Algebraic Topology, Exercise sheet No 2

Prove the following facts:

1. The interval $I = [0, 1]$ is connected.
2. A path connected space is connected.
3. A homotopy equivalence $f : X \to Y$ induces an isomorphism $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ of the respective sets of path components. Deduce that a contractible space is path connected.
4. Let $D^n = \{x = (x_1, \ldots, x_n) \mid \sum_i x_i^2 \leq 1\}$ be the closed n-ball. (D stands for 'disc'.) Assume there is a continuous map $f : D^n \to D^n$ such that $f(x) \neq x$ for all $x \in D_n$. Use this to construct a retraction $p : D^n \to \partial D^n = S^{n-1}$.
5. Use the facts $\pi_1(D^2, x_0) = \{e\}$ (D^2 is contractible) and $\pi_1(S^1, x_0) \cong \mathbb{Z}$ to prove Brouwers fixpoint theorem for $n = 2$:

   0.1 Theorem *Let $f : D^n \to D^n$ be continuous. Then there exists a fixpoint, i.e. $x \in D^n$ such that $f(x) = x$.*

   Remark: For $n \geq 2$, the theorem will follow from the existence of homotopy invariant functors $H_n : \mathrm{Top}_* \to \mathrm{Ab}$ such that
   $$H_n(S^n) \cong \mathbb{Z},$$
   since homotopy invariance implies $H_m(D^n) = \{e\}$ and exactly the same argument works.

6. Let $f : S^1 \to S^1$ be continuous. Define the degree of f, $\deg(f) \in \mathbb{Z}$, as follows: Consider the sequence
   $$\pi_1(S^1, x_0) \xrightarrow{f_*} \pi_1(S^1, f(x_0)) \xrightarrow{i_*} \pi_1(S^1, x_0),$$
   where the second map is induced by a path $i : I \to S^1$ with $\gamma(0) = x_0,\ \gamma(1) = f(x_0)$. Pick a generator $[\gamma] \in \pi_1(S^1, x_0)$ and define the degree of f to be $(\gamma_* \circ f_*)([\gamma]) := \deg(f)$.

   i) Show that this is well defined, i.e. independent of the choice of path i.

   ii) Show that it is homotopy invariant; $f \sim_h g \Rightarrow \deg(f) = \deg(g)$.

   iii) Calculate the degree of the identity map and the map $x \mapsto x^n$ where x is a complex number of norm 1.

7. Prove the fundamental theorem of algebra:

   0.2 Theorem *Let f be a polynomial with complex coefficients, i.e.*
   $$f(x) = x^n + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n,\ c_i \in \mathbb{C},\ n > 0.$$
   Then f has n roots, counted with multiplicities.

   (*hint:* Suppose f has no root. Then define $\hat{f} : S^1 \to S^1$ by
   $$\hat{f}(n) := f(x)/|f(x)|.$$
   Calculate $\deg(\hat{f})$ by considering both $h, g : S^1 \times I \to S^1$,
   $$h(x, t) = \hat{f}(tx),\ g(x, t) = \hat{f}(x/t),\ t \in I,$$
   and derive a contradiction. Finally, use induction to prove that there are n roots.)