Algebraic Topology, Exercise sheet No 6

1. Let G be a group acting on the space X in a properly discontinuous way. (I.e., every $x \in X$ has a neighborhood U such that the sets $gU,\ g \in G$ are pairwise disjoint.) Prove:
   (a) The quotient map $X \to X/G$ is a normal covering map.
   (b) If X is path-connected then G is the group of covering maps of $p : X \to X/G$.
   (c) If X is path-connected and locally path-connected then G is isomorphic to $\pi_1(X/G)/p_*(\pi_1(X))$.
2. Let G, H be non-trivial groups. Show that the only elements of $G \star H$ of finite order are the conjugates of the finite-order elements of G and H.
3. Prove the following *universal property* of $G \star_K H$ (w.r.t. given homomorphisms $\alpha_1 : K \to G, \alpha_2 : K \to H$): If there are homomorphisms $\beta_1 : G \to L$ and $\beta_2 : H \to L$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ then there exists a unique homomorphism $\gamma : G \star_K H \to L$ such that $\gamma \circ \iota_1 = \beta_1$ and $\gamma \circ \iota_2 = \beta_2$ (where ι_1, ι_2 are the embeddings of G, H into $G \star_K H$). Deduce that $G \star_K H$ is unique up to isomorphism.