But then the linearity of Λ shows that $\|\Lambda x\| < \epsilon$. Hence $\|\Lambda\| \leq \epsilon/\delta$, and (*c*) implies (*a*). ////

Consequences of Baire's Theorem

5.5 The manner in which the completeness of Banach spaces is frequently exploited depends on the following theorem about complete metric spaces, which also has many applications in other parts of mathematics. It implies two of the three most important theorems which make Banach spaces useful tools in analysis, the *Banach-Steinhaus theorem* and the *open mapping theorem*. The third is the *Hahn-Banach extension theorem*, in which completeness plays no role.

5.6 Baire's Theorem *If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.*

In particular (except in the trivial case $X = \varnothing$), the intersection is not empty. This is often the principal significance of the theorem.

PROOF Suppose $V_1, V_2, V_3, \ldots$ are dense and open in X. Let W be any open set in X. We have to show that $\bigcap V_n$ has a point in W if $W \neq \varnothing$.

Let ρ be the metric of X; let us write

$$S(x, r) = \{y \in X : \rho(x, y) < r\} \tag{1}$$

and let $\bar{S}(x, r)$ be the closure of $S(x, r)$. [*Note*: There exist situations in which $\bar{S}(x, r)$ does *not* contain all y with $\rho(x, y) \leq r$!]

Since V_1 is dense, $W \cap V_1$ is a nonempty open set, and we can therefore find x_1 and r_1 such that

$$\bar{S}(x_1, r_1) \subset W \cap V_1 \quad \text{and} \quad 0 < r_1 < 1. \tag{2}$$

If $n \geq 2$ and x_{n-1} and r_{n-1} are chosen, the denseness of V_n shows that $V_n \cap S(x_{n-1}, r_{n-1})$ is not empty, and we can therefore find x_n and r_n such that

$$\bar{S}(x_n, r_n) \subset V_n \cap S(x_{n-1}, r_{n-1}) \quad \text{and} \quad 0 < r_n < \frac{1}{n}. \tag{3}$$

By induction, this process produces a sequence $\{x_n\}$ in X. If $i > n$ and $j > n$, the construction shows that x_i and x_j both lie in $S(x_n, r_n)$, so that $\rho(x_i, x_j) < 2r_n < 2/n$, and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is a point $x \in X$ such that $x = \lim_{n \to \infty} x_n$.

Since x_i lies in the closed set $\bar{S}(x_n, r_n)$ if $i > n$, it follows that x lies in each $\bar{S}(x_n, r_n)$, and (3) shows that x lies in each V_n. By (2), $x \in W$. This completes the proof. ////

Corollary *In a complete metric space, the intersection of any countable collection of dense G_δ's is again a dense G_δ.*