Algebraic Topology, Exercise sheet No 3

Prove the following facts:

1. Recall that a space X is simply connected iff it is path connected and $\pi_1(X, x_0) = \{e\}$ for any (and thus all) $x_0 \in X$. Show that X is simply connected iff there is a unique homotopy class of paths connecting any pair of points.

2. If X_0 is the path component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

3. Given a space X and a path-connected subspace A containing the basepoint x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A.

4. Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

5. For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms γ_h depend only on the endpoints of the path h.

6. Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1, show that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\gamma_h"] \arrow[d, "f_*"] & \pi_1(X, x_1) \arrow[d, "f_*"'] \\
\pi_1(Y, f(x_0)) \arrow[r, "\gamma_{fh}"'] & \pi_1(Y, f(x_1))
\end{tikzcd}$$

7. Let G be a topological group with unit e. Show that $\pi_1(G, e)$ is abelian.