# GENERALIZATIONS OF THE SPRINGER CORRESPONDENCE AND CUSPIDAL LANGLANDS PARAMETERS

ANNE-MARIE AUBERT, AHMED MOUSSAOUI, AND MAARTEN SOLLEVELD

ABSTRACT. Let $\mathcal{H}$ be any reductive $p$-adic group. We introduce a notion of cuspidality for enhanced Langlands parameters for $\mathcal{H}$, which conjecturally puts supercuspidal $\mathcal{H}$-representations in bijection with such L-parameters. We also define a cuspidal support map and Bernstein components for enhanced L-parameters, in analogy with Bernstein's theory of representations of $p$-adic groups. We check that for several well-known reductive groups these analogies are actually precise.

Furthermore we reveal a new structure in the space of enhanced L-parameters for $\mathcal{H}$, that of a disjoint union of twisted extended quotients. This is an analogue of the ABPS conjecture (about irreducible $\mathcal{H}$-representations) on the Galois side of the local Langlands correspondence. Only, on the Galois side it is no longer conjectural. These results will be useful to reduce the problem of finding a local Langlands correspondence for $\mathcal{H}$-representations to the corresponding problem for supercuspidal representations of Levi subgroups of $\mathcal{H}$.

The main machinery behind this comes from perverse sheaves on algebraic groups. We extend Lusztig's generalized Springer correspondence to disconnected complex reductive groups $G$. It provides a bijection between, on the one hand, pairs consisting of a unipotent element $u$ in $G$ and an irreducible representation of the component group of the centralizer of $u$ in $G$, and, on the other hand, irreducible representations of a set of twisted group algebras of certain finite groups. Each of these twisted group algebras contains the group algebra of a Weyl group, which comes from the neutral component of $G$.

In 2025 an erratum was added, to repair Theorem 3.1.a.

## CONTENTS

## Introduction

As the title suggests, this paper consists of two parts. The first part is purely in complex algebraic geometry, and is accessible without any knowledge of the Langlands program or $p$-adic groups. We start with discussing the second part though, which is an application of and a motivation for the first part.

The local Langlands correspondence (LLC) predicts a relation between two rather different kinds of objects: on the one hand irreducible representations of reductive groups over a local field $F$, on the other hand some sort of representations of the Weil–Deligne group of $F$. According to the original setup [Bor, Lan], it should be possible to associate to every L-parameter a finite packet of irreducible admissible representations. Later this was improved by enhancing L-parameters [Lus1, KaLu], and the modern interpretation [ABPS6, Vog] says that the LLC should be a bijection (when formulated appropriately).

We consider only non-archimedean local fields $F$, and we speak of the Galois side versus the $p$-adic side of the LLC. The conjectural bijectivity makes it possible to transfer many notions and ideas from on side of the LLC to the other. Indeed, a main goal of this paper is to introduce an analogue, on the Galois side, of the Bernstein theory [BeDe] for smooth representations of reductive $p$-adic groups.

Bernstein's starting point is the notion of a supercuspidal representation. For a long time it has been unclear how to translate this to the Galois side. In [Mou, Def. 4.12] the second author discovered the (probably) correct notion for split reductive $p$-adic groups, which we generalize here.

For maximal generality, we adhere to the setup for L-parameters from [Art2]. Let $\mathbf{W}_F$ be the Weil group of $F$, let $\mathcal{H}$ be a connected reductive group over $F$ and let ${}^L\mathcal{H} = \mathcal{H}^\vee \rtimes \mathbf{W}_F$ be its dual L-group. Let $\mathcal{H}^\vee_{\mathrm{ad}}$ be the adjoint group of $\mathcal{H}^\vee$, and let $\mathcal{H}^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group of $\mathcal{H}^\vee_{\mathrm{ad}}$. Let $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{H}$ be an L-parameter, let $Z_{\mathcal{H}^\vee_{\mathrm{ad}}}(\phi(\mathbf{W}_F))$ be the centralizer of $\phi(\mathbf{W}_F)$ in $\mathcal{H}^\vee_{\mathrm{ad}}$ and let

$$G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))$$

be its inverse image in $\mathcal{H}^\vee_{\mathrm{sc}}$. To $\phi$ we associate the finite group $\mathcal{S}_\phi := \pi_0(Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi))$, where $Z^1$ is again defined via $\mathcal{H}^\vee_{\mathrm{ad}}$. We call any irreducible representation of $\mathcal{S}_\phi$ an *enhancement* of $\phi$ . The group $\mathcal{S}_\phi$ coincides with the group considered by both Arthur in [Art2] and Kaletha in [Kal, §4.6]. A remarkable fact is that the group $\mathcal{S}_\phi$ is isomorphic to the group $A_G(u_\phi) := \pi_0(Z_G(u_\phi))$, where $u_\phi := \phi(1, (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}))$.

We propose (see Definition 6.9) to call an enhanced L-parameter $(\phi, \rho)$ for $\mathcal{H}$ *cuspidal* if $u_\phi$ and $\rho$, considered as data for the complex reductive group $G$, form a cuspidal pair. By definition this means that the restriction of $\rho$ from $A_G(u) = \pi_0(Z_G(u))$ to $A_{G^\circ}(u)$ is a direct sum of cuspidal representations in Lusztig's sense [Lus2]. Intuitively, it says that $\rho$ or $\rho|_{A_{G^\circ}(u)}$ cannot be obtained (via an appropriate notion of parabolic induction) from any pair $(u', \rho')$ that can arise from a proper Levi subgroup of $G^\circ$. We emphasize that it is essential to use L-parameters enhanced with a representation of a suitable component group, for cuspidality cannot be detected from the L-parameter alone.

Let $\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{H})$ be the set of supercuspidal $\mathcal{H}$-representations (up to isomorphism) and let $\Phi_{\mathrm{cusp}}(\mathcal{H})$ be the set of $\mathcal{H}^\vee$-conjugacy classes of cuspidal L-parameters for

$\mathcal{H}$. It is known that in many cases such cuspidal L-parameters do indeed parametrize supercuspidal representations, and that moreover there is a nice bijection $\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{H}) \to \Phi_{\mathrm{cusp}}(\mathcal{H})$.

We call the enhanced L-parameters $(\phi, \rho)$ such that $\phi$ restricts trivially to the inertia group $\mathbf{I}_F$ *unipotent*. A representation $\pi$ of $\mathcal{H}$ is said to be unipotent (or sometimes to have unipotent reduction) if for some parahoric subgroup $\mathfrak{P}$ of $\mathcal{H}$, and the inflation $\sigma$ to $\mathfrak{P}$ of some unipotent cuspidal representation of its reductive quotient, the space $\mathrm{Hom}_{\mathfrak{P}}(\sigma, \pi)$ is nonzero. When $\mathcal{H}$ is simple of adjoint type, Lusztig has proved in [Lus3, Lus4] that the $\mathcal{H}^\vee$-conjugacy classes of unipotent enhanced L-parameters are in bijection with the equivalence classes of unipotent irreducible representations of $\mathcal{H}$. Under Lusztig's bijection the unipotent cuspidal enhanced L-parameters correspond to the unipotent supercuspidal irreducible representations of $\mathcal{H}$.

When $(\phi, \rho)$ is a unipotent enhanced L-parameter, the group $G$ coincides with the group $Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi(\mathrm{Frob}))$. In contrast, for $(\phi, \rho)$ arbitrary, the group $G$ is usually a proper subgroup of $Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi(\mathrm{Frob}))$.

Based on the notion of cuspidality that we have defined, we construct a *cuspidal support map* for L-parameters (Definition 7.7). It assigns to every enhanced L-parameter for $\mathcal{H}$ a Levi subgroup $\mathcal{L} \subset \mathcal{H}$ and a cuspidal L-parameter for $\mathcal{L}$, unique up to conjugation. We conjecture that this map is a precise analogue of Bernstein's cuspidal support map for irreducible $\mathcal{H}$-representations, in the sense that these cuspidal support maps commute with the respective local Langlands correspondences (assuming that these exist of course).

A result for $p$-adic groups which has already been transferred to the Galois side is the Langlands classification [SiZi]. On the $p$-adic side it reduces $\mathrm{Irr}(\mathcal{H})$ to the tempered duals of Levi subgroups of $\mathcal{H}$, while on the Galois side it reduces general (enhanced) L-parameters to bounded L-parameters for Levi subgroups. We show (Lemma 7.11) that our cuspidal support map factors through through the Langlands classification on the Galois side, just like Bernstein's cuspidal support map on the $p$-adic side.

Recall that a crucial role in the Bernstein decomposition is played by inertial equivalence classes of (super)cuspidal pairs for $\mathcal{H}$. These consist of a Levi subgroup $\mathcal{L} \subset \mathcal{H}$ and a supercuspidal representation thereof, up to equivalence by $\mathcal{H}$-conjugation and twists by unramified characters. Since the LLC for unramified characters is known, we can easily translate this to a notion of inertial equivalence classes of enhanced L-parameters (Definition 8.1). Using the cuspidal support map, we can also partition the set of enhanced L-parameters $\Phi_e(\mathcal{H})$ into countably many Bernstein components $\Phi_e(\mathcal{H})^{\mathfrak{s}^\vee}$, parametrized by the inertial equivalence classes $\mathfrak{s}^\vee$, see (115).

Let $\mathcal{L} \subset \mathcal{H}$ be a Levi subgroup, and let

$$W(\mathcal{H}, \mathcal{L}) = N_{\mathcal{H}}(\mathcal{L})/\mathcal{L}$$

be its "Weyl" group. In [ABPS6] it was shown to be naturally isomorphic to $N_{\mathcal{H}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee$, so it acts on both $\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$ and $\Phi_{\mathrm{cusp}}(\mathcal{L})$.

Our main result provides a complete description of the space of enhanced L-parameters $\Phi_e(\mathcal{H})$ in terms of cuspidal L-parameters for Levi subgroups, and the associated Weyl groups. It discovers a new structure in $\Phi_e(\mathcal{H})$, that of a union of

extended quotients. It improves on both the Langlands classification and the theory of the Bernstein centre (on the Galois side of the LLC).

Fix a character $\zeta_{\mathcal{H}}$ of $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})$ whose restriction to $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$ corresponds via the Kottwitz isomorphism to the class of $\mathcal{H}$ as an inner twist of its quasi-split inner form. We indicate the subset of enhanced L-parameters $(\phi, \rho)$ such that $\rho$ extends $\zeta_{\mathcal{H}}$ with a subscript $\zeta_{\mathcal{H}}$. This $\zeta_{\mathcal{H}}$ only plays a role when $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})$ is not fixed by $\mathbf{W}_F$, in particular it is redundant for inner twists of split groups.

**Theorem 1.** *(See Theorem 9.3)*
*Let $\mathfrak{Lev}(\mathcal{H})$ be a set of representatives for the conjugacy classes of Levi subgroups of $\mathcal{H}$. There exists a bijection*

$$\Phi_{e,\zeta_{\mathcal{H}}}(\mathcal{H}) \longleftrightarrow \bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \left(\Phi_{\mathrm{cusp},\zeta_{\mathcal{H}}}(\mathcal{L}) /\!/ W(\mathcal{H},\mathcal{L})\right)_{\kappa}.$$

Here $(\cdot/\!/\cdot)_{\kappa}$ denotes a twisted extended quotient, as defined in (13). The bijection is not entirely canonical, but we provide a sharp bound on the non-canonicity. We note that the bijection is not based on the earlier cuspidal support map, but rather on a modification thereof, which preserves boundedness of L-parameters.

We expect that Theorem 1 will turn out to be an analogue of the ABPS conjecture [ABPS6] on the Galois side of the LLC. To phrase this precisely in general, we need yet another ingredient.

**Conjecture 2.** *Let $\mathcal{H}$ be a connected reductive group over a local non-archimedean field, and let $\mathrm{Irr}(\mathcal{H})$ denote the set of its irreducible smooth representations. There exists a commutative bijective diagram*

$$
\begin{array}{ccc}
\mathrm{Irr}(\mathcal{H}) & \longleftrightarrow & \Phi_{e,\zeta_{\mathcal{H}}}(\mathcal{H}) \\
\updownarrow & & \updownarrow \\
\bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \left(\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L}) /\!/ W(\mathcal{H},\mathcal{L})\right)_{\kappa} & \longleftrightarrow & \bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \left(\Phi_{\mathrm{cusp},\zeta_{\mathcal{H}}}(\mathcal{L}) /\!/ W(\mathcal{H},\mathcal{L})\right)_{\kappa}
\end{array}
$$

*with the following maps:*

- *The right hand side is Theorem 1.*
- *The upper horizontal map is a local Langlands correspondence for $\mathcal{H}$.*
- *The lower horizontal map is obtained from local Langlands correspondences for $\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$ by applying $(\cdot /\!/ W(\mathcal{H},\mathcal{L}))_{\kappa}$.*
- *The left hand side is the bijection in the ABPS conjecture [ABPS6, §2].*

With this conjecture one can reduce the problem of finding a LLC for $\mathcal{H}$ to that of finding local Langlands correspondences for supercuspidal representations of its Levi subgroups. Conjecture 2 is currently known in the following cases:

- inner forms of $\mathrm{GL}_n(F)$ [ABPS5, Theorem 5.3],
- inner forms of $\mathrm{SL}_n(F)$ [ABPS5, Theorem 5.6],
- split classical groups [Mou, §5.3],
- principal series representations of split groups [ABPS4, §16].

In [AMS], we extensively use the results of the present paper in order to construct (twisted) graded Hecke algebras $\mathbb{H}$ based on a (possibly disconnected) complex reductive group $G$ and a cuspidal local system $\mathcal{L}$ on a unipotent orbit of a Levi subgroup $M$ of $G$, and to develop their representation theory. The algebras $\mathbb{H}$ generalize the graded Hecke algebras defined and investigated by Lusztig for connected $G$.

Now we come to the main technique behind the above: generalizations of the Springer correspondence. Let $G^\circ$ be a connected complex reductive group with a maximal torus $T$ and Weyl group $W(G^\circ, T)$. Recall that the original Springer correspondence [Spr] is a bijection between the irreducible representations of $W(G^\circ, T)$ and $G^\circ$-conjugacy classes of pairs $(u, \eta)$, where $u \in G^\circ$ is unipotent and $\eta$ is an irreducible representation of $A_{G^\circ}(u) = \pi_0(Z_{G^\circ}(u))$ which appears in the homology of the variety of Borel subgroups of $G^\circ$ containing $u$.

Lusztig [Lus2] generalized this to a setup which includes all pairs $(u, \eta)$ with $u \in G^\circ$ unipotent and $\eta \in \mathrm{Irr}(A_{G^\circ}(u))$. On the other side of the correspondence he replaced $\mathrm{Irr}(W(G^\circ, T))$ by a disjoint union $\sqcup_{\mathfrak{t}^\circ} \mathrm{Irr}(W_{\mathfrak{t}^\circ})$, where $\mathfrak{t}^\circ = [L, v, \epsilon]_{G^\circ}$ runs through cuspidal pairs $(v, \epsilon)$ for Levi subgroups $L$ of $G^\circ$, and $W_{\mathfrak{t}^\circ} = W(G^\circ, L)$ is the Weyl group associated to $\mathfrak{t}^\circ$.

More precisely, Lusztig first attaches to $(u, \eta)$ a cuspidal support $\mathfrak{t}^\circ = \Psi_{G^\circ}(u, \eta)$, and then he constructs a bijection $\Sigma_{\mathfrak{t}^\circ}$ between $\Psi_{G^\circ}^{-1}(\mathfrak{t}^\circ)$ and $\mathrm{Irr}(W_{\mathfrak{t}^\circ})$. In Section 2 we recall these constructions in more detail, and we prove:

**Theorem 3.** *The maps $\Psi_{G^\circ}$ and $\Sigma_{\mathfrak{t}^\circ}$ are equivariant with respect to algebraic automorphisms of the group $G^\circ$.*

Given a Langlands parameter $\phi$ for $\mathcal{H}$, we would like to apply this machinery to $G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))$. However, we immediately run into the problem that this complex reductive group is usually not connected. Thus we need a generalization of Lusztig's correspondence to disconnected reductive groups. Although there exist generalizations of the Springer correspondence in various directions [AcHe, AHJR, AcSa, Lus2, Lus5, LuSp, Sor], this particular issue has not yet been addressed in the literature.

We would like to have a version which transforms every pair $(u, \eta)$ for $G$ into an irreducible representation of some Weyl group. But this turns out to be impossible! The problem is illustrated by Example 3.2: we have to use twisted group algebras of groups $W_{\mathfrak{t}}$ which are not necessarily Weyl groups.

When $G$ is disconnected, we define the cuspidal support map by

$$\Psi_G(u, \eta) = \Psi_{G^\circ}(u, \eta^\circ)/G\text{-conjugacy},$$

where $\eta^\circ$ is any constituent of $\mathrm{Res}^{A_G(u)}_{A_{G^\circ}(u)} \eta$. This is well-defined by the $\mathrm{Ad}(G)$-equivariance of $\Psi_{G^\circ}$ from Theorem 3.

For a cuspidal support $\mathfrak{t} = [L, v, \epsilon]_G$ (where $L$ is a Levi subgroup of $G^\circ$), we put

$$W_{\mathfrak{t}} = N_G(L, v, \epsilon)/L \quad \text{and} \quad \mathfrak{t}^\circ = [L, v, \epsilon]_{G^\circ}.$$

Then $W_{\mathfrak{t}}$ contains $W_{\mathfrak{t}^\circ} = W(G^\circ, L)$ as a normal subgroup.

**Theorem 4.** *(See Theorem 4.7 and Proposition 4.5)*
*Let $\mathfrak{t} = [L, v, \epsilon]_G$ be a cuspidal support for $G$. There exist:*
- *a 2-cocycle $\natural_{\mathfrak{t}} \colon W_{\mathfrak{t}}/W_{\mathfrak{t}^\circ} \times W_{\mathfrak{t}}/W_{\mathfrak{t}^\circ} \to \mathbb{C}^\times$,*
- *a twisted group algebra $\mathbb{C}[W_{\mathfrak{t}}, \natural_{\mathfrak{t}}]$,*
- *a bijection $\Psi_G^{-1}(\mathfrak{t}) \to \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{t}}, \natural_{\mathfrak{t}}])$ which extends [Lus2].*

*Moreover the composition of the bijection with $\mathrm{Res}^{\mathbb{C}[W_{\mathfrak{t}}, \natural_{\mathfrak{t}}]}_{\mathbb{C}[W_{\mathfrak{t}^\circ}]}$ is canonical.*

Of course the proof of Theorem 4 starts with Lusztig's generalized Springer correspondence for $G^\circ$. Ultimately it involves a substantial part of the techniques and

objects from [Lus2], in particular we consider similar varieties and sheafs. In Section 3 we provide an expression for the 2-cocycle $\natural_{\mathfrak{t}}$, derived from the cuspidal case $L = G^\circ$.

Yet $\Psi_G$ and Theorem 4 still do not suffice for our plans with Langlands parameters. Namely, suppose that $(\phi, \rho)$ is an enhanced L-parameter for $\mathcal{H}$ and apply $\Psi_G$ with $G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))$ and $(u, \eta) = \big(\phi(1, (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})), \rho\big)$. We end up with $\mathfrak{t} = [L, v, \epsilon]_G$, where $L$ is a Levi subgroup of $G^\circ$. But the cuspidal support map for L-parameters should produce an enhanced L-parameter for a Levi subgroup $\mathcal{L}$ of $\mathcal{H}$, and that would involve a possibly disconnected group $Z^1_{\mathcal{L}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))$ instead of $L$.

To resolve this problem, we consider *quasi-Levi* subgroups of $G$. These are groups of the form $M = Z_G(Z(L)^\circ)$, where $L \subset G^\circ$ is a Levi subgroup (and hence $M^\circ = L$). With these one can define a *quasi-cuspidal* support, a triple $(M, v, q\epsilon)$ with $v \in M^\circ$ unipotent and $q\epsilon \in \mathrm{Irr}(A_M(v))$ such that $\mathrm{Res}^{A_M(v)}_{A_{M^\circ}(v)} q\epsilon$ is a sum of cuspidal representations. The cuspidal support map $\Psi_G$ can be adjusted to a canonical quasi-cuspidal support map $q\Psi_G$, see (65). It is this map that gives us the cuspidal support of enhanced L-parameters.

To a quasi-cuspidal support $q\mathfrak{t} = [M, v, q\epsilon]_G$ we associate the group $W_{q\mathfrak{t}} = N_G(M, v, q\epsilon)/M$, which (again) contains $W_{\mathfrak{t}^\circ} = N_{G^\circ}(M^\circ)/M^\circ$.

**Theorem 5.** *(See Theorem 5.5 and Lemma 5.4)*
*Theorem 4 also holds with quasi-Levi subgroups and with the quasi-cuspidal support $q\mathfrak{t}$ instead of $\mathfrak{t}$. It gives a bijection $q\Psi_G^{-1}(q\mathfrak{t}) \to \mathrm{Irr}(\mathbb{C}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}])$ which is canonical in the same degree as for $\mathfrak{t}$.*

The derivation of Theorem 5 from Theorem 4 relies to a large extent on (elementary) results about twisted group algebras, which we put in Section 1. The bijection from Theorem 5 is extensively used in Section 9, for Theorem 1.

## 1. Twisted group algebras and normal subgroups

Throughout this section $\Gamma$ is a finite group and $K$ is an algebraically closed field whose characteristic does not divide the order of $\Gamma$. Suppose that $\natural : \Gamma \times \Gamma \to K^\times$ is a 2-cocycle, that is,

$$(1) \qquad \natural(\gamma_1, \gamma_2 \gamma_3)\natural(\gamma_2, \gamma_3) = \natural(\gamma_1, \gamma_2)\natural(\gamma_1 \gamma_2, \gamma_3) \quad \forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma.$$

The $\natural$-twisted group algebra of $\Gamma$ is defined to be the $K$-vector space $K[\Gamma, \natural]$ with basis $\{T_\gamma : \gamma \in \Gamma\}$ and multiplication rules

$$(2) \qquad T_\gamma T_{\gamma'} = \natural(\gamma, \gamma')T_{\gamma\gamma'} \quad \gamma, \gamma' \in \Gamma.$$

Its representations can be considered as projective $\Gamma$-representations. Schur showed (see [CuRe, Theorem 53.7]) that there exists a finite central extension $\tilde{\Gamma}$ of $\Gamma$, such that

- char$(K)$ does not divide $|\tilde{\Gamma}|$,
- every irreducible projective $\Gamma$-representation over $K$ lifts to an irreducible $K$-linear representation of $\tilde{\Gamma}$.

Then $K[\Gamma, \natural]$ is a direct summand of $K[\tilde{\Gamma}]$, namely the image of a minimal idempotent in $K[\ker(\tilde{\Gamma} \to \Gamma)]$. The condition on $\mathrm{char}(K)$ ensures that $K[\tilde{\Gamma}]$ is semisimple, so $K[\Gamma, \natural]$ is also semisimple.

Let $N$ be a normal subgroup of $\Gamma$ and $(\pi, V_\pi)$ an irreducible representation of $N$ over $K$. We abbreviate this to $\pi \in \mathrm{Irr}_K(N)$. We want to analyse the set of irreducible $\Gamma$-representations whose restriction to $N$ contains $\pi$.

More generally, suppose that $\natural$ is a 2-cocycle of $\Gamma/N$. We identify it with a 2-cocycle $\Gamma \times \Gamma \to K^\times$ that factors through $(\Gamma/N)^2$. We also want to analyse the irreducible representations of $K[\Gamma, \natural]$ that contain $\pi$.

For $\gamma \in \Gamma$ we define $\gamma \cdot \pi \in \mathrm{Irr}_K(N)$ by

$$(3) \qquad (\gamma \cdot \pi)(n) = \pi(\gamma^{-1} n \gamma).$$

This determines an action of $\Gamma$ and of $\Gamma/N$ on $\mathrm{Irr}(N)$. Let $\Gamma_\pi$ be the isotropy group of $\pi$ in $\Gamma$. For every $\gamma \in \Gamma_\pi$ we choose a $I^\gamma = I_\pi^\gamma \in \mathrm{Aut}_K(V_\pi)$ such that

$$(4) \qquad I^\gamma \circ \pi(\gamma^{-1} n \gamma) = \pi(n) \circ I^\gamma \quad \forall n \in N.$$

Thus $I^\gamma \in \mathrm{Hom}_N(\gamma \cdot \pi, \pi)$. Given another $\gamma' \in \Gamma$, we can regard $I^\gamma$ also as an element of $\mathrm{Hom}_N(\gamma' \cdot \gamma \cdot \pi, \gamma' \cdot \pi)$, and then it can be composed with $I^{\gamma'} \in \mathrm{Hom}_N(\gamma' \cdot \pi, \pi)$. By Schur's lemma all these maps are unique up to scalars, so there exists a $\kappa_\pi(\gamma, \gamma') \in K^\times$ with

$$(5) \qquad I^{\gamma \gamma'} = \kappa_\pi(\gamma, \gamma') I^\gamma \circ I^{\gamma'}.$$

On comparing this with (1), one sees that $\kappa_\pi : \Gamma_\pi \times \Gamma_\pi \to K^\times$ is a 2-cocycle. Notice that the algebra $K[\Gamma_\pi, \kappa_\pi^{-1}]$ acts on $V_\pi$ by $T_\gamma \mapsto I^\gamma$. Let $[\Gamma_\pi/N] \subset \Gamma_\pi$ be a set of representatives for $\Gamma_\pi/N$. We may pick the $I^\gamma$ such that

$$(6) \qquad I^{\tilde{\gamma} n} = I^{\tilde{\gamma}} \circ \pi(n) \quad \forall \tilde{\gamma} \in [\Gamma_\pi/N], n \in N.$$

It follows from (4) that $I^{n \tilde{\gamma}} = \pi(n) \circ I^{\tilde{\gamma}}$ and that $\kappa_\pi$ factors as

$$\kappa_\pi : \Gamma_\pi \times \Gamma_\pi \to \Gamma_\pi/N \times \Gamma_\pi/N \to K^\times$$

Let $\natural : \Gamma/N \times \Gamma/N \to K^\times$ be a 2-cocycle. Thus we can construct the twisted group algebras $K[\Gamma, \natural]$ and $K[\Gamma_\pi/N, \natural\kappa_\pi]$. To avoid confusion we denote the standard basis elements of $K[\Gamma, \natural]$ by $S_\gamma$.

**Proposition 1.1.** *Let $(\tau, M)$ be a representation of $K[\Gamma_\pi/N, \natural\kappa_\pi]$.*

*(a) The algebra $K[\Gamma_\pi, \natural]$ acts on $M \otimes_K V_\pi$ by*

$$S_\gamma(m \otimes v) = \tau(T_{\gamma N}) m \otimes I^\gamma(v) \quad h \in K[\Gamma_\pi/N, \natural\kappa_\pi], v \in V_\pi.$$

*(b) The $K$-linear map*

$$\begin{array}{rccc} T : & \mathrm{ind}_{K[N]}^{K[\Gamma_\pi, \natural]}(V_\pi) & \to & K[\Gamma_\pi/N, \natural\kappa_\pi] \otimes_K V_\pi \\ & S_{\tilde{\gamma}} \otimes v & \mapsto & T_{\tilde{\gamma} N} \otimes I^{\tilde{\gamma}}(v) \qquad \tilde{\gamma} \in [\Gamma/N] \end{array}$$

*is an isomorphism of $K[\Gamma_\pi, \natural]$-representations.*

*(c) The map $M \mapsto \mathrm{ind}_{K[\Gamma_\pi, \natural]}^{K[\Gamma, \natural]}(T^{-1}(M \otimes V_\pi))$ is an equivalence between the following categories:*

- *subrepresentations of the left regular representation of $K[\Gamma_\pi/N, \natural\kappa_\pi]$;*
- *$K[\Gamma, \natural]$-subrepresentations of $\mathrm{ind}_{K[N]}^{K[\Gamma, \natural]}(V_\pi)$.*

(d) *We write* $\tau \ltimes \pi := \mathrm{ind}_{K[\Gamma_\pi]}^{K[\Gamma,\natural]}(M \otimes V_\pi)$. *For any representation* $V$ *of* $K[\Gamma,\natural]$ *there is an isomorphism*

$$\mathrm{Hom}_{K[\Gamma,\natural]}(\tau \ltimes \pi, V) \cong \mathrm{Hom}_{K[\Gamma_\pi/N,\natural\kappa_\pi]}(\tau, \mathrm{Hom}_N(\pi, V)).$$

*Proof.* (a) By (4) and (5)

$$\begin{aligned}
S_\gamma(S_{\gamma'}(m \otimes v)) &= S_\gamma(\tau(T_{\gamma'N})m \otimes I^{\gamma'}(v)) \\
&= \tau(T_{\gamma N}T_{\gamma'N})m \otimes I^{\gamma N} \circ I^{\gamma'}(v) \\
&= (\natural\kappa_\pi)(\gamma,\gamma')\tau(T_{\gamma\gamma'N})m \otimes \kappa_\pi(\gamma,\gamma')^{-1}I^{\gamma\gamma'}(v) \\
&= \natural(\gamma,\gamma')\tau(T_{\gamma\gamma'N})m \otimes I^{\gamma\gamma'}(v) \\
&= \natural(\gamma,\gamma')S_{\gamma\gamma'}(m \otimes v) = (S_\gamma S_{\gamma'})(m \otimes v).
\end{aligned}$$

(b) Since every $I^{\tilde\gamma} : V_\pi \to V_\pi$ is bijective, so is $T$. For any $n \in N$:

$$(7) \quad T(S_n(S_{\tilde\gamma} \otimes v)) = T(S_{\tilde\gamma} \otimes \pi(\tilde\gamma^{-1}n\tilde\gamma)v) = T_{\tilde\gamma N} \otimes I^{\tilde\gamma} \circ \pi(\tilde\gamma^{-1}n\tilde\gamma)(v) =$$
$$T_{\tilde\gamma N} \otimes \pi(n)I^{\tilde\gamma}(v) = S_n(T_{\tilde\gamma N} \otimes I^{\tilde\gamma}(v)) = S_n(T(S_{\tilde\gamma} \otimes v)),$$

so $T$ is $N$-equivariant. Let $\gamma_1 \in \Gamma$ and write $\gamma_1\tilde\gamma = n\tilde\gamma_2$ with $n \in N$ and $\tilde\gamma_2 \in [\Gamma/N]$. By (7)

$$\begin{aligned}
T(S_{\gamma_1}(S_{\tilde\gamma} \otimes v)) &= T(\natural(\gamma_1,\tilde\gamma)S_n S_{\tilde\gamma_2} \otimes v) = \natural(\gamma_1,\tilde\gamma)S_n T(S_{\tilde\gamma_2} \otimes v) \\
&= \natural(\gamma_1,\tilde\gamma)S_n(T_{\tilde\gamma_2 N} \otimes I^{\tilde\gamma_2}(v)) = \natural(\gamma_1,\tilde\gamma)T_{\tilde\gamma_2 N} \otimes \pi(n)I^{\tilde\gamma_2}(v) \\
&= \natural(\gamma_1,\tilde\gamma)T_{n\tilde\gamma_2 N} \otimes I^{n\tilde\gamma_2}(v) = \natural(\gamma_1,\tilde\gamma)T_{\gamma_1\tilde\gamma N} \otimes I^{\gamma_1\tilde\gamma}(v) \\
&= \kappa_\pi(\gamma_1,\tilde\gamma)^{-1}T_{\gamma_1 N}T_{\tilde\gamma N} \otimes \kappa_\pi(\gamma_1,\tilde\gamma)I^{\gamma_1}I^{\tilde\gamma}(v) = T_{\gamma_1 N}T_{\tilde\gamma N} \otimes I^{\gamma_1}I^{\tilde\gamma}(v) \\
&= S_{\gamma_1}(T_{\tilde\gamma N} \otimes I^{\tilde\gamma}(v)) = S_{\gamma_1}(T(S_{\tilde\gamma} \otimes v)).
\end{aligned}$$

(c) See [Sol2, Theorem 11.2.b]. The proof over there applies because we already have established parts (a) and (b).

(d) We already saw that all these algebras are semisimple. In particular $V$ is completely reducible. Let $V'$ be the $\pi$-isotypical component of $\mathrm{Res}_{K[N]}^{K[\Gamma,\natural]}(V)$. Every $K[\Gamma,\natural]$-homomorphism from $\tau \ltimes \pi$ has image in $K[\Gamma,\natural] \cdot V'$, so we may assume that $V = K[\Gamma,\natural] \cdot V'$. Then $V$ can be embedded in a direct sum of copies of $\mathrm{ind}_{K[N]}^{K[\Gamma,\natural]}(V_\pi)$. Hence it suffices to prove the claim in the case that $V = \mathrm{ind}_{K[N]}^{K[\Gamma,\natural]}(V_\pi)$.

By part (b) and the irreducibility of $\pi$

$$(8) \qquad \mathrm{Hom}_N\left(V_\pi, \mathrm{ind}_{K[N]}^{K[\Gamma,\natural]}(V_\pi)\right) = \mathrm{Hom}_N\left(V_\pi, \mathrm{ind}_{K[N]}^{K[\Gamma_\pi,\natural]}(V_\pi)\right) \cong K[\Gamma_\pi/N, \natural\kappa_\pi].$$

By Frobenius reciprocity

$$(9) \qquad \mathrm{Hom}_{K[\Gamma,\natural]}\left(\tau \ltimes \pi, \mathrm{ind}_{K[N]}^{K[\Gamma,\natural]}(V_\pi)\right) \cong \mathrm{Hom}_{K[\Gamma_\pi,\natural]}\left(M \otimes V_\pi, \mathrm{ind}_{K[N]}^{K[\Gamma,\natural]}(V_\pi)\right)$$

By (8) the right hand side simplifies to
(10)
$$\mathrm{Hom}_{K[\Gamma_\pi,\natural]}\left(M \otimes V_\pi, \mathrm{ind}_{K[N]}^{K[\Gamma_\pi,\natural]}(V_\pi)\right) \cong \mathrm{Hom}_{K[\Gamma_\pi,\natural]}\left(M \otimes V_\pi, K[\Gamma_\pi/N, \natural\kappa_\pi] \otimes V_\pi\right).$$

As we have seen in part (b), $K[N]$ acts only on the second tensor legs, so

$$(11) \quad \mathrm{Hom}_{K[N]}\left(M \otimes V_\pi, K[\Gamma_\pi/N, \natural\kappa_\pi] \otimes V_\pi\right) = \mathrm{Hom}_K\left(M, K[\Gamma_\pi/N, \natural\kappa_\pi]\right) \otimes K\mathrm{Id}_{V_\pi}.$$

An element $\phi = \phi' \otimes \mathrm{Id}_{V_\pi}$ of (11) is a $K[\Gamma_\pi, \natural]$-homomorphism if and only if commutes with the action described in part (a). On $V_\pi$ it automatically commutes with the $I^\gamma$, so the condition becomes that $\phi'$ commutes with left multiplication by $T_{\gamma N}$. In other words, $\phi'$ needs to be in $\mathrm{Hom}_{K[\Gamma_\pi/N, \natural\kappa_\pi]}(M, K[\Gamma_\pi/N, \natural\kappa_\pi])$. In view of (8), (9) is isomorphic with

$$\mathrm{Hom}_{K[\Gamma_\pi/N, \natural\kappa_\pi]}\Big(M, \mathrm{Hom}_N\big(V_\pi, \mathrm{ind}_{K[N]}^{K[\Gamma,\natural]}(V_\pi)\big)\Big). \qquad \square$$

This result leads to a version of Clifford theory. We will formulate it in terms of extended quotients, see [ABPS4, §2] or [ABPS5, Appendix B]. We briefly recall the necessary definitions.

Suppose that $\Gamma$ acts on some set $X$. Let $\kappa$ be a given function which assigns to each $x \in X$ a 2-cocycle $\kappa_x : \Gamma_x \times \Gamma_x \to \mathbb{C}^\times$, where $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$. It is assumed that $\kappa_{\gamma x}$ and $\gamma_* \kappa_x$ define the same class in $H^2(\Gamma_x, K^\times)$, where $\gamma_* : \Gamma_x \to \Gamma_{\gamma x}, \alpha \mapsto \gamma\alpha\gamma^{-1}$. Define

$$\widetilde{X}_\kappa := \{(x, \rho) : x \in X, \rho \in \mathrm{Irr}\, K[\Gamma_x, \kappa_x]\}.$$

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

(12) $$\phi_{\gamma, x} : K[\Gamma_x, \kappa_x] \to K[\Gamma_{\gamma x}, \kappa_{\gamma x}]$$

such that:

- $\phi_{\gamma, x}$ is inner if $\gamma x = x$;
- $\phi_{\gamma', \gamma x} \circ \phi_{\gamma, x} = \phi_{\gamma'\gamma, x}$ for all $\gamma', \gamma \in \Gamma, x \in X$.

We call these maps connecting homomorphisms, because they are reminiscent of a connection on a vector bundle. Then we can define $\Gamma$-action on $\widetilde{X}_\kappa$ by

$$\gamma \cdot (x, \rho) = (\gamma x, \rho \circ \phi_{\gamma, x}^{-1}).$$

We form the *twisted extended quotient*

(13) $$(X /\!/ \Gamma)_\kappa := \widetilde{X}_\kappa / \Gamma.$$

Let us return to the setting of Proposition 1.1.

**Theorem 1.2.** *Let $\kappa\natural$ be the family of 2-cocycles which assigns $\kappa_\pi\natural$ to $\pi \in \mathrm{Irr}_K(N)$. There is a bijection*

$$\begin{aligned} (\mathrm{Irr}_K(N) /\!/ \Gamma/N)_{\kappa\natural} &\to & \mathrm{Irr}(K[\Gamma, \natural]) \\ (\pi, \tau) &\mapsto & \tau \ltimes \pi := \mathrm{ind}_{K[\Gamma_\pi, \natural]}^{K[\Gamma, \natural]}(V_\tau \otimes V_\pi) \end{aligned}$$

*Proof.* With Proposition 1.1, [Sol2, Appendix] becomes valid in our situation. The theorem is a reformulation of parts (d) and (e) of [Sol2, Theorem 11.2]. For completeness we note that the connecting homomorphism

$$K[\Gamma_\pi/N, \kappa_\pi\natural] \to K[\Gamma_{\gamma \cdot \pi}/N, \kappa_{\gamma \cdot \pi}\natural]$$

is given by conjugation with $I_\pi^\gamma$, as in [ABPS4, (3)]. $\qquad \square$

For convenience we record the special case $\natural = 1$ of the above explicitly. It is very similar to [RaRa, p. 24] and [CuRe, §51].

(14) $$\mathrm{ind}_N^{\Gamma_\pi}(\pi) \cong K[\Gamma_\pi/N, \kappa_\pi] \otimes_K V_\pi \quad \text{as } \Gamma_\pi\text{-representations,}$$

(15) $$\mathrm{Irr}_K(\Gamma) \longleftrightarrow (\mathrm{Irr}_K(N) /\!/ \Gamma/N)_\kappa.$$

It will also be useful to analyse the structure of $K[\Gamma, \natural]$ as a bimodule over itself. Let $K[\Gamma, \natural]^{op}$ be the opposite algebra, and denote its standard basis elements by $S_\gamma$ ($\gamma \in \Gamma$).

**Lemma 1.3.** $(a)$ *There is a $K$-algebra isomorphism*

$$* : \quad K[\Gamma, \natural^{-1}] \quad \to \quad K[\Gamma, \natural]^{op}$$
$$T_\gamma \quad \mapsto \quad T_\gamma^* = S_\gamma^{-1}.$$

$(b)$ *There is a bijection*

$$\mathrm{Irr}(K[\Gamma, \natural]) \quad \to \quad \mathrm{Irr}(K[\Gamma, \natural^{-1}])$$
$$V \quad \mapsto \quad V^* = \mathrm{Hom}_K(V, K),$$

*where $(h \cdot \lambda)(v) = \lambda(h^* \cdot v)$ for $v \in V, \lambda \in V^*$ and $h \in K[\Gamma, \natural^{-1}]$.*
$(c)$ *Let $K[\Gamma, \natural] \oplus K[\Gamma, \natural^{-1}]$ act on $K[\Gamma, \natural]$ by $(a, h) \cdot b = abh^*$.*
    *As $K[\Gamma, \natural] \oplus K[\Gamma, \natural^{-1}]$-modules*

$$K[\Gamma, \natural] \cong \bigoplus\nolimits_{V \in \mathrm{Irr}(K[\Gamma, \natural])} V \otimes V^*.$$

*Proof.* (a) The map is $K$-linear by definition, and it clearly is bijective. For $\gamma, \gamma' \in \Gamma$:

$$T_\gamma^* \cdot T_{\gamma'}^* = S_\gamma^{-1} \cdot S_{\gamma'}^{-1} = (S_{\gamma'} \cdot S_\gamma)^{-1} = (\natural(\gamma, \gamma') S_{\gamma\gamma'})^{-1} = \natural(\gamma, \gamma')^{-1} T_{\gamma\gamma'}^*,$$

so * is an algebra homomorphism.
(b) Trivial, it holds for any finite dimensional algebra and its opposite.
(c) Let $\tilde{\Gamma}$ be a Schur extension of $\Gamma$, as on page 6. As a representation of $\tilde{\Gamma} \times (\tilde{\Gamma})^{op}$, $K[\tilde{\Gamma}]$ decomposes in the asserted manner. Hence the same holds for its direct factor $K[\Gamma, \natural]$. $\qquad \square$

## 2. The generalized Springer correspondence

Let $G$ be a connected complex reductive group. The generalized Springer correspondence for $G$ has been constructed by Lusztig. We will recall the main result of [Lus2], and then we prove that Lusztig's constructions are equivariant with respect to automorphisms of algebraic groups.

Let $l$ be a fixed prime number, and let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of $\mathbb{Q}_\ell$. For compatibility with the literature we phrase our results with $\overline{\mathbb{Q}_\ell}$-coefficients. However, by their algebro-geometric nature everything works just as well with coefficients in any other algebraically closed field of characteristic zero.

For $u$ a unipotent element in $G$, we denote by $A_G(u)$ the group of components $Z_G(u)/Z_G(u)^\circ$ of the centralizer in $G$ of $u$. We set

$$\mathcal{N}_G^+ := \{(u, \eta) : u \in G \text{ unipotent}, \eta \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_G(u))\}/G\text{-conjugacy}.$$

The set $\mathcal{N}_G^+$ is canonically in bijection with the set of pairs $(\mathcal{C}_u^G, \mathcal{F})$, where $\mathcal{C}_u^G$ is the $G$-conjugacy class of a unipotent element $u \in G$ and $\mathcal{F}$ is an irreducible $G$-equivariant local system on $\mathcal{C}_u^G$. The bijection associates to $(\mathcal{C}_u^G, \mathcal{F})$ an element $u \in \mathcal{C}_u^G$ and the representation of $A_G(u)$ on the stalk $\mathcal{F}_u$.

Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$, and let $L$ be a Levi factor of $P$. Let $v$ be a unipotent element in $L$. The group $Z_G(u) \times Z_L(v)U$ acts on the variety

$$(16) \qquad\qquad Y_{u,v} := \{y \in G \ : \ y^{-1}uy \in vU\}$$

by $(g,p) \cdot y = gyp^{-1}$, with $g \in Z_G(u)$, $p \in Z_L(v)U$, and $y \in Y_{u,v}$. We have

$$\dim Y_{u,v} \leq d_{u,v} := \frac{1}{2}(\dim Z_G(u) + \dim Z_L(v)) + \dim U.$$

The group $A_G(u) \times A_L(v)$ acts on the set of irreducible components of $Y_{u,v}$ of dimension $d_{u,v}$; we denote by $\sigma_{u,v}$ the corresponding permutation representation.

Let $\langle \, , \, \rangle_{A_G(u)}$ be the usual scalar product of the set of class functions on the finite group $A_G(u)$ with values in $\overline{\mathbb{Q}_\ell}$. An irreducible representation $\eta$ of $A_G(u)$ is called *cuspidal* (see [Lus2, Definition 2.4] and [LuSp, §0.4]) if

(17) $$\langle \eta, \sigma_{u,v} \rangle_{A_G(u)} \neq 0 \text{ implies that } P = G.$$

If $A_G(u)$ has a cuspidal representation, then [Lus2, Proposition 2.8] implies that $u$ is a distinguished unipotent element of $G$, i.e. not contained in any proper Levi subgroup of $G$. However, in general not every distinguished unipotent element supports a cuspidal representation. The set of irreducible cuspidal representations of $A_G(u)$ (over $\overline{\mathbb{Q}_\ell}$) is denoted by $\mathrm{Irr}_{\mathrm{cusp}}(A_G(u))$, and we write

$$\mathcal{N}_G^0 = \{(u,\eta) : u \in G \text{ unipotent}, \ \eta \in \mathrm{Irr}_{\mathrm{cusp}}(A_G(u))\}/G\text{-conjugacy}.$$

Given a pair $(u,\eta) \in \mathcal{N}_G^+$, there exists a triple $(P,L,v)$ as above and an

$$\epsilon \in \mathrm{Irr}_{\mathrm{cusp}}(A_L(v)) \quad \text{such that} \quad \langle \eta \otimes \epsilon^*, \sigma_{u,v} \rangle_{A_G(u) \times A_L(v)} \neq 0,$$

where $\epsilon^*$ is the dual of $\epsilon$ (see [Lus2, § 6.2] and [LuSp, §0.4]). Moreover $(P,L,v,\epsilon)$ is unique up to $G$-conjugation (see [Lus2, Prop. 6.3] and [LuSp]). We denote by $\mathfrak{t} := [L, \mathcal{C}_v^L, \epsilon]_G$ the $G$-conjugacy class of $(L,v,\epsilon)$ and we call it the *cuspidal support* of the pair $(u,\eta)$. The centre $Z(G)$ maps naturally to $A_G(u)$ and to $A_L(v)$. By construction [Lus2, Theorem 6.5.a]

(18) $$\eta \text{ and } \epsilon \text{ have the same } Z(G)\text{-character.}$$

Let $\mathcal{S}_G$ denote the set consisting of all triples $(L, \mathcal{C}_v^L, \epsilon)$ (up to $G$-conjugacy) where $L$ is a Levi subgroup of a parabolic subgroup of $G$, $\mathcal{C}_v^L$ is the $L$-conjugacy class of a unipotent element $v$ in $L$ and $\epsilon \in \mathrm{Irr}_{\mathrm{cusp}}(A_L(v))$. Let

(19) $$\Psi_G \colon \mathcal{N}_G^+ \to \mathcal{S}_G$$

be the map defined by sending the $G$-conjugacy class of $(u,\eta)$ to its cuspidal support. By (18) this map preserves the $Z(G)$-characters of the involved representations.

In [Lus2, §3.1], Lusztig defined a partition of $G$ in a finite number of irreducible, smooth, locally closed subvarieties, stable under conjugation. For all $g \in G$, we denote by $g_s$ the semisimple part of $g$. We say that $g \in G$ (or its conjugacy class) is isolated if $Z_G(g_s)^\circ$ is not contained in any proper Levi subgroup of $G$. In particular every unipotent conjugacy class is isolated.

Let $L$ be a Levi subgroup of $G$ and $S \subset L$ the inverse image of an isolated conjugacy class of $L/Z_L^\circ$ by the natural projection map $L \twoheadrightarrow L/Z_L^\circ$. Denote by

$$S_{\mathrm{reg}} = \{g \in S, \ Z_G(g_s)^\circ \subset L\}$$

the set of regular elements in $S$. Consider the irreducible, smooth, locally closed subvariety of $G$ defined by

$$Y_{(L,S)} = \bigcup_{g \in G} g S_{\mathrm{reg}} g^{-1} = \bigcup_{x \in S_{\mathrm{reg}}} \mathcal{C}_x^G.$$

We remark that $Y_{(L,S)}$ depends only on the $G$-conjugacy class of $(L,S)$.

Now, let $P = LU_P$ a parabolic subgroup of $G$ with Levi factor $L$, denote $\overline{\mathbf{c}} = (P, L, S)$, $\mathbf{c} = (L, S)$ and let

$$\widehat{X}_{\overline{\mathbf{c}}} = \{(g, x) \in G \times G, \ x^{-1}gx \in \overline{S} \cdot U_P\},$$
$$X_{\overline{\mathbf{c}}} = \{(g, xP) \in G \times G/P, \ x^{-1}gx \in \overline{S} \cdot U_P\},$$

where $\overline{S}$ is the closure of $S$. The subgroup $P$ acts freely by translation on right on the second coordinate of an element of $\widehat{X}_{\overline{\mathbf{c}}}$ and $\widehat{X}_{\overline{\mathbf{c}}}/P = X_{\overline{\mathbf{c}}}$. After [Lus2, 4.3], the projection on the first coordinate $\phi_{\overline{\mathbf{c}}} : X_{\overline{\mathbf{c}}} \longrightarrow G$ is proper and its image is $\overline{Y}_{\mathbf{c}}$.

The group $Z_L^\circ$ acts on $\overline{S}$ by translation and $L$ acts on $\overline{S}$ by conjugation. This gives rises to an action of $Z_L^\circ \times L$ on $\overline{S}$. The orbits form a stratification of $\overline{S}$, in which $S$ is the unique open stratum. Denote by $\sigma_{\overline{\mathbf{c}}} : \widehat{X}_{\overline{\mathbf{c}}} \longrightarrow \overline{S}$ the map which associates to $(g, x)$ the projection of $x^{-1}gx \in \overline{S} \cdot U_P$ on the factor $\overline{S}$ and $\varpi_P : \widehat{X}_{\overline{\mathbf{c}}} \longrightarrow X_{\overline{\mathbf{c}}}$ the map defined for all $(g, x) \in \widehat{X}_{\overline{\mathbf{c}}}$ by $\varpi_P(g, x) = (g, xP)$. To sum up, we have the following diagram:



By taking image inverse under $\sigma_{\overline{\mathbf{c}}}$, the stratification of $\overline{S}$ gives a stratification of $\widehat{X}_{\overline{\mathbf{c}}}$. The stratum $\widehat{X}_{\overline{\mathbf{c}}, \alpha}$ (corresponding to the open stratum $S$) is open and dense. We denote by $\sigma_{\overline{\mathbf{c}}, \alpha}$ the restriction of $\sigma_{\overline{\mathbf{c}}}$ to $\widehat{X}_{\overline{\mathbf{c}}, \alpha}$. Every stratum of $\widehat{X}_{\overline{\mathbf{c}}}$ is $P$-invariant and their images in $X_{\overline{\mathbf{c}}} = \widehat{X}_{\overline{\mathbf{c}}}/P$ form a stratification of $X_{\overline{\mathbf{c}}}$, with $X_{\overline{\mathbf{c}}, \alpha} = \widehat{X}_{\overline{\mathbf{c}}, \alpha}/P$ open and dense.

Let $\mathcal{E}$ be an irreducible $L$-equivariant cuspidal local system on $S$. Then $(\sigma_{\overline{\mathbf{c}}, \alpha})^*\mathcal{E}$ is a $G \times P$-equivariant local system on $\widehat{X}_{\overline{\mathbf{c}}, \alpha}$. There exists a unique $G$-equivariant local system on $X_{\overline{\mathbf{c}}, \alpha}$, denoted by $\overline{\mathcal{E}}$, such that $(\sigma_{\overline{\mathbf{c}}, \alpha})^*\mathcal{E} = (\varpi_P)^*\overline{\mathcal{E}}$.

We denote by $\widetilde{Y}_{\mathbf{c}} = \phi_{\overline{\mathbf{c}}}^{-1}(Y_{\mathbf{c}})$, $\pi_{\mathbf{c}} = \phi_{\overline{\mathbf{c}}}|_{\widetilde{Y}_{\mathbf{c}}}$, $\widetilde{\mathcal{E}} = \overline{\mathcal{E}}|_{\widetilde{Y}_{\mathbf{c}}}$ and

$$\mathcal{A}_{\mathcal{E}} = \mathrm{End}_{\mathcal{D}Y_{\mathbf{c}}}((\pi_{\mathbf{c}})_*\widetilde{\mathcal{E}}) \simeq \mathrm{End}_{\mathcal{D}_G Y_{\mathbf{c}}}((\pi_{\mathbf{c}})_*\widetilde{\mathcal{E}}),$$

where $\mathcal{D}Y_{\mathbf{c}}$ (resp. $\mathcal{D}_G Y_{\mathbf{c}}$) is the bounded derived category of $\overline{\mathbb{Q}}_\ell$-constructible sheaves (resp. $G$-equivariant) on $Y_{\mathbf{c}}$. We denote by $\mathrm{Irr}(\mathcal{A}_{\mathcal{E}})$ the set of (isomorphism classes of) simple $\mathcal{A}_{\mathcal{E}}$-modules and $\overline{\mathbb{Q}}_\ell$ the constant sheaf.

Let $K_{\overline{\mathbf{c}}} = \mathrm{IC}(X_{\overline{\mathbf{c}}}, \overline{\mathcal{E}})$ the intersection cohomology complex of Deligne–Goresky–MacPherson on $X_{\overline{\mathbf{c}}}$, with coefficients in $\overline{\mathcal{E}}$. Then $(\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}}$ is a complex on $\overline{Y}_{\mathbf{c}}$.

**Theorem 2.1.** [Lus2, Theorem 6.5]
*Let $\mathfrak{t} = [L, \mathcal{C}_v^L, \mathcal{E}] \in \mathcal{S}_G$, $(S, \mathcal{E}) = (Z_L^\circ \cdot \mathcal{C}_v^L, \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E})$ the corresponding cuspidal pair for $L$ and $P$ a parabolic subgroup of $G$ with Levi factor $L$. As before, we denote by $\overline{\mathbf{c}} = (P, L, S)$, $\mathbf{c} = (L, S)$ and $(\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}}$ the corresponding complex on $\overline{Y}_{\mathbf{c}}$.*

*(1) Let $(\mathcal{C}_u^G, \mathcal{F}) \in \mathcal{N}_G^+$. Then $\Psi_G(\mathcal{C}_u^G, \mathcal{F}) = (L, \mathcal{C}_v^L, \mathcal{E})$, if and only if the following conditions are satisfied :*
   *(a) $\mathcal{C}_u^G \subseteq \overline{Y}_{\mathbf{c}}$ ;*

(b) $\mathcal{F}$ is a direct summand of $R^{2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}}(f_{\overline{\mathbf{c}}})_!(\overline{\mathcal{E}})\big|_{\mathcal{C}_u^G}$, where $f_{\overline{\mathbf{c}}}$ is the restriction
    of $\phi_{\overline{\mathbf{c}}}$ to $X_{\overline{\mathbf{c}},\alpha} \subset X_{\overline{\mathbf{c}}}$, $d_{\mathcal{C}_u^G, \mathcal{C}_v^L} = (\nu_G - \frac{1}{2}\dim \mathcal{C}_u^G) - (\nu_L - \frac{1}{2}\dim \mathcal{C}_v^L)$, and
    $\nu_G$ (resp. $\nu_L$) is the number of positive roots of $G$ (resp. $L$).

(2) The natural morphism

$$R^{2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}}(f_{\overline{\mathbf{c}}})_!(\overline{\mathcal{E}})\big|_{\mathcal{C}_u^G} \longrightarrow \mathcal{H}^{2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}}((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})\big|_{\mathcal{C}_u^G}$$

given by the imbedding of $X_{\overline{\mathbf{c}},\alpha}$ into $X_{\overline{\mathbf{c}}}$ as an open subset, is an isomorphism.

(3) For all $\rho \in \mathrm{Irr}(\mathcal{A}_{\mathcal{E}})$, let $((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho$ the $\rho$-isotypical component of $(\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}}$,
    i.e.

$$(\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}} = \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{A}_{\mathcal{E}})} \rho \boxtimes ((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho.$$

Let $\overline{Y}_{\mathbf{c},\mathrm{un}}$ be the variety of unipotent elements in $\overline{Y}_{\mathbf{c}}$. There exists an unique
pair $(\mathcal{C}_u^G, \mathcal{F}) \in \mathcal{N}_G^+$ which satisfies the following conditions:

(a) $\mathcal{C}_u^G \subset \overline{Y}_{\mathbf{c}}$;

(b) $((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho\big|_{\overline{Y}_{\mathbf{c},\mathrm{un}}}$ is isomorphic to $\mathrm{IC}(\overline{\mathcal{C}_u^G}, \mathcal{F})[2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}]$ extended by $0$ on
    $\overline{Y}_{\mathbf{c},\mathrm{un}} - \overline{\mathcal{C}_u^G}$.

In particular, $\mathcal{F} = \mathcal{H}^{2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}}(((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho)\big|_{\mathcal{C}_u^G}$ and

$\rho = \mathrm{Hom}_G\big(\mathcal{F}, \mathcal{H}^{2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}}((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})\big|_{\mathcal{C}_u^G}\big)$. The map

$$\Sigma_{\mathfrak{t}} : \Psi_G^{-1}(\mathfrak{t}) \to \mathrm{Irr}(\mathcal{A}_{\mathcal{E}})$$

which associates $\rho$ to $(\mathcal{C}_u^G, \mathcal{F})$ is a bijection.

The relation of Theorem 2.1 with the classical Springer correspondence goes via $\mathcal{A}_{\mathcal{E}}$, which turns out to be isomorphic to the group algebra of a Weyl group. We define

(20) $$W_{\mathfrak{t}} := N_G(\mathfrak{t})/L = N_G(L, \mathcal{C}_v^L, \mathcal{E})/L.$$

**Theorem 2.2.** [Lus2, Theorem 9.2]

(a) $W_{\mathfrak{t}} = N_G(L)/L$.

(b) $N_G(L)/L$ is the Weyl group of the root system $R(G, Z(L)^\circ)$.

(c) There exists a canonical algebra isomorphism $\mathcal{A}_{\mathcal{E}} \cong \overline{\mathbb{Q}}_\ell[W_{\mathfrak{t}}]$. Together with Theorem 2.1.(3) this gives a canonical bijection $\Psi_G^{-1}(\mathfrak{t}) \to \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(W_{\mathfrak{t}})$.

In fact there exist two such canonical algebra isomorphisms, for one can always twist with the sign representation of $W_{\mathfrak{t}}$. When we employ generalized Springer correspondences in relation with the local Langlands correspondence, we will always use the isomorphism $\mathcal{A}_{\mathcal{E}} \cong \overline{\mathbb{Q}}_\ell[W_{\mathfrak{t}}]$ such that the trivial $W_{\mathfrak{t}}$-representation is the image of $(\mathcal{C}_v^G, \mathcal{E})$ under Theorems 2.1 and 2.2. (Here we extend $\mathcal{E}$ $G$-equivariantly to $\mathcal{C}_v^G$, compare with [Lus2, 9.5].)

Let $H$ be a group which acts on the connected complex reductive group $G$ by algebraic automorphisms. Then $H$ acts also on $\mathcal{N}_G^+$ and $\mathcal{S}_G$. Indeed, let $h \in H$, $(\mathcal{C}_u^G, \mathcal{F}) \in \mathcal{N}_G^+$, $\mathfrak{t} = [L, \mathcal{C}_v^L, \mathcal{E}]_G \in \mathcal{S}_G$ and $\rho \in \mathrm{Irr}(W_{\mathfrak{t}})$. Since $h(G) = G$, ${}^h\mathcal{C}_u^G = \mathcal{C}_{h \cdot u}^G$ is a unipotent orbit of $G$. Similarly, ${}^h L$ is a Levi subgroup of $G$, ${}^h\mathcal{C}_v^L$ is a unipotent orbit of ${}^h L$, etc.

We denote by $h^*$ the pullback of sheaves along the isomorphism $h^{-1} : G \to G$. Thus $h^*\mathcal{F}$ (resp. $h^*\mathcal{L}$) is a local system on $\mathcal{C}_{h \cdot u}^G$ (resp. ${}^h\mathcal{C}_v^L$). Keeping the above

notation, the action of $H$ on $\mathcal{N}_G^+$, $\mathcal{S}_G$ and $\mathrm{Irr}(W)$ is given by

$$h \cdot (\mathcal{C}_u^G, \mathcal{F}) = (\mathcal{C}_{h \cdot u}^G, h^* \mathcal{F}), \qquad h \cdot [L, \mathcal{C}_v^L, \mathcal{L}] = [{}^h L, \mathcal{C}_{h \cdot v}^{{}^h L}, h^* \mathcal{L}]$$

and $h \cdot \rho = \rho^h \in \mathrm{Irr}(W_{h \cdot \mathfrak{t}})$.

**Theorem 2.3.** *The Springer correspondence for $G$ is $H$-equivariant. More precisely, for all $h \in H$, the following diagrams are commutative:*

$$
\begin{array}{ccc}
\mathcal{N}_G^+ & \xrightarrow{\Psi_G} & \mathcal{S}_G^+ \\
\downarrow{h} & & \downarrow{h} \\
\mathcal{N}_G^+ & \xrightarrow{\Psi_G} & \mathcal{S}_G^+
\end{array}
\qquad
\begin{array}{ccc}
\Psi_G^{-1}(\mathfrak{t}) & \xrightarrow{\Sigma_\mathfrak{t}} & \mathrm{Irr}(W_\mathfrak{t}) \\
\downarrow{h} & & \downarrow{h} \\
\Psi_G^{-1}(h \cdot \mathfrak{t}) & \xrightarrow{\Sigma_{h \cdot \mathfrak{t}}} & \mathrm{Irr}(W_{h \cdot \mathfrak{t}})
\end{array}
$$

*In other words, for all $h \in H$, $(\mathcal{C}_u^G, \mathcal{F}) \in \Psi_G^{-1}(\mathfrak{t}) \subset \mathcal{N}_G^+$:*

$$\Psi_G(h \cdot (\mathcal{C}_u^G, \mathcal{F})) = h \cdot \Psi_G(\mathcal{C}_u^G, \mathcal{F}) \quad and \quad \Sigma_{h \cdot \mathfrak{t}}(h \cdot (\mathcal{C}_u^G, \mathcal{F})) = h \cdot \Sigma_\mathfrak{t}(\mathcal{C}_u^G, \mathcal{F}).$$

*Proof.* We keep the notations of Theorem 2.1. Let $h \in H$, $(\mathcal{C}_u^G, \mathcal{F}) \in \mathcal{N}_G^+$, $P$ a parabolic subgroup of $G$ with Levi factor $L$, $v \in L$ a unipotent element and $\mathcal{E}$ an irreducible cuspidal $L$-equivariant local system on $\mathcal{C}_v^L$ such that

$$\Psi_G(\mathcal{C}_u^G, \mathcal{F}) = [L, \mathcal{C}_v^L, \mathcal{E}] \in \mathcal{S}_G.$$

As in Theorem 2.1, let $(S, \mathcal{E}) = (Z_L^\circ \cdot \mathcal{C}_v^L, \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E})$ be the corresponding cuspidal pair for $L$ and $\overline{\mathbf{c}} = (P, L, S)$, $\mathbf{c} = (L, S)$. After (1) in Theorem 2.1, $\mathcal{C}_u^G \subset \overline{Y}_\mathbf{c}$, so ${}^h \mathcal{C}_u^G \subset {}^h \overline{Y}_\mathbf{c} = \overline{Y}_{h \cdot \mathbf{c}}$, where $h \cdot \mathbf{c} = ({}^h L, {}^h S)$. Consider the maps

$$
\begin{array}{ccc}
\widehat{X}_{h \cdot \overline{\mathbf{c}}} & \longrightarrow & \widehat{X}_{\overline{\mathbf{c}}} \\
(g, x) & \longmapsto & \left({}^{h^{-1}} g, {}^{h^{-1}} x\right)
\end{array}
,
\begin{array}{ccc}
X_{h \cdot \overline{\mathbf{c}}} & \longrightarrow & X_{\overline{\mathbf{c}}} \\
(g, x\,{}^h P) & \longmapsto & \left({}^{h^{-1}} g, {}^{h^{-1}} xP\right)
\end{array}
,
\begin{array}{ccc}
G & \longrightarrow & G \\
g & \longmapsto & {}^{h^{-1}} g
\end{array}
.
$$

and the following diagrams:

$$
\begin{array}{ccc}
\widehat{X}_{h \cdot \overline{\mathbf{c}}, \alpha} & \xrightarrow{h} & \widehat{X}_{\overline{\mathbf{c}}, \alpha} \\
\downarrow{\sigma_{h \cdot \overline{\mathbf{c}}, \alpha}} & & \downarrow{\sigma_{\overline{\mathbf{c}}, \alpha}} \\
{}^h S & \xrightarrow{h} & S
\end{array}
,
\begin{array}{ccc}
\widehat{X}_{h \cdot \overline{\mathbf{c}}, \alpha} & \xrightarrow{h} & \widehat{X}_{\overline{\mathbf{c}}, \alpha} \\
\downarrow{\varpi_{{}^h P}} & & \downarrow{\varpi_P} \\
X_{h \cdot \overline{\mathbf{c}}, \alpha} & \xrightarrow{h} & X_{\overline{\mathbf{c}}, \alpha}
\end{array}
,
\begin{array}{ccc}
X_{h \cdot \overline{\mathbf{c}}, \alpha} & \xrightarrow{h} & X_{\overline{\mathbf{c}}, \alpha} \\
\downarrow{f_{h \cdot \overline{\mathbf{c}}}} & & \downarrow{f_{\overline{\mathbf{c}}}} \\
\overline{Y}_{h \cdot \mathbf{c}} & \xrightarrow{h} & \overline{Y}_\mathbf{c}
\end{array}
.
$$

The first two commutative diagrams show that:

$$
\begin{aligned}
(\sigma_{h \cdot \overline{\mathbf{c}}, \alpha})^* (h^* \mathcal{E}) &= h^* (\sigma_{\overline{\mathbf{c}}, \alpha})^* (\mathcal{E}) \\
&= h^* (\varpi_P)^* (\overline{\mathcal{E}}) = (\varpi_{{}^h P})^* (h^* \overline{\mathcal{E}}).
\end{aligned}
$$

By unicity, this shows that $h^* \overline{\mathcal{E}} = \overline{h^* \mathcal{E}}$. The third cartesian diagram shows, by the proper base change theorem, that

$$h^* R^{2 d_C} (f_{\overline{\mathbf{c}}})_! (\overline{\mathcal{E}}) \cong R^{2 d_C} (f_{h \cdot \overline{\mathbf{c}}})_! (h^* \overline{\mathcal{E}}) = R^{2 d_C} (f_{h \cdot \overline{\mathbf{c}}}) (\overline{h^* \mathcal{E}}).$$

Because

$$
\begin{aligned}
0 \neq \mathrm{Hom}_{\mathcal{D} \mathcal{C}_u^G} (\mathcal{F}, R^{2 d_C} (f_{\overline{\mathbf{c}}})_! (\overline{\mathcal{E}})|_{\mathcal{C}_u^G}) &\cong \mathrm{Hom}_{\mathcal{D}^h \mathcal{C}_u^G} (h^* \mathcal{F}, h^* R^{2 d_C} (f_{\overline{\mathbf{c}}})_! (\overline{\mathcal{E}})|_{{}^h \mathcal{C}_u^G}) \\
&\cong \mathrm{Hom}_{\mathcal{D}^h \mathcal{C}_u^G} (h^* \mathcal{F}, R^{2 d_C} (f_{h \cdot \overline{\mathbf{c}}})_! (\overline{h^* \mathcal{E}})|_{{}^h \mathcal{C}_u^G}) \neq 0,
\end{aligned}
$$

with $d_C = d_{\mathcal{C}_u^G, \mathcal{C}_v^L} = d_{{}^h \mathcal{C}_u^G, {}^h \mathcal{C}_v^L}$. Thus $h^* \mathcal{F}$ is a direct summand of $R^{2d} (f_{h \cdot \overline{\mathbf{c}}})_! (\overline{h^* \mathcal{E}})|_{{}^h \mathcal{C}_u^G}$ and after Theorem 2.1, $\Psi_G$ is $H$-equivariant.

According to [GoMP, Proposition 5.4]

$$h^* K_{\overline{\mathbf{c}}} = h^* \mathrm{IC}(X_{\overline{\mathbf{c}}}, \overline{\mathcal{E}}) = \mathrm{IC}(h^* X_{\overline{\mathbf{c}}}, h^* \overline{\mathcal{E}}) = \mathrm{IC}(X_{h \cdot \overline{\mathbf{c}}}, \overline{h^* \mathcal{E}}) = K_{h \cdot \overline{\mathbf{c}}}.$$

Let $\rho \in \mathrm{Irr}(\mathcal{A}_{\mathcal{E}})$. By functoriality, $\mathcal{A}_{h^* \mathcal{E}} \simeq \mathcal{A}_{\mathcal{E}}$ and by considering the third commutative diagram, we get:

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{A}_{\mathcal{E}}}(\rho, (\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}}) &\cong \quad \mathrm{Hom}_{\mathcal{A}_{h^* \mathcal{E}}}(h^* \rho, h^* (\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}}) \\
&\cong \quad \mathrm{Hom}_{\mathcal{A}_{h^* \mathcal{E}}}(\rho^h, (\phi_{h \cdot \overline{\mathbf{c}}})_! K_{h \cdot \overline{\mathbf{c}}}) \\
h^* ((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho &\cong \qquad ((\phi_{h \cdot \overline{\mathbf{c}}})_! K_{h \cdot \overline{\mathbf{c}}})_{h^* \rho}
\end{aligned}
$$

Since $((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho |_{\overline{Y}_{\mathbf{c}, \mathrm{un}}} \simeq \mathrm{IC}(\overline{\mathcal{C}_u^G}, \mathcal{F})[2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}]$, we have

$$h^* ((\phi_{\overline{\mathbf{c}}})_! K_{\overline{\mathbf{c}}})_\rho |_{\overline{Y}_{\mathbf{c}, \mathrm{un}}} \cong h^* \mathrm{IC}(\overline{\mathcal{C}_u^G}, \mathcal{F})[2d_{\mathcal{C}_u^G, \mathcal{C}_v^L}]$$

$$((\phi_{h \cdot \overline{\mathbf{c}}})_! K_{h \cdot \overline{\mathbf{c}}})_{h^* \rho} \cong \mathrm{IC}(\overline{{}^h \mathcal{C}_u^G}, h^* \mathcal{F})[2d_{{}^h \mathcal{C}_u^G, {}^h \mathcal{C}_v^L}]$$

According to the characterization (3) of Theorem 2.1, this shows that $\Sigma_{\mathfrak{t}}$ is $H$-equivariant. $\qquad \square$

## 3. Disconnected groups: the cuspidal case

First we recall Lusztig's classification of unipotent cuspidal pairs for a connected reductive group. See Erratum in the appendix!

**Theorem 3.1.** *(Lusztig)*
*Let $G^\circ$ be a connected complex reductive group and write $Z = Z(G^\circ)/Z(G^\circ)^\circ$.*

(a) *Fix an $\mathrm{Aut}(G^\circ)$-orbit $X$ of characters $Z \to \overline{\mathbb{Q}}_\ell^\times$. There is at most one unipotent conjugacy class $\mathcal{C}_u^{G^\circ}$ which carries a cuspidal local system on which $Z$ acts as an element of $X$. Moreover $\mathcal{C}_u^{G^\circ}$ is $\mathrm{Aut}(G^\circ)$-stable and distinguished in $G^\circ$.*
(b) *Every cuspidal local system $\mathcal{E}$ on $\mathcal{C}_u^{G^\circ}$ is uniquely determined by the character by which $Z$ acts on it.*
(c) *The dimension of the cuspidal representation $\mathcal{E}_u$ of $A_{G^\circ}(u)$ is a power of two (possibly $2^0 = 1$). It is one if $G^\circ$ contains no factors which are isomorphic to spin or half-spin groups.*

*Proof.* In [Lus2, §2.10] it is explained how the classification can be reduced to simply connected, almost simple groups. Namely, first one notes that dividing out $Z(G^\circ)^\circ$ does not make an essential difference. Next everything is lifted to the simply connected cover $\tilde{G}$ of the semisimple group $G^\circ/Z(G^\circ)^\circ$. Since every automorphism of $G^\circ/Z(G^\circ)^\circ$ can be lifted to one of $\tilde{G}$, the canonical image of $X$ is contained in a unique $\mathrm{Aut}(\tilde{G})$-orbit $\tilde{X}$ on $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(\tilde{Z})$, where $\tilde{Z}$ is the $Z$ for $\tilde{G}$. Furthermore $\tilde{G}$ is a direct product of almost simple, simply connected groups, and $\tilde{X}$ decomposes as an analogous product. Therefore it suffices to establish the theorem for simple, simply connected groups $G_{\mathrm{sc}}$.

(a) and (b) are shown in the case-by-case calculations in [Lus2, §10 and §14–15]. But (a) is not made explicit there, so let us comment on it. There are only few cases in which one really needs an $\mathrm{Aut}(G_{\mathrm{sc}})$-orbit $X_{\mathrm{sc}}$ in $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(Z(G_{\mathrm{sc}}))$. Namely, only the spin groups $\mathrm{Spin}_N(\mathbb{C})$ where $N > 1$ is simultaneously a square and a triangular number. These groups have precisely two unipotent conjugacy classes, say $\mathcal{C}_+$ and $\mathcal{C}_-$, that carry a cuspidal local system. Let $\{1, -1\}$ be the kernel of $\mathrm{Spin}_N(\mathbb{C}) \to SO_N(\mathbb{C})$, a characteristic subgroup of $\mathrm{Spin}_N(\mathbb{C})$. Lusztig's classification shows that

$-1$ acts as $\epsilon$ on every cuspidal local system supported on $\mathcal{C}_\epsilon$. As $-1$ is fixed by $\mathrm{Aut}(G_{\mathrm{sc}})$, $X_{\mathrm{sc}}$ determines a unique character of $\{1, -1\}$ and thus specifies $\mathcal{C}_+$ or $\mathcal{C}_-$.

This proves the first part of (a). For the second part, we notice that every algebraic automorphism of $G^\circ$ maps a cuspidal local system on a unipotent conjugacy class in $G^\circ$ with $Z$-character in $X$ to another such local system. By [Lus2, Proposition 2.8] every such conjugacy class is distinguished in $G^\circ$.

(c) is obvious in types $A_n, C_n$ and $E_6$, for then $A_{G_{\mathrm{sc}}}(u)$ is abelian. For the root systems $E_8, F_4$ and $G_2$, $A_{G_{\mathrm{sc}}}(u)$ is a symmetric group and $\mathcal{E}_u$ is the sign representation [Lus2, §15]. In type $E_7$ [Miz, Table 9] shows that $A_{G_{\mathrm{sc}}}(u) \cong S_3 \times S_2$. According to [Lus2, §15.6], $(\mathcal{E})_u$ again has dimension one (it is the tensor product of the sign representations of $S_3$ and $S_2$).

In types $B_n$ and $D_n$, $G_{\mathrm{sc}} = \mathrm{Spin}_N(\mathbb{C})$ is a spin group. All the cuspidal local systems $\mathcal{E}$ for which the action of $Z(G_{\mathrm{sc}})$ factors through $Z(\mathrm{SO}_N(\mathbb{C}))$ are one-dimensional, for $A_{\mathrm{SO}_N(\mathbb{C})}(u)$ is abelian. If the character by which $Z(G_{\mathrm{sc}})$ acts on $\mathcal{E}$ is not of this kind, then [Lus2, Proposition 14.4] says that $\dim(\mathcal{E}_u)$ is a power of two. In that case the original $G^\circ$ has an almost direct factor isomorphic to $\mathrm{Spin}_N(\mathbb{C})$ or to a half-spin group $\mathrm{HSpin}_N(\mathbb{C}) = \mathrm{Spin}_N(\mathbb{C})/\{1, \omega\}$ (here $N \in 4\mathbb{N}$ and $\omega \in Z(\mathrm{Spin}_N(\mathbb{C})) \setminus \{1, -1\}$). $\qquad\square$

Let $G$ be a disconnected complex reductive group with neutral component $G^\circ$. We want to classify unipotent cuspidal pairs for $G$ in terms of those for $G^\circ$.

First we define them properly. For $u \in G^\circ$ we call an irreducible representation of $A_G(u)$ cuspidal if its restriction to $A_{G^\circ}(u)$ is a direct sum of irreducible cuspidal $A_{G^\circ}(u)$-representations. The set of irreducible cuspidal representations of $A_G(u)$ (over $\overline{\mathbb{Q}_\ell}$) is denoted by $\mathrm{Irr}_{\mathrm{cusp}}(A_G(u))$. We write

$$\mathcal{N}_G^0 = \{(u, \eta) : u \in G \text{ unipotent}, \eta \in \mathrm{Irr}_{\mathrm{cusp}}(A_G(u))\}/G\text{-conjugacy}.$$

Notice that the unipotency forces $u \in G^\circ$. Every $(u, \eta) \in \mathcal{N}_G^0$ gives rise to a unique $G$-equivariant local system $\mathcal{F}$ on $\mathcal{C}_u^G$. We call any $G$-equivariant local system on $\mathcal{C}_u^G$ cuspidal if and only if it arises in this way. Thus we may identify $\mathcal{N}_G^0$ with the set of pairs $(\mathcal{C}_u^G, \mathcal{F})$ where $\mathcal{C}_u^G$ is a unipotent conjugacy class in $G$ and $\mathcal{F}$ is a cuspidal local system on it. For example, if $G^\circ$ is a torus, then $u = 1$ and every irreducible representation of $A_G(u) = G/G^\circ$ is cuspidal.

It follows from (15) that there is a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(A_G(u)) \longleftrightarrow \Big(\mathrm{Irr}_{\mathrm{cusp}}(A_{G^\circ}(u))//A_G(u)/A_{G^\circ}(u)\Big)_\kappa.$$

So we want to identify the 2-cocycles $\kappa_\epsilon$ for $\epsilon \in \mathrm{Irr}_{\mathrm{cusp}}(A_{G^\circ}(u))$.

We note that there are natural isomorphisms

$$(21) \qquad\qquad A_G(u)/A_{G^\circ}(u) \leftarrow Z_G(u)/Z_{G^\circ}(u) \rightarrow G/G^\circ.$$

In fact Theorem 3.1.a implies that $\mathcal{C}_u^G = \mathcal{C}_u^{G^\circ}$, which accounts for the surjectivity of the map to the right.

Recall from [ABPS4, Lemma 4.2] that the short exact sequence

$$(22) \qquad 1 \rightarrow \pi_0(Z_{G^\circ}(u)/Z(G^\circ)) \rightarrow \pi_0(Z_G(u)/Z(G^\circ)) \rightarrow G/G^\circ \rightarrow 1$$

is split. However, the short exact sequence

$$(23) \qquad 1 \rightarrow \pi_0(Z_{G^\circ}(u)/Z(G^\circ)^\circ) \rightarrow \pi_0(Z_G(u)/Z(G^\circ)^\circ) \rightarrow G/G^\circ \rightarrow 1$$

need not be split. We choose a map

$$(24) \qquad\qquad s : G/G^\circ \to Z_G(u)$$

such that the induced map $G/G^\circ \to \pi_0(Z_G(u)/Z(G^\circ))$ is a group homomorphism that splits (22). The proof of [ABPS4, Lemma 4.2] shows that we can take $s(gG^\circ)$ in $Z_G(G^\circ)$ whenever the conjugation action of $g$ on $G^\circ$ is an inner automorphism of $G^\circ$. For all $\gamma, \gamma' \in G/G^\circ$

$$s(\gamma)s(\gamma')s(\gamma\gamma')^{-1} \in Z(G^\circ)Z_{G^\circ}(u)^\circ,$$

because it represents the neutral element of $\pi_0(Z_G(u)/Z(G^\circ))$.

Let $(\mathcal{C}_u^{G^\circ}, \mathcal{E}) \in \mathcal{N}_{G^\circ}^0$. The group $Z_{G^\circ}(u)^\circ$ acts trivially on $\epsilon = \mathcal{E}_u$ and by cuspidality $Z(G^\circ) \subset Z(L)$ acts according to a character. Therefore

$$(25) \qquad\qquad \natural_\mathcal{E}(\gamma, \gamma') := \epsilon\big(s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}\big)$$

lies in $\overline{\mathbb{Q}_\ell}^\times$. Comparing with (1), one checks easily that

$$(26) \qquad\qquad \natural_\mathcal{E} : G/G^\circ \times G/G^\circ \to \overline{\mathbb{Q}_\ell}^\times$$

is a 2-cocycle. We note that another element $u' \in \mathcal{C}_u^G$ would give the same cocycle: just conjugate $s$ with a $g \in G^\circ$ such that $gug^{-1} = u'$ and use the same formulas. Although $\natural_\mathcal{E}$ depends on the choice of $s$, its class in $H^2(G/G^\circ, \overline{\mathbb{Q}_\ell}^\times)$ does not. Indeed, suppose that $s'$ is another splitting as in (24). Since $s'(\gamma)$ and $s(\gamma)$ represent the same element of $\pi_0(Z_G(u)/Z(G^\circ))$, there exist

$$z(\gamma) \in Z(G^\circ) \text{ such that } s'(\gamma)s(\gamma)^{-1} \in z(\gamma)Z_{G^\circ}(u)^\circ.$$

As $Z_{G^\circ}(u)^\circ$ is normal in $Z_G(u)$ and contained in the kernel of $\epsilon$,

$$\epsilon\big(s'(\gamma)s'(\gamma')s'(\gamma\gamma')^{-1}\big) = \epsilon\big(s(\gamma)z(\gamma)s(\gamma')z(\gamma')s(\gamma\gamma')^{-1}z(\gamma\gamma')^{-1}\big) =$$
$$\natural_\mathcal{E}(\gamma, \gamma')\,\epsilon(z(\gamma))\epsilon(z(\gamma'))\epsilon\big(z(\gamma\gamma')^{-1}\big).$$

Therefore $s'$ gives rise to a 2-cocycle that differs from (26) by a coboundary, and the cohomology class of $\natural_\mathcal{E}$ depends only on $\mathcal{E}$. Via the isomorphism (21) we also get a 2-cocycle

$$\natural_\mathcal{E} : A_G(u)/A_{G^\circ}(u) \times A_G(u)/A_{G^\circ}(u) \to \overline{\mathbb{Q}_\ell}^\times.$$

It will turn out that the 2-cocycles $\natural_\mathcal{E}$ are trivial in many cases, in particular whenever $Z(G^\circ)$ acts trivially on $\mathcal{E}$. But sometimes their cohomology class is nontrivial.

**Example 3.2.** Consider the following subgroup of $\mathrm{SL}_2(\mathbb{C})^5$:

$$Q = \Big\{(\pm I_2) \times I_8, \left(\begin{smallmatrix} \pm i & 0 \\ 0 & \mp i \end{smallmatrix}\right) \times I_4 \times -I_4, \left(\begin{smallmatrix} 0 & \pm i \\ \pm i & 0 \end{smallmatrix}\right) \times I_2 \times -I_2 \times I_2 \times -I_2, \left(\begin{smallmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{smallmatrix}\right) \times I_2 \times -I_4 \times I_2 \Big\}$$

It is isomorphic to the quaternion group of order 8. We take $G = N_{\mathrm{SL}_{10}(\mathbb{C})}(Q)$. Then

$$G^\circ = Z_{\mathrm{SL}_{10}(\mathbb{C})}(Q) = \big(Z(\mathrm{GL}_2(\mathbb{C})) \times \mathrm{GL}_2(\mathbb{C})^4\big) \cap \mathrm{SL}_{10}(\mathbb{C}),$$

$$Z(G^\circ) = \Big\{(z_j)_{j=1}^5 \in Z(\mathrm{GL}_2(\mathbb{C}))^5 : \prod\nolimits_{j=1}^5 z_j^2 = 1\Big\}.$$

By [Lus2, §10.1–10.3] there exists a unique cuspidal pair for $G^\circ$, namely $(u = I_2 \times \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^{\otimes 4}, \epsilon)$ with $\epsilon$ the nontrivial character of

$$A_{G^\circ}(u) = Z(G^\circ)/Z(G^\circ)^\circ \cong \{\pm 1\}.$$

We note that the canonical map $Q \to A_G(u)$ is an isomorphism and that

$$G/G^\circ \cong A_G(u)/A_{G^\circ}(u) \cong Q/\{\pm 1\} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

There is a unique irreducible representation of $A_G(u)$ whose restriction to $A_{G^\circ}(u)$ contains $\epsilon$, and it has dimension 2.

The group $S_5$ acts on $\mathrm{GL}_2(\mathbb{C})^5$ by permutations. Let $P_\sigma \in \mathrm{GL}_{10}(\mathbb{C})$ be the matrix corresponding to a permutation $\sigma \in S_5$. Representatives for $G/G^\circ$ in $Z_G(u)$ are

$$(27) \qquad \left\{ 1, \left( \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right) P_{(23)(45)}, \left( \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix} \right) P_{(24)(35)}, \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) P_{(25)(34)} \right\}.$$

The elements (27) provide a splitting of (22), but (23) is not split in this case. Then $\natural_{\mathcal{E}}$ is the nontrivial cocycle of $G/G^\circ$ determined by the 2-dimensional projective representation with image $\left\{ 1, \left( \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right\}$.

The twisted group algebra $\overline{\mathbb{Q}_\ell}[G/G^\circ, \natural_{\mathcal{E}}]$ is isomorphic with $M_2(\overline{\mathbb{Q}_\ell})$. In particular it has precisely one irreducible representation. This agrees with the number of representations of $A_G(u)$ that we want to obtain. Notice that, without the twisting, $\overline{\mathbb{Q}_\ell}[G/G^\circ]$ would have four inequivalent irreducible representations, too many for this situation.

We return to our general setup. Let $G_{\mathcal{E}}$ be the subgroup of $G$ that stabilizes $\mathcal{E}$ (up to isomorphism). It contains $G^\circ$ and by Theorem 3.1.b it coincides with the stabilizer of the $Z(G^\circ)$-character of $\mathcal{E}$. By (21) there are group isomorphisms

$$(28) \qquad A_G(u)_\epsilon / A_{G^\circ}(u) \leftarrow Z_G(u)_\epsilon / Z_{G^\circ}(u) \rightarrow G_{\mathcal{E}}/G^\circ.$$

**Lemma 3.3.** *Let $(u, \epsilon) \in \mathcal{N}_{G^\circ}^0$. Then we can take $\kappa_\epsilon = \natural_{\mathcal{E}}^{-1}$ as 2-cocycles of $A_G(u)_\epsilon / A_{G^\circ}(u)$.*

*Proof.* With (28) we translate the lemma to a statement about cocycles of $Z_G(u)_\epsilon / Z_{G^\circ}(u)$. For $g \in Z_G(u)_\epsilon$ we have to find $I_\epsilon^g : V_\epsilon \rightarrow V_\epsilon$ such that

$$(29) \qquad I_\epsilon^g \circ \epsilon(h) \circ (I_\epsilon^g)^{-1} = \epsilon(ghg^{-1}) \qquad \forall h \in Z_{G^\circ}(u).$$

Since $Z_G(u) = s(G/G^\circ) Z_{G^\circ}(u)$, it suffices to find $I_\epsilon^{s(\gamma)}$ for $\gamma \in G_{\mathcal{E}}/G^\circ$. Namely, then we can put $I_\epsilon^{s(\gamma)h} = I_\epsilon^{s(\gamma)} \circ \epsilon(h)$ for $h \in Z_{G^\circ}(u)$, as in (6).

Let us consider $(\mathcal{C}_u^{G^\circ}, \mathcal{E})$ as a cuspidal local system for the simply connected cover $G_{\mathrm{sc}}$ of $G^\circ / Z(G^\circ)^\circ$. The action of $G$ on $G^\circ$ by conjugation lifts to an action on $G_{\mathrm{sc}}$ and $Z(G^\circ)^\circ$ acts trivially on $\epsilon$. Hence it suffices to construct $I_\epsilon^{s(\gamma)}$ for $\epsilon$ as a representation of $A_{G_{\mathrm{sc}}}(u)$.

Then $(A_{G_{\mathrm{sc}}}(u), \epsilon)$ decomposes as a direct product over almost simple factors of $G_{\mathrm{sc}}$. Factors with different cuspidal local systems have no interaction, so we may assume that $G_{\mathrm{sc}} = H^n, \epsilon = \sigma^{\otimes n}$ with $H$ simply connected and almost simple. The conjugation action of $G$ on $H^n$ is a combination of permutations of $\{1, 2, \ldots, n\}$ and automorphisms of $H$. If $g \in G$ permutes the factors of $H^n$ according to $\tau \in S_n$, then we can construct $I_{\sigma^{\otimes n}}^g$ as the permutation $\tau$ of $V_\sigma^{\otimes n}$, combined with some automorphisms of the vector space $V_\sigma$. In this way we reduce to the case where $G_{\mathrm{sc}}$ is almost simple.

Whenever $\epsilon$ is one-dimensional, we simply put

$$(30) \qquad I_\epsilon^\gamma = I_\epsilon^g = \mathrm{Id}_{V_\epsilon} \quad \text{for} \quad g = s(\gamma) \in s(G_{\mathcal{E}}/G^\circ).$$

To deal with the remaining cases, we recall from Theorem 3.1.c that in all those instances $G_{\mathrm{sc}} = \mathrm{Spin}_N(\mathbb{C})$ is a spin group and that the action of $Z(G_{\mathrm{sc}})$ on $\mathcal{E}$ does not factor through $Z(\mathrm{SO}_N(\mathbb{C}))$.

Suppose first that $N \geq 3$ is odd. Then $G_{\mathrm{sc}}$ is of type $B_{(N-1)/2}$ and all its automorphisms are inner. As explained after (24), we can take $s(G_{\mathcal{E}}/G^\circ)$ in $Z_{G_{\mathcal{E}}}(G^\circ)$. Thus (29) can be fulfilled by defining $I_\epsilon^g = \mathrm{Id}_{V_\epsilon}$.

Next we suppose that $N$ is even, so $G_{\mathrm{sc}}$ is of type $D_{N/2}$. By [Lus2, Proposition 14.6] $N = j(j+1)/2$ for some $j \geq 2$, and in particular $G_{\mathrm{sc}}$ is not isomorphic to the group $\mathrm{Spin}_8(\mathbb{C})$ of type $D_4$. Let us write $Z(G_{\mathrm{sc}}) = \{1, -1, \omega, -\omega\}$, where

$$\{1, -1\} = \ker\big(\mathrm{Spin}_N(\mathbb{C}) \to \mathrm{SO}_N(\mathbb{C})\big).$$

Our assumptions entail that $\epsilon(-1) \neq 1$. For both characters of $Z(G_{\mathrm{sc}})$ with $\epsilon(-1) = -1$ there is exactly one cuspidal pair $(\mathcal{C}_u^{G_{\mathrm{sc}}}, \mathcal{E})$ on which $Z(G_{\mathrm{sc}})$ acts in this way [Lus2, Proposition 14.6]. The group of outer automorphisms of $G_{\mathrm{sc}}$ has precisely two elements. It interchanges $\omega$ and $-\omega$, and hence it interchanges the two cuspidal pairs in question. Therefore the conjugation action of $G_{\mathcal{E}}$ on $G_{\mathrm{sc}}$ is by inner automorphisms of $G_{\mathrm{sc}}$. Now the same argument as in the $N$ odd case shows that we may take $I_\epsilon^g = \mathrm{Id}_{V_\epsilon}$.

Thus (30) works in all cases under consideration. The defining property of $s$ entails that

$$(31) \qquad I_\epsilon^\gamma \circ I_\epsilon^{\gamma'} = \natural_{\mathcal{E}}(\gamma, \gamma') I_\epsilon^{\gamma\gamma'}.$$

Together (28) this shows that the lemma holds when $G_{\mathrm{sc}}$ is almost simple. In view of our earlier reduction steps, that implies the general case. $\qquad \square$

Notice that $Y := \mathcal{C}_u^G Z(G^\circ)^\circ$ is a union of $G$-conjugacy classes in $G^\circ$. Tensoring $\mathcal{E}$ with the constant sheaf on $Z(G^\circ)^\circ$, we obtain a $G^\circ$-equivariant cuspidal local system on $Y$. We also denote that by $\mathcal{E}$.

Next we build a $G$-equivariant local system on $Y$ which contains every extension of $\epsilon$ to an irreducible representation of $A_G(u)$. The construction is the same as in [Lus2, §3.2], only for a disconnected group. Via the map

$$(32) \qquad Y \times G \to Y : (y, g) \mapsto g^{-1} y g$$

we pull $\mathcal{E}$ back to a local system $\hat{\mathcal{E}}$ on $Y \times G$. It is $G \times G^\circ$-equivariant for the action

$$(33) \qquad (h_1, h_0) \cdot (y, g) = (h_1 y h_1^{-1}, h_1 g h_0^{-1}).$$

The $G^\circ$ action is free, so we can divide it out and obtain a $G$-equivariant local system $\tilde{\mathcal{E}}$ on $Y \times G/G^\circ$ such that its pull back under the natural quotient map is isomorphic to $\hat{\mathcal{E}}$, see [BeLu, 2.6.3]. Let $\pi : Y \times G/G^\circ \to Y$ be the projection on the first coordinate. It is a $G$-equivariant fibration, so the direct image $\pi_* \tilde{\mathcal{E}}$ is a $G$-equivariant local system on $Y$. With (21) we see that its stalk at $y \in Y$ is isomorphic, as $Z_{G^\circ}(y)$-representation, to

$$\bigoplus_{g \in G/G^\circ} (\tilde{\mathcal{E}})_{y, gG^\circ} \cong \bigoplus_{g \in Z_G(y)/Z_{G^\circ}(y)} (\mathcal{E}_{g^{-1}yg}) \cong \bigoplus_{g \in Z_G(y)/Z_{G^\circ}(y)} g \cdot (\mathcal{E})_y.$$

The elements of $Z_G(y)$ permute these subspaces $\mathcal{E}_y$ in the expected way, so

$$(34) \qquad (\pi_* \tilde{\mathcal{E}})_y \cong \mathrm{ind}_{Z_{G^\circ}(y)}^{Z_G(y)} (\mathcal{E})_y \quad \text{as } Z_G(y)\text{-representations.}$$

In other words, we can consider $\pi_* \tilde{\mathcal{E}}$ as the induction of $(\mathcal{E})_Y$ from $G^\circ$ to $G$.

**Lemma 3.4.** *The $G$-endomorphism algebra of $\pi_* \tilde{\mathcal{E}}$ is isomorphic with $\overline{\mathbb{Q}_\ell}[G_{\mathcal{E}}/G^\circ, \natural_{\mathcal{E}}]$. Once $\natural_{\mathcal{E}}$ has been chosen, the isomorphism is canonical up to twisting by characters of $G_{\mathcal{E}}/G^\circ$.*

*Proof.* By [Lus2, Proposition 3.5], which applies also in the disconnected case, $\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})$ is canonically a direct sum of one-dimensional subspaces $\mathcal{A}_\gamma$ with $\gamma \in G_\mathcal{E}/G^\circ$. We need to specify one element in each of these subspaces to obtain a twisted group algebra. Recall the isomorphisms (28) and the map $s$ from (24). For $g = s(\gamma) \in s(G_\mathcal{E}/G^\circ)$ we define

$$I_\epsilon^\gamma = I_\epsilon^g : (\mathcal{E})_u \to (\mathcal{E})_u$$

as in the proof of Lemma 3.3. We already saw in (31) that the $I_\epsilon^\gamma$ span an algebra isomorphic to $\overline{\mathbb{Q}_\ell}[G_\mathcal{E}/G^\circ, \natural_\mathcal{E}]$. Each $I_\epsilon^\gamma$ extends uniquely to an isomorphism of $G$-equivariant local systems

(35)  $$I_\mathcal{E}^\gamma : (\mathcal{E})_Y \to \mathrm{Ad}(\gamma)^*(\mathcal{E})_Y.$$

We can consider this as a family of $\overline{\mathbb{Q}_\ell}$-linear maps

$$I_\mathcal{E}^\gamma : (\hat{\mathcal{E}})_{(y,g)} = (\mathcal{E})_{g^{-1}yg} \to (\hat{\mathcal{E}})_{(y,g\gamma^{-1})} = (\mathcal{E})_{\gamma g^{-1} yg\gamma^{-1}}.$$

Consequently the $I_\mathcal{E}^\gamma$ induce automorphisms of $\hat{\mathcal{E}}$, of $\tilde{\mathcal{E}}$ and of $\pi_*\tilde{\mathcal{E}}$. The latter automorphism belongs to $\mathcal{A}_\gamma$ and we take it as element of the required basis of $\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})$.

Any other choice of an isomorphism as in the lemma would differ from the first one by an automorphism of $\overline{\mathbb{Q}_\ell}[G_\mathcal{E}/G^\circ, \natural_\mathcal{E}]$ which stabilizes each of the subspaces $\overline{\mathbb{Q}_\ell}T_\gamma$. Every such automorphism is induced by a character of $G_\mathcal{E}/G^\circ$.  □

We note that the isomorphism in Lemma 3.4 is in general not canonical, because $s$ and the constructions in the proof of Lemma 3.3 are not. In the final result of this section, we complete the classification of unipotent cuspidal local systems on $G$.

**Proposition 3.5.** *There exists a canonical bijection*

$$\begin{array}{ccc}
\mathrm{Irr}(\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})) & \longleftrightarrow & \{\mathcal{F} : (\mathcal{C}_u^G, \mathcal{F}) \in \mathcal{N}_G^0, \ \mathrm{Res}_{G^\circ}^G\mathcal{F} \ contains \ \mathcal{E}\} \\
\rho & \mapsto & \mathrm{Hom}_{\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})}(\rho, \pi_*\tilde{\mathcal{E}}) \\
\mathrm{Hom}_G(\mathcal{F}, \pi_*\tilde{\mathcal{E}}) & \hookleftarrow & \mathcal{F}
\end{array}$$

*Upon choosing an isomorphism as in Lemma 3.4, we obtain a bijection*

$$\mathrm{Irr}(\overline{\mathbb{Q}_\ell}[G_\mathcal{E}/G^\circ, \natural_\mathcal{E}]) \longleftrightarrow \{(u, \eta) \in \mathcal{N}_G^0 : \mathrm{Res}_{A_{G^\circ}(u)}^{A_G(u)}\eta \ contains \ (\mathcal{E})_u\}.$$

*Proof.* The first map is canonical because its definition does not involve any arbitrary choices. To show that it is a bijection, we fix an isomorphism as in Lemma 3.4. By $G$-equivariance, it suffices to consider the claims at the stalk over $u$. Then we must look for irreducible $A_G(u)$-representations that contain $\epsilon$. By (28), (34) and Proposition 1.1.b

$$(\pi_*\tilde{\mathcal{E}})_u \cong \mathrm{ind}_{A_{G^\circ}(u)}^{A_G(u)}(\mathcal{E})_u \cong \mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\overline{\mathbb{Q}_\ell}[A_G(u)_\epsilon/A_{G^\circ}(u), \kappa_\epsilon] \otimes \epsilon).$$

By Lemma 3.3 the right hand side is

(36)  $$\mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\overline{\mathbb{Q}_\ell}[A_G(u)_\epsilon/A_{G^\circ}(u), \natural_\mathcal{E}^{-1}] \otimes \epsilon).$$

By Frobenius reciprocity and the definition of $A_G(u)_\epsilon$, the $A_G(u)$-endomorphism algebra of (36) is

(37)  $$\mathrm{End}_{A_G(u)_\epsilon}(\overline{\mathbb{Q}_\ell}[A_G(u)_\epsilon/A_{G^\circ}(u), \natural_\mathcal{E}^{-1}] \otimes \epsilon).$$

The description of the $A_G(u)_\epsilon$-action in Proposition 1.1.a shows that it is $\overline{\mathbb{Q}_\ell}[A_G(u)_\epsilon/A_{G^\circ}(u), \natural_\mathcal{E}^{-1}]^{op}$, acting by multiplication from the right. By (28) and

Lemma 1.3.a, (37) can be identified with $\overline{\mathbb{Q}}_\ell[G_\mathcal{E}/G^\circ, \natural_\mathcal{E}]$. Lemma 3.4 shows that this matches precisely with $\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})$. With (36) and Lemma 1.3.c it follows that

$$(38) \qquad \left(\pi_*\tilde{\mathcal{E}}\right)_u \cong \bigoplus\nolimits_{\rho \in \mathrm{Irr}(\mathrm{End}_G(\pi_*\tilde{\mathcal{E}}))} \rho \otimes \mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\rho^* \otimes \epsilon),$$

$$\mathrm{Hom}_{\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})}(\rho, (\pi_*\tilde{\mathcal{E}})_u) \cong \mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\rho^* \otimes \epsilon),$$

where $\rho^* \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[A_G(u)_\epsilon/A_{G^\circ}(u), \kappa_\epsilon])$ is the contragredient of $\rho$. By Lemma 1.3.b and Proposition 1.1.c every irreducible $A_G(u)$-representation containing $\epsilon$ is of the form $\mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\rho^* \otimes \epsilon)$, for a unique $\rho \in \mathrm{Irr}(\mathrm{End}_G(\pi_*\tilde{\mathcal{E}}))$. Hence the maps from left to right in the statement are bijective.

Let $(\mathcal{C}_u^G, \mathcal{F}) \in \mathcal{N}_G^0$ be such that $\mathrm{Res}_{G^\circ}^G \mathcal{F}$ contains $\mathcal{E}$. By what we have just shown, $\mathcal{F}_u \cong \mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\rho^* \otimes \epsilon)$, for a unique $\rho$. By (38)

$$\mathrm{Hom}_G(\mathcal{F}, \pi_*\tilde{\mathcal{E}}) = \mathrm{Hom}_{Z_G(u)}\left(\mathcal{F}_u, (\pi_*\tilde{\mathcal{E}})_u\right) = \mathrm{Hom}_{A_G(u)}\left(\mathrm{ind}_{A_G(u)_\epsilon}^{A_G(u)}(\rho^*\otimes\epsilon), (\pi_*\tilde{\mathcal{E}})_u\right) \cong \rho,$$

which provides the formula for the inverse of the above bijection. $\qquad\square$

## 4. DISCONNECTED GROUPS: THE NON-CUSPIDAL CASE

We would like to extend the generalized Springer correspondence for $G^\circ$ to $G$. First define the source and target properly.

**Definition 4.1.** For $\mathcal{N}_G^+$ we use exactly the same definition as in the connected case:

$$\mathcal{N}_G^+ = \{(u, \eta) : u \in G \text{ unipotent}, \eta \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(A_G(u))\}/G\text{-conjugacy}.$$

As $\mathcal{S}_G$ we take the same set as for $G^\circ$, but now considered up to $G$-conjugacy:

$$\mathcal{S}_G = \{\text{unipotent cuspidal supports for } G^\circ\}/G\text{-conjugacy}.$$

For $\mathfrak{t} = [L, \mathcal{C}_v^L, \mathcal{E}]_G \in \mathcal{S}_G$, let $N_G(\mathfrak{t})$ be the stabilizer of $(L, \mathcal{C}_v^L, \mathcal{E})$ in $G$. We define $W_\mathfrak{t}$ as the component group of $N_G(\mathfrak{t})$.

In the above notations, the group $L$ stabilizes $(L, \mathcal{C}_v^L, \mathcal{E})$ and any element of $G$ which stabilizes $(L, \mathcal{C}_v^L, \mathcal{E})$ must normalize $L$. Hence $L$ is the neutral component of $N_G(\mathfrak{t})$ and $W_\mathfrak{t} = N_G(\mathfrak{t})/L$ is a subgroup of $W(G, L) = N_G(L)/L$.

As in the connected case, $\mathcal{N}_G^+$ is canonically in bijection with the set of pairs $(\mathcal{C}_u^G, \mathcal{F})$, where $\mathcal{C}_u^G$ is the $G$-conjugacy class of a unipotent element $u$ and $\mathcal{F}$ is an irreducible $G$-equivariant local system on $\mathcal{C}_u^G$.

We define a map $\Psi_G : \mathcal{N}_G^+ \to \mathcal{S}_G$ in the following way. Let $(u, \eta) \in \mathcal{N}_G^+$. With Theorem 1.2 we can write $\eta = \eta^\circ \rtimes \tau$ with $\eta^\circ \in \mathrm{Irr}(A_{G^\circ}(u))$. Moreover $\eta^\circ$ is uniquely determined by $\eta$ up to $A_G(u)$-conjugacy. Then $(u, \eta^\circ) \in \mathcal{N}_{G^\circ}^+$. Using (19) we put

$$(39) \qquad\qquad \Psi_G(u, \eta) := \Psi_{G^\circ}(u, \eta^\circ)/G\text{-conjugacy}$$

By the $G$-equivariance of $\Psi_{G^\circ}$ (Theorem 2.3), $\Psi_G(u, \eta)$ does not depend on the choice of $\eta^\circ$. Write

$$\mathfrak{t}^\circ = [L, \mathcal{C}_v^L, \mathcal{E}]_{G^\circ}$$

and consider $\Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ) \in \mathrm{Irr}(W_{\mathfrak{t}^\circ})$. Just as in (39), $(L, \mathcal{C}_v^L, \mathcal{E}, \Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ))$ is uniquely determined by $(u, \eta)$, up to $G$-conjugacy.

We would like to define $\Sigma_\mathfrak{t}$ such that $\Sigma_\mathfrak{t}(u, \eta)$ is a representation of $W_\mathfrak{t}$ whose restriction to $W_{\mathfrak{t}^\circ}$ contains $\Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ)$. However, in general this does not work. It

turns out that we have to twist the group algebra $\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}}]$ with a certain 2-cocycle, which is trivial on $W_{\mathfrak{t}^\circ}$. In fact we have already seen this in Section 3. Over there $L = G^\circ$, $W_{\mathfrak{t}^\circ} = 1$, $W_{\mathfrak{t}} = G_{\mathcal{E}}/G^\circ$ and in Example 3.2 the group algebra of $W_{\mathfrak{t}}$ had to be twisted by a nontrivial 2-cocycle.

This twisting by nontrivial cocycles is only caused by the relation between irreducible representations of $A_{G^\circ}(u)$ and $A_G(u)$. The next two results show that the group $W_{\mathfrak{t}}$, considered on its own, would not need such twisting.

**Lemma 4.2.** *There exists a subgroup $\mathfrak{R}_{\mathfrak{t}} \subset W_{\mathfrak{t}}$ such that $W_{\mathfrak{t}} = \mathfrak{R}_{\mathfrak{t}} \ltimes W_{\mathfrak{t}^\circ}$.*

*Proof.* Thanks to Theorem 2.2, we know that $W_{\mathfrak{t}^\circ}$ equals $W(G^\circ, L) = N_{G^\circ}(L)/L$. On the other hand, $W_{\mathfrak{t}} \subset W(G, L)$ acts on the root system $R(G^\circ, Z(L)^\circ)$. Fix a positive subsystem and let $\mathfrak{R}_{\mathfrak{t}}$ be its stabilizer in $W_{\mathfrak{t}}$. Since $W(G^\circ, L)$ is the Weyl group of the root system $R(G^\circ, Z(L)^\circ)$ (see Theorem 2.2), it acts simply transitively on the collection of positive systems in $R(G^\circ, Z(L)^\circ)$. As $W(G^\circ, L)$ is normal in $W(G, L)$, we obtain the decomposition of $W_{\mathfrak{t}}$ as a semidirect product. $\square$

**Proposition 4.3.** *Let $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(W_{\mathfrak{t}^\circ})$. The cohomology class of $\kappa_\pi$ in $H^2(W_{\mathfrak{t},\pi}/W_{\mathfrak{t}^\circ}, \overline{\mathbb{Q}_\ell}^\times)$ is trivial.*

*Proof.* This is the statement of [ABPS4, Proposition 4.3], which is applicable by Lemma 4.2. $\square$

Let $N_G^+(\mathfrak{t})$ be the inverse image of $\mathfrak{R}_{\mathfrak{t}}$ in $N_G(\mathfrak{t}) \subset N_G(L)$. Then $L = N_G^+(\mathfrak{t})^\circ$ and $\mathfrak{R}_{\mathfrak{t}} \cong N_G^+(\mathfrak{t})/N_G^+(\mathfrak{t})^\circ$. Thus $(\mathcal{C}_v^L, \mathcal{E})$ can be considered as a cuspidal pair for $N_G^+(\mathfrak{t})^\circ$. In (31) we constructed a 2-cocycle $\natural_{\mathcal{E}} : \mathfrak{R}_{\mathfrak{t}} \times \mathfrak{R}_{\mathfrak{t}} \to \overline{\mathbb{Q}_\ell}^\times$. With Lemma 4.2 we can also consider it as a 2-cocycle of $W_{\mathfrak{t}}$, trivial on $W_{\mathfrak{t}^\circ}$:

$$(40) \qquad \natural_{\mathcal{E}} : W_{\mathfrak{t}}/W_{\mathfrak{t}^\circ} \times W_{\mathfrak{t}}/W_{\mathfrak{t}^\circ} \to \overline{\mathbb{Q}_\ell}^\times.$$

**Lemma 4.4.** *Let $\mathcal{F}^\circ$ be the $G^\circ$-equivariant local system on $\mathcal{C}_u^{G^\circ}$ corresponding to $\eta^\circ \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_{G^\circ}(u))$. There are natural isomorphisms*

$$W_{\mathfrak{t},\Sigma_{\mathfrak{t}^\circ}(u,\eta^\circ)}/W(G^\circ, L) \to G_{(\mathcal{C}_u^{G^\circ}, \mathcal{F}^\circ)}/G^\circ \leftarrow Z_G(u)_{\eta^\circ}/Z_{G^\circ}(u) \to A_G(u)_{\eta^\circ}/A_{G^\circ}(u).$$

*Proof.* There is a natural injection

$$(41) \qquad W_{\mathfrak{t}}/W(G^\circ, L) \cong N_G(L, \mathcal{C}_v^L, \mathcal{E})/N_{G^\circ}(L) \to G/G^\circ.$$

By Theorem 2.3 an element of $W_{\mathfrak{t}}/W(G^\circ, L)$ stabilizes $\Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ) = \Sigma_{\mathfrak{t}^\circ}(\mathcal{C}_u^{G^\circ}, \mathcal{F}^\circ)$ if and only if its image in $G/G^\circ$ stabilizes $(\mathcal{C}_u^{G^\circ}, \mathcal{F}^\circ)$. The second isomorphism is a direct consequence of the relation between $\mathcal{F}^\circ$ and $\eta^\circ$. $\square$

With this lemma we transfer (40) to a 2-cocycle

$$(42) \qquad \natural_{\mathcal{E}} : A_G(u)_{\eta^\circ}/A_{G^\circ}(u) \times A_G(u)_{\eta^\circ}/A_{G^\circ}(u) \to \overline{\mathbb{Q}_\ell}^\times.$$

Our construction of $\Sigma_{\mathfrak{t}}$ will generalize that of $\Sigma_{\mathfrak{t}^\circ}$ in [Lus2], in particular we use similar equivariant local systems. Recall that $(L, \mathcal{C}_v^L, \mathcal{E})$ is a cuspidal support. As in [Lus2, §3.1] we put $S = \mathcal{C}_v^L Z(L)^\circ$ and we extend $\mathcal{E}$ to a local system on $S$. We say that an element $y \in S$ is regular if $Z_G(y_s)^\circ$, the connected centralizer of the semisimple part of $y$, is contained in $L$. Consider the variety $Y = Y_{(L,S)}$ which is the union of all conjugacy classes in $G$ that meet the set of regular elements $S_{\mathrm{reg}}$. We build equivariant local systems $\hat{\mathcal{E}}$ on

$$\hat{Y} := \{(y, g) \in Y \times G : g^{-1} y g \in S_{\mathrm{reg}}\}$$

and $\tilde{\mathcal{E}}$ on $\tilde{Y} := \hat{Y}/L$ as in (32) and (33), only with $L$ instead of $G^\circ$. The projection map

$$\pi : \tilde{Y} \to Y, \ (y,g) \mapsto y$$

is a fibration with fibre $N_G(L)/L$, so

(43) $\qquad \pi : \tilde{Y} \to Y$ gives a $G$-equivariant local system $\pi_* \tilde{\mathcal{E}}$ on $Y$.

By Theorem 3.1.a $N_G(L)$ stabilizes $\mathcal{C}_v^L$, so $N_G(L)/L \cong Z_{N_G(L)}(v)/Z_L(v)$. The stalk of $\pi_* \tilde{\mathcal{E}}$ at $y \in S_{\text{reg}}$ is isomorphic, as representation of $Z_L(y) = Z_L(v)$, to

(44) $\quad (\pi_* \tilde{\mathcal{E}})_y \cong \bigoplus_{g \in N_G(L)/L} (\tilde{\mathcal{E}})_{y,gL} \cong \bigoplus_{g \in Z_{N_G(L)}(v)/Z_L(v)} \mathcal{E}_{g^{-1}yg} \cong \bigoplus_{g \in Z_{N_G(L)}(v)/Z_L(v)} g \cdot \mathcal{E}_y.$

On the part $Y^\circ$ of $Y$ that is $G^\circ$-conjugate to $S_{\text{reg}}$, $\pi_* \tilde{\mathcal{E}}$ can also be considered as a $G^\circ$-equivariant local system. As such $(\pi_* \tilde{\mathcal{E}})_{Y^\circ}$ contains the analogous local system $\pi_* \tilde{\mathcal{E}}^\circ$ for $G^\circ$ as a direct summand.

The following result generalizes Lemma 3.4.

**Proposition 4.5.** *The $G$-endomorphism algebra of $\pi_* \tilde{\mathcal{E}}$ is isomorphic with $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t}}, \natural_{\mathcal{E}}]$. Once $\natural_{\mathcal{E}}$ has been chosen via (25), the isomorphism is canonical up to twisting by characters of $W_{\mathfrak{t}}/W_{\mathfrak{t}^\circ}$.*

*Proof.* First we note that the results and proofs of [Lus2, §3] are also valid for the disconnected group $G$. By [Lus2, Proposition 3.5] $\text{End}_{\overline{\mathbb{Q}}_\ell}(\pi_* \tilde{\mathcal{E}}) = \text{End}_G(\pi_* \tilde{\mathcal{E}})$, and according to [Lus2, Remark 3.6] it is a twisted group algebra of $W_{\mathfrak{t}}$. It remains to determine the 2-cocycle. Again by [Lus2, Proposition 3.5], $\text{End}_G(\pi_* \tilde{\mathcal{E}})$ is naturally a direct sum of one-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces $\mathcal{A}_{\mathcal{E},w}$ ($w \in W_{\mathfrak{t}}$). An element of $\mathcal{A}_{\mathcal{E},w}$ consists of a system of $\overline{\mathbb{Q}}_\ell$-linear maps

(45) $\qquad \tilde{\mathcal{E}}_{y,g} = \mathcal{E}_{g^{-1}yg} \to \tilde{\mathcal{E}}_{y,gw^{-1}} = \mathcal{E}_{wg^{-1}ygw^{-1}}$

and is determined by a single $L$-intertwining map $\mathcal{E} \to \text{Ad}(w)^* \mathcal{E}$.

For $w \in W_{\mathfrak{t}^\circ}$ any element $b_w \in \mathcal{A}_{\mathcal{E},w}$ also acts on $\pi_* \tilde{\mathcal{E}}^\circ$. In [Lus2, Theorem 9.2.d] a canonical isomorphism

(46) $\qquad\qquad\qquad \text{End}_{G^\circ}(\pi_* \tilde{\mathcal{E}}^\circ) \cong \overline{\mathbb{Q}}_\ell[W_{\mathfrak{t}^\circ}],$

was constructed. Via this isomorphism we pick the $b_w$ ($w \in W_{\mathfrak{t}^\circ}$), then

(47) $\qquad w \mapsto b_w$ is a group homomorphism $W_{\mathfrak{t}^\circ} \to \text{Aut}_G(\pi_* \tilde{\mathcal{E}})$.

In view of Lemma 4.2, we still to have find suitable $b_\gamma \in \mathcal{A}_{\mathcal{E},\gamma}$ for $\gamma \in \mathfrak{R}_{\mathfrak{t}}$. Let $n_\gamma \in N_G^+(\mathfrak{t})$ be a lift of $\gamma \in N_G^+(\mathfrak{t})/L$. By [Lus2, §3.4-3.5] the choice of $b_\gamma$ is equivalent to the choice of an automorphism $I_{\mathcal{E}}^\gamma$ of $(\mathcal{E})_S$ that lifts the map

$$S \to S : g \mapsto n_\gamma g n_\gamma^{-1}.$$

Precisely such an automorphism was constructed (with the group $N_G^+(\mathfrak{t})$ in the role of $G$) in (35). We pick the unique $b_\gamma \in \mathcal{A}_{\mathcal{E},\gamma}$ corresponding to this $I_{\mathcal{E}}^\gamma$. Then the multiplication rules for the $b_\gamma$ are analogous to those for the $I_{\mathcal{E}}^\gamma$, so by Lemma 3.4 we get

(48) $\qquad\qquad b_\gamma \cdot b_{\gamma'} = \natural_{\mathcal{E}}(\gamma, \gamma') b_{\gamma\gamma'} \qquad \gamma, \gamma' \in \mathfrak{R}_{\mathfrak{t}}.$

Using Lemma 4.2 we define $b_{\gamma w} = b_\gamma b_w$ for $\gamma \in \mathfrak{R}_{\mathfrak{t}}, w \in W_{\mathfrak{t}^\circ}$. Now (47) and (48) imply that $b_w \cdot b_{w'} = \natural_{\mathcal{E}}(w, w') b_{ww'}$ for all $w, w' \in W_{\mathfrak{t}}$.

The only noncanonical part in the construction of the above isomorphism is the choice of the $b_\gamma \in \mathcal{A}_{\mathcal{E},\gamma}$ with $\gamma \in \mathfrak{R}_\mathfrak{t}$. Any other choice would differ from the above by an automorphism of $\overline{\mathbb{Q}_\ell}[\mathfrak{R}_\mathfrak{t}, \natural_\mathcal{E}]$ which stabilizes each of the one-dimensional subspaces $\mathcal{A}_{\mathcal{E},w}$. Every such automorphism is induced by a character of $\mathfrak{R}_\mathfrak{t} \cong W_\mathfrak{t}/W_{\mathfrak{t}^\circ}$. $\qquad\square$

Let $(u, \eta^\circ) \in \mathcal{N}_{G^\circ}^+$. Recall the cocycle $\kappa_{\eta^\circ}$ of $A_G(u)_{\eta^\circ}/A_{G^\circ}(u)$ constructed from $\eta^\circ \in \mathrm{Irr}(A_{G^\circ}(u))$ in (5). Like $\natural_\mathcal{E}$ it depends on some choices, but its cohomology class does not.

**Lemma 4.6.** *We can choose $\kappa_{\eta^\circ}$ equal to $\natural_\mathcal{E}^{-1}$ from (42).*

*Proof.* Let $s : G_{\mathcal{C}_u^{G^\circ}}/G^\circ \to Z_G(u)$ be as in (24). As a $G^\circ$-equivariant local system on $Y^\circ$,

$$(\pi_* \tilde{\mathcal{E}})_{Y^\circ} = \bigoplus\nolimits_{\gamma \in s(G_{\mathcal{C}_u^{G^\circ}}/G^\circ)} \mathrm{Ad}(\gamma)^*(\pi_* \tilde{\mathcal{E}}^\circ).$$

Every summand is of the same type as $\pi_* \tilde{\mathcal{E}}^\circ$, so we can apply all the constructions of [Lus2] to $\pi_* \tilde{\mathcal{E}}$. In particular we can build

$$(49) \quad \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})\big)|\mathcal{C}_u^{G^\circ} \cong \bigoplus\nolimits_{\gamma \in s(G_{\mathcal{C}_u^{G^\circ}}/G^\circ)} \mathrm{Ad}(\gamma)^* \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y^\circ}, \pi_* \tilde{\mathcal{E}}^\circ)\big)|\mathcal{C}_u^{G^\circ},$$

see [Lus2, Theorem 6.5]. Write

$$\rho^\circ = \Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ) \in \mathrm{Irr}(W_{\mathfrak{t}^\circ}).$$

Let $d_C = d_{\mathcal{C}_u^{G^\circ}, \mathcal{C}_v^L}$ be as in Theorem 2.1. Then $A_{G^\circ}(u)$ acts on

$$\mathrm{Ad}(\gamma)^* V_{\eta^\circ} = \mathrm{Ad}(\gamma)^* \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y^\circ}, \pi_* \tilde{\mathcal{E}}^\circ)_{\rho^\circ}\big)|\mathcal{C}_u^{G^\circ}$$

as $\gamma \cdot \eta^\circ$. Let $r(\gamma) \in \mathfrak{R}_\mathfrak{t} \cong W_\mathfrak{t}/W_{\mathfrak{t}^\circ}$ correspond to $\gamma G^\circ \in G/G^\circ$ under Lemma 4.4. By construction $b_{r(\gamma)} \in \mathrm{End}_G(\pi_* \tilde{\mathcal{E}})$ maps the $G^\circ$-local system $\mathrm{Ad}(\gamma)^*(\pi_* \tilde{\mathcal{E}}^\circ)$ to $\pi_* \tilde{\mathcal{E}}^\circ$. Suppose that $\gamma$ stabilizes $\eta^\circ$. For $I_{\eta^\circ}^\gamma$ we take the map

$$\mathrm{Ad}(\gamma)^* \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y^\circ}, \pi_* \tilde{\mathcal{E}}^\circ)_{\rho^\circ}\big)|\mathcal{C}_u^{G^\circ} \to \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y^\circ}, \pi_* \tilde{\mathcal{E}}^\circ)_{\rho^\circ}\big)|\mathcal{C}_u^{G^\circ}$$

induced by $b_{r(\gamma)}$. It commutes with the action of $Z_G(u)$, so it can be regarded as an element of $\mathrm{Hom}_{A_{G^\circ}(u)}(\gamma \cdot \eta^\circ, \eta^\circ)$. Then

$$\kappa_{\eta^\circ}^{-1}(\gamma, \gamma') = I_{\eta^\circ}^\gamma \circ I_{\eta^\circ}^{\gamma'} \circ (I_{\eta^\circ}^{\gamma\gamma'})^{-1} = b_{r(\gamma)} b_{r(\gamma')} b_{r(\gamma\gamma')}^{-1} = \natural_\mathcal{E}(r(\gamma), r(\gamma')) = \natural_\mathcal{E}(\gamma, \gamma'),$$

where we used (48) for the third equality. $\qquad\square$

Now we can state the main result of the first part of the paper.

**Theorem 4.7.** *Let $\mathfrak{t} = [L, \mathcal{C}_v^L, \mathcal{E}]_G \in \mathcal{S}_G$. There exists a canonical bijection*

$$\Sigma_\mathfrak{t} : \quad \Psi_G^{-1}(\mathfrak{t}) \quad \to \qquad \mathrm{Irr}(\mathrm{End}_G(\pi_* \tilde{\mathcal{E}}))$$
$$(\mathcal{C}_u^G, \mathcal{F}) \quad \mapsto \quad \mathrm{Hom}_G\Big(\mathcal{F}, \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})\big)|\mathcal{C}_u^G\Big).$$

*Suppose that $\rho \in \mathrm{Irr}\big(\mathrm{End}_G(\pi_* \tilde{\mathcal{E}})\big)$ contains $\rho^\circ \in \mathrm{Irr}\big(\mathrm{End}_{G^\circ}(\pi_* \tilde{\mathcal{E}}^\circ)\big)$ and that the unipotent conjugacy class of $\Sigma_{\mathfrak{t}^\circ}^{-1}(\rho^\circ)$ is represented by $u \in G^\circ$. Then*

$$\Sigma_\mathfrak{t}^{-1}(\rho) = \Big(\mathcal{C}_u^G, \mathcal{H}^{2d_C}\big(\mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})_\rho\big)|\mathcal{C}_u^G\Big),$$

*where $d_C$ is as in Theorem 2.1.*

*Upon choosing an isomorphism as in Proposition 4.5, we obtain a bijection*

$$\Psi_G^{-1}(\mathfrak{t}) \to \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}, \natural_\mathcal{E}]).$$

*Proof.* First we show that there exists a bijection $\Sigma_\mathfrak{t}$ between the indicated sets. To this end we may fix an isomorphism

(50) $$\mathrm{End}_G(\pi_* \tilde{\mathcal{E}}) \cong \overline{\mathbb{Q}_\ell}[W_\mathfrak{t}, \natural_\mathcal{E}]$$

as in Proposition 4.5. In particular it restricts to

$$\mathrm{End}_{G^\circ}(\pi_* \tilde{\mathcal{E}}^\circ) \cong \overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}^\circ}].$$

Let us compare $\Psi_G^{-1}(\mathfrak{t})$ with $\Psi_{G^\circ}^{-1}(\mathfrak{t}^\circ)$. For every $(u, \eta^\circ) \in \Psi_{G^\circ}^{-1}(\mathfrak{t}^\circ)$ we can produce an element of $\Psi_G^{-1}(\mathfrak{t})$ by extending $\eta^\circ$ to an irreducible representation $\eta$ of $A_G(u)$. By Lemma 4.6 and Proposition 1.1.c the only way to do so is taking $\eta$ of the form

(51) $$\eta^\circ \rtimes \tau' = \mathrm{ind}_{A_G(u)_{\eta^\circ}}^{A_G(u)}(\eta^\circ \otimes \tau') \text{ with } \tau' \in \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[A_G(u)_{\eta^\circ}/A_{G^\circ}(u), \natural_\mathcal{E}^{-1}]).$$

In view of Theorem 1.2 and Lemma 4.4 that yields a bijection

(52) $$\begin{array}{ccc} \left(\Psi_{G^\circ}^{-1}(\mathfrak{t}^\circ) // W_\mathfrak{t}/W_{\mathfrak{t}^\circ}\right)_{\natural_\mathcal{E}^{-1}} & \longleftrightarrow & \Psi_G^{-1}(\mathfrak{t}) \\ ((u, \eta^\circ), \tau') & \mapsto & (u, \eta^\circ \rtimes \tau'). \end{array}$$

By Lemma 1.3.b there is a bijection

(53) $$\begin{array}{ccc} \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}/W_{\mathfrak{t}^\circ}, \natural_\mathcal{E}]) & \longleftrightarrow & \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}/W_{\mathfrak{t}^\circ}, \natural_\mathcal{E}^{-1}]) \\ V & \mapsto & V^* = \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}}(V, \overline{\mathbb{Q}_\ell}). \end{array}$$

Recall from Proposition 4.3 that for any $\rho^\circ \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(W_{\mathfrak{t}^\circ})$ the cohomology class of $\kappa_{\rho^\circ}$ in $H^2(W_{\mathfrak{t}, \rho^\circ}/W_{\mathfrak{t}^\circ}, \overline{\mathbb{Q}_\ell}^\times)$ is trivial. With Theorem 1.2 we get a bijection

(54) $$\begin{array}{ccc} \left(\mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(W_{\mathfrak{t}^\circ}) // W_\mathfrak{t}/W_{\mathfrak{t}^\circ}\right)_{\natural_\mathcal{E}} & \longleftrightarrow & \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}, \natural_\mathcal{E}]) \\ (\rho^\circ, \tau) & \mapsto & \rho^\circ \rtimes \tau. \end{array}$$

From (52), (53) and (54) we obtain a bijection

(55) $$\begin{array}{ccc} \Psi_G^{-1}(\mathfrak{t}) & \longleftrightarrow & \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}, \natural_\mathcal{E}]) \\ (u, \eta^\circ \rtimes \tau') & \mapsto & \Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ) \rtimes \tau'^* \\ \Sigma_{\mathfrak{t}^\circ}^{-1}(\rho^\circ) \rtimes \tau^* & \longleftarrow & \rho^\circ \rtimes \tau \end{array}$$

Together with (50) we get a candidate for $\Sigma_\mathfrak{t}$, and we know that this candidate is bijective. To prove that it is canonical, it suffices to see that it satisfies the given formula for $\Sigma_\mathfrak{t}^{-1}(\rho)$. That formula involves a $G$-equivariant local system on $\mathcal{C}_u^G$. Since $\mathrm{End}_G(\pi_* \tilde{\mathcal{E}})$ is semisimple, we only have to determine its stalk at $u$, as a $A_G(u)$-representations. It follows from (49) that this stalk is

$$\mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})\right)_u \cong \mathrm{ind}_{A_{G^\circ}(u)}^{A_G(u)} \mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y^\circ}, \pi_* \tilde{\mathcal{E}}^\circ)\right)_u.$$

We abbreviate $\Sigma(u) = \{\rho^\circ = \Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ) : (u, \eta^\circ) \in \Psi_{G^\circ}^{-1}(\mathfrak{t}^\circ)\}$. Decomposing $\mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y^\circ}, \pi_* \tilde{\mathcal{E}}^\circ)\right)_u$ as a representation of $\overline{\mathbb{Q}_\ell}[A_{G^\circ}(u)] \times \mathrm{End}_{G^\circ}(\pi_* \tilde{\mathcal{E}}^\circ))$, like in [Lus2, §3.7], the right hand side becomes

(56) $$\mathrm{ind}_{A_{G^\circ}(u)}^{A_G(u)}\left(\bigoplus_{\rho^\circ \in \Sigma(u)} V_{\eta^\circ} \otimes V_{\rho^\circ}\right).$$

By Proposition 1.1 and Lemma 4.6 this is isomorphic to

$$(57) \qquad \bigoplus_{\rho^\circ \in \Sigma(u)} \mathrm{ind}_{A_G(u)_{\eta^\circ}}^{A_G(u)} \left( \overline{\mathbb{Q}}_\ell[A_G(u)_{\eta^\circ}/A_{G^\circ}(u), \natural_{\mathcal{E}}^{-1}] \otimes V_{\eta^\circ} \otimes V_{\rho^\circ} \right) := \bigoplus_{\rho^\circ \in \Sigma(u)} B_{\rho^\circ}.$$

(This equality defines $B_{\rho^\circ}$.) Let us analyse the action of $\mathrm{End}_G(\pi_* \tilde{\mathcal{E}})$ on (57). By (50) and Lemma 4.4 there is a subalgebra $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}, \natural_{\mathcal{E}}]$, which stabilizes $B_{\rho^\circ}$. Moreover, by Lemma 1.3.a

$$\overline{\mathbb{Q}}_\ell[A_G(u)_{\eta^\circ}/A_{G^\circ}(u), \natural_{\mathcal{E}}^{-1}] \cong \overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}/W_{\mathfrak{t}^\circ}, \natural_{\mathcal{E}}^{-1}] \cong \overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}/W_{\mathfrak{t}^\circ}, \natural_{\mathcal{E}}]^{op}.$$

By Lemma 1.3.c it decomposes as

$$(58) \quad \overline{\mathbb{Q}}_\ell[A_G(u)_{\eta^\circ}/A_{G^\circ}(u), \natural_{\mathcal{E}}^{-1}] \cong \bigoplus_{\tau' \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}/W_{\mathfrak{t}^\circ}, \natural_{\mathcal{E}}^{-1}])} V_{\tau'} \otimes V_{\tau'}^* \cong \bigoplus_{\tau \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}/W_{\mathfrak{t}^\circ}, \natural_{\mathcal{E}}])} V_\tau^* \otimes V_\tau.$$

Recall that $W_{\mathfrak{t}^\circ}$ is a normal subgroup of $W_\mathfrak{t}$ and that $W_\mathfrak{t}$ acts on $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(W_{\mathfrak{t}^\circ})$, as in (3). Then $W_\mathfrak{t}/W_{\mathfrak{t},\rho^\circ}$ is in bijection with the $W_\mathfrak{t}$-orbit of $\rho^\circ$, so

$$(59) \qquad \bigoplus_{\rho^i \in W_\mathfrak{t} \cdot \rho^\circ} B_{\rho^i} = \overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_{\mathcal{E}}] B_{\rho^\circ} \cong \mathrm{ind}_{\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}, \natural_{\mathcal{E}}]}^{\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_{\mathcal{E}}]} B_{\rho^\circ}.$$

It follows from (57), (58) and (59) that (56) is isomorphic to

$$(60) \qquad \bigoplus_{\rho^\circ \in \Sigma(u)/W_\mathfrak{t}} \mathrm{ind}_{\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}, \natural_{\mathcal{E}}]}^{\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_{\mathcal{E}}]} \mathrm{ind}_{A_G(u)_{\eta^\circ}}^{A_G(u)} \left( \bigoplus_{\tau \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}/W_{\mathfrak{t}^\circ}, \natural_{\mathcal{E}}])} V_\tau^* \otimes V_\tau \otimes V_{\eta^\circ} \otimes V_{\rho^\circ} \right) =$$

$$\bigoplus_{\rho^\circ \in \Sigma(u)/W_\mathfrak{t}} \bigoplus_{\tau \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}/W_{\mathfrak{t}^\circ}, \natural_{\mathcal{E}}])} \mathrm{ind}_{A_G(u)_{\eta^\circ}}^{A_G(u)} (V_\tau^* \otimes V_{\eta^\circ}) \otimes \mathrm{ind}_{\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}, \natural_{\mathcal{E}}]}^{\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_{\mathcal{E}}]} (V_\tau \otimes V_{\rho^\circ}).$$

Let $\rho = \rho^\circ \rtimes \tau = \mathrm{ind}_{\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}, \natural_{\mathcal{E}}]}^{\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_{\mathcal{E}}]} (V_\tau \otimes V_{\rho^\circ})$. By (60)

$$\mathcal{H}^{2d_C}\left( \mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})_\rho) \right)_u = \mathrm{Hom}_{\overline{\mathbb{Q}}_l[W_\mathfrak{t}, \natural_{\mathcal{E}}]}\left( \rho, \mathcal{H}^{2d_C}\left( \mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}}) \right)_u \right) \cong$$

$$\mathrm{ind}_{A_G(u)_{\eta^\circ}}^{A_G(u)} (V_\tau^* \otimes V_{\eta^\circ}) = \tau^* \ltimes \eta^\circ = \Sigma_{\mathfrak{t}^\circ}^{-1}(\rho^\circ) \rtimes \tau^*.$$

Hence the formula for $\Sigma_{\mathfrak{t}}^{-1}$ given in the theorem agrees with the bijection (55).

Let us also compare the given formula for $\Sigma_\mathfrak{t}$ with the above constructions. By Theorem 1.2 $\mathcal{F}_u \cong \eta^\circ \rtimes \tau'$ with $\eta^\circ \in \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(A_{G^\circ}(u))$. We rewrite

$$\mathrm{Hom}_G\left( \mathcal{F}, \mathcal{H}^{2d_C}\left( \mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})) | \mathcal{C}_u^G \right) = \mathrm{Hom}_{A_G(u)}\left( \eta^\circ \rtimes \tau', \mathcal{H}^{2d_C}\left( \mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})) \right)_u \right).$$

By (60) this equals

$$\mathrm{ind}_{\overline{\mathbb{Q}}_\ell[W_{\mathfrak{t},\rho^\circ}, \natural_{\mathcal{E}}]}^{\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_{\mathcal{E}}]} (V_{\tau'}^* \otimes V_{\rho^\circ}) = \tau'^* \ltimes \rho^\circ = \Sigma_{\mathfrak{t}^\circ}(u, \eta^\circ) \rtimes \tau'^*.$$

Hence $\Sigma_\mathfrak{t}$ as given agrees with (55).                                    □

The maps $\Psi_G$ and $\Sigma_\mathfrak{t}$ are compatible with restriction to Levi subgroups, in the following sense. Let $H \subset G$ be an algebraic subgroup such that $H \cap G^\circ$ is a Levi subgroup of $G^\circ$. Suppose that $u \in H^\circ$ is unipotent. By [Ree1, §3.2]

$$(61) \qquad Z_G(u)^\circ \cap H = Z_{G^\circ}(u)^\circ \cap H \quad \text{equals} \quad Z_{H^\circ}(u)^\circ = Z_H(u)^\circ.$$

Hence the natural map $A_H(u) \to A_G(u)$ is injective and we can regard $A_H(u)$ as a subgroup of $A_G(u)$. Let $\pi_*^H \widehat{\mathcal{E}_H}$ be the $H$-equivariant local system on $\mathcal{C}_u^H$ constructed

like $\pi_* \tilde{\mathcal{E}}$ but for the group $H$. By Proposition 4.5 $\mathrm{End}_H(\pi_*^H \widetilde{\mathcal{E}_H})$ is naturally a subalgebra of $\mathrm{End}_G(\pi_* \tilde{\mathcal{E}})$.

**Theorem 4.8.** *Let $\eta \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_G(u))$ and $\eta_H \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_H(u))$.*

(a) *If $\eta_H$ appears in $\mathrm{Res}_{A_H(u)}^{A_G(u)}(\eta)$, then $\Psi_G(u,\eta) = \Psi_H(u,\eta_H)/G$-conjugacy.*

(b) *Suppose that $\Psi_G(u,\eta) = \Psi_H(u,\eta_H)/G$-conjugacy. Then $\Sigma_{\Psi_H(u,\eta_H)}(u,\eta_H)$ is a constituent of $\mathrm{Res}_{\mathrm{End}_H(\pi_*^H \widetilde{\mathcal{E}_H})}^{\mathrm{End}_G(\pi_* \tilde{\mathcal{E}})} \Sigma_{\mathfrak{t}}(u,\eta)$ if and only if $\eta_H$ is a constituent of $\mathrm{Res}_{A_H(u)}^{A_G(u)}(\eta)$.*

**Remark.** In [Lus2, §8] both parts were proven in the connected case, for $G^\circ$ and $H^\circ$. As in this source, it is likely that in part (b) the multiplicity of $\eta_H$ in $\eta$ equals the multiplicity of $\Sigma_{\Psi_H(u,\eta_H)}(u,\eta_H)$ in $\Sigma_{\mathfrak{t}}(u,\eta)$. However, it seems difficult to prove that with the current techniques. We will return to this issue in [AMS].

*Proof.* (a) Let $\eta_H^\circ$ be an irreducible constituent of $\mathrm{Res}_{A_{H^\circ}(u)}^{A_H(u)}(\eta_H)$ and let $\eta^\circ$ be an irreducible constituent of $\mathrm{Res}_{A_{G^\circ}(u)}^{A_G(u)}(\eta)$ which contains $\eta_H^\circ$. By the definition (39) and by [Lus2, Theorem 8.3.a] there are equalities up to $G$-conjugacy:

$$\Psi_G(u,\eta) = \Psi_{G^\circ}(u,\eta) = \Psi_{H^\circ}(u,\eta_H^\circ) = \Psi_H(u,\eta_H).$$

(b) Write $\eta = \eta^\circ \rtimes \tau^*$ as in (51). Similarly, we can write any irreducible representation of $A_H(u)$ as $\eta_H = \eta_H^\circ \rtimes \tau_H^*$ with $\eta_H^\circ \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_{H^\circ}(u))$ and $\tau_H^* \in \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}_H,\eta_H^\circ}/W_{\mathfrak{t}_H}^\circ, \natural_{\mathcal{E}}^{-1}])$. As representations of $A_{G^\circ}(u)$:

$$(62) \qquad \eta = \mathrm{ind}_{A_G(u)_{\eta^\circ}}^{A_G(u)}(V_{\eta^\circ} \otimes V_\tau^*) \cong \bigoplus_{a \in A_G(u)/A_G(u)_{\eta^\circ}} (a \cdot \eta^\circ) \otimes (a \cdot V_\tau^*),$$

where $A_{G^\circ}(u)$ acts trivially on the parts $a \cdot V_\tau^*$. Using Proposition 1.1.d and (62) we compute

$$\mathrm{Hom}_{A_H(u)}(\eta_H, \eta) \cong \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[A_H(u)_{\eta_H^\circ}/A_{H^\circ}(u), \natural_{\mathcal{E}}^{-1}]}\left(\tau_H^*, \mathrm{Hom}_{A_{H^\circ}(u)}(\eta_H^\circ, \eta)\right) \cong$$

$$\bigoplus_{a \in A_G(u)/A_G(u)_{\eta^\circ}} \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[A_H(u)_{\eta_H^\circ}/A_{H^\circ}(u), \natural_{\mathcal{E}}^{-1}]}\left(\tau_H^*, \mathrm{Hom}_{A_{H^\circ}(u)}(\eta_H^\circ, a \cdot \eta^\circ) \otimes a \cdot V_\tau^*\right).$$

Here $\overline{\mathbb{Q}_\ell}[A_H(u)_{\eta_H^\circ}/A_{H^\circ}(u), \natural_{\mathcal{E}}^{-1}]$ does not act on $\mathrm{Hom}_{A_{H^\circ}(u)}(\eta_H^\circ, a \cdot \eta^\circ)$, so we can rearrange the last line as

$$(63) \qquad \bigoplus_{a \in A_G(u)/A_G(u)_{\eta^\circ}} \mathrm{Hom}_{A_{H^\circ}(u)}(\eta_H^\circ, a \cdot \eta^\circ) \otimes \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[A_H(u)_{\eta_H^\circ}/A_{H^\circ}(u), \natural_{\mathcal{E}}^{-1}]}(\tau_H^*, a \cdot V_\tau^*).$$

Notice that $\eta \cong a \cdot \eta \cong a \cdot \eta^\circ \rtimes a \cdot \tau^*$. We conclude from (63) that $\mathrm{Hom}_{A_H(u)}(\eta_H, \eta)$ is nonzero if and only if $\eta \cong \eta^\circ \rtimes \tau^*$ where $\mathrm{Hom}_{A_{H^\circ}(u)}(\eta_H^\circ, \eta^\circ) \neq 0$ and $\mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[A_H(u)_{\eta_H^\circ}/A_{H^\circ}(u), \natural_{\mathcal{E}}^{-1}]}(\tau_H^*, \tau^*) \neq 0$.

Write $\rho = \Sigma_{\mathfrak{t}}(u,\eta)$ and let $\rho_H = \rho_H^\circ \rtimes \tau_H \in \mathrm{Irr}(\mathrm{End}_H(\pi_*^H \widetilde{\mathcal{E}_H}))$. Just as in (63) one shows that $\mathrm{Hom}_{\mathrm{End}_H(\pi_*^H \widetilde{\mathcal{E}_H})}(\rho_H, \rho)$ is nonzero if and only if $\rho \cong \rho^\circ \rtimes \tau$ with $\mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}_H^\circ}]}(\rho_H^\circ, \rho^\circ) \neq 0$ and $\mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}_H,\rho_H^\circ}/W_{\mathfrak{t}_H^\circ}, \natural_{\mathcal{E}}]}(\tau_H, \tau) \neq 0$.

Write $\mathfrak{t}_H = \Psi_H(u,\eta_H)$, $\mathfrak{t}_H^\circ = \Psi_{H^\circ}(u,\eta_H^\circ)$ and consider $\rho_H^\circ = \Sigma_{\mathfrak{t}_H^\circ}(u,\eta_H^\circ)$. Then

$$\rho_H = \rho_H^\circ \rtimes \tau_H \quad \text{equals} \quad \Sigma_{\mathfrak{t}_H}(u,\eta_H) = \Sigma_{\mathfrak{t}_H}(u,\eta_H^\circ \rtimes \tau_H^*).$$

By [Lus2, Theorem 8.3.b]

$$\dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Hom}_{A_{H^\circ}(u)}(\eta_H^\circ, \eta^\circ) = \dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}_H^\circ}]}(\rho_H^\circ, \rho^\circ)$$

and from Lemmas 1.3 and 4.4 we see that

$$\dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[A_H(u)_{\eta_H^\circ}/A_{H^\circ}(u),\natural_{\mathcal{E}}^{-1}]}(\tau_H^*, \tau^*) = \dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}_H,\rho_H^\circ}/W_{\mathfrak{t}_H^\circ},\natural_{\mathcal{E}}]}(\tau_H, \tau).$$

The above observations entail that $\mathrm{Hom}_{A_H(u)}(\eta_H, \eta)$ is nonzero if and only if $\mathrm{Hom}_{\mathrm{End}_H(\pi_*^H \widetilde{\mathcal{E}_H})}(\rho_H, \rho)$ is nonzero.                                      $\square$

## 5. A VERSION WITH QUASI-LEVI SUBGROUPS

For applications to Langlands parameters we need a version of the generalized Springer correspondence which involves a disconnected version of Levi subgroups. Recall that every Levi subgroup $L$ of $G^\circ$ is of the form $L = Z_{G^\circ}(Z(L)^\circ)$.

**Definition 5.1.** Let $G$ be a possibly disconnected complex reductive algebraic group, and let $L \subset G^\circ$ be a Levi subgroup. Then we call $Z_G(Z(L)^\circ)$ a quasi-Levi subgroup of $G$.

Notice that $Z_G(Z(L)^\circ)$ also has neutral component $L$ and connected centre $Z(L)^\circ$. Hence there is canonical bijection between Levi subgroups and quasi-Levi subgroups of $G$. We will also need some variations on other previous notions.

**Definition 5.2.** A unipotent cuspidal quasi-support for $G$ is a triple $(M, v, q\epsilon)$ where $M \subset G$ is a quasi-Levi subgroup, $v \in M^\circ$ is unipotent and $q\epsilon \in \mathrm{Irr}_{\mathrm{cusp}}(A_M(v))$. We write

$$q\mathcal{S}_G = \{\text{cuspidal unipotent quasi-supports for } G\}/G\text{-conjugacy}.$$

Like before, we will also think of unipotent cuspidal quasi-supports as triples $(M, \mathcal{C}_v^M, q\mathcal{E})$, where $q\mathcal{E}$ is a cuspidal local system on $\mathcal{C}_v^M$. We want to define a canonical map

$$q\Psi_G : \mathcal{N}_G^+ \to q\mathcal{S}_G,$$

and to analyse its fibers. Of course this map should just be $\Psi_G$ if $G$ is connected.

Let $\mathfrak{t} = [M^\circ, \mathcal{C}_v^{M^\circ}, \mathcal{E}]_G$ and suppose that $(u, \eta) \in \mathcal{N}_G^+$ with $\Psi_G(u, \eta) = \mathfrak{t}$. Obviously, the cuspidal quasi-support of $(u, \eta)$ will involve the quasi-Levi subgroup $M = Z_G(Z(M^\circ)^\circ)$. From Theorem 4.7 we get

$$\rho = \Sigma_{\mathfrak{t}}(u, \eta) \in \mathrm{Irr}(\mathrm{End}_G(\pi_* \tilde{\mathcal{E}})).$$

Let $\pi_*^M \widetilde{\mathcal{E}_M}$ be the $M$-equivariant local system on $\mathcal{C}_v^M$ built from $\mathcal{E}$ in the same way as $\pi_* \tilde{\mathcal{E}}$, only with $M$ instead of $G$. From Proposition 4.5 we see that $\mathrm{End}_G(\pi_* \tilde{\mathcal{E}}) \cong \overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}}, \natural_{\mathcal{E}}]$ naturally contains a subalgebra

$$\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M}) \cong \overline{\mathbb{Q}_\ell}[M_{\mathcal{E}}/M^\circ, \natural_{\mathcal{E}}].$$

As $M_{\mathcal{E}}/M^\circ$ is normal in $W_{\mathfrak{t}} = N_G(\mathfrak{t})/M^\circ$, the latter group acts on $\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M})$ by conjugation in $\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}}, \natural_{\mathcal{E}}]$. Let $\rho_M \in \mathrm{Irr}(\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M}))$ be a constituent of $\rho$ as $\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M})$-representation. By the irreducibility of $\rho$ as $\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}}, \natural_{\mathcal{E}}]$-representation, $\rho_M$ is unique up to conjugation by $W_{\mathfrak{t}}$. Let us write $\mathfrak{t}_M = [M^\circ, \mathcal{C}_v^{M^\circ}, \mathcal{E}]_M \in \mathcal{N}_M^0$. By Proposition 3.5

$$(64) \qquad q\mathcal{E} := \mathrm{Hom}_{\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M})}(\rho_M, \pi_*^M \widetilde{\mathcal{E}_M}) = (\pi_*^M \tilde{\mathcal{E}})_{\rho_M}$$

is a cuspidal local system on $\mathcal{C}_v^M = \mathcal{C}_v^{M\circ}$, and $\Sigma_{\mathfrak{t}_M}(\mathcal{C}_v^M, q\mathcal{E}) = \rho_M$. Since any other choice $\rho_M'$ is conjugate to $\rho_M$ by an element of $N_G(\mathfrak{t})$, $(M, \mathcal{C}_v^M, q\mathcal{E})$ is determined by $(u, \eta)$, up to $G$-conjugacy. Thus we can canonically define

$$(65) \qquad q\Psi_G(u, \eta) = [M, \mathcal{C}_v^M, q\mathcal{E}]_G.$$

Let $\mathcal{F}$ be the irreducible local system on $\mathcal{C}_u^G$ with $\mathcal{F}_u = \eta$ and let $Y = Y_{(M\circ, S)} \subset G$, where $S = \mathcal{C}_v^M Z(M)^\circ$. From Theorem 4.7 we see that $\mathcal{F} = \mathcal{H}^{2d_C}(\mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})_\rho)|\mathcal{C}_u^G$. To $(M, \mathcal{C}_v^M, q\mathcal{E})$ we can apply the same constructions as to $(L, \mathcal{C}_v^L, \mathcal{E})$ in (43), in particular there is a version $q\pi_*$ of $\pi_*$ based on $G$ and $M$. Distinguishing the various operations $\pi_*$, we get usual isomorphisms of local systems on $Y$:

$$(66) \qquad q\pi_*(\widetilde{q\mathcal{E}}) = q\pi_*\left((\widetilde{\pi_*^M \mathcal{E}_M})_{\rho_M}\right) \cong (\pi_* \tilde{\mathcal{E}})_{\rho_M}.$$

Since $\pi_* \tilde{\mathcal{E}}$ is semisimple, $\mathrm{IC}(\overline{Y}, (\pi_* \tilde{\mathcal{E}})_{\rho_M}) \cong \mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})_{\rho_M}$. Hence $\mathcal{F}$ is also a direct summand of

$$(67) \qquad \mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y}, q\pi_*(\widetilde{q\mathcal{E}}))\right)|\mathcal{C}_u^G \cong \mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y}, \pi_* \tilde{\mathcal{E}})_{\rho_M}\right)|\mathcal{C}_u^G.$$

It follows that the characterization of $\Psi_G$ given in [Lus2, Theorem 6.5] remains valid for $q\Psi_G$. In particular $\mathcal{F}$ and $q\mathcal{E}$ have the same $Z(G)$-character and

$$(68) \qquad q\Psi_G \text{ preserves } Z(G)\text{-characters.}$$

We abbreviate

$$q\mathfrak{t} = [M, \mathcal{C}_v^M, q\mathcal{E}]_G.$$

Let $N_G(q\mathfrak{t})$ be the stabilizer of $(M, \mathcal{C}_v^M, q\mathcal{E})$ in $G$. It normalizes $M$ and contains $M$, because $(q\mathcal{E})_v \in \mathrm{Irr}(A_M(v))$. Every element of $N_G(q\mathfrak{t})$ maps $\mathcal{E}$ to a $M$-associate local system on $\mathcal{C}_v^{M\circ}$, because $q\mathcal{E}$ is a $M$-equivariant local system which, as $M^\circ$-equivariant sheaf, contains $\mathcal{E}$. Hence $N_G(q\mathfrak{t}) = N_G(\mathfrak{t}, q\mathcal{E})M$.

Analogous to $W_\mathfrak{t} = N_G(\mathfrak{t})/M^\circ$, we define

$$(69) \qquad W_{q\mathfrak{t}} = N_G(q\mathfrak{t})/M.$$

There are natural isomorphisms

$$(70) \qquad W_{q\mathfrak{t}} \cong N_G(\mathfrak{t}, q\mathcal{E})M/M \cong N_G(\mathfrak{t}, q\mathcal{E})/M_\mathcal{E} \cong \mathrm{Stab}_{W_\mathfrak{t}}(q\mathcal{E})/(M_\mathcal{E}/M^\circ).$$

The group $W_\mathfrak{t} = N_{G\circ}(M^\circ)/M^\circ$ is isomorphic to $N_{G\circ}(\mathfrak{t})M/M$, and there is a natural injection

$$(71) \qquad W_\mathfrak{t} \cong N_{G\circ}(\mathfrak{t})M/M \to W_{q\mathfrak{t}}.$$

**Lemma 5.3.** *There exists a 2-cocycle $\kappa_{q\mathfrak{t}}$ of $W_{q\mathfrak{t}}$ such that:*

*(a) there is a bijection*

$$q\Psi_G^{-1}(q\mathfrak{t}) \to \mathrm{Irr}\left(\overline{\mathbb{Q}}_\ell[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}]\right),$$

*(b) $\kappa_{q\mathfrak{t}}$ factors through $W_{q\mathfrak{t}}/W_\mathfrak{t}$ and $\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}]$ is canonically embedded in $\overline{\mathbb{Q}}_\ell[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}]$.*

*Proof.* (a) Recall the bijection $\Psi_G^{-1}(\mathfrak{t}) \to \mathrm{Irr}\left(\overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_\mathcal{E}]\right)$ from Theorem 4.7. With [CuRe, §53] we can find a central extension $\widetilde{W}_\mathfrak{t}$ of $W_\mathfrak{t}$ and a minimal idempotent $p_\mathcal{E} \in \overline{\mathbb{Q}}_\ell[\ker(\widetilde{W}_\mathfrak{t} \to W_\mathfrak{t})]$, such that

$$(72) \qquad \overline{\mathbb{Q}}_\ell[W_\mathfrak{t}, \natural_\mathcal{E}] \cong p_\mathcal{E} \overline{\mathbb{Q}}_\ell[\widetilde{W}_\mathfrak{t}].$$

Let $N \subset \widetilde{W_\mathfrak{t}}$ be the inverse image of $M_\mathcal{E}/M^\circ \subset W_\mathfrak{t}$. It is a normal subgroup of $\widetilde{W_\mathfrak{t}}$ because $M_\mathcal{E} = M \cap N_G(\mathfrak{t})$ is normal in $N_G(\mathfrak{t})$. We note that

(73) $$\widetilde{W_\mathfrak{t}}/N \cong W_\mathfrak{t}/(M_\mathcal{E}/M^\circ) \cong N_G(\mathfrak{t})/M_\mathcal{E}.$$

As a consequence of (72)

(74) $$\overline{\mathbb{Q}_\ell}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}] \cong p_\mathcal{E} \overline{\mathbb{Q}_\ell}[N].$$

By Theorem 1.2 there is a bijection

$$\begin{array}{ccc} \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(\widetilde{W_\mathfrak{t}}) & \longleftrightarrow & \big(\mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(N)//\widetilde{W_\mathfrak{t}}/N\big)_\kappa \\ \pi \rtimes \sigma & \longleftrightarrow & (\pi, \sigma). \end{array}$$

With Proposition 1.1.c we can restrict it to representations on which $p_\mathcal{E}$ acts as the identity. With (72) that yields a bijection

(75) $$\mathrm{Irr}\big(\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}, \natural_\mathcal{E}]\big) \longleftrightarrow \big(\mathrm{Irr}\big(\overline{\mathbb{Q}_\ell}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}]\big)//W_\mathfrak{t}/(M_\mathcal{E}/M^\circ)\big)_\kappa.$$

Under the bijections from Theorem 4.7 and (75), the set $q\Psi_G^{-1}(q\mathfrak{t}) \subset \Psi_G^{-1}(\mathfrak{t})$ is mapped to the fiber of $W_\mathfrak{t} \cdot \rho_M$ (with respect to the map from the extended quotient on the right hand side of (75) to the corresponding ordinary quotient). By the definition of extended quotients, this fiber is in bijection with $\mathrm{Irr}\big(\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t},\rho_M}/(M_\mathcal{E}/M^\circ), \kappa_{\rho_M}]\big)$. By the equivariance of the Springer correspondence, the stabilizer of $\Sigma_{\mathfrak{t}_M}(\mathcal{C}_v^M, q\mathcal{E}) = \rho_M$ in $W_\mathfrak{t}/(M_\mathcal{E}/M^\circ)$ is $\mathrm{Stab}_{W_\mathfrak{t}}(q\mathcal{E})/(M_\mathcal{E}/M^\circ)$, which by (70) is isomorphic with $W_{q\mathfrak{t}}$. Thus the composition of Theorem 4.7 and (75) provides the required bijection, with $\kappa_{\rho_M}$ as cocycle.

(b) Consider $w \in W_\mathfrak{t}$ with preimage $\tilde{w} \in \widetilde{W_\mathfrak{t}}$. Since $M_\mathcal{E}/M^\circ \cong M_\mathcal{E}G^\circ/G^\circ$, $w$ commutes with $M_\mathcal{E}/M^\circ$. As $\natural_\mathcal{E}$ is trivial on $W_\mathfrak{t}$, moreover for all $m \in M_\mathcal{E}/M^\circ$

(76) $$T_w T_m (T_w)^{-1} = T_m \qquad \text{in } \overline{\mathbb{Q}_\ell}[W_\mathfrak{t}, \natural_\mathcal{E}].$$

Hence $W_\mathfrak{t}$ stabilizes $\rho_M$ and

$$W_\mathfrak{t} \cong W_\mathfrak{t}(M_\mathcal{E}/M^\circ)/(M_\mathcal{E}/M^\circ) \quad \text{is contained in} \quad W_{\mathfrak{t},\rho_M}/(M_\mathcal{E}/M^\circ).$$

It also follows from (76) that we can take $I_{\rho_M}^w = \mathrm{Id}_{V_{\rho_M}}$. In view of Proposition 1.1.a, the 2-cocycle $\kappa_{\rho_M}$ on $W_{\mathfrak{t}^\circ}$ agrees with $\natural_\mathcal{E}|_{W_\mathfrak{t}} = 1$. Via (71) we consider $W_\mathfrak{t}$ as a subgroup of $W_{q\mathfrak{t}}$. Then the subalgebra of $\overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}]$ spanned by the $T_w$ with $w \in W_\mathfrak{t}$ is simply $\overline{\mathbb{Q}_\ell}[W_\mathfrak{t}]$. $\qquad \square$

We will make the bijection of Lemma 5.3.a canonical, by replacing $\overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}]$ with the endomorphism algebra of the equivariant local system (66).

**Lemma 5.4.** *Let $\kappa_{q\mathfrak{t}}$ be as in Lemma 5.3. There is an isomorphism*

$$\mathrm{End}_G\big(q\pi_*(\widetilde{q\mathcal{E}})\big) \cong \overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}],$$

*and it is canonical up to automorphisms of the right hand side which come from characters of $W_{q\mathfrak{t}}/W_{\mathfrak{t}^\circ}$. Under this isomorphism $\overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}^\circ}]$ to corresponds to $\mathrm{End}_{G^\circ}(\pi_* \widetilde{\mathcal{E}})$, which acts on $q\pi_*(\widetilde{q\mathcal{E}})$ via (66).*

*Proof.* Like Proposition 4.5, the larger part of this result follows from [Lus2, §3]. The constructions over there apply equally well to quasi-Levi subgroups of the possibly disconnected group $G$. These arguments show that, as a $\overline{\mathbb{Q}_\ell}$-vector space, $\mathrm{End}_G(q\pi_*(\widetilde{q\mathcal{E}}))$ is in a canonical way a direct sum of one-dimensional subspaces $q\mathcal{A}_w$ indexed by $W_{q\mathfrak{t}}$. Moreover, as an algebra it is a twisted group algebra of $W_{q\mathfrak{t}}$,

with respect to some 2-cocycle. To analyse the 2-cocycle, we relate it to objects appearing in the proof of Lemma 5.3.a.

By (66) $V_{\rho_M} \otimes q\pi_*(\widetilde{q\mathcal{E}})$, with $G$ acting trivially on $V_{\rho_M}$, is a direct summand of $\pi_*(\tilde{\mathcal{E}})$. By [Lus2, Proposition 3.5] the $G$-equivariant local system $\pi_*\tilde{\mathcal{E}}$ is semisimple. Therefore

$$(77) \qquad \mathrm{End}_G\big(V_{\rho_M} \otimes q\pi_*(\widetilde{q\mathcal{E}})\big) \cong \mathrm{End}_G\big(q\pi_*(\widetilde{q\mathcal{E}})\big) \otimes \mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V_{\rho_M})$$

is a subalgebra of $\mathrm{End}_G(\pi_*\tilde{\mathcal{E}})$. The basis elements $b_w \in \mathrm{End}_G(\pi_*\tilde{\mathcal{E}})$, $w \in W_{\mathfrak{t}}$, as constructed in Proposition 4.5, permute the subsystems of $\pi_*\tilde{\mathcal{E}}$ corresponding to different $\rho'_M \in \mathrm{Irr}\big(\mathrm{End}_M(\pi_*^M\widehat{\mathcal{E}_M})\big)$. More precisely, $b_w$ stabilizes $V_{\rho_M} \otimes q\pi_*(\widetilde{q\mathcal{E}})$ if and only if $w$ stabilizes $\rho_M$. Together with Proposition 4.5 this shows that (77) is spanned (over $\overline{\mathbb{Q}_\ell}$) by the $b_w$ with $w \in \mathrm{Stab}_{W_{\mathfrak{t}}}(\rho_M) = \mathrm{Stab}_{W_{\mathfrak{t}}}(q\mathcal{E})$.
In view of the description of $(\pi_*\tilde{\mathcal{E}})_y$ in (44), the stalk of $V_{\rho_M} \otimes q\pi_*(\widetilde{q\mathcal{E}})$ at $y \in S_{\mathrm{reg}}$ is

$$(78) \qquad \bigoplus_{z \in Z_{N_G(M^\circ)}(v)/Z_M(v)} z \cdot \big((\pi_*^M\widehat{\mathcal{E}_M})_{\rho_M}\big)_y \otimes V_{\rho_M}.$$

As concerns the index set for the sum, by Theorem 3.1.a the canonical map $Z_{N_G(M^\circ)}(v)/Z_M(v) \to N_G(M^\circ)/M$ is bijective.

The $b_w$ with $w \in M_{\mathcal{E}}/M^\circ$ act only on the second tensor factor of (78), and by the irreducibility of the $\mathrm{End}_M(\pi_*^M\widehat{\mathcal{E}_M})$-module $\rho_M$ they span the algebra $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V_{\rho_M})$. Let $[W_{q\mathfrak{t}}] \subset W_{\mathfrak{t}}$ be a set of representatives for $\mathrm{Stab}_{W_{\mathfrak{t}}}(q\mathcal{E})/(M_{\mathcal{E}}/M^\circ)$. By (71) we may assume that it contains $W_{\mathfrak{t}^\circ}$. From (45) we see that the $b_w$ with $w \in [W_{q\mathfrak{t}}]$ permute the direct summands of (78) according to the inclusion

$$W_{q\mathfrak{t}} = N_G(q\mathfrak{t})/M \to N_G(M^\circ)/M \cong Z_{N_G(M^\circ)}(v)/Z_M(v).$$

In particular $\{b_w : w \in [W_{q\mathfrak{t}}]\}$ is linearly independent over $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V_{\rho_M})$. Since (77) is spanned by the $b_w$ with $w \in \mathrm{Stab}_{W_{\mathfrak{t}}}(\rho_M)$, it follows that in fact $\{b_w : w \in [W_{q\mathfrak{t}}]\}$ is a basis of (77) over $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V_{\rho_M})$.

We want to modify these $b_w$ to endomorphisms of $q\pi_*(\widetilde{q\mathcal{E}})$, say to $qb_w \in q\mathcal{A}_w$. For $w \in W_{\mathfrak{t}^\circ}$ there is an easy canonical choice, as (76) shows that $W_{\mathfrak{t}^\circ}$ commutes with $\overline{\mathbb{Q}_\ell}[M/M^\circ, \natural_{\mathcal{E}}]$. Hence $b_w$ fixes $\rho_M \in \mathrm{Irr}(\overline{\mathbb{Q}_\ell}[M/M^\circ, \natural_{\mathcal{E}}])$ pointwise. Therefore we can take $qb_w = b_w$ for $w \in W_{\mathfrak{t}^\circ}$. By Theorem 2.2 these elements span the algebra $\mathrm{End}_{G^\circ}(\pi_*\tilde{\mathcal{E}}) \cong \overline{\mathbb{Q}_\ell}[W_{\mathfrak{t}^\circ}]$.

For general $w \in [W_{q\mathfrak{t}}]$ the description given in (45) shows that the action of $b_w$ on (78) consists of a permutation of the direct factors combined with a linear action on $V_{\rho_M}$. Let $\widetilde{W_{\mathfrak{t}}}$ and $N$ be as in (72) and (74). Then (78) can be embedded in a sum of copies of $\mathrm{ind}_N^{\widetilde{W_{\mathfrak{t}}}}(V_{\rho_M})$.

Now Proposition 1.1.b shows that there is a unique $qb_w \in q\mathcal{A}_w$ such that the action of $b_w$ on (78) can be identified with $qb_w \otimes I^w$, where $I^w$ is as in (4). We may choose the same $I^w$ as we did (implicitly) in the last part of the proof of Lemma 5.3.a, where we used them to determine the cocycle $\kappa_{\rho_M} = \kappa_{q\mathfrak{t}}$. Then Proposition 1.1.b shows also that these $qb_w$ multiply as in the algebra $\overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}}, \kappa_{\rho_M}]$.

Finally, the claim about the uniqueness follows in the same way as in the last part of the proof of Propostion 4.5. $\qquad\square$

Some remarks about the 2-cocycle $\kappa_{qt}$ are in order. If $W_{qt}$ is cyclic then $\kappa_{qt}$ is trivial because $H^2(W_{qt}, \overline{\mathbb{Q}_\ell}^\times) = \{1\}$. Furthermore

$$\text{if } M_\mathcal{E} = M^\circ, \text{ then } \text{End}_G(q\pi_*(\widetilde{q\mathcal{E}})) = \text{End}_G(\pi_*(\tilde{\mathcal{E}})) \cong \overline{\mathbb{Q}_\ell}[W_t, \natural_\mathcal{E}]$$

by Proposition 4.5. However, in contrast with the cocycle $\natural_\mathcal{E}$ appearing in Sections 3 and 4, it is in general rather difficult to obtain explicit information about $\kappa_{qt}$. One reason for this is that the classification of cuspidal local systems on disconnected reductive groups, as achieved in Theorem 3.1 and in Proposition 3.5, leaves many possibilities. In particular the groups $G_\mathcal{E}/G^\circ$ can be very large.

**Theorem 5.5.** $(a)$ *There exists a canonical bijection*

$$
\begin{array}{rccc}
q\Sigma_{qt}: & q\Psi_G^{-1}(qt) & \to & \text{Irr}\big(\text{End}_G(q\pi_*(\widetilde{q\mathcal{E}}))\big) \\
& (\mathcal{C}_u^G, \mathcal{F}) & \mapsto & \text{Hom}_G\Big(\mathcal{F}, \mathcal{H}^{2d_C}\big(\text{IC}(\overline{Y}, q\pi_*(\widetilde{q\mathcal{E}}))\big)|\mathcal{C}_u^G\Big).
\end{array}
$$

*It can be defined by the condition*

$$q\Sigma_{qt}^{-1}(\tau) = (\mathcal{C}_u^G, \mathcal{F}) \quad \Longleftrightarrow \quad \mathcal{F} = \mathcal{H}^{2d_C}\Big(\text{IC}(\overline{Y}, q\pi_*(\widetilde{q\mathcal{E}})_\tau)\Big)|\mathcal{C}_u^G.$$

$(b)$ *The restriction of $\mathcal{F}$ to a $G^\circ$-equivariant local system on $\mathcal{C}_u^{G^\circ}$ is $\bigoplus_i \Sigma_{t^\circ}^{-1}(\tau_i)$, where $t^\circ = [M^\circ, \mathcal{C}_v^{M^\circ}, \mathcal{E}]_{G^\circ}$ and $\tau = \bigoplus_i \tau_i$ is a decomposition into irreducible $\text{End}_{G^\circ}(\pi_* \tilde{\mathcal{E}})$-subrepresentations.*

$(c)$ *Upon choosing an isomorphism as in Lemma 5.4, we obtain the bijection*

$$q\Psi_G^{-1}(qt) \to \text{Irr}\big(\overline{\mathbb{Q}_\ell}[W_{qt}, \kappa_{qt}]\big)$$

*from Lemma 5.3.*

*Proof.* (c) Write $t_M = \Psi_M(\mathcal{C}_v^M, q\mathcal{E})$ and recall that $\rho_M = \Sigma_{t_M}(\mathcal{C}_v^M, q\mathcal{E})$. By Lemma 3.4

$$\text{End}_M\big(\pi_*^M(\widetilde{\mathcal{E}_M})\big) \cong \overline{\mathbb{Q}_\ell}[M_\mathcal{E}/M^\circ] \cong p_\mathcal{E} \overline{\mathbb{Q}_\ell}[N],$$

and by Lemma 5.4

$$\text{End}_G(q\pi_*(\widetilde{q\mathcal{E}})) \cong \overline{\mathbb{Q}_\ell}[W_{t,\rho_M}/(M_\mathcal{E}/M^\circ), \kappa_{\rho_M}].$$

From a $\tau$ as in the theorem we obtain, using (75),

$$\rho_M \rtimes \tau \in \text{Irr}(p_\mathcal{E} \overline{\mathbb{Q}_\ell}[\widetilde{W_t}]) = \text{Irr}(\overline{\mathbb{Q}_\ell}[W_t, \natural_\mathcal{E}]).$$

By Theorem 4.7 the bijection from Lemma 5.3 maps $(\mathcal{C}_u^G, \mathcal{F})$ to $\tau$ if and only if

(79) $$\mathcal{F} \quad \text{equals} \quad \mathcal{H}^{2d_C}\Big(\text{IC}(\overline{Y}, \pi_*(\tilde{\mathcal{E}}))_{\rho_M \rtimes \tau}\Big)|\mathcal{C}_u^G.$$

Recall from (66) that $\text{Hom}_N(\rho_M, q\pi_* \tilde{\mathcal{E}}) \cong \pi_*(\widetilde{q\mathcal{E}})$. We apply Proposition 1.1.d to $\widetilde{W_t}, N$ and the representation $\pi_* \tilde{\mathcal{E}}$, and we find that the right hand side of (79) is isomorphic with $\mathcal{H}^{2d_C}\big(\text{IC}(\overline{Y}, q\pi_*(\widetilde{q\mathcal{E}})_\tau)|\mathcal{C}_u^G\big)$. Since $q\pi_*(\widetilde{q\mathcal{E}})$ is semisimple, taking the $\tau$-Hom-space commutes with forming an intersection cohomology complex. Hence the bijection from Lemma 5.3 satisfies exactly the condition given in the theorem.
(a) This condition clearly is canonical, so with part (a) it determines a canonical bijection $q\Sigma_{qt}^{-1}$.

It remains to check that the given formula for $q\Sigma_{q\mathfrak{t}}$ agrees with the above construction. Let $(\mathcal{C}_u^G, \mathcal{F}) \in q\Psi_G^{-1}(q\mathfrak{t})$ and write $\mathcal{F}_u = \eta^\circ \rtimes \tau'$ as in the proof of Theorem 4.7. Then, by (66)

$$\mathrm{Hom}_G\left(\mathcal{F}, \mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y}, q\pi_*(\widetilde{q\mathcal{E}}))\right)|\mathcal{C}_u^G\right) = \mathrm{Hom}_G\left(\mathcal{F}, \mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y}, \pi_*(\tilde{\mathcal{E}}))_{\rho_M}\right)|\mathcal{C}_u^G\right)$$

$$(80) \qquad\qquad = \mathrm{Hom}_{A_G(u)}\left(\eta^\circ \rtimes \tau', \mathcal{H}^{2d_C}\left(\mathrm{IC}(\overline{Y}, \pi_*(\tilde{\mathcal{E}}))_{\rho_M}\right)_u\right).$$

As the actions of $A_G(u)$ and

$$\overline{\mathbb{Q}_l}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}] \cong \mathrm{End}_M(\pi_*^M(\widetilde{\mathcal{E}_M})) \subset \mathrm{End}_G(\pi_*(\tilde{\mathcal{E}}))$$

commute, (60) shows that (80) is isomorphic to

$$\mathrm{Hom}_{\overline{\mathbb{Q}_l}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}]}\left(\rho_M, \mathrm{ind}_{\overline{\mathbb{Q}_l}[W_{\mathfrak{t},\rho^\circ}, \natural_\mathcal{E}]}^{\overline{\mathbb{Q}_l}[W_\mathfrak{t}, \natural_\mathcal{E}]}(V_{\tau'}^* \otimes V_{\rho^\circ})\right) = \mathrm{Hom}_{\overline{\mathbb{Q}_l}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}]}\left(\rho_M, \tau'^* \ltimes \rho^\circ\right).$$

From (75) and the subsequent argument we see that $\tau'^* \ltimes \rho^\circ = \rho_M \rtimes \tau$ for a unique irreducible representation $\tau$ of (using Lemma 5.4)

$$\overline{\mathbb{Q}_l}[W_{\mathfrak{t},\rho_M}/(M_\mathcal{E}/M^\circ), \natural_\mathcal{E}]) \cong \overline{\mathbb{Q}_l}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}] \cong \mathrm{End}_G\left(q\pi_*(\widetilde{q\mathcal{E}})\right).$$

In view of all this, (80) becomes

$$\mathrm{Hom}_{\overline{\mathbb{Q}_l}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}]}\left(\rho_M, \rho_M \rtimes \tau\right) = \mathrm{Hom}_{\overline{\mathbb{Q}_l}[M_\mathcal{E}/M^\circ, \natural_\mathcal{E}]}\left(\rho_M, \mathrm{ind}_{\overline{\mathbb{Q}_l}[W_{\mathfrak{t},\rho_M}, \natural_\mathcal{E}]}^{\overline{\mathbb{Q}_l}[W_\mathfrak{t}, \natural_\mathcal{E}]}(V_{\rho_M} \otimes V_\tau)\right) = V_\tau.$$

This means that $q\Sigma_{q\mathfrak{t}}(\mathcal{C}_u^G, \mathcal{F})$ as given in the statement is isomorphic with the outcome of the bijection via part (c).

(b) The behaviour of the restriction of $q\Sigma_{q\mathfrak{t}}^{-1}(\tau)$ to $\mathcal{C}_u^{G^\circ}$ follows from comparing the characterization with Theorem 2.1.(3). $\qquad\qquad\square$

By Theorem 4.7 and (75), $q\Sigma_{q\mathfrak{t}}(u, \eta)$ is also given by

$$(81) \qquad\qquad \Sigma_\mathfrak{t}(\mathcal{C}_u^G, \mathcal{F}) = \Sigma_{\mathfrak{t}_M}(\mathcal{C}_v^M, q\mathcal{E}) \rtimes q\Sigma_{q\mathfrak{t}}(\mathcal{C}_u^G, \mathcal{F}).$$

However, it is hard to make sense of this $\rtimes$-sign in a completely canonical way, without using the isomorphisms from Proposition 4.5 and Lemma 5.4.

There is also an analogue of Theorem 4.8 with quasi-Levi subgroups. Assume that $H \subset G$ is an algebraic subgroup such that $H \cap G^\circ$ is a Levi subgroup of $G^\circ$ and $H$ contains the quasi-Levi subgroup $Z_G(Z(G^\circ \cap H)^\circ)$. Let $u \in H^\circ$ be unipotent. We saw in (61) that $A_H(u)$ can be regarded as a subgroup of $A_G(u)$.

**Proposition 5.6.** *In the above setting, let $\eta \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_G(u))$ and $\eta_H \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(A_H(u))$.*

(a) *If $\eta_H$ appears in $\mathrm{Res}_{A_H(u)}^{A_G(u)}(\eta)$, then $q\Psi_G(u, \eta) = q\Psi_H(u, \eta_H)/G$-conjugacy.*

(b) *There is a natural inclusion of algebras $\mathrm{End}_H(q\pi_*^H(\widetilde{q\mathcal{E}_H})) \to \mathrm{End}_G(q\pi_*(\widetilde{q\mathcal{E}}))$.*
   *Suppose that $q\Psi_G(u, \eta) = q\Psi_H(u, \eta_H)/G$-conjugacy. Then $q\Sigma_{q\Psi_H(u,\eta_H)}(u, \eta_H)$ is a constituent of $\mathrm{Res}_{\mathrm{End}_H(q\pi_*^H(\widetilde{q\mathcal{E}_H}))}^{\mathrm{End}_G(q\pi_*(\widetilde{q\mathcal{E}}))} q\Sigma_{q\mathfrak{t}}(u, \eta)$ if and only if $\eta_H$ is a constituent of $\mathrm{Res}_{A_H(u)}^{A_G(u)}(\eta)$.*

*Proof.* (a) From Theorem 4.8.a we know that $[M^\circ, \mathcal{C}_v^{M^\circ}, \mathcal{E}]_G = \Psi_G(u, \eta)$ equals $\Psi_H(u, \eta)$ up to $G$-conjugacy. In particular $M^\circ \subset H^\circ$ and, by the assumptions on $H$, $M \subset H$. It follows that $\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M})$ is also a subalgebra of $\mathrm{End}_H(\pi_*^H \widetilde{\mathcal{E}_H})$. By Theorem 4.8.b we may choose $\rho_M$ (used in (64) to construct $q\mathcal{E}$) to be a constituent of $\rho_H = \Sigma_{\mathfrak{t}_H}(u, \eta_H)$. Then $\Psi_H(u, \eta_H) = [M, \mathcal{C}_v^M, q\mathcal{E}]_H$, which agrees with

(65).

(b) By Lemma 5.4 $\mathrm{End}_H\big(q\pi_*^H(\widetilde{q\mathcal{E}_H})\big) \cong \overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}_H}, \kappa_{q\mathfrak{t}_H}]$. Here

$$W_{q\mathfrak{t}_H} = W_{\mathfrak{t}_H,\rho_M}/(M_\mathcal{E}/M^\circ) = N_H(M^\circ, \mathcal{C}_v^{M^\circ}, \mathcal{E})/M_\mathcal{E}$$

is a subgroup of

$$N_G(M^\circ, \mathcal{C}_v^{M^\circ}, \mathcal{E})/M_\mathcal{E} = W_{\mathfrak{t},\rho_M}/(M_\mathcal{E}/M^\circ) = W_{q\mathfrak{t}}.$$

The 2-cocycle $\kappa_{q\mathfrak{t}_H}$ is just the restriction of $\kappa_{q\mathfrak{t}}$, because both are based on the same representation $\rho_M$ of $\mathrm{End}_M(\pi_*^M \widetilde{\mathcal{E}_M})$. This gives an injection

$$\overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}_H}, \kappa_{q\mathfrak{t}_H}] \to \overline{\mathbb{Q}_\ell}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}].$$

With Lemma 5.4 we get an injection

$$\mathrm{End}_H(q\pi_*^H(\widetilde{q\mathcal{E}_H})) \to \mathrm{End}_G(q\pi_*(\widetilde{q\mathcal{E}})).$$

It is natural because every basis element $qb_w$ ($w \in W_{q\mathfrak{t}_H}$) of $\mathrm{End}_G(q\pi_*(\widetilde{q\mathcal{E}}))$ constructed in the proof of Lemma 5.4 stabilizes the subset $q\pi_*^H(\widetilde{q\mathcal{E}_H})$ and hence naturally determines an automorphism of that sheaf.

For the group $H$ (81) says

$$\Sigma_{\mathfrak{t}_H}(u, \eta_H) = \rho_M \rtimes q\Sigma_{q\mathfrak{t}_H}(u, \eta_H).$$

By Theorem 4.8.b $\Sigma_{\mathfrak{t}_H}(u, \eta_H)$ appears in $\Sigma_{\mathfrak{t}}(u, \eta)$ if and only if $\eta_H$ appears in $\eta$. With Proposition 1.1.c and (81) we see that this is also equivalent to $q\Sigma_{q\mathfrak{t}_H}(u, \eta_H)$ appearing in $q\Sigma_{q\mathfrak{t}}(u, \eta)$. □

This concludes the part of the paper which deals exclusively with Springer correspondences. We remark once more that all the results from Sections 2–5 also hold with $\mathbb{C}$ instead of $\overline{\mathbb{Q}_\ell}$.

## 6. Cuspidal Langlands parameters

We will introduce a notion of cuspidality for enhanced L-parameters. Before we come to that, we recall some generalities about Langlands parameters and Levi subgroups. For more background we refer to [Bor, Vog, ABPS6].

Let $F$ be a local non-archimedean field with Weil group $\mathbf{W}_F$. Let $\mathcal{H}$ be a connected reductive algebraic group over $F$ and let $\mathcal{H}^\vee$ be its complex dual group. The data for $\mathcal{H}$ provide an action of $\mathbf{W}_F$ on $\mathcal{H}^\vee$ which preserves a pinning, and that gives the Langlands dual group $^L\mathcal{H} = \mathcal{H}^\vee \rtimes \mathbf{W}_F$. (All these objects are determined by $F$ and $\mathcal{H}$ up to isomorphism.)

**Definition 6.1.** Let $T \subset \mathcal{H}^\vee$ be a torus such that the projection $Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(T) \to \mathbf{W}_F$ is surjective. Then we call $^L L = Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(T)$ a Levi L-subgroup of $^L\mathcal{H}$.

We remark that in [Bor] such groups are called Levi subgroups of $^L\mathcal{H}$. We prefer to stick to the connectedness of Levi subgroups.

Choose a $\mathbf{W}_F$-stable pinning for $\mathcal{H}^\vee$. This defines the notion of standard Levi subgroups of $\mathcal{H}^\vee$. An alternative characterization of Levi L-subgroups of $^L\mathcal{H}$ is as follows.

**Lemma 6.2.** *Let $^L L$ be a Levi L-subgroup of $^L\mathcal{H}$. There exists a $\mathbf{W}_F$-stable standard Levi subgroup $\mathcal{L}^\vee$ of $\mathcal{H}^\vee$ such that $^L L$ is $\mathcal{H}^\vee$-conjugate to $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ and $L := {}^L L \cap \mathcal{H}^\vee$ is conjugate to $\mathcal{L}^\vee$.*

*Conversely, every $\mathcal{H}^\vee$-conjugate of this $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ is a Levi L-subgroup of $^L\mathcal{H}$.*

*Proof.* By [Bor, Lemma 3.5] there exists a parabolic subgroup $P \subset \mathcal{H}^\vee$ such that

- $N_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(P) \to \mathbf{W}_F$ is surjective;
- $L$ is a Levi factor of $P$;
- $^L L = N_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(L) \cap N_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(P)$.

To construct such a $P$, choose a $\mathbb{Z}$-linear function $X^*(T) \to \mathbb{Z}$ in generic position (i.e. not orthogonal to any coroot). Then we can define $P$ as the subgroup of $\mathcal{H}^\vee$ generated by $L$ and by all root subgroups associated to positive (with respect to this linear function) cocharacters of $T$.

Let $P_I = L_I \rtimes U_I$ be the unique standard parabolic subgroup of $\mathcal{H}^\vee$ conjugate to $P$. Here $U_I$ denotes the unipotent radical of $P_I$, and $L_I$ its standard Levi factor. Then $N_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(P_I) \to \mathbf{W}_F$ is still surjective, so $P_I$ is $\mathbf{W}_F$-stable. Pick $h \in \mathcal{H}^\vee$ with $P_I = hPh^{-1}$. Then $hLh^{-1}$ is a Levi factor of $P_I$ and

$$h\,^L L h^{-1} = N_{P_I \rtimes \mathbf{W}_F}(h\,^L L h^{-1})$$

is a complement to $U_I$ in $P_I \rtimes \mathbf{W}_F$. All Levi factors of $P_I$ are $U_I$-conjugate, so there exists a $u \in U_I$ with $uh\,^L L h^{-1} u^{-1} = L_I$. Then

$$uh\,^L L h^{-1} u^{-1} = N_{P_I \rtimes \mathbf{W}_F}(L_I) = L_I \rtimes \mathbf{W}_F.$$

For the converse, let $\mathcal{L}^\vee$ be a $\mathbf{W}_F$-stable standard Levi subgroup of $\mathcal{H}^\vee$. Denote the standard maximal torus of $\mathcal{H}^\vee$ by $L_\emptyset$ and consider the root system $R := R(\mathcal{H}^\vee, L_\emptyset)$. By assumption $\mathbf{W}_F$ acts on $R$ and stabilizes a basis $\Delta$. Let $T \subset L_\emptyset$ be the neutral component of $Z(\mathcal{L}^\vee)^{\mathbf{W}_F}$. This is a $\mathbf{W}_F$-fixed torus which commutes with $\mathcal{L}^\vee$ and

$$\alpha(t) = (w \cdot \alpha)(t) \qquad \forall t \in T, \alpha \in R, w \in \mathbf{W}_F.$$

The Lie algebra $\mathfrak{l}_{\text{der}}$ of $L_\emptyset \cap \mathcal{H}^\vee_{\text{der}}$ is spanned by $\Delta^\vee$ and $\mathbf{W}_F$-stable. Let $I$ be the set of simple roots in $R(\mathcal{L}^\vee, L_\emptyset)$. Since $\mathcal{L}^\vee$ is $\mathbf{W}_F$-stable, so are $I$ and $\Delta \setminus I$. Let $X \in \mathfrak{l}_{\text{der}}$ be an element which annihilites $I$ and takes the same value in $\mathbb{R}_{>0}$ on all simple roots not in $I$. Then $X \in \text{Lie}(\mathcal{L}^\vee)$ is fixed by $\mathbf{W}_F$. This gives an element $\exp(X) \in T$ with $\alpha(\exp X) = \exp(\alpha(X)) > 1$ for all positive roots in $R \setminus R(\mathcal{L}^\vee, L_\emptyset)$.

In general, the $\mathcal{H}^\vee$-centralizer of the torus $T \subset Z(\mathcal{L}^\vee)^\circ$ is generated by $\mathcal{L}^\vee$ and by the root subgroups $U_\alpha$ for which $\alpha$ becomes trivial on $T$. With the above elements $\exp(X)$ we deduce that

$$Z_{\mathcal{H}^\vee}(T) = \mathcal{L}^\vee \quad \text{and} \quad Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(T) = \mathcal{L}^\vee \rtimes \mathbf{W}_F.$$

This means that $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ is a Levi L-subgroup of $^L\mathcal{H}$ in the sense of Definition 6.1. For any $h \in \mathcal{H}^\vee$:

$$h(\mathcal{L}^\vee \rtimes \mathbf{W}_F)h^{-1} = Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(hTh^{-1}).$$

This group contains $h\mathbf{W}_F h^{-1}$, so it projects onto $\mathbf{W}_F$. Hence it is again a Levi L-subgroup of $^L\mathcal{H}$. $\qquad \square$

**Remark 6.3.** Most Levi L-subgroups of $^L\mathcal{H}$ are not quasi-Levi, and conversely. For example, let $\mathcal{U} = \mathrm{U}(n, E/F)$ be a $p$-adic unitary group ($E$ is a quadratic extension of $F$) and let $^L\mathcal{U}$ be its dual L-group. The group $\mathbf{W}_F$ acts on $\mathcal{U}^\vee = \mathrm{GL}(n, \mathbb{C})$ via an outer automorphism which preserves the diagonal torus $T$ and the standard Borel subgroup $B$. Then $T \rtimes \mathbf{W}_F$ is a Levi L-subgroup of $^L\mathcal{U}$: it is the centralizer of $T^{\mathbf{W}_F}$ in $^L\mathcal{U}$. However, it is not quasi-Levi. Namely $Z_{L\mathcal{U}}(T) = T \rtimes \mathbf{W}_E$, which is an index two subgroup of $T \rtimes \mathbf{W}_F$.

The following definitions are well-known, we repeat them here to facilitate comparison with later generalizations.

**Definition 6.4.** A L-parameter for $^L\mathcal{H}$ is a continuous group homomorphism $\phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{H}$ such that:

- $\phi(w) \in \mathcal{H}^\vee w$ for all $w \in \mathbf{W}_F$;
- $\phi(w)$ is semisimple for all $w \in \mathbf{W}_F$;
- $\phi|_{\mathrm{SL}_2(\mathbb{C})}\colon \mathrm{SL}_2(\mathbb{C}) \to \mathcal{H}^\vee$ is a homomorphism of complex algebraic groups.

Recall that all inner forms of $\mathcal{H}$ share the same Langlands dual group $^L\mathcal{H}$, so the group $\mathcal{H}$ is not determined by the target $^L\mathcal{H}$ of a L-parameter. Let us specify which L-parameters are relevant for $\mathcal{H}$, and which are bounded or discrete.

**Definition 6.5.** Let $\phi\colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{H}$ be a L-parameter. We say that $\phi$ is bounded if $\phi(\mathrm{Frob}) = (h, \mathrm{Frob})$ with $h$ in some compact subgroup of $\mathcal{H}^\vee$.

Suppose that $^L L$ is a Levi L-subgroup of $^L\mathcal{H}$ and that

- $^L L$ contains the image of $\phi$;
- there is no smaller Levi L-subgroup of $^L\mathcal{H}$ with this property.

Then we call $\phi$ relevant for $\mathcal{H}$ if and only if the conjugacy class of $^L L$ is relevant for $\mathcal{H}$, that is, it corresponds to a conjugacy class of Levi subgroups of $\mathcal{H}$.

In this case we also say that $\phi$ is a discrete L-parameter for $^L L$, and for any Levi subgroup $\mathcal{L} \subset \mathcal{H}$ in the associated class. In particular $\phi$ is discrete for $\mathcal{H}$ if and only if there is no proper Levi L-subgroup of $^L\mathcal{H}$ containing the image of $\phi$.

The group $\mathcal{H}^\vee$ acts on the set of relevant L-parameters for $\mathcal{H}$. We denote the set of relevant $L$-parameters modulo $\mathcal{H}^\vee$-conjugation by $\Phi(\mathcal{H})$. The subset of bounded L-parameters (up to conjugacy) is denoted by $\Phi_{\mathrm{bdd}}(\mathcal{H})$. The local Langlands correspondence predicts that $\mathrm{Irr}(\mathcal{H})$ is partitioned into finite L-packets $\Pi_\phi(\mathcal{H})$, parametrized by $\Phi(\mathcal{H})$. Under this correspondence $\Phi_{\mathrm{bdd}}(\mathcal{H})$ should give rise to L-packets consisting entirely of tempered representations, and that should account for the entire tempered dual of $\mathcal{H}$.

It is expected (and established in many cases) that the following conditions are equivalent for $\phi \in \Phi(\mathcal{H})$:

- $\phi$ is discrete;
- $\Pi_\phi(\mathcal{H})$ contains an essentially square-integrable representation;
- all elements of $\Pi_\phi(\mathcal{H})$ are essentially square-integrable.

In other words, "discrete" (respectively "bounded") is the correct translation of "essentially square-integrable" (respectively "tempered") under the local Langlands correspondence.

However, it is more difficult to characterize when $\Pi_\phi(\mathcal{H})$ contains supercuspidal $\mathcal{H}$-representations. Of course $\phi$ has to be discrete, but even then. Sometimes $\Pi_\phi(\mathcal{H})$ consists entirely of supercuspidal representations, for example when $\mathcal{H} = \mathrm{SL}_2(F)$ and $\phi$ comes from an irreducible representation $\mathbf{W}_F \to \mathrm{GL}_2(\mathbb{C})$. In other cases $\Pi_\phi(\mathcal{H})$ contains only non-supercuspidal essentially square-integrable representations, for example when $\mathcal{H} = \mathrm{SL}_2(F)$, $\phi|_{\mathbf{W}_F} = \mathrm{id}_{\mathbf{W}_F}$ and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is an irreducible two-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

Moreover there are mixed cases, where $\Pi_\phi(\mathcal{H})$ contains both supercuspidal and non-supercuspidal representations. An example is formed by a Langlands parameter for a group of type $G_2$, with $\phi\big(1, \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)\big)$ a subregular unipotent element of $G_2(\mathbb{C})$. Then $\Pi_\phi(G_2(F))$ has a unique supercuspidal element and contains two representations from the principal series of $G_2(F)$, see [Lus3].

To parametrize the representations in a given L-packet, we need more information then just the Langlands parameter itself. Let $Z_{\mathcal{H}^\vee}(\phi)$ be the centralizer of $\phi(\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}))$ in $\mathcal{H}^\vee$. This is a complex reductive group, in general disconnected. We write

$$\mathfrak{R}_\phi := \pi_0\big(Z_{\mathcal{H}^\vee}(\phi)/Z(\mathcal{H}^\vee)^{\mathbf{W}_F}\big). \tag{82}$$

It is expected that $\Pi_\phi(\mathcal{H})$ is in bijection with $\mathrm{Irr}(\mathfrak{R}_\phi)$ if $\mathcal{H}$ is quasi-split. However, for general $\mathcal{H}$ this is not good enough, and we follow Arthur's setup [Art2].

Let $\mathcal{H}^\vee_{\mathrm{sc}}$ be the simply connected cover of the derived group $\mathcal{H}^\vee_{\mathrm{der}}$ of $\mathcal{H}^\vee$. The conjugation action of $\mathcal{H}^\vee_{\mathrm{der}}$ lifts to an action of $\mathcal{H}^\vee_{\mathrm{sc}}$ on $\mathcal{H}$ by conjugation. The action of $\mathbf{W}_F$ on $\mathcal{H}^\vee_{\mathrm{der}}$ lifts to an action on $\mathcal{H}^\vee_{\mathrm{sc}}$, because the latter group is simply connected. Thus we can form the group $\mathcal{H}^\vee_{\mathrm{sc}} \rtimes \mathbf{W}_F$. In this semidirect product we can compute $hwh^{-1}$ for $h \in \mathcal{H}^\vee_{\mathrm{sc}}$ and $w \in \mathbf{W}_F$. Dividing out the normal subgroup $\ker(\mathcal{H}^\vee_{\mathrm{sc}} \to \mathcal{H}^\vee_{\mathrm{der}})$, we can interpret $hwh^{-1}$ as an element of $\mathcal{H}^\vee_{\mathrm{der}} \rtimes \mathbf{W}_F$.

Together with the above this provides a conjugation action of $\mathcal{H}^\vee_{\mathrm{sc}}$ on $\mathcal{H}^\vee \rtimes \mathbf{W}_F$. Hence $\mathcal{H}^\vee_{\mathrm{sc}}$ also acts on the set of Langlands parameters for $\mathcal{H}$ and we can form $Z_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$.

Since $Z_{\mathcal{H}^\vee}(\phi) \cap Z(\mathcal{H}^\vee) = Z(\mathcal{H}^\vee)^{\mathbf{W}_F}$,

$$Z_{\mathcal{H}^\vee}(\phi)/Z(\mathcal{H}^\vee)^{\mathbf{W}_F} \cong Z_{\mathcal{H}^\vee}(\phi)Z(\mathcal{H}^\vee)/Z(\mathcal{H}^\vee). \tag{83}$$

The right hand side can be considered as a subgroup of the adjoint group $\mathcal{H}^\vee_{\mathrm{ad}}$. Let $Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$ be its inverse image under the quotient map $\mathcal{H}^\vee_{\mathrm{sc}} \to \mathcal{H}^\vee_{\mathrm{ad}}$. We can also characterize it as

$$Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi) = \big\{h \in \mathcal{H}^\vee_{\mathrm{sc}} : h\phi h^{-1} = \phi\, a_h \text{ for some } a_h \in B^1(\mathbf{W}_F, Z(\mathcal{H}^\vee))\big\} \tag{84}$$
$$= \big\{h \in Z_{\mathcal{H}^\vee_{\mathrm{sc}}}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big) : h\phi|_{\mathbf{W}_F}h^{-1} = \phi|_{\mathbf{W}_F}\, a_h \text{ for some } a_h \in B^1(\mathbf{W}_F, Z(\mathcal{H}^\vee))\big\}$$
$$= Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}) \cap Z_{\mathcal{H}^\vee_{\mathrm{sc}}}\big(\phi(\mathrm{SL}_2(\mathbb{C}))\big).$$

Here $B^1(\mathbf{W}_F, Z(\mathcal{H}^\vee))$ is the set of 1-coboundaries for group cohomology, that is, maps $\mathbf{W}_F \to Z(\mathcal{H}^\vee)$ of the form $w \mapsto zwz^{-1}w^{-1}$ with $z \in Z(\mathcal{H}^\vee)$. The neutral component of $Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$ is $Z_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)^\circ$, so it is a complex reductive group.

The difference between $Z_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$ and $Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$ is caused by the identification (83), which as it were includes $Z(\mathcal{H}^\vee)$ in $Z_{\mathcal{H}^\vee}(\phi)$. We note that $Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi) = Z_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$ whenever $Z(\mathcal{H}^\vee_{\mathrm{sc}})^{\mathbf{W}_F} = Z(\mathcal{H}^\vee_{\mathrm{sc}})$, in particular if $\mathcal{H}$ is an inner twist of a split group.

Given $\phi$, we form the finite group

$$\mathcal{S}_\phi := \pi_0\big(Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)\big). \tag{85}$$

Via (83), the map $\mathcal{H}^\vee_{\mathrm{sc}} \to \mathcal{H}^\vee_{\mathrm{ad}}$ induces a homomorphism $\mathcal{S}_\phi \to \mathfrak{R}_\phi$. In fact, $\mathcal{S}_\phi$ is a central extension of $\mathfrak{R}_\phi$ by $\mathcal{Z}_\phi := Z(\mathcal{H}^\vee_{\mathrm{sc}})/Z(\mathcal{H}^\vee_{\mathrm{sc}}) \cap Z_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)^\circ$ [ABPS6, Lemma 1.7]:

$$1 \to \mathcal{Z}_\phi \to \mathcal{S}_\phi \to \mathfrak{R}_\phi \to 1. \tag{86}$$

Since $\mathcal{H}^\vee_{\mathrm{sc}}$ is a central extension of $\mathcal{H}^\vee_{\mathrm{ad}} = \mathcal{H}^\vee/Z(\mathcal{H}^\vee)$, the conjugation action of $\mathcal{H}^\vee_{\mathrm{sc}}$ on itself and on $\mathcal{S}_\phi$ descends to an action of $\mathcal{H}^\vee_{\mathrm{ad}}$. Via the canonical quotient map, also $\mathcal{H}^\vee$ acts on $\mathcal{S}_\phi$ by conjugation.

An enhancement of $\phi$ is defined to be an irreducible complex representation $\rho$ of $\mathcal{S}_\phi$. We refer to [Art2, ABPS6] for a motivation of this particular kind of enhancements. We let $\mathcal{H}^\vee$ and $\mathcal{H}^\vee_{\mathrm{sc}}$ act on the set of enhanced L-parameters by

$$h \cdot (\phi, \rho) = (h\phi h^{-1}, h \cdot \rho) \quad \text{where} \quad (h \cdot \rho)(g) = \rho(h^{-1}gh). \tag{87}$$

We note that both groups acting in (87) yield the same orbit space.

The notion of relevance for enhanced L-parameters is more subtle. Firstly, we must specify $\mathcal{H}$ not only as an inner form of a quasi-split group $\mathcal{H}^*$, but even as an inner twist. That is, we must fix an isomorphism $\mathcal{H} \to \mathcal{H}^*$ of algebraic groups, defined over a separable closure of $F$. The inner twists of $\mathcal{H}$ are parametrized by the Galois cohomology group $H^1(F, \mathcal{H}_{\mathrm{ad}})$, where $\mathcal{H}_{\mathrm{ad}}$ denotes the adjoint group of $\mathcal{H}$ (considered as an algebraic group defined over $F$). The parametrization is canonically determined by requiring that $\mathcal{H}^*$ corresponds to the trivial element of $H^1(F, \mathcal{H}_{\mathrm{ad}})$. Kottwitz [Kot, Theorem 6.4] found a natural group isomorphism

$$(88) \qquad\qquad H^1(F, \mathcal{H}_{\mathrm{ad}}) \cong \mathrm{Irr}_{\mathbb{C}}\big(Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}\big).$$

(When $F$ has positive characteristic, see [Tha, Theorem 2.1].) In this way every inner twist of $\mathcal{H}$ is associated to a unique character of $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F} = Z(\mathcal{H}_{\mathrm{sc}}^{\vee} \rtimes \mathbf{W}_F)$. The functoriality of the Kottwitz homomorphism implies that this parametrization behaves well with respect to Levi subgroups. To make this statement precise, let $\mathcal{L}$ be a Levi $F$-subgroup of $\mathcal{H}$. Via $\mathcal{H} \to \mathcal{H}^*$, we regard $\mathcal{L}$ as an inner twist of a quasi-split Levi subgroup $\mathcal{L}^*$ of $\mathcal{H}^*$. Let $\mathcal{L}_c^{\vee}$ be the inverse image of $\mathcal{L}^{\vee}$ under $\mathcal{H}_{\mathrm{sc}}^{\vee} \to \mathcal{H}$. It contains $\mathcal{L}_{\mathrm{sc}}^{\vee}$ as the derived subgroup of $\mathcal{L}_c^{\vee}$. The next lemma is a variation on [KMSW, Lemma 0.4.9], tailored for our purposes.

**Lemma 6.6.** (a) *The centers of $\mathcal{H}_{\mathrm{sc}}^{\vee}, \mathcal{L}_c^{\vee}$ and $\mathcal{L}_{\mathrm{sc}}^{\vee}$ are related by*

$$Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F} Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F, \circ} = Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F} \supset Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}.$$

(b) *The character of $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$ determined by (88) is trivial on $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F} \cap Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F, \circ}$. Using part (a) we extend it to $Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F}$, trivially on $Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F, \circ}$. Then the character of $Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$ obtained by restriction equals the character of $Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$ associated to $\mathcal{L}$ by (88).*

*Proof.* (a) See [Art1, Lemma 1.1].
(b) The morphisms of reductive $F$-groups $\mathcal{H}_{\mathrm{ad}} \leftarrow \mathcal{L}/Z(\mathcal{H}) \to \mathcal{L}_{\mathrm{ad}}$ induce the following commutative diagram:

$$
\begin{array}{ccccc}
H^1(F, \mathcal{H}_{\mathrm{ad}}) & \longleftarrow & H^1(F, \mathcal{L}/Z(\mathcal{H})) & \longrightarrow & H^1(F, \mathcal{L}_{\mathrm{ad}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Irr}\big(Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}\big) & \longleftarrow & \mathrm{Irr}\big(\pi_0(Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F})\big) & \longrightarrow & \mathrm{Irr}\big(Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}\big).
\end{array}
$$

All the vertical arrows are isomorphisms, and according to [Art1, p. 217] the left horizontal arrows are injective. Since $\mathcal{L}$ is a Levi $F$-subgroup of $\mathcal{H}$, the element of $H^1(F, \mathcal{H}_{\mathrm{ad}})$ which parametrizes $\mathcal{H}$ can be represented by a Galois cocycle with values in the Levi subgroup $\mathcal{L}/Z(\mathcal{H})$ of $\mathcal{H}_{\mathrm{ad}}$. This cocycle maps naturally to an element of $H^1(F, \mathcal{L}_{\mathrm{ad}})$, which then parametrizes the inner twist $\mathcal{L}$ of $\mathcal{L}^*$.

On the bottom line of the diagram, the associated character of $Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$ must come from a character of $\pi_0(Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F})$, via the map $Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F} \to \pi_0(Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F})$ from part (a). Hence this character is trivial on $Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F} \cap Z(\mathcal{L}_c^{\vee})^{\mathbf{W}_F, \circ}$. The character of $Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$ associated to $\mathcal{L}$ is then obtained by applying the lower right map in the diagram. This works out as restriction to $Z(\mathcal{L}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$, in the indicated way. $\qquad\square$

Given any Langlands parameter $\phi$ for $^L\mathcal{H}$, there is a natural group homomorphism $Z(\mathcal{H}_{sc}^\vee)^{\mathbf{W}_F} \to Z(\mathcal{S}_\phi)$. The centre of $\mathcal{S}_\phi$ acts by a character on any $\rho \in \operatorname{Irr}_\mathbb{C}(\mathcal{S}_\phi)$, so any enhancement $\rho$ of $\phi$ determines a character $\zeta_\rho$ of $Z(\mathcal{H}_{sc}^\vee)^{\mathbf{W}_F}$.

**Definition 6.7.** Let $(\phi, \rho)$ be an enhanced L-parameter for $^L\mathcal{H}$. We say that $(\phi, \rho)$ or $\rho$ is $\mathcal{H}$-relevant if $\zeta_\rho$ parametrizes the inner twist $\mathcal{H}$ via (88).

By the next result, Definition 7.7 fits well with the earlier notion of relevance of $\phi$, as in Definition 6.5.

**Proposition 6.8.** *Let $\mathcal{H}$ be an inner twist of a quasi-split group and let $\zeta \in \operatorname{Irr}_\mathbb{C}\big(Z(\mathcal{H}_{sc}^\vee)^{\mathbf{W}_F}\big)$ be the associated character. Let $\phi$ be a Langlands parameter for $^L\mathcal{H}$. The following are equivalent:*

*(1) $\phi$ is relevant for $\mathcal{H}$;*
*(2) $Z(\mathcal{H}_{sc}^\vee)^{\mathbf{W}_F} \cap Z_{\mathcal{H}_{sc}^\vee}(\phi)^\circ \subset \ker \zeta$;*
*(3) there exists a $\rho \in \operatorname{Irr}_\mathbb{C}(\mathcal{S}_\phi)$ with $\zeta_\rho = \zeta$, that is, such that $(\phi, \rho)$ is $\mathcal{H}$-relevant.*

*Proof.* For the equivalence of (1) and (2) see [HiSa, Lemma 9.1] and [Art1, Corollary 2.2]. The equivalence of (2) and (3) is easy, it was already noted in [ABPS6, Proposition 1.6]. $\qquad\square$

Let us remark here that the usage of $\mathcal{H}_{sc}^\vee$ and the above relevance circumvents some of the problems in [Vog, §2]. In particular it removes the need to consider variations such as "pure inner forms" or "pure inner twists".

We denote the set of $\mathcal{H}^\vee$-equivalence classes of enhanced relevant L-parameters for $\mathcal{H}$ by $\Phi_e(\mathcal{H})$. Following [Art2] we choose an extension $\zeta_\mathcal{H}$ of $\zeta$ to a character of $Z(\mathcal{H}_{sc}^\vee)$. We define

$$(89) \qquad \Phi_{e,\zeta_\mathcal{H}}(\mathcal{H}) = \{(\phi, \rho) \in \Phi_e(\mathcal{H}) : \zeta_\mathcal{H} \operatorname{id}_{V_\rho} = \rho \circ (Z(\mathcal{H}_{sc}^\vee) \to \mathcal{S}_\phi)\},$$

where $V_\rho$ is the vector space underlying $\rho$. According to [Art2, §4]

$$Z(\mathcal{H}_{sc}^\vee) \cap Z_{\mathcal{H}_{sc}^\vee}(\phi)^\circ = Z(\mathcal{H}_{sc}^\vee)^{\mathbf{W}_F} \cap Z_{\mathcal{H}_{sc}^\vee}(\phi)^\circ.$$

Hence every extension of $\zeta$ to a character of $Z(\mathcal{H}_{sc}^\vee)$ is eligible if $\phi$ is $\mathcal{H}$-relevant. Of course we take $\zeta_\mathcal{H} = \operatorname{triv}$ if $\mathcal{H}$ is quasi-split. Since $\mathcal{S}_\phi/\mathcal{Z}_\phi \cong \mathfrak{R}_\phi$, we obtain

$$\Phi_{e,\operatorname{triv}}(\mathcal{H}) = \{(\phi, \rho) : \phi \in \Phi(\mathcal{H}), \rho \in \operatorname{Irr}(\mathfrak{R}_\phi)\} \qquad \text{if } \mathcal{H} \text{ is quasi-split.}$$

It is conjectured [Art2, ABPS6] that the local Langlands correspondence for $\mathcal{H}$ can be enhanced to a bijection

$$\operatorname{Irr}(\mathcal{H}) \longleftrightarrow \Phi_{e,\zeta_\mathcal{H}}(\mathcal{H}).$$

Recall that by the Jacobson–Morosov theorem any unipotent element $u$ of $Z_{\mathcal{H}_{sc}^\vee}(\phi(\mathbf{W}_F))^\circ$ can be extended to a homomorphism of algebraic groups $\operatorname{SL}_2(\mathbb{C}) \to Z_{\mathcal{H}_{sc}^\vee}(\phi(\mathbf{W}_F))^\circ$ taking the value $u$ at $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, by [Kos, Theorem 3.6] this extension is unique up to conjugation. Hence any element $(\phi, \rho) \in \Phi_e(\mathcal{H})$ is already determined by $\phi|_{\mathbf{W}_F}, u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)$ and $\rho$. More precisely, the map

$$(90) \qquad\qquad \phi \mapsto \Big(\phi|_{\mathbf{W}_F}, u_\phi = \phi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big)\Big)$$

provides a bijection between $\Phi(\mathcal{H})$ and the $\mathcal{H}^\vee$-conjugacy classes of pairs $(\phi|_{\mathbf{W}_F}, u_\phi)$. The inclusion $Z_{\mathcal{H}_{sc}^\vee}^1(\phi) \to Z_{\mathcal{H}_{sc}^\vee}^1(\phi|_{\mathbf{W}_F}) \cap Z_{\mathcal{H}_{sc}^\vee}(u_\phi)$ induces a group isomorphism

$$(91) \qquad\qquad \mathcal{S}_\phi \to \pi_0\big(Z_{\mathcal{H}_{sc}^\vee}^1(\phi|_{\mathbf{W}_F}) \cap Z_{\mathcal{H}_{sc}^\vee}(u_\phi)\big).$$

We will often identify $\Phi_e(\mathcal{H})$ with the set of $\mathcal{H}^\vee$-equivalence classes of such triples $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$. Another way to formulate (91) is

$$(92) \qquad \mathcal{S}_\phi \cong \pi_0(Z_G(u_\phi)) \quad \text{where} \quad G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}) \text{ and } u_\phi = \phi\big(1, \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)\big).$$

We note also that there is a natural bijection between the set of unipotent elements in $\mathcal{H}^\vee$ and those in $\mathcal{H}^\vee_{\mathrm{sc}}$, so we may take $u_\phi$ in either of these groups.

Based on many examples we believe that the following kind of enhanced L-parameters should parametrize supercuspidal representations.

**Definition 6.9.** An enhanced L-parameter $(\phi, \rho)$ for ${}^L\mathcal{H}$ is cuspidal if $\phi$ is discrete and $(u_\phi, \rho)$ is a cuspidal pair for $G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F})$. Here $\rho$ is considered as a representation of $\pi_0(Z_G(u_\phi))$ via (92).

We denote the set of $\mathcal{H}^\vee$-equivalence classes of $\mathcal{H}$-relevant cuspidal L-parameters by $\Phi_{\mathrm{cusp}}(\mathcal{H})$. When $\zeta_\mathcal{H}$ is as in (89), we put $\Phi_{\mathrm{cusp},\zeta_\mathcal{H}}(\mathcal{H}) = \Phi_{\mathrm{cusp}}(\mathcal{H}) \cap \Phi_{e,\zeta_\mathcal{H}}$.

It is easy to see that every group $\mathcal{H}$ has cuspidal L-parameters. Let $\phi \in \Phi(\mathcal{H})$ be a discrete parameter which is trivial on $\mathrm{SL}_2(\mathbb{C})$. Then $u_\phi = 1$ and $Z_{\mathcal{H}^\vee}(\phi)^\circ = Z(\mathcal{H}^\vee)^{\mathbf{W}_F, \circ}$. Hence $G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi)$ is finite and every enhancement $\rho$ of $\phi$ is cuspidal. By Proposition 6.8 we can choose a $\mathcal{H}$-relevant $\rho$.

In the case of quasi-split groups we can also use enhanced L-parameters of the form $(\phi, \rho)$ with $\rho \in \mathrm{Irr}(\mathfrak{R}_\phi)$, where $\mathfrak{R}_\phi$ is as in (82). Such a parameter is cuspidal if and only if $(u_\phi, \rho)$ is a cuspidal pair for $Z_{\mathcal{H}^\vee}(\phi)$.

**Conjecture 6.10.** *Let $\mathcal{H}$ be any reductive p-adic group, and choose a character $\zeta_\mathcal{H}$ of $Z(\mathcal{H}^\vee_{\mathrm{sc}})$ whose restriction to $Z(\mathcal{H}^\vee_{\mathrm{sc}})^{\mathbf{W}_F}$ parametrizes $\mathcal{H}$ via the Kottwitz homomorphism. Under the local Langlands correspondence, $\Phi_{\mathrm{cusp},\zeta_\mathcal{H}}(\mathcal{H})$ is in bijection with the set of supercuspidal irreducible smooth $\mathcal{H}$-representations (up to isomorphism).*

Now we check that, in many cases where a local Langlands correspondence is known, Conjecture 6.10 holds.

**Example 6.11.** Let $F$ be a p-adic field, $D$ a division algebra over $F$ such that $\dim_F D = d^2$ and $\mathcal{H} = \mathrm{GL}_m(D)$. Then $\mathcal{H}$ is an inner form of $\mathrm{GL}_n(F)$ with $n = md$. Let $(\phi, \rho) \in \Phi_{\mathrm{cusp}}(\mathcal{H})$. We have $\mathcal{H}^\vee_{\mathrm{sc}} = \mathrm{SL}_n(\mathbb{C})$ and ${}^L\mathcal{H} = \mathrm{GL}_n(\mathbb{C}) \times \mathbf{W}_F$. Since $\phi$ is discrete, it is an irreducible representation of $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C})$ and

$$\mathcal{S}_\phi = \pi_0(Z_{\mathrm{SL}_n(\mathbb{C})}(\phi)) = Z(\mathrm{SL}_n(\mathbb{C})) \cong \mathbb{Z}/n\mathbb{Z}.$$

Because $(\phi, \rho)$ is relevant for $\mathcal{H}$, $\rho$ is a character of $\mathcal{S}_\phi$ of order $d$. Furthermore $\phi$ decomposes as

$$\phi = \pi \boxtimes S_\pi \text{ with } \pi \in \mathrm{Irr}(\mathbf{W}_F), S_\pi \in \mathrm{Irr}(\mathrm{SL}_2(\mathbb{C})).$$

Let $d'$ denote the dimension of $S_\pi$. We will use same argument as in [Lus2, p. 247]. Choose an isomorphism $M_n(\mathbb{C}) \cong M_{n/d'}(\mathbb{C}) \otimes M_{d'}(\mathbb{C})$ and let $1_{n/d'}$ be the multiplicative unit of the matrix algebra $M_{n/d'}(\mathbb{C})$. Then

$$G = Z_{\mathrm{SL}_n(\mathbb{C})}(\phi(\mathbf{W}_F)) \simeq \big(1_{n/d'} \otimes \mathrm{GL}_{d'}(\mathbb{C})\big) \cap \mathrm{SL}_n(\mathbb{C}).$$

Since we assume that $(\phi, \rho)$ is cuspidal, this implies that $u_\phi$ is in the regular unipotent class of $\mathrm{GL}_{d'}(\mathbb{C})$, and $Z(\mathrm{SL}_{d'}(\mathbb{C}))$ acts on $\rho$ by a character of order $d'$. The kernel of the $Z(\mathrm{SL}_n(\mathbb{C}))$-character $\rho$ consists precisely of the $d$-th powers in $Z(\mathrm{SL}_n(\mathbb{C}))$. This is possible if and only if no such $d$-th power is a nontrivial element of $1_{n/d'} \otimes Z(\mathrm{SL}_{d'}(\mathbb{C}))$. Thus the only additional condition on $d'$ becomes: $\mathrm{lcm}(d, n/d') = n$.

By the local Langlands correspondence for $\mathrm{GL}_m(D)$ (see [HiSa, §11] and [ABPS2, §2]) $\phi$ is associated to a unique essentially square integrable representation $\pi_\phi$ of $\mathrm{GL}_m(D)$. According to [DKV, Théorème B.2.b] $\pi_\rho$ is supercuspidal if and only if $\mathrm{lcm}(d, n/d') = n$. Consequently the LLC for $\mathrm{GL}_m(D)$ restricts to a bijection between $\Phi_{\mathrm{cusp}}(\mathrm{GL}_m(D))$ and $\mathrm{Irr}_{\mathrm{cusp}}(\mathrm{GL}_m(D))$.

We recover the case $\mathrm{GL}_n(F)$ when $D = F$ and $\phi = \pi$ is an irreducible representation of $\mathbf{W}_F$. An other case is when $\mathcal{H} = \mathrm{GL}_1(D)$ with $d = 2$. We find that the cuspidal L-parameters of $\mathrm{GL}_1(D)$ come in two forms:

- $(\pi, \mathrm{id}_{Z(\mathrm{SL}_2(\mathbb{C}))})$ with $\pi$ an irreducible two-dimensional representation of $\mathbf{W}_F$;
- $(\chi \boxtimes S_2, \mathrm{id}_{Z(\mathrm{SL}_2(\mathbb{C}))})$, with $\chi$ a character of $\mathbf{W}_F$ and $S_2$ the irreducible two-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

The Langlands parameter in the latter case corresponds to the character $\hat{\chi} \circ \mathrm{Nrd}$ of $GL_1(D)$ and to the $GL_2(F)$-representation $\hat{\chi} \circ \det \otimes \mathrm{St}_{\mathrm{GL}_2(F)}$. These two representations are connected by the Jacquet–Langlands correspondence.

**Example 6.12.** Let $F$ be a $p$-adic field, and let $\mathcal{H}$ be a symplectic group $\mathrm{Sp}_{2n}(F)$ or a split special orthogonal group $\mathrm{SO}_m(F)$. We have $^L\mathcal{H} = \mathcal{H}^\vee \times \mathbf{W}_F$. Then [Mou, Proposition 4.14] shows, using results of Arthur and Mœglin, that the supercuspidal irreducible representations of $\mathcal{H}$ correspond, via the local Langlands correspondence, to cuspidal enhanced L-parameters.

**Example 6.13.** Let $F$ be a $p$-adic field and $E$ a quadratic extension of $F$. Let $\mathcal{H} = U_n(F)$ be the quasi-split unitary group defined over $F$ and split over $E$. We have $^L\mathcal{H} = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$. Let $\phi \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L\mathcal{H}$ be a discrete Langlands parameter and fix $\sigma \in \mathbf{W}_F$ such that $\mathbf{W}_F/\mathbf{W}_E \simeq \langle \sigma \rangle$. We use the notions of conjugate-dual, conjugate-orthogonal and conjugate-symplectic defined in [GGP, §3]. We can decompose the restriction of $\phi$ to $\mathbf{W}_E$ as an $n$-dimensional representation:

$$(93) \qquad \phi|_{\mathbf{W}_E} = \bigoplus_{\pi \in I_{\mathrm{O}}^E} m_\pi \pi \ \oplus \ \bigoplus_{\pi \in I_{\mathrm{S}}^E} m_\pi \pi \ \oplus \ \bigoplus_{\pi \in I_{\mathrm{GL}}^E} m_\pi \left( \pi \oplus {}^\sigma \pi^\vee \right),$$

where

- $I_{\mathrm{O}}^E$ is a set of irreducible conjugate-orthogonal representations of $\mathbf{W}_E$ ;
- $I_{\mathrm{S}}^E$ is a set of irreducible conjugate-symplectic representations of $\mathbf{W}_E$ ;
- $I_{\mathrm{GL}}^E$ is a set of irreducible representations of $\mathbf{W}_E$ which are not conjugate-dual.

Then, by [GGP, p.15]

$$Z_{\mathcal{H}^\vee}(\phi(\mathbf{W}_F)) \simeq \prod_{\pi \in I_{\mathrm{O}}^E} \mathrm{O}_{m_\pi}(\mathbb{C}) \times \prod_{\pi \in I_{\mathrm{S}}^E} \mathrm{Sp}_{m_\pi}(\mathbb{C}) \times \prod_{\pi \in I_{\mathrm{GL}}^E} \mathrm{GL}_{m_\pi}(\mathbb{C}).$$

Every term $m_\pi \pi$ in (93) can be decomposed as $\oplus_a \pi \boxtimes S_a$, where $S_a$ denotes the $a$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Here $a$ runs through some subset of $\mathbb{N}$ – every $a$ appears at most once because $\phi$ is discrete. For every such $(\pi, a)$ we choose an element $z_{\pi,a} \in A_{\mathrm{GL}_n(\mathbb{C})}(\phi)$ which acts as $-1$ on $\pi \boxtimes S_a$ and as the identity on all other factors $\pi' \boxtimes S_{a'}$.

From now on we assume that $\phi$ can be enhanced to a cuspidal L-parameter. The above and the classification of cuspidal pairs in [Lus2] show that $u_\phi = (u_{\phi,\pi})$ satisfies:

- if $\pi \in I_O^E$, then the partition associated to $u_{\phi,\pi}$ is $(1, 3, \ldots, 2d_\pi - 1)$,
  $A_{\mathrm{O}_{m_\pi}(\mathbb{C})}(u_{\phi,\pi}) = \prod_{a=1}^{d_\pi} \langle z_{\pi,2a-1} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{d_\pi}$ and $\varepsilon \in \mathrm{Irr}(A_{\mathrm{O}_{m_\pi}(\mathbb{C})}(u_{\phi,\pi}))$ is
  given by $\varepsilon(z_{\pi,2a-1}) = (-1)^a$ or $\varepsilon(z_{\pi,2a-1}) = (-1)^{a+1}$;
- if $\pi \in I_S^E$, then the partition associated to $u_{\phi,\pi}$ is $(2, 4, \ldots, 2d_\pi)$,
  $A_{\mathrm{Sp}_{m_\pi}(\mathbb{C})}(u_{\phi,\pi}) = \prod_{a=1}^{d_\pi} \langle z_{\pi,2a} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{d_\pi}$ and $\varepsilon \in \mathrm{Irr}(A_{\mathrm{Sp}_{m_\pi}(\mathbb{C})}(u_{\phi,\pi}))$ is
  given by $\varepsilon(z_{\pi,2a}) = (-1)^a$;
- if $\pi \in I_{\mathrm{GL}}^E$, then $m_\pi = 1$ and $u_{\phi,\pi} = 1$.

Because $\phi$ is discrete, $I_{\mathrm{GL}}^E$ is empty. Hence

$$(94) \qquad \phi|_{\mathbf{W}_E \times \mathrm{SL}_2(\mathbb{C})} = \bigoplus_{\pi \in I_O^E} \bigoplus_{a=1}^{d_\pi} \pi \boxtimes S_{2a-1} \ \oplus \bigoplus_{\pi \in I_S^E} \bigoplus_{a=1}^{d_\pi} \pi \boxtimes S_{2a}.$$

Moreover, in [Moe, Théorème 8.4.4], Mœglin classified the supercuspidal representations in an Arthur packet. In particular, for tempered Langlands parameters (which are Arthur parameters trivial on the second copy of $\mathrm{SL}_2(\mathbb{C})$), the description is given in term of a Jordan block and a character defined by this Jordan block. Here the Jordan block $\mathrm{Jord}(\phi)$ of the Langlands parameter $\phi$ of a supercuspidal representation of $\mathcal{H}$ is the set of pairs $(\pi, a)$, where $\pi$ is an irreducible representation of $\mathbf{W}_E$ stable under the action of the composition of inverse-transpose and $\sigma$, and $a$ is an integer such that $\pi \boxtimes S_a$ is a subrepresentation of $\phi|_{\mathbf{W}_E}$.

The condition on the Jordan block says that it has no holes (or is without jumps). More explicitly, for all $a > 2$, if $(\pi, a) \in \mathrm{Jord}(\phi)$ then $(\pi, a-2) \in \mathrm{Jord}(\phi)$. The shape of $\phi$ is then as (94). Moreover, the alternated characters are exactly the cuspidal ones. More precisely, [Moe, p.194] gives the definition $z_{\pi,a}$ as our $z_{\pi,a} z_{\pi,a-2}$ (or $z_{\pi,2}$ in the case of $a = 2$). But the cuspidal characters are exactly the characters which are alternated, i.e. such that $\varepsilon(z_{\pi,a} z_{\pi,a-2}) = -1$.

**Example 6.14.** Let $\phi$ be a relevant discrete L-parameter which is trivial on the wild inertia subgroup $\mathfrak{P}_F$ of the inertia group $\mathbf{I}_F$, and such that the centralizer of $\phi(\mathbf{I}_F)$ in $\mathcal{H}^\vee$ is a torus. The latter condition forces $\phi$ to be trivial on $\mathrm{SL}_2(\mathbb{C})$. Hence $u_\phi = 1$, and any enhancement of $\phi$ gives a cuspidal L-parameter. Let

$$C_\phi = \pi_0\big(Z_{\mathcal{H}^\vee}(\phi)/Z({}^L\mathcal{H})^\circ\big)$$

and let $\rho \in \mathrm{Irr}(C_\phi)$. It is known from [DeRe] that these enhanced L-parameters $(\phi, \rho)$ correspond to the depth-zero generic supercuspidal irreducible representations of $\mathcal{H}$, in the case where $\mathcal{H}$ is a pure inner form of an unramified reductive $p$-adic group. We note that the component group $C_\phi$ is a quotient of our $\mathcal{S}_\phi$, namely by the kernel of $\mathcal{H}_{\mathrm{sc}}^\vee \to \mathcal{H}^\vee$. A priori in these references only a subset of our enhancements of $\phi$ is considered. However, it boils down to the same, because the $p$-adic group $\mathcal{H}$ is chosen such that $\rho$ is relevant for it [DeRe, §2].

**Example 6.15.** Let $(\phi, \rho)$ be a relevant enhanced L-parameter such that $\phi$ is discrete and trivial on $\mathfrak{P}_F^{(r+1)}$ and nontrivial on $\mathfrak{P}_F^{(r)}$ for some integer $r > 0$, and such that the centralizer in $\mathcal{H}^\vee$ of $\phi(\mathfrak{P}_F^{(r)})$ is a maximal torus of $\mathcal{H}^\vee$. Again any such $(\phi, \rho)$ is cuspidal. The same argument as in Example 6.14 shows that the result of Reeder in [Ree2, §6] implies that these enhanced L-parameters correspond to the depth $r$ generic supercuspidal irreducible representations of $\mathcal{H}$, when $\mathcal{H}$ is a pure inner form of an unramified reductive $p$-adic group.

## 7. The cuspidal support of enhanced L-parameters

In the representation theory of $p$-adic groups Bernstein's cuspidal support map (see [BeDe, §2] or [Ren, VI.7.1]) plays an important role. It assigns to every irreducible smooth $\mathcal{H}$-representation $\pi$ a Levi subgroup $\mathcal{L}$ of $\mathcal{H}$ and a supercuspidal $\mathcal{L}$-representation $\sigma$, such that $\pi$ is contained in the normalized parabolic induction of $\sigma$. This condition determines $(\mathcal{L}, \sigma)$ uniquely up to $\mathcal{H}$-conjugacy. It is common to call $(\mathcal{L}, \sigma)$ a cuspidal pair for $\mathcal{H}$. The cuspidal support of $\pi \in \mathrm{Irr}(\mathcal{H})$ is a $\mathcal{H}$-conjugacy class of cuspidal pairs, often denoted by $\mathbf{Sc}(\pi)$.

It is expected that $\mathbf{Sc}$ relates very well to the LLC. In fact this is a special case of a conjecture about the relation with parabolic induction, see [Hai, Conjecture 5.22] and [ABPS6, §1.5]. Suppose that $\mathcal{P} = \mathcal{L}\mathcal{U}_{\mathcal{P}}$ is a parabolic subgroup of $\mathcal{H}$, that $\phi \in \Phi(\mathcal{L})$ and $\sigma \in \Pi_{\phi}(\mathcal{L})$. Then the L-packet $\Pi_{\phi}(\mathcal{H})$ should consist of constituents of the normalized parabolic induction $I_{\mathcal{P}}^{\mathcal{H}}(\sigma)$.

We will define an analogue of $\mathbf{Sc}$ for enhanced L-parameters. In this setting a cuspidal pair for $^L\mathcal{H}$ should become a triple $(\mathcal{L}^{\vee} \rtimes \mathbf{W}_F, \phi, \rho)$, where $\mathcal{L}^{\vee} \rtimes \mathbf{W}_F$ is the L-group of a Levi subgroup $\mathcal{L} \subset \mathcal{H}$ and $(\phi, \rho)$ is a cuspidal L-parameter for $\mathcal{L}$. However, the collection of such objects is not stable under $\mathcal{H}^{\vee}$-conjugation, because $h\mathcal{L}^{\vee}h^{-1}$ need not be $\mathbf{W}_F$-stable. To allow $\mathcal{H}^{\vee}$ to act on these triples, we must generalize Definition 6.9 in a less restrictive way.

**Definition 7.1.** Let $^LL$ be a Levi L-subgroup of $^L\mathcal{H}$. A Langlands parameter for $^LL$ is a group homomorphism $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {^LL}$ satisfying the requirements of Definition 6.4. An enhancement of $\phi$ is an irreducible representation $\rho$ of $\pi_0(Z_{L_{\mathrm{sc}}}^1(\phi))$, where $L_{\mathrm{sc}}$ is the simply connected cover of the derived group of $L = {^LL} \cap \mathcal{H}^{\vee}$. The group $L$ acts on the collection of enhanced L-parameters for $^LL$ by (87).

We say that $(\phi, \rho)$ is cuspidal for $^LL$ if $\phi$ is discrete for $^LL$ and $\left(u_{\phi} = \phi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right), \rho\right)$ is a cuspidal pair for $Z_{L_{\mathrm{sc}}}^1(\phi|_{\mathbf{W}_F})$. We denote the set of $L$-orbits by $\Phi_e(^LL)$ and the subset of cuspidal $L$-orbits by $\Phi_{\mathrm{cusp}}(^LL)$.

We remark that in this definition it is not specified for which $p$-adic group an enhanced L-parameter for $^LL$ is relevant. Hence $\Phi_e(^L\mathcal{L})$ is in general strictly larger than $\Phi_e(\mathcal{L})$, it also contains enhanced L-parameters for inner forms of $\mathcal{L}$.

Let $L_c$ be the pre-image of $L$ under under $\mathcal{H}_{\mathrm{sc}}^{\vee} \to \mathcal{H}^{\vee}$. Since $L$ is a Levi subgroup of $\mathcal{H}^{\vee}$, the derived group of $L_c$ is the simply connected cover of $L_{\mathrm{der}}$. Thus we identify $L_{\mathrm{sc}}$ with the inverse image of $L_{\mathrm{der}}$ under $\mathcal{H}_{\mathrm{sc}}^{\vee} \to \mathcal{H}^{\vee}$.

**Definition 7.2.** A cuspidal datum for $^L\mathcal{H}$ is a triple $(^LL, \phi, \rho)$ as in Definition 7.1, such that $(\phi, \rho)$ is cuspidal for $^LL$. It is relevant for $\mathcal{H}$ if

- $\rho = \zeta$ on $L_{\mathrm{sc}} \cap Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F}$, where $\zeta \in \mathrm{Irr}(Z(\mathcal{H}_{\mathrm{sc}}^{\vee})^{\mathbf{W}_F})$ parametrizes the inner twist $\mathcal{H}$ via the Kottwitz isomorphism (88).
- $\rho = 1$ on $L_{\mathrm{sc}} \cap Z(L_c)^{\circ}$.

For $h \in \mathcal{H}_{\mathrm{sc}}^{\vee}$ the conjugation action

$$L \to hLh^{-1} : l \mapsto hlh^{-1}$$

stabilizes the derived group of $L$ and lifts to $L_{\mathrm{sc}} \to (hLh^{-1})_{\mathrm{sc}}$. Using this, $\mathcal{H}_{\mathrm{sc}}^{\vee}$ and $\mathcal{H}^{\vee}$ act naturally on cuspidal data for $^L\mathcal{H}$ by

$$h \cdot (^LL, \phi, \rho) = (h\,^LLh^{-1}, h\phi h^{-1}, h \cdot \rho).$$

By Lemma 6.2 every cuspidal datum for ${}^L\mathcal{H}$ is $\mathcal{H}^\vee$-conjugate to one of the form $(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi, \rho)$, where $\mathcal{L}^\vee$ is a $\mathbf{W}_F$-stable standard Levi subgroup of $\mathcal{H}^\vee$. For $\zeta_\mathcal{H} \in \mathrm{Irr}(Z(\mathcal{H}^\vee_{\mathrm{sc}}))$ we write

$$\Phi_{e,\zeta_\mathcal{H}}({}^L L) = \{(\phi,\rho) \in \Phi_e({}^L L) : \rho = \zeta_\mathcal{H} \text{ on } L_{\mathrm{sc}} \cap Z(\mathcal{H}^\vee_{\mathrm{sc}}), \rho = 1 \text{ on } L_{\mathrm{sc}} \cap Z(L_c)^\circ\},$$

$$\Phi_{\mathrm{cusp},\zeta_\mathcal{H}}({}^L L) = \Phi_{\mathrm{cusp}}({}^L L) \cap \Phi_{e,\zeta_\mathcal{H}}({}^L L).$$

This depends only on the restriction of $\zeta_\mathcal{H}$ to the subgroup $Z(L_{\mathrm{sc}}) \subset Z(\mathcal{H}^\vee_{\mathrm{sc}})$.

Often we will be interested in cuspidal data up to $\mathcal{H}^\vee$-conjugacy. Upon fixing the first ingredient of $({}^L L, \phi, \rho)$, we can consider $(\phi, \rho)$ as a cuspidal L-parameter for ${}^L L$, modulo $L$-conjugacy. Recall from (90) that $\phi$ is determined up to $L$-conjugacy by $\phi|_{\mathbf{W}_F}$ and $u_\phi$. Hence the quadruple

$$(95) \qquad ({}^L L, \phi|_{\mathbf{W}_F}, u_\phi, \rho)$$

determines a unique $\mathcal{H}^\vee$-conjugacy class of cuspidal data. Therefore we will also regard quadruples of the form (95) as cuspidal data for ${}^L\mathcal{H}$.

Let $\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$ be the set of supercuspidal $\mathcal{L}$-representations and let $\sigma_1, \sigma_2 \in \mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$. We note that the cuspidal pairs $(\mathcal{L}, \sigma_1)$ and $(\mathcal{L}, \sigma_2)$ are $\mathcal{H}$-conjugate if and only if $\sigma_1$ and $\sigma_2$ are in the same orbit under

$$(96) \qquad W(\mathcal{H}, \mathcal{L}) = N_\mathcal{H}(\mathcal{L})/\mathcal{L}.$$

Recall from [ABPS6, Proposition 3.1] that there is a canonical isomorphism

$$(97) \qquad W(\mathcal{H}, \mathcal{L}) \cong N_{\mathcal{H}^\vee}(\mathcal{L}^\vee \rtimes \mathbf{W}_F)/\mathcal{L}^\vee.$$

Motivated by (97) we write, for any Levi L-subgroup ${}^L L$ of ${}^L\mathcal{H}$:

$$W({}^L\mathcal{H}, {}^L L) := N_{\mathcal{H}^\vee}({}^L L)/L.$$

This group acts naturally on the collection of cuspidal data for ${}^L\mathcal{H}$ with first ingredient ${}^L L$. Two cuspidal data

$$(98) \quad ({}^L L, \phi_1, \rho_1) \text{ and } ({}^L L, \phi_2, \rho_2) \text{ are } \mathcal{H}\text{-conjugate} \qquad \Longleftrightarrow$$

$$(\phi_1, \rho_1), (\phi_2, \rho_2) \in \Phi_{\mathrm{cusp}}({}^L L) \text{ are in the same orbit for the action of } W({}^L\mathcal{H}, {}^L L).$$

In the notation of (92), we use Section 5 (with complex representations and sheafs) to write

$$q\Psi_G(u_\phi, \rho) = [M, v, q\epsilon]_G, \text{ where } G = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F}).$$

**Proposition 7.3.** *Let $(\phi, \rho) \in \Phi_e(\mathcal{H})$.*

(a) *$(Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ), \phi|_{\mathbf{W}_F}, v, q\epsilon)$ is a $\mathcal{H}$-relevant cuspidal datum for ${}^L\mathcal{H}$.*

(b) *Upon replacing $(\phi, \rho)$ by a $\mathcal{H}^\vee$-conjugate representative L-parameter, there exists a Levi subgroup $\mathcal{L}$ of $\mathcal{H}$ such that:*
- *$Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ) = \mathcal{L}^\vee \rtimes \mathbf{W}_F$,*
- *$q\epsilon$ and $\rho$ yield the same character of $Z(\mathcal{H}^\vee_{\mathrm{sc}})Z(\mathcal{L}^\vee_c)^\circ$. It is trivial on $Z(\mathcal{L}^\vee_c)^\circ$ and determined by its restriction to $Z(\mathcal{L}^\vee_{\mathrm{sc}})$.*
- *$(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ is a cuspidal L-parameter for $\mathcal{L}$.*

(c) *The $\mathcal{H}^\vee$-conjugacy class of $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ is uniquely determined by $(\phi, \rho)$.*

*Proof.* (a) and (b) The torus $Z(M)^\circ$ commutes with $M$, so $Z_{\mathcal{H}^\vee}(Z(M)^\circ)$ is a Levi subgroup of $\mathcal{H}^\vee$ which contains the image of $M$ in $\mathcal{H}^\vee$. As $Z(M)^\circ \subset Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F})$, $Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ)$ is a Levi L-subgroup of $\mathcal{H}^\vee \rtimes \mathbf{W}_F$.

In view of Lemma 6.2 this implies that, upon conjugating $(\phi, \rho)$ with a suitable element of $\mathcal{H}^\vee$, we may assume that the above construction yields a $\mathbf{W}_F$-stable standard Levi subgroup $\mathcal{L}^\vee := Z_{\mathcal{H}^\vee}(Z(M)^\circ)$ with

$$\phi(\mathbf{W}_F) \subset Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ) = \mathcal{L}^\vee \rtimes \mathbf{W}_F.$$

Its pre-image $\mathcal{L}_c^\vee$ in $\mathcal{H}_{\mathrm{sc}}^\vee$ satisfies

$$(99) \qquad G \cap \mathcal{L}_c^\vee = Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi|_{\mathbf{W}_F}) \cap Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(Z(M)^\circ) = M.$$

Moreover $\mathcal{L}^\vee$ contains $v$ (or rather its image in $\mathcal{H}^\vee$, which we also denote by $v$). Suppose that $^L L$ is another Levi L-subgroup of $^L \mathcal{H}$ which contains $\phi(\mathbf{W}_F) \cup \{v\}$. Let $L_c$ be the inverse image of $L = {}^L L \cap \mathcal{H}^\vee$ in $\mathcal{H}_{\mathrm{sc}}^\vee$. Since $(v, q\epsilon)$ is a cuspidal pair for $M$, $M^\circ$ is a Levi subgroup of $G^\circ$ minimally containing $v$ (see [Lus2, Proposition 2.8] or Theorem 3.1.a). Hence $L_c \cap G$ contains a $Z_G(v)$-conjugate of $M^\circ$, say $zM^\circ z^{-1}$. Then $Z(L_c)^\circ \subset zZ(M)^\circ z^{-1}$, so

$$(100) \qquad L_c = Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(Z(L_c)^\circ) \supset Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(zZ(M)^\circ z^{-1}) = z\mathcal{L}_c^\vee z^{-1}.$$

Thus $L$ contains a conjugate of $\mathcal{L}^\vee$. Equivalently $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ minimally contains $\phi(\mathbf{W}_F) \cup \{v\}$. Hence $(\phi|_{\mathbf{W}_F}, v)$ is a discrete L-parameter for $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ and for some $F$-group $\mathcal{L}$ with complex dual $\mathcal{L}^\vee$.

By (91) $\rho \in \mathrm{Irr}\big(\pi_0(Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi))\big)$ can be regarded as a representation of $\pi_0(Z_G(u_\phi))$, and by (68) it has the same $Z(\mathcal{H}_{\mathrm{sc}}^\vee)$-character, say $\zeta$, as $q\epsilon \in \mathrm{Irr}(\pi_0(Z_M(v)))$.

Because $Z(M)^\circ$ becomes the trivial element in $\pi_0(Z_M(v))$, $\zeta$ is trivial on $Z(M)^\circ \cap Z(\mathcal{H}_{\mathrm{sc}}^\vee)$, We note that

$$(101) \qquad G \cap Z(\mathcal{L}_c^\vee)^\circ = G \cap Z\big(Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(Z(M)^\circ)\big)^\circ \subset G \cap Z(M)^\circ = Z(M)^\circ.$$

But by construction $Z(M)^\circ \subset Z(\mathcal{L}_c^\vee)^\circ$, so (101) is actually an equality. As $Z(\mathcal{H}_{\mathrm{sc}}^\vee) \subset G$, it follows that $\zeta$ is also trivial on $Z(\mathcal{L}_c^\vee)^\circ \cap Z(\mathcal{H}_{\mathrm{sc}}^\vee)$. In particular it extends uniquely to a character (still denoted $\zeta$) of $Z(\mathcal{H}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ$, which is trivial on $Z(\mathcal{L}_c^\vee)^\circ$.

Furthermore $\mathcal{L}_c^\vee$ is a connected Lie group, so $\mathcal{L}_c^\vee = \mathcal{L}_{\mathrm{sc}}^\vee Z(\mathcal{L}_c^\vee)^\circ$. From this we see that $\zeta$ is determined by its restriction to $\mathcal{L}_{\mathrm{sc}}^\vee \cap Z(\mathcal{H}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ$. By [Art1, Lemma 1.1] (see also Lemma 6.6.a), that group can be simplified to

$$\mathcal{L}_{\mathrm{sc}}^\vee \cap Z(\mathcal{H}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ = \mathcal{L}_{\mathrm{sc}}^\vee \cap Z(\mathcal{L}_c^\vee) = Z(\mathcal{L}_{\mathrm{sc}}^\vee).$$

Although $Z_{\mathcal{L}_{\mathrm{sc}}^\vee}(\phi) = Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi) \cap \mathcal{L}_{\mathrm{sc}}^\vee$, the inclusion $Z_{\mathcal{L}_{\mathrm{sc}}^\vee}^1(\phi) \supset Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) \cap \mathcal{L}_{\mathrm{sc}}^\vee$ can be strict, as the definitions of the two $Z^1$'s are different. Nevertheless, always

$$(102) \qquad Z_{\mathcal{L}_{\mathrm{sc}}^\vee}^1(\phi) \subset (Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) \cap \mathcal{L}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ.$$

Hence the relevant centralizers for $(\mathcal{L}, \phi|_{\mathbf{W}_F}, v)$ are

$$Z_{\mathcal{L}_{\mathrm{sc}}^\vee}^1(\phi|_{\mathbf{W}_F}) \cap Z_{\mathcal{L}_{\mathrm{sc}}^\vee}(v) \subset (G \cap Z_{\mathcal{L}_{\mathrm{sc}}^\vee}(v))Z(\mathcal{L}_c^\vee)^\circ = Z_{M_{\mathrm{der}}}(v)Z(\mathcal{L}_c^\vee)^\circ.$$

Since $q\epsilon \in \mathrm{Irr}(A_M(v))$ is trivial on $Z(M)^\circ = Z(\mathcal{L}_c^\vee)^\circ \cap M$, it can be considered as a representation of $\pi_0\big(Z_{\mathcal{L}_{\mathrm{sc}}^\vee}^1(\phi|_{\mathbf{W}_F}) \cap Z_{\mathcal{L}_{\mathrm{sc}}^\vee}(v)\big)$ which is trivial on $Z(\mathcal{L}_{\mathrm{sc}}^\vee) \cap Z(\mathcal{L}_c^\vee)^\circ$. We conclude that $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ is a cuspidal Langlands parameter for some inner form of $\mathcal{L}$. The $Z(\mathcal{L}_{\mathrm{sc}}^\vee)$-character of $q\epsilon$ is obtained from that of $\rho$ via extension to $Z(\mathcal{H}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ$ and then restriction. Comparing with Lemma 6.6.b, and recalling that $(\phi, \rho)$ is relevant for $\mathcal{H}$, we see that $(\phi|_{\mathbf{W}_F}, v, q\epsilon)$ is relevant for a Levi subgroup $\mathcal{L}$ of $\mathcal{H}$.

By Definition 7.2 relevance of cuspidal data can be read off from their $Z(\mathcal{H}_{\mathrm{sc}}^\vee)^{\mathbf{W}_F}$-characters. The same comparison involving $\zeta$ says that $(\mathcal{L}^\vee \rtimes \mathbf{W}_F, \phi|_{\mathbf{W}_F}, v, q\epsilon)$ is

also $\mathcal{H}$-relevant.

(c) Suppose that $^L L$ is as above and that it minimally contains $\phi(\mathbf{W}_F) \cup \{v\}$. From (100) or [Bor, Proposition 8.6] we see that $^L L$ is $\mathcal{H}^\vee$-conjugate to $\mathcal{L}^\vee \rtimes \mathbf{W}_F$. Hence the L-Levi subgroup $\mathcal{L}^\vee \rtimes \mathbf{W}_F$ is uniquely determined up to conjugation. $\qquad\square$

Before we continue with the cuspidal support map, we work out some consequences of the above proof.

**Lemma 7.4.** (a) *The exists a character $\zeta_\mathcal{H} \in \mathrm{Irr}(Z(\mathcal{H}_{\mathrm{sc}}^\vee))$ such that:*
- $\zeta_\mathcal{H}|_{Z(\mathcal{H}_{\mathrm{sc}}^\vee)^{\mathbf{W}_F}}$ *parametrizes the inner twist $\mathcal{H}$ via the Kottwitz isomorphism (88),*
- $\zeta_\mathcal{H} = 1$ *on $Z(\mathcal{H}_{\mathrm{sc}}^\vee) \cap Z(\mathcal{L}_c^\vee)^\circ$, for every Levi subgroup $\mathcal{L}$ of $\mathcal{H}$.*

(b) *Let $\mathcal{L} \subset \mathcal{H}$ be a Levi subgroup and let $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{L}$ be a Langlands parameter for $\mathcal{L}$. There exists a natural injection $\mathfrak{R}_\phi^\mathcal{L} \to \mathfrak{R}_\phi$.*

(c) *In the setting of parts (a) and (b), extend $\zeta_\mathcal{H}$ to a character of $Z(\mathcal{H}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ$ which is trivial on $Z(\mathcal{L}_c^\vee)^\circ$. Let $\zeta_\mathcal{H}^\mathcal{L}$ be the restriction of the latter character to $Z(\mathcal{L}_{\mathrm{sc}}^\vee)$. Let $p_{\zeta_\mathcal{H}} \in \mathbb{C}[\mathcal{Z}_\phi]$ and $p_{\zeta_\mathcal{H}^\mathcal{L}} \in \mathbb{C}[\mathcal{Z}_\phi^\mathcal{L}]$ be the central idempotents associated to these characters. Then there is a canonical injection*

$$p_{\zeta_\mathcal{H}^\mathcal{L}}\mathbb{C}[\mathcal{S}_\phi^\mathcal{L}] \to p_{\zeta_\mathcal{H}}\mathbb{C}[\mathcal{S}_\phi].$$

*Proof.* (a) Let $\mathcal{L}$ be a minimal Levi subgroup of $\mathcal{H}$ and let $\phi \in \Phi(\mathcal{L})$ be a discrete Langlands parameter which is trivial on $\mathrm{SL}_2(\mathbb{C})$. Then $\phi$ is $\mathcal{H}$-relevant, so by Proposition 6.8 there exists an enhancement $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$ such that the character $\zeta_\rho$ of $Z(\mathcal{H}_{\mathrm{sc}}^\vee)$ determined by $\rho$ parametrizes $\mathcal{H}$ via the Kottwitz isomorphism. Then

$$G^\circ = Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi)^\circ = \left(Z(\mathcal{L}_c^\vee)^{\mathbf{W}_F}\right)^\circ$$

is a torus, so every element of $\mathcal{N}_G^+$ is cuspidal. It follows that

$$q\Psi_G(u_\phi = 1, \rho) = [G, v = 1, q\epsilon]_G.$$

Now Proposition 7.3.b yields the desired condition for $\mathcal{L}$.

Then the same condition holds for any Levi subgroup $\mathcal{M}$ of $\mathcal{H}$ containing $\mathcal{L}$, for $Z(\mathcal{M}_c^\vee)^\circ \subset Z(\mathcal{L}_c^\vee)^\circ$. Moreover $\zeta_\mathcal{H}$ is invariant under conjugation, because it lives only on the centre. So the condition even holds for all $\mathcal{H}_{\mathrm{sc}}^\vee$-conjugates of $\mathcal{M}_c^\vee$, which means that it is satisfied for all Levi subgroups of $\mathcal{H}$.

(b) There is an obvious map

$$(103) \qquad Z_{\mathcal{L}^\vee}(\phi) \to \mathfrak{R}_\phi = Z_{\mathcal{H}^\vee}(\phi)\big/Z_{\mathcal{H}^\vee}(\phi)^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F}.$$

Its kernel equals

$$(104) \quad Z_{\mathcal{L}^\vee}(\phi) \cap Z_{\mathcal{H}^\vee}(\phi)^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F} = Z_{\mathcal{H}^\vee}(Z(\mathcal{L}^\vee)^\circ) \cap Z_{\mathcal{H}^\vee}(\phi)^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F}$$

$$= \left(Z_{\mathcal{H}^\vee}(Z(\mathcal{L}^\vee)^\circ) \cap Z_{\mathcal{H}^\vee}(\phi)^\circ\right)Z(\mathcal{H}^\vee)^{\mathbf{W}_F} = Z_{\mathcal{L}^\vee}(\phi)^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F}.$$

For the last equality we used that taking centralizers with tori preserves connectedness. We note that $Z_{\mathcal{L}^\vee}(\phi)^\circ \subset Z_{\mathcal{H}^\vee}(\phi)^\circ$. By [Art1, Lemma 1.1]

$$Z(\mathcal{L}^\vee)^{\mathbf{W}_F} = (Z(\mathcal{L}^\vee)^{\mathbf{W}_F})^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F},$$

which is contained in $Z_{\mathcal{H}^\vee}(\phi)^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F}$. Hence (103) factors through

$$\mathfrak{R}_\phi^\mathcal{L} = Z_{\mathcal{L}^\vee}(\phi)\big/Z_{\mathcal{L}^\vee}(\phi)^\circ Z(\mathcal{L}^\vee)^{\mathbf{W}_F}.$$

By (104) the kernel of the just constructed map $\mathfrak{R}_\phi^{\mathcal{L}} \to \mathfrak{R}_\phi$ is the image of $Z_{\mathcal{L}^\vee}(\phi)^\circ Z(\mathcal{H}^\vee)^{\mathbf{W}_F}$ in $\mathfrak{R}_\phi^{\mathcal{L}}$, which is only the neutral element.

(c) Lemma 6.6.a (for the trivial $\mathbf{W}_F$-action) shows that $\zeta_{\mathcal{H}}^{\mathcal{L}}$ is well-defined. By (86) every system of representatives for $\mathfrak{R}_\phi \cong \mathcal{S}_\phi / \mathcal{Z}_\phi$ in $\mathcal{S}_\phi$ provides a basis of $p_{\zeta_{\mathcal{H}}} \mathbb{C}[\mathcal{S}_\phi]$. Similarly

(105) $$p_{\zeta_{\mathcal{H}}^{\mathcal{L}}} \mathbb{C}[\mathcal{S}_\phi^{\mathcal{L}}] \cong \mathbb{C}[\mathfrak{R}_\phi^{\mathcal{L}}] \text{ as vector spaces.}$$

We have to find an appropriate variation on $\mathbb{C}[\mathfrak{R}_\phi^{\mathcal{L}}] \to \mathbb{C}[\mathfrak{R}_\phi]$. Recall from (102) that

(106) $$Z_{\mathcal{L}_{\mathrm{sc}}^\vee}^1(\phi) = (Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) \cap \mathcal{L}_{\mathrm{sc}}^\vee)(Z(\mathcal{L}_c^\vee)^\circ \cap \mathcal{L}_{\mathrm{sc}}^\vee).$$

This gives a group homomorphism

(107) $$\lambda : Z_{\mathcal{L}_{\mathrm{sc}}^\vee}^1(\phi) \to Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) / \big( Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) \cap Z(\mathcal{L}_c^\vee)^\circ \cap \mathcal{L}_{\mathrm{sc}}^\vee \big)$$

which lifts $\mathfrak{R}_\phi^{\mathcal{L}} \to \mathfrak{R}_\phi$. Consider the diagram

$$
\begin{array}{ccc}
p_{\zeta_{\mathcal{H}}^{\mathcal{L}}} \mathbb{C}[\mathcal{S}_\phi^{\mathcal{L}}] & \lhook\joinrel\longrightarrow & \mathbb{C}[\mathcal{S}_\phi^{\mathcal{L}}] \\
\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\
p_{\zeta_{\mathcal{H}}} \mathbb{C}[\mathcal{S}_\phi] & \lhook\joinrel\longrightarrow & \mathbb{C}[Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) / \big( Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(\phi) \cap Z(\mathcal{L}_c^\vee)^\circ \cap \mathcal{L}_{\mathrm{sc}}^\vee \big)].
\end{array}
$$

The lower arrow exists because $\zeta_{\mathcal{H}} = 1$ on $Z(\mathcal{H}_{\mathrm{sc}}^\vee) \cap Z(\mathcal{L}_c^\vee)^\circ$. The image $\lambda(p_{\zeta_{\mathcal{H}}^{\mathcal{L}}} \mathbb{C}[\mathcal{S}_\phi^{\mathcal{L}}])$ is contained in $p_{\zeta_{\mathcal{H}}} \mathbb{C}[\mathcal{S}_\phi]$ by the relation between $\zeta_{\mathcal{H}}$ and $\zeta_{\mathcal{H}}^{\mathcal{L}}$, which gives the left vertical arrow. Since (107) is a lift of $\mathfrak{R}_\phi^{\mathcal{L}} \to \mathfrak{R}_\phi$ and by (105), this arrow is injective. $\qquad\square$

It turns out that the cuspidal datum constructed in Proposition 7.3.a need not have the same infinitesimal character as $\phi$ (in the sense of [Hai, Vog]). Since this would be desirable for a cuspidal support map, we now work out some constructions which compensate for this. See (108) for their effect.

Recall from (65) that the unipotent element $v$ in $q\Psi_G(u_\phi, \rho)$ also appears as $\Psi_{G^\circ}(u_\phi, \rho^\circ) = (M^\circ, v, \epsilon)$, where $\rho^\circ$ is an irreducible $A_{G^\circ}(u_\phi)$-constituent of $\rho$. The construction of $\Psi_{G^\circ}$, which already started in (16), entails that there exists a parabolic subgroup $P$ of $G^\circ$ such that

- $M^\circ$ is a Levi factor of $P$,
- $u_\phi = v u_P$ with $u_P$ in the unipotent radical $U_P$ of $P$.

Upon conjugating $\phi$ with a suitable element of $Z_{G^\circ}(u_\phi)$, we may assume that $M^\circ$ contains $\phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)$ for all $z \in \mathbb{C}^\times$. (Alternatively, one could conjugate $M^\circ$ inside $G^\circ$.) Since the $G^\circ$-conjugacy class of $(M^\circ, v)$ matters most, this conjugation is harmless.

**Lemma 7.5.** *Suppose that* $\phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big) \in M^\circ$ *for all* $z \in \mathbb{C}^\times$. *Then* $\phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big) v \phi\big(1, \big(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\big)\big) = v^{z^2}$ *for all* $z \in \mathbb{C}^\times$.

*Proof.* The condition on $M^\circ$ entails that

$$\mathrm{Ad} \circ \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)(v) \in M^\circ \quad \text{and} \quad \mathrm{Ad} \circ \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)(u_P) \in U_P.$$

Hence $\mathrm{Ad} \circ \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)(v)$ is the image of

$$\mathrm{Ad} \circ \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)(v u_P) = \mathrm{Ad} \circ \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)(u_\phi)$$

under $P/U_P \xrightarrow{\sim} M^\circ$. Since $\phi|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to G^\circ$ is an algebraic group homomorphism,

$$\mathrm{Ad} \circ \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)(u_\phi) = \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big) \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big) \big(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\big)\big)$$
$$= \phi\big(1, \big(\begin{smallmatrix} 1 & z^2 \\ 0 & 1 \end{smallmatrix}\big)\big) = u_\phi^{z^2} = (vu_P)^{z^2}.$$

By the unipotency of $vu_P$ there are unique $X \in \mathrm{Lie}(M^\circ), Y \in \mathrm{Lie}(U_P)$ such that $vu_P = \exp_P(X+Y)$. As $\mathrm{Lie}\,(U_P)$ is an ideal of $\mathrm{Lie}\,(P)$, $\exp_P(X+Y) \in \exp_{M^\circ}(X)U_P$, and hence $X = \log_{M^\circ}(v)$. Similarly we compute

$$(vu_P)^{z^2} = \exp_P(\log_P(vu_P)^{z^2}) = \exp_P(z^2(X+Y)) \in \exp_{M^\circ}(z^2 X)U_P.$$

Consequently the image of $(vu_P)^{z^2}$ under $P/U_P \xrightarrow{\sim} M^\circ$ is $\exp_{M^\circ}(z^2 X) = v^{z^2}$.  $\square$

In the setting of Lemma 7.5, [KaLu, §2.4] shows that there exists an algebraic group homomorphism $\gamma_v : \mathrm{SL}_2(\mathbb{C}) \to M^\circ$ such that

- $\gamma_v \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big) = v$,
- $\gamma_v(\mathrm{SL}_2(\mathbb{C}))$ commutes with $\phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)\gamma_v \big(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\big)$ for all $z \in \mathbb{C}^\times$.

Moreover $\gamma_v$ is unique up to conjugation by $Z_{M^\circ}\big(v, \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)\big)$, for any $z \in \mathbb{C}^\times$ of infinite order. We will say that a homomorphism $\gamma_v$ satisfying these conditions is adapted to $\phi$.

**Lemma 7.6.** *Let $(\phi, \rho)$ be an enhanced L-parameter for $\mathcal{H}$ and write $q\Psi_G(u_\phi, \rho) = [M, v, q\epsilon]_G$, using (92). Up to $G$-conjugacy there exists a unique $\gamma_v : \mathrm{SL}_2(\mathbb{C}) \to M^\circ$ adapted to $\phi$. Moreover the cocharacter*

$$\chi_{\phi,v} \colon z \mapsto \phi\big(1, \big(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\big)\big)\gamma_v \big(\begin{smallmatrix} z^{-1} & 0 \\ 0 & z \end{smallmatrix}\big)$$

*has image in $Z(M)^\circ$.*

*Proof.* Everything except the last claim was already checked above. Since $(v, q\epsilon)$ is cuspidal, Theorem 3.1.a says that $v$ is distinguished. This means that it does not lie in any proper Levi subgroup of $M^\circ$. In other words, every torus of $M^\circ$ which centralizes $v$ is contained in $Z(M^\circ)^\circ$. Finally we note that, as $M$ is a quasi-Levi subgroup, $Z(M)^\circ = Z(M^\circ)^\circ$.  $\square$

Notice that the image of the cocharacter $\chi_{\phi,v} : \mathbb{C}^\times \to Z(M)^\circ$ commutes not only with $\gamma_v(\mathrm{SL}_2(\mathbb{C}))$ but also with $\phi(\mathbf{W}_F)$, because $M^\circ \subset G^\circ \subset Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi|_{\mathbf{W}_F})$.

**Definition 7.7.** In the setting of Lemma 7.6 we put

$$^L\Psi(\phi, \rho) = (Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ), \phi|_{\mathbf{W}_F}, v, q\epsilon),$$

a $\mathcal{H}$-relevant cuspidal datum for $^L\mathcal{H}$.

Let $\|\cdot\| : \mathbf{W}_F \to \mathbb{R}_{>0}$ be the group homomorphism with $\|w\| = q$ if $w(f) = f^q$ for all $f$ in the algebraic closure of the residue field of $F$.

We define a L-parameter $\varphi_v \colon \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ)$ by

$$\varphi_v(w, x) = \phi(w)\chi_{\phi,v}(\|w\|^{1/2})\gamma_v(x).$$

The cuspidal support of $(\phi, \rho)$ is

$$\mathbf{Sc}(\phi, \rho) = (Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(M)^\circ), \varphi_v, q\epsilon),$$

another $\mathcal{H}$-relevant cuspidal datum for $^L\mathcal{H}$.

By parts (a) and (c) of Proposition 7.3 the map $^L\Psi$ is canonical in the sense that its image is unique up to conjugation. By Lemma 7.6 **Sc** is also canonical. Furthermore, the images of $^L\Psi$ and **Sc** are $\mathcal{H}$-relevant by Proposition 7.3.b. In view of Proposition 7.3.c, we can always represent $^L\Psi(\phi,\rho)$ and $\mathbf{Sc}(\phi,\rho)$ by a cuspidal L-parameter for a Levi subgroup of $\mathcal{H}$.

An advantage of $\varphi_v$ over $(\phi|_{\mathbf{W}_F}, v)$ is that

(108)
$$\varphi_v\big(w, \big(\begin{smallmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{smallmatrix}\big)\big) = \phi(w)\chi_{\phi,v}(\|w\|^{1/2})\gamma_v\big(\begin{smallmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{smallmatrix}\big)$$
$$= \phi\big(w, \big(\begin{smallmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{smallmatrix}\big)\big).$$

In the terminology from [Hai, Vog], this says that the cuspidal support map for enhanced L-parameters preserves infinitesimal characters. It is interesting to compare the fibres of **Sc** with the variety constructed in [Vog, Corollary 4.6]. Vogan considers the set of all L-parameters for $^L\mathcal{H}$ with a fixed infinitesimal character (up to conjugation). In [Vog, Proposition 4.5] he proves that this set has the structure of a complex affine variety, on which $\mathcal{H}^\vee$ acts naturally, with only finitely many orbits. The same picture can be obtained from a fibre of **Sc**, upon neglecting all enhancements of L-parameters.

More or less by definition Bernstein's cuspidal support map for $\mathrm{Irr}(\mathcal{H})$ preserves infinitesimal characters. That property is slightly less strong for our **Sc** on the Galois side, for enhanced L-parameters with different cuspidal support can have the same infinitesimal character. The map

$$^L\Psi : \Phi_e(\mathcal{H}) \to \{\text{cuspidal data for } \mathcal{H}\}/\mathcal{H}^\vee\text{-conjugacy}$$

is an analogue of a modified version, say $\widetilde{\mathbf{Sc}}$, of Bernstein's cuspidal support map for $\mathrm{Irr}(\mathcal{H})$. Neither $^L\Psi$ nor $\widetilde{\mathbf{Sc}}$ preserve infinitesimal characters, but they have other advantages that the cuspidal support maps lack. For $^L\Psi$ this will become clear in the Section 9, while the importance of $\widetilde{\mathbf{Sc}}$ stems from its role in the ABPS conjecture.

To enable a comparison, we recall its definition from [ABPS6, §2.5]. Let $\mathcal{P} = \mathcal{M}\mathcal{U}_\mathcal{P}$ be a parabolic subgroup of $\mathcal{H}$ and let $\omega \in \mathrm{Irr}(\mathcal{M})$ be square-integrable modulo centre. Suppose that $\pi_t \in \mathrm{Irr}(\mathcal{H})$ is tempered and that it is a direct summand of the normalized parabolic induction $I_\mathcal{P}^\mathcal{H}(\omega)$. Let $(\mathcal{L}, \sigma)$ be the cuspidal support of $\omega$. Then $\sigma$ can be written uniquely as $\sigma = \sigma_u \otimes \nu$, with $\nu : \mathcal{L} \to \mathbb{R}_{>0}$ an unramified character and $\sigma_u \in \mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$ unitary (and hence tempered). One defines

$$\widetilde{\mathbf{Sc}}(\pi_t) = (\mathcal{L}, \sigma_u)/\mathcal{H}\text{-conjugacy}.$$

Notice that $\widetilde{\mathbf{Sc}}$ preserves temperedness of representations, in contrast with **Sc**.

More generally, by [Sol1, Theorem 2.15] every $\pi \in \mathrm{Irr}(\mathcal{H})$ can be written (in an essentially unique way) as a Langlands quotient of $I_\mathcal{P}^\mathcal{H}(\omega \otimes \chi)$, where $\mathcal{P} = \mathcal{M}\mathcal{U}_\mathcal{P}$ and $\omega$ are as above and $\chi \in X_{\mathrm{nr}}(\mathcal{M})$. Then $\chi$ restricts to an unramified character of $\mathcal{L}$ and the cuspidal support of $\omega \otimes \chi$ is $(\mathcal{L}, \sigma \otimes \chi)$. In this case one defines

$$\widetilde{\mathbf{Sc}}(\pi) = (\mathcal{L}, \sigma_u \otimes \chi)/\mathcal{H}\text{-conjugacy}.$$

We note that the only difference with $\mathbf{Sc}(\pi)$ is $\nu|_\mathcal{L}$, an unramified character $\mathcal{L} \to \mathbb{R}_{>0}$ which represents the absolute value of the infinitesimal central character of $\sigma$.

It has been believed for a long time that the (enhanced) L-parameters of $\pi \in \mathrm{Irr}(\mathcal{H})$ and $\mathbf{Sc}(\pi)$ are always related, but it was not clear how. With our new

notions we can make this precise. Let $\mathfrak{Lev}(\mathcal{H})$ be a set of representatives for the conjugacy classes of Levi subgroups of $\mathcal{H}$, and recall (96) and (98).

**Conjecture 7.8.** *Assume that a local Langlands correspondence exists for $\mathcal{H}$ and for supercuspidal representations of its Levi subgroups. The following diagram should commute:*

$$
\begin{array}{ccc}
\mathrm{Irr}(\mathcal{H}) & \xleftarrow{\quad\quad\mathrm{LLC}\quad\quad} & \Phi_e(\mathcal{H}) \\
\Big\downarrow{\scriptstyle\mathbf{Sc}} & & \Big\downarrow{\scriptstyle\mathbf{Sc}} \\
\bigsqcup_{\mathcal{L}\in\mathfrak{Lev}(\mathcal{H})}\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})/W(\mathcal{H},\mathcal{L}) & \xleftarrow{\quad\mathrm{LLC}\quad} & \bigsqcup_{\mathcal{L}\in\mathfrak{Lev}(\mathcal{H})}\Phi_{\mathrm{cusp}}(\mathcal{L})/W(\mathcal{H},\mathcal{L}).
\end{array}
$$

Conjecture 7.8 is known to hold for many of the groups for which a LLC has been established.

- For $\mathrm{GL}_n(F)$ it is a consequence of the Bernstein–Zelevinsky classification of $\mathrm{Irr}(\mathrm{GL}_n(F))$ [Zel] and the way it is used in the local Langlands correspondence for $\mathrm{GL}_n(F)$, see [Hen, §2].
- Irreducible representations of inner forms $\mathrm{GL}_m(D)$ of $\mathrm{GL}_n(F)$ can also be classified via a Zelevinsky-like scheme, see [Tad]. This is used in the LLC in the same way as for $\mathrm{GL}_n(F)$ [ABPS2, §2], so the conjecture also holds for these groups.
- The local Langlands correspondence for an inner form $\mathrm{SL}_m(D)$ of $\mathrm{SL}_n(F)$ is derived directly from that for $\mathrm{GL}_m(D)$: on the Galois side one lifts L-parameters $\mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})$, whereas on the $p$-adic side one restricts irreducible representations of $\mathrm{GL}_m(D)$ to $\mathrm{SL}_m(D)$ to construct L-packets. These two operations do not really change the infinitesimal central characters of L-parameters or smooth representations, only on $Z(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{C}^\times$ or $Z(\mathrm{GL}_m(D)) \cong F^\times$, respectively. Therefore Conjecture 7.8 for $\mathrm{GL}_m(D)$ implies it for $\mathrm{SL}_m(D)$.
- For the split classical groups $\mathrm{Sp}_{2n}(F)$ and $\mathrm{SO}_m(F)$ when $F$ is a $p$-adic field. The support cuspidal map specializes to the map defined in [Mou, Théorème 4.27], and the commutativity of the diagram follows from [Mou, Théorème 5.9].
- For principal series representations of split groups see [ABPS4, Theorem 15.1].
- For unipotent representations of simple $p$-adic groups $\mathcal{H}$ of adjoint type we refer to [Lus4]. Although it is not so easy to see, the essence is that Lusztig uses the element $f = \phi\Big(\mathrm{Frob}, \begin{pmatrix} \|\mathrm{Frob}\|^{1/2} & 0 \\ 0 & \|\mathrm{Frob}\|^{-1/2} \end{pmatrix}\Big)$ to parametrize the central character of a representation of a suitable affine Hecke algebra [Lus4, §9.3]. By construction this also parametrizes the infinitesimal central character of the associated representation of $\mathcal{H}$.

To support Conjecture 7.8, we check that the cuspidal support map is compatible with the Langlands classification for L-parameters. The latter is a version of the Langlands classification for $\mathrm{Irr}(\mathcal{H})$ on the Galois side of the LLC, it stems from [SiZi].

We will describe first a Galois side analogue for unramified characters. Let $\mathbf{I}_F \subset \mathbf{W}_F$ be as above the inertia subgroup and let $\mathrm{Frob} \in \mathbf{W}_F$ be a Frobenius element.

Recall from [Hai, §3.3.1] that there is a canonical isomorphism of complex tori

$$(109) \qquad X_{\mathrm{nr}}(\mathcal{L}) \cong \left(Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}\right)^{\circ}_{\mathrm{Frob}} = Z(\mathcal{L}^{\vee} \rtimes \mathbf{I}_F)^{\circ}_{\mathbf{W}_F/\mathbf{I}_F} = Z(\mathcal{L}^{\vee} \rtimes \mathbf{I}_F)^{\circ}_{\mathcal{L}^{\vee} \rtimes \mathbf{W}_F}.$$

The group $X_{\mathrm{nr}}(\mathcal{L})$ acts on $\mathrm{Irr}(\mathcal{L})$ by tensoring. This corresponds to an action of $\left(Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}\right)^{\circ}_{\mathrm{Frob}}$ on $\Phi(\mathcal{L})$ and on $\Phi_e(\mathcal{L})$. Namely, let $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{L}$ be a relevant L-parameter and let $z \in Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}$. We define $z\phi \in \Phi(\mathcal{L})$ by

$$(110) \qquad (z\phi)\big|_{\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})} = \phi\big|_{\mathbf{I}_F \times \mathrm{SL}_2(\mathbb{C})} \quad \text{and} \quad (z\phi)(\mathrm{Frob}) = z\,\phi(\mathrm{Frob}).$$

Notice that $z\phi \in \Phi(\mathcal{L})$ because $z \in Z(\mathcal{L}^{\vee} \rtimes \mathbf{I}_F)$. Suppose that $z' \in Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}$ represents the same element of $\left(Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}\right)_{\mathrm{Frob}}$. Then $z^{-1}z' = x^{-1}\mathrm{Frob}(x)$ for some $x \in Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}$, and

$$z'\phi = x^{-1}\mathrm{Frob}(x)z\phi = x^{-1}z\phi x.$$

Hence $z'\phi = z\phi$ in $\Phi(\mathcal{L})$ and we obtain an action of $\left(Z(\mathcal{L}^{\vee})^{\mathbf{I}_F}\right)_{\mathrm{Frob}}$ on $\Phi(\mathcal{L})$. As $z$ commutes with $\mathcal{L}^{\vee}$, $\mathcal{S}_{z\phi} = \mathcal{S}_{\phi}$. This enables us to lift the action to $\Phi_e(\mathcal{L})$ by

$$(111) \qquad z(\phi, \rho) = (z\phi, \rho).$$

To allow $\mathcal{H}^{\vee}$ to act on the above objects, we also have to define them for Levi L-subgroups ${}^L L$ of ${}^L\mathcal{H}$. Generalizing (109), we put

$$(112) \qquad X_{\mathrm{nr}}\big({}^L L\big) = Z\big(\mathcal{H}^{\vee} \rtimes \mathbf{I}_F \cap {}^L L\big)^{\circ}_{\mathrm{Frob}}.$$

This group plays the role of unramified characters for ${}^L L$, we will sometimes refer to it as the unramified twists of ${}^L L$. By the formula (110), $X_{\mathrm{nr}}({}^L L)$ acts on Langlands parameters with image in ${}^L L$. As in (111), that extends to an action on enhanced L-parameters for ${}^L L$.

The following notion replaces the data in the Langlands classification for $\mathcal{H}$.

**Definition 7.9.** Fix a pinning of $\mathcal{H}$ and a $\mathbf{W}_F$-stable pinning of $\mathcal{H}^{\vee}$. A standard triple for $\mathcal{H}$ consists of:

- a standard Levi subgroup $\mathcal{L}$ of $\mathcal{H}$;
- a bounded L-parameter $\phi_t \in \Phi_{\mathrm{bdd}}(\mathcal{L})$;
- an unramified twist $z \in X_{\mathrm{nr}}({}^L\mathcal{L})$, which is strictly positive with respect to the standard parabolic subgroup $\mathcal{P}$ with Levi factor $\mathcal{L}$.

The last condition means that $\alpha^{\vee}(z) > 1$ for every root $\alpha$ of $(\mathcal{U}_{\mathcal{P}}, Z(\mathcal{L})^{\circ})$, where $\mathcal{U}_{\mathcal{P}}$ denotes the unipotent radical of $\mathcal{P}$.

An enhancement of a standard triple $(\mathcal{L}, \phi_t, z)$ is an $\mathcal{L}$-relevant irreducible representation $\rho_t$ of $\mathcal{S}^{\mathcal{L}}_{z\phi_t}$. Let $\zeta_{\mathcal{H}}$ and $\zeta^{\mathcal{L}}_{\mathcal{H}}$ be as in Lemma 7.4. We say that $(\mathcal{L}, \phi_t, z, \rho_t)$ is an enhanced standard triple for $(\mathcal{H}, \zeta_{\mathcal{H}})$ if $\rho_t|_{Z(\mathcal{L}^{\vee}_{\mathrm{sc}})} = \zeta^{\mathcal{L}}_{\mathcal{H}}$.

**Theorem 7.10.** (a) *There exists a canonical bijection from the set of standard triples of $\mathcal{H}$ to $\Phi(\mathcal{H})$. It sends $(\mathcal{L}, \phi_t, z)$ to $z\phi_t$ (up to $\mathcal{H}^{\vee}$-conjugacy).*
(b) *The natural map*

$$p_{\zeta^{\mathcal{L}}_{\mathcal{H}}} \mathbb{C}[\mathcal{S}^{\mathcal{L}}_{\phi_t}] = p_{\zeta^{\mathcal{L}}_{\mathcal{H}}} \mathbb{C}[\mathcal{S}^{\mathcal{L}}_{z\phi_t}] \to p_{\zeta_{\mathcal{H}}} \mathbb{C}[\mathcal{S}^{\mathcal{H}}_{z\phi_t}]$$

*from Lemma 7.4 is an isomorphism. Hence part (a) can be enhanced to a canonical bijection*

$$\begin{aligned} \{\text{enhanced standard triples for } (\mathcal{H}, \zeta_{\mathcal{H}})\} &\longleftrightarrow \Phi_{e, \zeta_{\mathcal{H}}}(\mathcal{H}) \\ (\mathcal{L}, \phi_t, z, \rho_t) &\mapsto (z\phi_t, \rho_t). \end{aligned}$$

*Proof.* (a) See [SiZi, Theorem 4.6]. The differences are only notational: we replaced a standard parabolic subgroup $\mathcal{P}$ of $\mathcal{H}$ by its standard Levi factor $\mathcal{L}$ and we used $z \in X_{\mathrm{nr}}({}^L\mathcal{L})$ instead of the presentation of $X_{\mathrm{nr}}(\mathcal{L})$ by elements of $\mathfrak{a}_\mathcal{L}^*$. The regularity of $\nu \in \mathfrak{a}_\mathcal{L}^*$ in [SiZi] means that it lies in the open Weyl chamber of $\mathfrak{a}_\mathcal{L}^*$ determined by $\mathcal{P}$. This translates to $z$ being strictly positive with respect to $\mathcal{P}$.

(b) Since $z \in Z(\mathcal{L}^\vee)$, $\phi_t$ and $z\phi_t$ have the same $\mathcal{S}$-groups for $\mathcal{L}$. In [SiZi, Proposition 7.1] it is shown that the natural map $\mathfrak{R}^\mathcal{L}_{z\phi_t} \to \mathfrak{R}_{z\phi_t}$ is a bijection. In Lemma 7.4 we constructed a natural injection

$$p_{\zeta^\mathcal{L}_\mathcal{H}}\mathbb{C}[\mathcal{S}^\mathcal{L}_{z\phi_t}] \to p_{\zeta_\mathcal{H}}\mathbb{C}[\mathcal{S}^\mathcal{H}_{z\phi_t}].$$

The dimensions of these spaces are, respectively, $|\mathfrak{R}^\mathcal{L}_\phi|$ and $|\mathfrak{R}_\phi|$. These are equal by [SiZi, Proposition 7.1], so the above map is an algebra isomorphism.  $\square$

The maps ${}^L\Psi$ and $\mathbf{Sc}$ from Definition 7.7 are compatible with Theorem 7.10 in the sense that they factor through this Langlands classification.

**Lemma 7.11.** *Let $(\phi, \rho) \in \Phi_{e,\zeta_\mathcal{H}}(\mathcal{H})$ and let $(\mathcal{L}, \phi_t, z, \rho_t)$ be the enhanced standard triple associated to it by Theorem 7.10. Then*

$$\begin{aligned}
{}^L\Psi^\mathcal{H}(\phi, \rho) &= {}^L\Psi^\mathcal{L}(z\phi_t, \rho_t) &= z \cdot {}^L\Psi^\mathcal{L}(\phi_t, \rho_t), \\
\mathbf{Sc}^\mathcal{H}(\phi, \rho) &= \mathbf{Sc}^\mathcal{L}(z\phi_t, \rho_t) &= z \cdot \mathbf{Sc}^\mathcal{L}(\phi_t, \rho_t).
\end{aligned}$$

*Proof.* Because all the maps are well-defined on conjugacy classes of enhanced L-parameters, we may assume that $\phi = z\phi_t$ and $\rho = \rho_t$. By definition ${}^L\Psi^\mathcal{H}(z\phi_t, \rho_t)$ is given in terms of $q\Psi_G(u_\phi, \rho) = [M, v, q\epsilon]_G$, as $(Z_{{}^L\mathcal{H}}(Z(M)^\circ), \phi|_{\mathbf{W}_F}, v, q\epsilon)$. Consider

$$G_1 := Z_G(Z(\mathcal{L}_c^\vee)^\circ) = Z^1_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi|_{\mathbf{W}_F}) \cap \mathcal{L}_c^\vee$$

Since $\mathcal{L}_c^\vee = Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(Z(\mathcal{L}_c^\vee)^\circ)$ is a Levi subgroup of $\mathcal{H}_{\mathrm{sc}}^\vee$, $G_1$ is a quasi-Levi subgroup of $G = Z^1_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi|_{\mathbf{W}_F})$. Furthermore

$$G_1^\circ = \left(Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi|_{\mathbf{W}_F}) \cap \mathcal{L}_c^\vee\right)^\circ = Z_{\mathcal{L}_c^\vee}(\phi|_{\mathbf{W}_F})^\circ.$$

By Proposition 5.6

$$q\Psi_{G_1}(u_\phi, \rho) = q\Psi_G(u_\phi, \rho).$$

Write $G_2 = Z^1_{\mathcal{L}_{\mathrm{sc}}^\vee}(\phi|_{\mathbf{W}_F})Z(\mathcal{L}_c^\vee)^\circ$ and abbreviate $G_3 = Z^1_{\mathcal{L}_{\mathrm{sc}}^\vee}(\phi|_{\mathbf{W}_F})$. From

$$Z(\mathcal{H}_{\mathrm{sc}}^\vee) \subset Z(\mathcal{L}_c^\vee) = Z(\mathcal{L}_{\mathrm{sc}}^\vee)Z(\mathcal{L}_c^\vee)^\circ$$

and the description of $G_1^\circ$ we see that $G_1$ is a finite index subgroup of $G_2$. Since the quasi-cuspidal support of $(u_\phi, \rho)$ (for $G_1$) is derived from the cuspidal support (for $G_1^\circ = G_2^\circ$), $Z(M)^\circ$ is the same for $G_2$ and $G_1$. Hence

(113)          $q\Psi_{G_2}(u_\phi, \rho) = [M_2, v, q\epsilon_2]_{G_2}, \qquad M_2 = Z_{G_2}(Z(M)^\circ),$

where $q\epsilon_2$ is an extension of $q\epsilon \in \mathrm{Irr}(A_M(v))$ to $A_{M_2}(v)$. By (106) and Proposition 7.3.b $q\epsilon_2$ is obtained from $q\epsilon$ by setting it equal to 1 on a suitable central subgroup. When we replace $G_2$ by $G_3$ we only omit a part of its connected centre, which does not make a real difference for quasi-cuspidal supports. Concretely, (113) entails

$$q\Psi_{G_3}(u_\phi, \rho) = [M_3, v, q\epsilon_3]_{G_3}, \qquad M_3 = Z_{G_3}(Z(M)^\circ),$$

where the inflation of $q\epsilon_3$ to a function on $Z_{M_3}(v)$ agrees with the inflation of $q\epsilon_2$ to $Z_{M_3}(v)$. As explained in the proof of Proposition 7.3 after (102), this means that as cuspidal data

$${}^L\Psi^\mathcal{L}(\phi, \rho) = (Z_{{}^L\mathcal{L}}(Z(M)^\circ), \phi|_{\mathbf{W}_F}, v, q\epsilon_3) = (Z_{{}^L\mathcal{L}}(Z(M)^\circ), \phi|_{\mathbf{W}_F}, v, q\epsilon).$$

Now ${}^L\mathcal{L}$ is a Levi L-subgroup of ${}^L\mathcal{H}$ containing $\phi(\mathbf{W}_F) \cup \{v\}$. In the proof of Proposition 7.3.a we checked that $\mathcal{L}^\vee \rtimes \mathbf{W}_F \supset Z_{{}^L\mathcal{H}}(Z(M)^\circ)$. Hence

$$Z_{{}^L\mathcal{H}}(Z(M)^\circ) = Z_{{}^L\mathcal{L}}(Z(M)^\circ) \quad \text{and} \quad {}^L\Psi^\mathcal{L}(\phi, \rho) = {}^L\Psi^\mathcal{H}(\phi, \rho).$$

As $Z(\mathcal{L}_c^\vee) \subset Z(M)$, the element

$$z \in X_{\mathrm{nr}}({}^L\mathcal{L}) = (Z(\mathcal{L}^\vee)^{\mathbf{I}_F})^\circ_{\mathrm{Frob}}$$

also lies in $X_{\mathrm{nr}}(Z_{{}^L\mathcal{H}}(Z(M)^\circ))$. Since $z$ commutes with $\mathcal{L}_c^\vee$, $G_t = Z^1_{\mathcal{H}_{\mathrm{sc}}^\vee}(z\phi_t|_{\mathbf{W}_F}) \cap \mathcal{L}_c^\vee$ equals $Z^1_{\mathcal{H}_{\mathrm{sc}}^\vee}(\phi_t|_{\mathbf{W}_F}) \cap \mathcal{L}_c^\vee$. Now Definition 7.7 shows that

$${}^L\Psi^\mathcal{L}(z\phi_t, \rho_t) = (Z_{{}^L\mathcal{H}}(Z(M)^\circ), z\phi_t|_{\mathbf{W}_F}, v, q\epsilon)$$
$$= z \cdot (Z_{{}^L\mathcal{H}}(Z(M)^\circ), \phi_t|_{\mathbf{W}_F}, v, q\epsilon) = z \cdot {}^L\Psi^\mathcal{L}(\phi_t, \rho_t).$$

The construction of $\chi_{\phi,v}$ in Lemma 7.6 depends only on $q\Psi_{G_t}(u_\phi, \rho)$, so $\mathbf{Sc}^\mathcal{H}(\phi, \rho) = \mathbf{Sc}^\mathcal{L}(z\phi_t, \rho_t) = z \cdot \mathbf{Sc}^\mathcal{L}(\phi_t, \rho_t)$ as well. $\qquad\square$

## 8. INERTIAL EQUIVALENCE CLASSES OF L-PARAMETERS

In important ingredient in Bernstein's theory of representations of $p$-adic groups are inertial equivalence classes. Let $\mathcal{L} \subset \mathcal{H}$ be a Levi subgroup and let $X_{\mathrm{nr}}(\mathcal{L})$ be the group of unramified characters $\mathcal{L} \to \mathbb{C}^\times$. Two cuspidal pairs $(\mathcal{L}_1, \sigma_1)$ and $(\mathcal{L}_2, \sigma_2)$ are said to be inertially equivalent if there exist an unramified character $\chi_1$ of $\mathcal{L}_1$ and an element $h \in \mathcal{H}$ such that

$$h\mathcal{L}_2 h^{-1} = \mathcal{L}_1 \quad \text{and} \quad h \cdot \sigma_2 = \sigma_1 \otimes \chi_1.$$

We denote a typical inertial equivalence of cuspidal pairs by $\mathfrak{s} = [\mathcal{L}, \sigma]_\mathcal{H}$, we let $\mathfrak{B}(\mathcal{H})$ be the set of such classes. With every $\mathfrak{s} \in \mathfrak{B}(\mathcal{H})$ one can associate a set of irreducible smooth $\mathcal{H}$-representations:

$$\mathrm{Irr}(\mathcal{H})^\mathfrak{s} = \{\pi \in \mathrm{Irr}(\mathcal{H}) : \text{the cuspidal support of } \pi \text{ lies in } \mathfrak{s}\}.$$

A (weak) version of the Bernstein decomposition says that

$$(114) \qquad\qquad \mathrm{Irr}(\mathcal{H}) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(\mathcal{H})} \mathrm{Irr}(\mathcal{H})^\mathfrak{s}.$$

We will establish a similar decomposition for enhanced Langlands parameters.

Our notion of inertial equivalence generalizes [Hai, Definition 5.33] from homomorphisms $\mathbf{W}_F \to {}^L\mathcal{H}$ to enhanced L-parameters.

**Definition 8.1.** Let $({}^LL, \phi_v, q\epsilon)$ and $({}^LL', \phi_v', q\epsilon')$ be two cuspidal data for ${}^L\mathcal{H}$. They are inertially equivalent if there exist $z \in X_{\mathrm{nr}}({}^LL)$ and $h \in \mathcal{H}^\vee$ such that

$$h\,{}^LL'h^{-1} = {}^LL \quad \text{and} \quad (z\phi_v, q\epsilon) = (h\phi_v'h^{-1}, h \cdot q\epsilon').$$

The class of $({}^LL, \phi_v, q\epsilon)$ modulo $X_{\mathrm{nr}}({}^LL)$ is denoted $[{}^LL, \phi_v, q\epsilon]_{{}^LL}$, and its inertial equivalence class is denoted $[{}^LL, \phi_v, q\epsilon]_{{}^L\mathcal{H}}$. We say that $[{}^LL, \phi_v, q\epsilon]_{{}^L\mathcal{H}}$ is $\mathcal{H}$-relevant if any of its elements is so. We write $\mathfrak{B}^\vee({}^L\mathcal{H})$ for the set of inertial equivalence classes of cuspidal pairs for ${}^L\mathcal{H}$, and $\mathfrak{B}^\vee(\mathcal{H})$ for its subset of $\mathcal{H}$-relevant classes.

Given an inertial equivalence class $\mathfrak{s}^\vee$ for ${}^L\mathcal{H}$, we write, using Definition 7.7,

$$\Phi_e({}^L\mathcal{H})^{\mathfrak{s}^\vee} = \{(\phi, \rho) \in \Phi_e({}^L\mathcal{H}) : \text{the cuspidal support of } (\phi, \rho) \text{ lies in } \mathfrak{s}^\vee\}.$$

When $\mathfrak{s}^\vee$ is $\mathcal{H}$-relevant, we put

$$\Phi_e(\mathcal{H})^{\mathfrak{s}^\vee} = \{(\phi, \rho) \in \Phi_e(\mathcal{H}) : \text{the cuspidal support of } (\phi, \rho) \text{ lies in } \mathfrak{s}^\vee\}.$$

We note that $\mathfrak{s}^\vee$ as above determines a character of $Z(L_{\mathrm{sc}})$. In view of Proposition 7.3.b, it extends in a unique way to a character of $Z(\mathcal{H}_{\mathrm{sc}}^\vee)$ trivial on $Z(L_c)^\circ$.

The above construction yields partitions analogous to (114):

$$(115) \qquad \Phi_e(^L\mathcal{H}) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(^L\mathcal{H})} \Phi_e(^L\mathcal{H})^{\mathfrak{s}^\vee} \quad \text{and} \quad \Phi_e(\mathcal{H}) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(\mathcal{H})} \Phi_e(\mathcal{H})^{\mathfrak{s}^\vee}.$$

In this sense we consider $\Phi_e(^L\mathcal{H})^{\mathfrak{s}^\vee}$ as Bernstein components in the space of enhanced L-parameters for $^L\mathcal{H}$. We note that by Lemma 7.6 the difference between $\mathbf{Sc}(\phi, \rho)$ and $^L\Psi(\phi, \rho)$, namely the homomorphism

$$\mathbf{W}_F \to Z(M)^\circ : w \mapsto \chi_{\phi,v}\big(\|w\|^{1/2}\big),$$

can be considered as an unramified twist of $^LL$. Hence $\mathbf{Sc}(\phi, \rho)$ and $^L\Psi(\phi, \rho)$ belong to the same inertial equivalence class, and we could equally well have used $^L\Psi$ to define $\Phi_e(^L\mathcal{H})^{\mathfrak{s}^\vee}$ and $\Phi_e(\mathcal{H})^{\mathfrak{s}^\vee}$.

We return to $p$-adic groups, to consider other aspects of Bernstein's work. Bernstein associated to each inertial equivalence class $\mathfrak{s} = [\mathcal{L}, \sigma]_\mathcal{H} \in \mathfrak{B}(\mathcal{H})$ a finite group $W_\mathfrak{s}$. Let $W(\mathcal{H}, \mathcal{L}) = N_\mathcal{H}(\mathcal{L})/\mathcal{L}$, the "Weyl" group of $\mathcal{L}$. It acts on $\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$, which induces an action on the collection of inertial equivalence classes $[\mathcal{L}, \omega]_\mathcal{L}$ with $\omega \in \mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})$. Notice that

$$(\mathcal{L}, \omega_1), (\mathcal{L}, \omega_2) \text{ are } \mathcal{H}\text{-conjugate} \iff \text{there is a } w \in W(\mathcal{H}, \mathcal{L}) \text{ with } w \cdot \omega_1 \cong \omega_2.$$

The group $W_\mathfrak{s}$ is defined to be the stabilizer of $[\mathcal{L}, \sigma]_\mathcal{L}$ in $W(\mathcal{H}, \mathcal{L})$. It keeps track of which elements of $[\mathcal{L}, \sigma]_\mathcal{L}$ are $\mathcal{H}$-conjugate. This group plays an important role in the Bernstein centre.

Let $\mathrm{Rep}(\mathcal{H})$ be the category of smooth complex $\mathcal{H}$-representations, and let $\mathrm{Rep}(\mathcal{H})^\mathfrak{s}$ be its subcategory generated by $\mathrm{Irr}(\mathcal{H})^\mathfrak{s}$. The strong form of the Bernstein decomposition says that

$$\mathrm{Rep}(\mathcal{H}) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(\mathcal{H})} \mathrm{Rep}(\mathcal{H})^\mathfrak{s}.$$

By [BeDe, Proposition 3.14] the centre of the category $\mathrm{Rep}(\mathcal{H})^\mathfrak{s}$ is canonically isomorphic to $\mathcal{O}([\mathcal{L}, \sigma]_\mathcal{L}/W_\mathfrak{s})$. Here $[\mathcal{L}, \sigma]_\mathcal{L}$ is regarded as a complex affine variety via the transitive action of $X_{\mathrm{nr}}(\mathcal{L})$. The centre of $\mathrm{Rep}(\mathcal{H})$ is isomorphic to

$$\bigoplus_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \bigoplus_{\mathfrak{s}=[\mathcal{L},\sigma]_\mathcal{H} \in \mathfrak{B}(\mathcal{H})} Z(\mathrm{Rep}(\mathcal{H})^\mathfrak{s}) \cong \bigoplus_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \bigoplus_{\mathfrak{s}=[\mathcal{L},\sigma]_\mathcal{H} \in \mathfrak{B}(\mathcal{H})} \mathcal{O}([\mathcal{L},\sigma]_\mathcal{L}/W_\mathfrak{s})$$

$$= \bigoplus_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \mathcal{O}(\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})/W(\mathcal{H}, \mathcal{L})).$$

In other words, there are canonical bijections

$$(116) \qquad \begin{array}{rcl} \mathrm{Irr}\big(Z(\mathrm{Rep}(\mathcal{H})^\mathfrak{s})\big) & \longleftrightarrow & [\mathcal{L}, \sigma]_\mathcal{L}/W_\mathfrak{s}, \\ \mathrm{Irr}\big(Z(\mathrm{Rep}(\mathcal{H}))\big) & \longleftrightarrow & \bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})/W(\mathcal{H}, \mathcal{L}). \end{array}$$

We want identify the correct analogue of $W_\mathfrak{s}$ on the Galois side. From (112) we see that $N_{\mathcal{H}^\vee}(^LL)$ stabilizes $X_{\mathrm{nr}}(^LL)$ and that $L$ fixes $X_{\mathrm{nr}}(^LL)$ pointwise. Therefore $W(^L\mathcal{H}, {}^LL)$ also acts on classes $[^LL, \phi, \rho]_{{}^LL}$ of cuspidal data modulo unramified twists. We note that, like (96) and (98),

$$(117) \qquad \begin{array}{l} [^LL, \phi_v, q\epsilon]_{{}^L\mathcal{H}} = [^LL, \phi'_v, q\epsilon']_{{}^L\mathcal{H}} \qquad \Longleftrightarrow \\ \text{there is a } w \in W(^L\mathcal{H}, {}^LL) \text{ such that } w \cdot [^LL, \phi_v, q\epsilon]_{{}^LL} = [^LL, \phi'_v, q\epsilon']_{{}^LL}. \end{array}$$

Given any inertial equivalence class $\mathfrak{s}^\vee = [^LL, \phi_v, q\epsilon]_{L\mathcal{H}}$ with underlying class $\mathfrak{s}_L^\vee = [^LL, \phi_v, q\epsilon]_{LL}$, we define

$$W_{\mathfrak{s}^\vee} := \text{ the stabilizer of } \mathfrak{s}_L^\vee \text{ in } W(^L\mathcal{H}, {}^LL).$$

Now we approach this group from the Galois side. From $(^LL, \phi_v, q\epsilon)$ we can build

$$(118) \qquad v = \phi_v\big(1, \big(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\big)\big) \quad \text{and} \quad G = Z^1_{\mathcal{H}_{sc}^\vee}(\phi_v|_{\mathbf{W}_F}).$$

Let $L_c$ be the inverse image of $L$ in $\mathcal{H}_{sc}^\vee$ and consider the cuspidal quasi-support

$$(119) \qquad q\mathfrak{t} = [G \cap L_c, v, q\epsilon]_G = \big[G \cap L_c, \mathcal{C}_v^{G\cap L_c}, q\mathcal{E}\big]_G.$$

From (69) we get

$$(120) \qquad W_{q\mathfrak{t}} = N_G\big(G \cap L_c, \mathcal{C}_v^{G\cap L_c}, q\mathcal{E}\big)/(G \cap L_c).$$

**Lemma 8.2.** *$W_{q\mathfrak{t}}$ is canonically isomorphic to the isotropy group of $(^LL, \phi_v, q\epsilon)$ in $W(^L\mathcal{H}, {}^LL)$ and in $W_{\mathfrak{s}^\vee}$.*

*Proof.* Since $X_{nr}(^LL)$ is stable under $W(^L\mathcal{H}, {}^LL)$, any element of the latter group which fixes $(^LL, \phi_v, \rho)$ automatically stabilizes $\mathfrak{s}_L^\vee$. Therefore it does not matter whether we determine the isotropy group in $W(^L\mathcal{H}, {}^LL)$ or in $W_{\mathfrak{s}^\vee}$.

The proof of Proposition 7.3, with $G \cap L_c$ in the role of $M$, shows that $Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(G \cap L_c)^\circ)$ is a Levi L-subgroup of $^L\mathcal{H}$ minimally containing the image of $\phi$. As $Z(G \cap L_c)^\circ \supset Z(L_c)^\circ$,

$$Z_{\mathcal{H}^\vee}(Z(G \cap L_c)^\circ) \subset Z_{\mathcal{H}^\vee}(Z(L_c^\circ)) = L.$$

But $^LL$ also contains the image of $\phi$ minimally, so

$$(121) \qquad {}^LL = Z_{\mathcal{H}^\vee \cap \mathbf{W}_F}(Z(G \cap L_c)^\circ).$$

Suppose that $n \in N_{\mathcal{H}^\vee}(^LL)$ fixes $[\phi_v, q\epsilon]_L$. Then it lies in $N_{\mathcal{H}^\vee}\big(G \cap L_c, \mathcal{C}_{u_\phi}^{G\cap L_c}, q\mathcal{E}\big)$. The kernel of $\mathcal{H}_{sc}^\vee \to \mathcal{H}_{der}^\vee$ is contained in $L_c$, so in view of (120) $n$ lifts to a unique element of $W_{q\mathfrak{t}}$. This induces an injection

$$\text{Stab}_{W(^L\mathcal{H}, {}^LL)}([\phi_v, q\epsilon]_L) \cong \text{Stab}_{N_{\mathcal{H}^\vee}(^LL) \cap Z_{\mathcal{H}^\vee}(\phi(\mathbf{W}_F))}\big(\mathcal{C}_v^{G\cap L_c}, q\mathcal{E}\big)/Z_L(\phi(\mathbf{W}_F))$$

$$(122) \qquad \cong \text{Stab}_{N_{\mathcal{H}_{sc}^\vee}(^LL) \cap G}\big(\mathcal{C}_v^{G\cap L_c}, q\mathcal{E}\big)/(G \cap L_c) \to W_{q\mathfrak{t}}.$$

The only difference between the last two terms is that on the left hand side elements of $G$ have to normalize $^LL$, whereas on the right hand side they only have to normalize $G \cap L_{sc}$. Consider any $g \in N_G(G \cap L_c)$. It normalizes $Z(G \cap L_c)$, so it also normalizes $Z_{\mathcal{H}^\vee \rtimes \mathbf{W}_F}(Z(G \cap L_c)^\circ)$, which by (121) equals $^LL$. Therefore (122) is also surjective. $\qquad \square$

Assume for the remainder of this section that $Z(\mathcal{L}_{sc}^\vee)$ is fixed by $\mathbf{W}_F$, for every Levi subgroup $\mathcal{L} \subset \mathcal{H}$. (The general case is similar and can be obtained by including characters $\zeta_{\mathcal{H}}^{\mathcal{L}}$ as in Lemma 7.4.) In view of Lemma 8.2, the analogue of the Bernstein centre (116) becomes

$$(123) \qquad \bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \Phi_{cusp}(\mathcal{L})/W(^L\mathcal{H}, {}^L\mathcal{L}) = \bigsqcup_{\mathfrak{s}^\vee = [^L\mathcal{L}, \phi_v, q\epsilon]_{L\mathcal{H}} \in \mathfrak{B}^\vee(\mathcal{H})} \mathfrak{s}_{\mathcal{L}}^\vee/W_{\mathfrak{s}^\vee}.$$

Thus we interpret the "Bernstein centre of $\Phi_e(\mathcal{H})$" as the quotient along the map

$$\mathbf{Sc} : \Phi_e(\mathcal{H}) \to \bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \Phi_{cusp}(\mathcal{L})/W(^L\mathcal{H}, {}^L\mathcal{L}).$$

Let us agree that two enhanced L-parameters in the same Bernstein component are inseparable if they have the same infinitesimal character. Then (123) can be regarded as a maximal separable quotient of $\Phi_e(\mathcal{H})$. This fits nicely with the Dauns–Hofmann theorem, which says that for many noncommutative algebras $A$ the operation of taking the maximal separable quotient of $\mathrm{Irr}(A)$ is dual to restriction from $A$ to its centre $Z(A)$.

## 9. Extended quotients and L-parameters

The ABPS-conjecture from [ABPS1, §15] and [ABPS6, Conjecture 2] refines (116). In its roughest form it asserts that it can be lifted to a bijection

$$(124) \qquad \mathrm{Irr}(\mathcal{H})^{\mathfrak{s}} \longleftrightarrow ([\mathcal{L},\sigma]_{\mathcal{L}}//W_{\mathfrak{s}})_{\natural},$$

for a suitable family of 2-cocycles $\natural$. Equivalently, this can be formulated as a bijection

$$(125) \qquad \mathrm{Irr}(\mathcal{H}) \longleftrightarrow \bigsqcup_{\mathcal{L}\in\mathfrak{Lev}(\mathcal{H})} \big(\mathrm{Irr}_{\mathrm{cusp}}(\mathcal{L})//W(\mathcal{H},\mathcal{L})\big)_{\natural}.$$

The main goal of this section is to prove an analogue of (124) and (125) for enhanced Langlands parameters, which refines (123).

Fix a $\mathcal{H}$-relevant cuspidal datum $(^L L, \phi_v, q\epsilon)$ for $^L\mathcal{H}$, and write, in addition to the notations (118) and (119),

$$(126) \qquad q\mathfrak{t} = [G\cap L_c, v, q\epsilon]_G, \quad \mathfrak{t}^\circ = [G^\circ\cap L_c, \mathcal{C}_v^{G^\circ\cap L_c}, \mathcal{E}]_{G^\circ}.$$

The next result is a version of the generalized Springer correspondence with enhanced L-parameters.

**Proposition 9.1.** (*a*) *There is a bijection*

$$
\begin{array}{rccc}
^L\Sigma_{q\mathfrak{t}}: & {}^L\Psi^{-1}(^L L,\phi_v,q\epsilon) & \longleftrightarrow & \mathrm{Irr}(\mathbb{C}[W_{q\mathfrak{t}},\kappa_{q\mathfrak{t}}]) \\
& (\phi,\rho) & \mapsto & q\Sigma_{q\mathfrak{t}}(u_\phi,\rho) \\
& (\phi|_{\mathbf{W}_F}, q\Sigma_{q\mathfrak{t}}^{-1}(\tau)) & \hookleftarrow & \tau
\end{array}
$$

*It is canonical up to the choice of an isomorphism as in Lemma 5.4.*

(*b*) *Recall that Theorems 2.1.(3) and 2.2.c give a canonical bijection $\Sigma_{\mathfrak{t}^\circ}$ between $\mathrm{Irr}_{\mathbb{C}}(W_{\mathfrak{t}^\circ}) = \mathrm{Irr}(\mathrm{End}_{G^\circ}(\pi_*\tilde{\mathcal{E}}))$ and $\Psi_{G^\circ}^{-1}(\mathfrak{t}^\circ)\subset \mathcal{N}_{G^\circ}^+$. It relates to part (a) by*

$$
{}^L\Sigma_{q\mathfrak{t}}(\phi,\rho)|_{W_{\mathfrak{t}^\circ}} = \bigoplus_i \Sigma_{\mathfrak{t}^\circ}(u_\phi,\rho_i),
$$

*where $\rho = \bigoplus_i \rho_i$ is a decomposition into irreducible $A_{G^\circ}(u_\phi)$-subrepresentations.*

(*c*) *The $\mathcal{H}^\vee$-conjugacy class of $(\phi|_{\mathbf{W}_F}, u_\phi, \rho_i)$ is determined by any irreducible $\mathbb{C}[W_{\mathfrak{t}^\circ}]$-subrepresentation of $^L\Sigma_{q\mathfrak{t}}(\phi,\rho)$.*

*Proof.* (a) By Theorem 5.5 (with $\mathbb{C}$-coefficients) every $(\phi,\rho)\in {}^L\Psi^{-1}(^L L,\phi_v,q\epsilon)$ determines a unique irreducible representation $q\Sigma_{q\mathfrak{t}}(u_\phi,\rho)$ of $\mathbb{C}[W_{q\mathfrak{t}},\kappa_{q\mathfrak{t}}]$. Conversely, every $\tau \in \mathrm{Irr}(\mathbb{C}[W_{q\mathfrak{t}},\kappa_{q\mathfrak{t}}])$ gives rise to a unique $q\Sigma_{q\mathfrak{t}}^{-1}(\tau) = (u_\phi,\rho)\in\mathcal{N}_G^+$, and that determines an enhanced L-parameter $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$ for $^L\mathcal{H}$. It remains to see that $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$ is $\mathcal{H}$-relevant. By (68) $\rho$ has the same $Z(\mathcal{H}_{\mathrm{sc}}^\vee)^{\mathbf{W}_F}$-character as $q\epsilon$. By the assumed $\mathcal{H}$-relevance of $q\epsilon$, $\rho$ is $\mathcal{H}$-relevant. By Definition 6.7, $(\phi|_{\mathbf{W}_F}, u_\phi, \rho)$ is also $\mathcal{H}$-relevant.

(b) This is a direct consequence of Theorem 5.5.b.

(c) By the irreducibility of $\rho$, all the $\rho_i$ are $Z_G(\phi)$-conjugate. Similarly the irreducibility of the $\mathbb{C}[W_{q\mathfrak{t}},\kappa_{q\mathfrak{t}}]$-representation $\tau = {}^L\Sigma_{q\mathfrak{t}}(\phi,\rho)$ implies (with Theorem 1.2) that

all the irreducible $\mathbb{C}[W_{\mathfrak{t}^\circ}]$-constituents $\tau_i$ of $\tau$ are $W_{q\mathfrak{t}}$-conjugate. By part (b) $\tau_i$ determines a pair $(u_\phi, \rho_i)$ up to $G^\circ$-conjugacy. Hence it determines $(\phi|_{\mathbf{W}_F}, u_\phi, \rho_i)$ up to $\mathcal{H}^\vee$-conjugacy. $\qquad\square$

We will promote Proposition 9.1 to a statement involving extended quotients. By Lemma 8.2 $W_{\mathfrak{s}^\vee, \phi_v, q\epsilon} = W_{q\mathfrak{t}}$, so we can regard $\kappa_{q\mathfrak{t}}$ as a 2-cocycle $\kappa_{\phi_v, q\epsilon}$ of $W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}$. Then we can build

$$\widetilde{\mathfrak{s}_L^\vee} = \left([{}^L L, \phi_v, q\epsilon]_{{}^L L}\right)^{\widetilde{~}}_\kappa$$
$$= \left\{\left(({}^L L, z\phi_v, q\epsilon), \rho\right) : z \in X_{\mathrm{nr}}({}^L L), \rho \in \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}])\right\}.$$

Comparing with (13), we see that we still need an action on $W_{\mathfrak{s}^\vee}$ on this set.

**Lemma 9.2.** *Let $w \in W_{\mathfrak{s}^\vee}$ and $z \in X_{\mathrm{nr}}({}^L L)$ with $w(\phi_v, q\epsilon) \cong (z\phi, q\epsilon)$. There exists a family of algebra isomorphisms (for various such $w, z$)*

$$\psi_{w, \phi_v, q\epsilon} : \mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}] \to \mathbb{C}[W_{\mathfrak{s}^\vee, z\phi_v, q\epsilon}, \kappa_{z\phi_v, q\epsilon}]$$

*such that:*

(a) *The family is canonical up to the choice of isomorphisms $\mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}] \cong \mathrm{End}_G(\pi_*(\widetilde{q\mathcal{E}}))$ as in Lemma 5.4.*
(b) *$\psi_{w, \phi_v, q\epsilon}$ is conjugation with $T_w$ if $w \in W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}$.*
(c) *$\psi_{w', z\phi_v, q\epsilon} \circ \psi_{w, \phi_v, q\epsilon} = \psi_{w'w, \phi_v, q\epsilon}$ for all $w' \in W_{\mathfrak{s}^\vee}$.*
(d) *${}^L\Sigma_{q\mathfrak{t}}^{-1}(\rho) \cong {}^L\Sigma_{w(q\mathfrak{t})}^{-1}(\rho \circ \psi_{w, \phi_v, q\epsilon}^{-1})$ for all $\rho \in \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}])$.*

*Proof.* (a) Recall from Lemma 5.4 that

$$(127) \qquad \mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}] \cong \mathbb{C}[W_{q\mathfrak{t}}, \kappa_{q\mathfrak{t}}] \cong \mathrm{End}_G(\pi_*(\widetilde{q\mathcal{E}})).$$

We fix such isomorphisms. For any $n \in N_{\mathcal{H}^\vee}({}^L L)$ representing $w$:

$$(128) \qquad \begin{aligned} \mathbb{C}[W_{\mathfrak{s}^\vee, w(\phi_v)}, w(q\epsilon), \kappa_{w(\phi_v), w(q\epsilon)}] &\cong \mathbb{C}[W_{n(q\mathfrak{t})}, \kappa_{n(q\mathfrak{t})}] \cong \mathrm{End}_G(\pi_*(\widetilde{n \cdot q\mathcal{E}})), \\ \mathbb{C}[W_{\mathfrak{s}^\vee, z\phi_v, q\epsilon}, \kappa_{z\phi_v, q\epsilon}] &\cong \mathrm{End}_{nGn^{-1}}(\pi_*(\widetilde{q\mathcal{E}})) \cong \mathrm{End}_G(\pi_*(\widetilde{\mathrm{Ad}(n)^* q\mathcal{E}})), \end{aligned}$$

where $\pi_*(\widetilde{q\mathcal{E}})$ now denotes a sheaf on $nYn^{-1}$. By assumption there exists a $L_{\mathrm{sc}}$-intertwining map

$$(129) \qquad q\mathcal{E} \to \mathrm{Ad}(n)^* q\mathcal{E},$$

and by the irreducibility of $q\mathcal{E}$ it is unique up to scalars. In the same way as in [Lus2, §3] and in the proof of Lemma 5.4, it gives rise to an isomorphism of $G$-equivariant local systems

$$qb_w : \pi_*(\widetilde{q\mathcal{E}}) \to \pi_*(\widetilde{\mathrm{Ad}(n)^* q\mathcal{E}}).$$

In view of (128) and the essential uniqueness of (129), conjugation by $qb_w$ gives a canonical algebra isomorphism

$$\tilde{\psi}_{w, \phi_v, q\epsilon} : \mathrm{End}_G(\pi_*(\widetilde{q\mathcal{E}})) \to \mathrm{End}_{nGn^{-1}}(\pi_*(\widetilde{q\mathcal{E}})).$$

We define $\psi_{w, \phi_v, q\epsilon}$ as the composition of (127), $\tilde{\psi}_{w, \phi_v, q\epsilon}$ and (128).
(b) For $w \in W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}$ we thus obtain conjugation by the image of $qb_w$ which by construction (see the proof of Lemma 5.4) is $T_w$.
(c) The canonicity ensures that

$$\tilde{\psi}_{w', z\phi_v, q\epsilon} \circ \tilde{\psi}_{w, \phi_v, q\epsilon} = \tilde{\psi}_{w'w, \phi_v, q\epsilon}$$

which automatically leads to (c).

(d) By Theorem 5.5.a

$$n \cdot q\Sigma_{qt}^{-1}(\tilde{\rho}) \cong q\Sigma_{w(qt)}^{-1}(\tilde{\rho} \circ \tilde{\psi}_{w,\phi_v,q\epsilon}^{-1}) \quad \text{for} \quad \tilde{\rho} \in \mathrm{Irr}\big(\mathrm{End}_G(\pi_*(\widetilde{q\mathcal{E}}))\big).$$

Since $^L\Sigma_{qt}$ was defined using (127), we obtain property (d). □

We could have characterized $\psi_{w,\phi_v,q\epsilon}^{-1}$ also with property (d) of Lemma 9.2 only, that would suffice for our purposes. However, then one would not see so readily that the map is exactly as canonical as our earlier constructions.

**Theorem 9.3.** (*a*) *Let* $\mathfrak{s}_L^\vee = [^LL, \phi_v, q\epsilon]_{LL}$ *be an* $\mathcal{H}$-*relevant inertial equivalence class for the Levi L-subgroup* $^LL$ *of* $^L\mathcal{H}$ *and recall the notations* (126). *The maps* $^L\Sigma_{qt}$ *from Proposition 9.1.a combine to a bijection*

$$
\begin{array}{ccc}
\Phi_e(^L\mathcal{H})^{\mathfrak{s}^\vee} & \longleftrightarrow & \big(\Phi_e(^LL)^{\mathfrak{s}_L^\vee}/\!/W_{\mathfrak{s}^\vee}\big)_\kappa \\
(\phi, \rho) & \mapsto & \big(^L\Psi(\phi, \rho), q\Sigma_{qt}(u_\phi, \rho)\big) \\
\big(\phi_v|_{\mathbf{W}_F}, q\Sigma_{qt}^{-1}(\tau)\big) & \hookleftarrow & (^LL, \phi_v, q\epsilon, \tau)
\end{array}
$$

(*b*) *The bijection from part* (*a*) *has the following properties:*
- *It preserves boundedness of* (*enhanced*) *L-parameters.*
- *It is canonical up to the choice of isomorphisms as in* (127).
- *The restriction of* $\tau$ *to* $W_{\mathfrak{t}^\circ}$ *canonically determines the* (*non-enhanced*) *L-parameter in* $^L\Sigma_{qt}(\tau)$.
- *Let* $z, z' \in X_{\mathrm{nr}}(^LL)$ *and let* $\Gamma \subset W_{\mathfrak{s}^\vee, z\phi_v, q\epsilon}$ *be a subgroup. Suppose that* $\Gamma = \overline{\Gamma}/L \cong \overline{\Gamma}_c/L_c$, *where*

$$\overline{\Gamma} \subset N_{\mathcal{H}^\vee}(^LL) \cap Z_{\mathcal{H}^\vee}^1(z'\phi|_{\mathbf{W}_F}) \quad \text{with preimage} \quad \overline{\Gamma}_c \subset Z_{\mathcal{H}_{\mathrm{sc}}^\vee}(z'\phi(\mathbf{W}_F))^\circ.$$

  *Then the 2-cocycle* $\kappa_{\mathfrak{s}^\vee, z\phi_v, q\epsilon}$ *is trivial on* $\Gamma$.

(*c*) *Let* $\zeta_\mathcal{H} \in \mathrm{Irr}(Z(\mathcal{H}_{\mathrm{sc}}^\vee))$ *and* $\zeta_\mathcal{H}^\mathcal{L}$ *be as in Lemma 7.4. We write*

$$\Phi_{e,\zeta_\mathcal{H}}(\mathcal{H}, \mathcal{L}) = \{(\phi, \rho) \in \Phi_{e,\zeta_\mathcal{H}}(\mathcal{H}) : \mathbf{Sc}(\phi, \rho) \in \Phi_{\mathrm{cusp}}(\mathcal{L})\}.$$

*The bijections from part* (*a*) *give a bijection*

$$\Phi_{e,\zeta_\mathcal{H}}(\mathcal{H}, \mathcal{L}) \longleftrightarrow \big(\Phi_{\mathrm{cusp}, \zeta_\mathcal{H}^\mathcal{L}}(\mathcal{L})/\!/W(\mathcal{H}, \mathcal{L})\big)_\kappa.$$

(*d*) *Let* $\mathfrak{Lev}(\mathcal{H})$ *be a set of representatives for the conjugacy classes of Levi subgroups of* $\mathcal{H}$. *The maps from part* (*c*) *combine to a bijection*

$$\Phi_{e,\zeta_\mathcal{H}}(\mathcal{H}) \longleftrightarrow \bigsqcup_{\mathcal{L} \in \mathfrak{Lev}(\mathcal{H})} \big(\Phi_{\mathrm{cusp}, \zeta_\mathcal{H}^\mathcal{L}}(\mathcal{L})/\!/W(\mathcal{H}, \mathcal{L})\big)_\kappa.$$

(*e*) *Assume that* $Z(\mathcal{L}_{\mathrm{sc}}^\vee)$ *is fixed by* $\mathbf{W}_F$ *for every Levi subgroup* $\mathcal{L} \subset \mathcal{H}$. (*E.g.* $\mathcal{H}$ *is an inner twist of a split group.*) *Let* $\mathcal{H}^u$ *be the inner twist of* $\mathcal{H}$ *determined by* $u \in H^1(F, \mathcal{H}_{\mathrm{ad}}) \cong \mathrm{Irr}_\mathbb{C}\big(Z(\mathcal{H}_{\mathrm{sc}}^\vee)^{\mathbf{W}_F}\big)$. *The union of part* (*d*) *for all such* $u$ *is a bijection*

$$\Phi_e(^L\mathcal{H}) \longleftrightarrow \bigsqcup_{u \in H^1(F, \mathcal{H}_{\mathrm{ad}})} \bigsqcup_{\mathcal{L}^u \in \mathfrak{Lev}(\mathcal{H}^u)} \big(\Phi_{\mathrm{cusp}}(\mathcal{L}^u)/\!/W(\mathcal{H}^u, \mathcal{L}^u)\big)_\kappa.$$

*Proof.* (a) Proposition 9.1.a gives a bijection

$$(130) \qquad {}^L\Psi^{-1}(^LL, \phi_v, q\epsilon) \longleftrightarrow \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}]) \longleftrightarrow$$

$$\bigsqcup_{W_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}} \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{s}^\vee, w(\phi_v), w(q\epsilon)}, \kappa_{w(\phi_v), w(q\epsilon)}])/W_{\mathfrak{s}^\vee} = \big(W_{\mathfrak{s}^\vee} \cdot (^LL, \phi_v, q\epsilon)/\!/W_{\mathfrak{s}^\vee}\big)_\kappa.$$

For $z \in X_{\mathrm{nr}}(^L L)$ the pre-images $^L\Psi^{-1}(^L L, \phi_v, q\epsilon)$ and $^L\Psi^{-1}(^L L, z\phi_v, q\epsilon)$ intersect in $\Phi_e(^L\mathcal{H})$ if and only if their $L$-conjugacy classes differ by an element of $W_{\mathfrak{s}^\vee}$. Hence the maps (130) combine to the desired bijection.

(b) It preserves boundedness because it does not change $\phi|_{\mathbf{W}_F}$. The second and third properties follow from Proposition 9.1.c.

Write $G_{z'} = Z_{\mathcal{H}_{\mathrm{sc}}^\vee}^1(z'\phi|_{\mathbf{W}_F})$ and consider $\Gamma$ as a subgroup of $W_{z'\mathfrak{t}^\circ} = N_{G_{z'}^\circ}(z'\mathfrak{t}^\circ)/L_{\mathrm{sc}}$. Let $\pi_*(\widetilde{\mathcal{E}}_{z'})$ be the $G_{z'}$-equivariant sheaf constructed like $\pi_*(\widetilde{\mathcal{E}})$, but with $G_{z'}$ instead of $G$. For $\gamma \in \Gamma$ the proof of Lemma 5.4 provides a canonical element $qb'_\gamma \in \mathrm{End}_{G_{z'}}(\pi_*(\widetilde{\mathcal{E}}_{z'}))$, such that

$$\Gamma \to \mathrm{Aut}_{G_{z'}}(\pi_*(\widetilde{\mathcal{E}}_{z'})) : \gamma \mapsto qb'_\gamma$$

is a group homomorphism. Let $n \in G_{z'}^\circ \cap G_z$ be a lift of $\gamma$. Then $qb'_\gamma$ restricts to an isomorphism

$$(131) \qquad q\mathcal{E} = z'q\mathcal{E} \to \mathrm{Ad}(n)^*(z'q\mathcal{E}) = \mathrm{Ad}(n)^*(q\mathcal{E}).$$

As in the proof of Lemma 9.2, (131) gives rise to an element $qb_\gamma \in \mathrm{End}_{G_z}(\pi_*(\widetilde{\mathcal{E}}_z))$. We can choose the basis element

$$T_\gamma \quad \text{of} \quad \mathbb{C}[W_{zq\mathfrak{t}}, \kappa_{zq\mathfrak{t}}] = \mathbb{C}[W_{\mathfrak{s}^\vee, z\phi_v, q\epsilon}, \kappa_{z\phi_v, q\epsilon}]$$

to be the image of $qb_\gamma$ under (127). Then the $\mathbb{C}$-span of $\{T_\gamma : \gamma \in \Gamma\}$ is isomorphic to $\mathbb{C}[\Gamma]$, which shows that $\kappa_{z\phi_v, q\epsilon}|_{\Gamma \times \Gamma} = 1$.

(c) The union of the instances of part (a) with $^L L = {}^L\mathcal{L}$ and $q\epsilon|_{Z(\mathcal{L}_{\mathrm{sc}}^\vee)} = \zeta_{\mathcal{H}}^{\mathcal{L}}$ yields a surjection

$$(132) \qquad \bigsqcup_{(\phi_v, q\epsilon) \in \Phi_{\mathrm{cusp}, \zeta_{\mathcal{H}}^{\mathcal{L}}}(\mathcal{L})/X_{\mathrm{nr}}(^L\mathcal{L})} \left(\Phi_e(\mathcal{L})^{\mathfrak{s}_{\mathcal{L}}^\vee}//W_{\mathfrak{s}^\vee}\right)_\kappa \to \Phi_{e, \zeta_{\mathcal{H}}}(\mathcal{H}, \mathcal{L}).$$

Two elements $(\phi_v, q\epsilon, \tau)$ and $(\phi'_v, q\epsilon', \tau')$ on the left hand side can only have the same image in $\Phi_{e, \zeta_{\mathcal{H}}}(\mathcal{H}, \mathcal{L})$ if they have the same cuspidal support modulo unramified twists, for the map in Proposition 9.1.a preserves that. By (117) the inertial equivalence classes of $(\phi_v, \tau)$ and $(\phi'_v, \tau')$ differ only by an element of $W(^L\mathcal{H}, {}^L\mathcal{L}) \cong W(\mathcal{H}, \mathcal{L})$. We already know that the restriction of (132) to one inertial equivalence class is injective. Hence every fiber of (132) is in bijection with $W(^L\mathcal{H}, {}^L\mathcal{L})/W_{\mathfrak{s}^\vee}$ for some $\mathfrak{s}^\vee$.

By Lemma 8.2 $\widetilde{\Phi_{\mathrm{cusp}}}(\mathcal{L})$ (with respect to $W(^L\mathcal{H}, {}^L\mathcal{L})$) equals the disjoint union $\bigsqcup_{\mathfrak{s}_{\mathcal{L}}^\vee} \widetilde{(\mathfrak{s}_{\mathcal{L}}^\vee)}_\kappa$. In view of part (a), there is a unique way to extend the action of $W_{\mathfrak{s}^\vee}$ on $\widetilde{(\mathfrak{s}_{\mathcal{L}}^\vee)}_\kappa$ (for various $\mathfrak{s}_{\mathcal{L}}^\vee = [^L\mathcal{L}, \phi'_v, q\epsilon']_{^L\mathcal{L}}$) to an action of $W(^L\mathcal{H}, {}^L\mathcal{L})$ on $\widetilde{\Phi_{\mathrm{cusp}}}(\mathcal{L})$ such that maps from part (a) become constant on $W(^L\mathcal{H}, {}^L\mathcal{L})$-orbits. Then

$$\left(\Phi_{\mathrm{cusp}, \zeta_{\mathcal{H}}^{\mathcal{L}}}(\mathcal{L})//W(^L\mathcal{H}, {}^L\mathcal{L})\right)_\kappa = \widetilde{\Phi_{\mathrm{cusp}, \zeta_{\mathcal{H}}^{\mathcal{L}}}}(\mathcal{L})/W(^L\mathcal{H}, {}^L\mathcal{L}) \to \Phi_{e, \zeta_{\mathcal{H}}}(\mathcal{H}, \mathcal{L})$$

is the desired bijection.

(d) This is a direct consequence of part (c).

(e) By (88) and Definition 6.7

$$\Phi_e(^L\mathcal{H}) = \bigsqcup_{u \in H^1(F, \mathcal{H}_{\mathrm{ad}})} \Phi_e(\mathcal{H}^u).$$

By the assumption $\Phi_{\mathrm{cusp}}(\mathcal{L}^u) = \Phi_{\mathrm{cusp}, \zeta_u}(\mathcal{L}^u)$ for every extension $\zeta_u$ of the Kottwitz parameter of $\mathcal{L}^u$ to a character of $Z(\mathcal{L}_{\mathrm{sc}}^\vee)$, for there is nothing to extend to. Now apply part (d). $\qquad \square$

The canonicity in part (b) can be expressed as follows. Given $(\mathcal{L}, \phi_v, q\epsilon, \tau^\circ)$ with $\tau^\circ \in \mathrm{Irr}(W_{\mathfrak{t}^\circ})$, the set

$$\left\{ (\phi_v|_{\mathbf{W}_F}, q\Sigma_{qt}^{-1}(\tau)) \in \Phi_e(^L\mathcal{H})^{\mathfrak{s}^\vee} : \tau \in \mathrm{Irr}(\mathbb{C}[W_{\mathfrak{s}^\vee, \phi_v, q\epsilon}, \kappa_{\phi_v, q\epsilon}]) \text{ contains } \tau^\circ \right\}$$

is canonically determined.

It would be interesting to know when the above 2-cocycles $\kappa$ are trivial on $W_{\mathfrak{s}^\vee}$. Theorem 9.3.b shows that this happens quite often, in particular whenever $W_{\mathfrak{s}^\vee}$ fixes a point $(^L L, z'\phi_v, q\epsilon) \in \mathfrak{s}^\vee$ and at the same time $W_{\mathfrak{s}^\vee}$ equals the Weyl group $W(G_{z'}^\circ, L)$, where $G_{z'} = Z^1_{\mathcal{H}^\vee_{\mathrm{sc}}}(z'\phi|_{\mathbf{W}_F})$.

**Example 9.4.** Yet there are also cases in which $\kappa$ is definitely not trivial. Take $\mathcal{H} = \mathrm{SL}_5(D)$, where $D$ is a quaternion division algebra over $F$. This is an inner form of $\mathrm{SL}_{10}(F)$ and $^L\mathcal{H} = \mathrm{PGL}_{10}(\mathbb{C}) \times \mathbf{W}_F$.

We will rephrase Example 3.2 with L-parameters. We can ignore the factor $\mathbf{W}_F$ of $^L\mathcal{H}$, because it acts trivially on $\mathcal{H}^\vee$. Let $\overline{\phi} : \mathbf{W}_F \to \mathrm{SL}_2(\mathbb{C})^5$ be a group homomorphism whose image is the group $Q$ from Example 3.2. It projects to a homomorphism $\phi|_{\mathbf{W}_F} : \mathbf{W}_F \to \mathrm{PGL}_{10}(\mathbb{C})$. Let $u$ and $\epsilon$ be as in the same example. These data determine an enhanced L-parameter $(\phi, \epsilon)$ for $\mathcal{H}$. The group

$$G = Z_{\mathrm{SL}_{10}(\mathbb{C})}(Q) = Z_{\mathcal{H}^\vee_{\mathrm{sc}}}(\phi(\mathbf{W}_F))$$

was considered in Example 3.2. We checked over there that $W_{\mathfrak{s}^\vee} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and that its 2-cocycle $\kappa_{\mathfrak{s}^\vee, \phi, \epsilon}$ is nontrivial. We remark that this fits with the non-triviality of the 2-cocycle in [ABPS3, Example 5.5], which is essentially the same example, but on the $p$-adic side of the LLC.

Just like $^L\Psi$ in Lemma 7.11, the maps from Theorem 9.3 are compatible with the Langlands classification for L-parameters from Theorem 7.10.

**Lemma 9.5.** *Let $(\phi, \rho) \in \Phi_e(\mathcal{H})$ and let $(\mathcal{L}, \phi_t, z, \rho_t)$ be the enhanced standard triple associated to it by Theorem 7.10.*

(a) *Write $q\Psi_G(u_\phi, \rho) = q\mathfrak{t} = [M, \mathcal{C}_v^M, q\epsilon]_G$, $G_2 = Z^1_{\mathcal{L}^\vee_{\mathrm{sc}}}(\phi|_{\mathbf{W}_F})$, $M_2 = Z_{G_2}(Z(M)^\circ)$ and $q\mathfrak{t}_\mathcal{L} = [M_2, \mathcal{C}_v^{M_2}, q\epsilon_2]_{G_t}$ as in the proof of Lemma 7.11. Then $W_{q\mathfrak{t}} \cong W_{q\mathfrak{t}_\mathcal{L}}$.*

(b) *The image of $(\phi, \rho) \in \Phi_e(\mathcal{H})$ under Theorem 9.3.a equals the image of $(z\phi_t, \rho_t) \in \Phi_e(\mathcal{L})$. The latter can be expressed as $z \cdot (^L\Psi^\mathcal{L}(\phi_t, \rho_t), q\Sigma_{q\mathfrak{t}_\mathcal{L}}(u_{\phi_t}, \rho_t))$.*

*Proof.* Because all the maps are well-defined on conjugacy classes of enhanced L-parameters, we may assume that $\phi = z\phi_t$ and $\rho = \rho_t$.
(a) Recall from Lemma 8.2 that

$$(133) \qquad \begin{aligned} W_{q\mathfrak{t}} &\cong W(^L\mathcal{H}, Z_{^L\mathcal{H}}(Z(M)^\circ))_{\phi_v, q\epsilon}, \\ W_{q\mathfrak{t}_\mathcal{L}} &\cong W(^L\mathcal{L}, Z_{^L\mathcal{H}}(Z(M)^\circ))_{\phi_v, q\epsilon_2}. \end{aligned}$$

The argument following (113) shows that we may replace $q\epsilon_2$ by $q\epsilon$ here. Let $L_\emptyset$ be the unique minimal standard Levi subgroup of $\mathcal{H}^\vee$. Then

$$W(^L\mathcal{H}, Z_{^L\mathcal{H}}(Z(M)^\circ)) \cong N_{W(^L\mathcal{H}, {}^LL_\emptyset)}(Z(M)^\circ)/W(Z_{^L\mathcal{H}}(Z(M)^\circ), {}^LL_\emptyset),$$

$$W(^L\mathcal{L}, Z_{^L\mathcal{H}}(Z(M)^\circ)) \cong N_{W(^L\mathcal{L}, {}^LL_\emptyset)}(Z(M)^\circ)/W(Z_{^L\mathcal{H}}(Z(M)^\circ), {}^LL_\emptyset).$$

Recall from Definition 7.9 that $\phi_v|_{\mathbf{W}_F}\phi|_{\mathbf{W}_F} = z\phi_t|_{\mathbf{W}_F}$, where $z \in X_{\mathrm{nr}}(^L\mathcal{L}) = (Z(\mathcal{L}^\vee)^{\mathbf{I}_F})^\circ_{\mathrm{Frob}}$ is strictly positive with respect to the standard parabolic subgroup $\mathcal{P}$ of $\mathcal{H}$ having $\mathcal{L}$ as Levi factor. Hence the isotropy group of $z$ in the Weyl group

$W(^L\mathcal{H}, {}^LL_\emptyset)$ is the group generated by the reflections that fix $z$, which is precisely $W(^L\mathcal{L}, {}^LL_\emptyset)$.

Since $\phi_t$ is bounded and $z$ determines an unramified character of $\mathcal{L}$ with values in $\mathbb{R}_{>0}$, every element of $W_{qt}$ must fix both $\phi_t|_{\mathbf{W}_F}$ and $z$. By the above $W_{qt} \subset W(^L\mathcal{L}, {}^LL_\emptyset)$, and then (133) shows that $W_{qt} = W_{qt_\mathcal{L}}$.

(b) From Lemma 7.11 we know that $^L\Psi^\mathcal{H}(\phi, \rho) = {}^L\Psi^\mathcal{L}(\phi, \rho)$. By Proposition 5.6

$$q\Sigma_{qt_\mathcal{L}}(u_\phi, \rho) \quad \text{is a constituent of} \quad \text{Res}^{\text{End}_G(\pi_*(\widetilde{q\mathcal{E}}))}_{\text{End}_{G_2}(\pi_*^{G_2}(\widetilde{q\mathcal{E}_{G_2}}))} q\Sigma_{qt}(u, \eta).$$

By Lemma 5.4

$$\text{End}_G(\pi_*(\widetilde{q\mathcal{E}})) \cong \mathbb{C}[W_{qt}, \kappa_{qt}] \quad \text{and} \quad \text{End}_{G_t}(\pi_*^{G_t}(\widetilde{q\mathcal{E}_{G_t}})) \cong \mathbb{C}[W_{qt_\mathcal{L}}, \kappa_{qt_\mathcal{L}}].$$

By Proposition 5.6.b $\kappa_{qt}|_{W_{qt_\mathcal{L}}} = \kappa_{qt_\mathcal{L}}$, so in view of part (a) these two algebras are equal. Thus $q\Sigma_{qt_\mathcal{L}}(u_\phi, \rho) = q\Sigma_{qt}(u_\phi, \rho)$. Together with Lemma 7.11 this shows that the image of $(\phi, \rho)$ under Theorem 9.3.a is the same for $\mathcal{H}$ and for $\mathcal{L}$.

Since $z$ lifts to a central element of $\mathcal{L}^\vee$, $qt_\mathcal{L}$ is the same for $(\phi_t, \rho_t)$ and $(z\phi_t, \rho_t)$. That goes also for $q\Sigma_{qt_\mathcal{L}}(u_{\phi_t}, \rho_t)$. In combination with Lemma 7.11 we find that

$$\left({}^L\Psi^\mathcal{L}(z\phi_t, \rho_t), q\Sigma_{qt_\mathcal{L}}(u_{\phi_t}, \rho_t)\right) = z \cdot \left({}^L\Psi^\mathcal{L}(\phi_t, \rho_t), q\Sigma_{qt_\mathcal{L}}(u_{\phi_t}, \rho_t)\right). \qquad \square$$

## Appendix A. Erratum (2025)

In 2025 Simon Riche pointed out to us that Theorem 3.1.a does not hold in the stated generality.

Namely, $G^\circ$ could be a group $\text{Spin}_N(\mathbb{C}) \times \text{Spin}_N(\mathbb{C})$ such that $N$ is both a square and a triangular number. On the first copy of $\text{Spin}_N(\mathbb{C})$ one can take the cuspidal local system $\mathcal{L}_+$ supported on the unipotent class $\mathcal{C}_+$ (as in the proof of Theorem 3.1), the second copy one can take cuspidal local system $\mathcal{L}_-$ supported on the unipotent class $\mathcal{C}_-$. Then $\mathcal{L} = \mathcal{L}_+ \boxtimes \mathcal{L}_-$ is a cuspidal local system supported on the unipotent class $\mathcal{C}_+ \times \mathcal{C}_-$ in $G^\circ$.

Let $\tau$ be the automorphism of $G^\circ$ that swaps the two copies of $\text{Spin}_N(\mathbb{C})$. Then $\tau(\mathcal{C}_+ \times \mathcal{C}_-) = \mathcal{C}_- \times \mathcal{C}_+$ is another unipotent class in $G^\circ$ that supports a cuspidal local system, namely $\tau^*(\mathcal{L}) = \mathcal{L}_- \boxtimes \mathcal{L}_+$. The central characters of $\mathcal{L}$ and $\tau^*(\mathcal{L})$ differ, but they are in the same $\text{Aut}(G^\circ)$-orbit.

Below we repair this, by settling for a somewhat weaker statement of Theorem 3.1.a. This means that in general the unipotent class $\mathcal{C}_u^{G^\circ}$ is not stable under all automorphisms of $G^\circ$, and it need not be equal to $\mathcal{C}_u^G$. Although an equality of the form $\mathcal{C}_u^G = \mathcal{C}_u^{G^\circ}$ is used several times in the paper, that is mainly for convenience, with the below corrections it can be avoided. Most of the time it suffices to replace $G$ by $G_{\mathcal{C}_u^{G^\circ}}$, the stabilizer of $\mathcal{C}_u^{G^\circ}$.

**Corrections.**
- p.15, second line of Theorem 3.1: add "Let $\text{Aut}^s(G^\circ)$ be the subgroup of $\text{Aut}(G^\circ)$ that stabilizes every simple factor of $G^\circ$."
- p.15, statement of Theorem 3.1.a: replace "$\text{Aut}(G^\circ)$" by "$\text{Aut}^s(G^\circ)$"
- p.15, sixth line of proof of Theorem 3.1: replace "Aut" by "$\text{Aut}^s$"
- p.15, seventh line of proof of Theorem 3.1: replace "$\bar{X}$ decomposes" by "$\bar{X}$ and $\text{Aut}^s(\tilde{G})$ decompose"
- p.16, remove lines -6 and -5.

- p.16–18, replace every occurrence of "$G/G^{\circ}$" by "$G_{\mathcal{C}_u^{G^{\circ}}}/G^{\circ}$" (In Example 3.2 this does not make a difference.)
- p.19, line 19, replace "$Y$" by "$Y^{\circ} = \mathcal{C}_u^{G^{\circ}} Z(G^{\circ})^{\circ}$"
- p.19, (32), replace "$Y \times G$" by "$\widehat{Y^{\circ}} = \{(y,g) \in Y \times G : g^{-1}yg \in Y^{\circ}\}$"
- p.19, lines 24–27, replace "$Y \times G$" by "$\widehat{Y^{\circ}}$" (three times)
- p.19, line 28, after "fibration", add "with fibre $G_{\mathcal{C}_u^{G^{\circ}}}/G^{\circ}$"
- p.19, line 31, replace "$G/G^{\circ}$" by "$G_{\mathcal{C}_u^{G^{\circ}}}/G^{\circ}$"
- p.23, lines 3–7, replace "$N_G(L)/L$" by "$N_G(L)_{\mathcal{C}_v^L}/L$" (three times)
- p.24, line 6, replace "By Theorem 3.1.a $N_G(L)$ stabilizes $\mathcal{C}_v^L$" by "Recall from (21) that"
- p.29, line 1, remove "$= \mathcal{C}_v^{M^{\circ}}$"
- p.29, line 6, replace "$S = \mathcal{C}_v^M Z(M)^{\circ}$" by "$S = \mathcal{C}_v^{M^{\circ}} Z(M)^{\circ}$"
- p.31, line 14, remove "by Theorem 3.1.a"
- p.31, line 15 and 21, replace "$N_G(M^{\circ})/M$" by "$N_G(M^{\circ})_{\mathcal{C}_v^{M^{\circ}}} M/M$"

## References

[AcHe] P.N. Achar, A. Henderson, "Geometric Satake, Springer correspondence and small representations", Selecta Math. (N.S.) **19.4** (2013), 949–986.

[AHJR] P.N. Achar, A. Henderson, D. Juteau, S. Riche, "Weyl group actions on the Springer sheaf", Proc. Lond. Math. Soc. (3) **108.6** (2014), 1501–1528.

[AcSa] P.N. Achar, D.S. Sage, "On special pieces, the Springer correspondence, and unipotent characters", Amer. J. Math. **130.5** (2008), 1399–1425.

[Art1] J. Arthur, "On the transfer of distributions: weighted orbital integrals", Duke Math. J. **99.2** (1999), 209–283.

[Art2] J. Arthur, "A note on L-packets", Pure Appl. Math. Quaterly **2.1** (2006), 199–217.

[ABPS1] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "Geometric structure in smooth dual and local Langlands correspondence", Japan. J. Math. **9** (2014), 99–136.

[ABPS2] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "The local Langlands correspondence for inner forms of $\mathrm{SL}_n$", Research in the Mathematical Sciences **3.32** (2016), DOI: 10.1186/s40687-016-0079-4.

[ABPS3] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "Hecke algebras for inner forms of $p$-adic special linear groups", J. Inst. Math. Jussieu **16.2** (2017), 351–419.

[ABPS4] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "The principal series of $p$-adic groups with disconnected centre", Proc. London Math. Soc. **114.5** (2017), 798–854.

[ABPS5] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "The noncommutative geometry of inner forms of $p$-adic special linear groups", arXiv:1505.04361, 2015.

[ABPS6] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "Conjectures about $p$-adic groups and their noncommutative geometry", Contemp. Math. **691** (2017), 15–51.

[AMS] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Graded Hecke algebras for disconnected reductive groups", arXiv:1607.02713, 2016 (to appear in proceedings of the Simons symposium "Geometric Aspects of the Trace Formula", 2016)

[BeDe] J. Bernstein, P. Deligne, "Le centre de Bernstein", pp. 1–32 in: *Représentations des groupes réductifs sur un corps local*, Travaux en cours, Hermann, 1984.

[BeLu] J. Bernstein, V. Lunts, "*Equivariant sheaves and functors*", Lecture Notes in Mathematics **1578**, Springer-Verlag, 1994.

[Bor] A. Borel, "Automorphic L-functions", Proc. Symp. Pure Math **33.2** (1979), 27–61.

[CuRe] C.W. Curtis, I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **11**, John Wiley & Sons, 1962.

[DeRe] S. DeBacker, M. Reeder, "Depth-zero supercuspidal L-packets and their stability", Ann. of Math. (2) **169.3** (2009), 795–901.

[DKV] P. Deligne, D. Kazhdan, M.-F. Vigneras, "Représentations des algèbres centrales simples $p$-adiques", pp. 33–117 in: *Représentations des groupes réductifs sur un corps local* Travaux en cours, Hermann, 1984.

[GGP] W.-T. Gan, B. Gross, D. Prasad, "Symplectic local root numbers, central critical $L$ values, and restriction problems in the representation theory of classical groups", Astérisque **346** (2012), 1–109.

[GoMP] M. Goresky, R. MacPherson, "Intersection homology II", Invent. Math. **72** (1983), 77–129.

[Hai] T.J. Haines, "The stable Bernstein center and test functions for Shimura varieties", pp. 118–186 in: *Automorphic forms and Galois representations*, London Math. Soc. Lecture Note Ser. **415**, Cambridge University Press, 2014.

[Hen] G. Henniart, "Une caractérisation de la correspondance de Langlands locale pour GL(n)", Bull. Soc. math. France **130.4** (2002), 587–602.

[HiSa] K. Hiraga, H. Saito, "On L-packets for inner forms of $SL_n$", Mem. Amer. Math. Soc. **1013**, Vol. **215** (2012).

[Kal] T. Kaletha, "Global rigid inner forms and multiplicities of discrete automorphic representations", arXiv:1501.01667, 2015.

[KMSW] T. Kaletha, A. Minguez, S.-W. Shin, P.-J. White, "Endoscopic Classification of Representations: Inner Forms of Unitary Groups", arXiv:1409.3731v3, 2014

[KaLu] D. Kazhdan, G. Lusztig, "Proof of the Deligne–Langlands conjecture for Hecke algebras", Invent. Math. **87** (1987), 153–215.

[Kos] B. Kostant, "The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group" Amer. J. Math. **81** (1959), 973–1032.

[Kot] R.E. Kottwitz, "Stable trace formula: cuspidal tempered terms", Duke Math. J. **51.3** (1984), 611–650.

[Lan] R.P. Langlands, "On the classification of irreducible representations of real algebraic groups", pp. 101–170 in: *Representation theory and harmonic analysis on semisimple Lie groups*, Math. Surveys Monogr. **31**, American Mathematical Society, 1989.

[Lus1] G. Lusztig, "Some examples of square-integrable representations of semisimple $p$-adic groups", Trans. Amer. Math. Soc. **277.2** (1983), 623–653.

[Lus2] G. Lusztig, "Intersection cohomology complexes on a reductive group", Invent. Math. **75.2** (1984), 205–272.

[Lus3] G. Lusztig, "Classification of unipotent representations of simple $p$-adic groups", Int. Math. Res. Notices **11** (1995), 517–589.

[Lus4] G. Lusztig, "Classification of unipotent representations of simple p-adic groups. II", Represent. Theory **6** (2002), 243–289.

[Lus5] G. Lusztig, "Character sheaves on disconnected groups, II", Represent. Theory **8** (2004), 72–124.

[LuSp] G. Lusztig, N. Spaltenstein, "On the generalized Springer correspondence for classical groups", pp. 289–316 in: *Algebraic Groups an Related Topics*, Advanced Studies in Pure Math. **6**, North Holland, 1985.

[Miz] K. Mizuno, "The conjugate classes of unipotent elements of the Chevalley groups $E_7$ and $E_8$", Tokyo J. Math. **3** (1980), 391–461.

[Moe] C. Mœglin, "Classification et changement de base pour les séries discrètes des groupes unitaires $p$-adiques", Pacific J. Math. **233** (2007), 159–204.

[Mou] A. Moussaoui, "Centre de Bernstein stable et conjecture d'Aubert–Baum–Plymen–Solleveld", PhD. thesis, Université Pierre et Marie Curie, 2015.

[RaRa] A. Ram, J. Rammage "Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory" pp. 428–466 in: *A tribute to C.S. Seshadri (Chennai 2002)*, Trends in Mathematics, Birkhäuser, 2003.

[Ree1] M. Reeder, "Euler-Poincaré pairings and elliptic representations of Weyl groups and $p$-adic groups", Compos. Math. **129** (2001), 149–181.

[Ree2] M. Reeder, "Supercuspidal L-packets of positive depth and twisted Coxeter elements", J. reine angew. Math. **620** (2008), 1–33.

[Ren] D. Renard, *Représentations des groupes réductifs p-adiques*, Cours spécialisés **17**, Société Mathématique de France, 2010.

[SiZi] A.J. Silberger, E.-W. Zink, "Langlands Classification for L-Parameters", arXiv:1407.6494, 2014.

[Sol1] M. Solleveld, "Periodic cyclic homology of reductive $p$-adic groups", J. Noncommutative Geometry **3.4** (2009), 501–558.

[Sol2]  M. Solleveld, "Parabolically induced representations of graded Hecke algebras", Algebras and
        Representation Theory **15.2** (2012), 233–271.

[Sor]   K. Sorlin, "Springer correspondence in non connected reductive groups", J. reine angew. Math.
        **568** (2004), 197–234.

[Spr]   T.A. Springer, "A construction of representations of Weyl groups", Invent. Math. **44.3** (1978),
        279-293.

[Tad]   M. Tadić, "Induced representations of $GL(n, A)$ for $p$-adic division algebras $A$", J. reine
        angew. Mathe **405** (1990), 48–77.

[Tha]   Thăńg, Nguyêñ Quôć, "On Galois cohomology and weak approximation of connected reduc-
        tive groups over fields of positive characteristic", Proc. Japan Acad. Ser. A Math. Sci. **87.10**
        (2011), 203–208.

[Vog]   D. Vogan "The local Langlands conjecture", pp. 305–379 in: *Representation theory of groups
        and algebras*, Contemp. Math. **145**, American Mathematical Society, 1993.

[Zel]   A.V. Zelevinsky, "Induced representations of reductive p-adic groups II. On irreducible repre-
        sentations of GL(n)", Ann. Sci. École Norm. Sup. (4) **13.2** (1980), 165–210.

Institut de Mathématiques de Jussieu – Paris Rive Gauche, U.M.R. 7586 du C.N.R.S.,
U.P.M.C., 4 place Jussieu 75005 Paris, France
    *Email address*: anne-marie.aubert@imj-prg.fr

Department of Mathematics and Statistics, University of Calgary, 2500 University
Drive NW, Calgary, Alberta, Canada
    *Email address*: ahmed.moussaoui@ucalgary.ca

IMAPP, Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, the
Netherlands
    *Email address*: m.solleveld@science.ru.nl