Algebraic Topology, Exercise sheet No 4

Prove the following facts:

1. If X is locally path connected then its path components are clopen (=closed and open).

2. If X is connected and locally path connected then it is path connected. Consider the subspace of $\mathbb{R}^2$ given by
$$X = \{0\} \times [-1,1] \ \cup \ \left\{ \left(x, \sin \frac{1}{x} \right), \quad x \in (0,1] \right\}.$$
Prove that X is connected but not path-connected. Prove directly that X is not locally path-connected.

3. Consider the following subspace of $\mathbb{R}^2$ and prove that it is path-connected (thus connected) but not locally path-connected.

(ask Hessel)

4. Let X be a space and assume G acts on X totally discontinuously. (I.e., for every $x \in X$ there exists an open $U \ni x$ such that $gU \cap U \neq \emptyset \ \Rightarrow \ g = e$.) Consider the orbit space X/G (=the quotient space $X/\simeq$, where $\simeq$ is the equivalence relation $x \simeq y \ \Leftrightarrow \ \exists g \in G$ s. th. $gx = y$). Prove that the quotient map $\pi : X \to X/G$ is a covering map.