E 2.4.12. Show that every uniformly convex Banach space is reflexive.

Hint: By assumption there is to each $\varepsilon > 0$ a δ such that $\|\frac{1}{2}(x + y)\| > 1 - \delta$ implies $\|x - y\| < \varepsilon$ for all vectors x, y in $\mathfrak{B}(0, 1)$; cf. E 2.4.9. Take z in $\mathfrak{X}^{**}$ with $\|z\| = 1$ and choose φ in $\mathfrak{X}^*$ with $\|\varphi\| = 1$ and $|\langle z, \varphi\rangle - 1| < \delta$. Put

$$\mathfrak{C} = \{x \in \mathfrak{B}(0, 1) \mid |\langle x, \varphi\rangle - 1| < \delta\}$$

and use E 2.4.5 to show that z belongs to the weak closure of $\mathfrak{C}$. Note that $\|x - y\| < \varepsilon$ for all x and y in $\mathfrak{C}$ and conclude that $\|z - x\| \leq \varepsilon$ for some x in $\mathfrak{C}$.

E 2.4.13. Show that the Banach spaces $L^p(X)$, defined in 2.1.14 (or 2.1.15), are uniformly convex (E 2.4.9) for $p \geq 2$.

Hint: Use the inequality

$$|s + t|^p + |s - t|^p \leq 2^{p-1}(|s|^p + |t|^p),$$

valid for all real numbers s and t.

E 2.4.14. Given Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, such that $\mathfrak{X}$ is reflexive and $T: \mathfrak{Y} \to \mathfrak{X}^*$ is an isometry of $\mathfrak{Y}$ on a w^*-dense subspace of $\mathfrak{X}^*$. Show that $T(\mathfrak{Y}) = \mathfrak{X}^*$, so that also $\mathfrak{Y}$ is reflexive with $\mathfrak{Y}^* = T^*(\mathfrak{X})$.

E 2.4.15. Show that the Banach spaces $L^p(X)$, defined in 2.1.14 (or 2.1.15), are reflexive, and that $L^p(X)^* = L^q(X)$ whenever $p^{-1} + q^{-1} = 1$.

Hint: Take $p \geq 2$, and use the Hölder inequality (6.4.6) to construct an isometry T of $L^q(X)$ into $L^p(X)^*$. Now use E 2.4.13 and E 2.4.12 to see that the assumptions in E 2.4.14 are satisfied.

2.5. w^*-Compactness

Synopsis. Alaoglu's theorem. Krein–Milman's theorem. Examples of extremal sets. Extremal probability measures. Krein–Smulian's theorem. Vector-valued integration. Exercises.

2.5.1. In this section we consider a normed space $\mathfrak{X}$ and its dual space $\mathfrak{X}^*$. We shall be particularly interested in the closed unit ball of $\mathfrak{X}^*$, which we denote by $\mathfrak{B}^*$; but also other convex (compact) sets may occur. Note that if $\mathfrak{X}^*$ is given the w^*-topology (2.4.8), then $\mathfrak{B}^*$ is a w^*-closed subset of $\mathfrak{X}^*$, but in general not a w^*-neighborhood of 0 [unless $\dim(\mathfrak{X}) < \infty$]. The following easy, but fundamental, result is known as *Alaoglu's theorem*.

2.5.2. Theorem. *For each normed space* $\mathfrak{X}$, the unit ball $\mathfrak{B}^*$ *of* $\mathfrak{X}^*$ *is* w^**-compact.*

PROOF. Let $(\varphi_\lambda)_{\lambda \in \Lambda}$ be a universal net in $\mathfrak{B}^*$ (cf. 1.3.7). For every x in $\mathfrak{X}$ we have $|\varphi_\lambda(x)| \leq \|x\|$, so that the image net $(\varphi_\lambda(x))_{\lambda \in \Lambda}$ is contained in a compact subset of $\mathbb{F}$, and thus convergent to a number $\varphi(x)$ by 1.6.2(iv). For all x and y in $\mathfrak{X}$ and α in $\mathbb{F}$ we have, by straightforward computations with limits, that

$$|\varphi(x)| \leq \|x\|, \qquad \varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(\alpha x) = \alpha\varphi(x).$$

It follows that we have constructed a φ in $\mathfrak{B}^*$ such that $\varphi_\lambda \to \varphi$ (in w^*-topology). Thus every universal net is convergent, whence $\mathfrak{B}^*$ is compact by 1.6.2. □

2.5.3. A *face* of a convex subset $\mathfrak{C}$ in a vector space $\mathfrak{X}$ is a nonempty, convex subset $\mathfrak{F}$ of $\mathfrak{C}$ with the property that $\lambda x + (1-\lambda)y \in \mathfrak{F}$ implies $x \in \mathfrak{F}$ and $y \in \mathfrak{F}$, for all x and y in $\mathfrak{C}$ and $0 < \lambda < 1$.

An *extreme point* in $\mathfrak{C}$ is a one-point face, i.e., a point in $\mathfrak{C}$ that cannot be expressed as a nontrivial convex combination of elements from $\mathfrak{C}$. The *extremal boundary* of $\mathfrak{C}$ is the set of extreme points in $\mathfrak{C}$, denoted by $\partial\mathfrak{C}$ (and not to be confused with the equisymbolized topological boundary).

We are primarily interested in proving the *Krein–Milman theorem* (2.5.4) for convex, w^*-compact subsets in the dual space $\mathfrak{X}^*$ of a normed space $\mathfrak{X}$. However, it is convenient to have the more general result at our disposal.

2.5.4. Theorem. *Consider a vector space $\mathfrak{X}$ equipped with the weak topology induced by a separating space $\mathfrak{X}^*$ of functionals on $\mathfrak{X}$. Then for each convex, compact subset $\mathfrak{C}$ of $\mathfrak{X}$, the convex hull of the extremal boundary $\partial\mathfrak{C}$ of $\mathfrak{C}$ is dense in $\mathfrak{C}$.*

PROOF. Take a closed face $\mathfrak{C}_0$ of $\mathfrak{C}$, and consider the set Λ of closed faces of $\mathfrak{C}_0$. From the definition of a face in 2.5.3, we see that faces of $\mathfrak{C}_0$ are also faces of $\mathfrak{C}$. We order Λ under reverse inclusion, and claim that the order is inductive. Indeed, if $\{\mathfrak{F}_j \mid j \in J\}$ is a totally ordered subset of faces of $\mathfrak{C}_0$, then $\bigcap \mathfrak{F}_j \neq \emptyset$ because $\mathfrak{C}$ is compact, and clearly $\bigcap \mathfrak{F}_j$ is a face in Λ majorizing every $\mathfrak{F}_j$. By Zorn's lemma (1.1.3) we conclude that $\mathfrak{C}_0$ contains a minimal face $\mathfrak{F}$.

Take φ in $\mathfrak{X}^*$ and put

$$s = \inf\{\operatorname{Re}\langle x, \varphi\rangle \mid x \in \mathfrak{F}\}.$$

Since the function $x \to \operatorname{Re}\langle x, \varphi\rangle$ is weakly continuous, it attains its minimal value s on $\mathfrak{F}$ [use e.g. 1.6.2(v)], so that

$$\mathfrak{F}_\varphi = \{x \in \mathfrak{F} \mid \operatorname{Re}\langle x, \varphi\rangle = s\} \neq \emptyset. \tag{$*$}$$

Evidently $\mathfrak{F}_\varphi$ is a face of $\mathfrak{F}$, hence of $\mathfrak{C}_0$, and since $\mathfrak{F}$ is minimal, we conclude that $\mathfrak{F}_\varphi = \mathfrak{F}$. Since $\mathfrak{X}^*$ separates points in $\mathfrak{X}$, this implies that $\mathfrak{F}$ must be a one-point set, i.e., $\mathfrak{F}$ is an extreme point. We have thus shown that $\partial\mathfrak{C} \cap \mathfrak{C}_0 \neq \emptyset$ for every closed face $\mathfrak{C}_0$ of $\mathfrak{C}$.

Now let $\operatorname{conv}\{\partial\mathfrak{C}\}$ denote the convex hull of $\partial\mathfrak{C}$, i.e. the set of convex combinations of points from $\partial\mathfrak{C}$. Then $\operatorname{conv}\{\partial\mathfrak{C}\}$ is a convex subset of $\mathfrak{C}$, and its closure $\mathfrak{B}$ is therefore also convex, since the vector operations are continuous. If $x \in \mathfrak{C}\setminus\mathfrak{B}$, there is an open, convex neighborhood $\mathfrak{A}$ of x disjoint from $\mathfrak{B}$, cf. 2.4.2. Applying 2.4.7 we find φ in $\mathfrak{X}^*$ and t in $\mathbb{R}$, such that

$$\operatorname{Re}\varphi(x) \in \operatorname{Re}\varphi(\mathfrak{A}) < t \leq \operatorname{Re}\varphi(\mathfrak{B}).$$

Thus $s < t$, where s denotes the minimum of $\operatorname{Re}\varphi$ on $\mathfrak{C}$, and the face $\mathfrak{F}_\varphi$ defined as in (*) above (with $\mathfrak{F}$ replaced by $\mathfrak{C}$) satisfies $\mathfrak{F}_\varphi \cap \mathfrak{B} = \emptyset$. In particular, $\mathfrak{F}_\varphi \cap \partial\mathfrak{C} = \emptyset$, which contradicts the first result in the proof. Consequently, $\mathfrak{B} = \mathfrak{C}$, as desired. □

2.5.5. The following strategy for the attack on a problem concerning a convex, compact set is often successful: First, find the extreme points of the set. These points are often simpler to handle, so that the problem can be solved for them. Now if the solution is stable under the formation of convex combinations, and stable under limits (i.e. continuous), then the Krein–Milman theorem asserts that the solution is valid on the whole set.

In order to use the strategy outlined above, it is necessary to have a catalogue of extremal boundaries for the most common convex sets, which are often unit balls in dual spaces. The catalogue follows. Except for the last, most important, item, the proofs are left to the reader.

2.5.6. Catalogue. (a) Consider $C(X)$, where X is a compact Hausdorff space. The unit ball in $C(X)$ under ∞-norm is not compact. Even so, the ball is often well supplied with extreme points. These are the functions f in $C(X)$, such that $|f(x)| = 1$ for every x in X. If $\mathbb{F} = \mathbb{R}$ and X is connected, there are only two extreme points. However, if $\mathbb{F} = \mathbb{C}$, the convex hull of the extreme points (the unitary functions) is uniformly dense in the ball.

(b) Consider $L^1(X)$, where $X \subset \mathbb{R}^n$ (cf. 2.1.14). The unit ball is not compact, and there are no extreme points.

(c) Consider $L^p(X)$ for $1 < p < \infty$. Since $L^p(X) = (L^q(X))^*$ if $p^{-1} + q^{-1} = 1$ by 6.5.11, we know from 2.4.2 that the unit ball is w^*-compact. In this case, however, the extreme boundary coincides with the topological boundary, so that every unit vector is an extreme point. This corresponds to the geometrical fact that p-norms give "uniformly round" balls with no edges.

(d) Consider $L^\infty(X) = (L^1(X))^*$ (cf. 6.5.11). The extreme points in the unit ball are the functions f such that $|f(x)| = 1$ for (almost) all x in X.

(e) Consider the convex set of monotone increasing functions $f\colon [0, 1] \to [0, 1]$, which is compact in the topology of pointwise convergence. The extreme points are those functions that only take the values 0 and 1.

(f) Consider the convex set of holomorphic functions f on an open subset Ω of $\mathbb{C}$, such that $\|f\|_\infty \leq 1$. The extreme points are the functions $f(z) = \alpha(z - z_0)^{-1}$, where $z_0 \notin \Omega$ and $|\alpha| = d(z_0, \Omega)$. The strategy in 2.5.5 is capitalized in the Cauchy integral formula.

(g) Consider $\mathbf{M}_n$—the $n \times n$-matrices over $\mathbb{F}$. Note that $\mathbf{M}_n^* = \mathbf{M}_n$, because $\dim(\mathbf{M}_n) = n^2$. Identifying $\mathbf{M}_n$ with $\mathbf{B}(\mathbb{F}^n)$, we obtain an operator norm on $\mathbf{M}_n$ corresponding to the 2-norm on $\mathbb{F}^n$. The extreme points in the unit ball of $\mathbf{M}_n$ are the isometries on $\mathbb{F}^n$. For $\mathbb{F} = \mathbb{R}$ these are the orthogonal matrices, for $\mathbb{F} = \mathbb{C}$ the unitary matrices.