CONJUGACY OF LEVI SUBGROUPS OF REDUCTIVE GROUPS AND A GENERALIZATION TO LINEAR ALGEBRAIC GROUPS

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Abstract. We investigate Levi subgroups of a connected reductive algebraic group $\mathcal{G}$, over a ground field K. We parametrize their conjugacy classes in terms of sets of simple roots and we prove that two Levi K-subgroups of $\mathcal{G}$ are rationally conjugate if and only if they are geometrically conjugate.

These results are generalized to arbitrary connected linear algebraic K-groups. In that setting the appropriate analogue of a Levi subgroup is derived from the notion of a pseudo-parabolic subgroup.

1. Introduction

Let $\mathcal{G}$ be a connected reductive group over a field K. It is well-known that conjugacy classes of parabolic K-subgroups correspond bijectively to set of simple roots (relative to K). Further, two parabolic K-subgroups are $\mathcal{G}(K)$-conjugate if and only if they are conjugate by an element of $\mathcal{G}(\overline{K})$. In other words, rational and geometric conjugacy classes coincide.

By a Levi K-subgroup of $\mathcal{G}$ we mean a Levi factor of some parabolic K-subgroup of $\mathcal{G}$. Such groups play an important role in the representation theory of reductive groups, via parabolic induction. Conjugacy of Levi subgroups, also known as association of parabolic subgroups, has been studied less. Although their rational conjugacy classes are known (see [Cas, Proposition 1.3.4]), it appears that so far these have not been compared with geometric conjugacy classes.

Let Δ_K be the set of simple roots for $\mathcal{G}$ with respect to a maximal K-split torus $\mathcal{S}$. For every subset $I_K \subset \Delta_K$ there exists a standard Levi K-subgroup $\mathcal{L}_{I_K}$. We will prove:

Theorem A. *Let $\mathcal{G}$ be a connected reductive K-group. Every Levi K-subgroup of $\mathcal{G}$ is $\mathcal{G}(K)$-conjugate to a standard Levi K-subgroup.*

For two standard Levi K-subgroups $\mathcal{L}_{I_K}$ and $\mathcal{L}_{J_K}$ the following are equivalent:

- *I_K and J_K are associate under the Weyl group $W(\mathcal{G},\mathcal{S})$;*
- *$\mathcal{L}_{I_K}$ and $\mathcal{L}_{J_K}$ are $\mathcal{G}(K)$-conjugate;*
- *$\mathcal{L}_{I_K}$ and $\mathcal{L}_{J_K}$ are $\mathcal{G}(\overline{K})$-conjugate.*

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The first claim and the first equivalence are folklore and not hard to show. The meat of Theorem A is the equivalence of $\mathcal{G}(K)$-conjugacy and $\mathcal{G}(\overline{K})$-conjugacy, that is, of rational conjugacy and geometric conjugacy. Our proof of that equivalence involves reduction steps and a case-by-case analysis for quasi-split absolutely simple groups. It occupies Section 2 of the paper.

Our main result is a generalization of Theorem A to arbitrary connected linear algebraic groups. There we replace the notion of a Levi subgroup by that of a *pseudo-Levi subgroup*. By definition, a pseudo-Levi K-subgroup of $\mathcal{G}$ is the intersection of two opposite pseudo-parabolic K-subgroups of $\mathcal{G}$. We refer to [CGP, §2.1] and the start of Section 3 for more background. For reductive groups, pseudo-Levi subgroups are the same as Levi subgroups. When $\mathcal{G}$ does not admit a Levi decomposition, these pseudo-Levi subgroups are the best analogues. In the representation theory of pseudo-reductive groups over local fields (of positive characteristic), these pseudo-Levi subgroups play a key role [Sol, §4.1].

We prove that Theorem A has a natural analogue in the "pseudo"-setting:

Theorem B. *Let $\mathcal{G}$ be a connected linear algebraic K-group. Every pseudo-Levi K-subgroup of $\mathcal{G}$ is $\mathcal{G}(K)$-conjugate to a standard pseudo-Levi K-subgroup.*

For two standard pseudo-Levi K-subgroups $\mathcal{L}_{I_K}, \mathcal{L}_{J_K}$ the following are equivalent:

- *I_K and J_K are associate under the Weyl group $W(\mathcal{G}, \mathcal{S})$;*
- *$\mathcal{L}_{I_K}$ and $\mathcal{L}_{J_K}$ are $\mathcal{G}(K)$-conjugate;*
- *$\mathcal{L}_{I_K}$ and $\mathcal{L}_{J_K}$ are $\mathcal{G}(\overline{K})$-conjugate.*

Our arguments rely mainly on the structure theory of linear algebraic groups and pseudo-reductive groups developed by Conrad, Gabber and Prasad [CGP, CP]. The first claim and the first equivalence in Theorem B are quickly dealt with in Lemma 8. Like for reductive groups, the hard part is the equivalence of rational and geometric conjugacy. The proof of that constitutes the larger part of Section 3, from Theorem 10 onwards. We make use of Theorem A and of deep classification results about absolutely pseudo-simple groups [CP].

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2. Connected reductive groups

Let K be field with an algebraic closure $\overline{K}$ and a separable closure $K_s \subset \overline{K}$. Let Γ_K be the Galois group of K_s/K.

Let $\mathcal{G}$ be a connected reductive K-group. Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$ with character lattice $X^*(\mathcal{T})$. let $\Phi(\mathcal{G}, \mathcal{T}) \subset X^*(\mathcal{T})$ be the associated root system. We also fix a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ containing $\mathcal{T}$, which determines a basis Δ of $\Phi(\mathcal{G}, \mathcal{T})$. For every $\gamma \in \Gamma_K$ there exists a $g_\gamma \in \mathcal{G}(K_s)$ such that

$$g_\gamma \gamma(\mathcal{T}) g_\gamma^{-1} = \mathcal{T} \quad \text{and} \quad g_\gamma \gamma(\mathcal{B}) g_\gamma^{-1} = \mathcal{B}.$$

One defines the μ-action of Γ_K on $\mathcal{T}$ by

$$\mu_{\mathcal{B}}(\gamma)(t) = \mathrm{Ad}(g_\gamma) \circ \gamma(t). \tag{1}$$

This also determines an action $\mu_{\mathcal{B}}$ of Γ_K on $\Phi(\mathcal{G}, \mathcal{T})$, which stabilizes Δ.

Let $\mathcal{S}$ be a maximal K-split torus in $\mathcal{G}$. By [Spr, Theorem 13.3.6.(i)] applied to $Z_{\mathcal{G}}(\mathcal{S})$, we may assume that $\mathcal{T}$ is defined over K and contains $\mathcal{S}$. Then $Z_{\mathcal{G}}(\mathcal{S})$ is a minimal K-Levi subgroup of $\mathcal{G}$. Let

$$\Delta_0 := \{\alpha \in \Delta : \mathcal{S} \subset \ker \alpha\}$$

be the set of simple roots of $(Z_{\mathcal{G}}(\mathcal{S}), \mathcal{T})$. It is known that Δ_0 is stable under $\mu_{\mathcal{B}}(\Gamma_K)$ [Spr, Proposition 15.5.3.i], so $\mu_{\mathcal{B}}$ can be regarded as a group homomorphism $\Gamma_K \to \mathrm{Aut}(\Delta, \Delta_0)$. The triple $(\Delta, \Delta_0, \mu_{\mathcal{B}})$ is called the index of $\mathcal{G}$ [Spr, §15.5.5].

Recall from [Spr, Lemma 15.3.1] that the root system $\Phi(\mathcal{G}, \mathcal{S})$ is the image of $\Phi(\mathcal{G}, \mathcal{T})$ in $X^*(\mathcal{S})$, without 0. The set of simple roots Δ_K of $(\mathcal{G}, \mathcal{S})$ can be identified with $(\Delta \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$. The Weyl group of $(\mathcal{G}, \mathcal{S})$ can be expressed in various ways:

$$\begin{aligned} W(\mathcal{G}, \mathcal{S}) &= N_{\mathcal{G}}(\mathcal{S})/Z_{\mathcal{G}}(\mathcal{S}) \cong N_{\mathcal{G}(K)}(\mathcal{S}(K))/Z_{\mathcal{G}(K)}(\mathcal{S}(K)) \\ &\cong N_{\mathcal{G}}(\mathcal{S}, \mathcal{T})/N_{Z_{\mathcal{G}}(\mathcal{S})}(\mathcal{T}) = \big(N_{\mathcal{G}}(\mathcal{S}, \mathcal{T})/\mathcal{T}\big)\big/\big(N_{Z_{\mathcal{G}}(\mathcal{S})}(\mathcal{T})/\mathcal{T}\big) \\ &\cong \mathrm{Stab}_{W(\mathcal{G}, \mathcal{T})}(\mathcal{S})/W(Z_{\mathcal{G}}(\mathcal{S}), \mathcal{T}). \end{aligned} \tag{2}$$

Let $\mathcal{P}_{\Delta_0} = Z_{\mathcal{G}}(\mathcal{S})\mathcal{B}$ the minimal parabolic K-subgroup of $\mathcal{G}$ associated to Δ_0. It is well-known [Spr, Theorem 15.4.6] that the following sets are canonically in bijection:

- $\mathcal{G}(K)$-conjugacy classes of parabolic K-subgroups of $\mathcal{G}$;
- standard (i.e. containing $\mathcal{P}_{\Delta_0}$) parabolic K-subgroups of $\mathcal{G}$;
- subsets of $(\Delta \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$;
- $\mu_{\mathcal{B}}(\Gamma_K)$-stable subsets of Δ containing Δ_0.

Comparing these criteria over K and over $\overline{K}$, we see that two parabolic K-subgroups of $\mathcal{G}$ are $\mathcal{G}(K)$-conjugate if and only if they are $\mathcal{G}(\overline{K})$-conjugate.

By a parabolic pair for $\mathcal{G}$ we mean a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P} \subset \mathcal{G}$ is a parabolic subgroup and $\mathcal{L}$ is a Levi factor of $\mathcal{P}$. We say that the pair is defined over K if both $\mathcal{P}$ and $\mathcal{L}$ are so. We say that $\mathcal{L}$ is a Levi K-subgroup of $\mathcal{G}$ if there is a parabolic pair $(\mathcal{P}, \mathcal{L})$ defined over K. Equivalently, a Levi K-subgroup of $\mathcal{G}$ is the centralizer of a K-split torus in $\mathcal{G}$. We note that there exist K-subgroups of $\mathcal{G}$ which are not Levi, but which become Levi over a field extension of K. Examples are non-split maximal K-tori and (when $\mathcal{G}$ is split) the centralizer of any non-split non-trivial torus.

With [Spr, Lemma 15.4.5] every $\mu_{\mathcal{B}}(\Gamma_K)$-stable subset $I \subset \Delta$ containing Δ_0 gives rise to a standard Levi K-subgroup $\mathcal{L}_I$ of $\mathcal{G}$, namely the group generated by $Z_{\mathcal{G}}(\mathcal{S})$ and the root subgroups for roots in $\mathbb{Z}I \cap \Phi(\mathcal{G}, \mathcal{T})$. By construction $\mathcal{L}_I$ is a Levi factor of the standard parabolic K-subgroup $\mathcal{P}_I$ of $\mathcal{G}$. In the introduction we denoted $\mathcal{L}_I$ by $\mathcal{L}_{I_K}$, where $I_K = (I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$.

Two parabolic K-subgroups of $\mathcal{G}$ are called associate if their Levi factors are $\mathcal{G}(K)$-conjugate. As Levi factors are unique up to conjugation (see the proof of Lemma 1.a below), there is a natural bijection between the set of $\mathcal{G}(K)$-conjugacy classes of Levi K-subgroups of $\mathcal{G}$ and the set of association classes of parabolic K-subgroups of $\mathcal{G}$. The explicit description of these sets is known, for instance from [Cas, Proposition 1.3.4]. Unfortunately we could not find a complete proof of these statements in the literature, so we provide it here.

Lemma 1. (*a*) *Every Levi K-subgroup of $\mathcal{G}$ is $\mathcal{G}(K)$-conjugate to a standard Levi K-subgroup of $\mathcal{G}$.*

(*b*) *For two standard Levi K-subgroups $\mathcal{L}_I$ and $\mathcal{L}_J$ the following are equivalent:*

(i) $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate;

(ii) $(I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ and $(J \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ are $W(\mathcal{G},\mathcal{S})$-associate.

Proof. (a) Let $\mathcal{P}$ be a parabolic K-subgroup of $\mathcal{G}$ with a Levi factor $\mathcal{L}$ defined over K. Since $\mathcal{P}$ is $\mathcal{G}(K)$-conjugate to a standard parabolic subgroup $\mathcal{P}_I$ [Spr, Theorem 15.4.6], $\mathcal{L}$ is $\mathcal{G}(K)$-conjugate to a Levi factor of $\mathcal{P}_I$. By [Spr, Proposition 16.1.1] any two such factors are conjugate by an element of $\mathcal{P}_I(K)$. In particular $\mathcal{L}$ is $\mathcal{G}(K)$-conjugate to $\mathcal{L}_I$.

(b) Suppose that (ii) is fulfilled, that is,

$$w(I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K) = (J \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K) \quad \text{for some } w \in W(\mathcal{G},\mathcal{S}).$$

Let $\bar{w} \in N_{\mathcal{G}(K)}(\mathcal{S}(K))$ be a lift of w. Then $\bar{w}\mathcal{L}_I\bar{w}^{-1}$ contains $Z_{\mathcal{G}}(\mathcal{S})$ and

$$\Phi(\bar{w}\mathcal{L}_I\bar{w}^{-1},\mathcal{S}) = w\Phi(\mathcal{L}_I,\mathcal{S}) = \Phi(\mathcal{L}_J,\mathcal{S}).$$

Hence $\bar{w}\mathcal{L}_I\bar{w}^{-1} = \mathcal{L}_J$, showing that (i) holds.

Conversely, suppose that (i) holds, so $g\mathcal{L}_I g^{-1} = \mathcal{L}_J$ for some $g \in \mathcal{G}(K)$. Then $g\mathcal{S}g^{-1}$ is a maximal K-split torus of $\mathcal{L}_J$. By [Spr, Theorem 15.2.6] there is a $l \in \mathcal{L}_J(K)$ such that $lg\mathcal{S}g^{-1}l^{-1} = \mathcal{S}$. Thus $(lg)\mathcal{L}_I(lg)^{-1} = \mathcal{L}_J$ and $lg \in N_{\mathcal{G}}(\mathcal{S})$. Let w_1 be the image of lg in $W(\mathcal{G},\mathcal{S})$. Then $w_1(\Phi(\mathcal{L}_I,\mathcal{S})) = \Phi(\mathcal{L}_J,\mathcal{S})$, so $w_1\big((I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)\big)$ is a basis of $\Phi(\mathcal{L}_J,\mathcal{S})$. Any two bases of a root system are associate under its Weyl group, so there exists a $w_2 \in W(\mathcal{L}_J,\mathcal{S}) \subset W(\mathcal{G},\mathcal{S})$ such that

$$w_2 w_1\big((I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)\big) = (J \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K). \qquad \square$$

When $\mathcal{G}$ is K-split, Δ_0 is empty and the action of Γ_K is trivial. Then Lemma 1 says that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate if and only if I and J are $W(\mathcal{G},\mathcal{T})$-associate. With $\overline{K}$ instead of K we would obtain the same criterion. In particular $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate if and only if they are $\mathcal{G}(\overline{K})$-conjugate.

We want to prove that rational conjugacy and geometric conjugacy of Levi subgroups are equivalent in general. More precisely:

Theorem 2. *Let $\mathcal{L},\mathcal{L}'$ be two Levi K-subgroups of $\mathcal{G}$. Then $\mathcal{L}$ and $\mathcal{L}'$ are $\mathcal{G}(K)$-conjugate if and only if they are $\mathcal{G}(\overline{K})$-conjugate.*

The proof consists of several steps:

- Reduction from reductive to quasi-split $\mathcal{G}$.
- Reduction from reductive (quasi-split) to absolutely simple (quasi-split) $\mathcal{G}$.
- Proof for absolutely simple, quasi-split groups.

The first of these three steps is due to Jean-Loup Waldspurger.

Let $\mathcal{G}^*$ be a quasi-split K-group with an inner twist $\psi : \mathcal{G} \to \mathcal{G}^*$. Thus ψ is an isomorphism of K_s-groups and there exists a map $u : \Gamma_K \to \mathcal{G}^*(K_s)$ such that

$$\psi \circ \gamma \circ \psi^{-1} = \mathrm{Ad}(u(\gamma)) \circ \gamma^* \qquad \forall \gamma \in \Gamma_K. \tag{3}$$

Here γ^* denotes the Γ_K-action which defines the K-structure of $\mathcal{G}^*$. We fix a Borel K-subgroup $\mathcal{B}^*$ of $\mathcal{G}^*$ and a maximal K-torus $\mathcal{T}^* \subset \mathcal{B}^*$ which is maximally K-split. In other words, $(\mathcal{B}^*,\mathcal{T}^*)$ is a minimal parabolic pair of $\mathcal{G}^*$, defined over K. In $\mathcal{G}^*$ we also have the parabolic pair

$$(\mathcal{P}^*_{\Delta_0},\mathcal{L}^*_{\Delta_0}) := (\psi(\mathcal{P}_{\Delta_0}),\psi(\mathcal{L}_{\Delta_0})),$$

which is defined over K_s. By [Spr, Theorem 15.4.6 and Proposition 16.1.1], applied to $\mathcal{G}^*(K_s)$ as in the proof of Lemma 1.a, there exists a $g_0 \in \mathcal{G}^*(K_s)$ such that

$$g_0 \psi(\mathcal{P}_{\Delta_0}) g_0^{-1} \supset \mathcal{B}^* \quad \text{and} \quad g_0 \psi(\mathcal{L}_{\Delta_0}) g_0^{-1} \supset \mathcal{T}^*.$$

Replacing ψ by $\mathrm{Ad}(g_0) \circ \psi$, we may assume that $\mathcal{P}^*_{\Delta_0} \supset \mathcal{B}^*$ and $\mathcal{L}^*_{\Delta_0} \supset \mathcal{T}^*$.

Lemma 3. *(a) The parabolic pair $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$ is defined over K.*
(b) $u(\gamma) \in \mathcal{L}^*_{\Delta_0}(K_s)$ for all $\gamma \in \Gamma_K$.
(c) Let $\mathcal{H}$ be a K_s-subgroup of $\mathcal{G}$ containing $\mathcal{L}_{\Delta_0}$. Then $\mathcal{H}$ is defined over K if and only if $\psi(\mathcal{H})$ is defined over K.

Proof. (a) Recall that a K_s-subgroup of $\mathcal{G}$ is defined over K if and only if it is Γ_K-stable. Applying that to $\mathcal{P}_{\Delta_0}$ and $\mathcal{L}_{\Delta_0}$, we see from (3) that $\mathrm{Ad}(u(\gamma)) \circ \gamma^*$ stabilizes $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$. In other words, $\mathrm{Ad}(u(\gamma))$ sends $(\gamma^* \mathcal{P}^*_{\Delta_0}, \gamma^* \mathcal{L}^*_{\Delta_0})$ to $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$. By the above setup both $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$ and $(\gamma^* \mathcal{P}^*_{\Delta_0}, \gamma^* \mathcal{L}^*_{\Delta_0})$ are standard, that is, contain $(\mathcal{B}^*, \mathcal{T}^*)$. But two conjugate standard parabolic pairs of $\mathcal{G}^*$ are equal, so γ^* stabilizes $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$. Hence this parabolic pair is defined over K.
(b) From the argument for part (a) we see that $\mathrm{Ad}(u(\gamma))$ stabilizes $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$. As every parabolic subgroup is its own normalizer:

$$u(\gamma) \in N_{\mathcal{G}^*(K_s)}(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0}) = N_{\mathcal{P}^*_{\Delta_0}(K_s)}(\mathcal{L}^*_{\Delta_0}) = \mathcal{L}^*_{\Delta_0}(K_s).$$

(c) By part (b) $\mathrm{Ad}(u(\gamma))$ stabilizes $\psi(\mathcal{H})$, for any $\gamma \in \Gamma_K$. From (3) we see now that γ stabilizes $\mathcal{H}$ if and only if it stabilizes $\psi(\mathcal{H})$. □

We thank Jean-Loup Waldspurger for showing us the proof of the next result.

Lemma 4. *Suppose that Theorem 2 holds for all quasi-split K-groups. Then it holds for all reductive K-groups $\mathcal{G}$.*

Proof. By Lemma 1.a it suffices to consider two standard Levi K-subgroups $\mathcal{L}_I, \mathcal{L}_J$ of $\mathcal{G}$. We assume that they are $\mathcal{G}(\overline{K})$-conjugate. By Lemma 1.b this depends only the Weyl group of $(\mathcal{G}, \mathcal{T})$, so we can pick $w \in N_{\mathcal{G}(K_s)}(\mathcal{T})$ with $w \mathcal{L}_I w^{-1} = \mathcal{L}_J$. We denote the images of these objects (and of $\mathcal{P}_I, \mathcal{P}_J$) under ψ by a *, e.g. $\mathcal{L}_I^* = \psi(\mathcal{L}_I)$. Then $w^* \mathcal{L}^*_I w^{*-1} = \mathcal{L}^*_J$ and by Lemma 3.c the parabolic pairs $(\mathcal{P}^*_I, \mathcal{L}^*_I)$ and $(\mathcal{P}^*_J, \mathcal{L}^*_J)$ are defined over K.

Using the hypothesis of the lemma for $\mathcal{G}^*$, we pick a $h^* \in \mathcal{G}^*(K)$ with $h^* \mathcal{L}^*_I h^{*-1} = \mathcal{L}^*_J$. Write $\mathcal{P}^* = h^* \mathcal{P}^*_I h^{*-1}$, $h = \psi^{-1}(h^*)$ and $\mathcal{P} := \psi^{-1}(\mathcal{P}^*)$. Here $\mathcal{P}^*$ is defined over K because $\mathcal{P}^*_I$ and h^* are. Furthermore

$$\mathcal{P}^* \supset \mathcal{L}^*_J \supset \mathcal{L}^*_{\Delta_0} \quad \text{and} \quad \mathcal{P} \supset \mathcal{L}_J \supset \mathcal{L}_{\Delta_0},$$

so by Lemma 3.c $\mathcal{P}$ is defined over K.

Thus the parabolic K-subgroups $\mathcal{P}_I$ and $\mathcal{P}$ of $\mathcal{G}$ are conjugate by $h \in \mathcal{G}(K_s)$. Hence they are also $\mathcal{G}(K)$-conjugate, say $g \mathcal{P} g^{-1} = \mathcal{P}_I$ with $g \in \mathcal{G}(K)$. Now $g \mathcal{L}_J g^{-1}$ is a Levi factor of $\mathcal{P}_I$ defined over K. By [Spr, Proposition 16.1.1] $g \mathcal{L}_J g^{-1}$ is $\mathcal{P}_I(K)$-conjugate to $\mathcal{L}_I$, so $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate. □

Lemma 5. *Suppose that Theorem 2 holds for all absolutely simple K-groups. Then it holds for all reductive K-groups $\mathcal{G}$.*

Similarly, if Theorem 2 holds for all absolutely simple, quasi-split K-groups, then it holds for all quasi-split reductive K-groups $\mathcal{G}$.

Proof. The set of standard Levi K-subgroups of $\mathcal{G}$ does not change when we replace $\mathcal{G}$ by its adjoint group $\mathcal{G}_{\mathrm{ad}}$, because it depends only Δ, Δ_0 and the Galois action on those. In Lemma 1 the criterion (ii) also does not change under this replacement, because $W(\mathcal{G}_{\mathrm{ad}}, \mathcal{S}_{\mathrm{ad}}) \cong W(\mathcal{G}, \mathcal{S})$ (where $\mathcal{S}_{\mathrm{ad}}$ denotes the image of $\mathcal{S}$ in $\mathcal{G}_{\mathrm{ad}}$). Therefore we may assume that $\mathcal{G}$ is of adjoint type.

Now $\mathcal{G}$ is a direct product of K-simple groups of adjoint type. If Theorem 2 holds for $\mathcal{G}'$ and $\mathcal{G}''$, then it clearly holds for $\mathcal{G}' \times \mathcal{G}''$. Thus we may further assume that $\mathcal{G}$ is K-simple and of adjoint type.

Then there are simple adjoint K_s-groups $\mathcal{G}_i$ such that

$$\mathcal{G} \cong \mathcal{G}_1 \times \cdots \times \mathcal{G}_d \qquad \text{as } K_s\text{-groups.} \tag{4}$$

Since $\mathcal{G}$ is K-simple, the action of Γ_K (which defines the K-structure) permutes the $\mathcal{G}_i$ transitively. Write $\mathcal{T}_i = \mathcal{T} \cap \mathcal{G}_i$, so that $\mathcal{T} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_d$ and

$$W(\mathcal{G}, \mathcal{T}) = W(\mathcal{G}_1, \mathcal{T}_1) \times \cdots \times W(\mathcal{G}_d, \mathcal{T}_d), \tag{5}$$

$$\Phi(\mathcal{G}, \mathcal{T}) = \Phi(\mathcal{G}_1, \mathcal{T}_1) \sqcup \cdots \sqcup \Phi(\mathcal{G}_d, \mathcal{T}_d). \tag{6}$$

Put $\Delta^i = \Delta \cap \Phi(\mathcal{G}_i, \mathcal{T}_i)$ and $\Delta_0^i = \Delta_0 \cap \Phi(\mathcal{G}_i, \mathcal{T}_i)$. Let Γ_i be the Γ_K-stabilizer of $\mathcal{G}_i$. By [Spr, Proposition 15.5.3] $\mu_{\mathcal{B}}(\Gamma_i)$ stabilizes Δ_0^i and $\mu_{\mathcal{B}}(\Gamma)\Delta_0^i = \Delta_0$.

Select $\gamma_i \in \Gamma_K$ with $\gamma_i(\mathcal{G}_1) = \mathcal{G}_i$ and $\gamma_1 = 1$. Note that $\mathcal{B}_i := \gamma_i(\mathcal{B} \cap \mathcal{G}_1)$ is a Borel subgroup of $\mathcal{G}_i$. To simplify things a little bit, we replace $\mathcal{B}$ by $\mathcal{B}_1 \times \cdots \times \mathcal{B}_d$. With this new $\mathcal{B}$:

$$\mu_{\mathcal{B}}(\gamma_i)\Delta^1 = \gamma_i(\Delta^1) = \Delta^i \quad \text{and} \quad \mu_{\mathcal{B}}(\gamma_i)\Delta_0^1 = \gamma_i(\Delta_0^1) = \Delta_0^i. \tag{7}$$

By Lemma 1.a it suffices to prove Theorem 2 for standard Levi K-subgroups $\mathcal{L}_I, \mathcal{L}_J$ of $\mathcal{G}$, where $\Delta_0 \subset I, J \subset \Delta$ and I, J are $\mu_{\mathcal{B}}(\Gamma)$-stable. We suppose that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(\overline{K})$-conjugate, and we have to show that they are also $\mathcal{G}(K)$-conjugate.

By (4) the groups $\mathcal{L}_I \cap \mathcal{G}_i$ and $\mathcal{L}_J \cap \mathcal{G}_i$ are $\mathcal{G}_i(K_s)$-conjugate, for $i = 1, \ldots, d$. The absolutely simple group $\mathcal{G}_i$ is defined over the field $K_i := K_s^{\Gamma_i}$. By the assumption of the current lemma, $\mathcal{L}_I \cap \mathcal{G}_i$ and $\mathcal{L}_J \cap \mathcal{G}_i$ are $\mathcal{G}_i(K_i)$-conjugate.

Let $\mathcal{S}_i$ be the maximal K_i-split torus of $\mathcal{G}_i$ such that

$$\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_d \qquad \text{as } K_s\text{-groups.}$$

Then Γ_i acts trivially on $W(\mathcal{G}_i, \mathcal{S}_i)$, because the latter is generated by Γ_i-invariant reflections [Spr, Lemma 15.3.7.ii]. Consider the $\mu_{\mathcal{B}}(\Gamma_i)$-stable sets $I^i = I \cap \Phi(\mathcal{G}_i, \mathcal{T}_i)$ and $J^i = J \cap \Phi(\mathcal{G}_i, \mathcal{T}_i)$. By Lemma 1.b the sets $I^i \setminus \Delta_0^i$ and $J^i \setminus \Delta_0^i$ are $W(\mathcal{G}_i, \mathcal{S}_i)$-associate. Pick $w_1 \in W(\mathcal{G}_1, \mathcal{S}_1)$ with

$$w_1(J^1 \setminus \Delta_0^1) = I^1 \setminus \Delta_0^1.$$

The analogue of (5) for $\mathcal{S}$ reads

$$W(\mathcal{G}, \mathcal{S}) = \big(W(\mathcal{G}_1, \mathcal{S}_1) \times \cdots \times W(\mathcal{G}_d, \mathcal{S}_d) \big)^{\Gamma_K}. \tag{8}$$

Put $w_i = \gamma_i(w_1) \in W(\mathcal{G}_i, \mathcal{S}_i)$. From (8) we see that $w := w_1 \times \cdots \times w_d$ lies in $W(\mathcal{G}, \mathcal{S})$. By (7) and by the $\mu_{\mathcal{B}}(\Gamma_K)$-stability of I and J:

$$w_i(J^i \setminus \Delta_0^i) = I^i \setminus \Delta_0^i \qquad \text{for } i = 1, \ldots, d.$$

Hence $w(J \setminus \Delta_0) = I \setminus \Delta_0$. Now Lemma 1.b says that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate.

Finally, we take a closer at the special case where the initial group $\mathcal{G}$ was quasi-split over K. Then the group $\mathcal{G}_i$ from (4) is quasi-split over K_i, for instance because it admits the Γ_i-stable Borel subgroup $\mathcal{B}_i$. So in the above proof of Theorem 2 for a quasi-split group $\mathcal{G}$, we only need to assume it for the quasi-split absolutely simple groups $\mathcal{G}_i$. □

When $\mathcal{G}$ is quasi-split over K, Δ_0 is empty and we can choose $\mathcal{B}$ and $\mathcal{T}$ defined over K, that is, Γ_K-stable. Then the μ-action of Γ_K agrees with the action defining the K-structure, and it is known from [SiZi, Proposition 2.4.2] that

$$W(\mathcal{G},\mathcal{S}) = W(\mathcal{G},\mathcal{T})^{\Gamma_K}. \tag{9}$$

In this case every Γ_K-stable subset I of Δ gives rise to standard Levi K-subgroup $\mathcal{L}_I$ of $\mathcal{G}$. Lemma 1.b says that $\mathcal{L}_I$ and $\mathcal{L}_J$ are

- $\mathcal{G}(\overline{K})$-conjugate if and only if I and J are $W(\mathcal{G},\mathcal{T})$-associate;
- $\mathcal{G}(K)$-conjugate if and only if I and J are $W(\mathcal{G},\mathcal{T})^{\Gamma_K}$-associate.

Lemma 6. *Theorem 2 holds when $\mathcal{G}$ is absolutely simple and quasi-split (over K).*

Proof. By Lemma 1 and the remarks after its proof, Theorem 2 holds for K-split reductive groups. Thus it suffices to consider quasi-split, non-split, absolutely simple K-groups. In view of Lemma 1.a, we may assume that $\mathcal{L} = \mathcal{L}_I$ and $\mathcal{L}' = \mathcal{L}_J$ are standard Levi K-subgroups of $\mathcal{G}$. By the above criteria for conjugacy, the only things that matter are the root system $\Phi(\mathcal{G},\mathcal{T})$, its Weyl group and the Galois action on those. These reductions make a case-by-case consideration feasible. In each case, we suppose that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(\overline{K})$-conjugate and we have to show that $wI = J$ for some $w \in W(\mathcal{G},\mathcal{S}) = W(\mathcal{G},\mathcal{T})^{\Gamma_K}$.

Type $A_n^{(2)}$. The Γ_K-stable subset $I \subset A_n^{(2)}$ has the form

$$A_{n_1}{}^2 \times \cdots \times A_{n_k}{}^2 \times A_{n_0}^{(2)},$$

where n_0 has the same parity as n and

$$n_1 + \cdots + n_k + k \leq (n - n_0)/2.$$

Here the connected component $A_{n_0}^{(2)}$ lies in the middle of the Dynkin diagram, and all the connected components A_{n_i} occur two times, symmetrically around the middle. Similarly J looks like

$$A_{m_1}{}^2 \times \cdots \times A_{m_l}{}^2 \times A_{m_0}^{(2)}.$$

Lemma 1.b tells us that I and J are associate by an element w of $W(\mathcal{G},\mathcal{T}) \cong S_{n+1}$. Hence the multisets $(n_1, n_1, \ldots, n_k, n_k, n_0)$ and $(m_1, m_1, \ldots, m_l, m_l, m_0)$ are equal. Only the element n_0 (resp. m_0) occurs with odd multiplicity, so $n_0 = m_0$. Composing w inside S_{n+1} with a suitable permutation on the components A_{n_0} of I, we may assume that w fixes the subset $A_{m_0}^{(2)} = A_{n_0}^{(2)}$ of $A_n^{(2)}$. In $A_{(n-n_0-2)/2}{}^2$, the complement of $A_{n_0}^{(2)}$ and the two adjacent simple roots, the sets

$$I' := (A_{n_1} \times \cdots \times A_{n_k})^2 \quad \text{and} \quad J' := (A_{m_1} \times \cdots \times A_{m_l})^2$$

are associated by w. In particular $k = l$. With the group $(S^2_{(n-n_0)/2})^{\Gamma_K} \cong S_{(n-n_0)/2}$ we can sort I' and J', so that $n_1 \geq \cdots \geq n_k$ and $m_1 \geq \cdots \geq m_k$. As I' and J' came from the same multiset, they become equal after sorting. This shows that $w'I' = J'$ for some $w' \in (S^2_{(n-n_0)/2})^{\Gamma_K} \subset W(\mathcal{G},\mathcal{T})^{\Gamma_K}$. In view of (9), this says $w'I = J$ with $w' \in W(\mathcal{G},\mathcal{S})$.

Type $D_n^{(2)}$. The Γ_K-stable subset $I \subset D_n^{(2)}$ has the type

$$A_{n_1} \times \cdots \times A_{n_k} \times D_{n_0}^{(2)} \quad \text{with } n_0 \geq 2 \text{ and } n_1 + \cdots + n_k + k + n_0 \leq n,$$

or (when $n_0 = 0$)

$$A_{n_1} \times \cdots \times A_{n_k} \quad \text{with } n_1 + \cdots + n_k + k + 1 \leq n.$$

Similarly we write

$$J = A_{m_1} \times \cdots \times A_{m_l} \times D_{m_0}^{(2)} \quad \text{with } m_0 \neq 1.$$

By assumption there exists a $w \in W(D_n)$ such that $w(I) = J$. Suppose that $n_0 \geq 2$ and $wD_{n_0}^{(2)}$ is a component A_{n_0} of J. In the standard construction of the root system D_n in $\mathbb{Z}^n$, the subset $D_{n_0}^{(2)}$ involves precisely n_0 coordinates, whereas A_{n_0} involves $n_0 + 1$ coordinates (irrespective of where it is located in the Dynkin diagram). As $W(D_n) \subset S_n \ltimes \{\pm 1\}^n$, applying w to a set of simple roots does not change the number of involved coordinates. This contradiction shows that w must map $D_{n_0}^{(2)}$ to $D_{m_0}^{(2)}$ if $n_0 \geq 2$.

For the same reason, if $m_0 \geq 2$, then $w^{-1}D_{m_0}^{(2)}$ must be contained in $D_{n_0}^{(2)}$. Hence $n_0 = m_0$ and $wD_{n_0}^{(2)} = D_{m_0}^{(2)}$ whenever $n_0 \geq 2$ or $m_0 \geq 2$. Obviously the same conclusion holds in the remaining case $n_0 = m_0 = 0$.

Consider the sets of simple roots

$$I' := A_{n_1} \times \cdots \times A_{n_k} \quad \text{and} \quad J' := A_{m_1} \times \cdots \times A_{m_l}$$

They are associated by $w \in W(D_n)$, so $(n_1, \ldots, n_k) = (m_1, \ldots, m_l)$ as multisets. Then there exists a $w' \in S_{n-n_0-1}$ (or in S_{n-2} if $n_0 = 0$) with $w'I' = J'$. Such a w' commutes with the diagram automorphism, so $w'I = J$ with $w' \in W(D_n)^{\Gamma_K} = W(\mathcal{G}, \mathcal{S})$.

Type $D_4^{(3)}$. The cardinality of I is 0, 1, 3 or 4, and for all these sizes there is a unique Γ_K-stable subset of the Dynkin diagram $D_4^{(3)}$. Hence $\mathcal{L}_I$ is completely characterized by its rank $|I| = \mathrm{rk}(\Phi(\mathcal{L}_I, \mathcal{T}))$. For each possible rank there is a unique $\mathcal{G}(K)$-conjugacy class of Levi K-subgroups, and those Levi subgroups definitely cannot be $\mathcal{G}(\overline{K})$-conjugate to Levi subgroups of other ranks.

Type $E_6^{(2)}$. We label the Dynkin diagram as

$$\begin{array}{ccccccccc} & & & & \alpha_2 & & & & \\ & & & & | & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 \end{array}$$

The nontrivial automorphism γ exchanges α_1 with α_6 and α_3 with α_5. Since $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(\overline{K})$-conjugate, they have the same rank $|I| = |J|$. When $|I| = 0$ or $|I| = 6$, this already shows that $J = I$.

For the remaining ranks, we will check that the $W(E_6)$-association classes of Γ_K-stable subsets of E_6 of that rank are exactly the $W(E_6)^{\Gamma_K}$-association classes. That suffices, for it implies that the $W(E_6)$-associate sets I and J are already associated by an element of $W(E_6)^{\Gamma_K}$.

For $|I| = 1$, the options are $\{\alpha_2\}$ and $\{\alpha_4\}$. These sets are associated by an element $w_2 \in \langle s_{\alpha_2}, s_{\alpha_4} \rangle \cong S_3$. As α_2 and α_4 are fixed by Γ_K, $w_2 \in W(E_6)^{\Gamma_K}$. Hence there is only one $W(E_6)^{\Gamma_K}$-association class of I's of rank 1.

When $|I| = 2$, the possible sets of simple roots are

$$I_{2,1} = \{\alpha_2, \alpha_4\},\ I_{2,2} = \{\alpha_3, \alpha_5\}\ I_{2,3} = \{\alpha_1, \alpha_6\}.$$

Among these $I_{2,1} \cong A_2$ is the only connected Dynkin diagram, so it is not $W(E_6)$-associate to the other two. Pick $w_1 \in \langle s_{\alpha_1}, s_{\alpha_3}\rangle \cong S_3$ with $w_1(\alpha_1) = \alpha_3$. Then $(\gamma(w_1))(\alpha_6) = \alpha_5$ and $w_1\gamma(w_1) \in W(E_6)^{\Gamma_K}$. We conclude the $W(E_6)$-association classes on
$\{I_{2,1}, I_{2,2}, I_{2,3}\}$ are exactly the $W(E_6)^{\Gamma_K}$-association classes.

In the case $|I| = 3$, the possibilities are

$$I_{3,1} = \{\alpha_3, \alpha_4, \alpha_5\},\ I_{3,2} = \{\alpha_2, \alpha_3, \alpha_5\},\ I_{3,3} = \{\alpha_1, \alpha_2, \alpha_6\},\ I_{3,4} = \{\alpha_1, \alpha_4, \alpha_6\}.$$

Among these $I_{3,1} \cong A_3$ is the only connected diagram, so it is not $W(E_6)$-associate to the other three. The sets $I_{3,2}$ and $I_{3,3}$ are associated via $w_1\gamma(w_1)$, while the sets $I_{3,3}$ and $I_{3,4}$ are associated via w_2 (as above). Hence $\{I_{3,2}, I_{3,3}, I_{3,4}\}$ forms one $W(E_6)^{\Gamma_K}$-association class and one $W(E_6)$-association class.

If I has rank 4, it is one of

$$\begin{aligned}
&\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\} \cong A_2 \times A_2,\\
&\{\alpha_1, \alpha_2, \alpha_4, \alpha_6\} \cong A_2 \times A_1 \times A_1,\\
&\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\} \cong A_4.
\end{aligned}$$

These three are mutually non-isomorphic, so they form three association classes, both for $W(E_6)$ and for $W(E_6)^{\Gamma_K}$.

When $|I| = 5$, we have the options

$$E_6 \setminus \{\alpha_2\} \cong A_5 \quad \text{and} \quad E_6 \setminus \{\alpha_4\} \cong A_2 \times A_2 \times A_1.$$

These are not isomorphic, so they form two association classes, both for $W(E_6)$ and for $W(E_6)^{\Gamma_K}$. $\square$

3. Connected linear algebraic groups

The previous results about reductive groups can be generalized to all linear algebraic groups. This relies mainly on the theory initiated by Borel and Tits [BoTi], and worked out much further by Conrad, Gabber and Prasad [CGP, CP].

Let $\mathcal{G}$ be a connected linear algebraic K-group. We recall from [Spr, Theorem 4.3.7] that $\mathcal{G}$ is irreducible and smooth as K-variety. In particular it is a smooth affine group – the terminology used in [CGP].

When $\mathcal{G}$ has a Levi decomposition, it is clear how Levi subgroups of $\mathcal{G}$ can be defined: as a Levi subgroup (in the sense of the previous section) of a Levi factor of $\mathcal{G}$. However, there exist linear algebraic groups that do not admit any Levi decomposition, even over $\overline{K}$ [CGP, Appendix A.6]. For those we do not know a good notion of Levi subgroups.

Instead we investigate a closely related kind of subgroups, already present in [Spr]. Fix a K-rational cocharacter $\lambda : GL_1 \to \mathcal{G}$ and put

$$\begin{aligned}
\mathcal{P}_{\mathcal{G}}(\lambda) &= \{g \in \mathcal{G} : \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists in } \mathcal{G}\},\\
\mathcal{U}_{\mathcal{G}}(\lambda) &= \{g \in \mathcal{G} : \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} = 1\},\\
\mathcal{Z}_{\mathcal{G}}(\lambda) &= \{g \in \mathcal{G} : \lambda(a) g \lambda(a)^{-1} = g \ \forall a \in GL_1\} = \mathcal{P}_{\mathcal{G}}(\lambda) \cap \mathcal{P}_{\mathcal{G}}(\lambda^{-1}).
\end{aligned}$$

These are K-subgroups of $\mathcal{G}$ [CGP, Lemma 2.1.5]. Moreover $\mathcal{U}_\mathcal{G}(\lambda)$ is K-split unipotent [CGP, Proposition 2.1.10], and there is a Levi-like decomposition [CGP, Proposition 2.1.8]

$$\mathcal{P}_\mathcal{G}(\lambda) = \mathcal{Z}_\mathcal{G}(\lambda) \ltimes \mathcal{U}_\mathcal{G}(\lambda). \tag{10}$$

By [CGP, Lemma 2.1.5] $Z_\mathcal{G}(\lambda)$ is the (scheme-theoretic) centralizer of $\lambda(GL_1)$, a K-split torus in $\mathcal{G}$. More generally, if $\mathcal{S}'$ is any K-split torus in $\mathcal{G}$, $Z_\mathcal{G}(\mathcal{S}')$ is of the form $Z_\mathcal{G}(\lambda)$. To see this, one can take a K-rational cocharacter $\lambda : GL_1 \to \mathcal{S}'$ whose image does not lie in the kernel of any of the roots of $(\mathcal{G}, \mathcal{S}')$.

Let $\mathcal{R}_{u,K}(\mathcal{G})$ denote the unipotent K-radical of $\mathcal{G}$. By definition, a pseudo-parabolic K-subgroup of $\mathcal{G}$ is a group of the form

$$\mathcal{P}_\lambda := \mathcal{P}_\mathcal{G}(\lambda)\mathcal{R}_{u,K}(\mathcal{G}) \quad \text{for some } K\text{-rational cocharacter } \lambda : GL_1 \to \mathcal{G}.$$

Similarly we define

$$\mathcal{L}_\lambda := Z_\mathcal{G}(\lambda)\mathcal{R}_{u,K}(\mathcal{G}) = \mathcal{P}_\lambda \cap \mathcal{P}_{\lambda^{-1}}.$$

We call $\mathcal{L}_\lambda$ a pseudo-Levi subgroup of $\mathcal{G}$. Just a like a Levi subgroup of a reductive group is intersection of a parabolic subgroup with an opposite parabolic, a pseudo-Levi subgroup is the interesection of a pseudo-parabolic subgroup with an opposite pseudo-parabolic. We note that $\mathcal{L}_\lambda$ contains the centralizer of the K-split torus $\lambda(GL_1)$, but it may be strictly larger than the latter.

Unfortunately the groups $\mathcal{P}_\lambda$ and $\mathcal{L}_\lambda$ do in general not fit in a decomposition like (10), because $\mathcal{U}_\mathcal{G}(\lambda)$ may intersect $\mathcal{R}_{u,K}(\mathcal{G})$ nontrivially. When $\mathcal{G}$ is pseudo-reductive over K (that is, $\mathcal{R}_{u,K}(\mathcal{G}) = 1$), the groups $\mathcal{P}_\lambda$ and $\mathcal{L}_\lambda$ coincide with $\mathcal{P}_\mathcal{G}(\lambda)$ and $Z_\mathcal{G}(\lambda)$, respectively. In view of the remarks after (10), the pseudo-Levi K-subgroups of a pseudo-reductive group are precisely the centralizers of the K-split tori in that group.

More specifically, when $\mathcal{G}$ is reductive, the $\mathcal{P}_\lambda$ are precisely the parabolic subgroups of $\mathcal{G}$ [CGP, Proposition 2.2.9], the $\mathcal{L}_\lambda$ are the Levi subgroups of $\mathcal{G}$ and (10) is an actual Levi decomposition of $\mathcal{P}_\lambda$. This justifies our terminology "pseudo-Levi subgroup".

The notions pseudo-parabolic and pseudo-Levi are preserved under separable extensions of the base field K [CGP, Proposition 1.1.9], but not necessarily under inseparable base-change. This is caused by the corresponding behaviour of the unipotent K-radical.

We consider the maximal quotient K-group of $\mathcal{G}$ which is pseudo-reductive:

$$\mathcal{G}' := \mathcal{G}/\mathcal{R}_{u,K}(\mathcal{G}).$$

Lemma 7. *There is a natural bijection between the sets of pseudo-parabolic K-subgroups of $\mathcal{G}$ and of $\mathcal{G}'$. It remains a bijection if we take K-rational conjugacy classes on both sides.*

Proof. The map sends $\mathcal{P}_\lambda$ to $\mathcal{P}'_\lambda := \mathcal{P}_\lambda/\mathcal{R}_{u,K}(\mathcal{G})$. It is bijective by [CGP, Proposition 2.2.10]. According to [CGP, Proposition 3.5.7] every pseudo-parabolic subgroup of $\mathcal{G}$ (or of $\mathcal{G}'$) is its own scheme-theoretic normalizer. Hence the variety of $\mathcal{G}(K)$-conjugates of $\mathcal{P}_\lambda$ is $\mathcal{G}(K)/\mathcal{P}_\lambda(K)$. By [CGP, Lemma C.2.1] this is isomorphic with $(\mathcal{G}/\mathcal{P}_\lambda)(K)$. Next [CGP, Proposition 2.2.10] tells us that the K-varieties $\mathcal{G}/\mathcal{P}_\lambda$ and $\mathcal{G}'/\mathcal{P}'_\lambda$ can be identified. We obtain

$$\mathcal{G}(K)/\mathcal{P}_\lambda(K) \cong (\mathcal{G}/\mathcal{P}_\lambda)(K) \cong (\mathcal{G}'/\mathcal{P}'_\lambda)(K) \cong \mathcal{G}'(K)/\mathcal{P}'_\lambda(K),$$

where the right hand side can be interpreted as the variety of $\mathcal{G}'(K)$-conjugates of $\mathcal{P}'_\lambda$. It follows that two pseudo-parabolic K-subgroups $\mathcal{P}_\lambda$ and $\mathcal{P}_\mu$ are $\mathcal{G}(K)$-conjugate if and only if $\mathcal{P}'_\lambda$ and $\mathcal{P}'_\mu$ are $\mathcal{G}'(K)$-conjugate. □

The setup from the start of Section 2 (with $\mathcal{S}, \mathcal{T}, \Delta_0, \ldots$) remains valid for the current $\mathcal{G}$, when we reinterpret $\mathcal{B}$ as a minimal pseudo-parabolic K_s-subgroup of $\mathcal{G}$. (Also, the K-group $Z_{\mathcal{G}}(\mathcal{S})$ is not always pseudo-Levi in $\mathcal{G}$, for that we still have to add $\mathcal{R}_{u,K}(\mathcal{G})$ to it.) We refer to [CGP, Proposition C.2.10 and Theorem C.2.15] for the proofs in this generality.

The set of simple roots Δ_K for $(\mathcal{G}, \mathcal{S})$ can again be identified with $(\Delta \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$. For every $\mu_{\mathcal{B}}(\Gamma_K)$-stable subset I of Δ containing Δ_0 we get a standard pseudo-parabolic K-subgroup $\mathcal{P}_I$ of $\mathcal{G}$. By Lemma 7 and [Spr, Theorem 15.4.6] every pseudo-parabolic K-subgroup is $\mathcal{G}(K)$-conjugate to a unique such $\mathcal{P}_I$. The unicity implies that two pseudo-parabolic K-subgroups of $\mathcal{G}$ are $\mathcal{G}(K)$-conjugate if and only if they are $\mathcal{G}(K_s)$-conjugate. (Recall that by [CGP, Proposition 3.5.2.ii] pseudo-parabolicity is preserved under base change from K to K_s.) By [CGP, Proposition 3.5.4] (which can only be guaranteed when the fields are separably closed, as pointed out to us by Gopal Prasad), $\mathcal{G}(K_s)$-conjugacy of pseudo-parabolic subgroups is equivalent to $\mathcal{G}(\overline{K})$-conjugacy.

Write $\mathcal{P}_I = \mathcal{P}_{\lambda_I}$ for some K-rational homomorphism $\lambda_I : GL_1 \to \mathcal{S}$. It is easy to see (from [Spr, Lemma 15.4.4] and Lemma 7) that $\mathcal{P}_{\lambda_I^{-1}}$ does not depend on the choice of λ_I, and we may denote it by $\mathcal{P}_{-I}$. Then we define

$$\mathcal{L}_I := \mathcal{P}_I \cap \mathcal{P}_{-I} = \mathcal{P}_{\lambda_I} \cap \mathcal{P}_{\lambda_I^{-1}} = \mathcal{L}_{\lambda_I}.$$

We call $\mathcal{L}_I$ a standard pseudo-Levi subgroup of $\mathcal{G}$. It is the inverse image, with respect to the quotient map $\mathcal{G} \to \mathcal{G}'$, of the (standard pseudo-Levi) K-subgroup of $\mathcal{G}'$ called L_I in [Spr, Lemma 15.4.5]. In the introduction we called this $\mathcal{L}_{I_K}$, which relates to $\mathcal{L}_I$ by $I_K = (I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$.

We are ready to generalize Lemma 1.

Lemma 8. (*a*) *Every pseudo-Levi K-subgroup of $\mathcal{G}$ is $\mathcal{G}(K)$-conjugate to a standard Levi K-subgroup of $\mathcal{G}$.*

(*b*) *For two standard pseudo-Levi K-subgroups $\mathcal{L}_I, \mathcal{L}_J$ the following are equivalent:*

(i) $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate;

(ii) $(I \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ and $(J \setminus \Delta_0)/\mu_{\mathcal{B}}(\Gamma_K)$ are $W(\mathcal{G}, \mathcal{S})$-associate.

Proof. (a) Let $\mathcal{L}_\lambda$ be a pseudo-Levi K-subgroup of $\mathcal{G}$. Because $\mathcal{P}_\lambda$ is $\mathcal{G}(K)$-conjugate to a standard pseudo-parabolic K-subgroup $\mathcal{P}_I$ of $\mathcal{G}$, we may assume that

$$\mathcal{L}_\lambda \subset \mathcal{P}_\lambda = \mathcal{P}_I.$$

Since all maximal K-split tori of $\mathcal{P}_I$ are $\mathcal{P}_I(K)$-conjugate [CGP, Theorem C.2.3], we may further assume that the image of λ is contained in $\mathcal{S}$. By [CGP, Corollary 2.2.5] the K-split unipotent radical $\mathcal{R}_{us,K}(\mathcal{P}_I)$ equals both $\mathcal{U}_{\mathcal{G}}(\lambda_I)\mathcal{R}_{u,K}(\mathcal{G})$ and $\mathcal{U}_{\mathcal{G}}(\lambda)\mathcal{R}_{u,K}(\mathcal{G})$. By [Spr, Lemma 15.4.4] the Lie algebra of $\mathcal{P}_I/\mathcal{R}_{u,K}(\mathcal{G})$ can be analysed in terms of the weights for the adjoint action $\mathrm{Ad}(\lambda)$ of GL_1 on the Lie algebra of $\mathcal{G}'$. Namely, $\mathcal{P}_I/\mathcal{R}_{u,K}(\mathcal{G})$ corresponds to the sum of the subspaces on which GL_1 acts by characters $a \mapsto a^n$ with $n \in \mathbb{Z}_{\geq 0}$. The Lie algebra of the subgroup

$$\mathcal{R}_{us,K}(\mathcal{P}_I)\mathcal{R}_{u,K}(\mathcal{G})/\mathcal{R}_{u,K}(\mathcal{G})$$

is the sum of the subspaces on which $\mathrm{Ad}(\lambda)$ acts as $a \mapsto a^n$ with $n \in \mathbb{Z}_{>0}$. From (10) inside $\mathcal{G}'$ we deduce that the Lie algebra of $\mathcal{L}_I/\mathcal{R}_{u,K}(\mathcal{G})$ is the direct sum of the Lie algebra of $Z_{\mathcal{G}'}(\mathcal{S})$ and the root spaces for roots $\alpha \in \Phi(\mathcal{G}', \mathcal{S})$ with $\langle \alpha, \lambda \rangle = 0$. This holds for both λ and λ_I, from which we conclude that $\mathcal{L}_\lambda = \mathcal{L}_I$.
(b) This can shown just as Lemma 1.b, using in particular that the natural map $N_{\mathcal{G}}(\mathcal{S})(K) \to W(\mathcal{G}, \mathcal{S})$ is surjective [CGP, Proposition C.2.10]. □

Lemma 9. *There is a natural bijection between the sets of pseudo-Levi K-subgroups of $\mathcal{G}$ and of $\mathcal{G}'$. It remains a bijection if we take K-rational conjugacy classes on both sides.*

Proof. The map sends $\mathcal{L}_\lambda$ to $\mathcal{L}'_\lambda := \mathcal{L}_\lambda/\mathcal{R}_{u,K}(\mathcal{G})$. This map is bijective for the same reason as in with pseudo-parabolic subgroups: $\mathcal{G}$ and $\mathcal{G}'$ have essentially the same tori, see [CGP, Proposition 2.2.10].

By [CGP, Theorem C.2.15] the K-groups $\mathcal{G}$ and $\mathcal{G}'$ have the same root system and the same Weyl group. Then Lemma 8.b says that set of the conjugacy classes of pseudo-Levi K-subgroups are parametrized by the same data for both groups. Hence the map $\mathcal{L}_I = \mathcal{L}_{\lambda_I} \mapsto \mathcal{L}'_{\lambda_I} = \mathcal{L}'_I$ also induces a bijection between these sets of conjugacy classes. □

In case $\mathcal{G}'$ is reductive, Lemmas 7 and 9 furnish bijections

$$
\begin{array}{ccc}
\{\text{parabolic } K\text{-subgroups of } \mathcal{G}'\} & \longleftrightarrow & \{\text{pseudo-parabolic } K\text{-subgroups of } \mathcal{G}\} \\
\mathcal{P}_\lambda/\mathcal{R}_{u,K}(\mathcal{G}) = P_{\mathcal{G}'}(\lambda) & \leftrightarrow & \mathcal{P}_\lambda \\
 & & \\
\{\text{Levi } K\text{-subgroups of } \mathcal{G}'\} & \longleftrightarrow & \{\text{pseudo-Levi } K\text{-subgroups of } \mathcal{G}\} \\
\mathcal{L}_\lambda/\mathcal{R}_{u,K}(\mathcal{G}) = Z_{\mathcal{G}'}(\lambda) & \leftrightarrow & \mathcal{L}_\lambda
\end{array}
\tag{11}
$$

which induce bijections between the K-rational conjugacy classes on both sides.

We will now start to work towards the main result of this section:

Theorem 10. *Let $\mathcal{G}$ be a connected linear algebraic K-group. Any two pseudo-Levi K-subgroups of $\mathcal{G}$ which are $\mathcal{G}(\overline{K})$-conjugate are already $\mathcal{G}(K)$-conjugate.*

The main steps of our argument are:

- Reduction from the general case to absolutely pseudo-simple K-groups with trivial (scheme-theoretic) centre.
- Proof when $\mathcal{G}$ quasi-split over K (i.e. Δ_0 is empty).
- Proof for absolutely pseudo-simple K-groups with trivial centre (using the quasi-split case).

Lemma 11. *Suppose that Theorem 10 holds for all absolutely pseudo-simple groups with trivial centre. Then it holds for all connected linear algebraic groups.*

Proof. By Lemma 9 we may just as well consider the pseudo-reductive group $\mathcal{G}' = \mathcal{G}/\mathcal{R}_{u,K}(\mathcal{G})$. The derived group $\mathcal{D}(\mathcal{G}')$ has the same root system and Weyl group as $\mathcal{G}'$, both over K and over K_s, by [CGP, Proposition 1.2.6 or Theorem C.2.15]. In view of Lemma 8, we may replace $\mathcal{G}'$ by $\mathcal{D}(\mathcal{G}')$. In particular $\mathcal{G}'$ is now pseudo-semisimple [CGP, Remark 11.2.3]. The K-group $\mathcal{G}'/Z(\mathcal{G}')$ is again pseudo-reductive [CP, Proposition 4.1.3]. Dividing out the scheme-theoretic centre preserves the variety of pseudo-Levi K-subgroups, as one can see from the proof of [CGP, Proposition 2.2.12.2] (which is the same statement for pseudo-parabolic subgroups). Thus we may assume that $\mathcal{G}'$ is pseudo-reductive and has trivial centre.

Let $\{\mathcal{G}'_j\}_j$ be the finite collection of normal pseudo-simple K-subgroups of $\mathcal{G}'$, as in [CGP, Proposition 3.1.8]. The root system and Weyl group of $\mathcal{G}'(K)$ decompose as products of these objects for the $\mathcal{G}'_j(K)$. Combining that with Lemma 8 we see that it suffices to prove the theorem for each of the $\mathcal{G}'_j$.

To simplify the notation, we assume from now on that $\mathcal{G}$ is a pseudo-simple K-group. Let $\{\mathcal{G}_i\}_i$ be the finite collection of normal pseudo-simple K_s-subgroups of $\mathcal{G}$. These subgroups generate $\mathcal{G}$ as K_s-group [CGP, Lemma 3.1.5] and Γ_K permutes them transitively. This serves as a slightly weaker analogue of (4). Next we can argue exactly as in the proof of Lemma 5, only replacing some parts by their previously established "pseudo"-analogues. As a consequence, it suffices to prove the theorem for the absolutely pseudo-simple groups $\mathcal{G}_i$ (over the field $K_i = K_s^{\Gamma_i}$). If necessary, we can still divide out the centre of $\mathcal{G}_i$, as observed above for $\mathcal{G}'$. □

Following [CP, §C.2] we say that a connected linear algebraic group $\mathcal{G}$ is quasi-split (over K) if a minimal pseudo-parabolic K-subgroup of $\mathcal{G}$ is also minimal as pseudo-parabolic K_s-subgroup. In view of the classification of conjugacy classes of pseudo-parabolic K_s-subgroups, this condition is equivalent to $\Delta_0 = \emptyset$.

Proposition 12. *Theorem 10 holds when $\mathcal{G}$ is quasi-split over K.*

Proof. In view of Lemma 9 we may assume that $\mathcal{G}$ is pseudo-reductive. Consider the reductive $\overline{K}$-group $\mathcal{G}^{\mathrm{red}} := \mathcal{G}/\mathcal{R}_u(\mathcal{G})$. The image of $\mathcal{T}$ in $\mathcal{G}^{\mathrm{red}}$ is a maximal torus of $\mathcal{G}^{\mathrm{red}}$. It is isomorphic to $\mathcal{T}$ via the projection map, and we may identify it with $\mathcal{T}$. Thus $\mathcal{G}^{\mathrm{red}}$ has a reduced (integral) root system $\Phi(\mathcal{G}^{\mathrm{red}}, \mathcal{T})$. The maximal K-torus $\mathcal{T}$ of $\mathcal{G}$ splits over K_s. In the terminology of [CGP, Definition 2.3.1], $\mathcal{G}$ is pseudo-split over K_s. This is somewhat weaker than split – the root system $\Phi(\mathcal{G}, \mathcal{T})$ is integral but not necessarily reduced. (It can only be non-reduced if K has characteristic 2.)

By [CGP, Proposition 2.3.10] the quotient map $\mathcal{G} \to \mathcal{G}^{\mathrm{red}}$ induces a bijection between $\Phi(\mathcal{G}^{\mathrm{red}}, \mathcal{T})$ and $\Phi(\mathcal{G}, \mathcal{T})$, provided that the latter is reduced. In general $\Phi(\mathcal{G}^{\mathrm{red}}, \mathcal{T})$ can be identified with the system of non-multipliable roots in $\Phi(\mathcal{G}, \mathcal{T})$. In particular these two root systems have the same Weyl group, and there is a $W(\mathcal{G}, \mathcal{T})$-equivariant bijection

$$\begin{array}{ccc} \{\text{parabolic subsystems of } \Phi(\mathcal{G}, \mathcal{T})\} & \longrightarrow & \{\text{parabolic subsystems of } \Phi(\mathcal{G}^{\mathrm{red}}, \mathcal{T})\} \\ R & \mapsto & R \cap \Phi(\mathcal{G}^{\mathrm{red}}, \mathcal{T}) \end{array}.$$

This induces a bijection between the sets of simple roots for these root systems, say $I \longleftrightarrow I^{\mathrm{red}}$. We note that

$$I, J \text{ are } W(\mathcal{G}, \mathcal{T})\text{-associate} \iff I^{\mathrm{red}}, J^{\mathrm{red}} \text{ are } W(\mathcal{G}^{\mathrm{red}}, \mathcal{T})\text{-associate.} \tag{12}$$

By Lemma 8.a it suffices to prove the lemma for standard pseudo-Levi K-subgroups $\mathcal{L}_I, \mathcal{L}_J$. We assume that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(\overline{K})$-conjugate. Then the pseudo-parabolic $\overline{K}$-subgroups

$$\mathcal{L}_I^{\mathrm{red}} = \mathcal{L}_I \mathcal{R}_u(\mathcal{G})/\mathcal{R}_u(\mathcal{G}) \quad \text{and} \quad \mathcal{L}_J^{\mathrm{red}} = \mathcal{L}_J \mathcal{R}_u(\mathcal{G})/\mathcal{R}_u(\mathcal{G})$$

of $\mathcal{G}^{\mathrm{red}}$ are conjugate. By Lemma 8.b the associated sets of simple roots I^{red} and J^{red} are $W(\mathcal{G}^{\mathrm{red}}, \mathcal{T})$-associate. Then (12) and Lemma 8.b entail that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K_s)$-conjugate.

As $\mathcal{G}$ is quasi-split over K, the root system $\Phi(\mathcal{G}, \mathcal{S})$ can be obtained by a simple form of Galois descent: it consists of the Γ_K-orbits in $\Phi(\mathcal{G}, \mathcal{T})$. We know from [Spr, Lemma 15.3.7] that $W(\mathcal{G}, \mathcal{S})$ is generated by the reflections s_α with $\alpha \in \Phi(\mathcal{G}, \mathcal{S})$.

Let $\mathcal{H}$ be a quasi-split reductive K-group $\mathcal{H}$ with the same root datum as $\mathcal{G}^{\mathrm{red}}$, and the same Γ_K-action on that. By [SiZi, Proposition 2.4.2] (applied to $\mathcal{H}$), the aforementioned reflections generate the subgroup $W(\mathcal{G},\mathcal{T})^{\Gamma_K}$ of $W(\mathcal{G},\mathcal{T})$. Thus (9) holds again.

We already showed that the Γ_K-stable subsets I and J of Δ are $W(\mathcal{G},\mathcal{T})$-associate. By Lemma 1.b the corresponding Levi K-subgroups $\mathcal{L}_I^{\mathcal{H}}, \mathcal{L}_J^{\mathcal{H}}$ of $\mathcal{H}$ are $\mathcal{H}(\overline{K})$-conjugate. Then Theorem 2 says that $\mathcal{L}_I^{\mathcal{H}}$ and $\mathcal{L}_J^{\mathcal{H}}$ are also $\mathcal{H}(K)$-conjugate. Again using Lemma 1.b, we deduce that I and J are associate under $W(\mathcal{G},\mathcal{T})^{\Gamma_K} = W(\mathcal{G},\mathcal{S})$. Finally Lemma 8.b tells us that $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate. □

To go beyond quasi-split linear algebraic groups, we would like to use arguments like Lemmas 3 and 4. However, the usual notion of an inner form (for reductive groups) is not flexible enough for pseudo-reductive groups [CP, Appendix C]. Better results are obtained by allowing inner twists involving a K-group of automorphisms called $(\mathrm{Aut}^{sm}_{\mathcal{D}(\mathcal{G})/K})^\circ$ in [CP, §C.2]. This leads to the notion of pseudo-inner forms of pseudo-reductive groups. Every pseudo-reductive K-group admits a quasi-split inner form, apart from some exceptions that can only occur if $\mathrm{char}(K) = 2$ and $[K : K^2] > 4$ [CP, Theorem C.2.10].

For $\alpha \in \Phi(\mathcal{G},\mathcal{S})$ the root subgroup $\mathcal{U}_\alpha$ is defined in [CGP, §C.2.21]. It is a connected unipotent K-group, whose Lie algebra is the sum of the weight spaces (with respect to the adjoint action of $\mathcal{S}$) for roots $n\alpha$ with $n \in \mathbb{Z}_{>0}$. The same applies with $\mathcal{T}$ instead of $\mathcal{S}$, but then we only get K_s-groups.

Lemma 13. *Let $\mathcal{G}$ be a pseudo-reductive K_s-group and let $\lambda : GL_1 \to \mathcal{G}$ be a K_s-rational cocharacter. Suppose that $\phi \in (\mathrm{Aut}^{sm}_{\mathcal{D}(\mathcal{G})/K_s})^\circ(K_s)$ stabilizes the K_s-subgroups $\mathcal{P}_\lambda$ and $\mathcal{L}_\lambda$.*

Let $\mathcal{H}$ be a K_s-subgroup of $\mathcal{G}$ which is generated by the union of $\mathcal{L}_\lambda$ and some $\mathcal{U}_\alpha$ with $\alpha \in \Phi(\mathcal{G},\mathcal{T})$. Then ϕ stabilizes $\mathcal{H}$.

Remark. We thank Gopal Prasad for sharing the next proof with us. It simplifies the argument from earlier versions.

Proof. By Lemma 8.a we may assume that $\mathcal{L}_\lambda$ and $\mathcal{P}_\lambda$ are standard. As $\mathcal{T}$ splits over K_s, $\mathcal{P}_\emptyset$ is defined of K_s. Then $\mathcal{P}_\emptyset \cap \mathcal{L}_\lambda$ is a minimal pseudo-parababolic K_s-subgroup of $\mathcal{L}_\lambda$ and

$$\mathcal{P}_\emptyset = (\mathcal{P}_\emptyset \cap \mathcal{L}_\lambda)\mathcal{R}_{u,K_s}(\mathcal{P}_\lambda). \tag{13}$$

By the $\mathcal{L}_\lambda(K_s)$-conjugacy of maximal K_s-tori of $\mathcal{L}_\lambda$ [CGP, Theorem C.2.3], we may assume that the image of λ lies in $\mathcal{T}$. Moreover, as conjugation by elements of $\mathcal{L}_\lambda(K_s)$ stabilizes $\mathcal{H}$ and $\mathcal{P}_\lambda$ (which contain $\mathcal{L}_\lambda$), we may adjust ϕ by such an inner automorphism. Combining that with the $\mathcal{L}_\lambda(K_s)$-conjugacy of minimal pseudo-parabolic K_s-subgroups of $\mathcal{L}_\lambda$ [CGP, Theorem C.2.5], we may assume that ϕ stabilizes $\mathcal{T}$ and $\mathcal{P}_\emptyset \cap \mathcal{L}_\lambda$. As ϕ also stabilizes the characteristic K_s-subgroup $\mathcal{R}_{u,K_s}(\mathcal{P}_\lambda)$ of $\mathcal{P}_\lambda$, it stabilizes the minimal pseudo-parabolic subgroup (13) of $\mathcal{G}$.

Consider the reductive $\overline{K}$-group $\mathcal{G}^{\mathrm{red}} = \mathcal{G}/\mathcal{R}_u(\mathcal{G})$. Notice that ϕ induces an automorphism ϕ^{red} of $\mathcal{G}^{\mathrm{red}}$ which stabilizes the images $\mathcal{T}^{\mathrm{red}}$ of $\mathcal{T}$ and $\mathcal{P}_\emptyset^{\mathrm{red}}$ of $\mathcal{P}_\emptyset$. By [CGP, Proposition 2.3.10], $\mathcal{P}_\emptyset^{\mathrm{red}}$ is a Borel subgroup of $\mathcal{G}^{\mathrm{red}}$, while $\mathcal{T}^{\mathrm{red}}$ is a maximal torus of $\mathcal{G}^{\mathrm{red}}$.

The image of $(\mathrm{Aut}^{sm}_{\mathcal{D}(\mathcal{G})/K_s})^\circ$ in $\mathrm{Aut}_{\mathcal{D}(\mathcal{G}^{\mathrm{red}})/\overline{K}}$ is contained in the identity component of the latter, which is just the $\overline{K}$-group of inner automorphisms of $\mathcal{D}(\mathcal{G}^{\mathrm{red}})$. Hence

$\phi^{\mathrm{red}} = \mathrm{Ad}(g^{\mathrm{red}})$ for some $g^{\mathrm{red}} \in \mathcal{G}^{\mathrm{red}}(\overline{K})$. Now g^{red} normalizes both $\mathcal{T}^{\mathrm{red}}$ and $\mathcal{P}_{\emptyset}^{\mathrm{red}}$, so $g^{\mathrm{red}} \in \mathcal{T}^{\mathrm{red}}(\overline{K})$ and $\phi^{\mathrm{red}} = \mathrm{Ad}(g^{\mathrm{red}})$ is the identity of $\mathcal{T}^{\mathrm{red}}$.

Since the canonical map $\mathcal{T} \to \mathcal{T}^{\mathrm{red}}$ is an isomorphism of $\overline{K}$-groups, ϕ restricts to the identity on $\mathcal{T}$. It follows that ϕ stabilizes every root subgroup $\mathcal{U}_\alpha$ with $\alpha \in \Phi(\mathcal{G}, \mathcal{T})$. In view of the particular structure of $\mathcal{H}$, this entails that ϕ stabilizes $\mathcal{H}$. □

Suppose that $\mathcal{G}^*$ is a quasi-split pseudo-reductive group and that $\psi : \mathcal{G} \to \mathcal{G}^*$ is a pseudo-inner twist. (This forces $\mathcal{G}$ to be pseudo-reductive as well.) The setup leading to Lemma 3 remains valid if we replace all objects by their pseudo-versions.

Lemma 14. *Let $\mathcal{G}$ be a pseudo-reductive K-group and let $\mathcal{H}$ be a K-subgroup of $\mathcal{G}$ which is generated by the union of $\mathcal{L}_{\Delta_0}$ and some root subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Phi(\mathcal{G}, \mathcal{S})$. Then $\mathcal{H}$ is defined over K if and only if $\psi(\mathcal{H})$ is defined over K.*

Proof. The very definition of root subgroups with respect to $\mathcal{S}$ [CGP, §C.2.21] entails that $\mathcal{H}$ as K_s-subgroup of $\mathcal{G}$ is generated by $\mathcal{L}_{\Delta_0}$ and the union of some root subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Phi(\mathcal{G}, \mathcal{T})$.

Exactly as in the proof of Lemma 3 one shows that $(\mathcal{P}^*_{\Delta_0}, \mathcal{L}^*_{\Delta_0})$ is defined over K and stable under $\mathrm{Ad}(u(\gamma))$ for all $\gamma \in \Gamma_K$. Next Lemma 13 says that $\mathrm{Ad}(u(\gamma)) \in (\mathrm{Aut}^{sm}_{\mathcal{D}(\mathcal{G})/K})^\circ(K_s)$ stabilizes $\psi(\mathcal{H})$. Finally (3) shows that $\psi(\mathcal{H})$ is Γ_K-stable if and only if $\mathcal{H}$ is Γ_K-stable. □

Now we can finish the proof of our main result.

Proposition 15. *Theorem 10 holds for absolutely pseudo-simple K-groups with trivial centre.*

Proof. By Lemma 8.a it suffices to consider two standard pseudo-Levi subgroups $\mathcal{L}_I, \mathcal{L}_J$ which are $\mathcal{G}(\overline{K})$-conjugate. As $\mathcal{G}$ becomes pseudo-split over K_s, Proposition 12 tells us that there exists a $w \in \mathcal{G}(K_s)$ with $w \mathcal{L}_I w^{-1} = \mathcal{L}_J$.

By [CP, Proposition 4.1.3 and Theorem 9.2.1] $\mathcal{G}$ is generalized standard, in the sense of [CP, Definition 9.1.7]. With [CP, Definition 9.1.5] we see that (at least) one of the following conditions holds:

(i) The characteristic of K is not 2, or $\mathrm{char}(K) = 2$ and $[K : K^2] \leq 4$.
(ii) The group $\mathcal{G}$ is standard [CP, Definition 2.1.3] or exotic [CP, Definitions 2.2.2 and 2.2.3].
(iii) The root system of $\mathcal{G}$ over K_s has type B_n or C_n or BC_n with $n \geq 1$.

(i) and (ii). When $\mathcal{G}$ standard, [CP, Theorem C.2.10] tells us that $\mathcal{G}$ has a quasi-split pseudo-inner form. If we are in case (ii) with $\mathcal{G}$ non-standard and $\mathrm{char}(K) = 2$, then $\mathcal{G}$ is an exotic pseudo-reductive group with root system (over K_s) of type B_n, C_n or F_4. By [CP, Proposition C.1.3] it has a pseudo-split K_s/K-form. Since the Dynkin diagram of $\mathcal{G}$ admits no nontrivial automorphisms, the group $\mathrm{Aut}^{sm}_{\mathcal{G}/K}$ is connected and every K_s/K-form of $\mathcal{G}$ is pseudo-inner [CP, Proposition 6.3.4]. Thus, in the cases (i) and (ii) $\mathcal{G}$ has a quasi-split pseudo-inner form.

The Bruhat decomposition [CGP, Theorem C.2.8] and (10) tell us that

$$\mathcal{L}_J(K) = \mathcal{P}_{\Delta_0}(K) W(\mathcal{L}_J, \mathcal{S}) \mathcal{P}_{\Delta_0}(K) = \mathcal{U}_{\mathcal{L}_J}(\lambda_{\Delta_0})(K) \mathcal{L}_{\Delta_0}(K) W(\mathcal{L}_J, \mathcal{S}) \mathcal{U}_{\mathcal{L}_J}(\lambda_{\Delta_0})(K).$$

By [CGP, Proposition C.2.26] the K-subgroup $\mathcal{U}_{\mathcal{L}_J}(\lambda_{\Delta_0})$ of $\mathcal{G}$ is generated by those root subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Phi(\mathcal{G}, \mathcal{S})$ which it contains. By [CGP, Proposition C.2.24] the root subgroups contained in $\mathcal{L}_J$ generate representatives for the entire

Weyl group $W(\mathcal{L}_J,\mathcal{S})$. Hence $\mathcal{L}_J$ is generated by the union of $\mathcal{L}_{\Delta_0}$ and the $\mathcal{U}_\alpha, \alpha \in \Phi(\mathcal{G},\mathcal{S})$ contained in $\mathcal{L}_J$. Knowing that, [CGP, Proposition C.2.26] also says that any pseudo-parabolic subgroup $\mathcal{P}_\lambda$ of $\mathcal{G}$ with $\mathcal{L}_\lambda = \mathcal{L}_J$ is generated by those root subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Phi(\mathcal{G},\mathcal{S})$ which it contains. That is, $\mathcal{L}_J$ and $\mathcal{P}_\lambda$ fulfill the conditions for $\mathcal{H}$ in Lemma 14.

Now we argue as in the proof of Lemma 4, using Lemma 14 instead of Lemma 3.c. The hypothesis in Lemma 4 is fulfilled for quasi-split pseudo-reductive groups, by Proposition 12. This shows that $\mathcal{L}_J$ is $\mathcal{G}(K)$-conjugate to a pseudo-Levi factor of $\mathcal{P}_I$. In the proof of Lemma 8.a we checked that all such pseudo-Levi factors are $\mathcal{P}_I(K)$-conjugate, so $\mathcal{L}_J$ is $\mathcal{G}(K)$-conjugate to $\mathcal{L}_I$.

(iii). The three types can be dealt with in the same way, so we only consider root systems $\Phi(\mathcal{G},\mathcal{T})$ of type B_n. Since this Dynkin diagram does not admit any nontrivial automorphisms, the action $\mu_\mathcal{B}$ of Γ_K is trivial.

Suppose first that $n \leq 2$. Then any two diferent subsets of Δ are not $W(\mathcal{G},\mathcal{T})$-associate, as is easily checked. Hence $I = J$ and $\mathcal{L}_I = \mathcal{L}_J$ in this case.

From now on we suppose that $\Phi(\mathcal{G},\mathcal{T})$ has type B_n with $n > 2$. We realize the root system of type B_n in the standard way in $\mathbb{Z}^n$. Let $\alpha_1,\ldots,\alpha_{n-1},\alpha_n$ be the vertices of Δ, where $\alpha_i = e_i - e_{i+1}$ for $i < n$ and $\alpha_n = e_n$ is the short simple root.

By Lemma 8.b there exists a $w \in W(\mathcal{G},\mathcal{T}) = W(B_n)$ with $wI = J$. When I or J equals Δ, we immediately obtain $I = J$. Hence we may assume that $I \subsetneq \Delta \supsetneq J$. Let $m \in \mathbb{Z}_{\geq 0}$ be the smallest number such that $\alpha_{n-m} \notin I$. For $j < m$, $\alpha_{n-j} \in I$ is the unique root in Δ which is connected to α_n by a string of length j. As α_n is the unique short simple root and $wI \subset \Delta$, it follows that $w(\alpha_{n-j}) = \alpha_{n-j}$ for all $j < m$. The same considerations apply to J and $wI = J$, so $\{\alpha_{n+1-m},\ldots,\alpha_n\} \subset I \cap J$ is fixed pointwise by w and $\alpha_{n-m} \neq I \cup J$. As

$$\mathrm{span}_\mathbb{Z}\{\alpha_{n+1-m},\ldots,\alpha_n\} = \mathrm{span}_\mathbb{Z}\{e_n, e_{n-1},\ldots,e_{n+1-m}\},$$

w must lie in $W(B_{n-m})$.

Write $\Delta' = \{\alpha_1,\ldots,\alpha_{n-1-m}\}$ and $\Delta'' = \{\alpha_{n+1-m},\ldots,\alpha_n\}$, two orthogonal sets of simple roots. The standard pseudo-Levi K-subgroup $\mathcal{L}_{\Delta'\cup\Delta''}$ of $\mathcal{G}$ contains $\mathcal{L}_I$ and $\mathcal{L}_J$. Decomposing its root system in irreducible components gives

$$\mathcal{L}_{\Delta'\cup\Delta''} = \mathcal{L}_{\Delta'}\mathcal{L}_{\Delta''}.$$

The index of $\mathcal{L}_{\Delta'}$ consists of $\Delta', \Delta' \cap \Delta_0$ and the trivial action of Γ_K. Here Δ' has type A_{n-1-m} and by [Spr, Lemma 15.5.8] the subset $\Delta'_0 := \Delta' \cap \Delta_0$ is stable under the nontrivial automorphism of A_{n-1-m}. As shown in [Tit, §3.3.2], this implies that there exists a divisor d of $n-m$ such that

$$\Delta' \setminus \Delta'_0 = \mathbb{Z}d \cap [1,\ldots,n-1-m].$$

With [Spr, §17.1] we see that $(\Delta', \Delta'_0, \mathrm{triv})$ is the index of an inner form $\mathcal{H}$ of GL_n. Explicitly, we can take $\mathcal{H}(\mathbb{Q}) = GL_{(n-m)/d}(D)$ where D is a division algebra whose centre equals the ground field $\mathbb{Q}$. As maximal $\mathbb{Q}$-split torus $\mathcal{S}^\mathcal{H}(\mathbb{Q})$ we take the diagonal matrices with entries in $\mathbb{Q}^\times$.

The isomorphism class of the Dynkin diagram $I' := I \cap \Delta$ determines the isomorphism class of the standard Levi $\mathbb{Q}$-subgroup $\mathcal{L}^\mathcal{H}_{I'}$ of $\mathcal{H}$. Namely, $\mathcal{L}^\mathcal{H}_{I'}(\mathbb{Q})$ is a direct product of groups $GL_{n_j}(D)$, where $\sum_j n_j = (n-m)/d$ and I' has connected components of sizes $dn_j - 1$.

The set of simple roots $J' := J \cap \Delta'$ is associate to I' by $w \in W(B_{n-m})$, so isomorphic to I' as Dynkin diagram. It follows that the standard Levi $\mathbb{Q}$-subgroups $\mathcal{L}^\mathcal{H}_{I'}$ and

$\mathcal{L}^{\mathcal{H}}_{J'}$ of $\mathcal{H}$ are isomorphic. That is, $\mathcal{L}^{\mathcal{H}}_{J'}$ is also a direct product of the groups $GL_{n_j}(D)$, but maybe situated in a different (standard) position inside $GL_{(n-m)/d}(D)$. With a permutation w' from $S_{(n-m)/d}$ we can bring them in the same position. Then $n\mathcal{L}^{\mathcal{H}}_{I'}n^{-1} = \mathcal{L}^{\mathcal{H}}_{J'}$ for some $n \in N_{\mathcal{H}(\mathbb{Q}}(\mathcal{S}^{\mathcal{H}}(\mathbb{Q}))$ and $w'I' = J'$, where w' is the image of n in $W(\mathcal{H}, \mathcal{S}^{\mathcal{H}}) \cong S_{(n-m)/d}$.

As $W(\mathcal{H}, \mathcal{S}^{\mathcal{H}}) = W(\mathcal{L}_{\Delta'}, \mathcal{S})$, we conclude that I' and J' are associate by an element of $W(\mathcal{L}_{\Delta'}, \mathcal{S}) \subset W(\mathcal{G}, \mathcal{S})$. Since Δ' and Δ'' are orthogonal, w' fixes $I \cap \Delta'' = J \cap \Delta''$ pointwise. Hence $w'I = J$ and by Lemma 8.b $\mathcal{L}_I$ and $\mathcal{L}_J$ are $\mathcal{G}(K)$-conjugate. $\square$

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