Errata and addenda to G. Murphy's "C^*-Algebras and Operator Theory"

Michael Müger (Univ. Nijmegen)

May 17, 2017

1 Errata

- p.7: Theorem 1.2.1: There is a small problem with the proof: If $\deg(p) = 0$, i.e. $p(z) = c_0$ then $p(a) = c_0\mathbf{1}$, thus $\sigma(p(a)) = \{c_0\}$. On the other hand, $p(\sigma(a)) = \{c_0\}$ **if** $\sigma(a) \neq \emptyset$, while otherwise $p(\sigma(a)) = \emptyset$! Thus the result is not true for all unital algebras (but it is for Banach algebras, by Theorem 1.2.5).
- p.10, l.-15: Replace "depending on λ" by "independent of λ".
- p.14, l.-10: Remove 'unital' (unnecessary assumption).
- p.15, l.8: Replace Theorem 1.3.4 by Theorem 1.3.3(i).
- p.80, Remark 3.1.2: Approximate units are not needed! Let $a \in A, b \in J^+$. Then $b^{1/2} \in J$, and $ab = ab^{1/2}b^{1/2}$. Since $I \subseteq A$ is an ideal and $b^{1/2} \in J \subseteq I$, we have $ab^{1/2} \in I$. And since $J \subseteq I$ is an ideal, we have $(ab^{1/2})b^{1/2} \in J$. Thus $J \subseteq A$ is an ideal.
- p.82, first paragraph: Again, approximate units are superfluous: If $a \in (I \cap J)^+$ then $a^{1/2} \in I \cap J$. Now $a = a^{1/2}a^{1/2} \in IJ$, since $a^{1/2}$ is in both I and J.

2 Addenda: Banach algebras

- p.6, l.8: While one needs completeness for the stronger results on $\mathrm{Inv}(A)$, the following always holds:

  2.1 Proposition *If A is a normed unital algebra,* $\mathrm{Inv}(A)$ *is a topological group (w.r.t. the norm topology).*

  Proof. It is clear that $\mathrm{Inv}(A)$ is a group and that multiplication is continuous, since multiplication $A \times A \to A$ is jointly continuous, as observed on p. 2. It remains to show that the inverse map $\sigma : \mathrm{Inv}(A) \to \mathrm{Inv}(A), a \mapsto a^{-1}$ is continuous. To this purpose, let $r, r+h \in \mathrm{Inv}(A)$ and put $(r+h)^{-1} = r^{-1}+k$. We must show that $\|h\| \to 0$ implies $\|k\| \to 0$. From $\mathbf{1} = (r^{-1} + k)(r + h) = \mathbf{1} + r^{-1}h + kr + kh$ we obtain $r^{-1}h + kr + kh = 0$. Multiplying this on the right by r^{-1} we have $r^{-1}hr^{-1} + k + khr^{-1} = 0$, thus $k = -r^{-1}hr^{-1} - khr^{-1}$. Therefore $\|k\| \leq \|r^{-1}\|^2\|h\| + \|k\|\|h\|\|r^{-1}\|$, which is equivalent to $\|k\|(1 - \|h\|\|r^{-1}\|) \leq \|r^{-1}\|^2\|h\|$. Since we are considering $\|h\| \to 0$, we may assume $\|h\|\|r^{-1}\| < 1$. Then

  $$\|k\| \leq \frac{\|r^{-1}\|^2}{1 - \|h\|\|r^{-1}\|}\|h\|,$$

  from which it is clear that $\|h\| \to 0$ implies $\|k\| \to 0$. ■

- p.6, Remark 1.2.1: Let $H = \ell^2(\mathbb{N})$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Define $U, V \in B(H)$ by $Ue_n = e_{n+1}, Ve_1 = 0$ and $Ve_n = e_{n-1}$ for $n \geq 2$. Then $VU = \mathbf{1}$ is invertible, but $UV = 1 - p_{e_1}$ is not. Thus $0 \in \sigma(UV), 0 \notin \sigma(VU)$. It is clear that U is injective, whereas V is surjective, but not injective. Thus invertibility of a product does not imply invertibility of the factors!

- p.7, proof of Theorem 1.2.1: Murphy could be understood as claiming that invertibility of $a_1a_2\cdots a_n$ implies invertibility of all a_i, which is not true in general, cf. the preceding comment. But in the situation at hand, all a_i commute with each other, so that one can use the following:

  2.2 Proposition *Let A be a unital algebra.*

  (i) If $ab=ba$ and ab has inverse c then a and b are invertible with $a^{-1}=cb=bc$ and $b^{-1}=ca=ac$.

  (ii) Let $a_1,\ldots,a_n\in A$ such that $a_ia_j=a_ja_i$ for all i,j. Then $\prod_i a_i$ is invertible if and only if all a_i are invertible.

  Proof. (i) We have $c=(ab)^{-1}=(ba)^{-1}$. Thus $abc=bac=cab=cba=1$. This implies that $r=bc$ and $l=cb$ are right and left, respectively, inverses for a, thus $la=1=ar$. But then $r=(la)r=l(ar)=l$. Thus a is invertible with inverse $r=l=cb=bc$. In the same way one shows that $ca=ac$ is the inverse of b.

  (ii) The 'if' direction is trivial. The 'only if' direction is proven by induction over n, the case $n=2$ just being (i). ■

- p.7, Theorem 1.2.2: Murphy's proof is quite sloppy and obscures the rôle of completeness. Then again, the following should be known from a course in functional analysis:

  2.3 Lemma *If V is a Banach space and $\{x_k\}\subseteq V$ is a sequence such that $\sum_{k=0}^\infty\|x_k\|<\infty$ then $\lim_{n\to\infty}S_n$, where $S_n=\sum_{k=0}^n x_k$, exists and is called $\sum_{k=0}^\infty x_k$.*

  Proof. Since $\sum_n\|x_n\|$ converges, the sequence $\{T_n\}$ with $T_n=\sum_{k=0}^n\|x_k\|$ is a Cauchy sequence. If $n>m$ then $\|S_n-S_m\|=\|\sum_{k=m+1}^n x_k\|\le\sum_{k=m+1}^n\|x_k\|\le T_n-T_m=|T_n-T_m|$ shows that $\{S_n\}$ is a Cauchy sequence and therefore convergent by completeness of V. ■

- p.6: To Remark 1.2.1 and Theorem 1.2.2 one might add the following: If A is unital Banach algebra and $a,b\in A$ with $\|a\|\|b\|<1$ then $\|ab\|,\|ba\|<1$, thus $1-ab$ and $1-ba$ are invertible by Theorem 1.2.2, and

  $$(1-ab)^{-1}=1+ab+abab+\cdots,\qquad(1-ba)^{-1}=1+ba+baba+\cdots.$$

  This implies the formula $(1-ba)^{-1}=1+b(1-ab)^{-1}a$, which is hard to guess. Now one can check by purely algebraic computations that this holds for any unital algebra whenever $1-ab$ is invertible.

- p.9, after Theorem 1.2.6: Calling an algebra element $a\in A$ *nilpotent* if $a^n=0$ for some $n\in\mathbb{N}$, we have:

  2.4 Lemma *Let A be a unital Banach algebra and $a\in A$. Then*

  $$a\text{ nilpotent}\quad\Rightarrow\quad r(a)=0\quad\Rightarrow\quad a\notin\mathrm{Inv}(A).$$

  Proof. Let $a\in A$ be nilpotent and $\lambda\neq0$. Then $x=\sum_{k=0}^\infty(a/\lambda)^k$ converges since it has only finitely many non-zero summands due to nilpotence of a. As in the proof of Theorem 1.2.2, x is an inverse of $1-\lambda^{-1}a$. Thus also $\lambda\mathbf{1}-a$ is invertible, thus $\lambda\notin\sigma(a)$. Thus $r(a)=0$.

  By Theorem 1.2.5, $\sigma(a)\neq\emptyset$. Thus if $r(a)=0$, we must have $\sigma(a)=\{0\}$, thus $a\notin\mathrm{Inv}(A)$. ■

  This motivates to call $a\in A$ *quasi-nilpotent* if $r(a)=0$.

- p.9-10: Theorems 1.2.5 and 1.2.7 are the only ones in this course that use some complex analysis (holomorphicity, the identity theorem for power series, Liouville's theorem). Here we give an alternative proof that dispenses not only with complex analysis, but also with the Hahn-Banach and Banach-Steinhaus theorems (which are needed in the textbook proofs to extend classical results about ($\mathbb{C}$-valued) holomorphic functions to Banach algebra-valued functions) and with differentiability of the map $\sigma:\mathrm{Inv}(A)\to\mathrm{Inv}(A), a\mapsto a^{-1}$ (Theorem 1.2.3). Instead of the latter, we only need continuity, for which we gave an elementary proof that does not even need completeness. Accordingly, the same is true for much of the following:

2.5 Theorem *Let A be a unital normed algebra and $a \in A$. Then (i) $\sigma(a) \neq \emptyset$ and*

$$r(a) \geq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}. \tag{2.1}$$

(ii) If A is complete (thus a Banach algebra) then equality holds in (2.1).

Proof. (i) For every $a \in A$ we trivially have

$$0 \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} \|a^n\|^{1/n} \leq \|a\| < \infty. \tag{2.2}$$

Abbreviating $\nu = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$, for every $\varepsilon > 0$ there is a k such that $\|a^k\|^{1/k} < \nu + \varepsilon$. Now every $m \in \mathbb{N}$ is of the form $m = sk + r$ with unique $s \in \mathbb{N}_0$ and $0 \leq r < k$. Then

$$\|a^m\| = \|a^{sk+r}\| \leq \|a^k\|^s \|a\|^r < (\nu+\varepsilon)^{sk}\|a\|^r,$$

$$\|a^m\|^{1/m} \leq (\nu+\varepsilon)^{\frac{sk}{sk+r}} \|a\|^{\frac{r}{sk+r}}.$$

Now $m \to \infty$ means $\frac{sk}{sk+r} \to 1$ and $\frac{r}{sk+r} \to 0$, so that $\limsup_{m\to\infty} \|a^m\|^{1/m} \leq \nu + \varepsilon$. Since this holds for every $\varepsilon > 0$, we have $\limsup_{m\to\infty} \|a^m\|^{1/m} \leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$. Together with (2.2) this implies that $\lim_{m\to\infty} \|a^m\|^{1/m}$ exists and equals $\inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$.

Assume $\nu = 0$ and $a \in \mathrm{Inv}(A)$. Then there is $b \in A$ such that $ab = ba = \mathbf{1}$. Then $\mathbf{1} = a^n b^n$, thus $1 \leq \|\mathbf{1}\| = \|a^n b^n\| \leq \|a^n\|\|b^n\| \leq \|a^n\|\|b\|^n$. Taking n-th roots, we have $1 \leq \|a^n\|^{1/n}\|b\|$, and taking the $\limsup$ gives the contradiction $1 \leq \nu\|b\| = 0$. Thus a is not invertible, so that $0 \in \sigma(a)$, thus $\sigma(a) \neq \emptyset$. Now (2.1) is obviously true. (One might disagree about the definition of $r(a)$ when $\sigma(a) = \emptyset$.) This proves (i) when $\nu = 0$.

From now on assume $\nu > 0$. If $\mu > \nu$, choose μ' such that $\nu < \mu' < \mu$. Then the definition of $\limsup$ implies that there is a n_0 such that $n \geq n_0 \Rightarrow \|a^n\|^{1/n} < \mu'$. For such n we have $\frac{\|a^n\|}{\mu^n} \leq (\mu'/\mu)^n$ which tends to 0 as $n \to \infty$ since $\mu' < \mu$. Thus for every $\mu > \nu$ we have that $(a/\mu)^n \to 0$ as $n \to \infty$. (This is of course trivial if $\mu > \|a\|$, but our hypothesis is weaker when $\nu < \|a\|$.) On the other hand, for all $n \in \mathbb{N}$ we have $\|a^n\|^{1/n} \geq \nu$. With $\nu > 0$ this implies $\|(a/\nu)^n\| \geq 1$, and therefore $(a/\nu)^n \not\to 0$. These two facts will be essential later.

Assume that there is no $\lambda \in \sigma(a)$ with $|\lambda| \geq \nu$. This implies that $(a - \lambda\mathbf{1})^{-1}$ exists for all $|\lambda| \geq \nu$ and is continuous in λ. The same holds (note $|\lambda| \geq \nu > 0$) for the slightly more convenient function

$$\phi(\lambda) = (\frac{a}{\lambda} - \mathbf{1})^{-1} \qquad (|\lambda| \geq \nu).$$

For $0 \neq \lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, put $\lambda_k = \lambda e^{\frac{2\pi i}{n}k}$, where $k = 1, \ldots, n$. (One should really write $\lambda_{n,k}$, but we suppress the n.) Then $\lambda_1, \ldots, \lambda_n$ are the solutions of $z^n = \lambda^n$, and we have $z^n - \lambda^n = \prod_k (z - \lambda_k)$, in particular, $\prod_k \lambda_k = \lambda^n$. Let $|\lambda| \geq \nu$ and $n \in \mathbb{N}$. Then our assumption ($|\lambda| \geq \nu \implies \lambda \notin \sigma(a)$) implies $\lambda_k \notin \sigma(a)$ for all $k = 1, \ldots, n$. Thus all $\frac{a}{\lambda_k} - \mathbf{1}$ are invertible, and so is $(\frac{a}{\lambda})^n - \mathbf{1} = \prod_k (\frac{a}{\lambda_k} - \mathbf{1})$. Direct computation proves

$$z^n - \lambda^n = (z - \lambda)(z^{n-1} + z^{n-2}\lambda + \cdots + z\lambda^{n-2} + \lambda^{n-1}), \tag{2.3}$$

and applying this with $z \leftarrow a/\lambda_k, \lambda \leftarrow 1$ and observing $\lambda_k^n = \lambda^n$, we have

$$(\frac{a}{\lambda})^n - \mathbf{1} = (\frac{a}{\lambda_k})^n - \mathbf{1} = (\frac{a}{\lambda_k} - \mathbf{1})(\mathbf{1} + \frac{a}{\lambda_k} + \cdots + (\frac{a}{\lambda_k})^{n-1})$$

and therefore

$$\phi(\lambda_k) = (\frac{a}{\lambda_k} - \mathbf{1})^{-1} = ((\frac{a}{\lambda})^n - \mathbf{1})^{-1}(\mathbf{1} + \frac{a}{\lambda_k} + \cdots + (\frac{a}{\lambda_k})^{n-1}).$$

If $l \in \{1, \ldots, n-1\}$ then $z = e^{\frac{2\pi i}{n}l}$ satisfies $z \neq 1$ and $z^n = 1$. Thus using (2.3) with $\lambda = 1$ we have

$$\sum_{k=1}^{n} e^{\frac{2\pi i}{n}kl} = e^{\frac{2\pi i}{n}l} \sum_{k=0}^{n-1} z^k = e^{\frac{2\pi i}{n}l}\,\frac{z^n - 1}{z - 1} = 0.$$

This implies $\sum_{k=1}^n (\frac{a}{\lambda_k})^l = (\frac{a}{\lambda})^l \sum_{k=1}^n e^{-\frac{2\pi i}{n}kl} = 0$ for $l = 1, \dots, n-1$ and therefore

$$\frac{1}{n}\sum_{k=1}^n \phi(\lambda_k) = \frac{1}{n}((\frac{a}{\lambda})^n - \mathbf{1})^{-1}\sum_{k=1}^n (\mathbf{1} + \frac{a}{\lambda_k} + \dots + (\frac{a}{\lambda_k})^{n-1}) = ((\frac{a}{\lambda})^n - \mathbf{1})^{-1}. \tag{2.4}$$

For any $\eta > \nu$, the annulus $\Lambda = \{\lambda \in \mathbb{C} \mid \nu \leq |\lambda| \leq \eta\}$ is compact. Thus the continuous map $\phi : \Lambda \to A$ is uniformly continuous, cf. e.g. [10]. I.e., for every $\varepsilon > 0$ we can find $\delta > 0$ such that $\lambda, \lambda' \in \Lambda,\ |\lambda - \lambda'| < \delta \Rightarrow \|\phi(\lambda) - \phi(\lambda')\| < \varepsilon$. If $\nu < \mu < \nu + \delta$, we have $|\nu_k - \mu_k| = |\nu - \mu| < \delta$ and therefore $\|\phi(\nu_k) - \phi(\mu_k)\| < \varepsilon$ for all $n \in \mathbb{N}$ and $k = 1, \dots, n$. Combining this with (2.4) we have

$$\|((\frac{a}{\nu})^n - \mathbf{1})^{-1} - ((\frac{a}{\mu})^n - \mathbf{1})^{-1}\| \leq \frac{1}{n}\sum_{k=1}^n \|\phi(\nu_k) - \phi(\mu_k)\| < \varepsilon \quad \forall n \in \mathbb{N}. \tag{2.5}$$

As we have shown before, $\mu > \nu$ implies $(a/\mu)^n \to 0$ as $n \to \infty$. By continuity of the inverse map, $((a/\mu)^n - \mathbf{1})^{-1} \to -\mathbf{1}$. Combining this with (2.5) we find that $\|((a/\nu)^n - \mathbf{1})^{-1} + \mathbf{1}\| < 2\varepsilon$ for n large enough. Since ε was arbitrary, we have $((a/\nu)^n - \mathbf{1})^{-1} \to -\mathbf{1}$ and therefore $(a/\nu)^n \to 0$. But this is false, as also proven above. This contradiction proves that our assumption that there is no $\lambda \in \sigma(a)$ with $|\lambda| \geq \nu$ is false. Existence of such a λ obviously gives $\sigma(a) \neq \emptyset$ and $r(a) \geq \nu$, completing the proof of (i).

(ii) Replacing z in (2.3) by $a \in A$, both factors on the r.h.s. commute. If $\lambda \in \sigma(a)$ then $a - \lambda$ is not invertible, thus by Proposition 2.2 above, $a^n - \lambda^n$ is not invertible, so that $\lambda^n \in \sigma(a^n)$. Thus $r(a) \leq \inf_{n \in \mathbb{N}} r(a^n)^{1/n}$. When A is complete, Lemma 1.2.4 applies and gives $r(a^n) \leq \|a^n\|\ \forall n$, whence $r(a) \leq \nu$. ■

The above proof is from [12] (which is written in terms of 'quasi-inverses', for which one does not need a unit). It is a shame that it is not better known. (Versions of it can be found in [13, 2, 9], the one in [2] being closest to the one above.) While the proof is not hard to follow, much of it seems non-trivial to discover. In fact it was preceded by (somewhat) simpler proofs [16, 8] of the Gelfand-Mazur theorem.

- p.9: Since $\sigma(a) \neq \emptyset$ holds for every element of a unital normed algebra, also the Gelfand-Mazur theorem (Theorem 1.2.6) is true for every normed division algebra! (Murphy doesn't really give a proof: Let $a \in A$. By the above Theorem 2.5, $\sigma(a) \neq \emptyset$. Picking $\lambda \in \sigma(a)$, we have $a - \lambda\mathbf{1} \notin \mathrm{Inv}(A)$. Since A is a division algebra, we have $a - \lambda\mathbf{1} = 0$. Thus $A = \mathbb{C}\mathbf{1}$.)

- p.11, Theorem 1.2.8: I skip this result since it is used only to prove Theorem 2.1.11, for which we'll later give a much more direct proof.

- p.12-13: Relationship between $\sigma_A(a)$ and $\sigma_{\widetilde{A}}(a)$ when A is unital.

When A is non-unital and $a \in A$, we **must** adjoin a unit in order to define $\sigma(a) := \sigma_{\widetilde{A}}(a)$. (An alternative is introducing 'quasi-inverses', cf. e.g. [13], but this is not very customary in the operator algebra literature.) But if A is unital, we can consider both $\sigma_A(a)$ and $\sigma_{\widetilde{A}}(a)$, so that it is natural to ask what the relation between the two spectra is. (This is a bit academic since there is no reason to consider $\widetilde{A}$ if A is unital. But it may be useful when we do not know a priori of our algebra A is unital.) They cannot just be equal since no $a \in A$ is invertible in $\widetilde{A}$, even if $a \in \mathrm{Inv}(A)$. Here is the answer. (Part (i) is stated without proof in Murphy, p.40, l.4-6.)

2.6 Proposition *Let A be a unital algebra and $\widetilde{A}$ defined as usual. Then:*

(i) The map $\phi : \widetilde{A} \to A \oplus \mathbb{C},\ (a, \alpha) \mapsto (a + \alpha\mathbf{1}_A, \alpha)$ is an algebra isomorphism (where $A \oplus \mathbb{C}$ has the algebra structure of a direct sum), sending $A \subseteq \widetilde{A}$ to $A \oplus 0$.

(ii) For every $a \in A$ we have $\sigma_{\widetilde{A}}(a) = \sigma_A(a) \cup \{0\}$.

Proof. (i) $\phi(A) = A \oplus 0$ is evident from the definition of ϕ. We have $\phi(\mathbf{1}_{\widetilde{A}}) = \phi((0, 1)) = (1_A, 1) = \mathbf{1}_{A \oplus \mathbb{C}}$, and an easy computation shows that ϕ is multiplicative. If $\phi((a, \alpha)) = \phi((b, \beta))$ then $a + \alpha\mathbf{1}_A = b + \beta\mathbf{1}_A$ and $\alpha = \beta$, from which we get $a = b$. Thus ϕ is injective. Surjectivity of ϕ follows from $\phi((a - \alpha\mathbf{1}_A, \alpha)) = (a, \alpha)$.

(ii) From (i) it is immediate that $\lambda \in \sigma_{\widetilde{A}}((a, \alpha))$ if and only if $\lambda \in \sigma_{A \oplus \mathbb{C}}(\phi(a, \alpha))$. If $a \in A$ then $a - \lambda\mathbf{1}_{\widetilde{A}} = (a, -\lambda)$. Thus $\phi(a - \lambda\mathbf{1}_{\widetilde{A}}) = (a - \lambda\mathbf{1}_A, -\lambda)$, which is invertible in $A \oplus \mathbb{C}$ if and only if

$a - \lambda \mathbf{1}_A \in \mathrm{Inv}(A)$ and $\lambda \neq 0$. This proves the claim. ∎

- p.16: The fact that $r(cd) \leq r(c)r(d)$ and $r(c+d) \leq r(c)+r(d)$ if c and d commute can be proven without using the Gelfand representation. Cf. [7, Proposition 3.2.10] or [13, Theorem (1.4.1)(v)].

- p.16, Theorem 1.3.7: The proof given here is a bit brief. Surjectivity of $\widehat{a} : \Omega(A) \to \sigma(a)$ was proven before (in the proof of Theorem 1.3.4(1)). For injectivity, assume that $\widehat{a}(\varphi_1) = \widehat{a}(\varphi_2)$, thus $\varphi_1(a) = \varphi_2(a)$. Since the φ_i are algebra homomorphisms, it follows that $\varphi_1(p(a)) = \varphi_2(p(a))$ for every $p \in \mathbb{C}[z]$. Now $\varphi_1 = \varphi_2$ follows from continuity of the characters and density of $\{p(a) \mid p \in \mathbb{C}[z]\} \subseteq A$. ∎

  For Example 1.3.1, one actually needs a small variation of Theorem 1.3.7: If A is a unital Banach algebra and $a \in \mathrm{Inv}(A)$ such that A is generated by $\{\mathbf{1}, a, a^{-1}\}$ then again $\widehat{a} : \Omega(A) \to \sigma(a)$ is a homeomorphism. Here we note that
  $$\varphi(a)\varphi(a^{-1}) = \varphi(aa^{-1}) = \varphi(\mathbf{1}) = 1 \quad \forall \varphi \in \Omega(A),$$
  thus $\varphi(a^{-1}) = \varphi(a)^{-1}$. Thus if $\varphi_1(a) = \varphi_2(a)$ then $\varphi_1(a^n) = \varphi_2(a^n)$ for all $n \in \mathbb{Z}$ (where a^n with $n < 0$ means $(a^{-1})^{-n}$), and $\varphi_1 = \varphi_2$ follows as before.

- p.18: Here is an example of a non-zero Banach algebra A such that *every* $a \in A$ is quasi-nilpotent.

  Let $\alpha : \mathbb{N} \to (0, \infty)$ be a map satisfying
  $$\alpha_{n+m} \leq \alpha_n \alpha_m \quad \forall n, m \in \mathbb{N}. \tag{2.6}$$
  For $f : \mathbb{N} \to \mathbb{C}$, define
  $$\|f\| = \sum_{n \in \mathbb{N}} \alpha_n |f(n)|,$$
  and $\ell^1(\mathbb{N}, \alpha) = \{f : \mathbb{N} \to \mathbb{C} \mid \|f\| < \infty\}$. It is clear that $\|\cdot\|$ is a norm on $\ell^1(\mathbb{N}, \alpha)$ making the space complete (since this is just $\ell^1(\mathbb{N}, \mu)$ w.r.t. a modified counting measure μ). The product $f \cdot g$ defined by $(f \cdot g)(n) = \sum_{\substack{u,v \in \mathbb{N} \\ u+v=n}} f(u)g(v)$ is bilinear, associative, commutative and admits no unit. Submultiplicativity of $\|\cdot\|$ is an easy consequence of (2.6):
  $$\begin{aligned} \|f \cdot g\| &= \sum_n \alpha_n \left| \sum_{u+v=n} f(u)g(v) \right| \leq \sum_n \alpha_n \sum_{u+v=n} |f(u)g(v)| \\ &= \sum_{u,v \in \mathbb{N}} \alpha_{u+v} |f(u)|\,|f(v)| \leq \sum_{u,v \in \mathbb{N}} \alpha_u \alpha_v |f(u)|\,|f(v)| = \|f\|\,\|g\|. \end{aligned}$$
  Thus $(\ell^1(\mathbb{N}, \alpha), \cdot, \|\cdot\|)$ is a Banach algebra. The element $a \in A := (\ell^1(\mathbb{N}, \alpha)$ defined by $a(1) = 1,\ a(k) = 0\,\forall k \geq 2$ satisfies $(a^n)(k) = \delta_{k,n}$, thus $\|a^n\| = \alpha_n$. The polynomials in a are the finitely supported functions $f : \mathbb{N} \to \mathbb{C}$, which are dense. Thus a generates A.

  If we now choose the α_k such that $\lim_{n\to\infty} \alpha_n^{1/n} = 0$ then $r(a) = \lim_{n\to\infty} \alpha_n^{1/n} = 0$. One such choice is $\alpha_n = e^{-n^2}$ since then $\alpha_{n+m} = e^{-(n+m)^2} < e^{-n^2} e^{-m^2} = \alpha_n \alpha_m$ and $\alpha_n^{1/n} = e^{-n} \to 0$.

  Since A is abelian, we have $r(cd) \leq r(c)r(d)$ and $r(c+d) \leq r(c)+r(d)$, cf. p.16, implying that $r(p(a)) = 0$ for every polynomial p. Since $r(a) \leq \|a\|$, $r : A \to [0, \infty)$ is continuous, thus the density in A of $\{p(a) \mid p \text{ a polynomial}\}$ implies $r \equiv 0$.

- As part of Theorem 1.3.3 we saw that $\Omega(A) \neq \emptyset$ for every unital commutative Banach algebra. A non-unital commutative Banach algebra A can have $\Omega(A) = \emptyset$ by the following result, which applies to the algebra $A = \ell^1(\mathbb{N}, \alpha)$ from the preceding item.

  2.7 PROPOSITION *Let A be a non-unital commutative Banach algebra. Then $\Omega(A) = \emptyset$ if and only if $r(a) = 0$ for all $a \in A$ (thus $A = \mathrm{Rad}(A)$).*

  Proof. By Theorem 1.3.4, $\sigma(a) = \{0\} \cup \{\varphi(a) \mid \varphi \in \Omega(A)\}$. Thus if $\Omega(A) = \emptyset$ then $\sigma(a) = \{0\}$, and therefore $r(a) = 0$, for all $a \in A$. If $\Omega(A) \neq \emptyset$, there is a non-zero character φ. The non-vanishing of φ means that there is an $a \in A$ such that $\varphi(a) \neq 0$. But then $\varphi(a) \in \sigma(a)$, and therefore $r(a) \geq |\varphi(a)| > 0$. ∎

3 Addenda: C^*-algebras

- p.36, l.9: The proof of $\mathbf{1}^* = \mathbf{1}$ is a bit short. This is better:

$$\mathbf{1} = (\mathbf{1}^*)^* = (\mathbf{1}\mathbf{1}^*)^* = \mathbf{1}^{**}\mathbf{1}^* = \mathbf{1}\mathbf{1}^* = \mathbf{1}^*.$$

(The first and fourth equality are $a^{**} = a$, the second and fifth are the unit property of $\mathbf{1}$, and the third is $(ab)^* = b^*a^*$.)

- p.36, l.-1: If A is a Banach algebra with norm $\|\cdot\|$ and $*$ is an involution such that $\|a^*a\| = \|a\|^2$ for all $a \in A$ then $\|a\|^2 = \|a^*a\| \le \|a^*\|\|a\|$. If $a \neq 0$ then we can cancel $\|a\| \neq 0$ and find $\|a\| \le \|a^*\|$ (which clearly also holds for $a = 0$). Replacing a by a^* gives the converse inequality, thus $\|a^*\| = \|a\|$. Thus this part of the definition of a Banach $*$-algebra holds automatically.

- p.37, Example 2.1.3: If H is a Hilbert space and $a : A \to A$ is a linear map, then

$$\|a\| = \sup_{\substack{x\in H\\ \|x\|\le 1}} \|ax\| = \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\le 1}} |(ax,y)|,$$

where the first identity is the definition of $\|\cdot\|$ and the second follows by Cauchy-Schwarz. Now

$$\|a^*a\| = \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\le 1}} |(a^*ax,y)| = \sup_{\substack{x,y\in H\\ \|x\|,\|y\|\le 1}} |(ax,ay)| = \sup_{\substack{x\in H\\ \|x\|\le 1}} (ax,ax) = \sup_{\substack{x\in H\\ \|x\|\le 1}} \|ax\|^2 = \|a\|^2.$$

- p.37, Theorem 2.1.1: The identity $r(a) = \|a\|$ more generally holds for all normal a. Murphy obtains this using the Gelfand representation applied to $C^*(a,a^*)$, cf. p.41, l.-10. Here is a simple direct proof: For normal a, we compute

$$\|a^{2n}\|^2 = \|(a^{2n})^*a^{2n}\| = \|(a^*a)^{2n}\| = \|(a^*a)^n(a^*a)^n\| = \|(a^*a)^n\|^2, \tag{3.1}$$

where the first and last equalities are due to the C^*-identity and the second to normality of a. Since a^*a is self-adjoint, $\|(a^*a)^{2^n}\| = \|a^*a\|^{2^n}$ as in Murphy. With this, (3.3) and Theorem 1.2.7 we have

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \lim_{n\to\infty} \|(a^*a)^{2^{n-1}}\|^{1/2^n} = \lim_{n\to\infty} (\|(a^*a)\|^{2^{n-1}})^{1/2^n} = \|a^*a\|^{1/2} = \|a\|.$$

- p.39, Theorem 2.1.6: I prefer to postpone the introduction of the multiplier algebra $M(A)$ (p.38-39) to some later moment. For this reason, we give a direct proof of the following:

3.1 THEOREM *Let A be a C^*-algebra. Then $\widetilde{A}$ admits a unique C^*-norm, given by*

$$\|(a,\alpha)\|_{\widetilde{A}} = \sup_{\substack{b\in A\\ \|b\|\le 1}} \|ab + \alpha b\|_A.$$

Proof. $\widetilde{A}$ is a $*$-algebra (with $(a,\alpha)^* = (a^*,\overline{\alpha})$). Thus by Coro. 2.1.2 there is at most one C^*-norm on $\widetilde{A}$ with respect to which $\widetilde{A}$ is complete. If $(a,\alpha) \in \widetilde{A}$ then $L_{(a,\alpha)} : A \to A, b \mapsto ab + \alpha b$ is a linear map. It is easy to see that $(a,\alpha) \mapsto L_{(a,\alpha)}$ is an algebra homomorphism $\widetilde{A} \to \mathrm{End}(A)$. $L_{(a,\alpha)}$ is bounded in view of $\|ab + \alpha b\| \le (\|a\| + \alpha)\|b\|$, and $\|(a,\alpha)\|_{\widetilde{A}}$ coincides with $\|L_{(a,\alpha)}\|_{B(A)}$. As an operator norm, $\|\cdot\|_{B(A)}$ is subadditive and submultiplicative, thus the same holds for $\|\cdot\|_{\widetilde{A}}$. For $a \in A$, we have

$$\|a\|_A = \left\| a\frac{a^*}{\|a\|}\right\|_A \le \|a\|_{\widetilde{A}} \le \|a\|_A,$$

where the first identity is due to the C^*-identity for A, thus $\|a\|_{\widetilde{A}} = \|a\|$ for $a \in A$, so that $\|\cdot\|_{\widetilde{A}}$ extends $\|\cdot\|_A$. Since the inclusion maps $A \hookrightarrow \widetilde{A} \hookrightarrow B(A)$ are isometries, $\widetilde{A} \subseteq B(A)$ is complete w.r.t. $\|\cdot\|_{\widetilde{A}}$.

It remains to prove that $\|\cdot\|_{\widetilde{A}}$ satisfies the C^*-identity. We compute

$$\begin{aligned}
\|(a,\alpha)\|_{\widetilde{A}}^2 &= (\sup_{\substack{b\in A\\ \|b\|\le 1}} \|ab+\alpha b\|)^2 = \sup_{\substack{b\in A\\ \|b\|\le 1}} \|ab+\alpha b\|^2 \\
&= \sup_{\substack{b\in A\\ \|b\|\le 1}} \|(ab+\alpha b)^*(ab+\alpha b)\| \\
&= \sup_{\substack{b\in A\\ \|b\|\le 1}} \|b^*a^*ab+\alpha b^*a^*b+\overline{\alpha}b^*ab+|\alpha|^2 b^*b\| \\
&\le \sup_{\substack{b\in A\\ \|b\|\le 1}} \|a^*ab+\alpha a^*b+\overline{\alpha}ab+|\alpha|^2 b\| \\
&= \|(a^*a+\alpha a^*+\overline{\alpha}a, |\alpha|^2)\|_{\widetilde{A}} \\
&= \|(a,\alpha)^*(a,\alpha)\|_{\widetilde{A}} \;\le\; \|(a,\alpha)^*\|_{\widetilde{A}}\|(a,\alpha)\|_{\widetilde{A}} \qquad (3.2)
\end{aligned}$$

Thus $\|(a,\alpha)^*\| \le \|(a,\alpha)\|$. By symmetry, $\|(a,\alpha)^*\| = \|(a,\alpha)\|$. Inserting this in the last term of (3.2), we have

$$\|(a,\alpha)\|^2 \le \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)\|^2$$

and thus $\|(a,\alpha)\|^2 = \|(a,\alpha)\|^2$. ■

- p.40: Here is an important application of Theorem 2.1.7: If A is a Banach $*$-algebra, a *$*$-representation* of A is a $*$-homomorphism $\pi : A \to B(H)$ for some Hilbert space H. Since each $B(H)$ is a C^*-algebra, we have $\|\pi(a)\| \le \|a\|\ \forall a \in A$. Thus $\|a\|' := \sup_\pi \|\pi(a)\|_{B(H)}$, where π runs through all $*$-representations of A, satisfies $\|a\|' \le \|a\| < \infty\ \forall a \in A$. Now

  $$\|a+b\|' = \sup_\pi \|\pi(a+b)\| = \sup_\pi \|\pi(a)+\pi(b)\| \le \sup_\pi(\|\pi(a)\|+\|\pi(b)\|) \le \|a\|'+\|b\|',$$

  and similarly for multiplication. Using the C^*-identity for $B(H)$ we have

  $$\|a^*a\|' = \sup_\pi \|\pi(a^*a)\| = \sup_\pi \|\pi(a)^*\pi(a)\| = \sup_\pi \|\pi(a)\|^2 = (\sup_\pi \|\pi(a)\|)^2 = (\|a\|')^2.$$

  Thus $\|\cdot\|'$ is a C^*-seminorm on A. (Note that $\|a\|' = 0$ if $\pi(a) = 0$ for every $*$-representation of A.) Thus dividing out the ideal $I = \{a \in A \mid \|a\|' = 0\}$ and completing A/I w.r.t. $\|\cdot\|'$ gives a C^*-algebra.

- p.40, Theorem 2.1.7: It is not at all true that every $*$-homomorphism between two Banach $*$-algebras is continuous, let alone every homomorphism between Banach algebras. This is the subject of much research, cf. [3] (more than 900 pages), touching also foundational questions like the continuum hypothesis.

- p.41, l.3: We can delete 'hermitian', since in abelian algebra every element is normal and we have proven $r(a) = \|a\|$ for normal elements, not only hermitian ones.

- p.41, Theorem 2.1.11: As promised before, we prove give a much more direct proof of the following:

  3.2 Theorem *Let A be a unital C^*-algebra, $B \subseteq A$ a C^*-subalgebra with $\mathbf{1} \in B$. Then*

  (i) $\mathrm{Inv}(B) = B \cap \mathrm{Inv}(A)$.

  (ii) For every $b \in B$ we have $\sigma_B(b) = \sigma_A(b)$.

  Proof. (i) It is obvious that $\mathrm{Inv}(B) \subseteq B \cap \mathrm{Inv}(A)$. To prove the converse inclusion, consider first $b \in B_{sa} \cap \mathrm{Inv}(A)$. Since b is self-adjoint, $\sigma_A(b)$ and $\sigma_B(b)$ are subsets of $\mathbb{R}$ by Theorem 2.1.8. Since $b \in \mathrm{Inv}(A)$ we have $0 \notin \sigma_A(b)$, thus $i\mathbb{R} \cap \sigma_A(b) = \emptyset$. Thus we have a continuous map $f : \mathbb{R} \to A, t \mapsto (b - it\mathbf{1})^{-1}$. In view of $\sigma_B(b) \subseteq \mathbb{R}$ and uniqueness of inverses, we have $(b - it\mathbf{1})^{-1} \in B$ for all $t \ne 0$. Now continuity of f together with closedness of $B \subseteq A$ gives $b^{-1} = f(0) = \lim_{t\to 0} f(t) \in B$. Thus b is invertible in B, so that $B_{sa} \cap \mathrm{Inv}(A) \subseteq \mathrm{Inv}(B)$.

  The rest is essentially as in Murphy, but merits more detail: If $b \in B \cap \mathrm{Inv}(A)$ then there is an $a \in A$ with $ab = ba = \mathbf{1}$. Then $b^*a^* = a^*b^* = \mathbf{1}$, so that $bb^*a^*a = ba = \mathbf{1}$, meaning that bb^* has a^*a as right inverse.

Similarly, $a^*abb^* = a^*b^* = \mathbf{1}$, thus a^*a is left inverse of bb^*. Thus bb^* is invertible in A, thus also in B by self-adjointness of bb^* and the first part of the proof. Thus there is a $c \in B$ such that $bb^*c = cbb^* = \mathbf{1}$. Thus b^*c is right inverse of b, and since $a \in A$ is a left inverse of b, the same argument as in the proof of Proposition 2.2(i) above proves $b^*c = a$. Thus $b^{-1} = a = b^*c \in B$, so that $B \cap \mathrm{Inv}(A) \subseteq \mathrm{Inv}(B)$.

(ii) By (i), $b - \lambda\mathbf{1}$ is invertible in B if and only if it is invertible in A, thus $\sigma_B(b) = \sigma_A(b)$. ■

3.3 Remark Now one can consider situations where A or B or both are non-unital: E.g., if both are non-unital, consider $B \subseteq A \subseteq \widetilde{A}$. Then $\widetilde{B} \to C^*(B, \mathbf{1}_{\widetilde{A}}) \subseteq \widetilde{A}$, $(b, \beta) \mapsto b + \beta\mathbf{1}_{\widetilde{A}}$ is a $*$-isomorphism sending B to B. Thus for every $b \in B$ we have

$$\sigma_B(b) = \sigma_{\widetilde{B}}(b) = \sigma_{C^*(B,\mathbf{1}_{\widetilde{A}})}(b) = \sigma_{\widetilde{A}}(b) = \sigma_A(b),$$

where the outer equalities hold by definition of the spectrum in non-unital algebras, the second due to the mentioned $*$-isomorphism and the third by the above result (ii).

For the cases where one of the two algebras is non-unital or $\mathbf{1}_B \neq \mathbf{1}_A$, cf. p.44, l.-3 through p.45, l.6 in Murphy. ($\sigma_B(b)\backslash\{0\} = \sigma_A(b)\backslash\{0\}$ always holds.) □

- p.41, after Theorem 2.1.10: Let A be a C^*-algebra. We have seen that $\sigma(u) \subseteq \mathbb{T}$ for unitary $u \in A$ (cf. p.36, l.-5) and $\sigma(a) \subseteq \mathbb{R}$ for self-adjoint $a \in A$ (Theorem 2.1.8). The converse implications are false! Since self-adjoint or unitary elements are normal, it is clear that for non-normal $a \in A$ it is not true that $\sigma(a) \subseteq \mathbb{R}$ or $\sigma(a) \subseteq \mathbb{T}$ imply self-adjointness, respectively unitarity, of a. Examples: Let $A = M_{2\times 2}(\mathbb{C}) \cong B(\mathbb{C}^2)$. With $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we have $\sigma(a) = \{0\} \subseteq \mathbb{R}$ (since a is nilpotent) and $\sigma(b) = \{1\} \subseteq \mathbb{T}$, but a, b are both non-normal, thus neither self-adjoint nor unitary.

  But using Theorem 2.1.10, we have the following:

  3.4 Proposition *Let A be a unital C^*-algebra and $a \in A$ normal. Then:*

  (i) If $\sigma(a) \subseteq \mathbb{R}$ then a is self-adjoint ($a^* = a$).

  (ii) If $\sigma(a) \subseteq \mathbb{T}$ then a is unitary ($a^* = a^{-1}$).

  Proof. Consider the smallest C^*-subalgebra $B = C^*(\mathbf{1}, a) \subseteq A$ containing $\mathbf{1}$ and a (thus also a^*!). Since a is normal, B is a commutative unital C^*-algebra. Thus the Gelfand transform $\pi : B \to C(X)$ is a $*$-isomorphism (for some compact Hausdorff space X). Thus $f = \pi(a) \in C(X)$ is a function with image $f(X) = \sigma(f) = \sigma(a)$. Thus in case (i), f is real-valued, implying $\overline{f(x)} = f(x)\ \forall x \in X$, thus $f^* = f$ and therefore $a = a^*$ (since π is $*$-isomorphism). In case (ii), f takes values in $\mathbb{T}$, thus $|f(x)|^2 = 1\ \forall x$, thus $f^*f = ff^* = \mathbf{1}$, whence $a^*a = aa^* = \mathbf{1}$. ■

- p.42: Theorem 2.1.12 is better proven as a simple application of Theorems 2.1.13 and 2.1.14 and the above Proposition 3.4:

  Let A be unital C^*-algebra, $u \in A$ unitary with $\sigma(u) \subsetneq \mathbb{T}$. As in Murphy's proof, we may assume $-1 \notin \sigma(u)$. Let $f : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ be the branch of the logarithm that is real on the positive reals. Since u is normal and $\sigma(u) \subseteq \mathbb{C}\backslash(-\infty, 0]$, we can define $b = f(u) \in A$ by continuous functional calculus as on p.43. Since f maps $\mathbb{T}\backslash\{-1\}$ to $i\mathbb{R}$, with the first half of Theorem 2.1.14 we have $\sigma(b) = f(\sigma(u)) \subseteq i\mathbb{R}$. Since $\exp(f(x)) = x\ \forall x \in \mathbb{C}\backslash(-\infty, 0]$, the second half of Theorem 2.1.14 gives $\exp(b) = (\exp \circ f)(u) = u$.

  Thus putting $c = b/i$, we have $u = \exp(ic)$ and $\sigma(c) \subseteq \mathbb{R}$. By construction, c is an element of the commutative C^*-algebra $C^*(\mathbf{1}, u) \subseteq A$, thus normal, so that Proposition 3.4(i) above gives $c = c^*$. ■

- p.42, l.-2: Proof: Let $\alpha : A \to B$ be a $*$-isomorphism of C^*-algebras. By Theorem 2.1.7, both α and $\alpha^{-1} : B \to A$ are norm-decreasing. Thus for all $a \in A$,

$$\|a\| = \|\alpha^{-1}(\alpha(a))\| \leq \|\alpha(a)\| \leq \|a\|,$$

thus $\|a\| = \|\alpha(a)\|\ \forall a \in A$, thus α is isometric.

At this point one can (and I did) prove the more general Theorem 3.1.5 according to which every injective unital $*$-homomorphism of C^*-algebras is isometric. This immediately implies that $\alpha(A) \subseteq B$ is closed (thus a C^*-algebra) for every injective unital $*$-homomorphism $\alpha : A \to B$.

- p.43, Theorem 2.1.13: In proving that $\widehat{a} : \Omega(B) \to \sigma(a)$, surjectivity as always follows from $\widehat{a}(\varphi) = \varphi(a) \in \sigma(a)$. Injectivity requires one more observation (as in the case of a subalgebra generated by an invertible element): If $\widehat{a}(\varphi_1) = \widehat{a}(\varphi_2)$, thus $\varphi_1(a) = \varphi_2(a)$, then $\varphi_1(a^*) = \overline{\varphi_1(a)} = \overline{\varphi_2(a)} = \varphi_2(a^*)$ since characters are $*$-homomorphisms. Thus φ_1 and φ_2 agree $a^n(a^*)^m$ for all $n, m \in \mathbb{N}_0$, thus on all of $C^*(\mathbf{1}, a)$ by linearity and density.

  Some more explanations may be helpful. It is clear that $\pi(a) = \widehat{a} \in C(\Omega(B))$, where $\widehat{a}(\varphi) = \varphi(a)$. Let $\iota : \sigma(a) \hookrightarrow \mathbb{C}$ be the inclusion map. Recall that if $f \in C(\sigma(a))$ then $\widehat{a}^t(f) = f \circ \widehat{a}$ in $C(\sigma(a))$. Putting $f = \iota$ this gives that $\widehat{a}^t(\iota)$ is the composite map $\Omega(B) \stackrel{\widehat{a}}{\to} \sigma(a) \stackrel{\iota}{\hookrightarrow} \mathbb{C}$, where we may as well forget the ι. Thus this map sends $\varphi \in \Omega(B)$ to $\varphi(a) = \widehat{a}(\varphi)$. Thus $\widehat{a}^t(\iota) = \widehat{a} = \pi(a)$. Thus we have a composite $*$-isomorphism $\alpha : C(\sigma(a)) \to B$ defined by $\alpha = \pi^{-1} \circ \widehat{a}$ that sends $\mathbf{1} \in C(\sigma(a))$ to $\mathbf{1}_B$ and ι to a. Composing α with the inclusion map $B \hookrightarrow A$ we have an injective $*$-homomorphism $C(\sigma(a)) \to A$ (whose image is $B = C^*(\mathbf{1}, a)$. For $f = 1$ we have $\alpha(f) = \mathbf{1}_B = \mathbf{1}_A$, for $f = \iota$, perhaps better denoted z, we have $\alpha(f) = a$. Since α is a homomorphism, for a polynomial p, we see that $\alpha(p)$ coincides with the result obtained by 'plugging a into p'. This justifies writing $f(a)$ instead of $\alpha(f)$ and also the term 'continuous functional calculus'.

- p.43, Theorem 2.1.13: Non-unital version of continuous functional calculus:

  Let A be non-unital C^*-algebra, $a \in A$ normal and $f \in C(\sigma(a))$. In view of $\sigma(a) := \sigma_{\widetilde{A}}(a)$, we can apply the continuous functional calculus, as defined for unitary algebras, to define $f(a) \in \widetilde{A}$. It is natural to ask whether $f(a) \in A$. Recall that if A is non-unital, we have $0 \in \sigma(a)$ for all $a \in A$.

  3.5 PROPOSITION *Let A be non-unital C^*-algebra, $a \in A$ normal and $f \in C(\sigma(a))$. Then*

  (i) $f(a) \in A$ holds if and only if $f(0) = 0$.

  (ii) There is a unique $*$-isomorphism $\{f \in C(\sigma(a)) \mid f(0) = 0\} \to C^*(a),\ f \mapsto f(a)$ such that $z(a) = a$.

  (iii) For every self-adjoint idempotent $p \in A$ commuting with a there is a $*$-isomorphism $C(\sigma(a)) \to C^*(a, p)$ sending z to a and 1 to p. Every injective $*$-homomorphism $C(\sigma(a)) \to A$ sending z to a is of this form.

  Proof. (i) Let $\widetilde{A}$ be the unitization of A. Then the C^*-subalgebra $B = C^*(\mathbf{1}, a) \subseteq \widetilde{A}$ is commutative and unital. Thus we have the Gelfand isomorphism $\psi : B \to C(\Omega(B))$. Since B is generated by a, the map $\widehat{a} : \Omega(B) \to \sigma_B(a)$ is a homeomorphism. Since B is a unital subalgebra of the unital $\widetilde{A}$, we have $\sigma_B(a) = \sigma_{\widetilde{A}}(a) =: \sigma_A(a)$. Thus for $f \in C(\sigma(a))$ we have $f \circ \widehat{a} \in C(\Omega(B))$, thus $f(a) := \psi^{-1}(f \circ \widehat{a}) \in B \subseteq \widetilde{A}$. Identifying B with the unitization of $C^*(a)$ as above, we have $C^*(a) = \{b \in B \mid \varphi_\infty(b) = 0\}$. In view of $\varphi_\infty(b) = \widehat{b}(\infty)$ (where we write ∞ for the infinite character $\varphi_\infty \in \Omega(B)$), we have $f(a) \in C^*(a) \subseteq A$ if and only if $\psi(f(a))(\infty) = 0$. But $\psi(f(a))(\infty) = (f \circ \widehat{a})(\infty) = f(0)$, since $\widehat{a}(\infty) = \varphi_\infty(a) = 0$. Thus $f(a) \in A \ \Leftrightarrow\ f(0) = 0$.

  (ii) In view of (i), this is just the restriction of the $*$-isomorphism $C(\sigma(a)) \to C^*(\mathbf{1}, a)$ to the closed ideal $\{f \in C(\sigma(a) \mid f(0) = 0\}$.

  (iii) Let p be a self-adjoint idempotent commuting with a. Then $C^*(a, p)$ has p as unit, thus the map $\widetilde{C^*(a)} \to C^*(a, p), (b, \beta) \mapsto b + \beta p$ is a $*$-isomorphism. Thus unital continuous functional calculus gives a $*$-isomorphism $C(\sigma(a)) \to C^*(a, p)$ sending z to a and 1 to p. For the converse, it suffices to note that any $*$-homomorphism $C(\sigma(a)) \to A$ sends 1 to a self-adjoint idempotent commuting with a. ■

- p.44, Theorem 2.1.15: The version of Stone-Weierstrass used in this proof is the following: If Ω is locally compact Hausdorff, $M \subseteq C_0(\Omega)$ is a closed $*$-subalgebra separating the points of Ω (thus for $x, x' \in \Omega,\ x \neq x'$ there exists an $f \in M$ with $f(x) \neq f(x')$) and if for every $x \in \Omega$ there is an $f \in M$ with $f(x) \neq 0$, then $A = C_0(\Omega)$. (Cf. e.g. [10, Corollary F.1.9].)

In the situation at hand, the subalgebra M is closed, self-adjoint and separates the points of Ω, but it is proper. Thus the remaining condition of the above version of Stone-Weierstrass must be false. Thus there is an $x \in \Omega$ with $f(x) = 0 \; \forall f \in M$.

- p.45, l.7: In view of Proposition 3.4(i) above we can also define positive elements as normal elements whose spectrum is contained in $[0, \infty)$.

- p.45, Theorem 2.2.1: Instead of explicit reference to the Gelfand isomorphism, we can also use continuous functional calculus to define $b = f(a)$, where f the restriction of the function $\sqrt{\cdot} : [0,\infty) \to [0,\infty)$ to $\sigma(a) \subseteq [0,\infty)$. (If A is non-unital, one needs Proposition 3.5 above, which applies since $\sqrt{0} = 0$. This presumably is the reason why Murphy adopts his ad-hoc approach.)

  In the second half of the proof ('Suppose that...'), the point is that c is not assumed to lie in $C^*(a)$. (Otherwise uniqueness of the square root would be immediate.)

- p.46, l.-13: Proof that $\leq$ is a partial order: It is obvious that $a \leq a$ for all $a \in A_{sa}$. If $a \leq b$ and $b \leq a$ then $b - a \geq 0$ and $a - b \geq 0$, thus $\sigma(b-a) \subset [0,\infty) \cap (-\infty, 0] = \{0\}$. Since $b - a \in A_{sa}$, Theorem 2.1.1 implies $\|b-a\| = 0$, thus $a = b$. If $a \leq b \leq c$ then $c - b \geq 0$ and $b - a \geq 0$. Thus $c - a = (c-b)+(b-a) \geq 0$ by Lemma 2.2.3, giving $a \leq c$.

- p.46, Theorem 2.2.5: (2) If $a \leq b$ then $b - a \in A^+$, thus $b - a = d^*d$ for some d. Then $c^*bc - c^*ac = c^*d^*dc = (dc)^*dc \geq 0$ by Theorem 2.2.4.

- p.47: A continuous function $f : [0,\infty) \to [0,\infty)$ is called operator monotone if $0 \leq a \leq b$ implies $f(a) \leq f(b)$ whenever $a, b \in A$ for a unital C^*-algebra A. (It suffices to prove this for the $B(H)$'s.) E.g., $t \mapsto t^\alpha$ is operator monotone if and only if $\alpha \in [0, 1]$, cf. [11, Prop. 1.3.11]. Operator monotone functions can be characterized and have been studied extensively, cf. e.g. [5].

- p.52-3, Theorem 2.3.5: We give some more details:

  If $a \in B(H)$ is positive in the C^*-sense then $a = c^*c$ for some $c \in B(H)$. Then $(ax, x) = (c^*cx, x) = (cx, cx) \geq 0$ for all $x \in H$. Thus the sesquilinear form $\alpha(x, y) = (ax, y)$ is positive. Now assume this form α is positive. Then it is self-adjoint (cf. the comment for p.88). Thus

$$(ay, x) = \alpha(y, x) = \overline{\alpha(x, y)} = \overline{(ax, y)} = (y, ax) \quad \forall x, y \in H,$$

  implying $a = a^*$. It remains to prove $\sigma(a) \subseteq [0,\infty)$. Let thus $\lambda < 0$. As in Murphy, we compute

$$\begin{aligned} \|(a-\lambda)x\|^2 &= ((a-\lambda)x, (a-\lambda)x) \\ &= \|ax\|^2 + \lambda^2\|x\|^2 - \lambda(x, ax) - \lambda(ax, x) \\ &= \|ax\|^2 + \lambda^2\|x\|^2 - 2\lambda(ax, x) \;\geq\; \lambda^2\|x\|, \end{aligned} \tag{3.3}$$

  since $(ax, x) \geq 0$ and $\lambda < 0$. Thus $\ker(a - \lambda) = \{0\}$, and $a - \lambda$ is injective. Let $\{y_i\}$ be a sequence in $(a-\lambda)H$ converging to $y \in H$. There are unique (by injectivity) x_i such that $(a-\lambda)x_i = y_i$. Now (3.3) gives $\|y_i - y_j\| = \|(a-\lambda)(x_i - x_j)\| \geq \lambda^2\|x_i - x_j\|$, which with $\lambda^2 \neq 0$ implies that $\{x_i\}$ is a Cauchy sequence, thus convergent to some $x \in H$. Since $(a-\lambda)$ is continuous, we have $(a-\lambda)x = \lim_i (a-\lambda)x_i = \lim y_i = y$. Thus $y \in (a-\lambda)H$, proving that $(a-\lambda)H$ is closed. We claim that $(a-\lambda)H = H$. Since $(a-\lambda)H$ is closed, it suffices to consider $x \in ((a-\lambda)H)^\perp$. Thus $((a-\lambda)y, x) = 0$ for all $y \in H$. Self-adjointness of $a-\lambda$ implies $(y, (a-\lambda)x) = 0 \; \forall y$, thus $(a-\lambda)x = 0$. Injectivity of $a-\lambda$ gives $x = 0$, thus $a - \lambda : H \to H$ is surjective. We now know that $a - \lambda$ is a continuous bijection from H to H, thus open by the open mapping theorem. Thus $a - \lambda$ has a bounded inverse. This proves $\sigma(a) \cap (-\infty, 0) = \emptyset$, to wit $\sigma(a) \subseteq [0,\infty)$. ■

- p.74, Exercise 8 (Fuglede's theorem): Another way of proving Fuglede's theorem is to first prove it for $B(H)$, typically using Borel functional calculus as in [14]. Then it immediately follows for all C^*-algebras by the Gelfand-Naimark Theorem 3.4.1.

- Since commutative C^*-algebras are much easier to work with than non-abelian ones, the following illustrates the usefulness of the Fuglede's theorem:

3.6 Corollary *Let A be a unital C^*-algebra and $a, b \in A$ commuting normal elements. Then $[a^*, b^*] = [a^*, b] = [a, b^*] = 0$, so that $C^*(a, b) \subseteq A$ is abelian. In particular ab is normal.*

Proof. By assumption we have $[a, a^*] = [b, b^*] = [a, b] = 0$. Applying $*$ to $[a, b] = 0$ gives $[a^*, b^*] = 0$. Since A is normal, Fuglede's theorem gives $[a^*, b] = 0$, and applying $*$ (or again Fuglede's theorem) gives $[a, b^*] = 0$. Thus the elements a, a^*, b, b^* all commute with each other. Thus the algebra of polynomials in these elements is abelian, and so is its closure $C^*(a, b)$. Since every element of an abelian C^*-algebra is normal, in particular ab is normal. ■

- p.74, Exercise 8: A slight modification of this argument also proves the Putnam's generalization of Fuglede's theorem. (Murphy deduces the former by reducing it to the latter, cf. Exercise 3.1.) Namely if A is a unital C^*-algebra, $a, b, c \in A$ with a, b normal and $ac = cb$, induction gives $a^k c = cb^k$ for all $k \in \mathbb{N}_0$, implying $e^{\lambda a} c = c e^{\lambda b}$. Using this and normality of a, b, one proves boundedness of $f(\lambda) = e^{\lambda a^*} c e^{-\lambda b^*}$. Then $f : \mathbb{C} \to A$ is differentiable and bounded, thus constant by Liouville's theorem, and from $f'(0) = 0$ one obtains $a^* c = cb^*$.

- As shown earlier, one can avoid the use of complex analysis in the proof of $\sigma(a) \neq \emptyset$ and the spectral radius formula, cf. Theorem 2.5 in these notes. The same applies to Exercise 2.8, since the function $f(\lambda)$ used there is already given as a power series of infinite convergence radius:

$$\begin{aligned} f(\lambda) &= e^{\lambda a^*} c e^{-\lambda b^*} = \left(\sum_{k=0}^{\infty} \frac{\lambda^k (a^*)^k}{k!} \right) c \left(\sum_{l=0}^{\infty} \frac{(-\lambda)^l (b^*)^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \lambda^n d_n, \quad \text{where} \quad d_n = \sum_{m=0}^{n} \frac{(-1)^m (a^*)^{n-m} c (b^*)^m}{(n-m)! m!} \in A. \end{aligned}$$

(The rearrangement is justified by absolute convergence.) In view of $d_1 = a^* c - cb^*$, it suffices to prove $d_1 = 0$. In fact, boundedness of f implies $d_n = 0$ for all $n \geq 1$:

3.7 Proposition *Let A be a unital C^*-algebra, $\{d_n\}_{n \in \mathbb{N}_0} \subseteq A$ such that $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n d_n$ converges for all $\lambda \in \mathbb{C}$ and defines a bounded function. Then $d_n = 0$ for all $n \geq 1$.*

Proof. Pick any $\tau \in A^*$. Then $\tau(f(\lambda)) = \sum_{n=0}^{\infty} \lambda^n \tau(d_n)$ for all λ by norm convergence of the series and continuity of τ. For $m \in \mathbb{N}$ and $r > 0$ we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} \tau(f(re^{i\phi})) d\phi &= \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} \left(\sum_{n=0}^{\infty} r^n e^{in\phi} \tau(d_n) \right) d\phi \\ &= \sum_{n=0}^{\infty} r^n \tau(d_n) \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\phi} = r^m \tau(d_m), \end{aligned}$$

the interchange of summation being justified by the uniform convergence of the series, and thanks to $\int_0^{2\pi} e^{i(n-m)\phi} d\phi = 2\pi \delta_{n,m}$. In view of $\|f(\lambda)\| \leq M \ \forall \lambda$ we have $|\tau(f(\lambda))| \leq M \|\tau\| \ \forall \lambda$. Thus

$$|\tau(d_m)| = \frac{1}{2\pi r^m} \left| \int_0^{2\pi} e^{-im\phi} \tau(f(re^{i\phi})) d\phi \right| \leq \frac{M \|\tau\|}{r^m} \qquad \forall m \in \mathbb{N}, r > 0.$$

Taking the limit $r \to +\infty$, we find $\tau(d_m) = 0$. Since $\tau \in A^*$ was arbitrary, we have $d_m = 0$ for all $m \geq 1$. ■

The inequality appearing in the proof of course is a Cauchy inequality. But complex analysis, in the guise of complex differentiability, contour integration or Cauchy's theorem played no rôle in the proof, which only used the simple fact $\int_0^{2\pi} e^{in\phi} d\phi = 2\pi \delta_{n,0}$ from harmonic analysis. Note the analogy with the identity $\sum_{k=1}^{n} e^{\frac{2\pi i}{n} kl} = 0$ for $l \not\equiv 0 \pmod{n}$ used in the proof of Theorem 2.5.

- p.78, Theorem 3.1.1: Some remarks may be in order. Extend $f = \varphi(a) \in C_0(\Omega)$ to $\Omega_\infty = \Omega(C^*(1, a))$ by putting $f(\infty) = 0$. This extension $\widetilde{f}$ is in $C(\Omega_\infty)$. Let $0 < \varepsilon < 1$ and $K = \{\omega \in \Omega_\infty \mid |f(\omega)| \geq \varepsilon\}$, which is closed in Ω_∞. Clearly $\infty \notin K$. Thus Urysohn's lemma, applied to the compact Hausdorff space Ω_∞ gives a function $\widetilde{g} \in C(\Omega_\infty)$ such that $\widetilde{g} \upharpoonright K = 1$ and $\widetilde{g} \upharpoonright U = 0$ on some open nbhd of ∞. Then $g = \widetilde{g} \upharpoonright \Omega$ is continuous and compactly supported.

  The claim $\|f - \delta g f\| \leq \varepsilon$ is proved as follows: At $\omega \in K$, we have $\varepsilon \leq |f(\omega)| \leq 1$ (recall $f = \varphi(a)$, where $a \in \Lambda$, thus $\|a\| < 1$. On K, $g(\omega) = 1$, thus $1 - \delta g = 1 - \delta < \varepsilon$. Thus $|f(1 - \delta g)| < \varepsilon$. And for $\omega \notin K$, we have $|f(\omega)| \leq \varepsilon$ by definition of K. With $0 \leq \delta g(\omega) < 1$, this gives $|f(1 - \delta g)| < \varepsilon$. Thus $|f(\omega)(1 - \delta g(\omega))| < \varepsilon \ \forall \omega$, proving $\|f - \delta g f\| \leq \varepsilon$. The rest of the proof is fairly clear. (Note $u_\lambda = \lambda$!)

- p.81, Theorem 3.1.7: This proof is badly explained. We follow the better one in [4, Coro. I.5.6]: That $B + I$ is a subalgebra follows from I being an ideal. B is self-adjoint by definition of a C^*-subalgebra, and I by Theorem 3.1.3. Thus $B + I \subseteq A$ is a $*$-subalgebra, and it remains to prove the closedness. Let $q : A \to A/I$ be the quotient homomorphism. If $\iota : B \to A$ is the inclusion map, $p = q \circ \iota : B \to A/I$ is a $*$-homomorphism of C^*-algebras. Thus $p(B) \subseteq A/I$ is closed by Theorem 3.1.6. Since q is continuous, $q^{-1}(p(B)) \subseteq A$ is closed. Now, $q^{-1}(p(B)) = q^{-1}(q(B)) = B + I$. Thus $B + I \subseteq A$ is closed. Standard algebraic reasoning gives that the quotient $B/(B \cap I)$ is $*$-isomorphic (thus also isometric) to $(B + I)/I$.

- p.83: We will see that hereditary subalgebras are a generalization of ideals. As for ideals, we have:

  3.8 Lemma *If A is unital C^*-algebra and $B \subseteq A$ is hereditary C^*-subalgebra such that $\mathbf{1}_A \in B$ then $B = A$.*

  Proof. Let $0 \leq a \leq b$, where $a \in A, b \in B$. Then $a \leq \|a\| \mathbf{1}_A$, which is in B since $\mathbf{1}_A \in B$. Since B is hereditary, we find $A^+ \subseteq B^+$, implying $A = B$. ■

  3.9 Corollary *Let $n \geq 2$, $A = M_{n \times n}(\mathbb{C}) \cong B(\mathbb{C}^n)$. Then the C^*-subalgebra $B \subseteq A$ consisting of diagonal matrices is not hereditary.*

  Proof. Follows from the Lemma since $\mathbf{1}_A \in B \neq A$. ■

- p.84: Corollary 3.2.3 follows very directly from Theorem 3.2.1 (instead of 3.2.2). Equally easily, one shows that the existence of hereditary subalgebras that are not ideals is a non-abelian phenomenon:

  3.10 Corollary *Let A be a C^*-algebra.*

  (i) Every closed 2-sided ideal in A is a hereditary C^*-subalgebra.

  (ii) If A is abelian then hereditary C^*-subalgebras and closed ideals are the same thing.

  Proof. (i) If $I \subseteq A$ is a closed 2-sided ideal then $I = I^*$ by Theorem 3.1.3. Since I is a left ideal, $I = I \cap I^*$ is a hereditary subalgebra by Theorem 3.2.1(1).

  (ii) By abelianness, every closed left ideal is two-sided, thus self-adjoint by Theorem 3.1.3. Thus the bijection $L \mapsto L \cap L^*$ between closed left ideals and hereditary subalgebras simplifies to $L \mapsto L$. ■

- p.88, l.1: Give $\mathbb{C}^n$ the inner product $(x, y) = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$. Then the vectors $e_1 = (1, 0, \ldots, 0), \ldots$ are orthonormal. Thus the dual basis $f^i \in (\mathbb{C}^n)^*$ (defined by $f^i(e_j) = \delta_{i,j}$) is given by $f^i(x) = (x, e_i)$. Thus

$$\mathrm{Tr}(a^* a) \equiv \sum_i f^i(a^* a e_i) = \sum_i (a^* a e_i, e_i) = \sum_i (a e_i, a e_i) \geq 0.$$

  Since every positive element of $B(\mathbb{C}^n)$ is for the form $a^* a$, this proves positivity of the trace.

- p.88, l.6: Here we need the following definitions:

– A sesquilinear form on a complex vector space is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ that is additive in each variable and satisfies $(\lambda a, b) = \lambda(a,b), (a, \lambda b) = \overline{\lambda}(a,b)$ for all $a, b \in V, \lambda \in \mathbb{C}$.

– For a sesquilinear form $(\cdot,\cdot)$, the following are equivalent: (i) $(b,a) = \overline{(a,b)}$ for all $a, b \in V$, (ii) $(a,a) \in \mathbb{R}$ for all $a \in V$.

Proof. (i)⇒(ii) is obvious. For (ii)⇒(i), let $a, b \in V$. Then by biadditivity we have $(a+b, a+b) = (a,a) + (b,b) + (a,b) + (b,a)$. By assumption (ii), the l.h.s. and the first two terms on the r.h.s. are real. Thus $(a,b) + (b,a) \in \mathbb{R}$, implying $\mathrm{Im}((b,a)) = -\mathrm{Im}((a,b))$. Replacing b by $-b$ and using sesquilinearity we have $i((b,a) - (a,b)) = -i(a,b) + i(b,a) = (a, ib) + (ib, a) \in \mathbb{R}$. This gives $\mathrm{Re}((b,a) - (a,b)) = 0$. Combining these two facts we have $(b,a) = \overline{(a,b)}$. ■

Sesquilinear forms satisfying these equivalent conditions are called hermitian.

– A sesquilinear form $(\cdot,\cdot)$ is called positive if $(a,a) \geq 0$ for all $a \in V$. Obviously, positivity implies (the second formulation of) hermitianness.

- p.89, l.-13: In this paragraph, Murphy implicitly uses the following result:

3.11 Exercise Let A be a C^*-algebra and $a, b \in A^+$ with $\|a\|, \|b\| \leq 1$. Prove that $\|a - b\| \leq 1$ (without assuming $ab = ba$!).

- p.92, Theorem 3.3.10: That every bounded self-adjoint functional τ is the difference $\tau^+ - \tau^-$ of two positive functionals can be proven intrinsically, avoiding the Riesz representation theorem and the Jordan decomposition of measures. We follow [11, Lemma 3.2.2], but we omit the proof of $\|\tau\| = \|\tau^+\| + \|\tau^-\|$ and of the uniqueness of $\tau^\pm$ (under the latter condition), cf. [11].

As Murphy shows, if $\tau \in (A^*)_{sa}$, i.e. $\tau = \tau^* \in A^*$ then $\tau(A_{sa}) \subseteq \mathbb{R}$, thus $\tau \restriction A_{sa}$ is a (real-)linear bounded functional on the real Banach space A_{sa}, and one has $\|\tau \restriction A_{sa}\| = \|\tau\|$. On the other hand, every $\tau \in (A_{sa})^*$ extends to a $\mathbb{C}$-linear functional on A by $a \mapsto \tau(\mathrm{Re}(a)) + i\tau(\mathrm{Im}(a))$. Thus the self-adjoint part $(A^*)_{sa,\leq 1}$ of the unit ball of A^* can be identified with the unit ball of $(A_{sa})^*$, thus is compact in the weak-* topology by Alaoglu's theorem.

The subset $\Omega \subseteq (A^*)_{sa,\leq 1}$ of positive functionals of norm ≤ 1 is convex and weak-* closed, thus also weak-* compact. (If A is unital then $S(A) = \{\varphi \in A^* \mid \|\varphi\| \leq 1, \varphi(1) = 1\}$, so that $S(A)$ is weak-* compact. But this fails for non-unital algebras, which is why we work with Ω, the set of 'quasi-states'.) Every $\varphi \in \Omega$ is of the form $\lambda\tau$, where $\tau \in S(A)$ is a state and $\lambda \in [0,1]$. Thus Ω equals the convex hull of $S(A) \cup \{0\}$. Thus if we call K the convex hull of $\Omega \cup -\Omega$, then K also is the convex hull of S and $-S$. Since $\Omega \cup -\Omega$ is weak-* compact, K is a weak-* compact subset of $(A^*)_{sa,\leq 1}$. (The convex hull of a finite number of compact convex sets in a topological vector space is compact, cf. Murphy's Theorem A.12.) If we prove $K = (A^*)_{sa,\leq 1}$, then it follows immediately that every bounded self-adjoint functional is a linear combination of bounded positive functionals.

Thus suppose $\psi \in (A^*)_{sa,\leq 1} \backslash K$. Then by Hahn-Banach there are $x \in A_{sa}$ and $\alpha \in \mathbb{R}$ such that $\psi(x) > \alpha$, but $\varphi(x) \leq \alpha$ for all $\varphi \in K$. In view of $K = -K$, this implies $|\varphi(x)| \leq \alpha$ for all $\varphi \in K$, in particular $|\tau(x)| \leq \alpha$ for all $\tau \in S(A)$. But then Murphy's Theorem 3.3.6 implies $\|x\| \leq \alpha$. In view of $\|\psi\| \leq 1$, this contradicts $\psi(x) > \alpha$. Thus $K = (A^*)_{sa,\leq 1}$. ■

- p.95, Theorem 3.4.3: There is a simpler proof:

3.12 Exercise Give an alternative proof of the 'if' direction in Theorem 3.4.3, using the Gelfand isomorphism for commutative C^*-algebras instead of the universal representation.

- p.115: The proof of the double commutant theorem (Lemma (!) 4.1.4 in Murphy) merits more detail:

3.13 Theorem *Let $A \subseteq B(H)$ be a $*$-subalgebra such that $\mathbf{1} \in A$. Then $A'' = \overline{A}^s$, where $\overline{A}^s$ denotes the closure of A in the strong topology.*

Proof. (1) We have $A \subseteq A''$, and taking strong closures gives $\overline{A}^s \subseteq \overline{A''}^s$. Since commutants are strongly closed, this gives $\overline{A}^s \subseteq A''$, so that we are left with proving $A'' \subseteq \overline{A}^s$. Thus every $u \in A''$ must be in the strong closure of A, which amounts to showing that that every strong neighborhood U of u contains an element of A. Since the strong topology on $B(H)$ is defined in terms of the seminorms $\|a\|_x = \|ax\|$, where $x \in H$, it suffices that for any $x_1, \ldots, x_n \in H$ and $\varepsilon > 0$ there is an $a \in A$ such that $\|(a-u)x_i\| < \varepsilon \ \forall i = 1, \ldots, n$.

(2) We first consider the case $n = 1$. Let $x = x_1 \in H$. Define $K = \overline{Ax}$. Since $\mathbf{1} \in A$, we have $x \in K$. Since $A(Ax) \subseteq Ax$, and every $a \in A$ is continuous, we have $A\overline{Ax} \subseteq \overline{Ax} = K$. Thus the closed subspace K is stable under the action of A. Let p be the orthogonal projection onto K. If $a \in A$ and $z \in H$ then $pz \in K$, thus $apz \in K$ by A-invariance of K, thus $papz = apz$. Since this holds for all $z \in H$, we have $pap = ap$. Since this holds for every $a \in A$, also $pa^*p = a^*p$, and taking adjoints gives $pap = pa$. Thus $ap = pap = pa$, proving $p \in A'$. Together with $u \in A''$, this implies $pu = up$. Thus u maps $K = pH$ into itself, and with $x \in K$ we have $ux \in K$. Since $Ax \subseteq K$ is dense (by definition of K), there is a sequence $\{a_m\} \subseteq A$ such that $a_m x \to ux$. Thus for m large enough, $\|(a_m - u)x\| < \varepsilon$, and $a = a_m \in A$ solves our problem.

(3) When $n > 1$, consider $H^{(n)}$, the direct sum of n copies of H. As seen earlier, cf. p. 94 in Murphy, we can canonically identify $B(H^{(n)})$ with the set $M_{n\times n}(B(H))$ of $n \times n$ matrices with entries in $B(H)$. Consider the map $\varphi : B(H) \to B(H^{(n)})$ sending v to the diagonal $n \times n$-matrix $\mathrm{diag}(v, \ldots, v)$. Clearly φ is an injective unital $*$-homomorphism. Thus $\varphi(A) \subseteq B(H^{(n)})$ is a $*$-subalgebra containing $\mathbf{1}_{H^{(n)}}$, since $\mathbf{1} \in A$. Let now $u \in A''$. We claim that $\varphi(u) \in \varphi(A)''$. Proving this amounts to showing that $\varphi(u) = \mathrm{diag}(u, \ldots, u)$ commutes with every matrix $M = (m_{ij}) \in M_{n\times n}(B(H))$ that commutes with every diagonal matrix $\mathrm{diag}(b, \ldots, b)$, where $b \in A$. Now, $\mathrm{diag}(b, \ldots, b)M = M\mathrm{diag}(b, \ldots, b)$ holds if and only if $bm_{ij} = m_{ij}b$ for all i, j. Thus $m_{ij} \in A' \ \forall i, j$. But since $u \in A''$, u commutes with all m_{ij}, thus $\varphi(u)$ commutes with $\varphi(A)'$, proving the claim $\varphi(u) \in \varphi(A)''$.

(4) Now let $x_1, \ldots, x_n \in H$, and write $x = (x_1, \ldots, x_n) \in H^{(n)}$. Applying the formalism of (2) to the situation where H is replaced by $H^{(n)}$, A by the (unital $*$-)algebra $\varphi(A)$ and u by $\varphi(u)$ (satisfying $\varphi(u) \in \varphi(A)''$ by (3)), we find that there is a sequence $\{b_m\} \subseteq \varphi(A)$ such that $b_m x \to \varphi(u)x$. Since φ is injective, there is a unique sequence $\{a_m\} \subseteq A$ such that $b_m = \varphi(a_m)$. Thus $\varphi(a_m)x \to \varphi(u)x$. This means that for m large enough we have $\|\varphi(a_m - u)x\| < \varepsilon$. Since for $y = (y_1, \ldots, y_n) \in H^{(n)}$ we have $\|y\|^2 = \sum_{k=1}^n \|y_k\|^2$, this is the same as $\sum_{k=1}^n \|(a_m - u)x_k\|^2 < \varepsilon^2$, which implies $\|(a_m - u)x_k\| < \varepsilon$ for each $k = 1, \ldots, n$. Thus again, $a = a_m \in A$ is as desired. ■

- p.194, Theorem 6.3.8: The closed ideal $I \subseteq A$ is self-adjoint, thus is a C^*-algebra (with the $*$-operation of A). Since units are self-adjoint, we have $p = p^*$. It is clear that p commutes with I, but that p commutes with all of A requires proof. If $a \in A$ then $ap \in I$, thus ap commutes with p. Thus for all $a \in A$ we have $ap = (ap)p = p(ap) = pap$. Starring this and using $p = p^*$ gives $pa = pap \ \forall a \in A$. Thus $pa = pap = ap$ for all $a \in A$, proving $p \in Z(A)$.

References

[1] B. Blackadar: *Operator algebras. Theory of C^*-algebras and von Neumann algebras.* Springer, 2006.

[2] F. F. Bonsall, J. Duncan: *Complete normed algebras.* Springer, 1973.

[3] H. G. Dales: *Banach algebras and automatic continuity.* Oxford University Press, 2000.

[4] K. R. Davidson: *C^*-Algebras by example.* American Mathematical Society, 1996.

[5] W. Donoghue: *Monotone matrix functions and analytic continuation.* Springer, 1974.

[6] I. Gelfand, D. Raikov, G. Shilov: *Commutative normed rings.* Chelsea Publishing Company, 1964.

[7] R. V. Kadison, J. R. Ringrose: *Fundamentals of the theory of operator algebras*, Vol. 1. Academic Press, 1983. American Mathematical Society, 1997.

[8] S. Kametani: An elementary proof of the fundamental theorem of normed fields. J. Math. Soc. Japan **4**, 96-99 (1952).

[9] E. Kaniuth: *A course in commutative Banach algebras.* Springer, 2009.

[10] M. Müger: *Topology for the working mathematician.* Book in progress. `http://www.math.ru.nl/~mueger/topology.pdf`

[11] G. Pedersen: C^**-Algebras and their automorphism groups.* Academic Press, 1979.

[12] C. E. Rickart: An elementary proof of a fundamental theorem in the theory of Banach algebras. Michigan Math. J. **5**, 75-78 (1958).

[13] C. E. Rickart: *General theory of Banach algebras.* Van Nostrand, 1960.

[14] V. S. Sunder: Fuglede's theorem. Ind. J. Pure Appl. Math. **46**, 415-417 (2015).

[15] M. Takesaki: *Theory of operator algebras I.* Springer, 1979.

[16] L. Tornheim: Normed fields over the real and complex fields. Michigan Math. J. **1**, 61-68 (1952).