of [Lus3, p. 193], the action of $\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}^{-1}]$ on K^* from (7) gives rise to an action $\tilde{\Delta}$ on the dual space of (18). With the aid of (5), the map

$$\Delta : \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}] \to \mathrm{End}_\mathbb{C}\big(H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})\big), \quad \Delta(N_w) = \tilde{\Delta}\big((N_w)^{-1}\big)^* \tag{19}$$

makes (18) into a graded $\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}]$-module.

We describe the action of $S(\mathfrak{t}^* \oplus \mathbb{C}) \cong H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}})$ in more detail. The inclusions

$$\{y\} \times \mathcal{P}_y \subset (G \times \mathbb{C}^\times) \cdot (\{y\} \times \mathcal{P}_y) \subset \dot{\mathfrak{g}}$$

give maps

$$H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}) \to H^*_{G \times \mathbb{C}^\times}(G \times \mathbb{C}^\times \cdot \{y\} \times \mathcal{P}_y) \to H^*_{M(y)}(\mathcal{P}_y). \tag{20}$$

Here $(G \times \mathbb{C}^\times) \cdot (\{y\} \times \mathcal{P}_y) \cong (G \times \mathbb{C}^\times) \times_{M(y)} \mathcal{P}_y$, so by [Lus3, 1.6] the second map in (20) is an isomorphism. Recall from [Lus3, 1.9] that

$$H^*_{M(y)}(\mathcal{P}_y) \cong H^*_{M(y)^\circ}(\mathcal{P}_y)^{M(y)/M(y)^\circ}.$$

The product

$$H^*_{M(y)^\circ}(\mathcal{P}_y) \otimes H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) \to H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) \tag{21}$$

gives an action of the graded algebras in (20) on the graded vector space $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$. We denote the operator associated to $\xi \in S(\mathfrak{t}^* \oplus \mathbb{C})$ by $\Delta(\xi)$.

The projection $\{y\}\times\mathcal{P}_y \to \{y\}$ induces an algebra homomorphism $H^*_{M(y)^\circ}(\{y\}) \to H^*_{M(y)^\circ}(\mathcal{P}_y)$. With (21) this also gives an action of $H^*_{M(y)^\circ}(\{y\})$ on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$. Furthermore $M(y)$ acts naturally on $H^*_{M(y)^\circ}(\{y\})$ and on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$, and this action factors through the finite group $\pi_0(M(y)) = M(y)/M(y)^\circ$.

Theorem 3.2. *[Lusztig]*

(*a*) *The above operators* $\Delta(w)$ *and* $\Delta(\xi)$ *make* $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$ *into a graded* $\mathbb{H}(G, L, \mathcal{L})$*-module.*

(*b*) *The actions of* $H^*_{M(y)^\circ}(\{y\})$ *and* $\mathbb{H}(G, L, \mathcal{L})$ *commute.*

(*c*) $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$ *is finitely generated and projective as* $H^*_{M(y)^\circ}(\{y\})$*-module.*

(*d*) *The action of* $\pi_0(M(y))$ *commutes with the* $\mathbb{H}(G, L, \mathcal{L})$*-action. It is semilinear with respect to* $H^*_{M(y)^\circ}(\{y\})$*, that is, for* $m \in \pi_0(M(y)), \mu \in H^*_{M(y)^\circ}(\{y\})$*,* $h \in \mathbb{H}(G, L, \mathcal{L})$ *and* $\eta \in H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$*:*

$$m \cdot (\mu \otimes \Delta(h)\eta) = (m \cdot \mu) \otimes \Delta(h)(m \cdot \eta) = \Delta(h)\big((m \cdot \mu) \otimes (m \cdot \eta)\big).$$

Proof. (b) The actions of $S(\mathfrak{t}^*\oplus\mathbb{C})$ and $H^*_{M(y)^\circ}(\{y\})$ both come from (21). The algebra $H^*_{M(y)^\circ}(\mathcal{P}_y)$ is graded commutative [Lus3, 1.3]. However, since $H_j^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) = 0$ for odd j [Lus3, Propostion 8.6.a], only the action of the subalgebra $H^{\mathrm{even}}_{M(y)^\circ}(\mathcal{P}_y)$ matters. Since this is a commutative algebra, the actions of $S(\mathfrak{t}^*\oplus\mathbb{C})$ and $H^*_{M(y)^\circ}(\{y\})$ commute.

Write $\tilde{\mathcal{O}} = (G \times \mathbb{C}^\times)/M(y)^\circ$ and define

$$h : \tilde{\mathcal{O}} \to \mathfrak{g}, \quad (g, \lambda) \mapsto \lambda^{-2}\mathrm{Ad}(g)y.$$

There are natural isomorphisms

$$H^*_{M(y)^\circ}(\{y\}) \cong H^*_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}),$$
$$H_j^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})^* \cong H^{2\dim(\tilde{\mathcal{O}})-j}_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}, h^*K^*).$$

The dual of the action of $H^*_{M(y)^\circ}(\{y\})$ on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$ becomes the product

$$H^*_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}) \otimes H^*_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}, h^*K^*) \to H^*_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}, h^*K^*).$$

From the proof of [Lus3, 4.4] one sees that this action commutes with the operators $\tilde{\Delta}(w)$. Hence the $\Delta(w)$ also commute with the $H^*_{M(y)^\circ}(\{y\})$-action.

(c) See [Lus3, Proposition 8.6.c].

(d) The semilinearity is a consequence of the functoriality of the product in equivariant homology. Since the action of $S(\mathfrak{t}^* \oplus \mathbb{C})$ factors via

$$H^*_{M(y)}(\mathcal{P}_y) \cong H^*_{M(y)^\circ}(\mathcal{P}_y)^{\pi_0(M(y))},$$

it commutes with the action of $\pi_0(M(y))$ on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$.

The algebra $\mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}^{-1}]$ acts on $(\mathfrak{g}, K^*)$ and on $(\tilde{\mathcal{O}}, h^*K^*)$ by $G \times \mathbb{C}^\times$-equivariant endomorphisms. In other words, the operators $\tilde{\Delta}(w)$ on $H^*_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}, h^*K^*)$ commute with the natural action of $M(y) \subset G \times \mathbb{C}^\times$. Consequently the operators $\Delta(w)$ on $H^*_{G\times\mathbb{C}^\times}(\tilde{\mathcal{O}}, h^*K^*)^* \cong H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$ commute with the action of $M(y)$.

(a) For $G = G^\circ$ this is [Lus3, Theorem 8.13]. That proof also works if G is disconnected. We note that it uses parts (b), (c) and (d). □

In the same way $H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$ becomes a $\mathbb{H}(G^\circ, L, \mathcal{L})$-module.

Lemma 3.3. *There is an isomorphism of* $\mathbb{H}(G, L, \mathcal{L})$*-modules*

$$H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) \cong \mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$$

Proof. Recall from (4) that

$$\mathbb{H}(G, L, \mathcal{L}) = \mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural_\mathcal{L}] \ltimes \mathbb{H}(G^\circ, L, \mathcal{L}).$$

It follows from (17) that

$$H_j^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) = \bigoplus\nolimits_{\gamma \in \mathfrak{R}_\mathcal{L}} H_j^{M(y)^\circ}(\mathcal{P}_y^\circ \gamma^{-1}, \dot{\mathcal{L}}).$$

In (9) we saw that the action of $\mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural]$ on $(\dot{\mathfrak{g}}, \dot{\mathcal{L}})$ lifts the action

$$w \cdot (x, gP) = (x, gw^{-1}P) \quad w \in \mathfrak{R}_\mathcal{L}, (x, gP) \in \dot{\mathfrak{g}}.$$

Hence, for all $w, \gamma \in \mathfrak{R}_\mathcal{L}$:

$$\Delta(w) H_j^{M(y)^\circ}(\mathcal{P}_y^\circ \gamma^{-1}, \dot{\mathcal{L}}) = H_j^{M(y)^\circ}(\mathcal{P}_y^\circ \gamma^{-1} w^{-1}, \dot{\mathcal{L}}). \tag{22}$$

Therefore the action map

$$\mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural_\mathcal{L}] \underset{\mathbb{C}}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}}) = \mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(G^\circ,L,\mathcal{L})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}}) \to H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$$

is an isomorphism of $\mathbb{H}(G, L, \mathcal{L})$-modules. □

From the natural isomorphism

$$H^*_{M(y)^\circ}(\{y\}) \cong \mathcal{O}(\mathrm{Lie}(M(y)^\circ))^{M(y)^\circ}$$

one sees that the left hand side is the coordinate ring of the variety V_y of semisimple adjoint orbits in

$$\mathrm{Lie}(M(y)^\circ) = \{(\sigma, r) \in \mathfrak{g} \oplus \mathbb{C} : [\sigma, y] = 2ry\}.$$

For any $(\sigma, r)/\sim \in V_y$ let $\mathbb{C}_{\sigma,r}$ be the one-dimensional $H_*^{M(y)^\circ}(\{y\})$-module obtained by evaluating functions at the $\mathrm{Ad}(M(y)^\circ$-orbit of (σ, r). We define

$$E_{y,\sigma,r} = \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}),$$

$$E^\circ_{y,\sigma,r} = \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}}).$$

These are $\mathbb{H}(G, L, \mathcal{L})$-modules (respectively $\mathbb{H}(G^\circ, L, \mathcal{L})$-modules). In general they are reducible and not graded (in contrast with Theorem 3.2.a). These modules, and those in Lemma 3.3, are compatible with parabolic induction in a sense which we will describe next.

Let $Q \subset G$ be an algebraic subgroup such that $Q \cap G^\circ$ is a Levi subgroup of G° and $L \subset Q^\circ = Q \cap G^\circ$. Assume that $y \in \mathfrak{q} = \mathrm{Lie}(Q)$. Let $\mathcal{P}_y^Q$ and $\mathcal{P}_y^{Q^\circ}$ be the versions of $\mathcal{P}_y$ for Q and Q°. The role of P is now played by $P \cap Q$. There is a natural map

$$\mathcal{P}_y^Q \to \mathcal{P}_y : g(P \cap Q) \mapsto gP. \tag{23}$$

By [Lus3, 1.4.b] it induces, for every $n \in \mathbb{Z}$, a map

$$H^{M(y)^\circ}_{n+2\dim \mathcal{P}_y^Q}(\mathcal{P}_y^Q, \dot{\mathcal{L}}) \ \to\ H^{M(y)^\circ}_{n+2\dim \mathcal{P}_y}(\mathcal{P}_y, \dot{\mathcal{L}}). \tag{24}$$

Theorem 3.4. *Let Q and y be as above, and let C be a maximal torus of $M^Q(y)^\circ$.*
(*a*) *The map* (23) *induces an isomorphism of $\mathbb{H}(G,L,\mathcal{L})$-modules*

$$\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^Q,\dot{\mathcal{L}}) \to H_*^C(\mathcal{P}_y,\dot{\mathcal{L}}),$$

which respects the actions of $H_C^*(\{y\})$.
(*b*) *Let $(\sigma,r)/\!\sim\, \in V_y^Q$. The map* (23) *induces an isomorphism of $\mathbb{H}(G,L,\mathcal{L})$-modules*

$$\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r} \to E_{y,\sigma,r},$$

which respects the actions of $\pi_0(M^Q(y))_\sigma$.

Erratum. The map in part (a) is usually not surjective, and for part (b) we need an extra condition $\epsilon(\sigma,r)\neq 0$. This condition holds for almost all parameters, see the appendix.

Proof. (a) It was noted in [Lus7, 1.16] that the map of the theorem is well-defined, $\mathbb{H}(G,L,\mathcal{L})$-linear and $H_C^*(\{y\})$-linear.

Let us consider the statement for G° and Q° first. In [Lus7, §2] a C-variety $\dot{\mathcal{A}}$, which contains $\mathcal{P}_y^\circ$, is studied. Consider the diagram of $\mathbb{H}(G^\circ,L,\mathcal{L})$-modules

(25)
$$\begin{array}{ccc} & \mathbb{H}(G^\circ,L,\mathcal{L}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^{Q^\circ},\dot{\mathcal{L}}) & \\ \swarrow & & \searrow \\ H_*^C(\mathcal{P}_y^\circ,\dot{\mathcal{L}}) & \longrightarrow & H_*^C(\dot{\mathcal{A}},\dot{\mathcal{L}}) \end{array}$$

with maps coming from the theorem, from $\mathcal{P}_y^\circ \to \dot{\mathcal{A}}$ and from [Lus7, 2.15.(c)]. According to [Lus7, 2.19] the diagram commutes, and by [Lus7, 2.8.(g)] the horizontal map is injective. Moreover [Lus7, Theorem 2.16] says that the right slanted map is an isomorphism of $\mathbb{H}(G^\circ,L,\mathcal{L})$-modules. Consequently the horizontal map of (25) is surjective as well, and the entire diagram consists of isomorphisms.

Combining this result with Lemma 3.3, we get isomorphisms

$$\begin{aligned} &\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^Q,\dot{\mathcal{L}}) \cong \\ &\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} \mathbb{H}(Q,L,\mathcal{L}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^{Q^\circ},\dot{\mathcal{L}}) \cong \\ &\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(G^\circ,L,\mathcal{L})}{\otimes} \mathbb{H}(G^\circ,L,\mathcal{L}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^{Q^\circ},\dot{\mathcal{L}}) \cong \\ &\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(G^\circ,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^\circ,\dot{\mathcal{L}}) \quad \cong \quad H_*^C(\mathcal{P}_y,\dot{\mathcal{L}}). \end{aligned}$$

(b) Since $(\sigma,r)\in \mathrm{Lie}(M^Q(y))$ is semisimple, we may assume that $(\sigma,r)\in\mathrm{Lie}(C)$. By [Lus3, Proposition 7.5] there exist natural isomorphisms

$$\mathbb{C}_{\sigma,r} \underset{H_C^*(\{y\})}{\otimes} H_*^C(\mathcal{P}_y,\dot{\mathcal{L}}) \cong \mathbb{C}_{\sigma,r} \underset{H_C^*(\{y\})}{\otimes} H_C^*(\{y\}) \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y,\dot{\mathcal{L}}) \cong$$
$$\mathbb{C}_{\sigma,r} \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y,\dot{\mathcal{L}}) = E_{y,\sigma,r}.$$

The actions of $\mathbb{H}(G,L,\mathcal{L})$ and $H^*_C(\{y\})$ commute, so we also get

$$\mathbb{C}_{\sigma,r} \underset{H^*_C(\{y\})}{\otimes} \mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^Q,\dot{\mathcal{L}}) \cong \mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} \mathbb{C}_{\sigma,r} \underset{H^*_C(\{y\})}{\otimes} H_*^C(\mathcal{P}_y^Q,\dot{\mathcal{L}}) = \mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r}.$$

Now we can apply part (a) to obtain the desired isomorphism. Since the map (23) is $M^Q(y)$-equivariant, this isomorphism preserves the $\pi_0(M^Q(y))_\sigma$-actions. □

It is possible to choose an algebraic homomorphism $\gamma_y : \mathrm{SL}_2(\mathbb{C}) \to G^\circ$ with $\mathrm{d}\gamma_y \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = y$. It will turn out that often it is convenient to consider the element

$$\sigma_0 := \sigma + \mathrm{d}\gamma_y \left(\begin{smallmatrix} -r & 0 \\ 0 & r \end{smallmatrix}\right) \in Z_{\mathfrak{g}}(y). \tag{26}$$

instead of σ.

Proposition 3.5. *Assume that $\mathcal{P}_y$ is nonempty.*
(*a*) $\mathrm{Ad}(G)(\sigma) \cap \mathfrak{t}$ *is a single $W_{\mathcal{L}}$-orbit in* $\mathfrak{t}$.
(*b*) *The $\mathbb{H}(G,L,\mathcal{L})$-module $E_{y,\sigma,r}$ admits the central character* $(\mathrm{Ad}(G)(\sigma) \cap \mathfrak{t}, r) \in \mathfrak{t}/W_{\mathcal{L}} \times \mathbb{C}$.
(*c*) *The pair (y,σ) is G°-conjugate to one with σ, σ_0 and $\mathrm{d}\gamma_y \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ all three in* $\mathfrak{t}$.

Correction. See Proposition 1.7 and Lemma 1.8 of [AMS2].

Proof. (a) and (b) According to [Lus5, 8.13.a] there is a canonical surjection

$$H^*_{G^\circ \times \mathbb{C}^\times}(\text{point}) \cong \mathcal{O}(\mathfrak{g} \oplus \mathbb{C})^{G^\circ \times \mathbb{C}^\times} = \mathcal{O}(\mathfrak{g})^{G^\circ} \otimes \mathbb{C}[\mathbf{r}] \to Z(\mathrm{End}_{\mathcal{D}^b_{G^\circ \times \mathbb{C}^\times}(\mathfrak{g})}(K^*)). \tag{27}$$

By [Lus5, Theorem 8.11] the endomorphism algebra of K^*, in a bounded derived category of $G^\circ \times \mathbb{C}^\times$-equivariant sheaves on $\mathfrak{g}$, is canonically isomorphic to $\mathbb{H}(G^\circ,L,\mathcal{L})$. Together with Lemma 2.3 it follows that the right hand side of (27) is

$$Z(\mathbb{H}(G^\circ,L,\mathcal{L})) \cong S(\mathfrak{t}^*)^{W_{\mathcal{L}}^\circ} \otimes \mathbb{C}[\mathbf{r}].$$

By [Lus5, 8.13.b] the surjection (27) corresponds to an injection

$$\mathfrak{t}/W_{\mathcal{L}}^\circ \to \mathrm{Irr}(\mathcal{O}(\mathfrak{g})^{G^\circ}),$$

where the right hand side is the variety of semisimple adjoint orbits in $\mathfrak{g}$. Hence $\mathrm{Ad}(G^\circ)(\sigma) \cap \mathfrak{t}$ is either empty or a single $W_{\mathcal{L}}^\circ$-orbit. By Condition 3.1 $G/G^\circ \cong W_{\mathcal{L}}/W_{\mathcal{L}}^\circ$, so all these statements remain valid if we replace G° by G.

The action of $S(\mathfrak{t}^*)^{W_{\mathcal{L}}^\circ} \otimes \mathbb{C}[\mathbf{r}]$ on $E_{y,\sigma,r}$ can be realized as

$$H^*_{G \times \mathbb{C}^\times}(\text{point}) \to H^*_{M(y)^\circ}(\{y\}) \to H^*_{M(y)^\circ}(\mathcal{P}_y)$$

and then the product (21). By construction $H^*_{M(y)^\circ}(\{y\})$ acts on $E_{y,\sigma,r}$ via the character $(\sigma,r)/\!\sim\, \in V_y$. Hence $H^*_{G \times \mathbb{C}^\times}(\text{point})$ acts via the character $\mathrm{Ad}(G \times \mathbb{C}^\times)(\sigma,r)$.

The assumption $\mathcal{P}_y \neq \emptyset$ implies that $H_*^{M(y)^\circ}(\mathcal{P}_y,\dot{\mathcal{L}})$ is nonzero. By Theorem 3.2.c, and because V_y is an irreducible variety, $E_{y,\sigma,r} \neq 0$ for all $(\sigma,r)/\!\sim\, \in V_y$. Thus the above determines a unique character of $Z(\mathbb{H}(G,L,\mathcal{L}))$ via (27), which must be $(\mathrm{Ad}(G)(\sigma) \cap \mathfrak{t}, r)$. In particular the intersection is nonempty and constitutes one $W_{\mathcal{L}}$-orbit.
(c) By part (b) with $r = 0$ we may assume that $\sigma_0 \in \mathfrak{t}$. Then M is contained in the reductive group $Z_G(\sigma_0)$, so we can arrange that the image of γ_y lies in $Z_G(\sigma_0)$.

Applying part (b) to this group, with $r \neq 0$, we see that there exists a $g \in Z_{G^\circ}(\sigma_0)$ such that

$$\mathrm{Ad}(g)\sigma = \sigma_0 + \mathrm{Ad}(g)\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \text{ lies in } \mathfrak{t}.$$

Now the pair $(\mathrm{Ad}(g)y, \mathrm{Ad}(g)\sigma)$ has the required properties. □

Let $\pi_0(M(y))_\sigma$ be the stabilizer of $(\sigma, r)/\sim \in V_y$ in $\pi_0(M(y))$. (It does not depend on r because $\mathbb{C}^\times$ is central in $G \times \mathbb{C}^\times$.) It follows from Theorem 3.2.d that $\pi_0(M(y))_\sigma$ acts on $E_{y,\sigma,r}$ by $\mathbb{H}(G, L, \mathcal{L})$-module homomorphisms. Similarly, let $\pi_0(M)_\sigma^\circ$ be the stabilizer of $(\sigma, r)/\sim$ in $\pi_0(M(y) \cap G^\circ)$. It acts on $E^\circ_{y,\sigma,r}$ by $\mathbb{H}(G^\circ, L, \mathcal{L})$-module maps. To analyse these components groups we use (26).

Lemma 3.6. *(a) There are natural isomorphisms*

$$\pi_0(M(y))_\sigma \cong \pi_0(Z_G(\sigma, y)) \cong \pi_0(Z_G(\sigma_0, y)).$$

(b) Fix $r \in \mathbb{C}$. The map $\sigma \mapsto \sigma_0$ and part (a) induce a bijection between
- *G-conjugacy classes of triples (y, σ, ρ) with $y \in \mathfrak{g}$ nilpotent, $(\sigma, r) \in \mathrm{Lie}(M(y))$ semisimple and $\rho \in \mathrm{Irr}(\pi_0(M(y))_\sigma)$;*
- *G-conjugacy classes of triples (y, σ_0, ρ) with $y \in \mathfrak{g}$ nilpotent, $\sigma_0 \in \mathfrak{g}$ semisimple, $[\sigma_0, y] = 0$ and $\rho \in \mathrm{Irr}(\pi_0(M(y))_{\sigma_0})$.*

Remark. Via the Jordan decomposition the second set in part (b) is canonically in bijection with the G-orbits of pairs (x, ρ) where $x \in \mathfrak{g}$ and $\rho \in \pi_0(Z_G(x))$. Although that is a more elegant description we prefer to keep the semisimple and nilpotent parts separate, because only the (y, σ_0) with $\mathcal{P}_y \neq \emptyset$ are relevant for $\mathbb{H}(G, L, \mathcal{L})$.

Proof. (a) By definition

$$\pi_0(M(y))_\sigma = \mathrm{Stab}_{\pi_0(M(y))}(\mathrm{Ad}(M(y)^\circ)(\sigma, y)) \cong Z_{M(y)}(\sigma, r)/Z_{M(y)^\circ}(\sigma, r).$$

Since (σ, r) is a semisimple element of $\mathrm{Lie}(G \times \mathbb{C}^\times)$, taking centralizers with (σ, r) preserves connectedness. Hence the right hand side is

$$\big(Z_{G\times\mathbb{C}^\times}(\sigma, r) \cap M(y)\big)/\big(Z_{G\times\mathbb{C}^\times}(\sigma, r) \cap M(y)\big)^\circ = \pi_0\big(Z_{G\times\mathbb{C}^\times}(\sigma, r) \cap M(y)\big). \tag{28}$$

We note that $Z_{G\times\mathbb{C}^\times}(\sigma, r) = Z_G(\sigma) \times \mathbb{C}^\times$ and that there is a homeomorphism

$$Z_G(y) \times \mathbb{C}^\times \to M(y) : (g, \lambda) \mapsto g\gamma_y \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right).$$

It follows that the factor $\mathbb{C}^\times$ can be omitted from (28) without changing the quotient, and we obtain

$$\pi_0(M(y))_\sigma \cong Z_G(\sigma, y)/Z_G(\sigma, y)^\circ = \pi_0(Z_G(\sigma, y)).$$

By [KaLu, §2.4] the inclusion maps

$$Z_G(\sigma, y) \leftarrow Z_G\big(\sigma, \mathrm{d}\gamma_y(\mathfrak{sl}_2(\mathbb{C}))\big) \rightarrow Z_G(\sigma_0, y)$$

induce isomorphisms on component groups.
(b) Again by [KaLu, §2.4], the $Z_G(y)$-conjugacy class of σ_0 is uniquely determined by σ. The reason is that the homomorphism $\mathrm{d}\gamma_y : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ is unique up to the adjoint action of $Z_G(y)$. By the same argument σ_0 determines the $Z_G(y)$-adjoint orbit of σ. Thus $\sigma \mapsto \sigma_0$ gives a bijection between adjoint orbits of pairs (σ, y) and of pairs (σ_0, y). The remainder of the asserted bijection comes from part (a). □

Applying Lemma 3.6 with G° instead of G gives natural isomorphisms

$$\pi_0(M(y))^\circ_\sigma \cong \pi_0(Z_{G^\circ}(\sigma, y)) \cong \pi_0(Z_{G^\circ}(\sigma_0, y)). \tag{29}$$

For $\rho \in \mathrm{Irr}(\pi_0(M(y))_\sigma)$ and $\rho^\circ \in \mathrm{Irr}(\pi_0(M(y))^\circ_\sigma)$ we write

$$E_{y,\sigma,r,\rho} = \mathrm{Hom}_{\pi_0(M(y))_\sigma}(\rho, E_{y,\sigma,r}),$$
$$E^\circ_{y,\sigma,r,\rho^\circ} = \mathrm{Hom}_{\pi_0(M(y))^\circ_\sigma}(\rho^\circ, E^\circ_{y,\sigma,r}).$$

It follows from Theorem 3.2.d that these vector spaces are modules for $\mathbb{H}(G, L, \mathcal{L})$, respectively for $\mathbb{H}(G^\circ, L, \mathcal{L})$. When they are nonzero, we call them standard modules.

Recall the cuspidal support map Ψ_G from [Lus1, AMS]. It associates a cuspidal support $(L', \mathcal{C}^{L'}_{v'}, \mathcal{L}')$ to every pair (x, ρ) with $x \in \mathfrak{g}$ nilpotent and $\rho \in \mathrm{Irr}(\pi_0(Z_G(x)))$.

Proposition 3.7. *The $\mathbb{H}(G^\circ, L, \mathcal{L})$-module $E^\circ_{y,\sigma,r,\rho^\circ}$ is nonzero if and only if $\Psi_{Z_{G^\circ}(\sigma_0)}(y, \rho^\circ)$ is G°-conjugate to $(L, \mathcal{C}^L_v, \mathcal{L})$. Here ρ° is considered as an irreducible representation of $\pi_0(Z_{Z_{G^\circ}(\sigma_0)}(y))$ via Lemma 3.6.*

Proof. Assume first that $r \neq 0$. Unravelling the definitions in [Lus5], one sees that K^* is called B in that paper. We point out that the proof of [Lus5, Proposition 10.12] misses a *-sign in equation (c), the correct statement involves the dual local system $\mathcal{L}^*$. We can simplify things a bit if we apply [Lus5, §10] with the roles of $\mathcal{L}$ and $\mathcal{L}^*$ exchanged. It implies that $E^\circ_{y,\sigma,r,\rho^\circ} \neq 0$ if and only if

$$\mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma,y))}\big(\rho^\circ, \bigoplus\nolimits_n \mathcal{H}^n(i_y^! \tilde{K})\big) \neq 0. \tag{30}$$

Here $i_y : \{y\} \to \tilde{\mathfrak{g}} = \{x \in \mathfrak{g} : [\sigma, x] = 2rx\}$ is the inclusion and $\tilde{K}$ is the restriction of K to $\tilde{\mathfrak{g}}$. In the notation of [Lus5, Corollary 8.18], (30) means that (y, ρ°) (or more precisely the associated local system on $\tilde{\mathfrak{g}}$) is an element of $\mathcal{M}_{o,\mathcal{F}}$. By [Lus5, Proposition 8.17], that is equivalent to the existence of a $\rho^\circ_G \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(y))\big)$ such that:

- $\rho^\circ_G\big|_{\pi_0(Z_{G^\circ}(\sigma,y))}$ contains ρ°,
- the local system $\mathcal{F}^\circ$ on $\mathcal{C}^{G^\circ}_y$ with fibre ρ°_G at y is a direct summand of $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}^n(K)|_{\mathcal{C}^{G^\circ}_y}$.

According to [Lus5, Proposition 8.16] there is a unique K', among the possible choices of $(L', \mathcal{C}^{L'}_{v'}, \mathcal{L}')$, such that $K = K'$ fulfills this condition. By [Lus1, Theorem 6.5] it can be fulfilled with the cuspidal support of $(\mathcal{C}^{G^\circ}_y, \mathcal{F}^\circ)$ and n equal to

$$2d_{\mathcal{C}^{G^\circ}_y, \mathcal{C}^L_v} := \dim Z_{G^\circ}(y) - \dim Z_L(v). \tag{31}$$

Hence we may restrict n to $2d_{\mathcal{C}^{G^\circ}_y, \mathcal{C}^L_v}$ without changing the last condition.

By [AMS, Proposition 5.6.a] the second condition of the proposition is equivalent to: there exists a $\rho^\circ_G \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(y))\big)$ such that $\rho^\circ_G\big|_{\pi_0(Z_{G^\circ}(\sigma,y))}$ contains ρ° and $\Psi_{G^\circ}(y, \rho^\circ_G) = (L, \mathcal{C}^L_v, \mathcal{L})$.

Let $\mathcal{F}^\circ$ be the local system on $\mathcal{C}^{G^\circ}_y$ with $(\mathcal{F}^\circ)_y = \rho^\circ_G$. By [Lus1, Theorem 6.5] its cuspidal support is $(L, \mathcal{C}^L_v, \mathcal{L})$ if and only if $\mathcal{F}^\circ$ is a direct summand of $\mathcal{H}^{2d}(K)|_{\mathcal{C}^{G^\circ}_y}$, where $d = d_{\mathcal{C}^{G^\circ}_y, \mathcal{C}^L_v}$. Hence the second condition of the propostion is equivalent to all the above conditions, if $r \neq 0$.

For any $r \in \mathbb{C}$, as $\mathbb{C}[W^\circ_{\mathcal{L}}]$-modules:

$$E^\circ_{y,\sigma,r,\rho^\circ} = \mathrm{Hom}_{\pi_0(M(y))^\circ_\sigma}\big(\rho^\circ, H_*(\mathcal{P}_y, \dot{\mathcal{L}})\big), \tag{32}$$

see [Lus5, 10.12.(d)]. Recall from (29) that $\pi_0(M(y))^\circ_\sigma \cong \pi_0(Z_{G^\circ}(\sigma_0, y))$. For any $t \in \mathbb{C}$ we obtain

$$E^\circ_{y,\sigma_0+r\mathrm{d}\gamma_y\left(\begin{smallmatrix} t & 0 \\ 0 & -t\end{smallmatrix}\right),tr,\rho^\circ} = \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma_0,y))}\big(\rho^\circ, H_*(\mathcal{P}_y, \dot{\mathcal{L}})\big).$$

The right hand side is independent of $t \in \mathbb{C}$ and for $tr \neq 0$ it is nonzero if and only if $\Psi_{Z_{G^\circ}(\sigma_0)}(y, \rho^\circ) = (L, \mathcal{C}_v^L, \mathcal{L})$ (up to G°-conjugacy). Hence the same goes for $E^\circ_{y,\sigma_0,0,\rho^\circ}$. We have $\sigma_0 = \sigma$ if $r = 0$, so this accounts for all $(\sigma, r)/\sim \, \in V_y$ with $r \in \mathbb{C}$. $\square$

3.2. **Representations annihilated by r.**

The representations of $\mathbb{H}(G, L, \mathcal{L})$ which are annihilated by $\mathbf{r}$ can be identified with representations of

$$\mathbb{H}(G, L, \mathcal{L})/(\mathbf{r}) = \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}] \ltimes S(\mathfrak{t}^*).$$

We will study the irreducible representations of this algebra in a straightforward way: we give ad-hoc definitions of certain modules, then we show that these exhaust $\mathrm{Irr}\big(\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}] \ltimes S(\mathfrak{t}^*)\big)$, and we provide a parametrization.

The generalized Springer correspondence [Lus1] associates to (y, ρ°) an irreducible representation M_{y,ρ° of a suitable Weyl group. It is a representation of $W^\circ_\mathcal{L}$ if the cuspidal support $\Psi_{G^\circ}(y, \rho^\circ)$ is $(L, \mathcal{C}_v^L, \mathcal{L})$. If that is the case and $\sigma_0 \in \mathrm{Lie}(Z(G^\circ))$, we let $M^\circ_{y,\sigma_0,0,\rho^\circ}$ be the irreducible $\mathbb{H}(G^\circ, L, \mathcal{L})$-module on which $S(\mathfrak{t}^* \oplus \mathbb{C})$ acts via the character $(\sigma_0, 0) \in \mathfrak{t} \oplus \mathbb{C}$ and

$$M^\circ_{y,\sigma_0,0,\rho^\circ} = M_{y,\rho^\circ} \text{ as } \mathbb{C}[W^\circ_\mathcal{L}]\text{-modules.} \tag{33}$$

For a general $\sigma_0 \in Z_\mathfrak{g}(y)$ we can define a similar $W^\circ_\mathcal{L} \ltimes S(\mathfrak{t}^*)$-module. We may assume that $\mathcal{P}^\circ_y$ is nonempty, for otherwise $H_*^{M(y)^\circ}(\mathcal{P}^\circ_y, \dot{\mathcal{L}}) = 0$. Upon replacing (y, σ_0) by a suitable G°-conjugate, we may also assume that L centralizes σ_0. Write $Q^\circ = Z_{G^\circ}(\sigma_0)$, a Levi subgroup of G° containing L. Notice that $W_\mathcal{L}^{Q^\circ} = W(Q^\circ, T)$ is a Weyl group, the stabilizer of σ_0 in $W_\mathcal{L}$. Then $\pi_0(M(y))^\circ_{\sigma_0} \cong \pi_0(Z_{Q^\circ}(y))$, so $(y, \sigma_0, \rho^\circ)$ determines the irreducible $\mathbb{H}(Q^\circ, L, \mathcal{L})$-module $M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$. We define

$$M^\circ_{y,\sigma_0,0,\rho^\circ} = \mathrm{ind}_{W_\mathcal{L}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}^{W^\circ_\mathcal{L} \ltimes S(\mathfrak{t}^*)} (M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}) = \mathrm{ind}_{\mathbb{H}(Q^\circ,L,\mathcal{L})}^{\mathbb{H}(G^\circ,L,\mathcal{L})} (M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}). \tag{34}$$

Proposition 3.8. *The map $(y, \sigma_0, \rho^\circ) \mapsto M^\circ_{y,\sigma_0,0,\rho^\circ}$ induces a bijection between:*

- *G°-conjugacy classes of triples $(y, \sigma_0, \rho^\circ)$ such that $y \in Z_\mathfrak{g}(\sigma_0)$ nilpotent, $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(\sigma_0, y))\big)$ and $\Psi_{Z_{G^\circ}(\sigma_0)}(y, \rho^\circ)$ is G°-conjugate to $(L, \mathcal{C}_v^L, \mathcal{L})$;*
- $\mathrm{Irr}(W^\circ_\mathcal{L} \ltimes S(\mathfrak{t}^*)) = \mathrm{Irr}(\mathbb{H}(G^\circ, L, \mathcal{L})/(\mathbf{r}))$.

Proof. By definition $S(\mathfrak{t}^*)$ acts on $M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$ via the character σ. For $w \in W^\circ_\mathcal{L}$ it acts on $w M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ} \subset M^\circ_{y,\sigma_0,0,\rho^\circ}$ as the character $w\sigma$. Since $W_\mathcal{L}^{Q^\circ}$ is the centralizer of σ in $W_\mathcal{L}$, the $S(\mathfrak{t}^*)$-weights $w\sigma$ with $w \in W^\circ_\mathcal{L}/W_\mathcal{L}^{Q^\circ}$ are all different. As vector spaces

$$M^\circ_{y,\sigma_0,0,\rho^\circ} = \mathrm{ind}_{W_\mathcal{L}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}^{W^\circ_\mathcal{L} \ltimes S(\mathfrak{t}^*)} \big(M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}\big) = \bigoplus\nolimits_{w \in W^\circ_\mathcal{L}/W_\mathcal{L}^{Q^\circ}} M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ},$$

and $M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$ is irreducible. With Frobenius reciprocity we see that $M^\circ_{y,\sigma_0,0,\rho^\circ}$ is also irreducible.

Recall that the generalized Springer correspondence [Lus1] provides a bijection between $\mathrm{Irr}(W_\mathcal{L}^{Q^\circ})$ and the Q°-conjugacy classes of pairs (y, ρ°) where $y \in \mathrm{Lie}(Q^\circ)$

is nilpotent and $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_{Q^\circ}(y))\big)$ such that $\Psi_{Q^\circ}(y,\rho^\circ) = (L, \mathcal{C}_v^L, \mathcal{L})$. We obtain a bijection between G°-conjugacy classes of triples (y,σ_0,ρ°) and $W_{\mathcal{L}}^\circ$-association classes of pairs (σ_0,π) with $\sigma_0 \in \mathfrak{t}$ and $(\pi, V_\pi) \in \mathrm{Irr}((W_{\mathcal{L}}^\circ)_{\sigma_0})$. It is well-known, see for example [Sol2, Theorem 1.1], that the latter set is in bijection with $\mathrm{Irr}(W_{\mathcal{L}}^\circ \times S(\mathfrak{t}^*))$ via

$$(\sigma_0,\pi) \mapsto \mathrm{ind}_{(W_{\mathcal{L}}^\circ)_{\sigma_0} \ltimes S(\mathfrak{t}^*)}^{W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*)} (\mathbb{C}_{\sigma_0} \otimes V_\pi).$$

In other words, the map of the proposition is a bijection. $\square$

We woud like to relate the above irreducible representations of $W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*)$ to the standard modules from the previous paragraph. To facilitate this we first exhibit some properties of standard modules, which are specific for the case $r = 0$.

Lemma 3.9. *Assume that $\Psi_{G^\circ}(y,\rho^\circ) = (L, \mathcal{C}_v^L, \mathcal{L})$. The standard $\mathbb{H}(G^\circ, L, \mathcal{L})$-module $E_{y,\sigma_0,0,\rho^\circ}^\circ$ is completely reducible and admits a module decomposition by homological degree:*

$$E_{y,\sigma_0,0,\rho^\circ}^\circ = \bigoplus\nolimits_n \mathrm{Hom}_{\pi_0(M(y))_\sigma^\circ}\big(\rho^\circ, H_n(\mathcal{P}_y^\circ, \dot{\mathcal{L}})\big).$$

Erratum. In fact this module is indecomposable, filtered by the homological degree and has a unique irreducible quotient. The problem below is that $S(\mathfrak{t}^* \oplus \mathbb{C})$ need not act semisimply.

Proof. First we assume that σ_0 is central in $\mathrm{Lie}(G^\circ)$. Then the action of $S(\mathfrak{t}^* \oplus \mathbb{C})$ simplifies. Indeed, from (21) we see that it given just by evaluation at $(\sigma_0, 0)$. Hence the structure of $E_{y,\sigma_0,0}^\circ$ as a $\mathbb{H}(G^\circ, L, \mathcal{L})$-module is completely determined by the action of $\mathbb{C}[W_{\mathcal{L}}^\circ]$. That is a semisimple algebra, so

$$E_{y,\sigma_0,0}^\circ \text{ is completely reducible.} \tag{35}$$

Then the direct summand $E_{y,\sigma_0,0,\rho^\circ}^\circ$ is also completely reducible.

By [Lus5, 10.12.(d)] $E_{y,\sigma_0,0}^\circ$ can be identified with $H_*(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$, as $W_{\mathcal{L}}^\circ$-representations. In (19) we observed that the action of $\mathbb{C}[W_{\mathcal{L}}^\circ]$ preserves the homological degree, so

$$E_{y,\sigma_0,0}^\circ \cong \bigoplus\nolimits_n H_n(\mathcal{P}_y^\circ, \dot{\mathcal{L}}) \text{ as } W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*)\text{-representations.} \tag{36}$$

This decomposition persists after applying $\mathrm{Hom}(\rho^\circ, ?)$.

Now we lift the condition on σ_0, and we consider the Levi subgroup $Q^\circ = Z_{G^\circ}(\sigma_0)$ of G°. As explained before Proposition 3.8, we may assume that $L \subset Q^\circ$. By [Lus7, Corollary 1.18] there is a natural isomorphism of $\mathbb{H}(G^\circ, L, \mathcal{L})$-modules

$$W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*) \underset{W_{\mathcal{L}}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}{\otimes} E_{y,\sigma_0,0}^{Q^\circ} = \mathbb{H}(G^\circ, L, \mathcal{L}) \underset{\mathbb{H}(Q^\circ, L, \mathcal{L})}{\otimes} E_{y,\sigma_0,0}^{Q^\circ} \longrightarrow E_{y,\sigma_0,0}^\circ. \tag{37}$$

We note that [Lus7, Corollary 1.18] is applicable because $r = 0$ and $\mathrm{ad}(\sigma_0)$ is an invertible linear transformation of $\mathrm{Lie}(U_{Q^\circ})$, where U_{Q° is the unipotent radical of a parabolic subgroup of G° with Levi factor Q°.

For later use we remark that the map (37) comes from a morphism $\mathcal{P}_y^{Q^\circ} \to \mathcal{P}_y^\circ$. Hence it changes all homological degrees by the same amount, namely $\dim \mathcal{P}_y^\circ - \dim \mathcal{P}_y^{Q^\circ}$.

In (35) we saw that the $W_{\mathcal{L}}^{Q^\circ} \ltimes S(\mathfrak{t}^*)$-module $E_{y,\sigma_0,0}^{Q^\circ}$ is completely reducible. Above we also showed that $S(\mathfrak{t}^* \oplus \mathbb{C})$ acts on $E_{y,\sigma_0,0,\rho^\circ}^{Q^\circ}$ via the character $(\sigma_0, 0)$. With the braid relation from Propostion 2.2 we see that

$$S(\mathfrak{t}^* \oplus \mathbb{C}) \text{ acts on } w E^{Q^\circ}_{y,\sigma_0,0} \text{ via the character } (w\sigma_0, 0). \tag{38}$$

As $W_{\mathcal{L}}^{Q^\circ}$ is the stabilizer of σ_0 in $W_{\mathcal{L}}$, this brings the reducibility question for $E^\circ_{y,\sigma_0,0}$ back to that for $E^{Q^\circ}_{y,\sigma_0,0}$, which we already settled. Thus

$$E^\circ_{y,\sigma_0,0} \text{ is completely reducible.} \tag{39}$$

This implies that the direct summand $E^\circ_{y,\sigma_0,0,\rho^\circ}$ is also completely reducible.

It follows from (36) and (38) that $E^\circ_{y,\sigma_0,0} = H_*(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$ and that the action of $W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*)$ preserves the homological degree. The same goes for the action of $\pi_0(M(y))^\circ_\sigma$, which yields the desired module decomposition of $E^\circ_{y,\sigma_0,0,\rho^\circ}$. □

In terms of Lemma 3.9 we can describe explicitly how a standard module for $\mathbb{H}(G, L, \mathcal{L})/(\mathbf{r})$ contains the irreducible module with the same parameter.

Lemma 3.10. *The $W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*)$-module $E^\circ_{y,\sigma_0,0,\rho^\circ}$ has a unique irreducible subquotient isomorphic to $M^\circ_{y,\sigma_0,0,\rho^\circ}$. It is the component of $E^\circ_{y,\sigma_0,0,\rho^\circ}$ in the homological degree*

$$\dim \mathcal{P}_y^\circ - \dim \mathcal{P}_y^{Z_G(\sigma_0)^\circ} + \dim Z_{G^\circ}(\sigma_0, y) - \dim Z_L(v).$$

Proof. For the moment we assume that σ_0 is central in $\mathrm{Lie}(G^\circ)$. According to [Lus3, Theorem 8.15] every irreducible $\mathbb{H}(G^\circ, L, \mathcal{L})$-module is a quotient of some standard module. The central character of $M^\circ_{y,\sigma_0,0,\rho^\circ}$ is $(\sigma_0, 0) \in \mathfrak{t}/W_{\mathcal{L}}^\circ \times \mathbb{C}$. In view of Proposition 3.5.b, $M^\circ_{y,\sigma_0,0,\rho^\circ}$ cannot be a subquotient of a standard module $E^\circ_{y,\sigma,r,\rho^\circ}$ with $(\sigma, r) \neq (\sigma_0, 0)$. Therefore it must be a quotient of $E^\circ_{y,\sigma_0,0,\rho'}$ for some

$$\rho' \in \mathrm{Irr}(\pi_0(M(y))^\circ_{\sigma_0} = \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(y))\big).$$

By [Lus5, Proposition 10.12.b], modified as explained in the proof of Proposition 3.7, $E^\circ_{y,\sigma_0,0}$ is isomorphic to $H^*(\{y\}, i_y^!(K))$, where $i_y : \{y\} \to \mathfrak{g}$ is the inclusion. From [Lus5, 1.3.(d) and 1.4.(a)] we see that

$$E^\circ_{y,\sigma_0,0} \cong H^*(\{y\}, i_y^!(K)) = H^*(\{y\}, i_y^*(K)) \cong \mathcal{H}^*(K)|_y. \tag{40}$$

The generalized Springer correspondence, which in [Lus1] comes from sheaves on subvarieties of G°, can also be obtained from sheaves on subvarieties of $\mathfrak{g}$, see [Lus3, 2.2]. In that version it is given by

$$(y, \rho^\circ) \mapsto \mathrm{Hom}_{\pi_0(Z_{G^\circ}(y))}\big(\rho^\circ, \mathcal{H}^{2d}(K)|_y\big),$$

where $d = d_{\mathcal{C}_y^{G^\circ}, \mathcal{C}_v^L}$ is as in (31). More precisely [Lus1, Theorem 6.5]:

$$\mathcal{H}^{2d}(K)|_y \cong \bigoplus\nolimits_{\rho'} V_{\rho'} \otimes M_{y,\rho'} \text{ as } \pi_0(Z_{G^\circ}(y)) \times W_{\mathcal{L}}^\circ\text{-representations,} \tag{41}$$

where the sum runs over all $(\rho', V_{\rho'}) \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(y))\big)$ with $\Psi_{G^\circ}(y, \rho') = (L, \mathcal{C}_v^L, \mathcal{L})$.

Let I denote the set of all pairs $i = (\mathcal{C}_y^{G^\circ}, \mathcal{F}^\circ)$ where $\mathcal{C}_y^{G^\circ}$ is the adjoint orbit of a nilpotent element y in $\mathfrak{g}$, and $\mathcal{F}^\circ$ is an irreducible G°-equivariant local system (given up to isomorphism) on $\mathcal{C}_y^{G^\circ}$. In [Lus2, Theorem 24.8], Lusztig has proved that for any $i = (\mathcal{C}_y^{G^\circ}, \mathcal{F}^\circ) \in I$:

- $\mathcal{H}^n\big(\mathrm{IC}(\overline{\mathcal{C}}_y^{G^\circ}, \mathcal{F}^\circ)\big) = 0$ if n is odd.

- for $i' = (\mathcal{C}_{y'}^{G^\circ}, \mathcal{F}^{\circ,\prime}) \in I$ the polynomial
$$\Pi_{i,i'} := \sum\nolimits_m \big(\mathcal{F}^{\circ,\prime} : \mathcal{H}^{2m} \big(\mathrm{IC}(\overline{\mathcal{C}}_y^{G^\circ}, \mathcal{F}^\circ) \big) |_{\mathcal{C}_{y'}^{G^\circ}} \big) \, \mathbf{q}^{\mathbf{m}},$$
in the indeterminate $\mathbf{q}$, satisfies $\Pi_{i,i} = 1$.

From the second bullet we obtain

$$\big(\mathcal{F}^\circ : \mathcal{H}^{2m} \big(\mathrm{IC}(\overline{\mathcal{C}}_y^{G^\circ}, \mathcal{F}^\circ) \big) |_{\mathcal{C}_y^{G^\circ}} \big) = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0. \end{cases} \tag{42}$$

By combining (42) with [Lus1, Theorem 6.5], where the considered complex is shifted in degree $2d_{\mathcal{C}_y^{G^\circ}, \mathcal{C}_v^L} = \dim Z_{G^\circ}(y) - \dim Z_L(v)$, we obtain that

$$E^\circ_{y,\sigma_0,0,\rho^\circ} \cong \mathrm{Hom}_{\pi_0(Z_{G^\circ}(y))} \big(\rho^\circ, \mathcal{H}^*(K)|_y \big)$$

contains M_{y,ρ° with multiplicity one, as the component in the homological degree $2d_{\mathcal{C}_y^{G^\circ}, \mathcal{C}_v^L}$.

Now consider a general $\sigma_0 \in \mathrm{Lie}(G^\circ)$, and we write $Q = Z_G(\sigma_0)$. By Lemma 3.6

$$\pi_0(M(y))^\circ_{\sigma_0} \cong \pi_0(Z_{G^\circ}(\sigma_0, y)) = \pi_0(Z_{Q^\circ}(\sigma_0, y)) = \pi_0(Z_{Q^\circ}(y)). \tag{43}$$

By Theorem 3.2.d the action of this group commutes with that of $\mathbb{H}(G^\circ, L, \mathcal{L})$, so (37) contains an isomorphism of $\mathbb{H}(G^\circ, L, \mathcal{L})$-modules

$$\mathbb{H}(G^\circ, L, \mathcal{L}) \underset{\mathbb{H}(Q^\circ, L, \mathcal{L})}{\otimes} E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ} \to E^\circ_{y,\sigma_0,0,\rho^\circ}. \tag{44}$$

The argument for the irreducibility of $M^\circ_{y,\sigma_0,0,\rho^\circ}$ in the proof of Proposition 3.8 also applies here, when we use Proposition 3.5.b. It shows that the $S(\mathfrak{t}^*)$-modules $wE^\circ_{y,\sigma_0,0,\rho^\circ}$ with $w \in W^\circ_{\mathcal{L}} / W^{Q^\circ}_{\mathcal{L}}$ contain only different $S(\mathfrak{t}^*)$-modules, so they have no common irreducible constituents. It follows that the functor $\mathrm{ind}_{\mathbb{H}(Q^\circ,L,\mathcal{L})}^{\mathbb{H}(G^\circ,L,\mathcal{L})}$ provides a bijection between $\mathbb{H}(Q^\circ, L, \mathcal{L})$-subquotients of $E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$ and $\mathbb{H}(G^\circ, L, \mathcal{L})$-subquotients of $E^\circ_{y,\sigma_0,0,\rho^\circ}$. Together with the statement of the lemma for (Q°, σ_0), we see that $E^\circ_{y,\sigma_0,0,\rho^\circ}$ has a unique quotient isomorphic to $M^\circ_{y,\sigma_0,0,\rho^\circ}$ and no other constituents isomorphic to that.

As remarked before, the maps (37) and (44) come from a morphism $\mathcal{P}_y^{Q^\circ} \to \mathcal{P}_y^\circ$, so they change all homological degrees by $\dim \mathcal{P}_y^\circ - \dim \mathcal{P}_y^{Q^\circ}$. With the result for (Q°, σ_0) at hand, it follows that the image of $W^\circ_{\mathcal{L}} \ltimes S(\mathfrak{t}^*) \underset{W^{Q^\circ}_{\mathcal{L}} \ltimes S(\mathfrak{t}^*)}{\otimes} M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$ is the full component of $E^\circ_{y,\sigma_0,0,\rho^\circ}$ in the stated homological degree. By definition this image is also (isomorphic to) $M^\circ_{y,\sigma_0,0,\rho^\circ}$. $\square$

3.3. **Intertwining operators and 2-cocycles.**

For $r \in \mathbb{C}$ we let $\mathrm{Irr}_r(\mathbb{H}(G, L, \mathcal{L}))$ be the set of (equivalence classes of) irreducible $\mathbb{H}(G, L, \mathcal{L})$-modules on which $\mathbf{r}$ acts as r.

The irreducible representations of $\mathbb{H}(G, L, \mathcal{L})$ are built from those of $\mathbb{H}(G^\circ, L, \mathcal{L})$. Let us collect some available information about the latter here.

Theorem 3.11. *Let $y \in \mathfrak{g}$ be nilpotent and let $(\sigma, r)/\sim \in V_y$ be semisimple. Let $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ}(\sigma, y))\big)$ be such that $\Psi_{Z_{G^\circ}(\sigma_0)}(y, \rho^\circ) = (L, \mathcal{C}_v^L, \mathcal{L})$ (up to G°-conjugation).*

(a) If $r \neq 0$, then $E^\circ_{y,\sigma,r,\rho^\circ}$ has a unique irreducible quotient $\mathbb{H}(G^\circ, L, \mathcal{L})$-module. We call it $M^\circ_{y,\sigma,r,\rho^\circ}$.

(*b*) *If* $r = 0$, *then* $E^\circ_{y,\sigma_0,r,\rho^\circ}$ *has a unique irreducible summand isomorphic to* $M^\circ_{y,\sigma_0,0,\rho^\circ}$.
(*c*) *Parts (a) and (b) set up a canonical bijection between* $\mathrm{Irr}_r(\mathbb{H}(G^\circ, L, \mathcal{L}))$ *and the* G°*-orbits of triples* (y, σ, ρ°) *as above.*
(*d*) *Every irreducible constituent of* $E^\circ_{y,\sigma,r,\rho^\circ}$, *different from* $M^\circ_{y,\sigma,r,\rho^\circ}$, *is isomorphic to a representation* $M^\circ_{y',\sigma',r,\rho'}$ *with* $\dim \mathcal{C}_y^{G^\circ} < \dim \mathcal{C}_{y'}^{G^\circ}$.

Erratum. In case (b) it should be an irreducible quotient.

Proof. (a) is [Lus7, Theorem 1.15.a].
(b) is a less precise version of Lemma 3.10.
(c) For $r \neq 0$ see [Lus7, Theorem 1.15.c] and for $r = 0$ see Proposition 3.8.
(d) As noted in [Ciu, §3], this follows from [Lus5, §10]. □

Our goal is to generalize Theorem 3.11 from G° to G. To this end we have to extend both ρ° and $M^\circ_{y,\sigma,\rho^\circ}$ to representations of larger algebras. That involves the construction of some intertwining operators, followed by Clifford theory for representations of crossed product algebras. Although all our intertwining operators are parametrized by some group, they typically do not arise from a group homomorphism. Instead they form twisted group algebras, and we will have to determine the associated group cocycles as well.

The group $\mathfrak{R}_\mathcal{L}$ acts on the set of $\mathbb{H}(G^\circ, L, \mathcal{L})$-representations π by

$$(w \cdot \pi)(h) = \pi(N_w^{-1} h N_w) \qquad w \in \mathfrak{R}_\mathcal{L}, h \in \mathbb{H}(G^\circ, L, \mathcal{L}).$$

Let $\mathfrak{R}_{\mathcal{L},y,\sigma}$ (respectively $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$) be the stabilizer of $E^\circ_{y,\sigma,r}$ (respectively $E^\circ_{y,\sigma,r\rho^\circ}$) in $\mathfrak{R}_\mathcal{L}$. Similarly the group $\pi_0(Z_G(\sigma, y)))$ acts the set of $\pi_0(Z_{G^\circ}(\sigma, y))$-representations. Let $\pi_0(Z_G(\sigma, y))_{\rho^\circ}$ be the stabilizer of ρ° in $\pi_0(Z_G(\sigma, y))$.

Lemma 3.12. *There are natural isomorphisms*
(*a*) $\mathfrak{R}_{\mathcal{L},y,\sigma} \cong \pi_0(Z_G(\sigma, y))/\pi_0(Z_{G^\circ}(\sigma, y)) \cong \pi_0(Z_G(\sigma_0, y))/\pi_0(Z_{G^\circ}(\sigma_0, y))$,
(*b*) $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ} \cong \pi_0(Z_G(\sigma, y))_{\rho^\circ}/\pi_0(Z_{G^\circ}(\sigma, y))$.

Proof. (a) Since all the constructions are algebraic and $\mathfrak{R}_\mathcal{L}$ acts by algebraic automorphisms, $w \cdot E^\circ_{y,\sigma,r} \cong E^\circ_{w(y),w(\sigma),r}$. By Theorem 3.11.c $E^\circ_{w(y),w(\sigma),r} \cong E^\circ_{y,\sigma,r}$ if and only if (y, σ) and $(w(y), w(\sigma))$ are in the same $\mathrm{Ad}(G^\circ)$-orbit. We can write this condition as $wG^\circ \subset G_{\mathrm{Ad}(G^\circ)}(y, \sigma)$. Next we note that

$$G_{\mathrm{Ad}(G^\circ)}(y, \sigma)/G^\circ \cong Z_G(\sigma, y)/Z_{G^\circ}(\sigma, y).$$

Since G/G° is finite, the right hand side is isomorphic to $\pi_0(Z_G(\sigma, y))/\pi_0(Z_{G^\circ}(\sigma, y))$. By Lemma 3.6.a we can replace σ by σ_0 without changing these groups.
(b) Consider the stabilizer of ρ° in $\pi_0(Z_G(\sigma, y))/\pi_0(Z_{G^\circ}(\sigma, y))$. By part (a) it is isomorphic to the stabilizer of ρ° in $\mathfrak{R}_{\mathcal{L},y,\sigma}$. As $E^\circ_{y,\sigma,r,\rho^\circ} = \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma,y))}(\rho^\circ, E^\circ_{y,\sigma,r})$ and $\mathfrak{R}_{\mathcal{L},y,\sigma}$ stabilizes $E^\circ_{y,\sigma,r}$, this results in the desired isomorphism. □

Next we parametrize the relevant representations of $\pi_0(Z_G(\sigma, y))$.

Lemma 3.13. *There exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_\mathcal{L}^{-1}]) & \to & \big\{\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big) : \rho|_{\pi_0(Z_{G^\circ}(\sigma,y))} \text{ contains } \rho^\circ\big\} \\ (\tau, V_\tau) & \mapsto & \tau \ltimes \rho^\circ \end{array}.$$

Here $\tau \ltimes \rho^\circ = \mathrm{ind}^{\pi_0(Z_{G^\circ}(\sigma,y))}_{\pi_0(Z_{G^\circ}(\sigma,y))_{\rho^\circ}}(V_\tau \otimes V_{\rho^\circ})$, *where* $V_\tau \otimes V_{\rho^\circ}$ *is the tensor product of two projective representations of the stabilizer of* ρ° *in* $\pi_0(Z_{G^\circ}(\sigma, y))$.

Proof. For $\gamma \in \pi_0(Z_G(\sigma,y))_{\rho^\circ}$ we choose $I^\gamma \in \mathrm{Aut}_{\mathbb C}(V_{\rho^\circ})$ such that

$$I^\gamma \circ \rho^\circ(\gamma^{-1} z \gamma) = \rho^\circ(z) \circ I^\gamma \qquad z \in \pi_0(Z_{G^\circ}(\sigma,y)). \tag{45}$$

To simplify things a little, we may and will assume that $I^{\gamma z} = I^\gamma \circ \rho^\circ(z)$ for all $\pi_0(Z_G(\sigma,y))_{\rho^\circ}, z \in \pi_0(Z_{G^\circ}(\sigma,y))$. Then (45) implies that also $I^{z\gamma} = \rho^\circ(z) \circ I^\gamma$. By Schur's lemma there exist unique $\kappa_{\rho^\circ}(\gamma,\gamma') \in \mathbb C^\times$ such that

$$I^{\gamma\gamma'} = \kappa_{\rho^\circ}(\gamma,\gamma') I^\gamma \circ I^{\gamma'} \qquad \gamma,\gamma' \in \pi_0(Z_G(\sigma,y))_{\rho^\circ}. \tag{46}$$

Then κ_{ρ° is a 2-cocycle of $\pi_0(Z_G(\sigma,y))_{\rho^\circ}$. The above assumption and Lemma 3.12.b implies that it factors via

$$\pi_0(Z_G(\sigma,y))_{\rho^\circ} / \pi_0(Z_{G^\circ}(\sigma,y))_{\rho^\circ} \cong \mathfrak R_{\mathcal L,y,\sigma,r,\rho^\circ}.$$

Let $\mathbb C[\mathfrak R_{\mathcal L,y,\sigma,r,\rho^\circ}, \kappa_{\rho^\circ}]$ be the associated twisted group algebra, with basis $\{T_\gamma : \gamma \in \mathfrak R_{\mathcal L,y,\sigma,r,\rho^\circ}\}$. Then $\pi_0(Z_G(\sigma,y))$ acts on

$$\mathbb C[\mathfrak R_{\mathcal L,y,\sigma,r,\rho^\circ}, \kappa_{\rho^\circ}] \otimes_{\mathbb C} V_{\rho^\circ} \quad \text{by} \quad \gamma \cdot (T_{\gamma'} \otimes v) = T_{\gamma\gamma'} \otimes I^\gamma(v).$$

By Clifford theory (see [AMS, §1]) there is a bijection

$$\begin{array}{ccc} \mathrm{Irr}(\mathbb C[\mathfrak R_{\mathcal L,y,\sigma,\rho^\circ}, \kappa_{\rho^\circ}]) & \to & \big\{\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma,y))\big) : \rho|_{\pi_0(Z_{G^\circ}(\sigma,y))} \text{ contains } \rho^\circ\big\} \\ (\tau, V_\tau) & \mapsto & \tau \ltimes \rho^\circ \end{array}.$$

It remains to identify κ_{ρ°. By Proposition 3.7 the cuspidal support of (y,ρ°) is $(L, \mathcal C_v^L, \mathcal L)$, which means that it is contained in $\mathcal H^*(K)|_y$. Hence the 2-cocycle $\natural_{\mathcal L}$, used to extend the action of $W_{\mathcal L}^\circ$ on K to $\mathbb C[W_{\mathcal L}, \natural_{\mathcal L}]$, also gives an action on V_{ρ°. Comparing the multiplication relations in $\mathbb C[W_{\mathcal L}, \natural_{\mathcal L}]$ with (46), we see that we can arrange that κ_{ρ° is the restriction of $\natural_{\mathcal L}^{-1}$ to $\mathfrak R_{\mathcal L,y,\sigma,\rho^\circ}$. $\square$

The analogue of 3.13 for $M^\circ_{y,\sigma,\rho^\circ}$ is more difficult, we need some technical preparations. Since $N_{G^\circ}(P) = P$, we can identify G°/P with a variety $\mathcal P^\circ$ of parabolic subgroups P' of G°. For $g \in G$ and $P' \in \mathcal P$ we write

$$\mathrm{Ad}(g)P' = gP'g^{-1}.$$

This extends the left multiplication action of G° on $\mathcal P^\circ$ and it gives rise to an action of G on $\dot{\mathfrak g}$ by

$$\mathrm{Ad}(g)(x,P') = (\mathrm{Ad}(g)x, gP'g^{-1}).$$

By Condition 3.1 every element of G stabilizes $\mathcal L$, so $\mathrm{Ad}(g)^* \dot{\mathcal L} \cong \dot{\mathcal L}$. Lift $\mathrm{Ad}(g)$ to an isomorphism of G°-equivariant sheaves

$$\mathrm{Ad}_{\mathcal L}(g) : \dot{\mathcal L} \to \mathrm{Ad}(g)^* \dot{\mathcal L}. \tag{47}$$

(Although there is more than one way to do so, we will see in Proposition 3.15 that in relevant situations $\mathrm{Ad}_{\mathcal L}(g)$ is unique up to scalars.) Thus $\mathrm{Ad}_{\mathcal L}(g)$ provides a system of linear bijections

$$(\dot{\mathcal L})_{(x,P')} \to (\dot{\mathcal L})_{(\mathrm{Ad}(g)x, \mathrm{Ad}(g)P')} \quad \text{such that} \quad \mathrm{Ad}_{\mathcal L}(g) \circ (g^{-1} g^\circ g) = g^\circ \circ \mathrm{Ad}_{\mathcal L}(g) \tag{48}$$

for all $g^\circ \in G^\circ$. (Here we denote the canonical action of g° on $\mathcal L$ simply by g°.) Of course we can choose these maps such that $\mathrm{Ad}_{\mathcal L}(g g^\circ) = \mathrm{Ad}_{\mathcal L}(g) \circ g^\circ$ for $g^\circ \in G^\circ$. Notice that

$$\mathrm{Ad}_{\mathcal L}(g^\circ) \text{ coincides with the earlier action of } G^\circ \text{ on } \dot{\mathcal L}. \tag{49}$$

For $g \in Z_G(\sigma,y)$, $\mathrm{Ad}(g)$ stabilizes $\mathcal P^\circ_y$, and the map $\mathrm{Ad}_{\mathcal L}(g)$ induces an operator $H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal L}(g))$ on $H_*^{M(y)^\circ}(\mathcal P^\circ_y, \dot{\mathcal L})$.

Lemma 3.14. *For all $h \in \mathbb{H}(G^\circ, L, \mathcal{L}), \gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma}$ and $g \in \gamma G^\circ \cap Z_G(\sigma, y)$:*

$$H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) \circ \Delta(h) = \Delta(N_\gamma h N_\gamma^{-1}) \circ H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) \in \mathrm{End}_{\mathbb{C}}\big(H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})\big).$$

Proof. By Theorem 3.2.d the map $H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))$ commutes with the action of $\mathbb{H}(G^\circ, L, \mathcal{L})$ on $H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$. Moreover $H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) = 1$ for g in the connected group $Z_G(\sigma, \rho)^\circ$. Thus we get a map

$$\pi_0(Z_G(\sigma, y))_{\rho^\circ} \to \mathrm{Aut}_{\mathbb{C}}\big(H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})\big)$$

which sends $\pi_0(Z_{G^\circ}(\sigma, y))$ to $\mathrm{Aut}_{\mathbb{H}(G^\circ,L,\mathcal{L})}\big(H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})\big)$. Recall from (21) that the action of $S(\mathfrak{t}^* \oplus \mathbb{C})$ on $H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$ comes from the product with $H^*_{M(y) \cap G^\circ}(\mathcal{P}_y^\circ)$. The functoriality of this product and (48) entail that

$$H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\delta \otimes \eta) = \mathrm{Ad}(g)\delta \otimes H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\eta)$$

for $\eta \in H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})$ and $\delta \in H^*_{M(y)^\circ \cap G^\circ}(\mathcal{P}_y^\circ)$. The operators $\mathrm{Ad}(g)$ on $H^*_{M(y) \cap G^\circ}(\mathcal{P}_y^\circ)$ are trivial for $g \in Z_{G^\circ}(\sigma, y) \subset M(y) \cap G^\circ$, they factor through

$$Z_G(\sigma, y)/Z_{G^\circ}(\sigma, y) \cong \mathfrak{R}_{\mathcal{L},y,\sigma}.$$

Similarly, the operators $\mathrm{Ad}(g)$ on $S(\mathfrak{t}^* \oplus \mathbb{C}) \cong H^*_{G^\circ \times \mathbb{C}^\times}(\dot{\mathfrak{g}}^\circ)$ factor through $Z_G(\sigma, y)/Z_{G^\circ}(\sigma, y)$ and become the natural action of $\mathfrak{R}_{\mathcal{L},y,\sigma}$. Hence

$$H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) \circ \Delta(\xi) = \Delta(\mathrm{Ad}(\gamma)\xi) \circ H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) \tag{50}$$

for $\gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma}, g \in \gamma G^\circ \cap Z_G(\sigma, y))$ and $\xi \in S(\mathfrak{t}^* \oplus \mathbb{C})$. By making the appropriate choices, we can arrange that the dual map of (47) is

$$\mathrm{Ad}_{\mathcal{L}^*}(g^{-1}) : \mathrm{Ad}(g)^* \dot{\mathcal{L}}^* \to \dot{\mathcal{L}}^*.$$

It induces $\mathrm{Ad}_{\mathcal{L}^*}(g^{-1}) : \mathrm{Ad}(g)^* K^\circ \to K^\circ$, where K° is K^* but for G°. The operators N_w $(w \in W_{\mathcal{L}}^\circ)$ from (7) are $G^\circ \times \mathbb{C}^\times$-equivariant, so the operator

$$\mathrm{Ad}_{\mathcal{L}^*}(g^{-1})^{-1} \circ N_w \circ \mathrm{Ad}_{\mathcal{L}^*}(g^{-1}) \in \mathrm{Aut}_{G^\circ \times \mathbb{C}^\times}(K^\circ) \tag{51}$$

depends only on the image of g^{-1} in G/G°. If $\gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma}$ and $g \in \gamma G^\circ$, then we see from the definition of N_w in [Lus1, 3.4] that (51) is a (nonzero) scalar multiple of $N_{\gamma w \gamma^{-1}}$. Consequently

$$H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}^*}(g^{-1}))^{-1} \circ \tilde{\Delta}(N_w) \circ H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}^*}(g^{-1})) = \lambda(w, \gamma) \tilde{\Delta}(N_\gamma N_w N_\gamma^{-1})$$

for some number $\lambda(w, \gamma) \in \mathbb{C}^\times$. Dualizing, we find that

$$H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) \circ \Delta(N_w) \circ H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}^*}(g))^{-1} = \lambda(w, \gamma) \Delta(N_\gamma N_w^{-1} N_\gamma^{-1}). \tag{52}$$

Let $\alpha_i \in R(G^\circ, T)$ be a simple root and let $s_i \in W_{\mathcal{L}}^\circ$ be the associated simple reflection. By the multiplication rules in $\mathbb{H}(G^\circ, L, \mathcal{L})$

$$0 = \Delta(N_{s_i} \alpha_i - {}^{s_i}\alpha_i N_{s_i} - c_i \mathbf{r}(\alpha_i - {}^{s_i}\alpha_i)/\alpha_i) = \Delta(N_{s_i} \alpha_i + \alpha_i N_{s_i} - 2 c_i \mathbf{r}). \tag{53}$$

Now we apply (50) and (52) and to this equality, and we find

$$\begin{aligned} 0 &= H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) \circ \Delta(N_{s_i} \alpha_i + \alpha_i N_{s_i} - 2 c_i \mathbf{r}) \circ H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))^{-1} \\ &= \Delta(\lambda(s_i, \gamma) N_{\gamma s_i \gamma^{-1}} \, {}^{\gamma}\alpha_i + \lambda(s_i, \gamma) \, {}^{\gamma}\alpha_i N_{\gamma s_i \gamma^{-1}} - 2 c_i \mathbf{r}). \end{aligned} \tag{54}$$

We note that $\alpha_j := {}^\gamma \alpha_i$ is another simple root, with reflection $s_j := \gamma s_i \gamma^{-1}$ and $c_j = c_i$. By (53) the second line of (54) becomes

$$\lambda(s_i,\gamma)\Delta(N_{s_j}\ \alpha_j + \alpha_j N_{s_j} - 2c_j \mathbf{r}) + 2c_i \Delta(\lambda(s_i,\gamma)\mathbf{r} - \mathbf{r}) = 2(\lambda(s_i,\gamma)-1)c_i\Delta(\mathbf{r}).$$

Recall from (12) that $c_i > 0$. As $\Delta(\mathbf{r}) = r$ is nonzero for some choices of (σ, r), we deduce that $\lambda(s_i,\gamma) = 1$ for all $\gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$. In view of (52) this implies $\lambda(w,\gamma) = 1$ for all $w \in W_{\mathcal{L}}^\circ, \gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$. Now (50) and (52) provide the desired equalities. □

Lemma 3.14 says that for $g \in \gamma G^\circ \cap Z_G(\sigma,y)$, $H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))$ intertwines the standard $\mathbb{H}(G^\circ,L,\mathcal{L})$-modules $E^\circ_{y,\sigma,r}$ and $\gamma \cdot E^\circ_{y,\sigma,r}$. However, it does not necessarily map the subrepresentation $E^\circ_{y,\sigma,r,\rho^\circ}$ to $\gamma \cdot E^\circ_{y,\sigma,r,\rho^\circ}$, even for $\gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$. Moreover $g \mapsto H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))$ need not be multiplicative, by the freedom in (47). In general it is not even possible to make it multiplicative by clever choices in (47). The next lemma takes care of both these inconveniences.

Proposition 3.15. *Let y, σ, r, ρ° be as in Theorem 3.11. There exists a group homomorphism*

$$\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ} \to \mathrm{Aut}_{\mathbb{C}}(E^\circ_{y,\sigma,r,\rho^\circ}) : \gamma \mapsto J^\gamma,$$

depending algebraically on (σ, r) and unique up to scalars, such that

$$J^\gamma(\Delta(N_\gamma^{-1} h N_\gamma)\phi) = \Delta(h) J^\gamma(\phi) \qquad h \in \mathbb{H}(G^\circ,L,\mathcal{L}), \phi \in E^\circ_{y,\sigma,r,\rho^\circ}.$$

Proof. For $g \in Z_G(\sigma,y)$ let I^g be as in (45) and let $H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))$ be as in Lemma 3.14. We define

$$J^g(\phi) = H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\phi \circ (I^g)^{-1}) \qquad \phi \in E^\circ_{y,\sigma,r,\rho^\circ} = \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma,y))}(\rho^\circ, E^\circ_{y,\sigma,r}).$$

For $v \in V_{\rho^\circ}$ and $z \in Z_{G^\circ}(\sigma,y)$ we calculate:

$$\begin{aligned} J^g(\rho^\circ(z)\phi) &= H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\phi \circ (I^g)^{-1}\rho^\circ(z)v) \\ &= H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\phi \circ \rho^\circ(g^{-1}zg)(I^g)^{-1}v) \\ &= H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))((g^{-1}zg)\phi \circ (I^g)^{-1}v) \\ &= H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g)) H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g^{-1}zg))(\phi \circ (I^g)^{-1}v) \\ &= H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(z)) H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\phi \circ (I^g)^{-1}v) = z \cdot J^g(\phi)(v). \end{aligned}$$

Thus J^g sends $E^\circ_{y,\sigma,r,\rho^\circ}$. It is invertible because $H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))$ and I^g are. By (49), (45) and the intertwining property of ϕ, $J^g(\phi) = J^{g'}(\phi)$ whenever $g^{-1}g' \in G^\circ$. Hence $J^g \in \mathrm{Aut}_{\mathbb{C}}(E^\circ_{y,\sigma,r,\rho^\circ})$ depends only on the image of g in $\mathfrak{R}_{\mathcal{L}} \cong G/G^\circ$, and may denote it by J^γ when $g \in \gamma G^\circ$.

As the $\pi_0(Z_{G^\circ}(\sigma,y))$-action commutes with that of $\mathbb{H}(G^\circ,L,\mathcal{L})$, we deduce from Lemma 3.14 that

$$\begin{aligned} J^\gamma(\Delta(N_\gamma^{-1} h N_\gamma)\phi) = H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))\Delta(N_\gamma^{-1} h N_\gamma)(\phi \circ (I^g)^{-1}) = \\ \Delta(h) H_*^{M(y)^\circ}(\mathrm{Ad}_{\mathcal{L}}(g))(\phi \circ (I^g)^{-1}) = \Delta(h) J^\gamma(\phi). \end{aligned} \tag{55}$$

By Lemma 3.6 and (32) all the vector spaces $E^\circ_{y,\sigma,r,\rho^\circ}$ can be identified with $\mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma_0,y))}(\rho^\circ, H_*(\mathcal{P}_y^\circ, \dot{\mathcal{L}}))$. In this sense J^γ depends algebraically on $(\sigma, r) = (\sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right), r)$.

Given $y, r \neq 0$, [Lus3, Theorem 8.17.b] implies that $E^\circ_{y,\sigma,r,\rho^\circ}$ is irreducible for all σ in a Zariski-open nonempty subset of $\{\sigma \in \mathfrak{g} : [\sigma, y] = 2ry\}$. For such σ (55) and Schur's lemma imply that J^γ is unique up to scalars. By the algebraic dependence on (σ, r), this holds for all (σ, r). Hence the choice of $\mathrm{Ad}_{\mathcal{L}}(g)$ in (47) is also unique up to scalars. If we can choose the $\mathrm{Ad}_{\mathcal{L}}(g)$ such that $\gamma \mapsto J^\gamma$ is multiplicative for at least one value of (σ, r), then the definition of J^G shows that it immediately holds for all (σ, r).

For $r = 0$ (55) says that J^γ intertwines $E^\circ_{y,\sigma,0,\rho^\circ}$ and $\gamma \cdot E^\circ_{y,\sigma,0,\rho^\circ}$. Then it also intertwines the quotients $M^\circ_{y,\sigma,0,\rho^\circ}$ and $\gamma \cdot M^\circ_{y,\sigma,0,\rho^\circ}$ from Lemma 3.10. Recall that

$$M^\circ_{y,\sigma_0,0,\rho^\circ} = \mathrm{ind}_{W_{\mathcal{L}}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}^{W_{\mathcal{L}}^\circ \ltimes S(\mathfrak{t}^*)} (M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ})$$

where $Q^\circ = Z_{G^\circ}(\sigma_0)$ and $S(\mathfrak{t}^*)$ acts on $wM^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$ via the character $w\sigma_0$. By (55)

$$J^\gamma(wM^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}) = (\gamma w \gamma^{-1}) M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ},$$

and in particular all the J^γ restrict to elements

$$J^\gamma_{Q^\circ} \in \mathrm{Aut}_{W_{\mathcal{L}}^{Q^\circ} \ltimes S(\mathfrak{t}^*)}(M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}) = \mathrm{Aut}_{W_{\mathcal{L}}^{Q^\circ}}(M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}) = \mathrm{Aut}_{W_{\mathcal{L}}^{Q^\circ}}(M^{Q^\circ}_{y,\rho^\circ}).$$

Here $W_{\mathcal{L}}^{Q^\circ}$ is the Weyl group of (Q°, T), a group normalized by $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$. By [ABPS2, Proposition 4.3] we can choose the J^γ (which we recall are still unique up to scalars) such that $\gamma \mapsto J^\gamma$ is a group homomorphism (for $r = 0$). As we noted before, this determines a choice of all the J^γ such that $\gamma \mapsto J^\gamma$ is multiplicative. $\square$

Recall from Theorem 3.11 that the quotient map $E^\circ_{y,\sigma,r,\rho^\circ} \to M^\circ_{y,\sigma,r,\rho^\circ}$ provides a bijection between standard modules and $\mathrm{Irr}(\mathbb{H}(G^\circ, L, \mathcal{L}))$. Therefore Proposition 3.15 also applies to all irreducible representations of $\mathbb{H}(G^\circ, L, \mathcal{L})$. It expresses a regularity property of geometric graded Hecke algebras: the group of automorphisms $\mathfrak{R}_{\mathcal{L}}$ of the Dynkin diagram of (G°, T) can be lifted to a group of intertwining operators between the appropriate irreducible representations.

With Clifford theory we can obtain a first construction and classification of all irreducible representations of $\mathbb{H}(G, L, \mathcal{L})$:

Lemma 3.16. *There exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}]) & \to & \big\{\pi \in \mathrm{Irr}\big(\mathbb{H}(G, L, \mathcal{L})\big) : \pi|_{\mathbb{H}(G^\circ,L,\mathcal{L})} \text{ contains } M^\circ_{y,\sigma,r,\rho^\circ}\big\} \\ (\tau, V_\tau) & \mapsto & \tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ} \end{array}.$$

Here $\tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ} = \mathrm{ind}_{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},\sigma,y,\rho^\circ},L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})}(V_\tau \otimes M^\circ_{y,\sigma,r,\rho^\circ})$*, where* $\mathbb{H}(G^\circ, L, \mathcal{L})$ *acts trivially on* V_τ *and*

$$N_\gamma \cdot (v \otimes m) = \tau(N_\gamma) v \otimes J^\gamma(m) \qquad \gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, v \in V_\tau, m \in M^\circ_{y,\sigma,r,\rho^\circ}.$$

Proof. Let the central extension $\mathfrak{R}_{\mathcal{L}}^+ \to \mathfrak{R}_{\mathcal{L}}$ and $p_{\natural_{\mathcal{L}}}$ be as in the proof of Proposition 2.2, and let $\mathfrak{R}^+$ be the inverse image of $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$ in $\mathfrak{R}_{\mathcal{L}}^+$. As in (4), $\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, L, \mathcal{L})$ is the direct summand

$$p_{\natural_{\mathcal{L}}} \mathbb{C}[\mathfrak{R}^+] \ltimes \mathbb{H}(G^\circ, L, \mathcal{L}) \quad \text{ of } \quad \mathfrak{R}^+ \ltimes \mathbb{H}(G^\circ, L, \mathcal{L}).$$

By Proposition 3.15 and Clifford theory (in the version [Sol1, Theorem 1.2] or [RaRa, p. 24]) there is a bijection

$$\begin{array}{ccc} \mathrm{Irr}(\mathfrak{R}^+) & \to & \big\{\pi \in \mathrm{Irr}\big(\mathfrak{R}^+ \ltimes \mathbb{H}(G^\circ, L, \mathcal{L})\big) : \pi|_{\mathbb{H}(G^\circ,L,\mathcal{L})} \text{ contains } M^\circ_{y,\sigma,r,\rho^\circ}\big\} \\ (\tau, V_\tau) & \mapsto & \mathrm{ind}_{\mathfrak{R}^+ \ltimes \mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathfrak{R}^+_{\mathcal{L}} \ltimes \mathbb{H}(G^\circ,L,\mathcal{L})} (V_\tau \otimes M^\circ_{y,\sigma,r,\rho^\circ}) \end{array}.$$

Restrict this to the modules that are not annihilated by the central idempotent $p_{\natural_{\mathcal{L}}}$. □

3.4. **Parametrization of irreducible representations.**

We start this paragraph with a few further preparatory results. Let (y, σ, ρ°) be as before.

Lemma 3.17. *There are isomorphisms of $\pi_0(Z_G(\sigma, y))_{\rho^\circ}$-representations*

$$\begin{aligned} \mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})} (V_{\rho^\circ} \otimes E^\circ_{y,\sigma,r,\rho^\circ}) &\cong \mathrm{ind}_{\pi_0(Z_{G^\circ}(\sigma,y))}^{\pi_0(Z_G(\sigma,y))_{\rho^\circ}} (V_{\rho^\circ} \otimes E^\circ_{y,\sigma,r,\rho^\circ}) \\ &\cong \mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}^{-1}] \otimes V_{\rho^\circ} \otimes E^\circ_{y,\sigma,r,\rho^\circ}. \end{aligned}$$

In the last line the action is

$$g \cdot (N_w \otimes v \otimes \phi) = N_g N_w \otimes I^g(v) \otimes \phi$$

for $g \in \pi_0(Z_G(\sigma, y))_{\rho^\circ}, w \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, v \in V_{\rho^\circ}$ and $\phi \in E^\circ_{y,\sigma,r,\rho^\circ}$.

Proof. Recall that $E_{y,\sigma,r} = \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$ and that

$$\mathcal{P}_y \cap (G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}/P) = \mathcal{P}_y^\circ \times \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}. \tag{56}$$

There are two projective actions of $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$ on $E_{y,\sigma,r}^{G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}}$. The first one comes from considering it as the group underlying $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}] \subset \mathbb{H}(G, L, \mathcal{L})$, and the second one from considering it as a quotient of $\pi_0(Z_G(\sigma, y))_{\rho^\circ}$. Both induce a simply transitive permutation of the copies of $\mathcal{P}_y^\circ$ in (56), the first action by right multiplication and the second action by left multiplication. This implies the first stated isomorphism.

The second claim is an instance of [AMS, Proposition 1.1.b]. Here we use Lemma 3.13 to identify the 2-cocycle. We note that there is some choice in the second isomorphism of the lemma, we can still twist it by a character of $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$. □

Notice that the twisted group algebras of $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$ appearing in Lemmas 3.13 and 3.16 are opposite, but not necessarily isomorphic. If $(\tau, V_\tau) \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}])$, then $(\tau^*, V_\tau^*) \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}^{-1}])$, where

$$\tau^*(N_\gamma)\lambda = \lambda \circ \tau(N_\gamma^{-1}) \qquad \gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \lambda \in V_\tau^*.$$

As noted in [AMS, Lemma 1.3], this sets up a natural bijection between $\mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}])$ and $\mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_{\mathcal{L}}^{-1}])$.

Lemma 3.18. *In the notations of Lemma 3.16, there is an isomorphism of $\mathbb{H}(G, L, \mathcal{L})$-modules $E_{y,\sigma,r,\rho^\circ \rtimes \tau^*} \cong \tau \ltimes E^\circ_{y,\sigma,r,\rho^\circ}$.*

Proof. By Lemma 3.3

$$E_{y,\sigma,r,\rho^\circ \rtimes \tau^*} = \mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\tau^* \ltimes \rho^\circ, \mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L}}^{\mathbb{H}(G,L,\mathcal{L})} E^\circ_{y,\sigma,r}\big).$$

By Frobenius reciprocity this is isomorphic to

$$\text{Hom}_{\pi_0(Z_G(\sigma,y))_{\rho^\circ}}\big(\tau^* \otimes \rho^\circ, \text{ind}_{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} \text{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})} E^\circ_{y,\sigma,r}\big). \tag{57}$$

The action of $\pi_0(Z_G(\sigma,y))_{\rho^\circ}$ can be constructed entirely within $G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ,L,\mathcal{L}}$, so we can move the first induction outside the brackets. Furthermore we only need the ρ°-isotypical part of $E^\circ_{y,\sigma,r}$, so (57) equals

(58)

$$\text{ind}_{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} \text{Hom}_{\pi_0(Z_G(\sigma,y))_{\rho^\circ}}\big(\tau^* \otimes \rho^\circ, \text{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})} V_{\rho^\circ} \otimes E^\circ_{y,\sigma,r,\rho^\circ}\big).$$

From Lemma 3.17 and [AMS, Proposition 1.1.d] we deduce that

$$\begin{aligned} \text{Hom}_{\pi_0(Z_G(\sigma,y))_{\rho^\circ}}\big(\tau^* \otimes \rho^\circ, \text{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})} V_{\rho^\circ} \otimes E^\circ_{y,\sigma,r,\rho^\circ}\big) = \\ \text{Hom}_{\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}]}\big(\tau^*, \text{Hom}_{\pi_0(Z_{G^\circ}(\sigma,y))}(\rho^\circ, \mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}] \otimes V_{\rho^\circ} \otimes E^\circ_{y,\sigma,r,\rho^\circ})\big) = \\ \text{Hom}_{\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}]}\big(\tau^*, \mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}] \otimes E^\circ_{y,\sigma,r,\rho^\circ}\big). \end{aligned} \tag{59}$$

Here $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}]$ fixes $E^\circ_{y,\sigma,r,\rho^\circ}$ pointwise. By [AMS, Lemma 1.3.c] there is an isomorphism of $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}] \times \mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}]$-modules

$$\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}] \cong \bigoplus_{\pi \in \text{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}])} V_\pi^* \otimes V_\pi. \tag{60}$$

Thus the $\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})$-module (59) becomes $V_\tau \otimes E^\circ_{y,\sigma,r,\rho^\circ}$, while (57) and (58) become

$$\text{ind}_{\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} (V_\tau \otimes E^\circ_{y,\sigma,r,\rho^\circ}). \tag{61}$$

The subalgebra $\mathbb{H}(G^\circ,L,\mathcal{L})$ fixes V_τ pointwise. To understand the above $\mathbb{H}(G,L,\mathcal{L})$-module, it remains to identify the action of $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}]$ on $V_\tau \otimes E^\circ_{y,\sigma,r,\rho^\circ}$. For that we return to the first line of (59). Taking into account that the actions of $\pi_0(Z_G(\sigma,y))_{\rho^\circ}$ and $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}] \subset \mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L})$ commute, [AMS, Proposition 1.1.d] says that it is isomorphic to

$$\text{Hom}_{\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}]}\big(\tau^*, \mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}] \otimes E^\circ_{y,\sigma,r,\rho^\circ}\big). \tag{62}$$

We have seen in (59) that $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}^{-1}]$ fixes $E^\circ_{y,\sigma,r,\rho^\circ}$ pointwise, and we know from Theorem 3.2.d that its action commutes with $\Delta(\mathbb{H}(G^\circ \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},L,\mathcal{L}))$. The proof of Lemma 3.17 entails that, up to a scalar which depends only on $\gamma \in \mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$,

$$N_\gamma \cdot (N \otimes \phi) = N N_\gamma^{-1} \otimes \phi \qquad N \in \mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}], \phi \in E^\circ_{y,\sigma,r,\rho^\circ}.$$

Since this formula already defines an action, the family of scalars (for various γ) must form a character of $\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}$. We can make this character trivial by adjusting the choice of the second isomorphism in Lemma 3.17, which means that $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}]$ in (62) becomes a bimodule in the standard manner. By (60) for $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}]$, (62) is isomorphic, as $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}]$-module, to $V_\tau \otimes E^\circ_{y,\sigma,r,\rho^\circ}$. Consequently the $\mathbb{H}(G,L,\mathcal{L})$ is endowed with the expected action of $\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_{\mathcal{L}}]$ on V_τ, which means that it can be identified with $\tau \ltimes E^\circ_{y,\sigma,r,\rho^\circ}$. □

It will be useful to improve our understanding of standard modules with $r = 0$, like in Lemma 3.9.

Lemma 3.19. *The* $\mathbb{H}(G,L,\mathcal{L})$*-module* $E_{y,\sigma,0,\rho}$ *is completely reducible and can be decomposed along the homological degree:*

$$E_{y,\sigma,0,\rho} = \bigoplus_n \mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\rho, H_n(\mathcal{P}_y,\dot{\mathcal{L}})\big).$$

Erratum. Like in Lemma 3.9, this module is only filtered by the homological degrees.

Proof. By Lemma 3.3

$$E_{y,\sigma,0} \cong \mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} E^\circ_{y,\sigma,0}. \tag{63}$$

From Lemma 3.9 we know that $E^\circ_{y,\sigma,0}$ is completely reducible. As $\mathbb{H}(G^\circ,L,\mathcal{L}) = \mathbb{C}[\mathfrak{R}_\mathcal{L},\natural_\mathcal{L}] \ltimes \mathbb{H}(G^\circ,L,\mathcal{L})$ where $\mathbb{C}[\mathfrak{R}_\mathcal{L},\natural_\mathcal{L}]$ is a twisted group algebra of a finite group acting on $\mathbb{H}(G^\circ,L,\mathcal{L})$, the induction in (63) preserves complete reducibility.

From Lemma 3.9 we know that $E^\circ_{y,\sigma,0} = \bigoplus_n H_n(\mathcal{P}^\circ_y,\dot{\mathcal{L}})$. The proof of Lemma 3.3 shows that

$$\mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} H_n(\mathcal{P}^\circ_y,\dot{\mathcal{L}}) \cong \mathbb{C}[\mathfrak{R}_\mathcal{L},\natural_\mathcal{L}] \otimes_\mathbb{C} H_n(\mathcal{P}^\circ_y,\dot{\mathcal{L}}) \cong H_n(\mathcal{P}_y,\dot{\mathcal{L}}),$$

so $E_{y,\sigma,0} = \bigoplus_n H_n(\mathcal{P}_y,\dot{\mathcal{L}})$ as $\mathbb{H}(G,L,\mathcal{L})$-modules. Since the action of $\pi_0(Z_G(\sigma,y))$ commutes with that of $\mathbb{H}(G,L,\mathcal{L})$, $E_{y,\sigma,0,\rho} = \mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}(\rho,E_{y,\sigma,0})$ is also completely reducible, and the decomposition according to homological degree persists in $E_{y,\sigma,0,\rho}$. □

We note that the definitions (33) and (34) also can be used with G instead of G°, provided that one involves the generalized Springer correspondence for disconnected groups from [AMS, §4]. In this way we define the $\mathbb{H}(G,L,\mathcal{L})$-module $M_{y,\sigma_0,0,\rho}$.

Now we are ready to prove the main result of this section. It generalizes [Lus5, Corollary 8.18] to disconnected groups G. Recall that Condition 3.1 is in force.

Theorem 3.20. *Let* $y \in \mathfrak{g}$ *be nilpotent and let* $(\sigma,r)/\!\sim\, \in V_y$ *be semisimple. Let* $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma,y))\big)$ *be such that* $\Psi_{Z_G(\sigma_0)}(y,\rho) = (L,\mathcal{C}^L_v,\mathcal{L})$ *(up to G-conjugation).*

(*a*) *If* $r \neq 0$*, then* $E_{y,\sigma,r,\rho}$ *has a unique irreducible quotient* $\mathbb{H}(G,L,\mathcal{L})$*-module. We call it* $M_{y,\sigma,r,\rho}$*.*

(*b*) *If* $r = 0$*, then* $E_{y,\sigma_0,r,\rho}$ *has a unique irreducible subquotient isomorphic to* $M_{y,\sigma_0,0,\rho}$*. This subquotient is the component of* $E_{y,\sigma_0,r,\rho}$ *in one homological degree (as in Lemma 3.19).*

(*c*) *Parts (a) and (b) set up a canonical bijection between* $\mathrm{Irr}_r(\mathbb{H}(G,L,\mathcal{L}))$ *and the G-orbits of triples* (y,σ,ρ) *as above.*

(*d*) *The two sets from part (c) are canonically in bijection with the collection of G-orbits of triples* (y,σ_0,ρ) *as in Proposition 3.8. (The only difference is that* $\sigma_0 \in Z_\mathfrak{g}(y)$ *instead of* $(\sigma,r) \in \mathrm{Lie}(Z_{G\times\mathbb{C}^\times}(y))$*. That is,* (y,σ_0,ρ) *is obtained from* (y,σ,ρ) *via Lemma 3.6.)*

Proof. Let ρ° be an irreducible constituent of $\rho|_{\pi_0(Z_{G^\circ}(\sigma,y))}$. By Lemma 3.13 there is a unique $\tau^* \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ},\natural_\mathcal{L}^{-1}])$ such that $\rho \cong \rho^\circ \rtimes \tau^*$.

(a) From Lemma 3.18 we know that

$$E_{y,\sigma,r,\rho} \cong \tau \ltimes E^\circ_{y,\sigma,r,\rho^\circ}.$$

By Lemma 3.16 it has the irreducible quotient

$$\tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ} = (\tau \ltimes E^\circ_{y,\sigma,r,\rho^\circ})/(\tau \ltimes N^\circ), \tag{64}$$

where $N^\circ = \ker(E^\circ_{y,\sigma,r,\rho^\circ} \to M^\circ_{y,\sigma,r,\rho^\circ})$. Hence $\tau \ltimes N^\circ$ is a maximal proper submodule of $E_{y,\sigma,r,\rho^\circ}$. We define

$$I_M = \{V \in \mathrm{Irr}_r(\mathbb{H}(G^\circ, L, \mathcal{L})) : V \text{ is a constituent of } \tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ}\},$$
$$I_N = \{V \in \mathrm{Irr}_r(\mathbb{H}(G^\circ, L, \mathcal{L})) : V \text{ is a constituent of } \tau \ltimes N^\circ\}.$$

Recall from Theorem 3.11.d that all the irreducible $\mathbb{H}(G^\circ, L, \mathcal{L})$-constituents of N° are of the form $M^\circ_{y',\sigma',r,\rho'^\circ}$, where $\dim \mathcal{C}^{G^\circ}_{y'} > \dim \mathcal{C}^{G^\circ}_{y}$. Since $\mathfrak{R}_\mathcal{L}$ acts by algebraic automorphisms on G°, the same holds for all $\mathbb{H}(G^\circ, L, \mathcal{L})$-constituents of $\tau \ltimes N^\circ$. Hence I_M and I_N are disjoint. Moreover these sets are finite, so by Wedderburn's theorem about irreducible representations the canonical map

$$\mathbb{H}(G^\circ, L, \mathcal{L}) \to \bigoplus\nolimits_{V \in I_M} \mathrm{End}(V) \oplus \bigoplus\nolimits_{V \in I_N} \mathrm{End}(V)$$

is surjective. In particular there exists an element of $\mathbb{H}(G^\circ, L, \mathcal{L})$ which annihilates all $V \in I_N$ and fixes all $V \in I_M$ pointwise. By Theorem 3.2.d $\tau \ltimes N^\circ$ has finite length, so a suitable power h° of that element annihilates $\tau \ltimes N^\circ$. Since $\tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ}$ is completely reducible as $\mathbb{H}(G^\circ, L, \mathcal{L})$-module, h° acts as the identity on it.

Choose a basis $\mathcal{B}$ of $\mathbb{C}[\mathfrak{R}_\mathcal{L}, \natural_\mathcal{L}] \underset{\mathbb{C}[\mathfrak{R}_{\mathcal{L},y,\sigma,\rho^\circ}, \natural_\mathcal{L}]}{\otimes} V_\tau$, consisting of elements of the form $b = N_\gamma \otimes v$ with $\gamma \in \mathfrak{R}_\mathcal{L}$ and $v \in V_\tau$. Since $\tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ}$ is irreducible, we can find for all $b, b' \in \mathcal{B}$ an element $h_{bb'} \in \mathbb{H}(G, L, \mathcal{L})$ which maps $b' M^\circ_{y,\sigma,r,\rho^\circ}$ bijectively to $b M^\circ_{y,\sigma,r,\rho^\circ}$ and annihilates all the other subspaces $b'' M^\circ_{y,\sigma,r,\rho^\circ}$.

Consider any $x \in E_{y,\sigma,r,\rho^{\mathrm{Hom}}_{\pi_0(Z_{G^\circ}(s,u))}(\rho^\circ, H_{d(u)}(\mathcal{B}^{s,u}_{G^\circ}, \mathbb{C}))^\circ} \setminus \tau \ltimes N^\circ$. Write it in terms of $\mathcal{B}$ as $x = \sum_{b \in \mathcal{B}} b \otimes x_b$ with $x_b \in E^\circ_{y,\sigma,r,\rho}$. For at least one $b' \in \mathcal{B}$, $x_{b'} \in E^\circ_{y,\sigma,r,\rho} \setminus N^\circ$. Then

$$h^\circ h_{bb'} x = b \otimes v' \text{ for some } v' \in E^\circ_{y,\sigma,r,\rho^\circ} \setminus N^\circ.$$

As $\mathbb{H}(G^\circ, L, \mathcal{L})$-representation

$$b E^\circ_{y,\sigma,r,\rho^\circ} = (N_\gamma \otimes v) E^\circ_{y,\sigma,r,\rho^\circ} \cong \gamma \cdot E^\circ_{y,\sigma,r,\rho^\circ},$$

which has the unique maximal proper submodule $\gamma \cdot N^\circ \cong b N^\circ$. Hence

$$\mathbb{H}(G^\circ, L, \mathcal{L}) h^\circ h_{bb'} x = b E^\circ_{y,\sigma,r,\rho^\circ}.$$

This works for every $b \in \mathcal{B}$, so $\mathbb{H}(G, L, \mathcal{L}) x = E_{y,\sigma,r,\rho}$. Consequently there is no other maximal proper submodule of $E_{y,\sigma,r,\rho}$ besides $\tau \ltimes N^\circ$.
(b) Put $Q = Z_G(\sigma_0)$. By Lemma 3.3 and (37)

$$E_{y,\sigma_0,0} \cong \mathrm{ind}^{\mathbb{H}(G,L,\mathcal{L})}_{\mathbb{H}(Q,L,\mathcal{L})} E^Q_{y,\sigma_0,0} = \mathrm{ind}^{\mathbb{H}(G,L,\mathcal{L})}_{\mathbb{H}(Q^\circ,L,\mathcal{L})} E^{Q^\circ}_{y,\sigma_0,0}.$$

Now Theorem 3.2.d and (43) (but for G) imply

$$E_{y,\sigma_0,0,\rho} \cong \mathrm{ind}^{\mathbb{H}(G,L,\mathcal{L})}_{\mathbb{H}(Q,L,\mathcal{L})} E^Q_{y,\sigma_0,0,\rho}.$$

By Lemma 3.18 $E^Q_{y,\sigma_0,0,\rho^\circ \rtimes \tau^*} \cong \tau \ltimes E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$, whereas [AMS, (54)] shows that $M^Q_{y,\rho^\circ \rtimes \tau^*} \cong \tau \ltimes M^{Q^\circ}_{y,\rho^\circ}$ as $\mathbb{C}[W^Q_\mathcal{L}, \natural^Q_\mathcal{L}]$-modules. Decreeing that $S(\mathfrak{t}^*)$ acts trivially on V_τ, we obtain an isomorphism of $\mathbb{C}[W^Q_\mathcal{L}, \natural^Q_\mathcal{L}] \ltimes S(\mathfrak{t}^*)$-modules

$$M^Q_{y,\sigma_0,0,\rho^\circ \rtimes \tau^*} \cong \tau \ltimes M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}. \tag{65}$$

From Lemma 3.10 we know that $E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$ has a direct summand isomorphic to $M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}$. Hence there is a surjective $\mathbb{H}(G,L,\mathcal{L})$-module map

$$E_{y,\sigma_0,0,\rho} \cong \mathrm{ind}_{\mathbb{H}(Q,L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})}(\tau \ltimes E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}) \to \mathrm{ind}_{\mathbb{H}(Q,L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})}(\tau \ltimes M^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}) \cong M_{y,\sigma_0,0,\rho}. \tag{66}$$

The same argument as for part (a) shows that there exists a $h^\circ \in \mathbb{H}(G^\circ,L,\mathcal{L})$ which annihilates $\ker(E_{y,\sigma_0,0,\rho} \to M_{y,\sigma_0,0,\rho})$ and acts as the identity on $M_{y,\sigma_0,0,\rho}$. Therefore $M_{y,\sigma_0,0,\rho}$ appears with multiplicity one in $E_{y,\sigma_0,0,\rho}$. By the complete reducibility from Lemma 3.19, it appears as a direct summand.

Recall from from Lemma 3.18 and (44) that there are isomorphisms of $\mathbb{H}(G,L,\mathcal{L})$-modules

$$E_{y,\sigma_0,0,\rho} \cong \tau \ltimes E^\circ_{y,\sigma_0,0,\rho^\circ} \cong \tau \ltimes \mathrm{ind}_{\mathbb{H}(Q^\circ,L,\mathcal{L})}^{\mathbb{H}(G,L,\mathcal{L})} E^{Q^\circ}_{y,\sigma_0,0,\rho^\circ}.$$

From these and (66) we deduce

$$M_{y,\sigma,0,\rho^\circ \rtimes \tau^*} \cong \tau \ltimes M^\circ_{y,\sigma,0,\rho^\circ}. \tag{67}$$

Combining these with Lemma 3.10, we see that $M_{y,\sigma,0,\rho^\circ \rtimes \tau^*}$ is the component of $E_{y,\sigma_0,0,\rho}$ in one homological degree.

(c) For $r \neq 0$, part (a) and Lemma 3.18 induce an isomorphism of $\mathbb{H}(G,L,\mathcal{L})$-modules

$$M_{y,\sigma,r,\rho^\circ \rtimes \tau^*} \cong \tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ}. \tag{68}$$

From Lemma 3.16 we see that the irreducible modules (68) and (67) exhaust $\mathrm{Irr}(\mathbb{H}(G,L,\mathcal{L}))$. By [AMS, Theorem 1.2] and [Sol1, Theorem 1.2] two such representations are isomorphic if and only if there is a $\gamma \in \mathfrak{R}_\mathcal{L}$ such that

$$M^\circ_{y,\sigma,r,\rho^\circ} \cong \gamma \cdot M^\circ_{y',\sigma',r,\rho'^\circ} \quad \text{and} \quad \tau \cong \gamma \cdot \tau'. \tag{69}$$

By Theorem 3.11.c the first isomorphism means that (y,σ,ρ°) and (y',σ',ρ'°) are G-conjugate, while the second is equivalent to τ^* and τ'^* being associated under the action of G/G°. With Lemma 3.13 we see that (69) is equivalent to:

$$(y,\sigma,\rho = \rho^\circ \rtimes \tau^*) \quad \text{and} \quad (y',\sigma',\rho' = \rho'^\circ \rtimes \tau'^*) \quad \text{are } G\text{-conjugate.}$$

This yields the bijection between $\mathrm{Irr}(\mathbb{H}(G,L,\mathcal{L}))$ and the indicated set of parameters. It is canonical because $M_{y,\sigma,r,\rho}$ does not depend on any arbitrary choices, in particular the 2-cocycles from the previous paragraph do not appear in ρ.

(d) Apply Lemma 3.6.b to part (c). □

Recall that all the above was proven under Condition 3.1. Now we want to lift this condition, so we consider a group G which does not necessarily equal $G^\circ N_G(P,\mathcal{L})$. In (16) we saw that $\mathbb{H}(G,L,\mathcal{L})$ remains as in this section, but the parameters for irreducible representations could change when we replace $G^\circ N_G(P,\mathcal{L})$ by G.

Lemma 3.21. *The parametrizations of* $\mathrm{Irr}_r(\mathbb{H}(G,L,\mathcal{L}))$ *obtained in Theorem 3.20 remain valid without Condition 3.1.*

Proof. By the definition of $N_G(P,\mathcal{L})$, no element of $G \setminus G^\circ N_G(P,\mathcal{L})$ can stabilize the $G^\circ N_G(P,\mathcal{L})$-orbit of $(L,\mathcal{C}^L_v,\mathcal{L})$. So, when we replace $G^\circ N_G(P,\mathcal{L})$ by G, the orbit of the cuspidal support $(L,\mathcal{C}^L_v,\mathcal{L})$ becomes $[G : G^\circ N_G(P,\mathcal{L})]$ times larger. More precisely, $G \cdot (L,\mathcal{C}^L_v,\mathcal{L})$ can be written as a disjoint union of $[G : G^\circ N_G(P,\mathcal{L})]$ orbits for $G^\circ N_G(P,\mathcal{L})$ with representatives $(L,\mathcal{C}^L_v,\mathcal{L}')$, where $\mathcal{L}' = \mathrm{Ad}(g)^*\mathcal{L}$ for some $g \in N_G(P,L)$.

Let (y, σ_0, ρ) be as in Theorem 3.20, for the group $G^\circ N_G(P,\mathcal{L})$. By Theorem 3.20.d the stabilizer of $G^\circ N_G(P,\mathcal{L}) \cdot (y, \sigma_0, \rho)$ in G equals that of $G^\circ N_G(P,\mathcal{L}) \cdot (L, \mathcal{C}_v^L, \mathcal{L})$, so it is $G^\circ N_G(P,\mathcal{L})$. In particular the $Z_G(\sigma_0, y)$-stabilizer of ρ is precisely $Z_{G^\circ N_G(P,\mathcal{L})}(\sigma_0, y)$, which implies that

$$\rho^+ = \mathrm{ind}_{Z_{G^\circ N_G(P,\mathcal{L})}(\sigma_0,y)}^{Z_G(\sigma_0,y)}(\rho)$$

is an irreducible $\pi_0(Z_G(\sigma_0, y))$-representation. By [AMS, Theorem 4.8.a]

$$\Psi_{Z_G(\sigma_0)}(y, \rho^+) = (L, \mathcal{C}_v^L, \mathcal{L}) \quad \text{up to } G\text{-conjugation.}$$

From $G \cdot (\sigma_0, y, \rho^+)$ we can recover $G^\circ N_G(P,\mathcal{L}) \cdot (y, \sigma_0, \rho)$ as the unique $G^\circ N_G(P,\mathcal{L})$-orbit contained in it with cuspidal support $(L, \mathcal{C}_v^L, \mathcal{L})$ up to $G^\circ N_G(P,\mathcal{L})$-conjugation.

Consequently the canonical map $(y, \sigma_0, \rho) \mapsto (y, \sigma_0, \rho^+)$ provides a bijection between the triples in Theorem 3.20.d for $G^\circ N_G(P,\mathcal{L})$, and the same triples for G. With Lemma 3.6 (which is independent of Condition 3.1) we can replace (y, σ_0, ρ^+) by (y, σ, ρ^+), obtaining the same triples as in Theorem 3.20.c, but for G. $\square$

In Theorem 3.4 we showed that the assignment $(\sigma, y, r) \mapsto E_{y,\sigma,r}$ is compatible with parabolic induction. That cannot be true for the modules $E_{y,\sigma,r,\rho}$, if only because ρ is not a correct part of the data when G is replaced by a Levi subgroup. Nevertheless a weaker version of Theorem 3.4 holds for $E_{y,\sigma,r,\rho}$ and $M_{y,\sigma,r,\rho}$.

Let $Q \subset G$ be an algebraic subgroup such that $Q \cap G^\circ$ is a Levi subgroup of G° and $L \subset Q^\circ$. Let y, σ, r, ρ be as in Theorem 3.20, with $\sigma, y \in \mathfrak{q} = \mathrm{Lie}(Q)$. By [Ree, §3.2] the natural map

$$\pi_0(Z_Q(\sigma, y)) = \pi_0(Z_{Q \cap Z_G(\sigma_0)}(y)) \to \pi_0(Z_{Z_G(\sigma_0)}(y)) = \pi_0(Z_G(\sigma, y)) \tag{70}$$

is injective, so we can consider the left hand side as a subgroup of the right hand side. Let $\rho^Q \in \mathrm{Irr}\big(\pi_0(Z_Q(\sigma, y))\big)$ be such that $\Psi_{Z_Q(\sigma_0)}(y, \rho^Q) = (L, \mathcal{C}_v^L, \mathcal{L})$. Then $E_{y,\sigma,r,\rho}, M_{y,\sigma,r,\rho}, E^Q_{y,\sigma,r,\rho^Q}$ and M^Q_{y,σ,r,ρ^Q} are defined.

Proposition 3.22. (*a*) *There is a natural isomorphism of $\mathbb{H}(G, L, \mathcal{L})$-modules*

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r,\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes E_{y,\sigma,r,\rho},$$

where the sum runs over all $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$ with $\Psi_{Z_G(\sigma_0)}(y, \rho) = (L, \mathcal{C}_v^L, \mathcal{L})$.

(*b*) *For $r = 0$ part (a) contains an isomorphism of $S(\mathfrak{t}^*) \rtimes \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}]$-modules*

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,0,\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes M_{y,\sigma,0,\rho}.$$

(*c*) *The multiplicity of $M_{y,\sigma,r,\rho}$ in $\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r,\rho^Q}$ is $[\rho^Q : \rho]_{\pi_0(Z_Q(\sigma,y))}$. It already appears that many times as a quotient, via $E^Q_{y,\sigma,r,\rho^Q} \to M^Q_{y,\sigma,r,\rho^Q}$. More precisely, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{H}(Q,L,\mathcal{L})}(M^Q_{y,\sigma,r,\rho^Q}, M_{y,\sigma,r,\rho}) \cong \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho)^*.$$

Erratum. For this proposition to hold (with the same proof) we need an extra condition $\epsilon(\sigma, r) \neq 0$, see the appendix.

Remark. When we set $(\sigma, r) = (0,0)$, part (b) gives a natural isomorphism of $\mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}]$-modules

$$\mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}] \underset{\mathbb{C}[W_{\mathcal{L}}^Q, \natural_{\mathcal{L}}]}{\otimes} M^Q_{y,\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(y))}(\rho^Q, \rho) \otimes M_{y,\rho}.$$

Consequently $[M^Q_{y,\rho^Q} : M_{y,\rho}]_{\mathbb{C}[W_{\mathcal{L}}^Q, \natural_{\mathcal{L}}]} = [\rho^Q : \rho]_{\pi_0(Z_Q(y))}$. As the modules $M_{y,\rho}$ and M^Q_{y,ρ^Q} are obtained with the generalized Springer correpondence for disconnected groups from [AMS, Theorem 4.7], this solves the issue with the multiplicities mentioned in [AMS, Theorem 4.8.b].

Proof. (a) By Theorem 3.4.b

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r,\rho^Q} = \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, E_{y,\sigma,r}). \tag{71}$$

With Frobenius reciprocity we can rewrite this as

$$\mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}\big(\mathrm{ind}_{\pi_0(Z_Q(\sigma,y))}^{\pi_0(Z_G(\sigma,y))} \rho^Q, E_{y,\sigma,r}\big) = \big(\big(\mathrm{ind}_{\pi_0(Z_Q(\sigma,y))}^{\pi_0(Z_G(\sigma,y))} V_{\rho^Q}\big)^* \otimes E_{y,\sigma,r}\big)^{\pi_0(Z_G(\sigma,y))}.$$

Similarly $E_{y,\sigma,r,\rho} = (V_\rho^* \otimes E_{y,\sigma,r})^{\pi_0(Z_G(\sigma,y))}$. Again by Frobenius reciprocity

$$\begin{aligned} &\mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(V_\rho^*, \big(\mathrm{ind}_{\pi_0(Z_Q(\sigma,y))}^{\pi_0(Z_G(\sigma,y))} V_{\rho^Q}\big)^*\big) = \\ &\qquad \mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\mathrm{ind}_{\pi_0(Z_Q(\sigma,y))}^{\pi_0(Z_G(\sigma,y))} \rho^Q, \rho\big) = \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho). \end{aligned} \tag{72}$$

(b) Now we assume that $r = 0$. From Theorem 3.20.c we know that $M_{y,\sigma,0,\rho}$ is the component of $E_{y,\sigma,0,\rho}$ in one homological degree. By Lemma 3.10 this degree, say n^G, does not depend on ρ. Similarly

$$M^Q_{y,\sigma,0,\rho^Q} = \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}\big(\rho^Q, H_{n^Q}(\mathcal{P}_y^Q, \dot{\mathcal{L}})\big).$$

The isomorphism in part (a) comes eventually from Theorem 3.4.b, so by (24) it changes all homological degrees by a fixed amount $d = \dim \mathcal{P}_y - \dim P_y^Q$. Thus part (a) restricts to

$$\begin{aligned} \mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho^Q} &= \mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}\big(\rho^Q, H_{n^Q}(\mathcal{P}_y^Q, \dot{\mathcal{L}})\big) \\ &\cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes \mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\rho, H_{n^Q+d}(\mathcal{P}_y, \dot{\mathcal{L}})\big) \end{aligned} \tag{73}$$

We want to show that $n^Q + d = n^G$, for then (73) becomes the desired isomorphism. This is easily seen from the explicit formula given in Lemma 3.10, but we prefer an argument that does not use [Lus2]. Since n^G does not depend on ρ, it suffices to consider one ρ. By [AMS, Theorem 4.8.a] we can pick ρ such that $\mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \neq 0$, while maintaining the condition on the cuspidal support.

By (65), (66) and (38) the $(\sigma, 0)$-weight space of $M_{y,\sigma,0,\rho^\circ \rtimes \tau^*}$ is

$$\tau \ltimes M_{y,\rho^\circ} \in \mathrm{Irr}(\mathbb{C}[W_{\mathcal{L},\sigma}, \natural_{\mathcal{L}}]).$$

For the same reasons the $(\sigma, 0)$-weight space of (73) is

$$\mathrm{ind}_{\mathbb{C}[W^Q_{\mathcal{L},\sigma}, \natural_{\mathcal{L}}]}^{\mathbb{C}[W_{\mathcal{L},\sigma}, \natural_{\mathcal{L}}]} (\tau^Q \ltimes M_{y,\rho^{Q\circ}}) \in \mathrm{Mod}(\mathbb{C}[W_{\mathcal{L},\sigma}, \natural_{\mathcal{L}}]).$$

Here $\tau \ltimes M_{y,\rho^\circ}$ is the representation attached to $(y, \rho = \rho^\circ \rtimes \tau^*)$ by the generalized Springer correspondence for $Z_G(\sigma)$ from [AMS, §4]. In the same way, only for $Z_Q(\sigma)$, $\tau^Q \ltimes M_{y,\rho^{Q^\circ}}$ is related to $(y, \rho^Q = \rho^{Q^\circ} \rtimes \tau^Q)$.

As ρ^Q appears in ρ, [AMS, Proposition 4.8.b] guarantees that $\tau^Q \ltimes M_{y,\rho^{Q^\circ}}$ appears in $\tau \ltimes M_{y,\rho^\circ}$. Hence the $\mathbb{C}[W_{\mathcal{L},\sigma}, \natural_{\mathcal{L}}]$-module $\tau \ltimes M_{y,\rho^\circ}$ appears in (73). In view of the irreducibility of $M_{y,\sigma,0,\rho}$, this implies that $M_{y,\sigma,0,\rho}$ is a quotient of

$$\mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\rho, H_{n^Q+d}(\mathcal{P}_y, \dot{\mathcal{L}})\big) \subset E_{y,\sigma,0,\rho}.$$

By Theorem 3.11.d and (67), this is only possible if $n^Q + d = n^G$.
(c) From Theorem 3.20 we know that $M_{y,\sigma,r,\rho}$ appears with multiplicity one in $E_{y,\sigma,r,\rho}$. It follows from [Lus5, Corollary 10.7 and Proposition 10.12] that all other irreducible constituents of the standard module $E_{y,\sigma,r,\rho}$ are of the form $M_{y',\sigma,r,\rho'}$, where $\mathcal{C}^G_{y'}$ is a nilpotent orbit of larger dimension then $\mathcal{C}^G_y$. Together with part (a) this shows the indicated multiplicity is

$$\dim \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) = [\rho^Q : \rho]_{\pi_0(Z_Q(\sigma,y))}.$$

Now we assume that $r \neq 0$, so that $M_{y,\sigma,r,\rho}$ is the unique irreducible quotient of $E_{y,\sigma,r,\rho}$.

For every ρ as with $\Psi_{Z_G(\sigma_0)}(y, \rho) = (L, \mathcal{C}^L_v, \mathcal{L})$ we choose an element $f_\rho \in E_{y,\sigma,r,\rho}$ with $f_\rho \neq 0$ in $M_{y,\sigma,r,\rho}$, and we choose a basis $\{b_{\rho,i}\}_i$ of $\mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho)$. The set $F := \{b_{\rho,i} \otimes f_\rho\}_{\rho,i}$ generates the right hand side of part (a) as a $\mathbb{H}(G, L, \mathcal{L})$-module, and no proper subset of it has the same property. Via the canonical isomorphism of part (a) we consider F as a subset of the left hand side. Suppose that one element $b_{\rho,i} \otimes f_\rho$ belongs to

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} \ker\big(E^Q_{y,\sigma,r,\rho^Q} \to M^Q_{y,\sigma,r,\rho^Q}\big). \tag{74}$$

The remaining elements of F generate $\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho^Q}$. Since M^Q_{y,σ,r,ρ^Q} is the unique irreducible quotient of E^Q_{y,σ,r,ρ^Q}, they also generate the modules in part (a). This contradiction shows that all elements of F are nonzero in $\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho^Q}$, and that (74) is contained in

$$\bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes \ker\big(E_{y,\sigma,r,\rho} \to M_{y,\sigma,r,\rho}\big).$$

Consequently the canonical surjection

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r,\rho^Q} \to \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes M_{y,\sigma,r,\rho}$$

factors through $\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho^Q}$. We deduce natural isomorphisms

$$\mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho)^* \cong \mathrm{Hom}_{\mathbb{H}(G,L,\mathcal{L})}\big(\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r,\rho^Q}, M_{y\sigma,r,\rho}\big) \cong$$

$$\mathrm{Hom}_{\mathbb{H}(G,L,\mathcal{L})}\big(\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho^Q}, M_{y\sigma,r,\rho}\big) \cong \mathrm{Hom}_{\mathbb{H}(Q,L,\mathcal{L})}(M^Q_{y,\sigma,r,\rho^Q}, M_{y,\sigma,r,\rho}).$$

For $r = 0$ we can apply the functor $\mathrm{Hom}_{\mathbb{H}(G,L,\mathcal{L})}(?, M_{y,\sigma,0,\rho})$ to part (b). A computation analogous to the above yields the desired result. □

Depending on the circumstances, it might be useful to present the parameters from Theorem 3.20 and Lemma 3.21 in another way. If one is primarily interested in the algebra $\mathbb{H}(G, L, \mathcal{L}) = \mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural_{\mathcal{L}})$, then it is natural to involve the Lie algebra $\mathfrak{t}$. On the other hand, for studying the parameter space some simplification can be achieved by combining y and σ in a single element of $\mathfrak{g}$. Of course that is done with the Jordan decomposition $x = x_S + x_N$, where x_S (respectively x_N) denotes the semisimple (respectively nilpotent) part of $x \in \mathfrak{g}$ [Spr, Theorem 4.4.20].

Corollary 3.23. *In the setting of Lemma 3.21, there exists a canonical bijection between the following sets:*

- $\mathrm{Irr}_r(\mathbb{H}(G, L, \mathcal{L}))$*;*
- $N_G(L)/L$*-orbits of triples* $(\sigma_0, \mathcal{C}, \mathcal{F})$ *where* $\sigma_0 \in \mathfrak{t}$*,* $\mathcal{C}$ *is a nilpotent* $Z_G(\sigma_0)$*-orbit in* $Z_{\mathfrak{g}}(\sigma_0)$ *and* $\mathcal{F}$ *is an irreducible* $Z_G(\sigma_0)$*-equivariant local system on* $\mathcal{C}$ *such that* $\Psi_{Z_G(\sigma_0)}(\mathcal{C}, \mathcal{F}) = (L, \mathcal{C}_v^L, \mathcal{L})$ *(up to* $Z_G(\sigma_0)$*-conjugacy);*
- G*-orbits of pairs* (x, ρ) *with* $x \in \mathfrak{g}$ *and* $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(x))\big)$ *such that* $\Psi_{Z_G(x_S)}(x_N, \rho) = (L, \mathcal{C}_v^L, \mathcal{L})$ *(up to* G*-conjugacy).*

Proof. By Proposition 3.5.c we may assume that σ and σ_0 lie in $\mathfrak{t}$. Upon requiring that, the G-orbit of σ (or σ_0) reduces to a $N_G(L)/L$-orbit in $\mathfrak{t}$. The nilpotent element y lies in $Z_{\mathfrak{g}}(\sigma_0)$, and only its $Z_G(\sigma_0)$-orbit matters. The data of $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma_0, y))\big)$ are equivalent to that of an irreducible $Z_G(\sigma_0)$-equivariant local system $\mathcal{F}$ on $\mathcal{C}_y^{Z_G(\sigma_0)}$. Now Theorem 3.20.d provides a canonical bijection between the first two sets.

We put $x = \sigma_0 + y \in \mathfrak{g}$. By the Jordan decomposition every element of $\mathfrak{g}$ is of this form, and $Z_G(x) = Z_G(y, \sigma_0)$. Again Theorem 3.20.d yields the desired bijection, between the first and third sets. $\square$

We note that the third set of Corollary 3.23 is included in the set of all G-orbits of pairs (x, ρ) with $x \in \mathfrak{g}$ and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(x))\big)$. It follows that the latter set is canonically in bijection with

$$\bigsqcup\nolimits_{(L, \mathcal{C}_v^L, \mathcal{L})} \mathrm{Irr}_r(\mathbb{H}(G, L, \mathcal{L})), \tag{75}$$

where the disjoint union runs over all cuspidal supports for G (up to G-conjugacy).

3.5. **Tempered representations and the discrete series.**

In this paragraph we study two analytic properties of $\mathbb{H}(G, L, \mathcal{L})$-modules, temperedness and discrete series. Of course these are well-known for representations of reductive groups over local fields, and the definition in our context is designed to mimick those notions.

The complex vector space $\mathfrak{t} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ has a canonical real form $\mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The decomposition of an element $x \in \mathfrak{t}$ along $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \oplus i\mathfrak{t}_{\mathbb{R}}$ will be written as $x = \Re(x) + i\Im(x)$. We define the positive cones

$$\begin{aligned} \mathfrak{t}_{\mathbb{R}}^{+} &:= \{x \in \mathfrak{t}_{\mathbb{R}} : \langle x \,,\, \alpha \rangle \geq 0 \;\forall \alpha \in R(P, T)\}, \\ \mathfrak{t}_{\mathbb{R}}^{*+} &:= \{\lambda \in \mathfrak{t}_{\mathbb{R}}^{*} : \langle \alpha^{\vee} \,,\, \lambda \rangle \geq 0 \;\forall \alpha \in R(P, T)\}. \end{aligned}$$

The antidual of $\mathfrak{t}_{\mathbb{R}}^{*+}$ is the obtuse negative cone

$$\mathfrak{t}_{\mathbb{R}}^{-} := \{x \in \mathfrak{t}_{\mathbb{R}} : \langle x \,,\, \lambda \rangle \leq 0 \;\forall \lambda \in \mathfrak{t}_{\mathbb{R}}^{*+}\}.$$

It can also be described as

$$\mathfrak{t}_{\mathbb{R}}^- = \big\{ \sum\nolimits_{\alpha \in R(P,T)} x_\alpha \alpha^\vee : x_\alpha \leq 0 \big\}.$$

The interior $\mathfrak{t}_{\mathbb{R}}^{--}$ of $\mathfrak{t}_{\mathbb{R}}^-$ is given by

$$\mathfrak{t}_{\mathbb{R}}^{--} = \left\{ \begin{array}{ll} \{\sum_{\alpha \in R(P,T)} x_\alpha \alpha^\vee : x_\alpha < 0\} & \text{if } R(G,T)^\vee \text{ spans } \mathfrak{t}_{\mathbb{R}} \\ \emptyset & \text{otherwise.} \end{array} \right. \tag{76}$$

Let (π, V) be a $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural)$-module. We call $x \in \mathfrak{t}$ a weight of V if there is a $v \in V \setminus \{0\}$ such that $\pi(\xi)v = \xi(x)v$ for all $\xi \in S(\mathfrak{t}^*)$. This is equivalent to requiring that the generalized weight space

$$V_x := \{v \in V : (\pi(\xi) - \xi(x))^n v = 0 \text{ for some } n \in \mathbb{N}\} \tag{77}$$

is nonzero. Since $S(\mathfrak{t}^*)$ is commutative, V is the direct sum of its generalized weight spaces whenever it has finite dimension. We denote the set of weights of (π, V) by $\mathrm{Wt}(\pi, V)$, $\mathrm{Wt}(V)$ or $\mathrm{Wt}(\pi)$.

Definition 3.24. Let (π, V) be a finite dimensional $\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural)$-module. We call it tempered if $\Re(\mathrm{Wt}(\pi, V)) \subset \mathfrak{t}_{\mathbb{R}}^-$. We call it discrete series if $\Re(\mathrm{Wt}(\pi, V)) \subset \mathfrak{t}_{\mathbb{R}}^{--}$.

Similarly we say that (π, V) is anti-tempered (respectively anti-discrete series) if $\Re(\mathrm{Wt}(\pi, V)) \subset -\mathfrak{t}_{\mathbb{R}}^-$ (respectively $\subset -\mathfrak{t}_{\mathbb{R}}^{--}$).

We denote the set of irreducible tempered representations of this algebra by $\mathrm{Irr}_{\mathrm{temp}}(\mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}, c\mathbf{r}, \natural))$.

Theorem 3.25. *Let y, σ, ρ be as in Corollary 3.23, with $\sigma, \sigma_0 \in \mathfrak{t}$.*

(*a*) *Suppose that $\Re(r) \leq 0$. The following are equivalent:*
  - *(a) $E_{y,\sigma,r,\rho}$ is tempered;*
  - *(b) $M_{y,\sigma,r,\rho}$ is tempered;*
  - *(c) $\sigma_0 \in i\mathfrak{t}_{\mathbb{R}}$.*

(*b*) *Suppose that $\Re(r) \geq 0$. Then part (a) remains valid if we replace tempered by anti-tempered.*

Proof. (a) Choose τ and $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$ as before, so that $\rho = \rho^\circ \rtimes \tau^*$. By Clifford theory and Lemmas 3.16 and 3.18

$$\mathrm{Wt}(M_{y,\sigma,r,\rho}) = \mathfrak{R}_{\mathcal{L}} \mathrm{Wt}(M^\circ_{y,\sigma,r,\rho^\circ}), \tag{78}$$

and similarly for $E^\circ_{y,\sigma,r,\rho^\circ}$. Since $\mathfrak{R}_{\mathcal{L}}$ stabilizes $\mathfrak{t}_{\mathbb{R}}^-$, it follows that $E_{y,\sigma,r,\rho}$ (respectively $M_{y,\sigma,r,\rho}$) is tempered if and only if $E^\circ_{y,\sigma,r,\rho^\circ}$ (respectively $M^\circ_{y,\sigma,r,\rho^\circ}$ is tempered. This reduces the claim to the case where G is connected.

From now on we assume that $\mathfrak{R}_{\mathcal{L}} = 1$. From Proposition 2.2 we see that

$$\mathbb{H}(G^\circ, L, \mathcal{L}) = \mathbb{H}(G^\circ_{\mathrm{der}}, L \cap G_{\mathrm{der}}, \mathcal{L}) \otimes S(Z(\mathfrak{g})^*).$$

Write $\sigma_0 = \sigma_{0,\mathrm{der}} + z_0$ with $\sigma_{0,\mathrm{der}} \in \mathrm{Lie}(G_{\mathrm{der}})$ and $z_0 \in Z(\mathfrak{g})$. By Proposition 3.5.b both $M^\circ_{y,\sigma,r,\rho^\circ}$ and $E^\circ_{y,\sigma,r,\rho^\circ}$ admit the $S(Z(\mathfrak{g})^*)$-character z_0. By definition $\mathfrak{t}_{\mathbb{R}}^- \cap Z(\mathfrak{g}) = \{0\}$. Thus $M^\circ_{y,\sigma,r,\rho^\circ}$ and $E^\circ_{y,\sigma,r,\rho^\circ}$ are tempered as $Z(\mathfrak{Z}(g)^*)$-modules if and only if $\Re(z_0) = 0$, or equivalently $z_0 \in X_*(Z(G^\circ)^\circ) \otimes_{\mathbb{Z}} i\mathbb{R}$. This achieves further reduction, to the case where $G = G^\circ$ is semisimple.

When $\Re(r) < 0$, we will apply [Lus7, Theorem 1.21]. It says that the following are equivalent:

(i)' $E^\circ_{y,\sigma,r,\rho^\circ}$ is τ-tempered (where τ refers to the homomorphism $\Re : \mathbb{C} \to \mathbb{R}$);

(ii)' $M^\circ_{y,\sigma,r,\rho^\circ}$ is τ-tempered;

(iii)' All the eigenvalues of $\mathrm{ad}(\sigma_0) : \mathfrak{g} \to \mathfrak{g}$ are purely imaginary.

As $\mathfrak{t}$-module, $\mathfrak{g}$ is the direct sum of the weight spaces $\mathfrak{g}_\alpha$ with $\alpha \in R(G,T) \cup \{0\}$. We note that $R(G,T) \cup \{0\} \subset X^*(T) \subset \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ and $R(G,T)$ spans $\mathfrak{t}_\mathbb{R}^*$ (for G is semisimple). Hence (iii)' is equivalent to (iii).

As $\Re(r) < 0$, the condition for τ-temperedness of a module E [Lus7, 1.20] becomes

$$\Re(\lambda) \leq 0 \text{ for any eigenvalue of } \xi_V \text{ on } E. \tag{79}$$

Here $\xi_V \in \mathfrak{t}^*$ is determined by an irreducible finite dimensional $\mathfrak{g}$-module V which contains a unique line $\mathbb{C}v$ annihilated by $\mathfrak{u}$. Then ξ_V is the character by which $\mathfrak{t}$ acts on $\mathbb{C}v$. When we vary V, ξ_V runs through a set of dominant weights which spans $\mathfrak{t}_\mathbb{R}^{*+}$ over $\mathbb{R}_{\geq 0}$. Hence the condition (79) is equivalent to $\Re(\mathrm{Wt}(E)) \subset \mathfrak{t}_\mathbb{R}^-$. In other words, temperedness is the same as τ-temperedness when $\Re(r) < 0$, (i) is the same as (i)' and (ii) is equivalent to (ii)'. Thus [Lus7, Theorem 1.21] is our required result in this setting.

It remains to settle the case $\Re(r) = 0, G = G^\circ$ semisimple. Assume (iii). When we vary r and keep σ_0 fixed, the weights of $(E^\circ_{y,\sigma,r,\rho^\circ})$ depend algebraically on r. We have already shown that $\Re(\mathrm{Wt}(E^\circ_{y,\sigma,r,\rho^\circ})) \subset \mathfrak{t}_\mathbb{R}^-$ when $\Re(r) < 0$. Clearly $\mathfrak{t}_\mathbb{R}^-$ is closed in $\mathfrak{t}_\mathbb{R}$, so by continuity this property remains valid when $\Re(r) = 0$. That proves (i), upon which (ii) follows immediately.

Conversely, suppose that (iii) does not hold. We may assume that $\gamma_y : \mathrm{SL}_2(\mathbb{C}) \to G$ has image in $Z_G(\sigma_0)$. Recall from Proposition 3.5.b that

$$\mathrm{Wt}(M^\circ_{y,\sigma,r,\rho^\circ}) \subset W_\mathcal{L}\big(\sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\big).$$

In particular we can find a $w \in W_\mathcal{L}$ such that $w\big(\sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\big)$ is a $S(\mathfrak{t}^*)$-weight of $M^\circ_{y,\sigma,r,\rho^\circ}$. The map $z \mapsto \gamma_y \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$ is a cocharacter of T and $r \in i\mathbb{R}$, so $\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) \in i\mathfrak{t}_\mathbb{R}$. By our assumption $\sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) \in \mathfrak{t} \setminus i\mathfrak{t}_\mathbb{R}$, and this does not change upon applying $w \in W_\mathcal{L}$. Hence $M^\circ_{y,\sigma,r,\rho^\circ}$ is not tempered. We proved that (ii) implies (iii) when $\Re(r) = 0$.

(b) This is completely analogous to part (a), when we interpret τ-tempered with $\tau = -\Re : \mathbb{C} \to \mathbb{R}$. □

From Definition 3.24 and (76) we immediately see that $\mathbb{H}(G, L, \mathcal{L})$ has no discrete series representations if $R(G,T)$ does not span $\mathfrak{t}_\mathbb{R}^*$. That is equivalent to $Z(\mathfrak{g}) \neq 0$. Therefore we only formulate a criterium for discrete series when G° is semisimple.

Theorem 3.26. *Let G° be semisimple. Let y, σ, ρ be as in Corollary 3.23, with $\sigma, \sigma_0 \in \mathfrak{t}$.*

(*a*) *Suppose that $\Re(r) < 0$. The following are equivalent:*
- *(a) $M_{y,\sigma,r,\rho}$ is discrete series;*
- *(b) y is distinguished in $\mathfrak{g}$, that is, it is not contained in any proper Levi subalgebra of $\mathfrak{g}$.*

Moreover, if these conditions are fulfilled, then $\sigma_0 = 0$ and $E_{y,\sigma,r,\rho} = M_{y,\sigma,r,\rho}$.

(*b*) *Suppose that $\Re(r) > 0$. Then part (a) remains valid if we replace (i) by: $M_{y,\sigma,r,\rho}$ is anti-discrete series.*

(*c*) *For $\Re(r) = 0$ there are no (anti-)discrete series representations on which* $\mathbf{r}$ *acts as r.*

Proof. (a) Since $[\sigma_0, y] = 0$ and $\mathfrak{g}$ is semisimple, $\sigma_0 = 0$ whenever y is distinguished.

In view of (78) it suffices to prove the equivalence of (i) and (ii) when G is connected, so we assume that for the moment. We can reformulate (ii) as:

$$\langle x\,,\,\alpha\rangle < 0 \text{ for all } x \in \mathrm{Wt}(M^\circ_{y,\sigma,r,\rho^\circ}) \text{ and all } \alpha \in R(P,T).$$

The same argument as for temperedness shows that this is equivalent to $M^\circ_{y,\sigma,r,\rho^\circ}$ being τ-square integrable with $\tau = \Re$, in the sense of [Lus7]. By [Lus7, Theorem 1.22] that in turn is equivalent to (i). The same result also shows that $E^\circ_{y,\sigma,r,\rho^\circ} = M^\circ_{y,\sigma,r,\rho^\circ}$ when (i) and (ii) hold.

The last statement can be lifted from G° to G by (64) and Lemma 3.18:

$$E_{y,\sigma,r,\rho} \cong \tau \ltimes E_{y,\sigma,r,\rho^\circ} = \tau \ltimes M^\circ_{y,\sigma,r,\rho^\circ} \cong M_{y,\sigma,r,\rho}.$$

(b) This can be shown in the same way as part (a), when we consider τ-square integrable with $\tau = -\Re : \mathbb{C} \to \mathbb{R}$.
(c) Suppose that V is a discrete series $\mathbb{H}(G,L,\mathcal{L})$-module on which $\mathbf{r}$ acts as $r \in i\mathbb{R}$. By definition $\dim V < \infty$, so V has an irreducible subrepresentation, say $M_{y,\sigma,r,\rho}$. Its weights are a subset of those of V, so it is also discrete series. By (78) this means that $M^\circ_{y,\sigma,r,\rho^\circ} \in \mathrm{Irr}(\mathbb{H}(G^\circ,L,\mathcal{L}))$ is discrete series.

In particular it is tempered, so by Theorem 3.25 $\sigma_0 \in i\mathfrak{t}_\mathbb{R}$. As $\gamma_y : \mathrm{SL}_2(\mathbb{C}) \to G^\circ$ is algebraic and $r \in i\mathbb{R}$, $\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) \in i\mathfrak{t}_\mathbb{R}$ as well. From Proposition 3.5.b we know that

$$\mathrm{Wt}(M^\circ_{y,\sigma,r,\rho^\circ}) \subset W^\circ_\mathcal{L}\sigma = W^\circ_\mathcal{L}\big(\sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\big) \subset i\mathfrak{t}_\mathbb{R}.$$

Consequently $\Re(x) = 0 \notin \mathfrak{t}_\mathbb{R}^{--}$ for every $x \in \mathrm{Wt}(M^\circ_{y,\sigma,r,\rho^\circ})$. This contradicts the definition of discrete series. $\square$

When $R(G,T)$ does not span $\mathfrak{t}^*_\mathbb{R}$, it is sometimes useful to relax the notion of the discrete series in the following way.

Definition 3.27. Let (π,V) be a finite dimensional $\mathbb{H}(\mathfrak{t},W_\mathcal{L},c\mathbf{r},\natural)$-module, and let $\mathfrak{t}' \subset \mathfrak{t}$ be the $\mathbb{C}$-span of the coroots for $W^\circ_\mathcal{L}$. We say that (π,V) is essentially (anti-) discrete series if its restriction to $\mathbb{H}(\mathfrak{t}',W_\mathcal{L},c\mathbf{r})$ is (anti-) discrete series.

Corollary 3.28. *Let $r \in \mathbb{C}$ with $\Re(r) < 0$, and let y,σ,ρ be as in Corollary 3.23, with $\sigma,\sigma_0 \in \mathfrak{t}$. Then $M_{y,\sigma,r,\rho}$ is essentially discrete series if and only if y is distinguished in $\mathfrak{g}$.*

When $\Re(r) > 0$, the same holds with essentially anti-discrete series.

Proof. Fix $r \in \mathbb{C}$ with $\Re(r) < 0$. Notice that for algebra under consideration $\mathfrak{t}'$ as in Definition (3.27) equals $\mathfrak{t} \cap \mathfrak{g}_\mathrm{der}$. Namely, the roots of the semisimple Lie algebra $\mathfrak{g}_\mathrm{der}$ span the linear dual of any Cartan subalgebra, and hence also of $\mathfrak{t}$.

Recall from (6) that

$$\mathbb{H}(G^\circ,L,\mathcal{L}) = \mathbb{H}(\mathfrak{t} \cap \mathfrak{g}_\mathrm{der}, W^\circ_\mathcal{L}, c\mathbf{r}) \otimes S(Z(\mathfrak{g})^*).$$

The restriction of $M_{y,\sigma,r,\rho} = M_{y,\sigma,r,\rho^\circ \rtimes \tau^*}$ to $\mathbb{H}(G^\circ,L,\mathcal{L})$ is

$$V_\tau \otimes M_{y,\sigma - z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0}, \text{ where } \sigma = (\sigma - z_0) + z_0 \in (\mathfrak{t} \cap \mathfrak{g}_\mathrm{der}) \oplus Z(\mathfrak{g}).$$

The action on V_τ is trivial and there is no condition on the character z_0 by which $S(Z(\mathfrak{g})^*)$ acts. Hence $M_{y,\sigma,r,\rho}$ is essentially discrete series if and only if

$$M_{y,\sigma - z_0,r,\rho^\circ} \in \mathrm{Irr}(\mathbb{H}(\mathfrak{t} \cap \mathfrak{g}_\mathrm{der}, W^\circ_\mathcal{L}, c\mathbf{r}))$$

is discrete series. By Theorem 3.26 that is equivalent to y being distinguished in $\mathfrak{g}_\mathrm{der}$. Since $\mathfrak{g} = \mathfrak{g}_\mathrm{der} \oplus Z(\mathfrak{g})$, that is the case if and only if y is distinguished in $\mathfrak{g}$.

The case $\Re(r) > 0$ can be shown in the same way. □

Unfortunately Theorems 3.25, 3.26 and Corollary 3.28 do not work as we would like them when $\Re(r) > 0$, the prefix "anti" should rather not be there. In the Langlands program r will typically be $\log(q)$, where q is cardinality of a finite field, so $r \in \mathbb{R}_{>0}$ is the default. This problem comes from [Lus7] and can be traced back to Lusztig's conventions for the generalized Springer correspondence in [Lus1], see also Remark 2.5.

To make the properties of $\mathbb{H}(G, L, \mathcal{L})$-modules fit with those of Langlands parameters, we need a small adjustment. Extend the sign representation of the Weyl group $W_{\mathcal{L}}^\circ$ to a character of $W_{\mathcal{L}} = W_{\mathcal{L}}^\circ \rtimes \mathfrak{R}_{\mathcal{L}}$ by means of the trivial representation of $\mathfrak{R}_{\mathcal{L}}$. Then $N_w \mapsto \mathrm{sign}(w) N_w$ extends linearly to an involution of $\mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}]$.

The Iwahori–Matsumoto involution of $\mathbb{H}(G, L, \mathcal{L})$ is defined as the unique algebra automorphism such that

$$\mathrm{IM}(N_w) = \mathrm{sign}(w) N_w, \qquad \mathrm{IM}(\mathbf{r}) = \mathbf{r}, \qquad \mathrm{IM}(\xi) = -\xi \quad (\xi \in \mathfrak{t}^*). \tag{80}$$

Notice that IM preserves the braid relation

$$N_{s_i} \xi - {}^{s_i}\xi N_{s_i} = c_i \mathbf{r} (\xi - {}^{s_i}\xi)/\alpha_i,$$

for α_i is also multiplied by -1. We also note that the Iwahori–Matsumoto involutions for various graded Hecke algebras are compatible with parabolic induction. Suppose that $Q \subset G$ is as in Proposition 3.22 and let V be any $\mathbb{H}(Q, L, \mathcal{L})$-module. There is a canonical isomorphism of $\mathbb{H}(G, L, \mathcal{L})$-modules

$$\begin{array}{ccc} \mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} \mathrm{IM}^*(V) & \to & \mathrm{IM}^*\big(\mathbb{H}(G,L,\mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} V\big) \\ h \otimes v & \mapsto & \mathrm{IM}(h) \otimes v \end{array} . \tag{81}$$

This allows us to identify the two modules, and then Proposition 3.22 remains valid upon composition with IM.

Clearly IM has the effect $x \leftrightarrow -x$ on $S(\mathfrak{t}^*)$-weights of $\mathbb{H}(G, L, \mathcal{L})$-representations. Hence IM exchanges tempered with anti-tempered representations, and discrete series with anti-discrete series representations. For $\Re(r) \geq 0$ Theorem 3.25 yields equivalences

$$\mathrm{IM}^* E_{y,\sigma,r,\rho} \text{ is tempered} \iff \mathrm{IM}^* M_{y,\sigma,r,\rho} \text{ is tempered} \iff \sigma_0 \in i\mathfrak{t}_{\mathbb{R}}. \tag{82}$$

For $\Re(r) > 0$ Corollary 3.28 says that

$$\mathrm{IM}^* M_{y,\sigma,r,\rho} \text{ is essentially discrete series} \iff y \text{ is distinguished in } \mathfrak{g}. \tag{83}$$

We note that IM^* changes central characters of these representations: by Proposition 3.5.b both $\mathrm{IM}^* E_{y,\sigma,r,\rho}$ and $\mathrm{IM}^* M_{y,\sigma,r,\rho}$

$$\text{admit the central character } (-W_{\mathcal{L}}\sigma, r) \in \mathfrak{t}/W_{\mathcal{L}} \times \mathbb{C}. \tag{84}$$

Composition with the Iwahori–Matsumoto involution corresponds to two changes in the previous setup:

- In (7) the action of $\mathbb{C}[W_{\mathcal{L}}, \natural_{\mathcal{L}}^{-1}]$ on K^* is twisted by the sign character of $W_{\mathcal{L}}$, that is, we use a normalization different from that of Lusztig in [Lus1].
- The action (21) of $\mathfrak{t}^* \subset H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}})$ on standard modules is adjusted by a factor -1.

To $r \in \mathbb{C}$ and a triple (y, σ_0, ρ) as in Theorem 3.20.c we will associate the irreducible representation $\mathrm{IM}^* M_{y, \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \sigma_0, r, \rho}$. This parametrization of $\mathrm{Irr}_r(\mathbb{H}(G, L, \mathcal{L}))$ is in some respects more suitable than that in Theorem 3.20, for example to study tempered representations.

We use it here to highlight the relation with extended quotients. Recall that $W_\mathcal{L}$ acts linearly on $\mathfrak{t}$ and that $\mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}] \subset \mathbb{H}(G, L, \mathcal{L})$. We write

$$\tilde{\mathfrak{t}}_{\natural_\mathcal{L}} = \{(x, \pi_x) : x \in \mathfrak{t}, \pi_x \in \mathrm{Irr}(\mathbb{C}[(W_\mathcal{L})_x, \natural_\mathcal{L}])\}.$$

The group $W_\mathcal{L}$ acts on $\tilde{\mathfrak{t}}_{\natural_\mathcal{L}}$ by

$$w \cdot (x, \pi_x) = (wx, w^* \pi_x) \text{ where } (w^* \pi_x)(N_v) = \pi_x(N_w^{-1} N_v N_w) \text{ for } v \in (W_\mathcal{L})_{wx}.$$

The twisted extended quotient of $\mathfrak{t}$ by $W_\mathcal{L}$ (with respect to $\natural_\mathcal{L}$) is defined as

$$(\mathfrak{t}/\!/W_\mathcal{L})_{\natural_\mathcal{L}} = \tilde{\mathfrak{t}}_{\natural_\mathcal{L}} / W_\mathcal{L}. \tag{85}$$

Theorem 3.29. *Let $r \in \mathbb{C}$. There exists a canonical bijection*

$$\mu_{G, L \mathcal{L}} : (\mathfrak{t}/\!/W_\mathcal{L})_{\natural_\mathcal{L}} \to \mathrm{Irr}_r(\mathbb{H}(G, L, \mathcal{L}))$$

such that:

- $\mu_{G,L,\mathcal{L}}(i\mathfrak{t}_\mathbb{R}/\!/W_\mathcal{L})_{\natural_\mathcal{L}} = \mathrm{Irr}_{r,\mathrm{temp}}(\mathbb{H}(G, L, \mathcal{L}))$ *when* $\Re(r) \geq 0$.
  For $\Re(r) \leq 0$ *it is the anti-tempered part of* $\mathrm{Irr}_r(\mathbb{H}(G, L, \mathcal{L}))$.
- *The central character of* $\mu_{G,L,\mathcal{L}}(x, \pi_x)$ *is* $\big(W_\mathcal{L}\big(x + \mathrm{d}\gamma \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\big), r\big)$ *for some algebraic homomorphism* $\gamma : \mathrm{SL}_2(\mathbb{C}) \to Z_G(x)^\circ$.

Remark. This establishes a version of the ABPS-conjectures [ABPS1, §15] for the twisted graded Hecke algebra $\mathbb{H}(G, L, \mathcal{L})$.

Proof. By [ABPS3, Lemma 2.3] there exists a canonical bijection

$$\begin{array}{ccc} (\mathfrak{t}/\!/W_\mathcal{L})_{\natural_\mathcal{L}} & \to & \mathrm{Irr}(S(\mathfrak{t}^*) \rtimes \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}]) \\ (x, \pi_x) & \mapsto & \mathbb{C}_x \rtimes \pi_x = \mathrm{ind}_{S(\mathfrak{t}^*) \rtimes \mathbb{C}[(W_\mathcal{L})_x, \natural_\mathcal{L}]}^{S(\mathfrak{t}^*) \rtimes \mathbb{C}[W_\mathcal{L}, \natural_\mathcal{L}]} (\mathbb{C}_x \otimes V_{\pi_x}) \end{array}.$$

We can consider $\mathbb{C}_x \rtimes \pi_x$ as an irreducible $\mathbb{H}(G, L, \mathcal{L})$-representation with central character $(W_\mathcal{L} x, 0)$. By Lemma 3.21 there are y, ρ, unique up to $Z_G(x)$-conjugation, such that $\mathbb{C}_x \rtimes \pi_x \cong \mathrm{IM}^* M_{y, -x, 0, \rho}$. Choose an algebraic homomorphism γ_y : $\mathrm{SL}_2(\mathbb{C}) \to Z_G(x)^\circ$ with $\mathrm{d}\gamma_y \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = y$. Now we can define

$$\mu_{G,L,\mathcal{L}}(x, \pi_x) = \mathrm{IM}^* M_{y, \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - x, r, \rho}.$$

This is canonical because all the above choices are unique up to conjugation. By (84) its central character is $\big(W_\mathcal{L}\big(x - \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\big), r\big)$. Define $\gamma : \mathrm{SL}_2(\mathbb{C}) \to Z_G(x)^\circ$ by

$$\gamma(g) = \gamma_y \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \gamma_y(g) \gamma_y \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),$$

it is associated to the unipotent element $\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. As $\mathrm{d}\gamma \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) = -\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$, the central character of $\mu_{G,L,\mathcal{L}}$ attains the desired form.

The claims about temperedness follow from Theorem 3.25 and (82). □

4. The twisted graded Hecke algebra of a cuspidal quasi-support

In disconnected reductive groups one sometimes has to deal with disconnected variations on Levi subgroups. Here we will generalize the results of the previous two sections to that setting.

Recall [AMS] that a quasi-Levi subgroup of G is a group of the form $M = Z_G(Z(L)^\circ)$, where L is a Levi subgroup of G°. Thus $Z(M)^\circ = Z(L)^\circ$ and $M \longleftrightarrow L = M^\circ$ is a bijection between quasi-Levi subgroups of G and the Levi subgroups of G°.

Definition 4.1. A cuspidal quasi-support for G is a triple $(M, \mathcal{C}_v^M, q\mathcal{L})$ where:

- M is a quasi-Levi subgroup of G;
- $\mathcal{C}_v^M$ is a nilpotent $\mathrm{Ad}(M)$-orbit in $\mathfrak{m} = \mathrm{Lie}(M)$.
- $q\mathcal{L}$ is a M-equivariant cuspidal local system on $\mathcal{C}_v^M$, i.e. as M°-equivariant local system it is a direct sum of cuspidal local systems.

We denote the G-conjugacy class of $(M, v, q\mathcal{L})$ by $[M, v, \mathcal{L}]_G$. With this cuspidal quasi-support we associate the groups

$$N_G(q\mathcal{L}) = \mathrm{Stab}_{N_G(M)}(q\mathcal{L}) \quad \text{and} \quad W_{q\mathcal{L}} = N_G(q\mathcal{L})/M. \tag{86}$$

Let $\mathcal{L}$ be an irreducible constituent of $q\mathcal{L}$ as M°-equivariant local system on $\mathcal{C}_v^{M^\circ} = \mathcal{C}_v^M$. Then

$$W_{\mathcal{L}}^\circ = N_{G^\circ}(M^\circ)/M^\circ \cong N_{G^\circ}(M^\circ)M/M$$

is a a subgroup of $W_{q\mathcal{L}}$. It is normal because G° is normal in G.

Let P° be a parabolic subgroup of G° with Levi decomposition $P^\circ = M^\circ \ltimes U$. The definition of M entails that it normalizes U, so

$$P := M \ltimes U$$

is a again a group. We put

$$\begin{aligned} N_G(P, q\mathcal{L}) &= N_G(P, M) \cap N_G(q\mathcal{L}), \\ \mathfrak{R}_{q\mathcal{L}} &= N_G(P, q\mathcal{L})/M. \end{aligned}$$

The same proof as for Lemma 2.1.b shows that

$$W_{q\mathcal{L}} = W_{\mathcal{L}}^\circ \rtimes \mathfrak{R}_{q\mathcal{L}}. \tag{87}$$

We define $\dot{\mathfrak{g}}$ as before, but with M instead of L, and with the new P. We put

$$K = (\mathrm{pr}_1)_! \dot{q\mathcal{L}} \quad \text{and} \quad K^* = (\mathrm{pr}_1)_! \dot{q\mathcal{L}}^*, \tag{88}$$

these are perverse sheaves on $\mathfrak{g}$. Considering $(\mathrm{pr}_1)_! \dot{q\mathcal{L}}$ as a local system on $\mathfrak{g}_{RS}$, [AMS, Lemma 5.4] says that

$$\mathrm{End}_G((\mathrm{pr}_1)_! \dot{q\mathcal{L}}) \cong \mathbb{C}[W_{q\mathcal{L}}, \natural_{q\mathcal{L}}],$$

where $\natural_{q\mathcal{L}} : (W_{q\mathcal{L}}/W_{\mathcal{L}}^\circ)^2 \to \mathbb{C}^\times$ is a suitable 2-cocycle. As in (8)

$$\mathrm{End}_G^+((\mathrm{pr}_1)_! \dot{q\mathcal{L}}) \cong \mathbb{C}[\mathfrak{R}_{q\mathcal{L}}, \natural_{q\mathcal{L}}]. \tag{89}$$

To $(M, \mathcal{C}_v^M, q\mathcal{L})$ we associate the twisted graded Hecke algebra

$$\mathbb{H}(G, M, q\mathcal{L}) := \mathbb{H}(\mathfrak{t}, W_{q\mathcal{L}}, c\mathbf{r}, \natural_{q\mathcal{L}}),$$

where the parameters c_i are as in (12). As in Lemma 2.8, we can consider it as

$$\mathbb{H}(G, M, q\mathcal{L}) = \mathbb{H}(\mathfrak{t}, W_{\mathcal{L}}^\circ, c\mathbf{r}) \rtimes \mathrm{End}_G^+((\mathrm{pr}_1)_! \dot{q\mathcal{L}}),$$

and then it depends canonically on $(G, M, q\mathcal{L})$. We note that (87) implies

$$\mathbb{H}(G^\circ N_G(P, q\mathcal{L}), M, q\mathcal{L}) = \mathbb{H}(G, M, q\mathcal{L}). \tag{90}$$

All the material from Proposition 2.6 up to and including Theorem 3.2, and the parts of [Lus3] on which it is based, extend to this situation with the above substitutions. We will use these results also for $\mathbb{H}(G, M, q\mathcal{L})$.

To generalize the remainder of Section 3 we need to assume that:

Condition 4.2. The group G equals $G^\circ N_G(P, q\mathcal{L})$.

By (90) this imposes no further restriction on the collection of twisted graded Hecke algebras under consideration. Let us write

$$\mathcal{P}_y^\circ = \{gP \in G^\circ M/P : \mathrm{Ad}(g^{-1})y \in \mathcal{C}_v^M + \mathfrak{u}\} = \mathcal{P}_y \cap G^\circ M/P.$$

Condition 4.2 guarantees that $\mathcal{P}_y = \mathcal{P}_y^\circ \times \mathfrak{R}_{q\mathcal{L}}$ as $M(y)$-varieties. With these minor modifications Lemma 3.3 also goes through: there is an isomorphism of $\mathbb{H}(G, M, q\mathcal{L})$-modules

$$H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{q\mathcal{L}}) \cong \mathrm{ind}_{\mathbb{H}(G^\circ M, M, q\mathcal{L})}^{\mathbb{H}(G,M,q\mathcal{L})} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{q\mathcal{L}}). \tag{91}$$

We note that $N_G(q\mathcal{L}) \cap G^\circ = N_{G^\circ}(M^\circ)$, for by [Lus1, Theorem 9.2] $N_G(M^\circ)$ stabilizes all M°-equivariant cuspidal local systems contained in $q\mathcal{L}$. Hence

$$N_{G^\circ M}(q\mathcal{L})/M \cong N_{G^\circ}(q\mathcal{L})/M^\circ = N_{G^\circ}(M^\circ)/M^\circ = W_\mathcal{L}^\circ. \tag{92}$$

Moreover the 2-cocycles $\natural_{q\mathcal{L}}$ and $\natural_\mathcal{L}$ are trivial on $W_\mathcal{L}^\circ$, so we can

$$\text{identify} \quad \mathbb{H}(G^\circ M, M, q\mathcal{L}) \quad \text{with} \quad \mathbb{H}(G^\circ, L, \mathcal{L}). \tag{93}$$

We already performed the construction and parametrization of $\mathbb{H}(G^\circ, L, \mathcal{L})$ in Theorem 3.11, but now we want it in terms of M and $q\mathcal{L}$. To this end we need to recall how $q\mathcal{L}$ can be constructed from $\mathcal{L}$. Let $M_\mathcal{L}$ be the stabilizer in M of $(\mathcal{C}_v^{M^\circ}, \mathcal{L})$. Let $(\mathrm{pr}_1)_! \dot{\mathcal{L}}_M$ be like $(\mathrm{pr}_1)_! \dot{\mathcal{L}}$, but for M. About this local system on $\mathfrak{m}_{RS} = \mathcal{C}_v^M + \mathfrak{t}_{\mathrm{reg}}$ [AMS, Proposition 4.5] says

$$\mathrm{End}_M((\mathrm{pr}_1)_! \dot{\mathcal{L}}_M) \cong \mathbb{C}[M_\mathcal{L}/M^\circ, \natural_\mathcal{L}].$$

By [AMS, (63)] there is a unique $\rho_M \in \mathrm{Irr}(\mathbb{C}[M_\mathcal{L}/M^\circ, \natural_\mathcal{L}^{-1}])$ such that

$$\text{the extension of } q\mathcal{L} \text{ from } \mathcal{C}_v^M \text{ to } \mathfrak{m}_{RS} \text{ is } \mathrm{Hom}_{\mathbb{C}[M_\mathcal{L}/M^\circ, \natural_\mathcal{L}]}(\rho_M^*, (\mathrm{pr}_1)_! \dot{\mathcal{L}}_M). \tag{94}$$

From the proof of [AMS, Proposition 3.5] we see that the stalk of (94) at $v \in \mathcal{C}_v^M$, considered as $Z_M(v)$-representation, is

$$(q\mathcal{L})_v \cong \mathrm{ind}_{Z_M(v)_{\mathcal{L}_v}}^{Z_M(v)} (\rho_M \otimes \mathcal{L}_v) = \mathrm{ind}_{Z_{M_\mathcal{L}}(v)}^{Z_M(v)} (\rho_M \otimes \mathcal{L}_v).$$

Here $Z_M(v)_{\mathcal{L}_v}$ denotes the stabilizer of $\mathcal{L}_v \in \mathrm{Irr}(Z_{M^\circ}(v))$ in $Z_M(v)$. The same holds for other elements in the M-conjugacy class of v, so as M-equivariant sheaves

$$q\mathcal{L} \cong \mathrm{ind}_{M_\mathcal{L}}^M (\rho_M \otimes \mathcal{L}). \tag{95}$$

We recall from [AMS, (64)] that the cuspidal support map Ψ_G has a "quasi" version $q\Psi_G$, which associates to every pair (y, ρ) with $y \in G^\circ$ unipotent and $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(y))\big)$ a cuspidal quasi-support.

Lemma 4.3. *Let $y \in \mathfrak{g}$ be nilpotent such that $\mathcal{P}_y$ is nonempty. Then M stabilizes the M°-orbit of y.*

Proof. From [Lus1, Theorem 6.5] and [AMS, (64)] we deduce that there exists a $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(y))\big)$ such that $q\Psi_G(y,\rho) = (M, \mathcal{C}_v^M, q\mathcal{L})$ (up to G-conjugacy). Now [AMS, Lemma 7.6] says that there exist algebraic homomorphisms $\gamma_y, \gamma_v : \mathrm{SL}_2(\mathbb{C}) \to M^\circ$ such that

$$\text{(96)}\quad \mathrm{d}\gamma_y \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = y, \quad \mathrm{d}\gamma_v \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = v \quad \text{and} \quad \mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) - \mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{Lie}(Z(M)^\circ).$$

In view of [Car, Proposition 5.6.4] the G°-conjugacy class of y (resp. the M°-orbit of v) is completely determined by the G°-conjugacy class of $\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (resp. the M°-conjugacy class of $\mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). By [AMS, Theorem 3.1.a] $\mathcal{C}_v^{M^\circ} = \mathcal{C}_v^M$. It follows that for every $m \in M$ there is a $m_0 \in M^\circ$ such that

$$\mathrm{Ad}(m)\mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \mathrm{Ad}(m_0)\mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

We calculate, using (96):

$$\begin{aligned} \mathrm{Ad}(m)\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) &= \mathrm{Ad}(m)\mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + \mathrm{Ad}(m)\big(\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) - \mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big) \\ \mathrm{Ad}(m)\mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) &+ \mathrm{Ad}(m)\big(\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) - \mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big) = \mathrm{Ad}(m_0)\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right). \end{aligned}$$

This implies that m stabilizes the M°-orbit of y. □

Lemma 4.4. *Let $\sigma_0 \in \mathfrak{t} = \mathrm{Lie}(Z(M)^\circ)$ be semisimple, write $Q = Z_{G^\circ}(\sigma_0)$ and let $y \in Z_{\mathfrak{g}}(\sigma_0) = \mathrm{Lie}(Q)$ be nilpotent. The map $\rho^\circ \mapsto \rho^\circ \rtimes \rho_M$ is a bijection between the following sets:*

$$\big\{\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_Q(y))\big) : \Psi_Q(y,\rho^\circ) = (M^\circ, \mathcal{C}_v^M, \mathcal{L}) \textit{ up to } G^\circ\textit{-conjugation}\big\},$$
$$\big\{\tau^\circ \in \mathrm{Irr}\big(\pi_0(Z_{QM}(y))\big) : q\Psi_{QM}(y,\tau^\circ) = (M, \mathcal{C}_v^M, q\mathcal{L}) \textit{ up to } G^\circ M\textit{-conjugation}\big\}.$$

Proof. Notice that $M^\circ \subset Q$, for $\sigma \in \mathfrak{t}$. By [Lus1, Theorem 9.2] there is a canonical bijection

$$\Sigma_{\mathcal{L}} : \Psi_Q^{-1}(M^\circ, \mathcal{C}_v^M, \mathcal{L}) \to \mathrm{Irr}(W_{\mathcal{L}}^Q).$$

Similarly, by [AMS, Lemma 5.3 and Theorem 5.5.a] there is a canonical bijection

$$q\Sigma_{q\mathcal{L}} : q\Psi_{QM}^{-1}(M, \mathcal{C}_v^M, q\mathcal{L}) \to \mathrm{Irr}(N_{QM}(M, q\mathcal{L})/M) \cong \mathrm{Irr}(W_{\mathcal{L}}^Q),$$

where we used (92) for the last identification. Composing these two, we obtain a bijection

$$\text{(97)}\qquad q\Sigma_{q\mathcal{L}}^{-1} \circ \Sigma_{\mathcal{L}} : \Psi_Q^{-1}(M^\circ, \mathcal{C}_v^M, \mathcal{L}) \to q\Psi_{QM}^{-1}(M, \mathcal{C}_v^M, q\mathcal{L}).$$

Since $\mathcal{L}$ is a subsheaf of $\mathcal{L}$ and the $W_{\mathcal{L}}^\circ$-action on $q\mathcal{L}$ extends that on $\mathcal{L}$, $\Sigma_{\mathcal{L}}(y,\rho^\circ)$ is contained in $q\Sigma_{q\mathcal{L}}(y,\tau^\circ)$ for some τ°. Hence (97) preserves the fibers over y. This provides a canonical bijection between the two sets figuring in the lemma.

The action of $W_{\mathcal{L}}^Q$ on $q\mathcal{L}$ and the sheafs associated to it for $\Sigma_{\mathcal{L}}$ and $q\Sigma_{q\mathcal{L}}$ comes from $Q \subset G^\circ$, so it fixes the part $\mathrm{ind}_{M_{\mathcal{L}}}^M \rho_M$ in (95). Now it follows from the descriptions of $\Sigma_{\mathcal{L}}$ and $q\Sigma_{q\mathcal{L}}$ in [AMS, §5] that

$$\text{(98)}\qquad q\Sigma_{q\mathcal{L}}^{-1} \circ \Sigma_{\mathcal{L}}(y,\rho^\circ) = \big(y, \big(\mathrm{ind}_{M_{\mathcal{L}}}^M(\rho_M \otimes \rho^\circ)\big)_y\big) = \big(y, \mathrm{ind}_{Z_{M_{\mathcal{L}}}(y)}^{Z_M(y)}(\rho_M \otimes \rho^\circ)\big).$$

For the same reasons the action of $\pi_0(Z_Q(y))$ on (98) fixes the $\mathrm{ind}_{M_{\mathcal{L}}}^M \rho_M$ part pointwise, and sees only ρ°. To analyse the right hand side as representation of

$\pi_0(Z_{QM}(y))$, we investigate $Z_M(y)/Z_{M^\circ}(y)$. Using Lemma 4.3 we find

$$\begin{aligned} \pi_0(Z_{QM}(y))/\pi_0(Z_Q(y)) = Z_{QM}(y)/Z_Q(y) \; &= \; Z_{QM}(y)/Z_{QM^\circ}(y) \\ \cong \mathrm{Stab}_{M/M^\circ}(\mathrm{Ad}(QM^\circ)y) \; &= \; M/M^\circ \\ = \mathrm{Stab}_{M/M^\circ}(\mathrm{Ad}(M^\circ)y) \; &\cong \; Z_M(y)/Z_{M^\circ}(y). \end{aligned} \tag{99}$$

With (99) we can identify the representation on the right hand side of (98) with

$$\mathrm{ind}_{\pi_0(Z_{QM_\mathcal{L}}(y))}^{\pi_0(Z_{QM}(y))}(\rho_M \otimes \rho^\circ). \tag{100}$$

We already knew that it is irreducible, so $\pi_0(Z_{QM_\mathcal{L}}(y))/\pi_0(Z_Q(y))$ must be the stabilizer of $\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_Q(y))\big)$ in $\pi_0(Z_{QM}(y))/\pi_0(Z_Q(y)) \cong M/M^\circ$. In other words, (100) equals $\rho_M \ltimes \rho^\circ \in \mathrm{Irr}\big(\pi_0(Z_{QM}(y))\big)$. □

Lemma 4.5. *Let σ_0, y, ρ° be as in Lemma 4.4, and define $\sigma \in \mathfrak{g}$ as in Lemma 3.6. With the identification (93), the $\mathbb{H}(G^\circ M, M, q\mathcal{L})$-module $E^\circ_{y,\sigma,r,\rho^\circ \rtimes \rho_M}$ is canonically isomorphic to the $\mathbb{H}(G^\circ, M^\circ, \mathcal{L})$-module $E^\circ_{y,\sigma,r,\rho^\circ}$.*

Proof. Let us recall that

$$\begin{aligned} E^\circ_{y,\sigma,r,\rho^\circ} \; &= \; \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma_0,y))}\big(\rho^\circ, \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})\big), \\ E^\circ_{y,\sigma,r,\rho^\circ \rtimes \rho_M} \; &= \; \mathrm{Hom}_{\pi_0(Z_{G^\circ M}(\sigma_0,y))}\big(\rho^\circ \rtimes \rho_M, \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{q\mathcal{L}})\big). \end{aligned} \tag{101}$$

Here the first $\mathcal{P}_y^\circ$ is a subset of G°/P°, whereas the second $\mathcal{P}_y^\circ$ is contained in $G^\circ M/P$. Yet they are canonically isomorphic via $G^\circ/P^\circ \xrightarrow{\sim} G^\circ P/P = G^\circ M/P$. By (95)

$$\begin{aligned} \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{q\mathcal{L}}) &\cong \mathrm{ind}_{M_\mathcal{L}}^M\big(\rho_M \otimes \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{\mathcal{L}})\big) \\ &= \mathrm{ind}_{M_\mathcal{L}}^M\big(\rho_M \otimes E^\circ_{y,\sigma,r}\big) \end{aligned}$$

From this and [AMS, Proposition 1.1.d] we see that

$$E^\circ_{y,\sigma,r,\rho^\circ \rtimes \rho_M} \cong \mathrm{Hom}_{\mathbb{C}[M_\mathcal{L}/M^\circ, \natural_\mathcal{L}^{-1}]}\Big(\rho_M, \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma_0,y))}\big(\rho^\circ, \mathrm{ind}_{M_\mathcal{L}}^M\big(\rho_M \otimes E^\circ_{y,\sigma,r}\big)\big)\Big).$$

Recall from Proposition 3.7 that ρ° only sees the cuspidal support $(M^\circ, v, \mathcal{L})$. In the above expression the part of $\mathrm{ind}_{M_\mathcal{L}}^M$ associated to $M \setminus M_\mathcal{L}$ gives rise to cuspidal supports $(M^\circ, v, m\cdot\mathcal{L})$ with $m\cdot\mathcal{L} \not\cong \mathcal{L}$, so this part does not contribute to $E^\circ_{y,\sigma,r,\rho^\circ \rtimes \rho_M}$. We conclude that

$$\begin{aligned} E^\circ_{y,\sigma,r,\rho^\circ \rtimes \rho_M} \cong \mathrm{Hom}_{\mathbb{C}[M_\mathcal{L}/M^\circ, \natural_\mathcal{L}^{-1}]}\Big(\rho_M, \mathrm{Hom}_{\pi_0(Z_{G^\circ}(\sigma_0,y))}\big(\rho^\circ, \rho_M \otimes E^\circ_{y,\sigma,r}\big)\Big) = \\ \mathrm{Hom}_{\mathbb{C}[M_\mathcal{L}/M^\circ, \natural_\mathcal{L}^{-1}]}\Big(\rho_M, \rho_M \otimes E^\circ_{y,\sigma,r,\rho^\circ}\Big) = E^\circ_{y,\sigma,r,\rho^\circ}. \end{aligned}$$ □

We note that, as a consequence of Lemmas 4.4, 4.5 and Theorem 3.11, Theorem 3.11 is also valid with G° replaced by $G^\circ M$, L by M and $\mathcal{L}$ by $q\mathcal{L}$. Knowing this and assuming Condition 4.2, we can use Clifford theory to relate $\mathrm{Irr}(\mathbb{H}(G, M, q\mathcal{L}))$ to $\mathrm{Irr}(\mathbb{H}(G^\circ M, M, q\mathcal{L}))$.

Thus all of Paragraphs 3.2–3.5 remains valid in the setting of the current section. Let us summarise the most important results, analogues of Theorem 3.20 and Corollary 3.23. In view of Lemma 3.21 we do not need Condition 4.2 anymore once we have obtained these results. Therefore we state them without assuming Condition 4.2.

Theorem 4.6. *Fix $r \in \mathbb{C}$.*

(a) *Let $y, \sigma \in \mathfrak{g}$ with y nilpotent, σ semisimple and $[\sigma, y] = 2ry$. Let $\tau \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma_0, y))\big)$ such that $q\Psi_{Z_G(\sigma_0)}(y, \tau) = (M, \mathcal{C}_v^M, q\mathcal{L})$ (up to G-conjugation). With these data we associate the $\mathbb{H}(G^\circ N_G(P, q\mathcal{L}), M, q\mathcal{L})$-module*

$$E_{y,\sigma,r,\tau} = \mathrm{Hom}_{\pi_0(Z_{G^\circ N_G(P,q\mathcal{L})}(\sigma_0,y))}\big(\tau, \mathbb{C}_{\sigma,r} \underset{H_*^{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y^\circ, \dot{q\mathcal{L}})\big).$$

Via (90) *we consider it also as a $\mathbb{H}(G, M, q\mathcal{L})$-module.*

Then the $\mathbb{H}(G, M, q\mathcal{L})$-module $E_{y,\sigma,r,\tau}$ has a distinguished irreducible quotient $M_{y,\sigma,r,\tau}$, which appears with multiplicity one in $E_{y,\sigma,r,\tau}$.

(b) *The map $M_{y,\sigma,r,\tau} \longleftrightarrow (y, \sigma, \tau)$ gives a bijection between $\mathrm{Irr}_r(\mathbb{H}(G, M, q\mathcal{L}))$ and G-conjugacy classes of triples as in part (a).*

(c) *The set $\mathrm{Irr}_r(\mathbb{H}(G, M, q\mathcal{L}))$ is also canonically in bijection with the following two sets:*
- *G-orbits of pairs (x, τ) with $x \in \mathfrak{g}$ and $\tau \in \mathrm{Irr}\big(\pi_0(Z_G(x))\big)$ such that $q\Psi_{Z_G(x_S)}(x_N, \tau) = (M, \mathcal{C}_v^M, q\mathcal{L})$ up to G-conjugacy.*
- *$N_G(M)/M$-orbits of triples $(\sigma_0, \mathcal{C}, \mathcal{F})$, with $\sigma_0 \in \mathfrak{t}$, $\mathcal{C}$ a nilpotent $Z_G(\sigma_0)$-orbit in $Z_{\mathfrak{g}}(\sigma_0)$ and $\mathcal{F}$ a $Z_G(\sigma_0)$-equivariant cuspidal local system on $\mathcal{C}$ such that $q\Psi_{Z_G(\sigma_0)}(\mathcal{C}, \mathcal{F}) = (M, \mathcal{C}_v^M, q\mathcal{L})$ up to G-conjugacy.*

Appendix A. Compatibility with parabolic induction

It has turned out that Theorem 3.4 is not correct as stated. The maps given there have almost all the claimed properties, the only problem is that usually they are not surjective. In this appendix (written in June 2018) we will repair that, by proving a weaker version of the Theorem.

Given Q as on page 16, PQ° is a parabolic subgroup of G° with Q° as Levi factor. The unipotent radical $\mathcal{R}_u(PQ^\circ)$ is normalized by Q°, so its Lie algebra $\mathfrak{u}_Q = \mathrm{Lie}(\mathcal{R}_u(PQ^\circ))$ is stable under the adjoint actions of Q° and $\mathfrak{q}$. In particular $\mathrm{ad}(y)$ acts on $\mathfrak{u}_Q$. We denote the cokernel of $\mathrm{ad}(y) : \mathfrak{u}_Q \to \mathfrak{u}_Q$ by ${}_y\mathfrak{u}_Q$. For $v \in \mathfrak{u}_Q$ and $(\sigma, r) \in \mathrm{Lie}(M(y)^\circ)$ we have

$$[\sigma, [y, v]] = [y, [\sigma, v]] + [[\sigma, y], v] = [y, [\sigma, v]] + [2ry, v] \in \mathrm{ad}(y)\mathfrak{u}_Q.$$

Hence $\mathrm{ad}(y)$ descends to a linear map ${}_y\mathfrak{u}_Q \to {}_y\mathfrak{u}_Q$. Following Lusztig [Lus7, §1.16], we define

$$\begin{array}{llll} \epsilon: & \mathrm{Lie}(M^Q(y)^\circ) & \to & \mathbb{C} \\ & (\sigma, r) & \mapsto & \det(\mathrm{ad}(\sigma) - 2r : {}_y\mathfrak{u}_Q \to {}_y\mathfrak{u}_Q) \end{array}.$$

It is easily seen that ϵ is invariant under the adjoint action of $M^Q(y)$, so it defines an element of $H^*_{M^Q(y)}(\{y\})$. For a given y, all the parameters (y, σ, r) for which parabolic induction from $\mathbb{H}(Q, L, \mathcal{L})$ to $\mathbb{H}(G, L, \mathcal{L})$ can behave problematically, are zeros of ϵ.

For any closed subgroup C of $M^Q(y)^\circ$, ϵ yields an element ϵ_C of $H_C^*(\{y\})$ (by restriction). We recall from [Lus3, Proposition 7.5] that for connected C there is a natural isomorphism

$$H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \cong H_C^*(\{y\}) \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}). \tag{102}$$

Here $H_C^*(\{y\})$ acts on the first tensor leg and $\mathbb{H}(G, L, \mathcal{L})$ acts on the second tensor leg. By Theorem 3.2.b these actions commute, and $H^C(\mathcal{P}_y, \dot{\mathcal{L}})$ becomes a module over $H_C^*(\{y\}) \otimes_{\mathbb{C}} \mathbb{H}(G, L, \mathcal{L})$.

Now we can formulate an improved version of Theorem 3.4.

Theorem A.1. *Let Q and y be as on page 16, and let C be a maximal torus of $M^Q(y)^\circ$.*

(*a*) *The map* (23) *induces an injection of $\mathbb{H}(G, L, \mathcal{L})$-modules*

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} H_*^C(\mathcal{P}_y^Q, \dot{\mathcal{L}}) \to H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}).$$

It respects the actions of $H_C^*(\{y\})$ and its image contains $\epsilon_C H_*^C(\mathcal{P}_y, \dot{\mathcal{L}})$.

(*b*) *Let $(\sigma, r)/{\sim} \in V_y^Q$ be such that $\epsilon(\sigma, r) \neq 0$. The map* (23) *induces an isomorphism of $\mathbb{H}(G, L, \mathcal{L})$-modules*

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r} \to E_{y,\sigma,r},$$

which respects the actions of $\pi_0(M^Q(y))_\sigma$.

Proof. (a) The given proof of Theorem 3.4 is valid with only one modification. Namely, the diagram (25) does not commute. A careful consideration of [Lus7, §2] shows failure to do so stems from the difference between certain maps $i_!$ and $(p^*)^{-1}$, where p is the projection of a vector bundle on its base space and i is the zero section of the same vector bundle. In [Lus7, Lemma 2.18] this difference is identified as multiplication by ϵ_C.
(b) For $(\sigma, r) \in \mathrm{Lie}(C)$ with $\epsilon(\sigma, r) \neq 0$, the proof of Theorem 3.4.b needs only one small adjustment. From (102) we get

$$\begin{aligned} \mathbb{C}_{\sigma,r} \underset{H_C^*(\{y\})}{\otimes} \epsilon_C H_*^C(\mathcal{P}_y, \dot{\mathcal{L}}) \cong \mathbb{C}_{\sigma,r} \underset{H_C^*(\{y\})}{\otimes} \epsilon_C H_C^*(\{y\}) \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) \cong \\ \mathbb{C}_{\sigma,r} \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}) = E_{y,\sigma,r}. \end{aligned}$$

The difference with before is the appearance of ϵ_C, with that and the above the proof of Theorem 3.4.b goes through. $\square$

There is just result in the paper that uses Theorem 3.4, namely Proposition 3.22. We have to replace that by a version which involves only the cases of Theorem 3.4 covered by Theorem A.1. Fortunately, under the extra condition $\epsilon(\sigma, r) \neq 0$ the proof of Proposition 3.22 goes through, when we replace the references to Theorem 3.4 by references to Theorem A.1.

Since ϵ is a polynomial function, its zero set is a subvariety of smaller dimension (say of V_y). Nevertheless, we also want to explicitly exhibit a large class of parameters (y, σ, r) on which ϵ does not vanish. By Proposition 3.5.c it suffices to do so under the assumption that $\sigma, \sigma_0 \in \mathfrak{t}$.

Let us call $x \in \mathfrak{t}$ (strictly) positive with respect to PQ° if $\Re(\alpha(t))$ is (strictly) positive for all $\alpha \in R(\mathcal{R}_u(PQ^\circ), T)$. We say that x is (strictly) negative with respect to PQ° if $-x$ is (strictly) positive.

Lemma A.2. *Let $y \in \mathfrak{q}$ be nilpotent and let $(\sigma, r) \in \mathfrak{t} \oplus \mathbb{C}$ with $[\sigma, y] = 2ry$. Suppose that $\sigma = \sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$ as in* (26)*, with $\sigma_0 \in Z_{\mathfrak{t}}(y)$. Assume furthermore that one of the following holds:*

- *$\Re(r) > 0$ and σ_0 is negative with respect to PQ°;*
- *$\Re(r) < 0$ and σ_0 is positive with respect to PQ°;*
- *$\Re(r) = 0$ and σ_0 is strictly positive or strictly negative with respect to PQ°.*

Then $\epsilon(\sigma, r) \neq 0$.

Proof. Via $\mathrm{d}\gamma_y : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{q}$, $\mathfrak{u}_Q$ becomes a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Since $\sigma_0 \in \mathfrak{t}$ commutes with y, it commutes with $\mathrm{d}\gamma_y(\mathfrak{sl}_2(\mathbb{C}))$. For any eigenvalue $\lambda \in \mathbb{C}$ of σ_0, let ${}_\lambda\mathfrak{u}_Q$ be the eigenspace in $\mathfrak{u}_Q$.

For $n \in \mathbb{Z}_{\geq 0}$ let $\mathrm{Sym}^n(\mathbb{C}^2)$ be the unique irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$. We decompose the $\mathfrak{sl}_2(\mathbb{C})$-module ${}_\lambda\mathfrak{u}_Q$ as

$$ {}_\lambda\mathfrak{u}_Q = \bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n(\mathbb{C}^2)^{\mu(\lambda,n)} \quad \text{with} \quad \mu(\lambda, n) \in \mathbb{Z}_{\geq 0}. $$

The cokernel of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathrm{Sym}^n(\mathbb{C}^2)$ is the lowest weight space W_{-n} in that representation, on which $\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$ acts as $-nr$. Hence σ acts on

$$ \mathrm{coker}(\mathrm{ad}(y) :_\lambda \mathfrak{u}_Q \to_\lambda \mathfrak{u}_Q) \cong \bigoplus_{n \geq 0} W_{-n}^{\mu(\lambda,r)} \quad \text{as} \quad \bigoplus_{n \geq 0} (\lambda - nr)\mathrm{Id}_{W_n^{\mu(\lambda,r)}}. $$

Consequently

$$ (\mathrm{ad}(\sigma) - 2r)\big|_{{}_\lambda\mathfrak{u}_Q} = \bigoplus_{\lambda \in \mathbb{C}} \bigoplus_{n \geq 0} (\lambda - (n+2)r)\mathrm{Id}_{W_n^{\mu(\lambda,r)}}. $$

By definition then

$$ \epsilon(\sigma, r) = \prod_{\lambda \in \mathbb{C}} \prod_{n \geq 0} (\lambda - (n+2)r)^{\mu(\lambda,n)}. $$

When $\Re(r) > 0$ and σ_0 is negative with respect to PQ°, $\Re(\lambda - (n+2)r) < 0$ for all eigenvalues λ of σ_0 on $\mathfrak{u}_Q$, and in particular $\epsilon(\sigma, r) \neq 0$.

Similarly, we see that $\epsilon(\sigma, r) \neq 0$ in the other two possible cases in the lemma. $\square$

As an application of Lemma A.2, we prove a result in the spirit of the Langlands classification for graded Hecke algebras [Eve]. It highlights a procedure to obtain irreducible $\mathbb{H}(G, L, \mathcal{L})$-modules from irreducible tempered modules of a parabolic subalgebra $\mathbb{H}(Q, L, \mathcal{L})$: twist by a central character which is strictly positive with respect to PQ°, induce parabolically and then take the unique irreducible quotient.

Proposition A.3. *Let y, σ, r, ρ be as in 3.20*

(*a*) *If $\Re(r) \neq 0$ and $\sigma_0 \in i\mathfrak{t}_\mathbb{R} + Z(\mathfrak{g})$, then $M_{y,\sigma,r,\rho} = E_{y,\sigma,r,\rho}$.*

(*b*) *Suppose that $\Re(r) > 0$ and $\sigma, \sigma_0 \in \mathfrak{t}$ such that $\Re(\sigma_0)$ is negative with respect to P. Then $\Re(\sigma_0)$ is strictly negative with respect to PQ°, where $Q = Z_G(\Re(\sigma_0))$. Further $M_{y,\sigma,r,\rho}$ is the unique irreducible quotient of $\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho}$.*

(*c*) *In the setting of part (b), $\mathrm{IM}^* M_{y,\sigma,r,\rho}$ is the unique irreducible quotient of*

$$ \mathrm{IM}^*\big(\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho}\big) \cong \mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} \mathrm{IM}^*(M^Q_{y,\sigma,r,\rho}). $$

Furthermore $\mathrm{IM}^*(M^Q_{y,\sigma,r,\rho})$ comes from the twist of a tempered $\mathbb{H}(Q^\circ, L, \mathcal{L})$-module by a strictly positive character of $S(Z(\mathfrak{q}^*))$.

Remark. By (83) the extra condition in part (a) holds for instance when $\Re(r) > 0$ and $\mathrm{IM}^*(M_{y,\sigma,r,\rho})$ is tempered. By Proposition 3.5 and Lemma 2.1 every parameter (y, σ) is G°-conjugate to one with the properties as in (b) and (c).

Proof. (a) Write $\sigma_0 = \sigma_{0,\mathrm{der}} + z_0$ with $\sigma_{0,\mathrm{der}} \in \mathfrak{g}_{\mathrm{der}}$ and $z_0 \in Z(\mathfrak{g})$. Then, as in the proof of 3.28,

$$E^\circ_{y,\sigma,r,\rho^\circ} = E^\circ_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0} \quad \text{and} \quad M^\circ_{y,\sigma,r,\rho^\circ} = M^\circ_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0}.$$

By [Lus7, Theorem 1.21] $E^\circ_{y,\sigma-z_0,r,\rho^\circ} = M^\circ_{y,\sigma-z_0,r,\rho^\circ}$ as $\mathbb{H}(G_{\mathrm{der}}, L \cap G_{\mathrm{der}}, \mathcal{L})$-modules, so $E^\circ_{y,\sigma,r,\rho^\circ} = M^\circ_{y,\sigma,r,\rho^\circ}$ as $\mathbb{H}(G^\circ, L, \mathcal{L})$-modules. Together with Lemma 3.18 and (63) this gives $E_{y,\sigma,r,\rho} = M_{y,\sigma,r,\rho}$.
(b) Notice that $Z_G(\sigma, y) = Z_Q(\sigma, y)$, so by [AMS, Theorem 4.8.a] ρ is a valid enhancement of the parameter (σ, y) for $\mathbb{H}(Q, L, \mathcal{L})$.

By construction $\Re(\sigma_0)$ is strictly negative with respect to PQ°. Now Lemma A.2 says that we may apply Proposition 3.22. That and part (a) yield

$$\mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} M^Q_{y,\sigma,r,\rho} = \mathbb{H}(G, L, \mathcal{L}) \underset{\mathbb{H}(Q,L,\mathcal{L})}{\otimes} E^Q_{y,\sigma,r,\rho} = E_{y,\sigma,r,\rho}.$$

Now apply Theorem 3.20.b.
(c) The first statement follows from part (b) and 81. Write

$$M^{Q^\circ}_{y,\sigma,r,\rho^\circ} = M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0} = M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \big(\mathbb{C}_{z_0 - \Re(z_0)} \otimes \mathbb{C}_{\Re(z_0)} \big)$$

as in the proof of part (a), with Q in the role of G. By Theorem 3.25.b $M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0-\Re(z_0)}$ is anti-tempered. The definition of Q entails that $\Re(z_0) = \Re(\sigma_0)$, which we know is strictly negative. Hence

$$\mathrm{IM}^*(M^{Q^\circ}_{y,\sigma,r,\rho^\circ}) = \mathrm{IM}^*(M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0-\Re(z_0)}) \otimes \mathbb{C}_{-\Re(\sigma_0)},$$

where the right hand side is the twist of a tempered $\mathbb{H}(Q^\circ, L, \mathcal{L})$-module by the strictly positive character $-\Re(\sigma_0)$ of $S(Z(\mathfrak{q}^*))$. By (80)

$$\mathrm{IM}^*(M^Q_{y,\sigma,r\rho}) = \tau \ltimes \mathrm{IM}^*(M^{Q^\circ}_{y,\sigma,r,\rho^\circ}). \tag{103}$$

□

We note that by Lemma 3.16 $S(Z(\mathfrak{q}^*))$ acts on (103) by the characters $\gamma(-\Re(\sigma_0))$ with $\gamma \in \mathfrak{R}^Q$. Since $\mathfrak{R}^Q$ normalizes PQ°, it preserves the strict positivity of $-\Re(\sigma_0)$. In this sense $\mathrm{IM}^*(M^Q_{y,\sigma,r\rho})$ is essentially the twist of a tempered $\mathbb{H}(Q, L, \mathcal{L})$-module by a strictly positive central character.

References

[Art] J. Arthur, "A note on L-packets", Pure Appl. Math. Quaterly **2.1** (2006), 199–217.

[ABPS1] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "Geometric structure in smooth dual and local Langlands correspondence", Japan. J. Math. **9** (2014), 99–136.

[ABPS2] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "The principal series of p-adic groups with disconnected centre", Proc. London Math. Soc. **114.5** (2017), 798–854.

[ABPS3] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "Conjectures about p-adic groups and their noncommutative geometry", Contemp. Math. **691** (2017), 15–51.

[AMS] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Generalizations of the Springer correspondence and cuspidal Langlands parameters", Manus. Math. (2018), 1–72.

[AMS2] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Affine Hecke algebras for Langlands parameters", arXiv:1701.03593, 2017.

[BeLu] J. Bernstein, V. Lunts, "Equivariant Sheaves and Functors", Lecture Notes in Mathematics **1578**.

[Bor] A. Borel, "Automorphic L-functions", Proc. Symp. Pure Math **33.2** (1979), 27–61.

[Car] R.W. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Pure and Applied Mathematics, John Wiley & Sons, New York NJ, 1985.

[Ciu] D. Ciubotaru, "On unitary unipotent representations of p-adic groups and affine Hecke algebras with unequal parameters", Representation Theory **12** (2008), 453–498.

[Eve] S. Evens, "The Langlands classification for graded Hecke algebras", Proc. Amer. Math. Soc. **124.4** (1996), 1285–1290.

[Hai] T.J. Haines, "The stable Bernstein center and test functions for Shimura varieties", pp. 118–186 in: *Automorphic forms and Galois representations*, London Math. Soc. Lecture Note Ser. **415**, Cambridge University Press, 2014.

[Kal] T. Kaletha, "Global rigid inner forms and multiplicities of discrete automorphic representations", arXiv:1501.01667, 2015.

[KaLu] D. Kazhdan, G. Lusztig, "Proof of the Deligne–Langlands conjecture for Hecke algebras", Invent. Math. **87** (1987), 153–215.

[Lus1] G. Lusztig, "Intersection cohomology complexes on a reductive group", Invent. Math. **75.2** (1984), 205–272.

[Lus2] G. Lusztig, "Character sheaves V", Adv. in Math. **61** (1986), 103–155.

[Lus3] G. Lusztig, "Cuspidal local systems and graded Hecke algebras", Publ. Math. Inst. Hautes Études Sci. **67** (1988), 145–202.

[Lus4] G. Lusztig, "Affine Hecke algebras and their graded version", J. Amer. Math. Soc **2.3** (1989), 599–635.

[Lus5] G. Lusztig, "Cuspidal local systems and graded Hecke algebras. II", pp. 217–275 in: *Representations of groups*, Canadian Mathematical Society Conference Proceedings **16**, 1995.

[Lus6] G. Lusztig, "Classification of unipotent representations of simple p-adic groups", Int. Math. Res. Notices **11** (1995), 517–589.

[Lus7] G. Lusztig, "Cuspidal local systems and graded Hecke algebras. III", Represent. Theory **6** (2002), 202–242.

[Lus8] G. Lusztig, "Classification of unipotent representations of simple p-adic groups. II", Represent. Theory **6** (2002), 243–289.

[RaRa] A. Ram, J. Ramagge, "Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory", pp. 428–466 in: *A tribute to C.S. Seshadri (Chennai 2002)*, Trends in Mathematics, Birkhäuser, 2003.

[Ree] M. Reeder, "Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations", Representation Theory **6** (2002), 101–126.

[Sol1] M. Solleveld, "Parabolically induced representations of graded Hecke algebras", Algebras and Representation Theory **15.2** (2012), 233–271.

[Sol2] M. Solleveld, "Homology of graded Hecke algebras", J. Algebra **323** (2010), 1622–1648.

[Spr] T.A. Springer, *Linear algebraic groups 2nd ed.*, Progress in Mathematics **9**, Birkhäuser, 1998.

[Wit] S. Witherspoon, "Twisted graded Hecke algebras", J. Algebra **317** (2007), 30–42.

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