Algebraic Topology, Exercise sheet No 9

Next week you are going to compute the homology groups of the non-trivial spheres $S^m, m \geq 1$:

$$H_n(S^m) = \begin{cases} \mathbb{Z} & n = 0, m \\ 0 & \text{otherwise} \end{cases}$$

(If we define S^0 to be a two-point set then you already know

$$H_n(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

You might consider this as a limit case of the above for $m = 0$.)
Prove the following:

1. Let $m \geq 2$. Show that there is no retraction from the closed ball D^m to its boundary $\partial D^m = S^{m-1}$.

2. Prove Brouwer's fixpoint theorem for $D^m, m \geq 2$: Every map $f : D^m \to D^m$ has a fixpoint x, i.e. $f(x) = x$.

Let $n \geq 1$. Every map $f : S^n \to S^n$ induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. Fixing an isomorphism $\alpha : H_n(S^n) \to \mathbb{Z}$, we get $\alpha \circ f_* \circ \alpha^{-1} : \mathbb{Z} \to \mathbb{Z}$. This map is given by multiplication with an element of $\mathbb{Z}$, which we call the degree $\deg f$ of f.

Consider the map $f : S^n \to S^n$ given by $(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$. Next week you will prove that $\deg f = -1$. Use this to prove the following:

3. For every map $f : S^{2n} \mapsto S^{2n}$ there exists x such that $f(x) = x$ or $f(x) = -x$. Proceed as follows:

   - If $f : S^n \to S^n$ is given by $x \mapsto -x$ then $\deg f = (-1)^{n+1}$.
   - The map $x \mapsto -x$ on S^{2n} is not homotopic to the identity map.
   - Assume $f : S^{2n} \mapsto S^{2n}$ is such that $f(x) \neq x$ and $f(x) \neq -x$ for all x. Then there are continuous maps

   $$\begin{aligned} F(x,t) &= \frac{tf(x) + (1-t)x}{\|tf(x) + (1-t)x\|}, \qquad S^{2n} \times I \to S^{2n}, \\ G(x,t) &= \frac{-tx + (1-t)f(x)}{\| -tx + (1-t)f(x)\|}, \qquad S^{2n} \times I \to S^{2n}, \end{aligned}$$

   which are homotopies from the identity map to f and from f to minus the identity map, respectively.

4. Let $f : S^{2n} \to \mathbb{R}^{2n+1}$ be such that $f(x)$ is orthogonal to x for all $x \in S^{2n}$. Prove that there exists $x \in S^{2n}$ such that $f(x) = 0$.

Remark. The last result is usually stated as follows: The even sphere S^{2n} does not admit an everywhere non-vanishing tangent vector field.