GRADED HECKE ALGEBRAS, CONSTRUCTIBLE SHEAVES AND THE p-ADIC KAZHDAN–LUSZTIG CONJECTURE

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Abstract. Graded Hecke algebras can be constructed in terms of equivariant cohomology and constructible sheaves on nilpotent cones. In earlier work, their standard modules and their irreducible modules where realized with such geometric methods.

We pursue this setup to study properties of module categories of (twisted) graded Hecke algebras, in particular what happens geometrically upon formal completion with respect to a central character. We prove a version of the Kazhdan–Lusztig conjecture for (twisted) graded Hecke algebras. It expresses the multiplicity of an irreducible module in a standard module as the multiplicity of an equivariant local system in an equivariant perverse sheaf.

This is applied to smooth representations of reductive p-adic groups. Under some conditions, we verify the p-adic Kazhdan–Lusztig conjecture from [Vog]. Here the equivariant constructible sheaves live on certain varieties of Langlands parameters. The involved conditions are checked for substantial classes of groups and representations.

Contents

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Introduction

The importance of graded Hecke algebras stems from the multitude of ways in which they can arise:

- in terms of generators and relations,
- as degenerations of affine Hecke algebras [Lus2],
- from harmonic analysis and differential operators on Lie algebras [Che, Opd],
- from progenerators for representations of reductive p-adic groups [Sol2],
- from constructible sheaves and equivariant cohomology [Lus1, Lus3, AMS2],
- from enhanced Langlands parameters for reductive p-adic groups [AMS3].

In the last three settings one naturally encounters slightly more general objects $\mathbb{H}$ called twisted graded Hecke algebras. The irreducible and standard modules of these algebras were classified and constructed geometrically in [Lus1, Lus3, AMS2]. The main goal of this paper is to apply that setup to compute the multiplicity of an irreducible $\mathbb{H}$-module in another $\mathbb{H}$-module (typically a standard module).

Via [Sol2] the analogous issues for smooth representations of a reductive p-adic group $\mathcal{G}(F)$ can be translated to modules over twisted graded Hecke algebras. In good cases, that can also be related to the geometry of varieties of enhanced Langlands parameters for $\mathcal{G}(F)$, via [AMS3]. That links our goals to versions of the Kazhdan–Lusztig conjecture for p-adic groups [Vog, Zel].

We hope that this paper may contribute to the long term project of geometrization and categorification of the (local) Langlands correspondence, for which we refer to [BCHN, FaSc, Hel, Zhu]. Roughly speaking, it is expected that a derived category of smooth $\mathcal{G}(F)$-representations is equivalent with some derived category of coherent sheaves on a variety or stack of Langlands parameters.

In our setting the complexes of sheaves are equivariant and constructible. We showed in [Sol6] that the associated derived categories are naturally equivalent with derived categories of differential graded modules over suitable twisted graded Hecke algebras $\mathbb{H}$. In particular, one can never detect all $\mathbb{H}$-modules or all smooth $\mathcal{G}(F)$-representations with such sheaves – most of them are not graded. Nevertheless our setup could provide a stepping stone to relate more appropriate sheaves to $\mathcal{G}(F)$-representations.

Main results.

Let G be a complex reductive algebraic group and let M be a Levi subgroup of G. For maximal generality we allow disconnected versions of G and M. Let $q\mathcal{E}$ be an M-equivariant cuspidal local system on a nilpotent orbit in Lie(M). These data give rise to a (twisted) graded Hecke algebra $\mathbb{H}(G, M, q\mathcal{E})$, see [Sol6, §2.1]. In the introduction and in most of the paper we assume that $N_G(M)$ stabilizes $q\mathcal{E}$, which by [AMS2] can be done without loss of generality.

Let $\mathfrak{g}_N$ be the nilpotent cone in $\mathfrak{g} = \mathrm{Lie}(G)$. Via some kind of parabolic induction, one constructs from $q\mathcal{E}$ a semisimple complex $K_N \in \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$. Here $\mathcal{D}^b_{G\times\mathbb{C}^\times}$ denotes an equivariant bounded derived category of constructible sheaves, from [BeLu]. This provides an isomorphism of graded algebras [Sol6, Theorem 2.2]:

$$\mathbb{H}(G, M, q\mathcal{E}) \cong \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N). \tag{1}$$

The algebra $\mathbb{H}(G, M, q\mathcal{E})$ comes with a finite "Weyl-like" group $W_{q\mathcal{E}}$ acting on $\mathfrak{t} = \mathrm{Lie}(Z(M))$. Its centre can be described as

$$Z(\mathbb{H}(G, M, q\mathcal{E})) \cong \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}} = \mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}].$$

Via a specific injection $\Sigma_v : \mathfrak{t} \oplus \mathbb{C} \to \mathfrak{m} \oplus \mathbb{C}$ (the identity on $\mathfrak{t}$, but in general not in $\mathbb{C}$), we can regard $Z(\mathbb{H}(G, M, q\mathcal{E}))$ as a quotient of $\mathcal{O}(\mathfrak{g} \oplus \mathbb{C})^G$. A semisimple element $(\sigma, r) \in \mathfrak{g} \oplus \mathbb{C}$ determines a unique central character of $\mathbb{H}(G, M, q\mathcal{E})$ if it lies in $\mathrm{Ad}(G)\Sigma_v(\mathfrak{t} \oplus \mathbb{C})$, and otherwise it is irrelevant for $\mathbb{H}(G, M, q\mathcal{E})$. We fix a relevant (σ, r) and we denote the corresponding formal completion of $Z(\mathbb{H}(G, M, q\mathcal{E}))$ by $\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma,r}$. In the process of localization, $G \times \mathbb{C}^\times$ will be replaced by $Z_G(\sigma) \times \mathbb{C}^\times$ and $\mathfrak{g}_N$ by

$$\mathfrak{g}_N^{\sigma,r} := \{y \in \mathfrak{g}_N : [\sigma, y] = 2ry\}.$$

A variation on the construction of K_N yields an object $K_{N,\sigma,r} \in \mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$. Since (σ, r) belongs to $\mathrm{Lie}(Z_G(\sigma) \times \mathbb{C}^\times)$, it defines a character of

$$H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt}) \cong \mathcal{O}\big(\mathrm{Lie}(Z_G(\sigma) \times \mathbb{C}^\times)\big)^{Z_G(\sigma)\times\mathbb{C}^\times},$$

and we can formally complete the latter algebra with respect to (σ, r).

Theorem A. *(see Theorem 2.4)*
There is a natural algebra isomorphism

$$\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma,r} \underset{Z(\mathbb{H}(G,M,q\mathcal{E}))}{\otimes} \mathbb{H}(G, M, q\mathcal{E}) \xrightarrow{\sim} \hat{H}^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r}).$$

This induces an equivalence of categories

$$\mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G, M, q\mathcal{E})) \cong \mathrm{Mod}_{\mathrm{fl},\sigma,r}\big(\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})\big).$$

Here $\mathrm{Mod}_{\mathrm{fl},\sigma,r}$ *denotes the category of finite length modules all whose irreducible subquotients admit the central character* (σ, r).

When G is connected and $r \neq 0$, Theorem A is due to Lusztig [Lus3]. Our investigations revealed a technical problem in the relevant part of [Lus3], which was resolved in collaboration with Lusztig, see [Lus7, # 121] and Appendix A.

Graded Hecke algebras coming from reductive p-adic groups have the variable $\mathbf{r} \in \mathbb{H}(G, M, q\mathcal{E})$ specialized to a positive real number. In that respect a version of Theorem C for $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r)$ is fitting. As $\mathbb{C}[\mathbf{r}] = \mathcal{O}(\mathrm{Lie}(\mathbb{C}^\times))$ comes from the $\mathbb{C}^\times$-actions, a natural attempt is to replace $G \times \mathbb{C}^\times$-equivariance by G-equivariance. That does not work well directly in (1), only after localization.

Theorem B. *(see Paragraph 2.2)*
Fix $r \in \mathbb{C}$. *Theorem A becomes valid with* $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r)$ *instead of* $\mathbb{H}(G, M, q\mathcal{E})$ *once we forget the* $\mathbb{C}^\times$*-equivariant structure everywhere.*

Theorems A and B could be helpful for a geometric construction of Ext-groups between objects of $\mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G, M, q\mathcal{E}))$ or $\mathrm{Mod}_{\mathrm{fl},\sigma}(\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r))$.

For more concrete results, it is necessary to understand standard and irreducible $\mathbb{H}(G, M, q\mathcal{E})$-modules better. They were obtained in [Lus1, AMS2] via the equivariant cohomology of certain flag varieties with local systems. They can be parametrized by the following data (considered up to G-conjugation):

$$\text{a semisimple } \sigma \in \mathfrak{g},\ r \in \mathbb{C},\ y \in \mathfrak{g}_N^{\sigma,r} \text{ and certain } \rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big).$$

The standard $\mathbb{H}(G, M, q\mathcal{E})$-module $E_{y,\sigma,r,\rho}$ has a distinguished (unique if $r \neq 0$) irreducible quotient $M_{y,\sigma,r,\rho}$. In Paragraph 3.1 we observe that the parametrization from [AMS2] is more complicated than necessary, and we make it more natural.

Next we provide several useful alternative constructions of $E_{y,\sigma,r,\rho}$:

Theorem C. *(see Proposition 3.5 and Lemma 4.3)*
Suppose that $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$ fulfills the condition to parametrize an $\mathbb{H}(G, M, q\mathcal{E})$-module. Let $i_y : \{y\} \to \mathfrak{g}_N^{\sigma,r}$ be the inclusion.

(*a*) *There is an isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules*

$$\mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\rho, H^*(i_y^! K_{N,\sigma,r})\big) \cong E_{y,\sigma,r,\rho}.$$

(*b*) *Denote the dual of a local system or representation by a $\vee$. There is an isomorphism of $\mathbb{H}(G, M, q\mathcal{E}^\vee)$-modules*

$$\mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\rho^\vee, H^*(i_y^* K_{N,\sigma,r})^\vee\big) \cong E_{y,\sigma,r,\rho^\vee}.$$

(*c*) *Let $\mathrm{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}\rho$ be the $Z_G(\sigma) \times \mathbb{C}^\times$-equivariant local system on $\mathrm{Ad}(Z_G(\sigma))y$ determined by ρ, and let $j_N : \mathrm{Ad}(Z_G(\sigma))y \to \mathfrak{g}_N^{\sigma,r}$ be the inclusion. Then*

$$\mathrm{Hom}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})}\big(K_{N,\sigma,r}, j_N^* \mathrm{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}\rho\big) \tag{2}$$

is a graded $\mathbb{H}(G, M, q\mathcal{E}^\vee)$-module,

$$\mathrm{Hom}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,r})}\big(K_{N,\sigma,r}, j_N^* \mathrm{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}\rho\big) \tag{3}$$

is a graded right $\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$-module and there are canonical surjections of $\mathbb{H}(G, M, q\mathcal{E}^\vee)$-modules

$$(2) \to (3) \to E_{y,\sigma,r,\rho^\vee}.$$

We note that $E_{y,\sigma,r,\rho^\vee}$ is not a graded $\mathbb{H}(G, M, q\mathcal{E}^\vee)$-module, because the surjection in Theorem C.c consists of dividing out the submodule generated by the inhomogeneous ideal

$$\ker(\mathrm{ev}_{\sigma,r}) \subset H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt}) \cong \mathcal{O}(Z_\mathfrak{g}(\sigma))^{Z_G(\sigma)} \otimes_\mathbb{C} \mathbb{C}[\mathbf{r}].$$

In Appendix B we prove that, under a mild condition, standard modules of twisted graded Hecke algebras are compatible with parabolic induction. That is relevant for parts of [AMS2, AMS3] which are related to Section 5, and it shows that our standard modules are really analogous to the standard representations for reductive groups studied by Langlands and others.

The original Kazhdan–Lusztig conjecture [KaLu1] concerned the multiplicities of irreducible modules in standard modules for semisimple complex Lie algebras. One can ask for the analogous multiplicities for any group or algebra with a good notion of standard modules, in particular for a (twisted) graded Hecke algebra. The next result relies on some properties of $(\mathfrak{g}_N^{\sigma,r}, K_{N,\sigma,r})$ from Section 4.

Theorem D. *(see Proposition 5.1)*
Suppose that (y, σ, r, ρ) *and* (y', σ, r, ρ') *parametrize* $\mathbb{H}(G, M, q\mathcal{E})$*-modules. The multiplicity of the irreducible module* $M_{y',\sigma,r,\rho'}$ *in the standard module* $E_{y,\sigma,r,\rho}$ *equals the multiplicity of the local system* $\mathrm{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}\rho$ *in the pullback to* $\mathrm{Ad}(Z_G(\sigma))y$ *of the cohomology sheaf* $\mathcal{H}^*\big(\mathrm{IC}_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{Z_G(\sigma,y')}^{Z_G(\sigma)}\rho')\big)$.

A version of the Kazhdan–Lusztig conjecture for the p-adic group $GL_n(F)$ appeared in [Zel], and Vogan [Vog] formulated it for all connected reductive groups over local fields. We point out that [Vog, Conjecture 8.11] contains some signs which are useful for real reductive groups but better omitted in the non-archimedean instances.

To transfer Theorem D to reductive p-adic groups and their Langlands parameters, we need to make several assumptions about aspects of the local Langlands correspondence. We refer to Section 5 for an explanation of the setup and the conditions.

Let q_F be the cardinality of the residue field of the non-archimedean local field F. For $r = \log(q_F)/2$ the variety

$$\mathfrak{g}_N^{\sigma,-r} = \{y \in \mathfrak{g}_N : \mathrm{Ad}(\exp \sigma)y = q_F^{-1} y\}. \tag{4}$$

can be identified with a set of unramified Langlands parameters $\phi : \mathbf{W}_F \rtimes \mathbb{C} \to G$, with $\exp(\sigma)$ the image of a Frobenius element of $\mathbf{W}_F$. This can be used to model varieties of arbitrary Langlands parameters for $\mathcal{G}(F)$.

Theorem E. *(see Theorem 5.4)*
Consider a Bernstein block in the category of smooth complex representations of a connected reductive p-adic group $\mathcal{G}(F)$*, coming from a supercuspidal representation* ω *of a Levi subgroup* $\mathcal{M}(F)$*. Suppose that the conditions from Section 5 hold.*

(*a*) *Let* $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^\omega$ *be the category of finite length* $\mathcal{G}(F)$*-representations all whose irreducible subquotients have cuspidal support* $(\mathcal{M}(F), \omega)$*.*
Then $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^\omega$ *is equivalent with a category of the form*

$$\mathrm{Mod}_{\mathrm{fl},\sigma}\big(\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,-r})}(K_{N,\sigma,-r})\big) \quad \textit{where} \quad r = \log(q_F)/2.$$

(*b*) *The p-adic Kazhdan–Lusztig conjecture (as in* [Vog, Conjecture 8.11] *without the signs) holds for irreducible and standard representations in* $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^\omega$*. It takes the form of Theorem D, where all objects live on a variety of Langlands parameters* (4)*.*
(*c*) *Parts (a) and (b) hold unconditionally in the following cases:*
- *inner forms of general linear groups,*
- *inner forms of special linear groups,*
- *principal series representations of quasi-split groups,*
- *unipotent representations (of arbitrary connected reductive groups over* F*),*
- *classical* F*-groups (not necessarily quasi-split).*

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1. The setup from [Sol6]

All our groups will be complex linear algebraic groups. We mainly work in the equivariant bounded derived categories of constructible sheaves from [BeLu]. For a group H acting on a space X, this category will be denoted $\mathcal{D}^b_H(X)$.

Let G be a complex reductive group, possibly disconnected. To construct a graded Hecke algebra geometrically, we need a cuspidal quasi-support $(M, \mathcal{C}_v^M, q\mathcal{E})$ for G [AMS1]. This consists of:

- a quasi-Levi subgroup M of G, which means that M° is a Levi subgroup of G° and $M = Z_G(Z(M^\circ)^\circ)$,
- $\mathcal{C}_v^M$ is a $\mathrm{Ad}(M)$-orbit in the nilpotent variety $\mathfrak{m}_N$ in the Lie algebra $\mathfrak{m}$ of M,
- $q\mathcal{E}$ is a M-equivariant cuspidal local system on $\mathcal{C}_v^M$.

We write $T = Z(M)^\circ$, $\mathfrak{t} = \mathrm{Lie}(T)$ and

$$W_{q\mathcal{E}} = \mathrm{Stab}_{N_G(M)}(q\mathcal{E})/M = N_G(M, q\mathcal{E})/M$$

To these data one associates a twisted graded Hecke algebra

$$\mathbb{H}(G, M, q\mathcal{E}) = \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}}], \tag{1.1}$$

see [Sol6, §2.1]. As vector space it is the tensor product of

- a polynomial algebra $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}]$,
- a twisted group algebra $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$,

and there are nontrivial cross relations between these two subalgebras.

Let $\mathfrak{g}_N$ be the nilpotent variety in the Lie algebra $\mathfrak{g}$ of G. The algebra (1.1) can be realized in terms of suitable equivariant sheaves on $\mathfrak{g}$ or $\mathfrak{g}_N$. We let $\mathbb{C}^\times$ act on $\mathfrak{g}$ and $\mathfrak{g}_N$ by $\lambda \cdot X = \lambda^{-2} X$. Then every M-equivariant local system on $\mathcal{C}_v^M$, and in particular $q\mathcal{E}$, is automatically $M \times \mathbb{C}^\times$-equivariant.

Let $P^\circ = M^\circ U$ be a parabolic subgroup of G° with Levi factor M° and unipotent radical U. Then $P = MU$ is a "quasi-parabolic" subgroup of G. Consider the varieties

$$\begin{aligned}\dot{\mathfrak{g}} &= \{(X, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\},\\ \dot{\mathfrak{g}}_N &= \dot{\mathfrak{g}} \cap (\mathfrak{g}_N \times G/P).\end{aligned}$$

We let $G \times \mathbb{C}^\times$ act on these varieties by

$$(g_1, \lambda) \cdot (X, gP) = (\lambda^{-2}\mathrm{Ad}(g_1)X, g_1 gP).$$

By [Lus1, Proposition 4.2] there are natural isomorphisms of graded algebras

$$H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}) \cong H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}_N) \cong \mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]. \tag{1.2}$$

Consider the maps

$$\begin{aligned}&\mathcal{C}_v^M \xleftarrow{f_1} \{(X, g) \in \mathfrak{g} \times G : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\} \xrightarrow{f_2} \dot{\mathfrak{g}},\\ &f_1(X, g) = \mathrm{pr}_{\mathcal{C}_v^M}(\mathrm{Ad}(g^{-1})X), \qquad\qquad f_2(X, g) = (X, gP).\end{aligned} \tag{1.3}$$

Let $\dot{q\mathcal{E}}$ be the unique $G \times \mathbb{C}^\times$-equivariant local system on $\dot{\mathfrak{g}}$ such that $f_2^* \dot{q\mathcal{E}} = f_1^* q\mathcal{E}$. Let $\mathrm{pr}_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$ be the projection on the first coordinate and define

$$K := \mathrm{pr}_{1,!} \dot{q\mathcal{E}} \qquad \in \mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}).$$

Let $\dot{q\mathcal{E}}_N$ be the pullback of $\dot{q\mathcal{E}}$ to $\dot{\mathfrak{g}}_N$ and put

$$K_N := \mathrm{pr}_{1,N,!} \dot{q\mathcal{E}}_N \qquad \in \mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}_N),$$

a semisimple complex isomorphic to the pullback of K to $\mathfrak{g}_N$ [Sol6, §2.2]. From [Sol6, Theorem 2.2], based on [Lus1, Lus3, AMS2], we recall:

Theorem 1.1. *There exist natural isomorphisms of graded algebras*

$$\mathbb{H}(G, M, q\mathcal{E}) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N).$$

Consider the subgroup $N_G(M, q\mathcal{E})G^\circ = N_G(P, M, q\mathcal{E})G^\circ$ of G. It is known from [AMS2, (90)] that

$$\mathbb{H}(G, M, q\mathcal{E}) = \mathbb{H}(N_G(P, M, q\mathcal{E})G^\circ, M, q\mathcal{E}).$$

Moreover, by [AMS2, Lemma 3.21] the relevant sets of parameters for these two algebras are in natural bijection. Therefore we may, and will, assume without loss of generality:

Condition 1.2. G equals $N_G(P, M, q\mathcal{E})G^\circ$, or equivalently $N_G(M)$ stabilizes $q\mathcal{E}$.

2. Formal completion at a central character

We want to complete $\mathbb{H}(G, M, q\mathcal{E}) \cong \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ with respect to (the kernel of) a central character. Recall from [AMS2, Lemma 2.3 and §4] that

$$Z(\mathbb{H}(G, M, q\mathcal{E})) \supset \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}} = \mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]. \tag{2.1}$$

In many cases (2.1) is an equality, namely whenever $W_{q\mathcal{E}}$ acts faithfully on $\mathfrak{t}$. To localize in a geometric way, we need to interpret (2.1) in terms of equivariant homology. By [Lus1, §1.11] there are natural isomorphisms

$$H^*_{G\times\mathbb{C}^\times}(\mathrm{pt}) \cong \mathcal{O}(\mathfrak{g} \oplus \mathbb{C})^{G\times\mathbb{C}^\times} \cong \mathcal{O}(\mathfrak{g}/\!/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]. \tag{2.2}$$

The algebra $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})$ acts naturally on $\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ by the product in equivariant homology [Lus3, §1.20]. That determines a homomorphism

$$H^*_{G\times\mathbb{C}^\times}(\mathrm{pt}) \to Z\big(\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)\big) \cong \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}}. \tag{2.3}$$

Let $\gamma_v : SL_2(\mathbb{C}) \to M$ be an algebraic homomorphism with $\mathrm{d}\gamma_v \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = v$. With $s_v := \mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{m}$ we define an injection

$$\begin{array}{cccc} \Sigma_v : & \mathfrak{t} \oplus \mathbb{C} & \to & \mathfrak{m} \oplus \mathbb{C} \\ & (\sigma_0, r) & \mapsto & (\sigma_0 + r\sigma_v, r) \end{array}.$$

Lemma 2.1. (*a*) *The map*

$$\mathfrak{t}/W_{q\mathcal{E}} \times \mathbb{C} = \mathrm{Irr}(\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}}) \to \mathrm{Irr}(H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})) = \mathfrak{g}/\!/G \times \mathbb{C}$$

dual to (2.3) *is injective and equals the map induced by* Σ_v.

(*b*) *The support of* $\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ *as* $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})$*-module is* $\mathrm{Ad}(G)\Sigma_v(\mathfrak{t}\oplus\mathbb{C})/\!/\mathrm{Ad}(G)$.

Proof. (a) This follows from [AMS3, Proposition 1.7] upon specializing all coordinates from $\vec{r}$ to r.

(b) Let T_M be a maximal torus of M°, whose Lie algebra $\mathfrak{t}_M$ contains $\mathfrak{t} \oplus \mathbb{C}\sigma_v$. By part (a)

$$(\mathfrak{g} \oplus \mathbb{C})/\!/G \cong (\mathfrak{t}_M \oplus \mathbb{C})/W(G^\circ, T_M).$$

Since $\Sigma_v(\mathfrak{t} \oplus \mathbb{C})$ is a closed subset of $\mathfrak{t}_M \oplus \mathbb{C}$, $\mathrm{Ad}(G)\Sigma_v(\mathfrak{t} \oplus \mathbb{C})/\!/\mathrm{Ad}(G)$ is closed in $(\mathfrak{g} \oplus \mathbb{C})/\!/G$. Now the statement is a consequence of (2.1), (2.2) and part (a). □

Lemma 2.1 entails that we can formally complete $\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ with respect to elements of $\mathfrak{g}//G\times\mathbb{C}$ that come from $\mathfrak{t}/W_{q\mathcal{E}}\times\mathbb{C}$. With the techniques from [Lus3, §4], that can be done geometrically. In Appendix A we discuss why these techniques apply in the setting of [Lus3, §8], which is a special case of our current setting.

Fix $(\sigma,r)\in\mathrm{Ad}(G)\mathfrak{t}\times\mathbb{C}$ and put

$$C = Z_{G\times\mathbb{C}^\times}(\sigma,r) = Z_G(\sigma)\times\mathbb{C}^\times .$$

The inclusion $\mathfrak{c}=\mathrm{Lie}(C)\subset\mathfrak{g}$ makes $H^*_C(\mathrm{pt})$ into a module for $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt}) = \mathcal{O}(\mathfrak{g})^G\otimes\mathbb{C}[\mathbf{r}]$. Further, any $G\times\mathbb{C}^\times$-equivariant sheaf can be regarded as a C-equivariant sheaf. Like in (2.3) we obtain a graded algebra homomorphism

$$H^*_C(\mathrm{pt})\to\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K).$$

That induces a graded algebra homomorphism

$$H^*_C(\mathrm{pt})\underset{H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes}\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)\longrightarrow H^*_C(\mathrm{pt})\underset{H^*_C(\mathrm{pt})}{\otimes}\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K). \tag{2.4}$$

We denote the completion of $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})$ with respect to the maximal ideal determined by (σ,r) as

$$\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} = \hat{\mathcal{O}}(\mathfrak{g}\oplus\mathbb{C}//G\times\mathbb{C}^\times)_{\sigma,r}$$

We define $\hat{H}^*_C(\mathrm{pt})_{\sigma,r}=\hat{\mathcal{O}}(\mathfrak{c}//C)_{\sigma,r}$ analogously.

Proposition 2.2. (*a*) *The natural map $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})\to H^*_C(\mathrm{pt})$ induces an algebra isomorphism $\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}\to\hat{H}^*_C(\mathrm{pt})_{\sigma,r}$.*

(*b*) *Part (a) and* (2.4) *induce an isomorphism of $\hat{H}^*_C(\mathrm{pt})_{\sigma,r}$-algebras*

$$\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}\underset{H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes}\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)\longrightarrow\hat{H}^*_C(\mathrm{pt})_{\sigma,r}\underset{H^*_C(\mathrm{pt})}{\otimes}\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K).$$

(*c*) *The graded algebra $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$ is Noetherian and only has terms in even degrees ≥ 0.*

(*d*) *Parts (b) and (c) also hold with $(\mathfrak{g}_N,K_N)$ instead of $(\mathfrak{g},K)$.*

Proof. (a) According to [Lus3, 4.3.(a)] this holds for connected groups, so for $G^\circ\times\mathbb{C}^\times$ and $C^\circ = Z_G(\sigma)^\circ\times\mathbb{C}^\times$. From $H^*_G(\mathrm{pt}) = H^*_{G^\circ}(\mathrm{pt})^{G/G^\circ}$ [Lus1, §1.9] we deduce that

$$\begin{aligned}\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} &= \Big(\bigoplus\nolimits_{g\in G/Z_G(\sigma)G^\circ}\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\mathrm{Ad}(g)\sigma,r}\Big)^{G/G^\circ}\\ &\cong (\hat{H}^*_{G^\circ\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r})^{Z_G(\sigma)G^\circ/G^\circ}\;\cong\;(\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r})^{Z_G(\sigma)}\\ &= (\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r})^{C/C^\circ}\;=\;\hat{H}^*_C(\mathrm{pt})_{\sigma,r}.\end{aligned}\tag{2.5}$$

(b) Part (a) and Proposition A.1 show this for connected algebraic groups: there is a natural C-equivariant isomorphism of $\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r}$-algebras

$$\hat{H}^*_{G^\circ\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}\underset{H^*_{G^\circ\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes}\mathrm{End}^*_{\mathcal{D}^b_{G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K)\to\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r}\underset{H^*_{C^\circ}(\mathrm{pt})}{\otimes}\mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g})}(K).$$

Let $J_{\sigma,r}$ be the kernel of $\mathrm{ev}_{\sigma,r}: H^*_{C^\circ}(\mathrm{pt})\to\mathbb{C}$, it is a C-stable ideal. For any $n\in\mathbb{N}$ we can divide out the submodules associated to $J^n_{\sigma,r}$ above, which yields another C-equivariant isomorphism

$$\mathrm{End}^*_{\mathcal{D}^b_{G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K)/J^n_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_{G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K)\to\mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g})}(K)/J^n_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g})}(K).\tag{2.6}$$

The finite group $C/C^\circ \cong Z_G(\sigma)G^\circ/G^\circ$ acts naturally on both sides of (2.6). Averaging over this group, we obtain an isomorphism

$$\begin{aligned}(2.7)\quad & \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K)/J^n_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K) \cong \\ & \Big(\mathrm{End}^*_{\mathcal{D}^b_{G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K)/J^n_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_{G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K)\Big)^{C/C^\circ} \longrightarrow \\ & \Big(\mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g})}(K)/J^n_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g})}(K)\Big)^{C/C^\circ} \cong \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)/J^n_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K).\end{aligned}$$

The inverse limit of the instances of (2.7) for $n \in \mathbb{N}$ is a natural isomorphism of $\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r}$-algebras

$$\begin{aligned}(2.8)\quad & \hat{H}^*_{Z_G(\sigma)G^\circ\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G(\sigma)G^\circ\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)G^\circ\times\mathbb{C}^\times}(\mathfrak{g})}(K) \\ & \longrightarrow \quad \hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K).\end{aligned}$$

A computation analogous to (2.5) shows that the left hand side of (2.8) is naturally isomorphic to

$$\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K).$$

(c) Clearly $\mathbb{H}(G, M, q\mathcal{E})$ has finite rank over the finitely generated algebra $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$. The latter is integral over $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}}$, so has finite rank over $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}}$. Lemma 2.1.a implies that (2.3) is surjective, so $\mathbb{H}(G, M, q\mathcal{E})$ also has finite rank as $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})$-module. Fix a finite set of generators F and map it to $\tilde{F} \subset \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$ by (2.4). That yields a finite rank $H^*_C(\mathrm{pt})$-submodule $H^*_C(\mathrm{pt})\tilde{F}$ of $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$. By parts (a) and (b)

$$H^*_C\hat{(}\mathrm{pt})_{\sigma,r}\tilde{F} \cong \hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}F \cong H^*_C\hat{(}\mathrm{pt})_{\sigma,r}\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$$

for all possible (σ, r). Since localization is an exact functor, this implies that

$$\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)/H^*_C(\mathrm{pt})\tilde{F}$$

localizes to zero everywhere. Hence this quotient is zero and $\tilde{F}$ generates $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$ as $H^*_C(\mathrm{pt})$-module. The image of

$$H^*_C(\mathrm{pt}) \to \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K).$$

is Noetherian because it is finitely generated and commutative. Hence $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$ is Noetherian as well.

By Theorem 1.1 the left hand side of part (b) only involves elements of even degrees ≥ 0. Hence so does the right hand side, and its subalgebra $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K)$.

(d) This can be shown in the same way as parts (b) and (c). □

2.1. **Localization on $\mathfrak{g}$ and $\mathfrak{g}_N$.**

Having completed $\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ with respect to a central character, we want to see how this affects the underlying variety $\mathfrak{g}$. Let $T_{\sigma,r}$ be the smallest algebraic torus in $G^\circ \times \mathbb{C}^\times$ whose Lie algebra contains (σ, r). Then $\mathfrak{g}^{\sigma,r} := \mathfrak{g}^{T_{\sigma,r}}$ is C-stable and

$$\begin{aligned} \mathfrak{g}^{\sigma,r} \;=\; \mathfrak{g}^{\exp(\mathbb{C}(\sigma,r))} &= \{X \in \mathfrak{g} : e^{-2zr}\mathrm{Ad}(\exp(z\sigma))X = X \;\forall z \in \mathbb{C}\} \\ &= \{X \in \mathfrak{g} : \mathrm{Ad}(\exp(z\sigma))X = e^{2zr}X \;\forall z \in \mathbb{C}\} \\ &= \{X \in \mathfrak{g} : \mathrm{ad}(\sigma)X = 2rX\}. \end{aligned} \tag{2.9}$$

Notice that $\mathfrak{g}^{\sigma,r}$ consists entirely of nilpotent elements (unless $r = 0$). We write

$$\begin{aligned} \dot{\mathfrak{g}} &= \{(X, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\}, \\ \dot{\mathfrak{g}}^{\sigma,r} &= \dot{\mathfrak{g}}^{T_{\sigma,r}} = \dot{\mathfrak{g}} \cap \big(\mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}\big). \end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccc} \dot{\mathfrak{g}}^{\sigma,r} & \xrightarrow{j_{\sigma,r}} & \dot{\mathfrak{g}} \\ \downarrow \mathrm{pr}_1 & & \downarrow \mathrm{pr}_1 \\ \mathfrak{g}^{\sigma,r} & \xrightarrow{j_{\sigma,r}} & \mathfrak{g} \end{array} , \tag{2.10}$$

where the vertical maps are inclusions. We define

$$K_{\sigma,r} = \mathrm{pr}_{1,!} j_{\sigma,r}^* (q\dot{\mathcal{E}}) \in \mathcal{D}^b_C(\mathfrak{g}^{\sigma,r}).$$

Since (2.9) is often not a pullback diagram, $K_{\sigma,r}$ need not be isomorphic to $j^*_{\sigma,r}(K) = j^*_{\sigma,r}\mathrm{pr}_{1,!}(q\dot{\mathcal{E}})$. Nevertheless $K_{\sigma,r}$ can be regarded as some kind of restriction of K to $\mathfrak{g}^{\sigma,r}$. According to [Lus3, §8.12]

$$K_{\sigma,r} = \mathrm{pr}_{1,!}\mathrm{IC}_{Z_G(\sigma)\times\mathbb{C}^\times}\big(\mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}, j^*_{\sigma,r} q\dot{\mathcal{E}}\big), \tag{2.11}$$

where now

$$\mathrm{pr}_1 : \mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)} \to \mathfrak{g}^{\sigma,r} \quad \text{is proper.} \tag{2.12}$$

As noted in [Lus3, §5.3] (where K is called B and pullbacks to $T_{\sigma,r}$-fixed subvarieties are indicated by a superscript tilde), this implies that $K_{\sigma,r}$ is a semisimple complex, that is, a direct sum of degree shifts of simple perverse sheaves on $\mathfrak{g}^{\sigma,r}$. Notice that for $(\sigma, r) = (0, 0)$ we have

$$\mathfrak{g}^{0,0} = \mathfrak{g}, \quad \dot{\mathfrak{g}}^{0,0} = \dot{\mathfrak{g}} \quad \text{and} \quad K_{0,0} = K. \tag{2.13}$$

Thus the objects in Theorem 1.1 are special cases of their localized versions in this paragraph. Write

$$\mathfrak{g}_N^{\sigma,r} = \mathfrak{g}^{\sigma,r} \cap \mathfrak{g}_N, \qquad \dot{\mathfrak{g}}_N^{\sigma,r} = \dot{\mathfrak{g}}_N^{T_{\sigma,r}} = \dot{\mathfrak{g}}_N \cap \big(\mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}\big).$$

Let $j_{N,\sigma,r} : \dot{\mathfrak{g}}_N^{\sigma,r} \to \dot{\mathfrak{g}}$ be the inclusion and define

$$K_{N,\sigma,r} = (\mathrm{pr}_{1,N})_! j^*_{N,\sigma,r}(q\dot{\mathcal{E}}_N) \in \mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r}).$$

From the diagram

$$\begin{array}{ccc} \dot{\mathfrak{g}}_N^{\sigma,r} & \xrightarrow{j_{N,\sigma,r}} & \dot{\mathfrak{g}}^{\sigma,r} \\ \Big\downarrow \mathrm{pr}_{1,N} & & \Big\downarrow \mathrm{pr}_1 \\ \mathfrak{g}_N^{\sigma,r} & \xrightarrow{j_{N,\sigma,r}} & \mathfrak{g}^{\sigma,r} \end{array}$$

we see with base change [BeLu, Theorem 3.4.3] that $K_{N,\sigma,r}$ is the pullback of $K_{\sigma,r}$ to $\mathfrak{g}_N^{\sigma,r}$. We record that

$$\begin{array}{lllllll} \mathfrak{g}_N^{\sigma,r} & = & \mathfrak{g}^{\sigma,r}, & \dot{\mathfrak{g}}_N^{\sigma,r} & = & \dot{\mathfrak{g}}^{\sigma,r}, & K_{N,\sigma,r} = K_{\sigma,r} \quad \text{for } r \neq 0, \\ \mathfrak{g}_N^{\sigma,0} & = & Z_{\mathfrak{g}}(\sigma)_N, & \dot{\mathfrak{g}}_N^{\sigma,0} & = & Z_{\dot{\mathfrak{g}}}(\sigma)_N, & K_{N,\sigma,0} = K_{N,\sigma} \quad \text{for } r = 0. \end{array}$$

As both $K_{\sigma,r}$ (see above) and $K_{N,\sigma}$ are semisimple complexes [Sol6, Lemma 2.8], $K_{N,\sigma,r}$ is always a semisimple object of $\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})$.

Proposition 2.3. (*a*) *There exists a natural isomorphism of $\hat{H}^*_C(\mathrm{pt})_{\sigma,r}$-algebras*

$$\hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \longrightarrow \hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K).$$

(*b*) *The algebras in part (a) are naturally isomorphic with*

$$\hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r}) \cong \hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N)}(K_N).$$

(*c*) *The graded algebras $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r})$ and $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$ are Noetherian and only have terms in even degrees.*

Proof. (a) In [Lus3, §4.9–4.10], which is applicable by Appendix A, this was proven under the assumption that G (and hence C) is connected. Explicitly, there exists an isomorphism of $\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r}$-algebras

$$\hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r} \underset{H^*_{C^\circ}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \longrightarrow \hat{H}^*_{C^\circ}(\mathrm{pt})_{\sigma,r} \underset{H^*_{C^\circ}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g})}(K).$$

With the same argument as in the proof of Proposition 2.2.b, we can take C/C°-invariants on both side, and that replaces all occurences of C° by C.
(b) By Proposition 2.2.c and Theorem 1.1 there is a natural isomorphism

$$\hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g})}(K) \cong \hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N)}(K_N).$$

When $r \neq 0$, the left hand sides of parts (a) and (b) are the same. When $r = 0$, we assume (as we may by Lemma 2.1) that $\sigma \in \mathfrak{t}$. In the notations from [Sol6, §2.3] we have

$$\mathfrak{g}^{\sigma,0} = Z_{\mathfrak{g}}(\sigma), \quad \dot{\mathfrak{g}}^{\sigma,0} = \dot{\mathfrak{g}}^\sigma \quad \text{and} \quad K_{\sigma,0} = K_\sigma.$$

Then [Sol6, Proposition 2.10] provides the final isomorphism

$$\hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,0})}(K_{\sigma,0}) \to \hat{H}^*_C(\mathrm{pt})_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,0})}(K_{N,\sigma,0}).$$

(c) This can be proven like in Proposition 2.2.c. □

We let $W_{q\mathcal{E}}$ act on $\Sigma_v(\mathfrak{t} \oplus \mathbb{C})$ by decreeing that Σ_v is $W_{q\mathcal{E}}$-equivariant. For $(\sigma, r) \in \mathrm{Ad}(G)\Sigma_v(\mathfrak{t} \oplus \mathbb{C})$, let $Z_{\sigma,r}$ be the maximal ideal of

$$\mathcal{O}(\Sigma_v(\mathfrak{t} \oplus \mathbb{C})/W_{q\mathcal{E}}) = \mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}} \times \mathbb{C}(\sigma_v, 1)) \cong \mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}} \times \mathbb{C}) \subset Z(\mathbb{H}(G, M, q\mathcal{E}))$$

determined by (σ, r) via Lemma 2.1. Every finite length module V of $\mathbb{H}(G, M, q\mathcal{E})$ can be decomposed as

$$V = \bigoplus_{(\sigma,r) \in \Sigma_v(\mathfrak{t} \oplus \mathbb{C})/W_{q\mathcal{E}}} V_{\sigma,r} \;,$$
$$V_{\sigma,r} = \{v \in V : Z_{\sigma,r}^n \cdot v = 0 \text{ for some } n \in \mathbb{N}\}.$$

Hence the category of finite length left modules is a direct sum

$$\mathrm{Mod}_{\mathrm{fl}}(\mathbb{H}(G,M,q\mathcal{E})) = \bigoplus\nolimits_{(\sigma,r)\in\Sigma_v(\mathfrak{t}\oplus\mathbb{C})/W_{q\mathcal{E}}} \mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G,M,q\mathcal{E})). \tag{2.14}$$

The same holds for right modules:

$$\mathbb{H}(G,M,q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fl}} = \bigoplus\nolimits_{(\sigma,r)\in\Sigma_v(\mathfrak{t}\oplus\mathbb{C})/W_{q\mathcal{E}}} \mathbb{H}(G,M,q\mathcal{E}) - \mathrm{Mod}_{\mathrm{fl},\sigma,r}. \tag{2.15}$$

Let $\hat{Z}(\mathbb{H}(G,M,q\mathcal{E}))_{\sigma,r}$ be the formal completion of $Z(\mathbb{H}(G,M,q\mathcal{E}))$ with respect to $Z_{\sigma,r}$. Then $\mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G,M,q\mathcal{E}))$ can be identified with the category of finite length left modules, continuous with respect to the adic topology, of the completed algebra

$$\hat{Z}(\mathbb{H}(G,M,q\mathcal{E}))_{\sigma,r} \underset{Z(\mathbb{H}(G,M,q\mathcal{E}))}{\otimes} \mathbb{H}(G,M,q\mathcal{E}).$$

We use a similar notation for the algebras $\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ and $\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r})$, with respect to the maximal ideals determined by (σ,r) in $H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})$ and in $H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})$.

Theorem 2.4. (*a*) *There are natural algebra isomorphisms*

$$\begin{aligned} &\hat{Z}(\mathbb{H}(G,M,q\mathcal{E}))_{\sigma,r} \underset{Z(\mathbb{H}(G,M,q\mathcal{E}))}{\otimes} \mathbb{H}(G,M,q\mathcal{E}) \xrightarrow{\sim} \\ &\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K) \xrightarrow{\sim} \\ &\hat{H}^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}). \end{aligned}$$

(*b*) *Part (a) induces equivalences of categories*

$$\begin{aligned} \mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G,M,q\mathcal{E})) &\cong \mathrm{Mod}_{\mathrm{fl},\sigma,r}\big(\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)\big) \\ &\cong \mathrm{Mod}_{\mathrm{fl},\sigma,r}\big(\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r})\big), \end{aligned}$$

and analogously with right modules.

(*c*) *Parts (a) and (b) also hold with $(\mathfrak{g}_N, K_N)$ instead of $(\mathfrak{g}, K)$.*

Proof. (a) is a consequence of Theorem 1.1 and Propositions 2.2, 2.3.
(b) follows directly from (a).
(c) The proof is completely analogous to that of parts (a) and (b). □

We point out that the data (σ,r) in Theorem 2.4 can be scaled by an arbitrary $z\in\mathbb{C}^\times$. Namely,

$$\mathfrak{g}^{z\sigma,zr} = \mathfrak{g}^{\sigma,r}, \quad \dot{\mathfrak{g}}^{z\sigma,zr} = \dot{\mathfrak{g}}^{\sigma,r}, \quad K_{z\sigma,zr} = K_{\sigma,r}$$

and similarly with subscripts N. The scaling by degree automorphism [Sol6, (1.7)] provides an isomorphism between one the algebras associated to (σ,r) and its analogue associated to $(z\sigma,zr)$.

2.2. **Twisted graded Hecke algebras with a fixed r.**

So far we mainly considered twisted graded Hecke algebras with a formal variable $\mathbf{r}$. Often we localized $\mathbf{r}$ at a complex number r, but still we allowed modules on which $\mathbf{r}$ did not act as a scalar. In the Hecke algebras that arise from reductive p-adic groups, $\mathbf{r}$ is always specialized to some $r \in \mathbb{R}$, see [Sol2]. That prompts us to find versions of our previous results for such algebras.

Fix $r \in \mathbb{C}$ and write

$$\mathbb{H}(G, M, q\mathcal{E}, r) = \mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r) = \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, cr, \natural_{q\mathcal{E}}).$$

The centre of $\mathbb{H}(G, M, q\mathcal{E}, r)$ contains $\mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}})$. It will be convenient to identify $\mathfrak{t}$ with $\mathfrak{t}_r = \mathfrak{t} + r\sigma_v$ via Σ_v, and to identify $\mathfrak{t}/W_{q\mathcal{E}}$ with $\mathfrak{t}_r/W_{q\mathcal{E}}$. This enables us to localize $\mathbb{H}(G, M, q\mathcal{E}, r)$ at $W_{q\mathcal{E}}\sigma \in \mathfrak{t}_r/W_{q\mathcal{E}}$, which is consistent with the previous paragraph. The irreducible and standard modules of $\mathbb{H}(G, M, q\mathcal{E}, r)$ have already been classified in [AMS2] they come from $\mathbb{H}(G, M, q\mathcal{E})$ by imposing that $\mathbf{r}$ acts as r. However, the Ext-groups of two $\mathbb{H}(G, M, q\mathcal{E}, r)$-modules are usually not isomorphic to their Ext-groups as $\mathbb{H}(G, M, q\mathcal{E})$-modules.

As $\mathbb{C}[\mathbf{r}] \cong H^*_{\mathbb{C}^\times}(\mathrm{pt})$ comes in from the $\mathbb{C}^\times$-actions on our varieties and sheaves, it is natural to try to replace $Z_G(\sigma) \times \mathbb{C}^\times$-equivariant sheaves by $Z_G(\sigma)$-equivariant sheaves in Paragraph 2.1. However, the $\mathbb{C}^\times$-actions are there for a reason. Without them, [Lus1] would just give

$$\mathrm{End}^*_{\mathcal{D}^b_G(\mathfrak{g})}(K) \cong \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}] \cong \mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r}),$$

and from there one would never get any r in the picture. Therefore we proceed more subtly, first we formally complete with respect to (σ, r) and only then we forget the $\mathbb{C}^\times$-actions. For $\sigma \in \mathfrak{t}_r$, Theorem 2.4.a implies

$$\begin{aligned}
&\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}, r))_\sigma \underset{Z(\mathbb{H}(G,M,q\mathcal{E},r))}{\otimes} \mathbb{H}(G, M, q\mathcal{E}, r) \;\cong \\
&\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma,r}/(\mathbf{r} - r) \underset{Z(\mathbb{H}(G,M,q\mathcal{E}))}{\otimes} \mathbb{H}(G, M, q\mathcal{E}) \;\cong \\
&\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}/(\mathbf{r} - r) \underset{H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g})}(K) \;\cong \\
&\hat{H}^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}/(\mathbf{r} - r) \underset{H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}).
\end{aligned} \tag{2.16}$$

It is much easier to analyse (2.16) when $r = 0$, so we settle that case first.

Lemma 2.5. *For $\sigma \in \mathfrak{t}$ there are natural algebra isomorphisms*

$$\begin{aligned}
&\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}, 0))_\sigma \underset{Z(\mathbb{H}(G,M,q\mathcal{E},0))}{\otimes} \mathbb{H}(G, M, q\mathcal{E}, 0) \;\cong \\
&\hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,0})}(K_{\sigma,0}) \cong \\
&\hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,0})}(K_{N,\sigma,0}).
\end{aligned}$$

Proof. The final line of (2.16) simplifies because

$$\hat{H}^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,0}/(\mathbf{r}) \cong \hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \otimes \hat{H}^*_{\mathbb{C}^\times}(\mathrm{pt})_0/(\mathbf{r}) \cong \hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \tag{2.17}$$

as $H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})$-modules. Further, by [Lus3, §4.11] there is a natural isomorphism

$$\hat{H}^*_{Z_G^\circ(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,0})}(K_{\sigma,0}) \cong \hat{H}^*_{Z_G^\circ(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G^\circ(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)}(\mathfrak{g}^{\sigma,0})}(K_{\sigma,0}). \tag{2.18}$$

Taking $\pi_0(Z_G(\sigma))$-invariants, as in the proof of Proposition 2.2, we obtain the analogue of (2.18) with $Z_G(\sigma)$ instead of $Z_G^\circ(\sigma)$. Combining that with (2.16) and (2.17) proves the first isomorphism.

The isomorphism between the first and third terms in the statement can be shown in the same way, starting from part (c) instead of part (a) of Theorem 2.4. □

Next we consider a nonzero r and we fix $\sigma \in \mathfrak{t}_r$. Notice that $\mathfrak{t}_r \subset \mathfrak{m}$, so σ commutes with $\mathfrak{t}$ and with $T = \exp(\mathfrak{t})$. Let T' be a maximal torus of $Z_G^\circ(\sigma)$ containing T.

Lemma 2.6. *There is a natural algebra isomorphism*

$$\hat{H}^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \mathbb{C}[[\mathbf{r}-r]] \otimes_{\mathbb{C}} \hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}).$$

Proof. From [Lus3, §4.11] and Proposition A.1 we get

$$\hat{H}^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \hat{H}^*_{T'\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{T'\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{T'\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}). \tag{2.19}$$

The subgroup $\exp(\mathbb{C}(\sigma,r)) \subset T' \times \mathbb{C}^\times$ fixes $\mathfrak{g}^{\sigma,r}$ pointwise and projects onto $\mathbb{C}^\times$ because $r \neq 0$. Hence

$$T' \times \mathbb{C}^\times = T' \times \exp(\mathbb{C}(\sigma,r)) \quad \text{in} \quad G^\circ \times \mathbb{C}^\times. \tag{2.20}$$

By connectedness $\exp(\mathbb{C}(\sigma,r))$ also acts trivially on $K_{\sigma,r}$, so we can further decompose according to (2.20):

$$\begin{aligned} &\mathrm{End}^*_{\mathcal{D}^b_{T'\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \mathrm{End}^*_{\mathcal{D}^b_{\exp(\mathbb{C}(\sigma,r))}(\mathrm{pt})}(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{End}^*_{\mathcal{D}^b_{T'}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \\ &H^*_{\exp(\mathbb{C}(\sigma,r))}(\mathrm{pt}) \otimes_{\mathbb{C}} \mathrm{End}^*_{\mathcal{D}^b_{T'}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \mathbb{C}[\mathbf{r}-r] \otimes_{\mathbb{C}} \mathrm{End}^*_{\mathcal{D}^b_{T'}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}). \end{aligned} \tag{2.21}$$

Then (2.19) and its analogue without $\mathbb{C}^\times$ yield

$$\begin{aligned} &\hat{H}^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \\ &\mathbb{C}[[\mathbf{r}-r]] \otimes \hat{H}^*_{T'}(\mathrm{pt})_\sigma \underset{H^*_{T'}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{T'}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \cong \\ &\mathbb{C}[[\mathbf{r}-r]] \otimes \hat{H}^*_{Z_G^\circ(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G^\circ(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G^\circ(\sigma)}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \end{aligned} \tag{2.22}$$

As in the proof of Proposition 2.2.b, taking $\pi_0(Z_G(\sigma))$-invariants in (2.22) replaces $Z_G^\circ(\sigma)$ by $Z_G(\sigma)$. □

Now we can prove our desired variation on Theorem 2.4.

Theorem 2.7. *Fix any* $r \in \mathbb{C}$ *and* $\sigma \in \mathfrak{t}_r$.
(*a*) *There exists a natural algebra isomorphism*

$$\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}, r))_\sigma \underset{Z(\mathbb{H}(G,M,q\mathcal{E},r))}{\otimes} \mathbb{H}(G, M, q\mathcal{E}, r) \; \cong \; \hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}).$$

(*b*) *This induces an equivalence of categories*

$$\mathrm{Mod}_{\mathrm{fl},\sigma}\big(\mathbb{H}(G, M, q\mathcal{E}, r)\big) \cong \mathrm{Mod}_{\mathrm{fl},\sigma}\big(\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r})\big).$$

(*c*) *Parts (a) and (b) also hold with* $(\mathfrak{g}^{\sigma,r}_N, K_{N,\sigma,r})$ *instead of* $(\mathfrak{g}^{\sigma,r}, K_{\sigma,r})$.

Proof. (b) follows directly from (a).
(a,c) For $r = 0$ this is Lemma 2.5, so we may assume $r \neq 0$. Then the subscripts N do not change anything. By (2.16) and Lemma 2.6

$$\begin{aligned}
&\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}, r))_\sigma \underset{Z(\mathbb{H}(G,M,q\mathcal{E},r))}{\otimes} \mathbb{H}(G, M, q\mathcal{E}, r) \; \cong \\
&\big(\mathbb{C}[[\mathbf{r} - r]] \otimes_{\mathbb{C}} \hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma\big)/(\mathbf{r} - r) \underset{H^*_{Z_G(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \; \cong \\
&\hat{H}^*_{Z_G(\sigma)}(\mathrm{pt})_\sigma \underset{H^*_{Z_G(\sigma)}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}).
\end{aligned}$$

□

3. Standard modules of twisted graded Hecke algebras

In [Lus1, AMS2] standard (left) modules for $\mathbb{H}(G, M, q\mathcal{E})$ were studied. We will quickly recall their construction and then we relate these standard modules to the previous section. Recall that Condition 1.2 is in force.

Let $y \in \mathfrak{g}$ be nilpotent and define

$$\mathcal{P}_y = \{gP \in G/P : \mathrm{Ad}(g^{-1})y \in \mathcal{C}_v^M + \mathfrak{u}\}.$$

The group

$$Z_{G\times\mathbb{C}^\times}(y) = \{(g_1, \lambda) \in G \times \mathbb{C}^\times : \mathrm{Ad}(g_1)y = \lambda^2 y\} \tag{3.1}$$

acts on $\mathcal{P}_y$ by $(g_1, \lambda) \cdot gP = g_1 gP$. This puts $\mathcal{P}_y$ in $Z_{G\times\mathbb{C}^\times}(y)$-equivariant bijection with $\{y\}\times\mathcal{P}_y \subset \dot{\mathfrak{g}}$. The local system $q\dot{\mathcal{E}}$ on $\dot{\mathfrak{g}}$ restricts to a local system on $\{y\}\times\mathcal{P}_y \cong \mathcal{P}_y$, still called $q\dot{\mathcal{E}}$. The action of $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$ on K from Theorem 1.1 induces an action on

$$H_*^{G\times\mathbb{C}^\times}(\mathfrak{g}, K) \cong H_*^{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}}, q\dot{\mathcal{E}}) \tag{3.2}$$

From [Sol6, (2.4)] and the product in equivariant cohomology, we get an action of $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ on (3.2). These can be pulled back to actions on

$$H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, q\dot{\mathcal{E}}),$$

making that vector space into a graded left module over $\mathbb{H}(G, M, q\mathcal{E})$ and over $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$. Further, $Z_{G\times\mathbb{C}^\times}(y)$ acts naturally on $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, q\dot{\mathcal{E}})$ and on $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$, and those actions factor through the component group $\pi_0(Z_{G\times\mathbb{C}^\times}(y))$.

Theorem 3.1. (see [Lus1, Theorem 8.13] and [AMS2, Theorem 3.2 and §4])

(*a*) *The actions of* $\mathbb{H}(G, M, q\mathcal{E})$ *and* $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$ *on* $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$ *commute.*

(*b*) *As* $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$*-module,* $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$ *is finitely generated and free.*

(*c*) *The action of* $\pi_0(Z_{G\times\mathbb{C}^\times}(y))$ *on* $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$ *commutes with the action of* $\mathbb{H}(G, M, q\mathcal{E})$ *and is semilinear with respect to* $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(y)$.

Proof. Comparing with the references, it only remains to see that in part (b) the module is free. This part ultimately relies on [Lus1, Proposition 7.2], where it is proven that the module is finitely generated and projective. However, that argument actually shows that the module is free. □

Recall from [Lus1, §1.11] that $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$ is the ring of invariant polynomials on the maximal reductive quotient of $\mathrm{Lie}(Z_{G\times\mathbb{C}^\times}(y))$. The characters of that ring are parametrized by the semisimple orbits in the reductive Lie algebra. We let $\mathfrak{g}\oplus\mathbb{C}$ act on $\mathfrak{g}$ by

$$(\sigma, r)\cdot X = [\sigma, X] - 2rX,$$

that is the derivative of the $G\times\mathbb{C}^\times$-action. Then we can write

$$\mathrm{Lie}(Z_{G\times\mathbb{C}^\times}(y)) = \{(\sigma, r)\in\mathfrak{g}\oplus\mathbb{C} : [\sigma, y] = 2ry\} = Z_{\mathfrak{g}\oplus\mathbb{C}}(y). \tag{3.3}$$

Thus every semisimple $(\sigma, r)\in Z_{\mathfrak{g}\oplus\mathbb{C}}(y)$ defines a unique character of $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$, which we denote $\mathbb{C}_{\sigma,r}$. This gives us a family of $\mathbb{H}(G, M, q\mathcal{E})$-modules

$$E_{y,\sigma,r} := \mathbb{C}_{\sigma,r} \underset{H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})}{\otimes} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \qquad \text{for semisimple } (\sigma, r)\in Z_{\mathfrak{g}\oplus\mathbb{C}}(y).$$

It is known from [AMS3, Proposition 1.7] that (when Condition 1.2 holds) $E_{y,\sigma,r}$ admits the central character $((\mathrm{Ad}(G)\sigma - r\sigma_v)\cap\mathfrak{t}, r)$. Via Lemma 2.1 this corresponds to $\mathrm{Ad}(G)(\sigma, r)\cap\Sigma_v(\mathfrak{t}\oplus\mathbb{C})$. Let

$$C_y = Z_C(y) = Z_{G\times\mathbb{C}^\times}(y, \sigma, r)$$

be the intersection of $Z_{G\times\mathbb{C}^\times}(y)$ from (3.1) and $C = Z_{G\times\mathbb{C}^\times}(\sigma, r)$ (with respect to the adjoint action). The component group $\pi_0(C_y)$ acts naturally on $E_{y,\sigma,r}$ by $\mathbb{H}(G, M, q\mathcal{E})$-intertwiners. For $\rho\in\mathrm{Irr}(\pi_0(C_y))$ we form the $\mathbb{H}(G, M, q\mathcal{E})$-module

$$E_{y,\sigma,r,\rho} = \mathrm{Hom}_{\pi_0(C_y)}(\rho, E_{y,\sigma,r}).$$

Choose an algebraic homomorphism $\gamma_y : SL_2(\mathbb{C})\to G^\circ$ with $\mathrm{d}\gamma_y\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) = y$. It is often convenient to involve the element

$$\sigma_0 := \sigma + \mathrm{d}\gamma_y\left(\begin{smallmatrix}-r&0\\0&r\end{smallmatrix}\right)\in Z_{\mathfrak{g}}(y). \tag{3.4}$$

For instance, by [AMS2, Lemma 3.6.a] there are natural isomorphisms

$$\pi_0(C_y)\cong\pi_0(Z_G(y,\sigma))\cong\pi_0(Z_G(y,\sigma_0)) \tag{3.5}$$

It was shown in [AMS2, Proposition 3.7 and §4] that $E_{y,\sigma,r\rho}$ is nonzero if and only if the cuspidal quasi-support $q\Psi_{Z_G(\sigma_0)}(y,\rho)$, for the group $Z_G(\sigma_0)$ and with ρ considered as representation of $\pi_0(Z_G(y,\sigma_0))$ via (3.5), is G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$. Equivalent conditions will be described in Proposition 4.1. For such ρ we call $E_{y,\sigma,r,\rho}$ a standard (geometric) $\mathbb{H}(G, M, q\mathcal{E})$-module.

Theorem 3.2. [AMS2, Theorem 4.6]

(*a*) *For $r \in \mathbb{C}^\times$, every standard $\mathbb{H}(G, M, q\mathcal{E})$-module $E_{y,\sigma,r,\rho}$ has a unique irreducible quotient $M_{y,\sigma,r,\rho}$.*

(*b*) *For $r = 0$, the standard module $E_{y,\sigma,0,\rho}$ has a unique distinguished irreducible summand, called $M_{y,\sigma,0,\rho}$.*

(*c*) *For any $r \in \mathbb{C}$, the correspondence $M_{y,\sigma,r,\rho} \longleftrightarrow (y, \sigma, \rho)$ provides a bijection between*

$$\mathrm{Irr}_r(\mathbb{H}(G, M, q\mathcal{E})) = \mathrm{Irr}(\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r))$$

and the G-association classes of triples (y, σ, ρ) as in (3.3).

3.1. **Relation with the Iwahori–Matsumoto and sign involutions.**

We would like to make the parametrization of $\mathrm{Irr}_r(\mathbb{H}(G, M, q\mathcal{E}))$ from Theorem 3.2 with $r \in \mathbb{R}_{>0}$ compatible with the analytic properties temperedness and (essentially) discrete series. When G is connected, this is worked out in [Lus5]. Unfortunately the outcome is not exactly what we want, it rather produces "anti-tempered" representations where we would like temperedness.

We recall from [AMS2, Lemma 2.1] that $W_{q\mathcal{E}} = W_{q\mathcal{E}}^\circ \rtimes \Gamma_{q\mathcal{E}}$, where $W_{q\mathcal{E}}^\circ$ is the Weyl group of a root system and $\Gamma_{q\mathcal{E}}$ is the stabilizer in $W_{q\mathcal{E}}$ of the set of positive roots. We extend the sign character of $W_{q\mathcal{E}}^\circ$ to $W_{q\mathcal{E}}$ by making it trivial on $\Gamma_{q\mathcal{E}}$. To improve the temperedness properties of standard $\mathbb{H}(G, M, q\mathcal{E})$-modules, one can use the Iwahori–Matsumoto involution, given by

$$\mathrm{IM}(N_w) = \mathrm{sgn}(w) N_w, \quad \mathrm{IM}(\mathbf{r}) = \mathbf{r}, \quad \mathrm{IM}(\xi) = -\xi \qquad w \in W_{q\mathcal{E}}, \xi \in \mathfrak{t}^\vee.$$

Here the sign character of $W_{q\mathcal{E}}$ is by definition trivial on $\Gamma_{q\mathcal{E}}$. Composing a $\mathbb{H}(G, M, q\mathcal{E})$-module with IM changes its $\mathcal{O}(\mathfrak{t})$-weights by a factor -1. To compensate for that, in [AMS2, AMS3] the authors associate to (y, σ, ρ, r) and (y, σ_0, ρ, r) the modules

$$\mathrm{IM}^* E_{y, \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \sigma_0, r, \rho} \qquad \text{and} \qquad \mathrm{IM}^* M_{y, \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \sigma_0, r, \rho}.$$

We note that $(\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \sigma_0, r) \in Z_{\mathfrak{g} \oplus \mathbb{C}}(y)$ and

$$(\mathrm{d}\gamma_y \left(\begin{smallmatrix} -r & 0 \\ 0 & r \end{smallmatrix}\right) - \sigma_0, -r) = (-\sigma, -r) \in Z_{\mathfrak{g} \oplus \mathbb{C}}(y).$$

The Iwahori–Matsumoto involution commutes with the sign automorphism:

$$\begin{aligned} & \mathrm{sgn}(N_w) = \mathrm{sgn}(w) N_w, \quad \mathrm{sgn}(\mathbf{r}) = -\mathbf{r}, \quad \mathrm{sgn}|_{\mathcal{O}(\mathfrak{t})} = \mathrm{id}_{\mathcal{O}(\mathfrak{t})}, \\ & \mathrm{sgn} \circ \mathrm{IM} = \mathrm{IM} \circ \mathrm{sgn} : \ \mathbb{H}(G, M, q\mathcal{E}) \to \mathbb{H}(G, M, q\mathcal{E}), \\ & \mathrm{sgn} \circ \mathrm{IM}(N_w) = N_w, \quad \mathrm{sgn} \circ \mathrm{IM}(\mathbf{r}) = -\mathbf{r}, \quad \mathrm{sgn} \circ \mathrm{IM}(\xi) = -\xi \qquad w \in W_{q\mathcal{E}}, \xi \in \mathfrak{t}^\vee. \end{aligned}$$

Composing with the involution $\mathrm{sgn} \circ \mathrm{IM}$ has an easy effect on standard modules:

Proposition 3.3. *Let $E_{y,\sigma,r,\rho}$ be a geometric standard $\mathbb{H}(G, M, q\mathcal{E})$-module, as in Theorem 3.2. There are canonical isomorphisms of $\mathbb{H}(G, M, q\mathcal{E})$-modules*

(*a*) $\mathrm{sgn}^* \mathrm{IM}^* E_{y,\sigma,r} \cong E_{y,-\sigma,-r}$,

(*b*) $\mathrm{sgn}^* \mathrm{IM}^* E_{y,\sigma,r,\rho} \cong E_{y,-\sigma,-r,\rho}$,

(*c*) $\mathrm{sgn}^* \mathrm{IM}^* M_{y,\sigma,r,\rho} \cong M_{y,-\sigma,-r,\rho}$.

Proof. (a) By [Lus1, Proposition 7.2] there is a natural isomorphism of $H^*_{Z^\circ_{G \times \mathbb{C}^\times}(y)}(\mathrm{pt})$-modules

$$H_*^{Z^\circ_{G \times \mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \cong H^*_{Z^\circ_{G \times \mathbb{C}^\times}(y)}(\mathrm{pt}) \otimes H_*(\mathcal{P}_y, \dot{q\mathcal{E}})$$

That gives a linear bijection

$$E_{y,\sigma,r} = \mathbb{C}_{\sigma,r} \underset{H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})}{\otimes} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \longrightarrow H_*(\mathcal{P}_y, \dot{q\mathcal{E}}). \tag{3.6}$$

The construction of the $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$-action on $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$ from [Lus1, p.193] comes from its action on K and can also be performed without the $Z^\circ_{G\times\mathbb{C}^\times}(y)$-equivariance. That renders (3.6) $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$-equivariant. We can do the same with $(y, -\sigma, -r)$, that gives a $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$-linear bijection $E_{y,-\sigma,-r} \to H_*(\mathcal{P}_y, \dot{q\mathcal{E}})$. Composing that with the inverse of (3.6), we obtain a natural isomorphism of $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$-modules

$$E_{y,-\sigma,-r} \to E_{y,\sigma,r}. \tag{3.7}$$

Via (3.6) we transfer the $\mathbb{H}(G, M, q\mathcal{E})$-module structure of $E_{y,-\sigma,-r}$ to $E_{y,\sigma,r}$, and we call the new module $E'_{y,\sigma,r}$. The action of $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ on $E_{y,\sigma,r}$ comes from the natural maps

$$\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}) \cong H^*_{G\times\mathbb{C}^\times}(\dot{\mathfrak{g}}) \to H^*_{Z_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y) \cong H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y)^{\pi_0(Z_{G\times\mathbb{C}^\times}(y))}$$

and the product

$$H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y) \otimes_{\mathbb{C}} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \longrightarrow H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}). \tag{3.8}$$

Similarly the $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(y)$-action comes from $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(y) \to H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y)$ and (3.8). It follows that (3.7) modifies the action of $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ by a factor -1 on $\mathfrak{t}^\vee$ and on $\mathbf{r}$. In other words, $E'_{y,\sigma,r} = \mathrm{sgn}^* \mathrm{IM}^* E_{y,\sigma,r}$ as $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$-modules. We already knew that for the $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$-action, so (3.7) induces the desired isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules.

(b) To enhance this picture with a ρ, we need more precise information. Recall from Theorem 3.1.b that $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$ is finitely generated and free over $H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$. The action map

$$H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt}) \otimes_{\mathbb{C}} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \to H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \tag{3.9}$$

respects degrees and $H^0_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt}) = \mathbb{C}$ while $H^n_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt}) = 0$ for $n < 0$. Let E_y be the unique complement to

$$H^{>0}_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt}) \otimes_{\mathbb{C}} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \quad \text{in} \quad H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$$

which is spanned by homogeneous elements (i.e. elements that live in one degree). Then (3.9) restricts to a linear bijection

$$H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt}) \otimes_{\mathbb{C}} E_y \to H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}). \tag{3.10}$$

Now (3.6) sends $\mathbb{C}_{\sigma,r} \otimes_{\mathbb{C}} E_y$ bijectively to $H_*(\mathcal{P}_y, \dot{q\mathcal{E}})$ for any (σ, r). In this way we can regard E_y as a canonical copy of $H_*(\mathcal{P}_y, \dot{q\mathcal{E}})$ in $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$.

The group C_y acts naturally on $H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}})$ and that induces the action of $\pi_0(C_y) \cong \pi_0(Z_G(y, \sigma))$ on $E_{y,\sigma,r}$ and on $E_{y,-\sigma,-r}$. The C_y-actions preserve the degrees in (3.9) and are the identity on $H^0_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathrm{pt})$. Hence they restrict to an action of C_y on E_y. Exactly the same action on E_y is obtained if we start with $(-\sigma,-r)$ instead of (σ,r).

Furthermore the isomorphism of $\mathbb{H}(G,M,q\mathcal{E})$-modules from part (a) factors as

$$E_{y,\sigma,r} \to E_y \to E_{y,-\sigma,-r},$$

so part (a) is also C_y-equivariant. In particular, for any $\rho \in \mathrm{Irr}(C_y)$, part (a) induces isomorphisms of $\mathbb{H}(G,M,q\mathcal{E})$-modules

(3.11)

$$\mathrm{sgn}^*\mathrm{IM}^* E_{y,\sigma,r,\rho} = \mathrm{Hom}_{C_y}(\rho, \mathrm{sgn}^*\mathrm{IM}^* E_{y,\sigma,r}) \cong \mathrm{Hom}_{C_y}(\rho, E_{y,-\sigma,-r}) = E_{y,-\sigma,-r,\rho}.$$

(c) When $r \neq 0$, (3.11) sends the unique irreducible quotient $\mathrm{sgn}^*\mathrm{IM}^* M_{y,\sigma,r,\rho}$ on the left to the unique irreducible quotient $M_{y,-\sigma,-r,\rho}$ on the right. When $r = 0$, the distinguished irreducible summand $\mathrm{sgn}^*\mathrm{IM}^* M_{y,\sigma,0,\rho}$ of $\mathrm{sgn}^*\mathrm{IM}^* E_{y,\sigma,0,\rho}$ is in the component of E_y in one particular homological degree [AMS2, Lemma 3.10 and Theorem 3.20]. As part (a) factors via E_y, it preserves these homological degrees. Hence it sends $\mathrm{sgn}^*\mathrm{IM}^* M_{y,\sigma,0,\rho}$ to the distinguished irreducible summand $M_{y,-\sigma,0,\rho}$ of $E_{y,-\sigma,0,\rho}$. □

With Proposition 3.3 at hand, we can reformulate the results of [AMS2] that use the Iwahori–Matsumoto involution in terms of the sign involution of $\mathbb{H}(G,M,q\mathcal{E})$. In particular we see that the module $\mathrm{sgn}^* E_{y,\sigma,-r,\rho}$ is isomorphic to $\mathrm{IM}^* E_{y,-\sigma,r,\rho}$. In [AMS2] the latter module was associated to the data

$$(y,\sigma_0,r,\rho) \quad \text{and} \quad (y, \sigma_0 + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right), r, \rho).$$

Similar statements holds without ρ and with E replaced by M. This is the class of modules that is standard in the analytic sense related to the Langlands classification, see [Sol1, §3.5]. To distinguish them from the earlier geometric standard modules, we refer to $\mathrm{sgn}^* E_{y,\sigma,-r,\rho}$ as an analytic standard $\mathbb{H}(G,M,q\mathcal{E})$-module.

Using the sign automorphism we can vary on [AMS2, Theorem 4.6]. Fix $r \in \mathbb{C}$ and consider triples (y,σ,ρ) such that:

- $y \in \mathfrak{g}$ is nilpotent,
- $\sigma \in \mathfrak{g}$ is semisimple and $[\sigma, y] = -2ry$,
- $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma,y))\big)$ and $q\Psi_{Z_G(\sigma_0)}(y,\rho) = (M, \mathcal{C}_v^M, q\mathcal{E})$ up to G-conjugacy.

By Theorem 3.2.c the map

$$(\sigma,y,\rho) \mapsto \mathrm{sgn}^*(M_{y,\sigma,-r,\rho}) \tag{3.12}$$

defines a bijection between the set of G-conjugacy classes of triples (y,σ,ρ) as above and $\mathrm{Irr}_r(\mathbb{H}(G,M,q\mathcal{E}))$. Notice that the central character of $\mathrm{sgn}^*(M_{y,\sigma,-r,\rho})$ is $(\sigma + r\sigma_v.r)$ when $\sigma \in \mathfrak{t}_{-r}$. This constitutes an improvement on [AMS2, §3.5] because our new parametrization of $\mathrm{Irr}_r(\mathbb{H}(G,M,q\mathcal{E}))$ has all the desired properties with respect to temperedness (see below) and is more natural – we do not have to involve $\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$ any more.

Theorem 3.4. [AMS2, Theorem 3.25, Theorem 3.26 and §4]
Consider an analytic standard $\mathbb{H}(G,M,q\mathcal{E})$*-module* $\mathrm{sgn}^* E_{y,\sigma,-r,\rho}$.

(a) *Suppose that* $\Re(r) \geq 0$. *The* $\mathbb{H}(G,M,q\mathcal{E})$*-modules* $\mathrm{sgn}^*(E_{y,\sigma,-r,\rho})$ *and* $\mathrm{sgn}^*(M_{y,\sigma,-r,\rho})$ *are tempered if and only if* σ_0 *lies in* $i\mathfrak{t}_\mathbb{R} = i\mathbb{R} \otimes_\mathbb{Z} X_*(T)$.
Here $\sigma_0 = \sigma + \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$ *is as in* (3.4), *but with* $-r$ *instead of* r.

(*b*) *Suppose that* $\Re(r) > 0$. *Then* $\mathrm{sgn}^*(E_{y,\sigma,-r,\rho})$ *and* $\mathrm{sgn}^*(M_{y,\sigma,-r,\rho})$ *are essentially discrete series if and only if* y *is distinguished in* $\mathfrak{g}$.

Moreover, when these conditions are fulfilled

$$\mathrm{sgn}^*(E_{y,\sigma,-r,\rho}) = \mathrm{sgn}^*(M_{y,\sigma,-r,\rho}) \in \mathrm{Irr}_r(\mathbb{H}(G,M,q\mathcal{E})).$$

In terms of Theorems 3.2 and 3.4, the bijection from [Sol6, Theorem 1.5] becomes

$$\begin{array}{ccc} \mathrm{Irr}_r(\mathbb{H}(G,M,q\mathcal{E})) & \longrightarrow & \mathrm{Irr}_0(\mathbb{H}(G,M,q\mathcal{E})) \\ \mathrm{sgn}^*(M_{y,\sigma,-r,\rho}) & \mapsto & \mathrm{sgn}^*(M_{y,\sigma,0,\rho}) \end{array} . \tag{3.13}$$

We would like to analyse the right $\mathbb{H}(G,M,q\mathcal{E})$-modules from Theorem 2.4 as left modules over the opposite algebra $\mathbb{H}(G,M,q\mathcal{E})^{op}$. This opposite algebra is easily identified via the isomorphism

$$\begin{array}{ccc} \mathbb{H}(G,M,q\mathcal{E})^{op} & \xrightarrow{\sim} & \mathbb{H}(G,M,q\mathcal{E}^\vee) \\ N_w \xi & \mapsto & \xi (N_w)^{-1} \end{array} \qquad w \in W_{q\mathcal{E}}, \xi \in \mathcal{O}(\mathfrak{t} \oplus \mathbb{C}), \tag{3.14}$$

see [AMS2, (14)]. That gives an equivalence of categories

$$\mathbb{H}(G,M,q\mathcal{E}) - \mathrm{Mod} \cong \mathrm{Mod} - \mathbb{H}(G,M,q\mathcal{E}^\vee). \tag{3.15}$$

The dual local system $q\mathcal{E}^\vee$ on $\mathcal{C}_v^M$ is also cuspidal, so all the previous results hold just as well for $\mathbb{H}(G,M,q\mathcal{E}^\vee)$. In particular we have a complete classification of its irreducible and its standard left modules.

3.2. Construction from $K_{\sigma,r}$ or $K_{N,\sigma,r}$.

We want to relate the standard modules of $\mathbb{H}(G,M,q\mathcal{E})$ (or its opposite) to Theorem 2.4. The vector spaces $H^*(\{y\}, i_y^! K_{\sigma,r})$ and $H^*(\{y\}, i_y^* K_{\sigma,r})$ become left $\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r})$-modules via the natural algebra homomorphism

$$\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})}(K_{\sigma,r}) \to \big(\mathrm{End}^*_{\mathcal{D}^b_{C_y^\circ}(\{y\})}(i_y^{!/*} K_{\sigma,r})\big)^{\pi_0(C_y)} \to \mathrm{End}^*_{\mathcal{D}^b(\{y\})}(i_y^{!/*} K_{\sigma,r}), \tag{3.16}$$

which specializes $H^*_{C_y^\circ}(\{y\})$ at (σ,r), see [Lus3, §10.2]. Via Theorem 2.4.b, $H^*(i_y^! K_{\sigma,r})$ and $H^*(i_y^* K_{\sigma,r})$ also become left $\mathbb{H}(G,M,q\mathcal{E})$-modules. By [Lus3, §10.4], and as in Proposition 4.1, they carry natural actions of $\pi_0(C_y)$, which commute with the $\mathbb{H}(G,M,q\mathcal{E})$-actions. The modules $H^*(\{y\}, i_y^! K_{\sigma,r})$ and $H^*(\{y\}, i_y^* K_{\sigma,r})$ are annihilated by $\mathbf{r} - r$, so they descend to $\mathbb{H}(G,M,q\mathcal{E},r)$-modules. From Theorem 2.7 we see that the action of $\mathbb{H}(G,M,q\mathcal{E},r)$ can also be constructed more directly, as in (3.16) with $Z_G(\sigma)$ instead of $C = Z_G(\sigma) \times \mathbb{C}^\times$.

Let $K^\vee_{\sigma,r} \in \mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})$ be the analogue of $K_{\sigma,r}$, but constructed from $q\mathcal{E}^\vee$.

Proposition 3.5. *Assume Condition 1.2 and denote the subvariety of* $\exp(\sigma)$*-fixed points in* $\mathcal{P}_y$ *by* $\mathcal{P}_y^\sigma$.

(*a*) *There are natural isomorphisms of* $\mathbb{H}(G,M,q\mathcal{E}) \times \pi_0(C_y)$*-representations*

$$\begin{array}{ccccc} H^*(\{y\}, i_y^! K_{\sigma,r}) & \cong & H_*(\mathcal{P}_y^\sigma, \dot{q\mathcal{E}}) & \cong & E_{y,\sigma,r}, \\ H^*(\{y\}, i_y^* K_{\sigma,r}) & \cong & \mathbb{C}_{\sigma,r} \underset{H^*_{C_y^\circ}(\{y\})}{\otimes} H^*_{C_y^\circ}(\mathcal{P}_y^\sigma, \dot{q\mathcal{E}}). & & \end{array}$$

(*b*) *There are natural isomorphisms of* $\mathbb{H}(G,M,q\mathcal{E}^\vee) \times \pi_0(C_y)$*-representations*

$$\begin{array}{ccccc} H^*(\{y\}, i_y^! K_{\sigma,r})^\vee & \cong & H^*(\{y\}, i_y^* K^\vee_{\sigma,r}) & \cong & E^\vee_{y,\sigma,r}, \\ H^*(\{y\}, i_y^* K_{\sigma,r})^\vee & \cong & H^*(\{y\}, i_y^! K^\vee_{\sigma,r}) & \cong & E_{y,\sigma,r}, \end{array}$$

not necessarily preserving the gradings.

(c) Parts (a) and (b) are also valid with $(\mathfrak{g}_N^{\sigma,r}, K_{N,\sigma,r})$ instead of $(\mathfrak{g}^{\sigma,r}, K_{\sigma,r})$.

Proof. (a) When G is connected, the isomorphisms with $H^*(\{y\}, i_y^! K_{\sigma,r})$ are shown in [Lus3, Proposition 10.12]. We generalize those arguments to our setting. Consider the pullback diagram

$$\begin{array}{ccc} \mathcal{P}_y^\sigma & \xrightarrow{k} & \dot{\mathfrak{g}}^{\sigma,r} \\ \downarrow \pi & & \downarrow \mathrm{pr}_1 \\ \{y\} & \xrightarrow{i_y} & \mathfrak{g}^{\sigma,r} \end{array}$$

where $k(gP) = (y, gP)$. From general results about derived sheaves, [BeLu, §1.4.6 and Theorem 1.8.ii], extended to the equivariant derived category in [BeLu, Theorem 3.4.3], it is known that

$$i_y^* \mathrm{pr}_{1,*} = \pi_* k^* \qquad \text{and} \qquad i_y^! \mathrm{pr}_{1,*} = \pi_* k^! \tag{3.17}$$

as functors $\mathcal{D}^b_C(\dot{\mathfrak{g}}^{\sigma,r}) \to \mathcal{D}^b_{C_y}(\{y\})$. With that we compute

$$\begin{aligned} H^*(\{y\}, i_y^! K_{\sigma,r}) &\cong H^*(\{y\}, i_y^! \mathrm{pr}_{1,*} q\dot{\mathcal{E}}) \cong H^*(\{y\}, \pi_* k^! q\dot{\mathcal{E}}) \\ &\cong H^*(\mathcal{P}_y^\sigma, k^! q\dot{\mathcal{E}}) \cong H^*(\mathcal{P}_y^\sigma, Dk^* Dq\dot{\mathcal{E}}) \\ &\cong H^*(\mathcal{P}_y^\sigma, Dk^* q\dot{\mathcal{E}}^\vee) = H^*(\mathcal{P}_y^\sigma, Dq\dot{\mathcal{E}}^\vee) = H_*(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}). \end{aligned}$$

The last part of the proof of [Lus3, Proposition 10.12] shows that

$$H_*(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}) \cong \mathbb{C}_{\sigma,r} \underset{H^*_{C_y^\circ}(\{y\})}{\otimes} H_*^{C_y^\circ}(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}) \cong \mathbb{C}_{\sigma,r} \underset{H^*_{C_y^\circ}(\{y\})}{\otimes} H_*^{C_y^\circ}(\mathcal{P}_y, q\dot{\mathcal{E}}) = E_{y,\sigma,r}$$

as $\mathbb{H}(G, M, q\mathcal{E}^\vee) \times \pi_0(C_y)$-representations. Similarly we use (3.17) to compute

$$\begin{aligned} H^*(\{y\}, i_y^* K_{\sigma,r}) &\cong H^*(\{y\}, i_y^* \mathrm{pr}_{1,*} q\dot{\mathcal{E}}) \cong H^*(\{y\}, \pi_* k^* q\dot{\mathcal{E}}) \\ &\cong H^*(\{y\}, \pi_* q\dot{\mathcal{E}}) \cong H^*(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}). \end{aligned}$$

Notice that $\mathcal{P}_y^\sigma$ is compact, so cohomology coincides with compactly supported cohomology here. The last part of the proof of [Lus3, Proposition 10.12] also shows that there is an isomorphism of $\mathbb{H}(G, M, q\mathcal{E}^\vee) \times \pi_0(C_y)$-representations

$$H^*(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}) \cong \mathbb{C}_{\sigma,r} \underset{H^*_{C_y^\circ}(\{y\})}{\otimes} H^*_{C_y^\circ}(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}) \cong \mathbb{C}_{\sigma,r} \underset{H^*_{C_y^\circ}(\{y\})}{\otimes} H^*_{C_y^\circ}(\mathcal{P}_y, q\dot{\mathcal{E}}).$$

(b) By (2.11), (2.12) and the properties of Verdier duality [Ach, §2.8 and Lemma 3.3.13] there are natural isomorphisms (which may shift the gradings by different amounts on different simple summands)

$$\begin{aligned} DK_{\sigma,r} &= \mathrm{pr}_{1,*} D\mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}, q\dot{\mathcal{E}}) \\ &\cong \mathrm{pr}_{1,!}\mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}, Dq\dot{\mathcal{E}}) \\ &\cong \mathrm{pr}_{1,!}\mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}, q\dot{\mathcal{E}}^\vee) = K^\vee_{\sigma,r}. \end{aligned} \tag{3.18}$$

From part (a) and (3.18) we see that

$$\begin{aligned} H^*(\{y\}, i_y^! K_{\sigma,r})^\vee &\cong H^{-*}(\{y\}, D i_y^! K_{\sigma,r}) \cong H^{-*}(\{y\}, i_y^* DK_{\sigma,r}) \\ &\cong H^{-*}(\{y\}, i_y^* K^\vee_{\sigma,r}) \cong H^{-*}(\{y\}, i_y^* K^\vee_{\sigma,r}). \end{aligned}$$

Here $-*$ means that initially the grading is reversed (by $\vee$), while in the last line the grading must also be adjusted to account for (3.18). Similarly there is a natural vector space isomorphism

$$H^*(\{y\}, i_y^* K_{\sigma,r})^\vee \cong H^{-*}(\{y\}, i_y^! K_{\sigma,r}^\vee).$$

The $\mathbb{H}(G, M, q\mathcal{E}) \times \pi_0(C_y)$-actions in part (a) become actions of $\mathbb{H}(G, M, q\mathcal{E})^{op} \cong \mathbb{H}(G, M, q\mathcal{E}^\vee)$ and of $\pi_0(C_y)$ upon taking vector space duals. We can reformulate the isomorphisms from part (a) as

$$\begin{array}{lll} H_*(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}})^\vee & \cong & H^{-*}(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}^\vee), \\ H^*(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}})^\vee & \cong & H_{-*}(\mathcal{P}_y^\sigma, q\dot{\mathcal{E}}^\vee). \end{array} \tag{3.19}$$

Using the explicit description of the actions given in [Sol6, §2.1] and in [AMS2], one checks readily that in (3.19) we have isomorphisms of $\mathbb{H}(G, M, q\mathcal{E}^\vee) \times \pi_0(C_y)$-representations.
(c) This can be shown in the same way as parts (a) and (b). □

With Proposition 3.5 we can henceforth interpret the geometric standard $\mathbb{H}(G, M, q\mathcal{E})$-module $E_{y,\sigma,r,\rho}$ as

$$\mathrm{Hom}_{\pi_0(C_y)}\big(\rho, H^*(\{y\}, i_y^! K_{N,\sigma,r})\big) \cong \mathbb{C}_{\sigma,r} \underset{H^*_{C_y^\circ}(\{y\})}{\otimes} \mathrm{Hom}^*_{\mathcal{D}^b_{C_y}(\{y\})}\big(\rho, i_y^! K_{N,\sigma,r}\big). \tag{3.20}$$

With Theorem 2.7 we can reformulate (3.20) as an isomorphism of $\mathbb{H}(G, M, q\mathcal{E}, r)$-modules:

$$E_{y,\sigma,r,\rho} \cong \mathbb{C}_\sigma \underset{H^*_{Z_G^\circ(\sigma)}(\{y\})}{\otimes} \mathrm{Hom}^*_{\mathcal{D}^b_{Z_G(\sigma,y)}(\{y\})}\big(\rho, i_y^! K_{N,\sigma,r}\big).$$

4. Structure of the localized complexes $K_{\sigma,r}$ and $K_{N,\sigma,r}$

With $K_{\sigma,r}$ and $K_{\sigma,r}^\vee$ we can give an alternative interpretation of the cuspidal quasi-supports involved in standard modules (see in particular Theorem 3.2.c).

Proposition 4.1. *Fix a nilpotent $y \in \mathfrak{g}^{\sigma,r}$ and let $i_y : \{y\} \to \mathfrak{g}^{\sigma,r}$ be the inclusion. For $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(y, \sigma_0))\big) = \mathrm{Irr}(\pi_0(C_y))$, the following are equivalent:*
(i) the cuspidal quasi-support $q\Psi_{Z_G(\sigma_0)}(y, \rho)$, with respect to the group $Z_G(\sigma_0)$, is G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$,
(ii) $\mathrm{Hom}_{\pi_0(C_y)}\big(\rho, H^*(\{y\}, i_y^! K_{\sigma,r})\big) \neq 0$,
(iii) $\mathrm{Hom}_{\pi_0(C_y)}\big(H^*(\{y\}, i_y^* K_{\sigma,r}), \rho\big) \neq 0$,
(iv) $\mathrm{Hom}_{\pi_0(C_y)}\big(\rho, H^*(\{y\}, i_y^! K_{N,\sigma,r})\big) \neq 0$,
(v) $\mathrm{Hom}_{\pi_0(C_y)}\big(H^*(\{y\}, i_y^* K_{N,\sigma,r}), \rho\big) \neq 0$.

Proof. Recall from Proposition 3.5.a that

$$H^*(\{y\}, i_y^! K_{\sigma,r}) \cong E_{y,\sigma,r}.$$

With that in mind, the equivalence of (i) and (ii) is shown in [AMS2, Proposition 3.7] when G is connected. With [AMS2, §4] those arguments can be extended to disconnected G and cuspidal *quasi*-supports. The equivalence of (ii) and (iii) follows from Proposition 3.5.b. By Proposition 3.5.(a,c)

$$H^*(\{y\}, i_y^! K_{\sigma,r}) \cong H^*(\{y\}, i_y^! K_{N,\sigma,r}) \quad \text{and} \quad H^*(\{y\}, i_y^* K_{\sigma,r}) \cong H^*(\{y\}, i_y^* K_{N,\sigma,r})$$

as $\mathbb{H}(G, M, q\mathcal{E}) \times \pi_0(C_y)$-representations. That proves the equivalence of (ii) with (iv) and of (iii) with (v). □

Cuspidal quasi-supports were defined [AMS1, §5], in relation with a Springer correspondence for disconnected reductive groups. In our context, it is more convenient to use the property (ii) or (iii) in Proposition 4.1: for a given triple (y, σ, ρ) that determines $(M, \mathcal{C}_v^M, q\mathcal{E})$ up to G-conjugacy.

In the opposite direction, Proposition 4.1 almost determines the semisimple complex $K_{\sigma,r}$. To work this out, let $\mathcal{O}_y = \mathrm{Ad}(C)y \subset \mathfrak{g}^{\sigma,r}$ be the C-orbit of y. Regarding ρ as a C_y-equivariant sheaf on $\{y\}$ and invoking the equivalence of categories

$$\mathrm{ind}_{C_y}^C : \mathcal{D}^b_{C_y}(\{y\}) \xrightarrow{\sim} \mathcal{D}^b_C(\mathcal{O}_y), \tag{4.1}$$

we obtain a C-equivariant local system $\mathrm{ind}_{C_y}^C(\rho)$ on $\mathcal{O}_y$. We form the equivariant intersection cohomology complex $\mathrm{IC}_C\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho)\big) \in \mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})$, which is supported on $\overline{\mathcal{O}_y}$. This is the usual intersection cohomology complex $\mathrm{IC}\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho)\big)$, only now considered with its C-equivariant structure.

Theorem 4.2. (*a*) *Fix $r \in \mathbb{C}^\times$. Every simple direct summand of $K_{\sigma,r}$ is isomorphic to $\mathrm{IC}_C\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho)\big)$, for data (y, σ, ρ) that fulfill the conditions in Proposition 4.1. Conversely, every such equivariant intersection cohomology complex is a direct summand of $K_{\sigma,r}$ (with multiplicity ≥ 1).*

(*b*) *For arbitrary $r \in \mathbb{C}$, part (a) becomes valid when we replace all involved sheaves by their versions for $\mathfrak{g}_N^{\sigma,r}$.*

Proof. (a) From [AMS2, (95)] we see that, as M°-equivariant local system on $\mathcal{C}_v^M = \mathcal{C}_v^{M^\circ}$, $q\mathcal{E}$ is a direct sum of M-conjugates of $\mathcal{E}$ and $(q\mathcal{E})_v \cong \mathcal{E}_v \rtimes \rho_M$ for a suitable representation ρ_M. Hence $q\mathcal{E} \in \mathcal{D}^b_{G^\circ}(\dot{\mathfrak{g}})$ is a direct sum of G-conjugates of $\dot{\mathcal{E}} \in \mathcal{D}^b_{G^\circ}(\dot{\mathfrak{g}}^\circ)$.

Then the diagram (2.10) shows that, as element of $\mathcal{D}^b_{Z_G^\circ(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})$, $K_{\sigma,r}$ is a direct sum of $Z_G(\sigma)$-conjugates of $K^\circ_{\sigma,r}$ – the version of $K_{\sigma,r}$ for $(G^\circ, M^\circ, \mathcal{E})$. By [Lus3, §5.3], $K^\circ_{\sigma,r}$ is a semisimple complex of sheaves. Further [Lus3, Proposition 8.17] (for which we need $r \neq 0$) and Proposition 4.1 entail that the simple direct summands of $K^\circ_{\sigma,r}$ are the $Z_G^\circ(\sigma) \times \mathbb{C}^\times$-equivariant intersection cohomology complexes

$$\mathrm{IC}_{C^\circ}\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{Z_{G^\circ}(\sigma,y)}^{C^\circ}(\rho^\circ)\big) \quad \text{with} \quad \mathrm{Hom}_{\pi_0(C_y)}(H^*(\{y\}, i_y^* K^\circ_{\sigma,r}), \rho^\circ) \neq 0. \tag{4.2}$$

More precisely, every such summand appears with a multiplicity ≥ 1. Then $K_{\sigma,r}$ is a direct sum of terms

$$\mathrm{IC}_{C^\circ}\big(\mathfrak{g}^{\sigma,r}, \mathrm{Ad}(g)^* \mathrm{ind}_{Z_{G^\circ}(\sigma,y)}^{C^\circ}(\rho^\circ)\big),$$

where $g \in Z_G(\sigma)$ and (y, ρ) are as in (4.2). Again every such summand appears with multiplicity ≥ 1 in $K_{\sigma,r}$.

On the other hand, we already knew that $K_{\sigma,r}$ is a C-equivariant semisimple complex of sheaves. We deduce that $K_{\sigma,r}$ is a direct sum of terms $\mathrm{IC}_C\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho')\big)$, where $\rho' \in \mathrm{Irr}(\pi_0(C_y))$ contains some ρ° as before. That settles the geometric structure of $K_{\sigma,r}$, it remains to identify exactly which ρ' occur.

The above works equally well with the group $G^\circ M$ instead of G. Let us assume that $\sigma_0, \sigma - r\sigma_v \in \mathfrak{t}$, as we may by [AMS3, Proposition 1.7.c]. Then [AMS2, Lemma 4.4] says that every ρ° as in (4.2) corresponds to a unique

$$\rho^\circ \rtimes \rho_M \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ M}(\sigma_0, y))\big)$$

with $q\Psi_{Z^\circ_G(\sigma_0)M}(y, \rho^\circ \rtimes \rho_M)$ conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$ – see also (B.12). A direct comparison of the constructions of $K^\circ_{\sigma,r}$ and of $K_{\sigma,r}$ for $G^\circ M$ shows that the latter equals the direct sum of the complexes

$$\mathrm{IC}_{Z_{G^\circ M}(\sigma)\times\mathbb{C}^\times}\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^{C}(\rho^\circ \rtimes \rho_M)\big),$$

with the same multiplicities as for $K^\circ_{\sigma,r}$.

The step from $K_{\sigma,r}$ for $G^\circ M$ to $K_{\sigma,r}$ for G is just induction, compare with (B.10). This induction preserves the cuspidal quasi-supports (for G) from [AMS1, §5], because those are based on what happens for objects coming from $G^\circ M$ (when this support comes from M). We conclude that $K_{\sigma,r}$ (for G) is a direct sum of terms

$$\mathrm{IC}_C\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^{C}\big(\mathrm{ind}_{C_y\cap G^\circ M}^{C_y}(\rho^\circ \rtimes \rho_M)\big)\big),$$

with multiplicities coming from (4.2). In particular $K_{\sigma,r}$ is also a direct sum of (degree shifts of) simple perverse sheaves

$$\mathrm{IC}_C\big(\mathfrak{g}^{\sigma,r}, \mathrm{ind}_{C_y}^{C}(\rho)\big) \quad \text{where} \quad q\Psi_{Z_G(\sigma_0)}(y,\rho) = [M, \mathcal{C}_v^M, q\mathcal{E}]_G. \tag{4.3}$$

By Frobenius reciprocity, applied to $\mathrm{ind}_{C_y\cap G^\circ M}^{C_y}(\rho^\circ \rtimes \rho_M)$, every term (4.3) appears with a multiplicity ≥ 1 in $K_{\sigma,r}$.
(b) This can be shown in the same way, if we the replace the crucial input from [Lus3, Proposition 8.17] by [Lus3, §9.5]. □

With the complexes $K_{\sigma,r}$, we can construct standard modules in yet another way.

Lemma 4.3. *Assume that (y,ρ) fulfills the equivalent conditions in Proposition 4.1 and let $j : \mathcal{O}_y \to \mathfrak{g}^{\sigma,r}$ be the inclusion.*

(*a*) *There are natural isomorphisms of $\mathbb{H}(G, M, q\mathcal{E}^\vee)$-modules*

$$\mathrm{Hom}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})}\big(K_{\sigma,r}, j_*\mathrm{ind}_{C_y}^{C}(\rho)\big) \cong \big(H^*_{C_y^\circ}(\{y\}) \otimes_{\mathbb{C}} H^*((i_y^* K_{\sigma,r})^\vee \otimes \rho)\big)^{\pi_0(C_y)}$$

$$\mathbb{C}_{\sigma,r} \underset{H^*_C(\mathrm{pt})}{\otimes} \mathrm{Hom}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})}\big(K_{\sigma,r}, j_*\mathrm{ind}_{C_y}^{C}(\rho)\big) \cong E_{y,\sigma,r,\rho^\vee}.$$

The former is an isomorphism of graded modules.

(*b*) *The isomorphisms from part (a) are also valid for $K_{N,\sigma,r}$ and the inclusion $j_N : \mathcal{O}_y \to \mathfrak{g}_N^{\sigma,r}$.*

Proof. (a) By adjunction and (4.1) there are natural isomorphisms

$$\begin{aligned} \mathrm{Hom}^*_{\mathcal{D}^b_C(\mathfrak{g}^{\sigma,r})}\big(K_{\sigma,r}, j_*\mathrm{ind}_{C_y}^{C}(\rho)\big) &\cong \mathrm{Hom}^*_{\mathcal{D}^b_C(\mathcal{O}_y)}\big(j^*K_{\sigma,r}, \mathrm{ind}_{C_y}^{C}(\rho)\big) \\ &\cong \mathrm{Hom}^*_{\mathcal{D}^b_{C_y}(\{y\})}(i_y^* K_{\sigma,r}, \rho). \end{aligned} \tag{4.4}$$

With [Lus3, §1.10] it can be rewritten as

$$\begin{aligned} \big(\mathrm{Hom}^*_{\mathcal{D}^b_{C_y^\circ}(\{y\})}(i_y^* K_{\sigma,r}, \rho)\big)^{\pi_0(C_y)} &\cong \big(H^*_{C_y^\circ}(\{y\}, D i_y^* K_{\sigma,r} \otimes_{\mathbb{C}} \rho)\big)^{\pi_0(C_y)} \\ &\cong \big(H^*_{C_y^\circ}(\{y\}, D i_y^* K_{\sigma,r}) \otimes_{\mathbb{C}} \rho\big)^{\pi_0(C_y)}. \end{aligned} \tag{4.5}$$

By [Lus3, §1.21], (4.5) is isomorphic with

$$\big(H^*_{C_y^\circ}(\{y\}) \otimes_{\mathbb{C}} H^*(\{y\}, D i_y^* K_{\sigma,r}) \otimes \rho\big)^{\pi_0(C_y)}, \tag{4.6}$$

which gives the first isomorphism of the statement.

Since C_y acts trivially on $\mathbb{C}_{\sigma,r}$, we can tensor the isomorphisms (4.4)–(4.6) with $\mathbb{C}_{\sigma,r}$ over $H^*_{C_y^\circ}(\{y\})$. That preserves the structure as left $\mathbb{H}(G, M, q\mathcal{E}^\vee)$-module or right $\mathbb{H}(G, M, q\mathcal{E})$-module, but it destroys the grading unless $(\sigma, r) = (0, 0)$. It does not matter whether we tensor with $\mathbb{C}_{\sigma,r}$ before or after taking $\pi_0(C_y)$-invariants. Thus it transforms (4.6) into

$$\big(H^*(\{y\}, Di_y^* K_{\sigma,r}) \otimes \rho\big)^{\pi_0(C_y)} \cong \big(H^{-*}(\{y\}, i_y^* K_{\sigma,r})^\vee \otimes \rho\big)^{\pi_0(C_y)}.$$

By Proposition 3.5 that is isomorphic with

$$\big(H^*(\{y\}, i_y^! K_{\sigma,r}^\vee)[\dim_{\mathbb{R}} \mathfrak{g}^{\sigma,r}] \otimes \rho\big)^{\pi_0(C_y)} \cong (E_{y,\sigma,r} \otimes \rho)^{\pi_0(C_y)}.$$

This can also be interpreted as $\mathrm{Hom}_{\pi_0(C_y)}(\rho^\vee, E_{y,\sigma,r}) = E_{y,\sigma,r,\rho^\vee}$.
(b) The same argument as for part (a) works. $\Box$

From Theorem 4.2.b we know that $K_{N,\sigma,r}$ is nonzero if and only if there exist $y \in \mathfrak{g}_N^{\sigma,r}$ and $y \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$ such that $q\Psi_{Z_G(\sigma_0)}(y, \rho) = [M, \mathcal{C}_v^M, q\mathcal{E}]_G$. Let $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r}, K_{N,\sigma,r})$ be the triangulated subcategory of $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$ generated by $K_{N,\sigma,r}$. In view of [Sol6, Theorem 3.5], it can be expected that $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$ decomposes as a direct sum of such subcategories.

By Theorem 4.2 with $(0, 0)$ in the role of (σ, r), every simple direct summand of $K_N = K_{N,0,0}$ is isomorphic to (a degree shift of)

$$\mathrm{IC}_{G\times\mathbb{C}^\times}\big(\mathfrak{g}_N, \mathrm{ind}_{Z_{G\times\mathbb{C}^\times}(y)}^{G\times\mathbb{C}^\times}(\rho)\big) \quad \text{with} \quad y \in \mathfrak{g}_N, \rho \in \mathrm{Irr}\big(\pi_0(Z_G(y))\big) \text{ and } q\Psi_G(y, \rho) = [M, \mathcal{C}_v^M, q\mathcal{E}]_G. \tag{4.7}$$

Lemma 4.4. *Fix a semisimple $(\sigma, r) \in \mathfrak{g} \oplus \mathbb{C}$ and let $j_{N,\sigma,r} : \mathfrak{g}_N^{\sigma,r} \to \mathfrak{g}_N$ be the inclusion. The simple direct summands of $K_{N,\sigma,r} \in \mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$ are (up to degree shifts) precisely those simple direct summands of $j_{N,\sigma,r}^* K_N$ that come from (4.7) with $y \in \mathfrak{g}_N^{\sigma,r}$.*

Proof. Let $y \in \mathfrak{g}_N^{\sigma,r}$. It is known from [KaLu2, §5.4] that $\mathrm{Ad}(G)y \cap \mathfrak{g}_N^{\sigma,r}$ consists of finitely many $Z_G(\sigma)$-orbits, all closed in this set. We enumerate these as $\mathrm{Ad}(Z_G(\sigma))y_j$, $j = 1, \ldots, n_y^\sigma$ and we write

$$\rho_j = g_j \cdot \rho \in \mathrm{Irr}\big(\pi_0(Z_G(y_j))\big) \text{ for a } g_j \in G \text{ with } \mathrm{Ad}(g_j)y = y_j.$$

Then $q\Psi_{Z_G(\sigma_0)}(y_j, \rho_j)$ is still G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$. Let C_j be the G-stabilizer of $\mathrm{Ad}(Z_G(\sigma))y_j$. Pullback of (4.7) along $j_{N,\sigma,r}$ gives

$$j_{N,\sigma,r}^* \mathrm{IC}_{G\times\mathbb{C}^\times}\big(\mathfrak{g}_N, \mathrm{ind}_{Z_{G\times\mathbb{C}^\times}(y)}^{G\times\mathbb{C}^\times}(\rho)\big) \cong \bigoplus\nolimits_{j=1}^{n_y^\sigma} \mathrm{IC}_{C_j}\big(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{Z_{G\times\mathbb{C}^\times}(y_j)}^{C_j}(\rho_j)\big). \tag{4.8}$$

Upon forgetting a part of the equivariant structure, the right hand side becomes

$$\bigoplus\nolimits_{j=1}^{n_y^\sigma} \mathrm{IC}_{Z_G(\sigma)\times\mathbb{C}^\times}\big(\mathfrak{g}_N, \mathrm{ind}_{Z_{G\times\mathbb{C}^\times}(y_j)}^{Z_G(\sigma)\times\mathbb{C}^\times}(\rho_j)\big). \tag{4.9}$$

From Theorem 4.2.b we see that every simple direct summand of $K_{N,\sigma,r}$ occurs as a summand of (4.9), and conversely. $\Box$

Next we derive the above expectation from its analogue for K_N.

Proposition 4.5. *$\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$ decomposes as an orthogonal direct sum of the triangulated subcategories $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r}, K_{N,\sigma,r})$, where the sum runs over G-conjugacy classes of cuspidal quasi-supports $(M, \mathcal{C}_v^M, q\mathcal{E})$.*

Proof. By [Sol6, Theorem 3.5] there is an orthogonal decomposition

$$\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N) = \bigoplus\nolimits_{[M,\mathcal{C}_v^M,q\mathcal{E}]_G} \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N, K_N). \tag{4.10}$$

Here K_N depends on $(M,\mathcal{C}_v^M,q\mathcal{E})$: by [Sol6, Proposition 2.3] and like in (2.11)

$$K_N \cong \mathrm{pr}_{1,N,!}\mathrm{IC}_{G\times\mathbb{C}^\times}(\mathfrak{g}_N \times G/P, q\dot{\mathcal{E}}_N).$$

The embedding $i : \mathrm{Ad}(G)\mathfrak{g}_N^{\sigma,r} \to \mathfrak{g}_N$ is locally closed, so by [Ach, Proposition 1.3.9] both $i^* i_*$ and $i^! i_!$ are equivalent to the identity on $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathrm{Ad}(G)\mathfrak{g}_N^{\sigma,r})$. Via i_* we regard this as a subcategory of $\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$. Then (4.10) implies

$$\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathrm{Ad}(G)\mathfrak{g}_N^{\sigma,r}) = \bigoplus\nolimits_{[M,\mathcal{C}_v^M,q\mathcal{E}]_G} \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathrm{Ad}(G)\mathfrak{g}_N^{\sigma,r}, i^* K_N). \tag{4.11}$$

Fix a simple direct summand (4.7) of K_N, with $y \in \mathfrak{g}_N^{\sigma,r}$. In (4.11) it generates a subcategory

$$\mathcal{D}^b_{G\times\mathbb{C}^\times}\big(\mathrm{Ad}(G)\mathfrak{g}_N^{\sigma,r}, \mathrm{IC}_{G\times\mathbb{C}^\times}\big(\mathrm{ind}_{Z_{G\times\mathbb{C}^\times}(y)}^{G\times\mathbb{C}^\times}(\rho)\big)\big).$$

In view of (4.8) and (4.9), forgetting a part of the equivariant structure results in

$$\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}\Big(\mathfrak{g}_N^{\sigma,r}, \bigoplus\nolimits_{j=1}^{n_y^\sigma} \mathrm{IC}_{Z_G(\sigma)\times\mathbb{C}^\times}\mathrm{ind}_{Z_{G\times\mathbb{C}^\times}(y_j)}^{Z_G(\sigma)\times\mathbb{C}^\times}(\rho_j)\Big). \tag{4.12}$$

The sum of the categories (4.12), over all eligible y and $[M,\mathcal{C}_v^M,q\mathcal{E}]_G$, already contains all simple perverse sheaves in $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$. Together with (4.11), that means that the summands of i^*K_N with $y \notin \mathrm{Ad}(G)\mathfrak{g}_N^{\sigma,r}$ give a contribution that is entirely contained in (4.12). Thus (4.11) induces an orthogonal decomposition

$$\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r}) = \bigoplus\nolimits_{[M,\mathcal{C}_v^M,q\mathcal{E}]_G} \mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r}, j^*_{N,\sigma,r}K_N), \tag{4.13}$$

where $j^*_{N,\sigma,r}K_N$ can also be replaced by (4.9). Now Lemma 4.4 allows us to replace $j^*_{N,\sigma,r}K_N$ by $K_{N,\sigma,r}$ while preserving (4.13). □

Remark 4.6. For any $y \in \mathfrak{g}_N^{\sigma,r}$

$$\mathbb{C}^\times y \text{ is contained in } \mathrm{Ad}(Z_G^\circ(\sigma))y, \tag{4.14}$$

because y is part of a $\mathfrak{sl}_2$-triple in $Z_{\mathfrak{g}}(\sigma)$. In particular $Z_G(\sigma)$ and $C = Z_G(\sigma)\times\mathbb{C}^\times$ have the same orbits on $\mathfrak{g}_N^{\sigma,r}$. Recall from [AMS2, Lemma 3.6.a] that $\pi_0(C_y) \cong \pi_0(Z_G(\sigma,y))$. For these reasons the results in Section 4 remain valid when we replace C by $Z_G(\sigma)$ everywhere.

5. The Kazhdan–Lusztig conjecture

The properties of $K_{N,\sigma,r}$ can be used to compute multiplicities between irreducible and standard modules. That enables us to investigate the Kazhdan–Lusztig conjecture [Vog, §8] for graded Hecke algebras. For $\pi \in \mathrm{Irr}(\mathbb{H}(G,M,q\mathcal{E}))$, write

$$\mu(\pi, E_{y,\sigma,r,\rho}) = \text{ multiplicity of } \pi \text{ in } E_{y,\sigma,r,\rho},$$

computed in the Grothendieck group of $\mathrm{Mod_{fl}}(\mathbb{H}(G,M,q\mathcal{E}))$. When $\mathbf{r}-r$ annihilates π, we can of course compute $\mu(\pi, E_{y,\sigma,r,\rho})$ just as well in the Grothendieck group of $\mathrm{Mod_{fl}}(\mathbb{H}(G,M,q\mathcal{E},r))$. In relation with the analytic standard modules from Theorem 3.4 we record the obvious equality

$$\mu(\mathrm{sgn}^*\pi, \mathrm{sgn}^*E_{y,\sigma,r,\rho}) = \mu(\pi, E_{y,\sigma,r,\rho}). \tag{5.1}$$

Let $y' \in \mathfrak{g}_N^{\sigma,r}$ and $\rho' \in \operatorname{Irr} \pi_0(Z_G(\sigma, y'))$. Then $\operatorname{ind}_{C_{y'}}^C(\rho')$ is an irreducible C-equivariant local system on $\mathcal{O}_{y'} = \operatorname{Ad}(C)y'$. We define

$$\mu\big(\operatorname{ind}_{C_y}^C(\rho), \operatorname{ind}_{C_{y'}}^C(\rho')\big) = \text{multiplicity of } \operatorname{ind}_{C_y}^C(\rho) \text{ in } \mathcal{H}^*\big(\operatorname{IC}_C(\mathfrak{g}_N^{\sigma,r}, \operatorname{ind}_{C_{y'}}^C(\rho'))\big)\big|_{\mathcal{O}_y}.$$

The notations on the right hand side mean that we build a C-equivariant intersection cohomology complex from ρ', we take its cohomology sheaves and we pull those back to $\mathcal{O}_y$. With Remark 4.6, we can also regard $\operatorname{ind}_{C_{y'}}^C(\rho')$ as the irreducible $Z_G(\sigma)$-equivariant local system $\operatorname{ind}_{Z_G(\sigma,y')}^{Z_G(\sigma)}(\rho')$ on $\mathcal{O}_{y'} = \operatorname{Ad}(Z_G(\sigma))y'$. Then we can define $\mu\big(\operatorname{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}(\rho), \operatorname{ind}_{Z_G(\sigma,y')}^{Z_G(\sigma)}(\rho')\big)$ as

$$\text{the multiplicity of } \operatorname{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}(\rho) \text{ in } \mathcal{H}^*\big(\operatorname{IC}_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,r}, \operatorname{ind}_{Z_G(\sigma,y')}^{Z_G(\sigma)}(\rho'))\big)\big|_{\mathcal{O}_y}.$$

Replacing C by $Z_G(\sigma)$ does not really change the involved equivariant intersection complexes, so we conclude that

$$\mu\big(\operatorname{ind}_{C_y}^C(\rho), \operatorname{ind}_{C_{y'}}^C(\rho')\big) = \mu\big(\operatorname{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}(\rho), \operatorname{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}(\rho')\big). \tag{5.2}$$

Proposition 4.1 says that, if (y', ρ') does not fulfill the conditions stated there:

$$\mu(\operatorname{ind}_{C_y}^C(\rho), \operatorname{ind}_{C_{y'}}^C(\rho')) = 0.$$

That is not surprising, because in that case (y', ρ') does not correspond to any $\mathbb{H}(G, M, q\mathcal{E})$-module.

Proposition 5.1. *In the above setup, assume that both (y, ρ) and (y', ρ') satisfy the equivalent conditions in Proposition 4.1.*

(*a*) $\mu(M_{y',\sigma,r,\rho'}, E_{y,\sigma,r,\rho}) = \mu\big(\operatorname{ind}_{C_y}^C(\rho), \operatorname{ind}_{C_{y'}}^C(\rho')\big)$.

(*b*) *The same holds if we replace the standard module $E_{y,\sigma,r,\rho}$ by the "costandard module"* $\operatorname{Hom}_{\pi_0(C_y)}\big(\rho, H^*(\{y\}, i_y^* K_{N,\sigma,r})\big)$.

Proof. In the cases where G and M are connected and $r \neq 0$, this is proven in [Lus3, §10.4–10.8]. With Proposition 4.1 and Theorem 4.2 available, these arguments from [Lus3] remain valid in our generality. We remark that, since we work with $\mathfrak{g}_N^{\sigma,r}$ instead $\mathfrak{g}^{\sigma,r}$, no extra problems arise when $r = 0$. □

Proposition 5.1 and (5.1) establish a version of the Kazhdan–Lusztig conjecture for (twisted) graded Hecke algebras of the form $\mathbb{H}(G, M, q\mathcal{E})$ or $\mathbb{H}(G, M, q\mathcal{E}, r)$. In view of (5.2), we may also interpret the geometric multiplicities as computed with $Z_G(\sigma)$-equivariant constructible sheaves. That fits well with Paragraph 2.2, in particular with Theorem 2.7.

From here we would like to establish cases of the Kazhdan–Lusztig conjecture for p-adic groups [Vog, Conjecture 8.11] (but without the sign involved over there, in our context such a sign would be superfluous). It remains to look for instances of a local Langlands correspondence which run via an algebra of the form $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r)$.

We will now discuss in which cases this is known, and the setup needed to get there. Let F be a non-archimedean local field and let $\mathcal{G}$ be a connected reductive group defined over F. Let $\mathcal{M}$ be a F-Levi subgroup of $\mathcal{G}$ and let $\tau \in \operatorname{Irr}(\mathcal{M}(F))$ be supercuspidal. This already gives rise to the category $\operatorname{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^\tau$ of finite length smooth $\mathcal{G}(F)$-representations all whose irreducible subquotients have cuspidal support conjugate to $(\mathcal{M}(F), \tau)$. Assume now that τ is tempered, write

$$X_{\mathrm{nr}}^+(\mathcal{M}(F)) = \mathrm{Hom}(\mathcal{M}(F), \mathbb{R}_{>0})$$

and let $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau+}$ be the category of all finite length smooth $\mathcal{G}(F)$-representations whose cuspidal support is contained in the $\mathcal{G}(F)$-orbit of $\big(\mathcal{M}(F), \tau X_{\mathrm{nr}}^+(\mathcal{M}(F))\big)$. The set of irreducible objects of $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau}$ will be denoted $\mathrm{Irr}(\mathcal{G}(F))^{\tau}$, and likewise with $\tau+$.

To the data $(\mathcal{G}(F), \mathcal{M}(F), \tau)$ one can associate a twisted graded Hecke algebra $\mathbb{H}_\tau$, such that there is an equivalence of categories

$$\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau+} \cong \mathbb{H}_\tau - \mathrm{Mod}_{\mathrm{fl},\mathfrak{a}}, \tag{5.3}$$

see [Sol2, Corollary 8.1]. Here $\mathrm{Mod}_{\mathrm{fl},\mathfrak{a}}$ means finite length right modules with all $\mathcal{O}(\mathfrak{t})$-weights in $\mathfrak{a}$, and one may identify

$$\mathfrak{a} = \mathrm{Lie}\big(X_{\mathrm{nr}}^+(\mathcal{M}(F))\big) = \mathrm{Hom}(\mathcal{M}(F), \mathbb{R}).$$

Theorem 5.2. *In the above setting, suppose that $\mathbb{H}_\tau^{op}$ is of the form $\mathbb{H}(G, M, q\mathcal{E}, r)$ for some $r \in \mathbb{R}$. Fix $\sigma_0 \in \mathfrak{a}$ and write $\sigma = \sigma_0 - r\sigma_v$.*

(*a*) *There is an equivalence of categories*

$$\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau+} \cong \mathrm{Mod}_{\mathrm{fl},\mathfrak{a}}(\mathbb{H}(G, M, q\mathcal{E}, r)).$$

(*b*) *There is an equivalence of categories*

$$\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau \otimes \exp(\sigma_0)} \cong \mathrm{Mod}_{\mathrm{fl},\sigma}\big(\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,-r})}(K_{N,\sigma,-r})\big).$$

(*c*) *The Kazhdan–Lusztig conjecture holds for $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau \otimes \exp(\sigma_0)}$, in the form*

$$\mu(\mathrm{sgn}^* M_{y',\sigma,-r,\rho'}, \mathrm{sgn}^* E_{y,\sigma,-r,\rho}) = \mu\big(\mathrm{ind}_{Z_G(\sigma,y)}^{Z_G(\sigma)}(\rho), \mathrm{ind}_{Z_G(\sigma,y')}^{Z_G(\sigma)}(\rho')\big)$$

where the right hand side is computed in $\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,-r})$.

Proof. (a) This is an obvious consequence of (5.3).
(b) Apply sgn^* and Theorem 2.7.c to part (a).
(c) This follows from (5.1), part (b), Proposition 5.1.a and (5.2). □

We note that by [Sol2, Sol5] the k-parameters of the algebras $\mathbb{H}_\tau$ are very often (conjecturally always) of the required kind. The analysis of the 2-cocycles of the group $W_{q\mathcal{E}}$ for $\mathbb{H}_\tau$ may be difficult sometimes, but fortunately these 2-cocycles are trivial in most cases. Therefore the assumption of Theorem 5.2 is fulfilled for large classes of groups $\mathcal{G}$ and representations, and we expect that it holds always.

Next we suppose that a local Langlands correspondence is known for sufficiently large classes of representations of $\mathcal{M}(F)$ and of $\mathcal{G}(F)$ that is, for some supercuspidal $\mathcal{M}(F)$-representations and for all the resulting Bernstein components of $\mathrm{Irr}(\mathcal{G}(F))$. Let $\mathcal{G}^\vee$ and $\mathcal{M}^\vee$ be the complex dual groups of $\mathcal{G}$ and $\mathcal{M}$. Let (ϕ, ρ) be the enhanced L-parameter of τ, so ϕ takes values in $\mathcal{M}^\vee \rtimes \mathbf{W}_F$. The group $X_{\mathrm{nr}}^+(\mathcal{M}(F))$ embeds naturally in $Z(\mathcal{M}^\vee)$, and the latter acts on the set of Langlands parameters for $\mathcal{M}(F)$ by adjusting the image of a Frobenius element.

To $(\mathcal{G}(F), \mathcal{M}(F), \phi, \rho)$ one can associate a triple $(G, M, q\mathcal{E})$ as throughout this paper [AMS3, §3.1], and a twisted graded Hecke algebra $\mathbb{H}_{\phi,\rho}$ of the form $\mathbb{H}(G, M, q\mathcal{E})$.

The involved group G is called $G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G})$ in [AMS3, (71)], it is a finite cover of $Z_{\mathcal{G}^\vee}(\phi(\mathbf{W}_F))$. An important property of this algebra is:

Theorem 5.3. [AMS3, Theorem 3.8]
For any $r \in \mathbb{R}$ there exists a canonical bijection between $\mathrm{Irr}_{\mathfrak{a}}(\mathbb{H}_{\phi,\rho}/(\mathbf{r} - r))$ and the set of enhanced L-parameters for $\mathcal{G}(F)$ whose cuspidal support is $\mathcal{G}^\vee$-conjugate to an element of $(\mathcal{M}^\vee, X_{\mathrm{nr}}^+(\mathcal{M}(F))\phi, \rho)$.

Last but not least, we assume that we have an algebra isomorphism

$$\mathbb{H}_\tau^{op} \cong \mathbb{H}_{\phi,\rho}/(\mathbf{r} - \log(q_F)/2) \tag{5.4}$$

such that Theorem 5.3 and the induced bijections

$$\mathrm{Irr}(\mathcal{G}(F))^{\tau+} \longleftrightarrow \mathrm{Irr}_{\mathfrak{a}}(\mathbb{H}_\tau^{op}) \longleftrightarrow \mathrm{Irr}_{\mathfrak{a}}\big(\mathbb{H}_{\phi,\rho}/(\mathbf{r} - \log(q_F)/2)\big) \tag{5.5}$$

realize a local Langlands correspondence for $\mathrm{Irr}(\mathcal{G}(F))^{\tau+}$. This involves the parametrization of irreducible and analytic standard modules from Theorem 3.4 and the translation to Langlands parameters in [AMS3, Theorem 3.8].

In this setting, for $r = \log(q_F)/2$:

$$\mathfrak{g}_N^{\sigma,-r} = \{y \in \mathfrak{g}_N : [\sigma, y] = -\log(q_F)y\} = \{y \in \mathfrak{g}_N : \mathrm{Ad}(\exp\sigma)y = q_F^{-1}y\}. \tag{5.6}$$

Here $(\exp\sigma, y)$ defines an unramified L-parameter $\phi : \mathbf{W}_F \rtimes \mathbb{C} \to G$, with $\exp(\sigma)$ the image of a geometric Frobenius element $\mathrm{Frob} \in \mathbf{W}_F$. Via the construction of G mentioned before Theorem 5.3 that gives rise to a Langlands parameter for $\mathcal{G}(F)$, namely ϕ with the image of Frob adjusted by $\exp(\sigma)$.

We note that the above is based on the construction of $\mathcal{G}$ as inner twist of a quasi-split F-group. Alternatively, one may work with $\mathcal{G}$ as rigid inner twist of a quasi-split group [Kal]. That requires some minor adjustments of the setup, which are discussed in [Sol4, §7]. In particular the above group G will then become the centralizer of $\phi(\mathbf{W}_F)$ in the complex dual group of $\mathcal{G}/Z(\mathcal{G}_{\mathrm{der}})$.

Theorem 5.4. *We fix $r = \log(q_F)/2$.*

(*a*) *Under the above assumptions, Theorem 5.2 holds for $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau+}$, where now $\mathfrak{g}_N^{\sigma,-r}$ is a variety of Langlands parameters associated to $\mathrm{Irr}(\mathcal{G}(F))^{\tau\otimes\exp(\sigma)}$.*
In particular the Kazhdan–Lusztig conjecture from [Vog, Conjecture 8.11] *holds for irreducible and standard representations in $\mathrm{Rep}_{\mathrm{fl}}(\mathcal{G}(F))^{\tau+}$.*

(*b*) *Part (a) holds unconditionally in the following cases:*
- *inner forms of general linear groups,*
- *inner forms of special linear groups,*
- *principal series representations of quasi-split groups,*
- *unipotent representations (of arbitrary reductive groups over F),*
- *classical F-groups – namely symplectic groups, (special) orthogonal groups, unitary groups and general (s)pin groups. Such a group need not be F-split, we only require that it is a pure inner form of a quasi-split group.*

Proof. (a) This is just a restatement of the above, taking into account that we omit the signs from [Vog, Conjecture 8.11].
(b) We need to check that the setup involving (5.4) and (5.5) is valid in the mentioned cases. It suffices to check that for an analogous setup with affine Hecke algebras, because that can always be reduced to graded Hecke algebras with [AMS3, §2].

For the inner forms of general/special linear groups, that was done in [ABPS2] and [AMS3, §5]. For unipotent representations we refer to [Lus4, Lus6, Sol3, Sol4].

For classical F-groups we use the LLC from [MoRe], the Hecke algebras for Bernstein components from [Hei] and the Hecke algebras for Langlands parameters as well as the comparison results from [AMS4].

The required properties of affine Hecke algebras for principal series representations of split groups were established in [ABPS1, Roc]. This was generalized to quasi-split groups in [Sol7]. □

Appendix A. Localization in equivariant cohomology

The fundamental localization theorem in equivariant (co)homology is [Lus3, Proposition 4.4]. It is analogous to theorems in equivariant K-theory [Seg, §4], [ChGi, §5.10] and in equivariant K-homology [KaLu2, 1.3.k]. In [Lus3] it is proven for equivariant local systems $\mathcal{L}$ on varieties X such that

$$H_c^{\mathrm{odd}}(X, \mathcal{L}^\vee) = 0. \tag{A.1}$$

During our investigations it transpired that the condition (A.1) is not always satisfied by $(\ddot{\mathfrak{g}}, \ddot{\mathcal{L}})$ in the setting of [Lus3, §8.12] - on which important parts of [Lus3] and other papers rely. Fortunately, this can be repaired by relaxing the conditions of [Lus3, Proposition 4.4], as professor Lusztig kindly explained us.

Proposition A.1. *Let G be a connected reductive complex group acting on an affine variety X. Let M be a Levi subgroup of G (i.e. the centralizer of a semisimple element of $\mathfrak{g}$). Then the natural map*

$$H_M^*(\mathrm{pt}) \underset{H_G^*(\mathrm{pt})}{\otimes} H_*^G(X, \mathcal{L}) \longrightarrow H_*^M(X, \mathcal{L})$$

is an isomorphism.

Proof. The proof of the analogous statement in equivariant K-homology [KaLu2, 1.8.(a)] can be translated to equivariant cohomology. The crucial point is the Künneth formula in equivariant K-homology [KaLu2, 1.3.(n3)], which is proven in [KaLu2, §1.5–1.6].

The conditions about simple connectedness in [KaLu2, §1] are only needed to ensure that the representation ring $R(T)$ is a free module over $R(G)$, for any maximal torus T of G. In equivariant cohomology this translates to $H_T^*(\mathrm{pt}) \cong \mathcal{O}(\mathfrak{t})$ being free over

$$H_G^*(\mathrm{pt}) \cong \mathcal{O}(\mathfrak{g}//G) \cong \mathcal{O}(\mathfrak{t})^{W(G,T)},$$

which is true for every connected reductive group G. □

For $s \in \mathfrak{g}$ we consider the inclusion $j : X^{\exp(\mathbb{C}s)} \to X$.

Proposition A.2. *In the setup of Proposition A.1, assume that $s \in \mathfrak{g}$ is central (and hence semisimple). The map*

$$\mathrm{id} \otimes j_! : \hat{H}_G^*(\mathrm{pt})_s \underset{H_G^*(\mathrm{pt})}{\otimes} H_*^G(X^{\exp(\mathbb{C}s)}, j^*\mathcal{L}) \longrightarrow \hat{H}_G^*(\mathrm{pt})_s \underset{H_G^*(\mathrm{pt})}{\otimes} H_*^G(X, \mathcal{L})$$

is an isomorphism.

Proof. Let $T \subset G$ be a maximal torus, so $s \in \mathrm{Lie}(T)$. The centrality of s and Chevalley's theorem [Var, §4.9] entail that $\hat{H}_T^*(\mathrm{pt})_s$ is a free module over $\hat{H}_G^*(\mathrm{pt})_s$.

By Proposition A.1

$$\hat{H}^*_T(\mathrm{pt})_s \underset{\hat{H}^*_G(\mathrm{pt})_s}{\otimes} \hat{H}^*_G(\mathrm{pt})_s \underset{H^*_G(\mathrm{pt})}{\otimes} H^G_*(X,\mathcal{L}) \;\cong\; \hat{H}^*_T(\mathrm{pt})_s \underset{H^*_G(\mathrm{pt})}{\otimes} H^G_*(X,\mathcal{L}) \;\cong$$
$$\hat{H}^*_T(\mathrm{pt})_s \underset{H^*_T(\mathrm{pt})}{\otimes} H^*_T(\mathrm{pt}) \underset{H^*_G(\mathrm{pt})}{\otimes} H^G_*(X,\mathcal{L}) \;\cong\; \hat{H}^*_T(\mathrm{pt})_s \underset{H^*_T(\mathrm{pt})}{\otimes} H^T_*(X,\mathcal{L}),$$

and similarly with $(X^{\exp(\mathbb{C}s)}, j^*\mathcal{L})$. In this way we reduce the issue from G to T. That case is shown in [Lus3, Proposition 4.4.a]. □

With Propositions A.1 and A.2 at hand, everything in [Lus3, §4] can be carried out without assuming that certain odd cohomology groups vanish. Instead we have to assume that the involved groups are reductive, but that assumption can be lifted with [Lus1, §1.h].

Problems with the condition (A.1) also entail that [Lus3, 5.1.(b,c,d)] are not necessarily isomorphisms in the setting of [Lus3, §8.12]. Professor Lusztig showed us that this can be overcome by rephrasing [Lus3, Proposition 5.2] in the category $\mathcal{D}^b_H(\tilde{X})$ instead of $\mathcal{D}^b(\tilde{X})$, see [Lus7, # 121]. The upshot is that all the proofs about representations of graded Hecke algebras in [Lus3, §8] can be fixed.

Appendix B. Compatibility with parabolic induction

The family of geometric standard modules $E_{y,\sigma,r\rho}$ from Section 3 behaves well under parabolic induction. However, it does not behave as well as claimed in [AMS2, Theorem 3.4]: that result is slightly too optimistic. Here we repair [AMS2, Theorem 3.4] by adding an extra condition, and we extend it from graded Hecke algebras associated to a cuspidal support to graded Hecke algebras associated to a cuspidal quasi-support.

Let P° be a parabolic subgroup of G° with a Levi factor L. Let $v \in \mathrm{Lie}(L)$ be nilpotent and let $\mathcal{E}$ be an L-equivariant cuspidal local system on $\mathcal{C}^L_v$. In [AMS2, §2], a twisted graded Hecke algebra $\mathbb{H}(G, L, \mathcal{E})$ is associated to the cuspidal support $(L, \mathcal{C}^L_v, \mathcal{E})$. Like in Condition 1.2, we assume without loss of generality that $G = G^\circ N_G(P^\circ, \mathcal{E})$.

Let Q be an algebraic subgroup of G such that $Q^\circ = Q \cap G^\circ$ is a Levi subgroup of G and $L \subset Q^\circ$. Then $P^\circ Q^\circ$ is a parabolic subgroup of G° with Q° as Levi factor. The unipotent radical $\mathcal{R}_u(P^\circ Q^\circ)$ is normalized by Q°, so its Lie algebra $\mathfrak{u}_Q = \mathrm{Lie}(\mathcal{R}_u(P^\circ Q^\circ))$ is stable under the adjoint actions of Q° and $\mathfrak{q}$. In particular, for $Y \in \mathfrak{q}$ $\mathrm{ad}(y)$ acts on $\mathfrak{u}_Q$. We denote the cokernel of $\mathrm{ad}(y) : \mathfrak{u}_Q \to \mathfrak{u}_Q$ by ${}_y\mathfrak{u}_Q$. For $N \in \mathfrak{u}_Q$ and $(\sigma, r) \in Z_{\mathfrak{g}\oplus\mathbb{C}}(y)$ we have

$$[\sigma, [y, N]] = [y, [\sigma, N]] + [[\sigma, y], N] = [y, [\sigma, N]] + [2ry, N] \in \mathrm{ad}(y)\mathfrak{u}_Q.$$

Hence $\mathrm{ad}(\sigma)$ descends to a linear map ${}_y\mathfrak{u}_Q \to {}_y\mathfrak{u}_Q$. Following Lusztig [Lus5, §1.16], we define

$$\begin{array}{llll} \epsilon : & Z_{\mathfrak{q}\oplus\mathbb{C}}(y) & \to & \mathbb{C} \\ & (\sigma, r) & \mapsto & \det(\mathrm{ad}(\sigma) - 2r : {}_y\mathfrak{u}_Q \to {}_y\mathfrak{u}_Q) \end{array}.$$

It is easily seen that ϵ is invariant under the adjoint action of $Z_{Q\times\mathbb{C}^\times}(y)$, so it defines an element of $H^*_{Z_{Q\times\mathbb{C}^\times}(y)}(\{y\})$. For a given nilpotent y, all the parameters (y, σ, r) for which parabolic induction from $\mathbb{H}(Q, L, \mathcal{E})$ to $\mathbb{H}(G, L, \mathcal{E})$ can behave problematically, are zeros of ϵ.

For any closed subgroup S of $Z_{Q\times\mathbb{C}^\times}(y)^\circ$, ϵ yields an element ϵ_S of $H_S^*(\{y\})$ (by restriction). We recall from [Lus1, Proposition 7.5] that for connected S there is a natural isomorphism

$$H_*^S(\mathcal{P}_y,\dot{\mathcal{E}}) \cong H_S^*(\{y\}) \underset{H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\{y\})}{\otimes} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y,\dot{\mathcal{E}}). \tag{B.1}$$

Here $H_S^*(\{y\})$ acts on the first tensor leg and $\mathbb{H}(G,L,\mathcal{E})$ acts on the second tensor leg. By [AMS2, Theorem 3.2.b] these actions commute, and $H_*^S(\mathcal{P}_y,\dot{\mathcal{E}})$ becomes a module over $H_S^*(\{y\}) \otimes_{\mathbb{C}} \mathbb{H}(G,L,\mathcal{E})$.

To indicate that an object is constructed with respect to the group Q (instead of G), we endow it with a superscript Q. For instance, we have the variety $\mathcal{P}_y^Q$, which admits a natural map

$$\mathcal{P}_y^Q \to \mathcal{P}_y : g(P^\circ \cap Q) \mapsto gP^\circ. \tag{B.2}$$

Now we can formulate an improved version of [AMS2, Theorem 3.4].

Theorem B.1. *Let S be a maximal torus of $Z_{Q\times\mathbb{C}^\times}(y)^\circ$.*

(a) *The map* (B.2) *induces an injection of $\mathbb{H}(G,L,\mathcal{E})$-modules*

$$\mathbb{H}(G,L,\mathcal{E}) \underset{\mathbb{H}(Q,L,\mathcal{E})}{\otimes} H_*^S(\mathcal{P}_y^Q,\dot{\mathcal{E}}) \to H_*^S(\mathcal{P}_y,\dot{\mathcal{E}}).$$

It respects the actions of $H_S^*(\{y\})$ and its image contains $\epsilon_S H_*^S(\mathcal{P}_y,\dot{\mathcal{E}})$.

(b) *Let $(\sigma,r) \in Z_{\mathfrak{q}\oplus\mathbb{C}}(y)$ be semisimple, such that $\epsilon(\sigma,r) \neq 0$. The map* (B.2) *induces an isomorphism of $\mathbb{H}(G,L,\mathcal{E})$-modules*

$$\mathbb{H}(G,L,\mathcal{E}) \underset{\mathbb{H}(Q,L,\mathcal{E})}{\otimes} E^Q_{y,\sigma,r} \to E_{y,\sigma,r},$$

which respects the actions of $\pi_0(M^Q(y))_\sigma \cong \pi_0(Z_Q(\sigma,y))$.

(c) *When $r=0$, part (b) holds for any semisimple $\sigma \in Z_{\mathfrak{q}}(y)$.*

Proof. (a) The given proof of [AMS2, Theorem 3.4] is valid with only one modification. Namely, the diagram [AMS2, (25)] does not commute. A careful consideration of [Lus5, §2] shows failure to do so stems from the difference between certain maps $i_!$ and $(p^*)^{-1}$, where p is the projection of a vector bundle on its base space and i is the zero section of the same vector bundle. In [Lus5, Lemma 2.18] this difference is identified as multiplication by ϵ_S.

(b) For $(\sigma,r) \in \mathrm{Lie}(S)$ with $\epsilon(\sigma,r) \neq 0$, the proof of [AMS2, Theorem 3.4.b] needs only one small adjustment. From (B.1) we get

$$\begin{aligned}\mathbb{C}_{\sigma,r} \underset{H_S^*(\{y\})}{\otimes} \epsilon_S H_*^S(\mathcal{P}_y,\dot{\mathcal{E}}) \;&\cong\; \mathbb{C}_{\sigma,r} \underset{H_S^*(\{y\})}{\otimes} \epsilon_S H_S^*(\{y\}) \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y,\dot{\mathcal{E}}) \\ &\cong\; \mathbb{C}_{\sigma,r} \underset{H^*_{M(y)^\circ}(\{y\})}{\otimes} H_*^{M(y)^\circ}(\mathcal{P}_y,\dot{\mathcal{E}}) \;=\; E_{y,\sigma,r}.\end{aligned}$$

The difference with before is the appearance of ϵ_S, with that and the above the proof of [AMS2, Theorem 3.4.b] goes through.

(c) As $\mathrm{ad}(\sigma)$ is invertible on $\mathrm{Lie}\big(\mathcal{R}(P^\circ Z_G^\circ(\sigma))\big)$, $\epsilon_{Z_G^\circ(\sigma)}(\sigma,r) \neq 0$. From part (b) we get a natural isomorphism

$$\mathbb{H}(G^\circ,L,\mathcal{E}) \underset{\mathbb{H}(Z_G^\circ(\sigma),L,\mathcal{E})}{\otimes} E^{Z_G^\circ(\sigma)}_{y,\sigma,0} \cong E^{G^\circ}_{y,\sigma,0}. \tag{B.3}$$

Let $Q_y \subset Z_Q^\circ(\sigma)$ be a Levi subgroup which is minimal for the property that it contains L and $\exp(y)$. Then $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ acts on both

$$E_{y,\sigma,0}^{Z_G^\circ(\sigma)} \quad \text{and} \quad \mathbb{H}(Z_G^\circ(\sigma), L, \mathcal{E}) \underset{\mathbb{H}(Q_y,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{Q_y} \tag{B.4}$$

by evaluation at $(\sigma, 0)$. Hence the structure of these two $\mathbb{H}(Z_G^\circ(\sigma), L, \mathcal{E})$-modules is completely determined by the action of $\mathbb{C}\big[W_\mathcal{E}^{Z_G^\circ(\sigma)}\big]$. But by [AMS2, Theorem 3.2.c]

$$E_{y,\sigma,r}^{Z_G^\circ(\sigma)} \quad \text{and} \quad \mathbb{H}(Z_G^\circ(\sigma), L, \mathcal{E}) \underset{\mathbb{H}(Q_y,L,\mathcal{E})}{\otimes} E_{y,\sigma,r}^{Q_y} \tag{B.5}$$

do not depend on (σ, r) as $\mathbb{C}\big[W_\mathcal{E}^{Z_G^\circ(\sigma)}\big]$-modules. From a case with $\epsilon(\sigma, r) \neq 0$ we see that these two $W_\mathcal{E}^{Z_G^\circ(\sigma)}$-representations are naturally isomorphic. Together with (B.3) that gives a natural isomorphism of $\mathbb{H}(G^\circ, L, \mathcal{E})$-modules

$$\mathbb{H}(G^\circ, L, \mathcal{E}) \underset{\mathbb{H}(Q_y,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{Q_y} \longrightarrow E_{y,\sigma,0}^{G^\circ}. \tag{B.6}$$

By the transitivity of induction, (B.6) entails that

$$\mathbb{H}(G^\circ, L, \mathcal{E}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{Q^\circ} \;\cong\; E_{y,\sigma,0}^{G^\circ}. \tag{B.7}$$

It was shown in [AMS2, Lemma 3.3] that

$$H_*^{Z_{G\times\mathbb{C}^\times}^\circ(y)}(\mathcal{P}_y, \mathcal{E}) \cong \mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{E})}^{\mathbb{H}(G,L,\mathcal{E})} H_*^{Z_{G\times\mathbb{C}^\times}^\circ(y)}(\mathcal{P}_y^{G^\circ}, \mathcal{E}). \tag{B.8}$$

From (B.7) and (B.8) we get natural isomorphisms of $\mathbb{H}(G^\circ, L, \mathcal{E})$-modules

$$\begin{aligned} E_{y,\sigma,0} &\cong \mathbb{H}(G, L, \mathcal{E}) \underset{\mathbb{H}(G^\circ,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{G^\circ} \cong \mathbb{H}(G, L, \mathcal{E}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{Q^\circ} \\ &\cong \mathbb{H}(G, L, \mathcal{E}) \underset{\mathbb{H}(Q,L,\mathcal{E})}{\otimes} \mathbb{H}(Q, L, \mathcal{E}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{Q^\circ} \cong \mathbb{H}(G, L, \mathcal{E}) \underset{\mathbb{H}(Q,L,\mathcal{E})}{\otimes} E_{y,\sigma,0}^{Q}. \end{aligned}$$

Here the composed isomorphism is still induced by (B.2), so just as in the case $\epsilon(\sigma, r) \neq 0$ it is $\pi_0(Z_Q(\sigma, y))$-equivariant. $\square$

There is just one result in [AMS2] that uses [AMS2, Theorem 3.4], namely [AMS2, Proposition 3.22]. It has to be replaced by a version that involves only the cases of [AMS2, Theorem 3.4] covered by Theorem B.1.b.

Now we set out to formulate and prove analogues of [AMS2, Theorem 3.4 and Proposition 3.22] in the setting of Section 1, so with a cuspidal quasi-support $(M, \mathcal{C}_v^M, q\mathcal{E})$. Also, Condition 1.2 remains in force.

For comparison with Theorem B.1 we assume that $M^\circ = L$ and that $\mathcal{E}$ is contained in the restriction of $q\mathcal{E}$ to $\mathcal{C}_v^L$. It is known from [AMS2, (93)] that

$$\mathbb{H}(G^\circ M, M, q\mathcal{E}) = \mathbb{H}(G^\circ, L, \mathcal{E}). \tag{B.9}$$

Taking this into account, [AMS2, (91)] provides a canonical isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules

$$\begin{aligned} H_*^{Z_{G\times\mathbb{C}^\times}^\circ(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) &\cong \mathrm{ind}_{\mathbb{H}(G^\circ M,M,q\mathcal{E})}^{\mathbb{H}(G,M,q\mathcal{E})} H_*^{Z_{G^\circ M\times\mathbb{C}^\times}^\circ(y)}(\mathcal{P}_y^{G^\circ M}, \dot{q\mathcal{E}}) \\ &\cong \mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{E})}^{\mathbb{H}(G,M,q\mathcal{E})} H_*^{Z_{G^\circ\times\mathbb{C}^\times}^\circ(y)}(\mathcal{P}_y^{G^\circ}, \dot{\mathcal{E}}). \end{aligned} \tag{B.10}$$

According to [AMS2, (95)] we can write $(q\mathcal{E})_v = \mathcal{E}_v \rtimes \rho_M$ for a unique

$$\rho_M \in \mathrm{Irr}(\mathbb{C}[M_{\mathcal{E}}/M^\circ, \natural_{\mathcal{E}}]) = \mathrm{Irr}(\mathbb{C}[W_{\mathcal{E}}/W_{\mathcal{E}}^\circ, \natural_{\mathcal{E}}]). \tag{B.11}$$

Next [AMS2, Lemma 4.4] says that, when $\sigma_0 \in \mathfrak{t}$, the sets

$$\begin{aligned} &\big\{\rho^\circ \in \mathrm{Irr}\big(\pi_0(Z(\sigma_0, y))\big) : \Psi_{Z_G^\circ(\sigma_0)}(y, \rho^\circ) = [L, \mathcal{C}_v^L, \mathcal{E}]_{G^\circ}\big\}, \\ &\big\{\tau^\circ \in \mathrm{Irr}\big(\pi_0(Z_{G^\circ M}(\sigma_0, y))\big) : q\Psi_{MZ_G^\circ(\sigma_0)}(y, \tau^\circ) = [M, \mathcal{C}_v^M, q\mathcal{E}]_{G^\circ M}\big\} \end{aligned} \tag{B.12}$$

are in bijection via $\rho^\circ \mapsto \rho^\circ \rtimes \rho_M$. Further, by [AMS2, Lemma 4.5] the identification (B.9) turns a standard $\mathbb{H}(G^\circ, L, \mathcal{E})$-module $E^{G^\circ}_{y,\sigma,r,\rho^\circ}$ into the standard $\mathbb{H}(G^\circ M, M, q\mathcal{E})$-module $E_{y,\sigma,r,\rho^\circ \rtimes \rho_M}$.

Theorem B.2. *Let Q be an algebraic subgroup of G such that $Q^\circ = G^\circ \cap Q$ is a Levi subgroup of G° and $M \subset Q$. Let $y \in \mathfrak{q}$ be nilpotent and let S be a maximal torus of $Z_{Q\times\mathbb{C}^\times}(y)$. Further, let $(\sigma, r) \in Z_{\mathfrak{q}\oplus\mathbb{C}}(y)$ be semisimple, such that $\epsilon(\sigma, r) \neq 0$ or $r = 0$.*

(*a*) *The map* (B.2) *induces an injection of $\mathbb{H}(G, M, q\mathcal{E})$-modules*

$$\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} H_*^S(\mathcal{P}_y^Q, \dot{q\mathcal{E}}) \to H_*^S(\mathcal{P}_y, \dot{q\mathcal{E}}).$$

It respects the actions of $H_S^*(\{y\})$ and its image contains $\epsilon_S H_*^S(\mathcal{P}_y, \dot{q\mathcal{E}})$.

(*b*) *The map* (B.2) *induces an isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules*

$$\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} E^Q_{y,\sigma,r} \to E_{y,\sigma,r},$$

which respects the actions of $\pi_0(Z_Q(\sigma, y))$.

Let $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$ with $q\Psi_{Z_G(\sigma_0)}(y, \rho) = [M, \mathcal{C}_v^M, q\mathcal{E}]_G$ and let $\rho^Q \in \mathrm{Irr}\big(\pi_0(Z_Q(\sigma, y))\big)$ with $q\Psi_{Z_Q(\sigma_0)}(y, \rho) = [M, \mathcal{C}_v^M, q\mathcal{E}]_Q$.

(*c*) *There is a natural isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules*

$$\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} E^Q_{y,\sigma,r,\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes E_{y,\sigma,r,\rho},$$

where the sum runs over all ρ as above.

(*d*) *For $r = 0$ part (c) restricts to an isomorphism of $\mathcal{O}(\mathfrak{t}\oplus\mathbb{C}) \rtimes \mathbb{C}[W_q\mathcal{E}, \natural_{q\mathcal{E}}]$-modules*

$$\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} M^Q_{y,\sigma,0,\rho^Q} \cong \bigoplus\nolimits_\rho \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho^Q, \rho) \otimes M_{y,\sigma,0,\rho}.$$

(*e*) *The multiplicity of $M_{y,\sigma,r,\rho}$ in $\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} E^Q_{y,\sigma,r,\rho^Q}$ is $[\rho^Q : \rho]_{\pi_0(Z_Q(\sigma,y))}$.*

It already appears that many times as a quotient, via $E^Q_{y,\sigma,r,\rho^Q} \to M^Q_{y,\sigma,r,\rho^Q}$. More precisely, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{H}(Q,L,\mathcal{L})}(M^Q_{y,\sigma,r,\rho^Q}, M_{y,\sigma,r,\rho}) \cong \mathrm{Hom}_{\pi_0(Z_Q(\sigma,y))}(\rho, \rho^Q).$$

Proof. (a) By (B.1) and [AMS2, Lemma 3.3 and §4] the right hand side of the statement is canonically isomorphic with

$$H_S^*(\{y\}) \underset{H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}}{\otimes} H_*^{Z^\circ_{G\times\mathbb{C}^\times}(y)}(\mathcal{P}_y, \dot{q\mathcal{E}}) \cong$$

$$H_S^*(\{y\}) \underset{H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}}{\otimes} \mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(G^\circ M,M,q\mathcal{E})}{\otimes} H_*^{Z^\circ_{G^\circ M\times\mathbb{C}^\times}(y)}(\mathcal{P}_y^{G^\circ M}, \dot{q\mathcal{E}}).$$

Via (B.10) that is canonically isomorphic with

$$H_S^*(\{y\}) \underset{H^*_{Z^\circ_{G\times\mathbb{C}^\times}(y)}}{\otimes} \mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ,L,\mathcal{E})}{\otimes} H_*^{Z^\circ_{G^\circ\times\mathbb{C}^\times}(y)}(\mathcal{P}_y^{G^\circ},\dot{\mathcal{E}}) \cong \mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ,L,\mathcal{E})}{\otimes} H_*^S(\mathcal{P}_y^{G^\circ},\dot{\mathcal{E}}).$$

For similar reasons the left hand side of the statement is canonically isomorphic with

$$\begin{aligned}
&\mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} \mathbb{H}(Q,M,q\mathcal{E}) \underset{\mathbb{H}(Q^\circ M,M,q\mathcal{E})}{\otimes} H_*^S(\mathcal{P}_y^{Q^\circ M},\dot{q\mathcal{E}}) \cong \\
&\mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ M,M,q\mathcal{E})}{\otimes} \mathbb{H}(G^\circ M,M,q\mathcal{E}) \underset{\mathbb{H}(Q^\circ M,M,q\mathcal{E})}{\otimes} H_*^S(\mathcal{P}_y^{Q^\circ M},\dot{q\mathcal{E}}) \cong \\
&\mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ,L,\mathcal{E})}{\otimes} \mathbb{H}(G^\circ,L,\mathcal{E}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{E})}{\otimes} H_*^S(\mathcal{P}_y^{Q^\circ},\dot{\mathcal{E}}).
\end{aligned}$$

Now we apply Theorem B.1.a for $G^\circ, Q^\circ, L, \mathcal{E}$ and use the exactness of $\mathrm{ind}_{\mathbb{H}(G^\circ,L,\mathcal{E})}^{\mathbb{H}(G,M,q\mathcal{E})}$.

(b) Like in part (a) there are canonical isomorphisms

$$E_{y,\sigma,r} \cong \mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ M,M,q\mathcal{E})}{\otimes} E_{y,\sigma,r}^{G^\circ M} \cong \mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ,L,\mathcal{E})}{\otimes} E_{y,\sigma,r}^{G^\circ},$$

$$\begin{aligned}
&\mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} E_{y,\sigma,r}^{Q} \\
&\quad\cong \mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ M,M,q\mathcal{E})}{\otimes} \mathbb{H}(G^\circ M,M,q\mathcal{E}) \underset{\mathbb{H}(Q^\circ M,M,q\mathcal{E})}{\otimes} E_{y,\sigma,r}^{Q^\circ M} \\
&\quad\cong \mathbb{H}(G,M,q\mathcal{E}) \underset{\mathbb{H}(G^\circ,L,\mathcal{E})}{\otimes} \mathbb{H}(G^\circ,L,\mathcal{E}) \underset{\mathbb{H}(Q^\circ,L,\mathcal{E})}{\otimes} E_{y,\sigma,r}^{Q^\circ}.
\end{aligned}$$

It remains to apply Theorem B.1.b.

(c,d,e) These can be shown in the same way as [AMS2, Proposition 3.22], with the following modifications:

- We use part (b) instead of [AMS2, Theorem 3.4.b].
- The references to [AMS1, §4] should be extended to the setting with cuspidal quasi-supports by means of [AMS1, §5].
- The references to [AMS2, §3] should be extended to the setting with cuspidal quasi-supports by invoking [AMS2, §4]. □

Since ϵ is a nonzero polynomial function, its zero set is a subvariety of smaller dimension (say of V_y). Still, we want to explicitly exhibit a large class of parameters (y,σ,r) on which ϵ does not vanish. With [AMS3, Proposition 1.7.c] we assume (via conjugation by an element of G°) that $\sigma_0, \sigma - r\sigma_v \in \mathfrak{t}$.

Let us call $x \in \mathfrak{t}$ (strictly) positive with respect to PQ° if $\Re(\alpha(t))$ is (strictly) positive for all $\alpha \in R(\mathcal{R}_u(PQ^\circ),T)$. We say that x is (strictly) negative with respect to PQ° if $-x$ is (strictly) positive.

Lemma B.3. *Let $y \in \mathfrak{q}$ be nilpotent and let $(\sigma,r) \in \Sigma_v(\mathfrak{t}\oplus\mathbb{C})$ with $[\sigma,y] = 2ry$. Write $\sigma = \sigma_0 + \mathrm{d}\gamma_y\left(\begin{smallmatrix} r & 0 \\ 0 & -r\end{smallmatrix}\right)$ as in (3.4), with $\sigma_0 \in Z_{\mathfrak{t}}(y)$. Assume that one of the following holds:*

- *$\Re(r) > 0$ and σ_0 is negative with respect to PQ°;*
- *$\Re(r) < 0$ and σ_0 is positive with respect to PQ°;*
- *$\Re(r) = 0$ and σ_0 is strictly positive or strictly negative with respect to PQ°.*

Then $\epsilon(\sigma,r) \neq 0$.

Proof. Via $\mathrm{d}\gamma_y : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{q}$, $\mathfrak{u}_Q$ becomes a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Since $\sigma_0 \in \mathfrak{t}$ commutes with y, it commutes with $\mathrm{d}\gamma_y(\mathfrak{sl}_2(\mathbb{C}))$. For any eigenvalue $\lambda \in \mathbb{C}$ of σ_0, let ${}_\lambda\mathfrak{u}_Q$ be the eigenspace in $\mathfrak{u}_Q$.

For $n \in \mathbb{Z}_{\geq 0}$ let $\mathrm{Sym}^n(\mathbb{C}^2)$ be the unique irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$. We decompose the $\mathfrak{sl}_2(\mathbb{C})$-module ${}_\lambda\mathfrak{u}_Q$ as

$$ {}_\lambda\mathfrak{u}_Q = \bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n(\mathbb{C}^2)^{\mu(\lambda,n)} \quad \text{with} \quad \mu(\lambda, n) \in \mathbb{Z}_{\geq 0}. $$

The cokernel of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathrm{Sym}^n(\mathbb{C}^2)$ is the lowest weight space W_{-n} in that representation, on which $\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$ acts as $-nr$. Hence σ acts on

$$ \mathrm{coker}(\mathrm{ad}(y) : {}_\lambda\mathfrak{u}_Q \to {}_\lambda\mathfrak{u}_Q) \cong \bigoplus\nolimits_{n \geq 0} W_{-n}^{\mu(\lambda,r)} \quad \text{as} \quad \bigoplus\nolimits_{n \geq 0} (\lambda - nr)\mathrm{Id}_{W_n^{\mu(\lambda,r)}}. $$

Consequently

$$ (\mathrm{ad}(\sigma) - 2r)\big|_{{}_\lambda\mathfrak{u}_Q} = \bigoplus\nolimits_{\lambda \in \mathbb{C}} \bigoplus\nolimits_{n \geq 0} (\lambda - (n+2)r)\mathrm{Id}_{W_n^{\mu(\lambda,r)}}. $$

By definition then

$$ \epsilon(\sigma, r) = \prod\nolimits_{\lambda \in \mathbb{C}} \prod\nolimits_{n \geq 0} (\lambda - (n+2)r)^{\mu(\lambda,n)}. $$

When $\Re(r) > 0$ and σ_0 is negative with respect to PQ°, $\Re(\lambda - (n+2)r) < 0$ for all eigenvalues λ of σ_0 on $\mathfrak{u}_Q$, and in particular $\epsilon(\sigma, r) \neq 0$.

Similarly, we see that $\epsilon(\sigma, r) \neq 0$ in the other two possible cases in the lemma. □

As an application of Lemma B.3, we prove a result in the spirit of the Langlands classification for graded Hecke algebras [Eve]. It highlights a procedure to obtain irreducible $\mathbb{H}(G, M, q\mathcal{E})$-modules from irreducible tempered modules of a parabolic subalgebra $\mathbb{H}(Q, M, q\mathcal{E})$: twist by a central character which is strictly positive with respect to PQ°, induce parabolically and then take the unique irreducible quotient.

Proposition B.4. *Let $y \in \mathfrak{g}$ be nilpotent, $(\sigma, r) \in Z_{\mathfrak{g} \oplus \mathbb{C}}(y)$ semisimple and let $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma, y))\big)$ with $\Psi_{Z_G(\sigma_0)}(y, \rho) = [M, \mathcal{C}_v^M, q\mathcal{E}]_G$.*

(*a*) *If $\Re(r) \neq 0$ and $\sigma_0 \in i\mathfrak{t}_\mathbb{R} + Z(\mathfrak{g})$, then $M_{y,\sigma,r,\rho} = E_{y,\sigma,r,\rho}$.*

(*b*) *Suppose that $\Re(r) > 0$ and $\sigma_0, \sigma - r\sigma_v \in \mathfrak{t}$ such that $\Re(\sigma_0)$ is negative with respect to P. Then $\Re(\sigma_0)$ is strictly negative with respect to PQ°, where $Q = Z_G(\Re(\sigma_0))$. Further $M_{y,\sigma,r,\rho}$ is the unique irreducible quotient of $\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} M^Q_{y,\sigma,r,\rho}$.*

(*c*) *In the setting of part (b), $\mathrm{IM}^*(M_{y,\sigma,r,\rho}) \cong \mathrm{sgn}^*(M_{y,-\sigma,-r,\rho})$ is the unique irreducible quotient of*

$$ \mathrm{IM}^*\big(\mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} M^Q_{y,\sigma,r,\rho}\big) \cong \mathbb{H}(G, M, q\mathcal{E}) \underset{\mathbb{H}(Q,M,q\mathcal{E})}{\otimes} \mathrm{IM}^*(M^Q_{y,\sigma,r,\rho}). $$

(*d*) *Let $(L, \mathcal{E})$ be related to $(M, q\mathcal{E})$ as in* (B.9). *Then $\mathrm{IM}^*(M^Q_{y,\sigma,r,\rho}) \cong \mathrm{sgn}^*(M^Q_{y,-\sigma,-r,\rho})$ comes from the twist of a tempered $\mathbb{H}(Q^\circ, L, \mathcal{E})$-module by a strictly positive character of $\mathcal{O}(Z(\mathfrak{q}))$.*

Remark. By [AMS2, (82)] the extra condition in part (a) holds for instance when $\Re(r) > 0$ and $\mathrm{IM}^*(M_{y,\sigma,r,\rho})$ is tempered. By [AMS2, Proposition 1.7] every parameter (y, σ) is G°-conjugate to one with the properties as in (b).

Proof. (a) Write $\sigma_0 = \sigma_{0,\mathrm{der}} + z_0$ with $\sigma_{0,\mathrm{der}} \in \mathfrak{g}_{\mathrm{der}}$ and $z_0 \in Z(\mathfrak{g})$. Then, as in the proof of [AMS2, Corollary 3.28],

$$E^{G^\circ}_{y,\sigma,r,\rho^\circ} = E^{G^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0} \quad \text{and} \quad M^{G^\circ}_{y,\sigma,r,\rho^\circ} = M^{G^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0}.$$

By [Lus5, Theorem 1.21] $E^{G^\circ}_{y,\sigma-z_0,r,\rho^\circ} = M^{G^\circ}_{y,\sigma-z_0,r,\rho^\circ}$ as $\mathbb{H}(G_{\mathrm{der}}, L \cap G_{\mathrm{der}}, \mathcal{E})$-modules, so $E^{G^\circ}_{y,\sigma,r,\rho^\circ} = M^{G^\circ}_{y,\sigma,r,\rho^\circ}$ as $\mathbb{H}(G^\circ, L, \mathcal{E})$-modules. Together with [AMS2, Lemma 3.18 and (63)] this gives $E_{y,\sigma,r,\rho} = M_{y,\sigma,r,\rho}$.

(b) Notice that $Z_G(\sigma, y) = Z_Q(\sigma, y)$, so by [AMS1, Theorem 4.8.a] ρ is a valid enhancement of the parameter (σ, y) for $\mathbb{H}(Q, L, \mathcal{E})$.

By construction $\Re(\sigma_0)$ is strictly negative with respect to PQ°. Now Lemma B.3 says that we may apply [AMS2, Proposition 3.22]. That and part (a) yield

$$\mathbb{H}(G, L, \mathcal{E}) \underset{\mathbb{H}(Q,L,\mathcal{E})}{\otimes} M^Q_{y,\sigma,r,\rho} = \mathbb{H}(G, L, \mathcal{E}) \underset{\mathbb{H}(Q,L,\mathcal{E})}{\otimes} E^Q_{y,\sigma,r,\rho} = E_{y,\sigma,r,\rho}.$$

Now apply [AMS2, Theorem 3.20.b].

(c) This follows from part (b) and the compatability of IM^* with parabolic induction, as in [AMS2, (81)].

(d) Write

$$M^{Q^\circ}_{y,\sigma,r,\rho^\circ} = M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0} = M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \big(\mathbb{C}_{z_0-\Re(z_0)} \otimes \mathbb{C}_{\Re(z_0)}\big)$$

as in the proof of part (a), with Q in the role of G. By [AMS2, Theorem 3.25.b], $M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0-\Re(z_0)}$ is anti-tempered. The definition of Q entails that $\Re(z_0)$ equals $\Re(\sigma_0)$, which we know is strictly negative. Hence

$$\mathrm{IM}^*\big(M^{Q^\circ}_{y,\sigma,r,\rho^\circ}\big) = \mathrm{IM}^*\big(M^{Q^\circ}_{y,\sigma-z_0,r,\rho^\circ} \otimes \mathbb{C}_{z_0-\Re(z_0)}\big) \otimes \mathbb{C}_{-\Re(\sigma_0)},$$

where the right hand side is the twist of a tempered $\mathbb{H}(Q^\circ, L, \mathcal{E})$-module by the strictly positive character $-\Re(\sigma_0)$ of $S(Z(\mathfrak{q})^*)$. By [AMS2, (68) and (80)]

$$\mathrm{IM}^*(M^Q_{y,\sigma,r\rho}) = \mathrm{IM}^*\big(\tau \ltimes M^{Q^\circ}_{y,\sigma,r,\rho^\circ}\big) = \tau \ltimes \mathrm{IM}^*\big(M^{Q^\circ}_{y,\sigma,r,\rho^\circ}\big). \tag{B.13}$$ □

We note that by [AMS2, Lemma 3.16] $\mathcal{O}(Z(\mathfrak{q}))$ acts on (B.13) by the characters $\gamma(-\Re(\sigma_0))$ with $\gamma \in \Gamma^Q_{q\mathcal{E}}$. Since $\Gamma^Q_{q\mathcal{E}}$ normalizes PQ°, it preserves the strict positivity of $-\Re(\sigma_0)$. In this sense $\mathrm{IM}^*(M^Q_{y,\sigma,r\rho})$ is essentially the twist of a tempered $\mathbb{H}(Q, L, \mathcal{E})$-module by a strictly positive central character.

References

[Ach] P.N. Achar, *Perverse sheaves and applications to representation theory*, Mathematical Surveys and Monographs **258**, American Mathematical Society, 2021

[Art] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, Colloquium Publications volume **61**, American Mathematical Society, 2013

[ABPS1] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "The principal series of p-adic groups with disconnected centre", Proc. London Math. Soc. **114.5** (2017), 798–854

[ABPS2] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "Hecke algebras for inner forms of p-adic special linear groups", J. Inst. Math. Jussieu **16.2** (2017), 351–419

[AMS1] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Generalizations of the Springer correspondence and cuspidal Langlands parameters", Manus. Math. **157** (2018), 121–192

[AMS2] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Graded Hecke algebras for disconnected reductive groups", pp. 23–84 in: *Geometric aspects of the trace formula, W. Müller, S. W. Shin, N. Templier (eds.)*, Simons Symposia, Springer, 2018

[AMS3] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Affine Hecke algebras for Langlands parameters", arXiv:1701.03593v4, 2023

[AMS4] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Affine Hecke algebras for classical p-adic groups", arXiv:2211.08196, 2022

[BCHN] D. Ben-Zvi, H. Chen, D. Helm, D. Nadler, "Coherent Springer theory and the categorical Deligne–Langlands correspondence", arXiv:2010.02321, 2021

[BeLu] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics **1578**, Springer-Verlag, Berlin, 1994

[Che] I. Cherednik, "A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras", Invent. Math. **106.2** (1991), 411–431

[ChGi] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, 1997

[Eve] S. Evens, "The Langlands classification for graded Hecke algebras", Proc. Amer. Math. Soc. **124.4** (1996), 1285–1290

[FaSc] L. Fargues, P. Scholze, "Geometrization of the local Langlands correspondence", arXiv:2102.13459, 2021

[Hei] V. Heiermann, "Local Langlands correspondence for classical groups and affine Hecke algebras", Math. Z. **287** (2017), 1029–1052

[Hel] E. Hellmann, "On the derived category of the Iwahori–Hecke algebra", Compos. Math. **159.5**, 1042–1110

[Kal] T. Kaletha, "Rigid inner forms of real and p-adic groups", Ann. Math. **184.2** (2016), 559–632

[KaLu1] D. Kazhdan, G. Lusztig, "Representations of Coxeter groups and Hecke algebras", Invent. Math. **53.2** (1979), 165–184

[KaLu2] D. Kazhdan, G. Lusztig, "Proof of the Deligne–Langlands conjecture for Hecke algebras", Invent. Math. **87** (1987), 153–215

[Lus1] G. Lusztig, "Cuspidal local systems and graded Hecke algebras", Publ. Math. Inst. Hautes Études Sci. **67** (1988), 145–202

[Lus2] G. Lusztig, "Affine Hecke algebras and their graded version", J. Amer. Math. Soc **2.3** (1989), 599–635

[Lus3] G. Lusztig, "Cuspidal local systems and graded Hecke algebras. II", pp. 217–275 in: *Representations of groups*, Canadian Mathematical Society Conference Proceedings **16**, 1995

[Lus4] G. Lusztig, "Classification of unipotent representations of simple p-adic groups", Int. Math. Res. Notices **11** (1995), 517-589

[Lus5] G. Lusztig, "Cuspidal local systems and graded Hecke algebras. III", Represent. Theory **6** (2002), 202–242

[Lus6] G. Lusztig, "Classification of unipotent representations of simple p-adic groups II", Represent. Theory **6** (2002), 243–289

[Lus7] G. Lusztig, "Comments on my papers", arXiv:1707.09368, 2021

[MoRe] C. Mœglin, D. Renard, "Sur les paquets d'Arthur des groupes classiques et unitaires non quasi-déployés", pp. 341–361 in: *Relative Aspects in Representation Theory, Langlands Functoriality and Automorphic Forms*, Lecture Notes in Mathematics **2221**, 2018

[Opd] E.M. Opdam, "Harmonic analysis for certain representations of graded Hecke algebras", Acta Math. **175** (1995), 75-121

[Roc] A. Roche, "Types and Hecke algebras for principal series representations of split reductive p-adic groups", Ann. Sci. École Norm. Sup. **31.3** (1998), 361–413

[Seg] G.B. Segal, "Equivariant K-theory", Publ. Math. Inst. Hautes Études Sci. **34** (1968), 129–151

[Sol1] M. Solleveld, "Affine Hecke algebras and their representations", Indagationes Mathematica **32.5** (2021), 1005–1082

[Sol2] M. Solleveld, "Endomorphism algebras and Hecke algebras for reductive p-adic groups", J. Algebra **606** (2022), 371–470

[Sol3] M. Solleveld, "A local Langlands correspondence for unipotent representations", Amer. J. Math. **145.3** (2023)

[Sol4] M. Solleveld, "On unipotent representations of ramified p-adic groups", Represent. Theory **27** (2023), 669–716

[Sol5] M. Solleveld, "Parameters of Hecke algebras for Bernstein components of p-adic groups", arXiv:2103.13113, 2021

[Sol6] M. Solleveld, "Graded Hecke algebras and equivariant constructible sheaves on the nilpotent cone", arXiv:2205.07490, 2022

[Sol7] M. Solleveld, "On principal series representations of quasi-split reductive p-adic groups", arXiv:2304.06418, 2023

[Var] V.S. Varadarajan, *Lie groups, Lie algebras and their representations*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs NJ, 1974

[Vog] D. Vogan, "The local Langlands conjecture", pp. 305–379 in: *Representation theory of groups and algebras*, Contemp. Math. **145**, American Mathematical Society, Providence RI, 1993

[Zel] A.V. Zelevinsky, "P-adic analog of the Kazhdan–Lusztig hypothesis", Functional Anal. Appl. **15.2** (1981), 83–92

[Zhu] X. Zhu, "Coherent sheaves on the stack of Langlands parameters", arXiv:2008.02998, 2021

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