Algebraic Topology, Exercise sheet No 10

Recall the Mayer-Vietoris theorem:

THEOREM 0.1 (MAYER-VIETORIS) *Let X be a space and $U, V \subset X$ subspaces such that $U^0 \cup V^0 = X$. Then there exists a long exact sequence*

$$\cdots\ H_{n+1}(X) \xrightarrow{\partial_{n+1}} H_n(U \cap V) \xrightarrow{i_{1*} \oplus i_{2*}} H_n(U) \oplus H_n(V) \xrightarrow{j_{1*} - j_{2*}} H_n(X) \xrightarrow{\partial_n} H_{n-1}(U \cap V) \ldots,$$

where $i_1 : U \cap V \hookrightarrow U,\ i_2 : U \cap V \hookrightarrow V$ and $j_1 : U \hookrightarrow X,\ j_2 : V \hookrightarrow X$ are the obvious inclusion maps.

1. Use the Mayer-Vietoris exact sequence to compute the homology groups $H_n(S^m)$ for $m \geq 1$:

$$H_n(S^m) = \begin{cases} \mathbb{Z} & n = 0,\, m \\ 0 & \text{otherwise} \end{cases}$$

2. Let $n \geq 1$. Every map $f : S^n \to S^n$ induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. Fixing an isomorphism $\alpha : H_n(S^n) \to \mathbb{Z}$, we get $\alpha \circ f_* \circ \alpha^{-1} : \mathbb{Z} \to \mathbb{Z}$. This map is given by multiplication with an element of $\mathbb{Z}$, which we call the degree $\deg f$ of f.

   Consider the map $f : S^n \to S^n$ given by $(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$. Prove that $\deg f = -1$.

Hints: Proceed along the following steps:

1. Compute $H_0(S^m)$ for $m \geq 1$.
2. Consider the covering $(0 < \varepsilon < 1)$

$$\begin{aligned} U &= \{(x_0, x_1) \in S^1 \mid x_1 \geq -\varepsilon\}, \\ V &= \{(x_0, x_1) \in S^1 \mid x_1 \leq +\varepsilon\} \end{aligned}$$

   of S^1 and use the end of the Mayer-Vietoris sequence to prove $H_1(S^1) \cong \mathbb{Z}$.
3. Prove $\deg f = -1$ for $f : S^1 \to S^1,\ (x_0, x_1) \mapsto (-x_0, x_1)$.
4. Prove $H_n(S^1) = 0$ for $n \geq 2$.
5. Consider the covering $(0 < \varepsilon < 1)$

$$\begin{aligned} U &= \{(x_0, \ldots, x_m) \in S^1 \mid x_m \geq -\varepsilon\}, \\ V &= \{(x_0, \ldots, x_m) \in S^1 \mid x_m \leq +\varepsilon\} \end{aligned}$$

   of S^m and prove $H_1(S^m) = 0$ for $m \geq 2$.
6. For $n, m \geq 2$, show that $H_n(S^m) \cong H_{n-1}(S^{m-1})$ and deduce part one of the theorem.
7. Prove $\deg f = -1$ for $f : S^m \to S^m$, where $m \geq 2$.