Example I.1.2 Let X be a locally compact Hausdorff space. Then $C_0(X)$, the space of all continuous functions on X vanishing at infinity, forms a C*-algebra with complex conjugation as the adjoint operation. Then

$$\|\overline{f}f\|_X = \sup_{x\in X} |\overline{f(x)}f(x)| = \sup_{x\in X} |f(x)|^2 = \|f\|_X^2.$$

Following the definitions for operators, we say that an element A of a C*-algebra $\mathfrak{A}$ is **self-adjoint** if $A^* = A$; N is **normal** if $N^*N = NN^*$; and U is **unitary** if $U^*U = I = UU^*$. We also define A to be **positive** if $A = A^*$ and the spectrum (see section I.2) $\sigma(A)$ is contained in the non-negative real line $[0,\infty)$.

It is often convenient to have a unit around, even when the algebra is not unital. So we show how to adjoin one while maintaining the C*-algebra structure. In fact, it is unique (see the exercises).

When a C*-algebra $\mathfrak{A}$ has an identity element I, compute

$$I^*A = (A^*I)^* = A^{**} = A \quad \text{for all} \quad A \in \mathfrak{A}.$$

So $I^* = I^*I = I$. Hence $\|I\|^2 = \|I^*I\| = \|I\|$. Since $I \neq 0$, this shows that $\|I\| = 1$.

Proposition I.1.3 *Every non-unital C*-algebra $\mathfrak{A}$ is contained in a unital C* algebra $\mathfrak{A}^\sim$ as a maximal ideal of codimension one.*

Proof. Form $\mathfrak{A}^\sim := \mathfrak{A} \oplus \mathbb{C}$ and define

$$\begin{aligned}(A,\lambda)(B,\mu) &:= (AB + \lambda B + \mu A, \lambda\mu)\\ (A,\lambda)^* &:= (A^*, \overline{\lambda})\\ \|(A,\lambda)\| &:= \sup_{\|B\|\leq 1} \|AB + \lambda B\|\end{aligned}$$

This makes $\mathfrak{A}^\sim$ into a $*$-algebra. The norm is a Banach algebra norm because it is the norm induced from the space $\mathcal{B}(\mathfrak{A})$ of bounded operators on $\mathfrak{A}$ given by the $*$-algebra of operators $\{L_A + \lambda I : A \in \mathfrak{A},\ \lambda \in \mathbb{C}\}$, where $L_A(B) = AB$ is a left multiplication operator. Thus this is a Banach $*$-algebra with unit $(0,1)$. By design, $\mathfrak{A}$ is a maximal ideal of co-dimension one. The imbedding of $\mathfrak{A}$ into $\mathfrak{A}^\sim$ is isometric because

$$\|A\| = \|A(A^*/\|A\|)\| \leq \|(A,0)\| = \sup_{\|B\|\leq 1} \|AB\| \leq \|A\|.$$

It remains to verify the C*-algebra norm condition.

$$\begin{aligned} \|(A,\lambda)\|^2 &= \sup_{\|B\|\le 1} \|AB+\lambda B\|^2 \\ &= \sup_{\|B\|\le 1} \|B^*A^*AB+\lambda B^*A^*B+\overline{\lambda}B^*AB+|\lambda|^2B^*B\| \\ &\le \sup_{\|B\|\le 1} \|A^*AB+\lambda A^*B+\overline{\lambda}AB+|\lambda|^2B\| \\ &= \|(A^*A+\lambda A^*+\overline{\lambda}A, |\lambda|^2)\| \\ &= \|(A,\lambda)^*(A,\lambda)\| \le \|(A,\lambda)^*\|\,\|(A,\lambda)\| \end{aligned}$$

Thus $\|(A,\lambda)\| \le \|(A,\lambda)^*\|$. By symmetry, we have $\|(A,\lambda)\| = \|(A,\lambda)^*\|$. Hence the inequality above is an equality, and so

$$\|(A,\lambda)\|^2 = \|(A,\lambda)^*(A,\lambda)\|$$

as claimed. ■

I.2 Banach Algebras Basics

For the convenience of the reader, we review the necessary background from Banach algebras that we need.

The **spectrum** of an element A of a unital Banach algebra $\mathfrak{A}$ is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}.$$

The complement of the spectrum is called the **resolvent**, and $R_A(\lambda) = (\lambda I - A)^{-1}$ is the **resolvent function.**

Theorem I.2.1 *In any unital Banach algebra $\mathfrak{A}$, the spectrum of each A in $\mathfrak{A}$ is a non-empty compact set; and the resolvent function is analytic on $\mathbb{C}\setminus\sigma(A)$.*

Proof. If $|\lambda| > \|A\|$, then $\|\lambda^{-n}A^n\| \le (|\lambda|^{-1}\|A\|)^n$ decreases geometrically fast; so the series

$$\sum_{n\ge 0} \lambda^{-n-1}A^n$$

is norm convergent. The limit is $(\lambda I - A)^{-1}$ since

$$(\lambda I - A)\sum_{n=0}^{k} \lambda^{-n-1}A^n = I - \lambda^{-k-2}A^{k+1}$$

which converges to I. Moreover, this shows that $R_A(\lambda)$ is analytic, and has a Laurent expansion about the point at infinity. Furthermore,

$$\lim_{|\lambda|\to\infty} \|R_A(\lambda)\| \le \lim_{|\lambda|\to\infty} |\lambda|^{-1}(1-|\lambda|^{-1}\|A\|)^{-1} = 0.$$